An odd thing about partitions

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Partitions

A *partition* of n is a way of writing n as a sum of positive integers, where the ordering of the integers is irrelevant

Partitions of 3

$$3$$

$$2 + 1$$

$$1 + 1 + 1$$

Partitions of 4

Partitions of 4

$$4$$

Partitions of 4

$$4$$

$$3+1$$

Partitions of 4

$$4$$

$$3+1$$

$$2+2$$

Partitions of 4

$$4$$

$$3+1$$

$$2+2$$

$$2+1+1$$

Partitions of 4

$$4$$

$$3 + 1$$

$$2 + 2$$

$$2 + 1 + 1$$

$$1 + 1 + 1 + 1$$

$p(n) :=$ number of partitions of n

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$$p(3) = 3$$

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$$p(4) = 5$$

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$$p(3) = 3$$

$$p(4) = 5$$

$$p(\pi) = 0$$

$p(n) :=$ number of partitions of n

$$p(3) = 3$$

$$p(4) = 5$$

$$p(\pi) = 0$$

$$p(0) = 1$$

n	$p(n)$
0	1
1	1
2	2
3	3
4	5

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	?

Partitions of 5

Partitions of 5

$$5$$

Partitions of 5

$$5$$

$$4+1$$

Partitions of 5

$$5$$

$$4+1$$

$$3+2$$

Partitions of 5

$$5$$

$$4+1$$

$$3+2$$

$$3+1+1$$

Partitions of 5

$$5$$

$$4+1$$

$$3+2$$

$$3+1+1$$

$$2+2+1$$

Partitions of 5

$$5$$

$$4+1$$

$$3+2$$

$$3+1+1$$

$$2+2+1$$

$$2+1+1+1$$

Partitions of 5

$$5$$

$$4+1$$

$$3+2$$

$$3+1+1$$

$$2+2+1$$

$$2+1+1+1$$

$$1+1+1+1+1$$

n	$p(n)$
0	1
1	1
2	2
3	3
4	5
5	7

n	$p(n)$
10	42
20	627
30	5604
40	37338
50	204226

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$$

$p(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$
$p(5)$	$p(6)$	$p(7)$	$p(8)$	$p(9)$
$p(10)$	$p(11)$	$p(12)$	$p(13)$	$p(14)$
$p(15)$	$p(16)$	$p(17)$	$p(18)$	$p(19)$
$p(20)$	$p(21)$	$p(22)$	$p(23)$	$p(24)$
$p(25)$	$p(26)$	$p(27)$	$p(28)$	$p(29)$
$p(30)$	$p(31)$	$p(32)$	$p(33)$	$p(34)$
$p(35)$	$p(36)$	$p(37)$	$p(38)$	$p(39)$
$p(40)$	$p(41)$	$p(42)$	$p(43)$	$p(44)$
$p(45)$	$p(46)$	$p(47)$	$p(48)$	$p(49)$

1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310
14883	17977	21637	26015	31185
37338	44583	53174	63261	75175
89134	105558	124754	147273	173525

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7	11	15	22	30
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$$p(4), p(9), p(14), p(19), \ldots, p(49)$$

are all divisible by 5

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are all divisible by 5

Is $p(5n+4)$ divisible by 5 for every $n \geq 0$?

Ramanujan congruences

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For every $n \geq 0$,

$$5 \mid p(5n+4),$$

Ramanujan congruences

For every $n \geq 0$,

$$5 \mid p(5n+4),$$

$$7 \mid p(7n+5),$$

Ramanujan congruences

For every $n \geq 0$,

$$5 \mid p(5n+4),$$

$$7 \mid p(7n+5),$$

$$11 \mid p(11n+6).$$

Other congruences for $p(n)$?

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(Atkin-O'Brien)

For every $n \geq 0$,

$$13 \mid p(157525693n + 111247)$$

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(Atkin-O'Brien)

For every $n \geq 0$,

$$13 \mid p(157525693n + 111247)$$

($p(111247)$ is a number with well over 300 digits)

(K. Ono)

For every prime $m \geq 5$, there exist positive integers A and B such that for every $n \geq 0$,

$$m \mid p(An + B).$$

Generating functions

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$$p(0) \quad p(1) \quad p(2) \quad p(3) \quad p(4) \quad \cdots$$

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$$p(0) \quad p(1) \quad p(2) \quad p(3) \quad p(4) \quad \cdots$$

$$p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \cdots$$

Generating functions

$$p(0) \quad p(1) \quad p(2) \quad p(3) \quad p(4) \quad \cdots$$

$$p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \cdots$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \ldots$$

Generating functions

$$p(0) \quad p(1) \quad p(2) \quad p(3) \quad p(4) \quad \cdots$$

$$p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \cdots$$

$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \ldots$$

$$= \sum_{n=0}^{\infty} p(n)x^n$$

Modularity

$$x := e^{2\pi i z}$$

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$$x := e^{2\pi i z}$$

$$\rho(z) := \sum_{n=0}^{\infty} p(n) x^{24n-1}$$

$$\rho(-1/z) = \sqrt{\frac{i}{z}}\,\rho(z)$$

$\rho(z)$ is a weight $-1/2$ modular form

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Ramanujan: $p(n)$ is divisible by 5 (7,11) at least $\frac{1}{5}$ $(\frac{1}{7}, \frac{1}{11})$ of the time

Atkin-O'Brien: $p(n)$ is divisible by 13 at least $\frac{1}{157525693}$ of the time

Ono: For any prime $m \geq 5$, $p(n)$ is divisible by m a positive proportion of the time

What about 2 and 3?

Restricted partition functions

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$$Q(4) = 2$$

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$$Q(4) = 2$$

$$4 \qquad 3+1 \qquad 2+2$$

$$2+1+1 \qquad 1+1+1+1$$

n	$Q(n)$
0	1
1	1
2	1
3	2
4	2
5	3

n	$Q(n)$
10	10
20	64
30	296
40	1113
50	3658

When is $Q(n)$ odd?

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n	$Q(n)$
0	1
1	1
2	1
3	2
4	2
5	3
6	4
7	5
8	6
9	8
10	10

When is $Q(n)$ odd?

n	$Q(n)$
0	1
1	1
2	1
3	2
4	2
5	3
6	4
7	5
8	6
9	8
10	10

When is $Q(n)$ odd?

n	$Q(n)$	$24n+1$
0	1	1
1	1	25
2	1	49
3	2	73
4	2	97
5	3	121
6	4	145
7	5	169
8	6	193
9	8	217
10	10	241

$Q(n)$ is odd

$$\iff$$

$24n + 1$ is a square

$Q(n)$ is odd

$$\Longleftrightarrow$$

$24n + 1$ is a square

$$\Longleftrightarrow$$

$$n \in \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, \ldots\}$$

Why?

$Q(n)$ is odd

$$\iff$$

$24n+1$ is a square

$$\iff$$

$$n = \frac{k(3k+1)}{2} \qquad (k \in \mathbb{Z})$$

$Q(n)$ is even 100% of the time

ℓ-regular partitions

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We call a partition ℓ-*regular* if none of its summands is divisible by ℓ

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$b_\ell(n) :=$ number of ℓ-regular partitions of n

$$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1$$

$$2+2+1 \qquad 2+1+1+1 \qquad 1+1+1+1+1$$

$$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1$$

$$2+2+1 \qquad 2+1+1+1 \qquad 1+1+1+1+1$$

$$b_2(5) = 3$$

$$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1$$

$$2+2+1 \qquad 2+1+1+1 \qquad 1+1+1+1+1$$

$$b_3(5) = 5$$

$b_5(4)$

$$b_5(4) = p(4)$$

$b_5(4) = p(4)$

$b_5(9)$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - ?$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(14) = p(14) - ?$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(14) = p(14) - p(9) - ?$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(14) = p(14) - p(9) - p(4)$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(14) = p(14) - p(9) - p(4)$$

$$b_5(19) = p(19) - p(14) - p(9) - p(4) + ?$$

$$b_5(4) = p(4)$$

$$b_5(9) = p(9) - p(4)$$

$$b_5(14) = p(14) - p(9) - p(4)$$

$$b_5(19) = p(19) - p(14) - p(9) - p(4) + p(4)$$

For every $n \geq 0$,

$$5 \mid b_5(5n+4).$$

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$b_5(n)$ is divisible by 5 at least $\frac{1}{5}$ of the time

Some curious results

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(P.)

$b_{11}(n)$ is divisible by 5 at least $4/5$ of the time

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(P.)

$b_{11}(n)$ is divisible by 5 at least $4/5$ of the time

$b_{23}(n)$ is divisible by 11 at least $10/11$ of the time

When is $b_5(n)$ odd?

When is $b_5(n)$ odd?

$b_5(n)$ is odd

$$\iff$$

$$n \in \{0, 1, 3, 4, 7, 8, 9, 15, 17, 19, 20, 21, 23, 27, 28, 29, \ldots\}$$

When is $b_5(n)$ odd?

$$b_5(n) \text{ is odd}$$

$$\Longleftrightarrow$$

$$n \in \{0, 1, 3, 4, 7, 8, 9, 15, 17, 19, 20, 21, 23, 27, 28, 29, \ldots\}$$

When is $b_5(2n)$ odd?

When is $b_5(2n)$ odd?

$$b_5(2n) \text{ is odd}$$

$$\iff$$

$$2n \in \{0, 4, 8, 20, 28, 48, 60, 88, \ldots\}$$

When is $b_5(2n)$ odd?

$$b_5(2n) \text{ is odd}$$

$$\iff$$

$$2n \in \{0, 4, 8, 20, 28, 48, 60, 88, \ldots\}$$

$$\iff$$

$$n/2 \in \{0, 1, 2, 5, 7, 12, 15, 22, \ldots\}$$

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

$b_5(2n)$ is odd

$$\Longleftrightarrow$$

$$n/2 = \frac{k(3k+1)}{2} \qquad (k \in \mathbb{Z})$$

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

$b_5(2n)$ is odd

$$\Longleftrightarrow$$

$$2n = 2k(3k+1) \qquad (k \in \mathbb{Z})$$

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

$b_5(2n)$ is odd

$$\Longleftrightarrow$$

$$2n = 2k(3k+1) \qquad (k \in \mathbb{Z})$$

$$\Longleftrightarrow$$

$12n + 1$ is a square

Combinatorial proof?

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

For every $n \geq 0$,

$$2 \mid b_5(20n + 5),$$

$$2 \mid b_5(20n + 13).$$

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

For every $n \geq 0$,

$$2 \mid b_5(20n+5),$$

$$2 \mid b_5(20n+13).$$

$b_5(n)$ is even at least 60% of the time

When is $b_{13}(2n)$ odd?

$$b_{13}(2n) \text{ is odd}$$

$$\iff$$

$$2n \in \{0, 4, 6, 12, 24, 40, 58, 60, 84, 112, 144, 162, 180, \ldots\}$$

When is $b_{13}(2n)$ odd?

$b_{13}(2n)$ is odd

$$\iff$$

$$2n \in \{0, 4, 12, 24, 40, 60, 84, 112, 144, \ldots\}$$

$$\cup \{6, 58, 162, 318, \ldots\}$$

When is $b_{13}(2n)$ odd?

$b_{13}(2n)$ is odd

$$\iff$$

$$n/2 \in \{0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots\}$$

or $\quad n - 6 \in \{0, 52, 156, 312, \ldots\}$

When is $b_{13}(2n)$ odd?

$b_{13}(2n)$ is odd

$$\iff$$

$$n/2 \in \{0, 1, 3, 6, 10, 15, 21, 28, 36, \ldots\}$$

or $\quad \dfrac{n-6}{52} \in \{0, 1, 3, 6, \ldots\}$

(N. Calkin, N. Drake, K. James, S. Law, P., J. Radder)

$b_{13}(2n)$ is odd

$$\Longleftrightarrow$$

$2n = 2k(k+1)$ or $26k(k+1)+6$ ($k \in \mathbb{Z}_{\geq 0}$)