Partition Numbers

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Outline

Find the pattern (Christmas trees?)

1	2	3	5	7	11
15	22	30	42	56	77
101	135	176	231	297	385
490	627	792	1002	1255	1575
1958	2436	3010	3718	4565	5604
6842	8349	10143	12310	14883	17977
21637	26015	31185	37338	44583	53174
63261	75175	89134	105558	124754	147273
173525	204226	239943	281589	329931	386155
451276	526823	614154	715220	831820	966467
1121505	1300156	1505499	1741630	2012558	2323520
2679689	3087735	3554345	4087968	4697205	5392783
6185689	7089500	8118264	9289091	10619863	12132164
13848650	15796476	18004327	20506255	23338469	26543660
30167357	34262962	38887673	44108109	49995925	56634173
64112359	72533807	82010177	92669720	104651419	118114304
133230930	150198136	169229875	190569292	214481126	241265379
271248950	304801365	342325709	384276336	431149389	483502844
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Definition

Partition number of $n \in \mathbb{Z}^+$, written as $p(n)$, is a way of writing n as sums of positive integers.

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Example

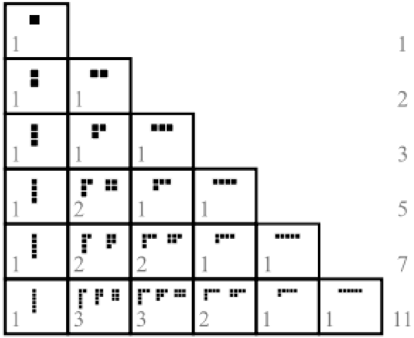
Take example of $n = 4$:

- 4
- 1 + 3
- 2 + 2
- 1 + 1 + 2
- 1 + 1 + 1 + 1

So $p(4) = 5$.

Some observations

- $p(n) \geq 1$.
- Strictly increasing.
- $p(n)$ is hard to compute for a given n.



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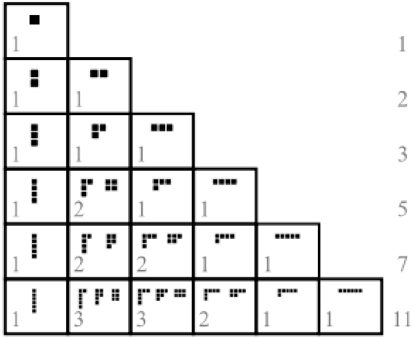

- Not too crazy to think that $p(n)$ can be related to $p(n-1)$, $p(n-2)$, ... $p(1)$. i.e. recursive formula!
- But it is not clear how we should add up these numbers to get $p(n)$.

Obvious observations

- Leonhard Euler: obviously

$$\sum_{n=1}^{\infty} p(n) \cdot x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}.$$

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- Leonhard Euler: again, obviously
$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots .$$
These numbers are pentagonal numbers, $g_k = \frac{1}{2}k(3k-1)$
`https://en.wikipedia.org/wiki/File:Pentagonal_number.gif`

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- Srinivasa Ramanujan: obviously

$$p\,(5n+4) \equiv 0 \pmod{5}.$$

Obviously not obvious results

1. How should we explore the structure of this sequence?
2. Why and how does the pentagonal numbers come up in the study of partition numbers?
3. Sufficient conditions for divisibility of $p(n)$.
4. An unified theorem that explains all of the above.

Outline

GF of partition numbers

What is the coefficient of x^4 in following expansion?

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)(1+x^7)\ldots$$

- Equal to the number of ways of getting x^4, since all coefficients are 1.

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- Equal to the number of ways of getting x^4, since all coefficients are 1.
- Consider each bracket, and choose a single term from each.
- Anything beyond $(1+x^4)$ will be irrelevant, so:
  - $1, 1, 1, x^4$
  - $x, 1, x^3, 1$
- Thus coefficient of x^4 is 2

GF of partition numbers

What is the coefficient of x^4 in following expansion?

$$\begin{aligned}
&(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\dots \\
=&(1+x^1+x^{1+1})(1+x^2+x^{2+2})(1+x^3+x^{3+3})(1+x^4+x^{4+4})\dots
\end{aligned}$$

GF of partition numbers

What is the coefficient of x^4 in following expansion?

$$=(1+x^1+x^{1+1})(1+x^2+x^{2+2})(1+x^3+x^{3+3})(1+x^{\color{red}4}+x^{4+4})\dots$$

- $x^{\color{red}4}$
- x^3x^1
- x^{2+2}
- x^2x^{1+1}

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- x^4
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- Each of these terms directly correspond to a partition of 4, with the **restriction** each part appears at most twice.

GF of partition numbers

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$$(1+x+x^2+\dots+x^m)(1+x^2+x^4+\dots+x^{2m})(1+x^3+x^6+\dots+x^{3m})(1+x^4+x^8+\dots+x^{4m})\dots$$

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- Using the same idea, each of these terms directly correspond to a partition of 4, with the **restriction** each part appears at most m times.
- Coefficient of x^n is equal to partition of n, with the **restriction** each part appears at most m times.

Let $m \to \infty$, we would remove this restriction and obtain the unrestricted partition number, $p(n)$ as the coefficient of x^n

GF of partition numbers

$$\begin{aligned}\sum_{n=1}^{\infty} p(n)\cdot x^n &= (1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots,\\ &= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\dots,\\ &= \prod_{k=1}^{\infty}\frac{1}{1-x^k}.\end{aligned}$$

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\end{aligned}
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- Euler is happy
- GF often allow extra insights into sequences
- This is not quite as satisfying, how about this?

$$
1 = \left(\sum_{n=1}^{\infty} p(n) \cdot x^n \right) \times \left(\prod_{k=1}^{\infty} 1 - x^k \right) \tag{1}
$$

Outline

How to break down the infinite product

- 1 hour lecture of contents.
- Euler's Pentagonal Number Theorem breaks down this product as:

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- The proof of EPNT is very easy... Provided that you can prove something stronger
- Jacobi's Triple Product Identity

$$\prod_{k=1}^{\infty} \left(1 + xq^k\right)\left(1 + x^{-1}q^{k-1}\right)\left(1 - q^k\right) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n+1)} x^n. \tag{3}$$

“Proof” of EPNT

Assume JTPI is true:

$$\prod_{k=1}^{\infty}\left(1+xq^{k}\right)\left(1+x^{-1}q^{k-1}\right)\left(1-q^{k}\right)=\sum_{n=-\infty}^{\infty}q^{\frac{1}{2}n(n+1)}x^{n}$$

Substituting $q \to q^3$ and $x \to -q^{-1}$

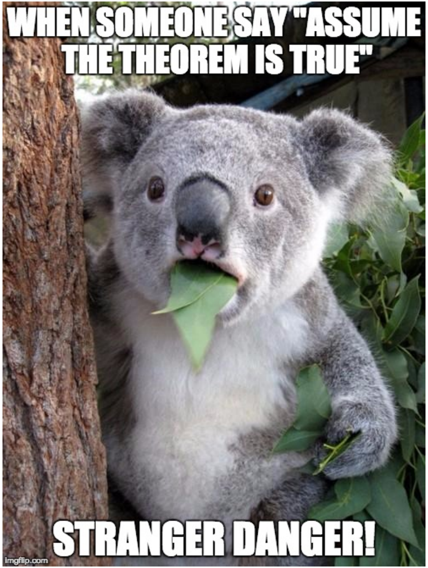
$$\prod_{k=1}^{\infty}(1-q^{3k-2})(1-q^{3k-1})(1-q^{3k})=\sum_{n=-\infty}^{\infty}(-1)^{n}\cdot q^{\frac{1}{2}n(3n-1)}.$$

But the LHS has consecutive exponents,

$$\prod_{k=1}^{\infty}\left(1-q^{k}\right)=\sum_{n=-\infty}^{\infty}(-1)^{n}\cdot q^{\frac{1}{2}n(3n-1)}.$$

Done!

Alert!


WHEN SOMEONE SAY "ASSUME THE THEOREM IS TRUE"
STRANGER DANGER!
imgflip.com

Outline of proof for JTPI

Jacobi's Triple Product Identity

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We will first define:

$$f(x):=\prod_{i=1}^{\infty}\left(1+xq^{i}\right)\times\prod_{j=1}^{\infty}\left(1+x^{-1}q^{j-1}\right). \tag{4}$$

The proof then follows this simple structure:

1. Collapsing $f(x)$ into

$$a_0(q)\times \text{RHS}$$

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3. Realise the formation of a_0 require cancellation of powers of q, which forms a special partition of m
4. Create a pictorial bijection between $p(n)$ and b_m for all n,m
5. Thus, $f(x)$ can be expressed as two infinite product **as well as** the power series of $p(n)$ (inverse of the third infinite product on LHS) and RHS.

Outline

Deriving the recurrence formula

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \sum_{n=1}^{\infty} p(n) \cdot q^n$$

$$1 = \left(\sum_{n=1}^{\infty} p(n) \cdot q^n \right) \times \left(\prod_{k=1}^{\infty} 1 - q^k \right)$$

$$1 = \left(\sum_{n=1}^{\infty} p(n) \cdot q^n \right) \times \left(\sum_{n=-\infty}^{\infty} (-1)^n \cdot q^{\frac{1}{2}n(3n-1)} \right) \qquad \text{(by applying EPNT)}$$

$$1 = \left(\sum_{n=1}^{\infty} p(n) \cdot q^n \right) \times \left(1 + \sum_{m=1}^{\infty} (-1)^m \left[q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}} \right] \right).$$

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But multiplying power series is easy!

$$\left(\sum_{n=0}^{\infty} a_n q^n \right) \left(\sum_{m=0}^{\infty} b_m q^m \right) = \sum_{t=0}^{\infty} \left(\sum_{i=0}^{t} a_{t-i} b_i \right) q^t.$$

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By multiplication of power series, and equating the coefficient for $q^t, t > 0$:

$$0 = p\left(t\right) + \sum_{m \geq 1} \left(-1\right)^m \cdot \left\{ p\left(t - \frac{m(3m-1)}{2} \right) + p\left(t - \frac{m(3m+1)}{2} \right) \right\}.$$

This equation becomes well-defined if we set $p(t) = 0$ for all $t < 0$.

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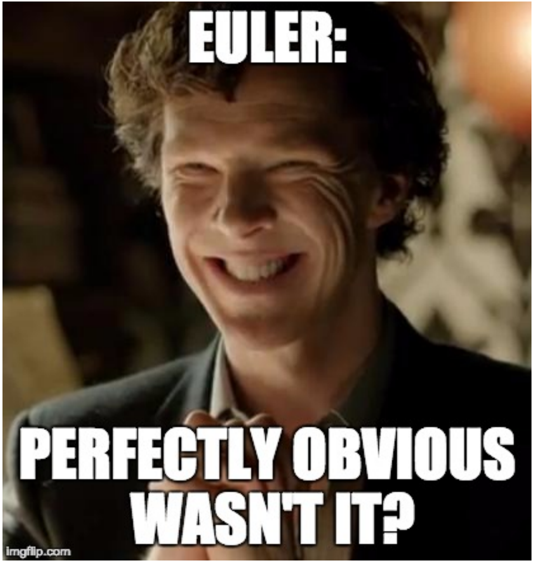
This equation becomes well-defined if we set $p(t) = 0$ for all $t < 0$. E.g. Consider $t = 6$,

$$\begin{aligned} p(6) &= \{p(6-1) + p(6-2)\} - \{p(6-5) + p(6-7)\} \\ &= p(5) + p(4) - p(1) - p(-1) \\ &= 7 + 5 - 1 - 0 \\ &= 11 \end{aligned}$$

Sure... That was obvious...



Sketch of proof for Ramanujan's congruences

- Start with JTPI, make specific substitutions like we did with EPNT
- The coefficient of x^{5m+5} in $x\left[\frac{(1-x^5)(1-x^{10})\ldots}{(1-x)(1-x^2)\ldots}\right]$ is divisible by 5 by modular arithmetic

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\begin{aligned}
x \times \prod_{k=1}^{\infty} \frac{1}{1-x^k} &= \frac{x}{(1-x)(1-x^2)\dots} \\
&= x\left[\frac{(1-x^5)(1-x^{10})\dots}{(1-x)(1-x^2)\dots}\right] \times \frac{1}{(1-x^5)}\frac{1}{(1-x^{10})}\dots \\
&= x\left[\frac{(1-x^5)(1-x^{10})\dots}{(1-x)(1-x^2)\dots}\right] \times \left(1+x^5+x^{10}+\dots\right)\left(1+x^{10}+x^{20}+\dots\right)\dots
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&= x\left[\frac{(1-x^5)(1-x^{10})\ldots}{(1-x)(1-x^2)\ldots}\right] \times \left(1+x^5+x^{10}+\ldots\right)\left(1+x^{10}+x^{20}+\ldots\right)\ldots
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$$x \times \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \frac{x}{(1-x)(1-x^2)\dots}$$

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$$= x\left[\frac{(1-x^5)(1-x^{10})\dots}{(1-x)(1-x^2)\dots}\right] \times \left(1+x^5+x^{10}+\dots\right)\left(1+x^{10}+x^{20}+\dots\right)\dots$$

Sketch of proof for Ramanujan's congruences

- Start with JTPI, make specific substitutions like we did with EPNT
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- Equating the power of x gives $5m+5 = n+1 \implies n = 5m+4$. So, $p\,(5n+4) \equiv 0 \pmod 5$

Final remarks

Theorem (Ramanujan's Congruences)

For $n \in \mathbb{N}$

$$\begin{aligned} p\left(5n+4\right) &\equiv 0 \pmod{5} \\ p\left(7n+5\right) &\equiv 0 \pmod{7} \\ p\left(11n+6\right) &\equiv 0 \pmod{11} \end{aligned}$$

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- $p(11^3 \cdot 13 \cdot k + 237) \equiv 0 \pmod{13}$

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$$p\,(5n+4) \equiv 0 \pmod{5}$$
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- $p(11^3 \cdot 13 \cdot k + 237) \equiv 0 \pmod{13}$
- In 2001, Ken Ono proved such congruence relation exist for every prime greater than 3
- In 2006, Ono further proved if N is a integer coprime to 6, then there are integers a and b such that:
$$p(am+b) \equiv 0 \pmod{N}. \tag{5}$$
- Partition numbers and their approximations can be used in quantum physics.

Outline

Outline of proof for JTPI

Jacobi's Triple Product Identity

$$\prod_{k=1}^{\infty} \left(1 + xq^{k}\right) \left(1 + x^{-1}q^{k-1}\right) \left(1 - q^{k}\right) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n+1)} x^{n}.$$

We will first define:

$$f(x) := \prod_{i=1}^{\infty} \left(1 + xq^{i}\right) \times \prod_{j=1}^{\infty} \left(1 + x^{-1}q^{j-1}\right). \tag{6}$$

The proof then follows this simple structure

1. Collapsing $f(x)$ into just one term
2. Staring at that one term very hard and apply the previous counting exponent idea

Symmetry of $f(x)$

$$
\begin{aligned}
f(x) &= \prod_{i=1}^{\infty} \left(1 + xq^{i}\right) \times \prod_{j=1}^{\infty} \left(1 + x^{-1} q^{j-1}\right) \\
&= (1 + xq)(1 + xq^{2})(1 + xq^{3}) \ldots \\
&\times (1 + x^{-1})(1 + x^{-1} q)(1 + x^{-1} q^{2}) \ldots
\end{aligned}
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\end{aligned}
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Via substitution,

$$
\begin{aligned}
f(xq) = {} & (1 + xq^{2})(1 + xq^{3})(1 + xq^{4}) \dots \\
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Via substitution,

$$
\begin{aligned}
f(xq) &= (1+xq^2)(1+xq^3)(1+xq^4)\dots \\
&\times \color{red}{(1+x^{-1}q^{-1})}(1+x^{-1})(1+x^{-1}q)\dots \\
&= \color{red}{(1+x^{-1}q^{-1})\frac{(1+xq)}{(1+xq)}} \cdot (1+xq^2)(1+xq^3)(1+xq^4)\dots \\
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&= (1 + x^{-1}q^{-1})\frac{(1 + xq)}{(1 + xq)} \cdot (1 + xq^2)(1 + xq^3)(1 + xq^4)\dots \\
&\times (1 + x^{-1})(1 + x^{-1}q)\dots \\
&= \frac{(1 + x^{-1}q^{-1})}{(1 + xq)} \cdot f(x)
\end{aligned} \tag{7}
$$

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$f(x)$ as a formal Laurent series

What is $f(x)$ if not just a bunch of x^n with coefficients at the front?

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \cdot x^n.$$

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Equating coefficient of x^{n-1}, we have:

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Equating coefficient of x^{n-1}, we have:

$$a_n = q^n a_{n-1}.$$

JTPI step 1 proven

$$\begin{aligned}
a_n &= q^n a_{n-1} \\
&= q^n \cdot \left(q^{n-1} a_{n-2}\right) \\
&= q^n q^{n-1} \cdot \left(q^{n-2} a_{n-3}\right) \\
&\vdots \\
&= q^{n+(n-1)+(n-2)+\cdots+1} \cdot a_0 \\
&= q^{\frac{1}{2}n(n+1)} a_0
\end{aligned}$$

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\end{aligned}
$$

Thus,

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} a_n \cdot x^n \\
&= a_0 \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n+1)} x^n .
\end{aligned}
$$

Sketch of Step 2

- a_0 is simply the coefficient of x^0 in the expansion of

$$\begin{aligned} f(x) &= \prod_{i=1}^{\infty}\left(1 + xq^i\right) \times \prod_{j=1}^{\infty}\left(1 + x^{-1}q^{j-1}\right) \\ &= (1+xq)(1+xq^2)(1+xq^3)(1+xq^4)\dots \\ &\times (1+x^{-1})(1+x^{-1}q)(1+x^{-1}q^2)(1+x^{-1}q^3)\dots \end{aligned}$$

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- As long as we have an **equal number** of terms from the first and second infinite product, we are guaranteed the cancellation of powers of x
- A power series in q will capture different selection of terms:

$$a_0 = \sum_{m=0}^{\infty} b_m \cdot q^m. \tag{8}$$

Sketch of Step 2

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f(x) &= \prod_{i=1}^{\infty}\left(1+xq^{i}\right)\times\prod_{j=1}^{\infty}\left(1+x^{-1}q^{j-1}\right)\\
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\end{aligned}
$$

Consider $m=4$. Keep in mind we need an **equal number** of terms from the first and second product in $f(x)$, the only possible ways to obtain $x^0\cdot q^4$ are:

- $(1+xq)\times(1+x^{-1}q^3)$
- $(1+xq^2)\times(1+x^{-1}q^2)$
- $(1+xq^3)\times(1+x^{-1}q)$
- $(1+xq^4)\times(1+x^{-1})$
- $(1+xq)(1+xq^2)\times(1+x^{-1})(1+x^{-1}q)$.

A special partition

In particular, the number m is partitioned into the sum of distinct elements from the positive integers plus an **equal number** of distinct elements from the natural numbers (including 0), i.e.

$$m = (i_1 + i_2 + \cdots + i_n) + (j_1 + j_2 + \cdots + j_n); \tag{9}$$

where: $i_1 < i_2 < \cdots < i_n \in \mathbb{Z}^+$, and $j_1 < j_2 < \cdots < j_n \in \mathbb{N}$.
The number of ways of achieving this partition for a fixed m is the same as $p(m)$.

Ferrers Diagram

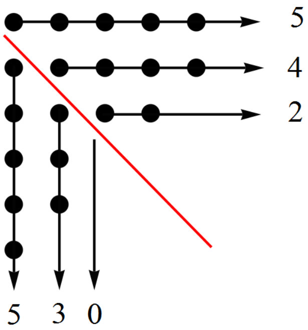

Figure: $19 = 5 + 5 + 4 + 2 + 2 + 1 = (5 + 4 + 2) + (5 + 3 + 0)$

- Strictly decreasing sequence is guaranteed by the diagonal
- Equal number of parts is guaranteed by the use of 0 in the lower half