# INTRODUCTION TO UNIVERSALITY

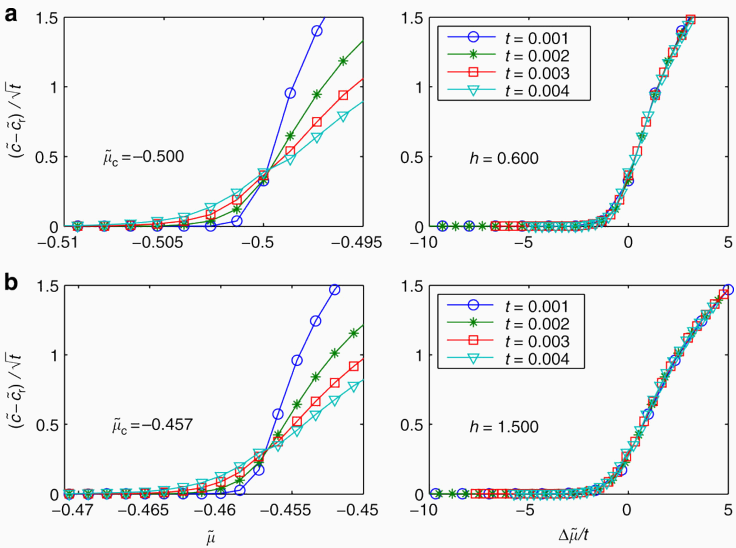
## Phase Transitions in Random Graphs

Alberto Pezzotta

July 3, 2015

# UNIVERSALITY
# IN PHASE TRANSITIONS

# AN EXAMPLE FROM COLD ATOMS PHYSICS




(Chen et al., “Critical behaviours of contact near phase transitions”, 2014)

# RANDOM GRAPHS

Newman et al., “Random graphs as models of networks”, 2002

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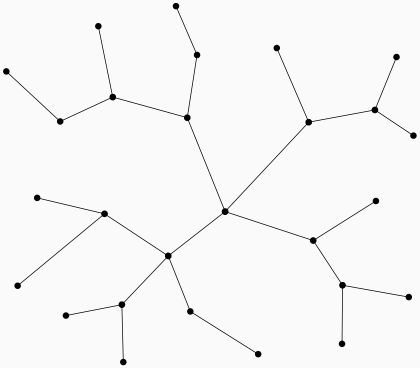
- Model real networks:
  - possibility of tailoring the random graph;
  - sacrifice some features (e.g., clustering coefficient);
  - interpolating between random graphs and networks (e.g., WS and NW small world)

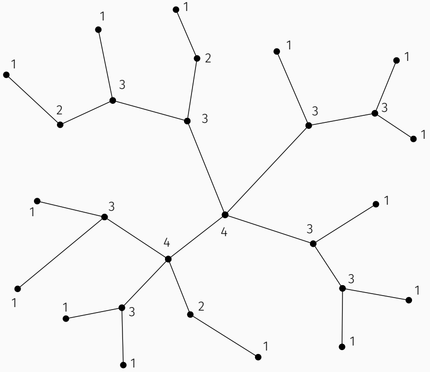
# WHY RANDOM GRAPHS?

Newman et al., “Random graphs as models of networks”, 2002

- Model real networks:
  - possibility of tailoring the random graph;
  - sacrifice some features (e.g., clustering coefficient);
  - interpolating between random graphs and networks (e.g., WS and NW small world)
- Make connection with statistical mechanics:
  - universal properties?
  - phase transitions $\rightsquigarrow$ giant component



Newman et al., 2001



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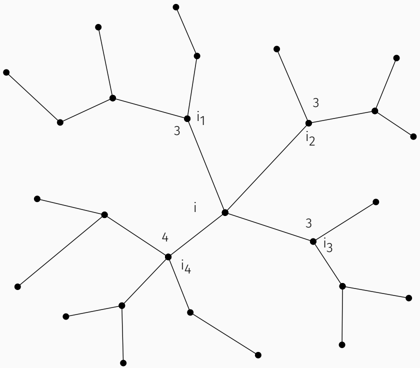
$p_k$ = degree dist.

$g_0(z)$ = generating function of $p_k = \sum_k p_k z^k$



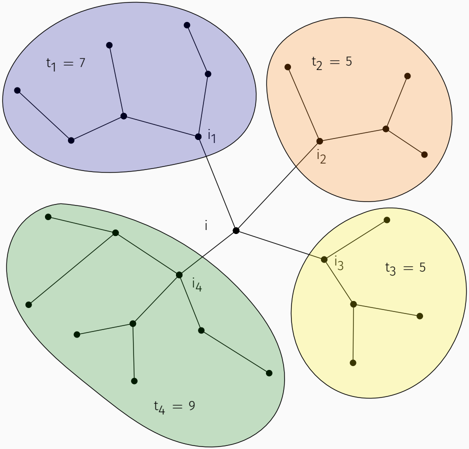

Newman et al., 2001

$p_k \rightsquigarrow g_0(z)$

$q_k$ = excess degree dist = $(k+1)p_{k+1}/\langle k \rangle_p$

$g_1(z)$ = generating function of $q_k$ = $\sum_k q_k z^k$

# DEGREE DISTRIBUTION AND ...

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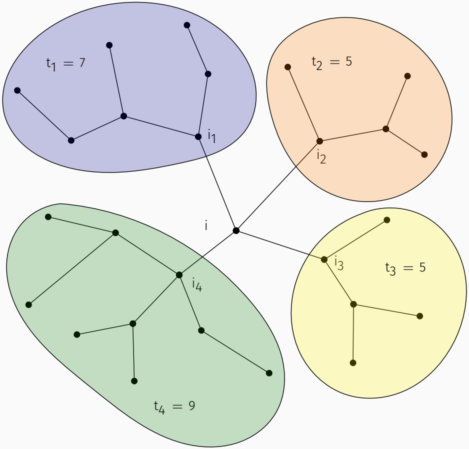
$$p_k \quad \rightsquigarrow \quad g_0(z)$$

$$q_k \quad \rightsquigarrow \quad g_1(z) = \frac{g_0'(z)}{g_0'(1)}$$

$\rho_t$ = branch size dist

$h_1(z)$ = generating function of $\rho_t$

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$$\pi_s = \text{component size dist} = \sum_k P(s|k) p_k$$

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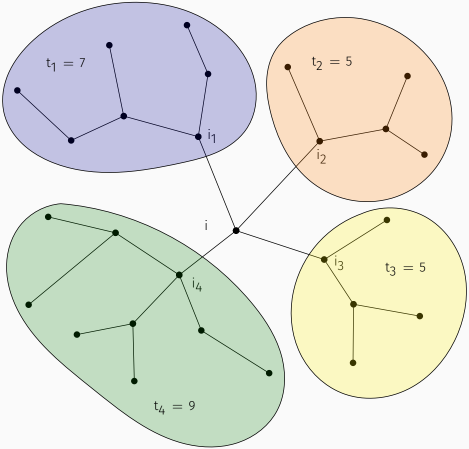


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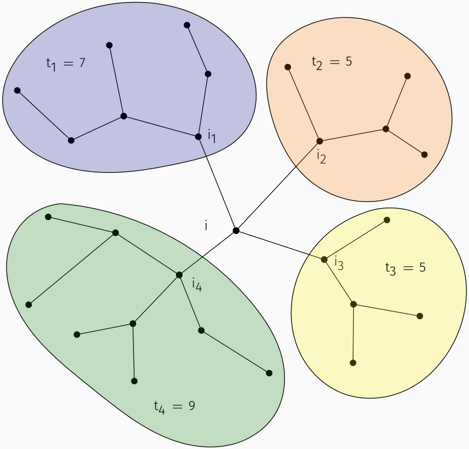


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TREE assumption

$$P(s|k) = \sum_{\{t_k\}} \delta(s-1, \sum_{m=1}^{k} t_m) \prod_{m=1}^{k} \rho_{t_m}$$

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$$\begin{cases} h_1(z) = z\, g_1(h_1(z)) \\ h_0(z) = z\, g_0(h_1(z)) \end{cases} \Rightarrow \begin{cases} h_1(1) = u = g_1(u) \\ h_0(1) = g_0(u) = 1 - S \end{cases}$$

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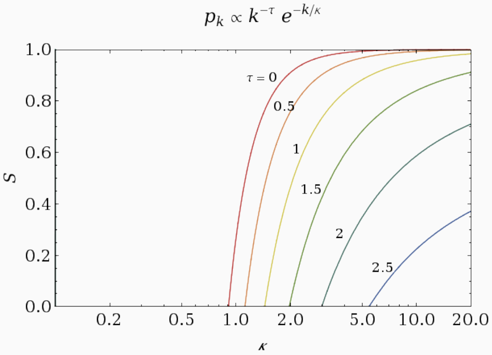
Critical point at $\kappa$ such that

$$g_1'(1) = 1 = \langle k \rangle_q$$



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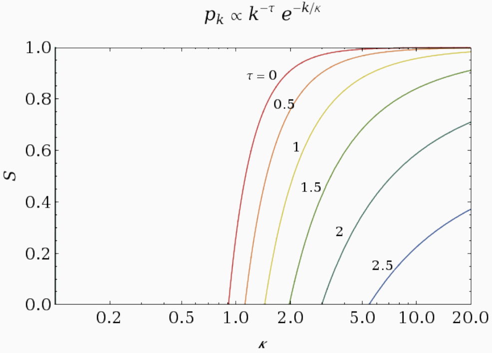


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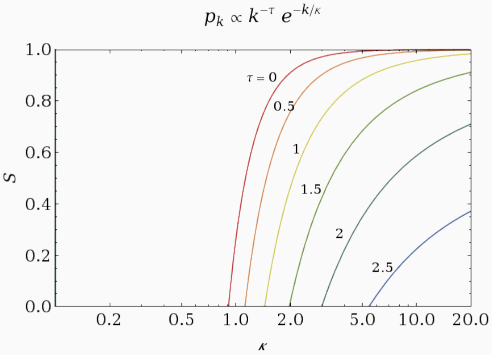


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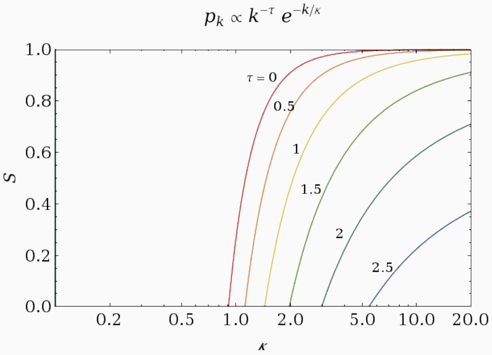


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- $\kappa \to \infty \equiv$ **power law** $\rightsquigarrow$ **always** giant component
- **linear increase** at criticality $\rightsquigarrow$ connection with **percolation** (??)

# A WELL UNDERSTOOD EXAMPLE

**Erdös–Renyi** random graph $G_{n,p}$ (Erdös and Rényi, 1960):

- $n$ vertices $\rightsquigarrow n(n-1)/2$ possible edges (links);
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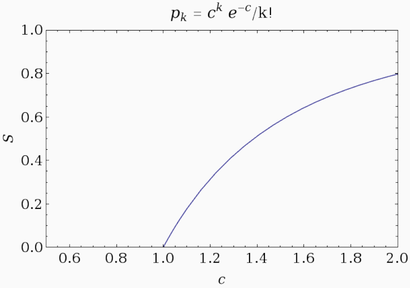

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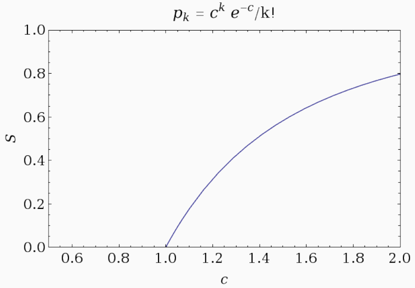


**Known properties**:

- **equivalency** to **Potts** model (at the level of **Large Deviations**, Engel et al., 2004);
- ph. transition in the universality class of **percolation**, $(q \to 1)$-Potts

# ERDÖS–RENYI & POTTS MODEL

$\varphi(G)$ = # conn. comps in $G/N$; $c$ = average degree

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Large deviations: rate and cumulant generating functions

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$$Z_N(q; T = 1/c) \simeq e^{Nc/2}\, Y_N(q; c) \qquad \text{or} \qquad f(q; 1/c) \simeq -\frac{1}{2} - \frac{1}{c} y(q; c)$$

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In terms of fractions $x(\sigma;\{\sigma_i\}) = \frac{1}{N}\sum_i \delta_{\sigma\sigma_i}$:

$$f(\{x(\sigma)\}_{\sigma=0}^{q-1};1/c) = -\sum_{\sigma=0}^{q-1}\left\{\frac{1}{2}x(\sigma)^2 - \frac{1}{c}x(\sigma)\ln x(\sigma)\right\}$$

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Equilibrium solution encoding **symmetry breaking**:

$$x(0) = [1 + (q-1)s]/q\ ; \quad x(\sigma > 0) = (1 - s)/q$$

Solutions in the $q \to 1$ limit:

$$\begin{aligned} c \leq 1 &: s^* = 0; \\ c > 1 &: s^* = 1 - e^{-cs^*} \rightsquigarrow \text{Erdös–Renyi equation} \end{aligned}$$

# OPEN PROBLEMS

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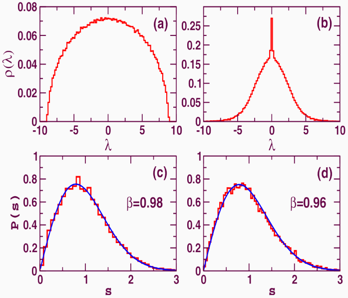
- Connection with Random Matrix Theory

- Connection with **Random Matrix Theory**, spectra of
  - **adjacency** matrix (Bandyopadhyay and Jalan, 2007; Kühn, 2008; Zhan et al., 2010)



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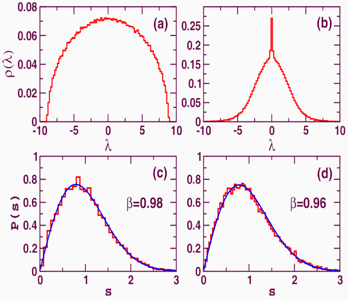

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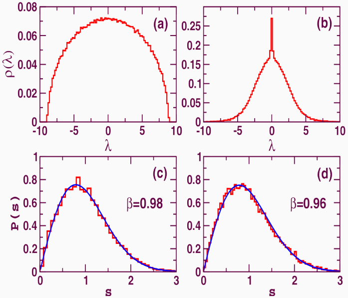
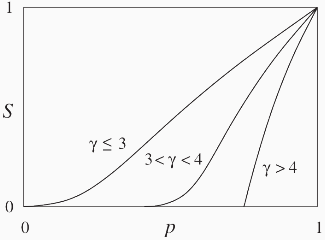



- Universality of phase transitions in generic degree distribution random graphs (Dorogovtsev et al., 2008; Cohen et al., 2002).



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