Final Project for Financial Securities and Markets

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1 Introduction

We would like to buy a contract paying at maturity T the amount in USD:

$$max\{0, (\frac{S(T)}{S(0)} - k) \cdot (k' - \frac{Libor(T-\Delta, T-\Delta, T)}{Libor(0, T-\Delta, T)})\} \tag{1}$$

- $S(t)$ is the Nikkei-225 spot price quantoed from JPY into USD
- $L(t, T-\Delta, T)$ is the 3-month USD LIBOR rate between $T-\Delta$ and T
- Δ is a period of 3 months (0.25 years)
- T is the expiration date: 3 years
- k, k' given relative strike prices

The objective is to create a pricing function that can calculate the price of a given contract based on its terms, such as T, k, and k', as well as market data such as volatility and correlation between the underlying assets. We can price the contract by calculating or simulating the values of $S(t)$ and the three-month USD Libor. We can use Geometric Brownian Motion Model for stock price and the Hull-White model for the three-month USD LIBOR. We would then use these simulations to calculate the expected payout at maturity. This pricing methodology can be further refined using techniques such as Monte Carlo simulation.

2 Assumptions

2.1 Underlying assumptions for Hull-White Model

- **Mean-Reverting Process**: The assumption of a mean-reverting process is motivated by the observation that interest rates tend to fluctuate around certain equilibrium levels. The assumption makes sense because it captures the tendency of interest rates to revert to a long-term average over time, reflecting market forces that drive rates towards equilibrium.

- **Deterministic Mean Reversion Level ($\Theta(t)$)**: Assuming a deterministic mean reversion level allows for calibration of the model to match observed bond prices in the market. This choice is based on the belief that the term structure of interest rates contains valuable information about future interest rate expectations. The assumption of a functional form for the mean reversion level allows for the incorporation of this information.

- **Speed of Mean Reversion (a)**: The parameter a represents the speed at which the short rate reverts back to its mean level. The choice of a is based on empirical evidence and market observations of interest rate dynamics. It reflects the belief that interest rates adjust to changes in market conditions at different speeds, with some rates reverting faster or slower than others.

- **Volatility of the Short Rate (σ)**: The assumption of short rate volatility captures the random fluctuations and uncertainty in interest rate movements. It is based on the understanding that interest rates are influenced by various economic factors, market events, and investor sentiment. The inclusion of volatility allows the model to capture the realistic behavior of interest rates and the associated risks.

- **Geometric Brownian Motion ($dW^{Q^d}(t)$)**: The assumption of a Geometric Brownian motion for the short rate differential reflects the widely used stochastic process in finance. It is based on the belief that financial variables, including interest rates, exhibit continuous-time dynamics and can be modeled using this mathematical framework. The Geometric Brownian motion incorporates randomness and is consistent with the efficient pricing and risk management techniques used in the financial industry.

- **No Jump Diffusion**: The exclusion of jump diffusion in the Hull-White model simplifies its mathematical formulation. It assumes that the short rate changes smoothly over time and does not experience sudden, discontinuous jumps. This assumption is based on the belief that interest rate movements can be adequately captured by the mean reversion and volatility components and that extreme, discontinuous events occur less frequently or can be approximated by continuous processes.

- **Risk-Neutral Measure**: The assumption of a risk-neutral measure, such as Q^d, simplifies pricing and valuation by assuming that investors are risk-neutral. It implies that investors do not require compensation for taking on risk, and all future uncertainties are accounted for in the dynamics of the short rate and its volatility.

- **Stationarity and Flexibility**: The Hull-White model assumes that the statistical properties of the short rate process are constant, but the mean reversion level can be time-dependent in order to accurately represent the observed term structure of interest rates.

- **No-Arbitrage**: The no-arbitrage assumption ensures that the Hull-White model is consistent with observed market prices. By calibrating the model to the current term structure of interest rates, it accurately replicates the prices of bonds and fixed-income securities. This assumption enables valuation and risk management of fixed-income securities and interest rate derivatives while maintaining consistency with market prices.

In summary, the Hull-White model is well-suited for pricing the LIBOR rate due to its ability to capture mean reversion, incorporate volatility, calibrate to market prices, and provide a framework for risk management. It allows market participants to accurately value LIBOR-based instruments and make informed decisions in the dynamic interest rate environment.

2.2 Underlying Assumption for Geometric Brownian Motion Model

We use the Geometric Brownian Motion (GBM) model to simulate the price dynamics of the stock price (NIKKEI-225 index) and the exchange rate (JPY to USD).The underlying assumptions behind the model include:

- **Continuous Time**: Assuming continuous time allows for a more mathematically tractable model. By assuming continuous time, the model implies that the underlying variable, such as asset prices or exchange rates, follows a continuous random walk. This assumption allows for the incorporation of both long-term trends and short-term fluctuations, capturing the idea that prices evolve smoothly.

- **Log-Normal Distribution**: GBM assumes that the logarithm of asset prices adheres to a normal distribution, and this choice is primarily motivated by the desirable statistical properties it offers in financial asset modeling. By using a log-normal distribution, GBM ensures that prices cannot be negative, which aligns with the nature of financial assets. Additionally, the log-normal distribution allows for the representation of substantial positive returns or movements that are frequently observed in financial markets. Under GBM, the logarithm of the variable being modeled is assumed to follow a Brownian motion characterized by a constant drift and volatility. This framework provides a mathematically tractable and widely employed model for simulating the dynamics of asset prices.

- **Independent and Identically Distributed (IID) Increments**: GBM assumes that the increments or changes in asset prices are independent of each other and identically distributed. This means that the future price changes depend solely on the current price and are not influenced by past price movements. This assumption simplifies the modeling process and facilitates analytical solutions.

- **Absence of Jumps**: GBM assumes the absence of sudden, discontinuous price movements or jumps. Instead, it assumes a smooth continuous path for asset prices. While jumps can occur in real markets, GBM disregards them to maintain simplicity and tractability.

- **No Transaction Costs or Taxes**: The model assumes the absence of transaction costs or taxes associated with buying or selling assets. This assumption simplifies the analysis by ignoring the impact of these factors on the model's outcomes.
- **No Arbitrage**: The assumption of no arbitrage opportunities implies that there are no opportunities for risk-free profits by exploiting price discrepancies. This assumption is fundamental in financial theory and is essential for pricing projects.
- **Risk Neutral Assumption**: Under the risk-neutral measure, the expected return on the asset is the risk-free interest rate. This assumption allows for the use of risk-neutral probabilities in pricing the NIKKEI-225 and exchange rate based on the GBM model. By transforming the probabilities used in the model to reflect the risk-neutral measure, the pricing process is simplified. This assumption ensures that the GBM model can be calibrated to produce prices that are consistent with no-arbitrage conditions.

GBM provides a relatively simple and tractable framework for capturing the dynamics of stock prices, as it allows for the representation of random fluctuations and long-term growth trends observed in equity markets. Furthermore, for liquid markets like the Nikkei-225 with continuous trading and no transaction costs, GBM can provide a reasonable approximation of price behavior.

3 Methodology

3.1 Hull-White Model for 3-month LIBOR rate

The Hull-White model assumes that the short rate follows a mean-reverting process, which is calibrated to the current term structure of interest rates. We aim to use the Hull-White model to simulate the interest rate in the U.S. Consider the Hull-White model under domestic risk-neutral measure Q^d We can write the differential equation for the short rate $r(t)$ as:

$$\mathrm{d}r(t) = [\theta(t) - ar(t)]\mathrm{d}t + \sigma \mathrm{d}W^{Q^d}(t) \tag{2}$$

- $\theta(t)$ is the mean reversion level of the short rate at time t. It is a deterministic function of time chosen in order to fit the theoretical bond prices $p(0,T); T > 0$ to the observed curve $p^*(0,T); T > 0$.
- a is the speed of mean reversion of the short rate
- σ is the volatility of the short rate
- $dW^{Q^d}(t)$ represents the differential of a Wiener process under the domestic risk-neutral measure Q^d.

3.1.1 $\Theta(t)$ and forward rate $f(t,T)$

By affine term structure, bond price

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)} \tag{3}$$

Let A and B be deterministic functions given by:

$$A(t,T) = \int_t^T \frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)ds$$

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

Now we want to fit the theoretical prices above to the observed prices and it is convenient to do this using the forward rates. Since there is a one-to-one correspondence between forward rates and bond prices, we may just as well fit the theoretical forward rate curve $\{f(0,T); T > 0\}$ to the observed curve $f^*(t,T) = -\frac{\partial \log p^*(t,T)}{\partial T}$.

By forward rates of affine model,

$$f(0,T) = B_T(0,T)r(0) - A_T(0,T) \tag{4}$$

$$f^*(0,T) = f(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\Theta(s)ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 \tag{5}$$

To solve (5) we write it as:

$$f^*(0,T) = x(T) - g(T)$$

$$\dot{x} = -ax(t) + \theta(t), x(0) = r(0)$$

$$g(t) = \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 = \frac{\sigma^2}{2}B^2(0,t)$$

We now have

$$\begin{aligned}\theta(T) &= \dot{x}(T) + ax(T) = f^*(0,T) + \dot{g}(T) + ax(T) \\ &= f_T^*(0,T) + \dot{g}(T) + a\{f^*(0,T) + g(T)\}\end{aligned} \tag{6}$$

3.1.2 $dW^{Q^d}(t)$

Since $dW^{Q^d}(t)$ represents the differential of a Wiener process under the domestic risk-neutral measure Q^d, Now we can simulate it by Cholesky decomposition:

$$dW^{Q^d}(t) = \rho_{sr}dW + \sqrt{1-\rho_{sr}^2}dW^{\perp} \tag{7}$$

where ρ_{sr} is the correlation between the short rate and the Nikkei-225 spot price, and dW and $dW^{\perp}$ are independent Brownian motions.

3.1.3 Zero-Coupon Bond price $p(t, T)$ and LIBOR rate $L(t, T - \Delta, T)$

By choosing Θ according to (6), we have, for a fixed choice of a and σ, determined our martingale measure, as it produces a term structure $\{p(0, T); T > 0\}$ such that $p(0, T) = p^*(0, T)$ for all $T > 0$. Now we would like to compute the theoretical bond prices under this martingale measure.

By The Hull-White term structure, we can compute

$$p(t, T) = \frac{p^\star(0, T)}{p^\star(0, t)} \exp\{B(t, T)f^\star(0, t) - \frac{\sigma^2}{4a} B^2(t, T)(1 - e^{-2at}) - B(t, T)r(t)\} \quad (8)$$

where $p^\star(0, T)$ and $p^\star(0, t)$ represent the risk-neutral price or risk-neutral value of a zero-coupon bond with maturities T and t, respectively. $f^*(0, t)$, the forward rate, which is the current expected future interest rate at time t based on historical market data. can be estimated using Monte Carlo simulations based on historical market data.

To calculate the contract payoff (1), we must find the 3-month USD LIBOR $L(t, T - \Delta, T)$. $L(0, T - \Delta, T)$ represents the observed 3-month USD LIBOR rate between $T - \Delta$ and T at time 0. L$(T - \Delta, T - \Delta, T)$ represents the observed 3-month USD LIBOR rate between $T - \Delta$ and T at time $T - \Delta$.

By the definition of the LIBOR rate,

$$L(t, S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(S, T)}$$

$$L(T - \Delta, T - \Delta, T) = \frac{p(T - \Delta, T) - 1}{\Delta \cdot p(T - \Delta, T)} \quad (9)$$

$$L(0, T - \Delta, T) = \frac{p(0, T) - p(0, T - \Delta)}{\Delta \cdot p(0, T)} \quad (10)$$

3.2 Geometric Brownian Motion for $S(t)$

Geometric Brownian Motion (GBM) is widely used to model equity prices such as the Nikkei-225 spot price, represented as $S(t)$. GBM is a stochastic process that assumes the natural logarithm of the asset price follows a Brownian motion with drift and volatility. This framework helps capture the random fluctuations and long-term growth trends observed in equity markets.

We make the assumption that both the stock price and the exchange rate are always positive, implying that the movements of the NIKKEI-225 index and the exchange rate follow the geometric Brownian motion. Let's use the symbol X_t to represent the exchange rate from JPY to USD at time t, where USD is the domestic currency and JPY is the foreign currency, and we define it as follows:

$$X = \frac{units\,of\,USD}{unit\,of\,JPY}$$

The dynamics of the exchange rate is

$$dX = X(r_d - r_f)dt + X\sigma_X dV^{Q^d} \tag{11}$$

where r_d is the domestic risk-free interest rate, r_f is the foreign risk-free interest rate, and σ_X is the volatility of the spot exchange rate. V^{Q^d} is a Wiener process under the domestic risk-neutral measure Q^d.

Now we model the stock price $S(t)$ in foreign currency using Wiener process under the foreign risk-neutral measure Q^f.

$$dS(t) = S(t)(r_f - q)dt + S(t)\sigma_S dW^{Q^f} \tag{12}$$

- $S(t)$ is NIKKEI-225 index price at time t in JPY price
- r_f is the risk-free rate in the foreign currency (JPY)
- q is the dividend yield of NIKKEI-225 index
- σ_S is the volatility of the Nikkei-225 spot price
- W^{Q^f} is a Brownian motion under the foreign risk-neutral measure Q^f, representing random fluctuations in the stock price.

Let $\tilde{S}(t)$ be the domestic price of the foreign stock, and $\tilde{S} = S \cdot X$. Let W^{Q^d} is another Brownian motion under the domestic risk-neutral measure Q^d, where σ_X is the volatility of the USD to NIKKEI-225 exchange rate, and ρ_{SX} is the correlation between the NIKKEI-225 spot price and the exchange rate.

$$d\tilde{S}(t) = \tilde{S}(t)(r_d - q)dt + \tilde{S}(t)\sqrt{\sigma_S^2 + \sigma_X^2 + 2\rho_{SX}\sigma_S\sigma_X}\,dW^{Q^d}$$

Since the equation for the Geometric Brownian Motion (GBM) dynamics of the stock price is

$$dS(t) = uS(t)dt + \sigma_S S(t)dW(t)$$

We can apply Itô's lemma and solve the stochastic differential equation to obtain the solution:

$$S(t) = S(0)\exp\{\left(u - \frac{\sigma_S^2}{2}\right)t + \sigma_S\sqrt{t}\epsilon\}$$

where ϵ follows a standard normal distribution with mean zero and variance one. Then,

$$(u - \frac{\sigma_S^2}{2}) + (r_d - r_f) - \frac{\sigma_X^2}{2} = (r_d - q) - \frac{1}{2}(\sigma_S^2 + \sigma_X^2 + 2\rho_{SX}\sigma_S\sigma_X)$$

We have $u = r_f - q - \rho_{SX}\sigma_S\sigma_X$. Therefore, the price dynamics satisfy

$$d\tilde{S}(t) = \tilde{S}(t)\left(r_f - q - \rho_{SX}\sigma_S\sigma_X\right)dt + \tilde{S}(t)\sigma_S dW^{Q^d} \tag{13}$$

We can simulate the price by:

$$\tilde{S}(t) = \tilde{S}(0) \cdot \exp\left(\left(r_f - q - \rho_{SX}\sigma_S\sigma_X - \frac{\sigma_S^2}{2}\right)t + \sigma_S\sqrt{t}\epsilon\right)$$

$$\tilde{S}(t+\Delta t) - \tilde{S}(t) = \tilde{S}(t) \cdot \left(r_f - q - \rho_{SX}\sigma_S\sigma_X - \frac{\sigma_S^2}{2}\right)\Delta t + \sigma_S\tilde{S}(t)\sqrt{\Delta t}\epsilon$$

where ϵ is follows a standard normal distribution with mean zero and variance one.

By Risk Neutral Valuation Formula, and discounting process is path-dependent and operates independently, we can deduce the following:

$$\Pi(T) = max\{0, (\frac{S(T)}{S(0)} - k) \cdot (k' - \frac{Libor(T-\Delta, T-\Delta, T)}{Libor(0, T-\Delta, T)})\}$$

$$V(0) = E^Q[e^{-\sum_{i=0}^{n-1} r_i \Delta t}\Pi(T)] \tag{14}$$

- $V(0)$ represents the present value of the derivative at time 0.
- Δt represents the time increment or interval.
- n is the number of intervals such that $n \cdot \Delta t = T$.
- E^Q denotes the expectation operator under the risk-neutral measure.
- r_i is the short rate simulated at time $(i+1) \cdot \Delta t$.

4 Calibration

4.1 Market data for the Hull-White model calibration

- Nikkei-225 stock price data: Nikkei-225 historical stock price data is used to estimate the correlation and volatility of the short rate and is incorporated into the simulation process. The data is from Yahoo Finance.
- 3-month LIBOR historical data: This data represents the observed 3-month LIBOR rates from iborate and is used in equations (9) and (10) to calculate the contract payoff involving the LIBOR rate.
- USD to JPY historicalv exchange rate data: This data is used to estimate the exchange rate volatility and correlation, which can be relevant in the Hull-White model simulation. We use data from Yahoo Finance.

- US Treasury-bond Yield Curve: The yield curve of US Treasury bonds with different maturities is required in equation (8) to compute the theoretical bond prices under the Hull-White martingale measure.

- Stock Dividend Rate: This rate is set as the 5-year average dividend rate and is used in the calculation procedure. We get data according to seeking Aplpha as 0.0183.

- Risk-free Interest Rate in Japan: This rate is used in calculating the LIBOR rate and relative quantoed equity. We get the data from Seek Alpha as 0.00505.

The provided data sets are used in the pricing routine as inputs, and there are global variables that have fixed values within the pricing routine. However, it's important to note that some of these fixed values may change over time. Therefore, when using the provided pricing routine, it is recommended to implement the updated values for the global variables to ensure accurate calculations.

The time series data plays a crucial role in the pricing routine, specifically in calculating the correlation between the underlying assets and their individual volatility. As market conditions and data change over time, it is essential to regularly update the time series data to reflect the most recent market information.

By incorporating the updated values for global variables and utilizing current time series data, the pricing routine will provide more accurate and relevant results for pricing calculations. This ensures that the pricing routine aligns with the latest market conditions and reflects any changes in the fixed values of the global variables.

5 Simulation

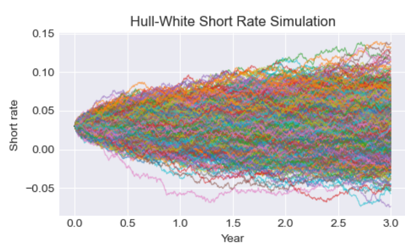
The Python implementation uses Monte Carlo simulation to obtain both the quantoed equity price and LIBOR. First, I did data extraction and correlation calculation using Python. The pricing routine in the Monte Carlo simulation primarily utilizes the Quantlib library in Python, with code iterations based on the work by Iliya Valchanov in Data Science Plus. The provided code routine for the Hull-White model also refers to the Hull-White Model by Mansoor Ahmed, with adjustments made to fit the specific problem. The following sections provide the specific Python implementation details for the methodology. The attached file contains detailed code implementation.

In the Hull-White model, it is necessary to calculate the correlation between the underlying assets and their individual volatilities. During each iteration of the simulation, two random numbers are generated from a normal distribution, using historical stock-exchange rate correlation data. The price simulation utilizes a one-step Monte Carlo approach to determine the prices of the quantoed equity and LIBOR. The simulation is performed with a step size of one month. Following the simulation, the prices of the LIBOR rate and quantoed equity undergo a discounting process.

Then, the simulation of stock prices uses the Geometric Brownian Motion (GBM) model. The simulation is based on the assumption that stock price changes follow a log-normal distribution. By incorporating parameters such as the initial stock price, risk-free rate, dividend yield, volatility, time horizon, and the number of Monte Carlo paths, the simulation generates a matrix of stock prices representing a range of possible future outcomes. The simulated stock prices provide insights into potential stock price paths and can be used for risk management, option pricing, and portfolio optimization. However, it is important to acknowledge the limitations of the GBM model and consider more advanced models for more accurate stock price simulations.

Next, the simulated values are used to calculate LIBOR rates and with the simulated results, the final payoff for the contract is determined and then discounted using the interest rate set in the previous section. Finally, the average of all simulated paths in the Monte Carlo simulation is taken to yield the resulting price for the contract.

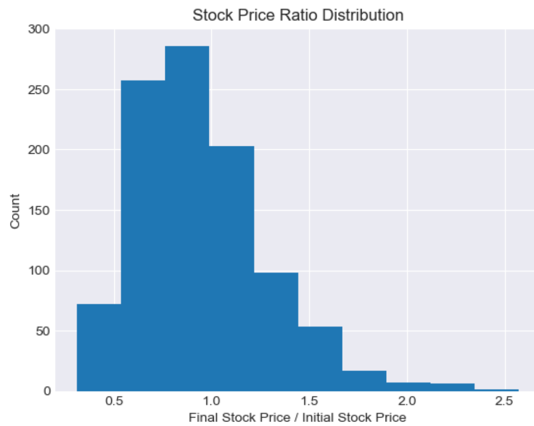
Using the calibration methods described above, we can obtain the calibrated parameters for the Hull-White model. In our model, we assume that the calibration parameters are known in advance, with a value of 0.04 for parameter a and σ is equal to 0.02. The result of the Hull-White short-rate simulation, and Stock Price simulation, conducted over a period of 3 years on a daily basis, is reflected below.



(a) Short Rate Simulation

(b) Stock Price Simulation

Additionally, we visualize the ST/S0 from 1000 simulations, and the average payoff is shown below.



(a) ST/S0

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Contract Average Price: 0.12923407405641155
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(b) Average Contract Price

In summary, the simulation process in the model involves generating short-rate prices that represent LIBOR values. These prices are then discounted and averaged to obtain the present value of the LIBOR rate. The Hull-White model offers more flexibility compared to the Vasicek model by allowing adjustments in the pricing simulation through modifications in maturity. Both the quantoed equity and LIBOR are evaluated under the domestic risk-neutral measure. The Hull-White model's closed-form and analytic routine make it a sound choice, although the calibration process may require additional effort.

6 References

[1] Tomas Bjork. Arbitrage theory in continuous time. Oxford university press, 2009.