Bayesian Poisson Regression - Complete Hand Calculations

1. Model Setup

Probabilistic Model

```
y_n | μ_n ~ Poisson(μ_n)
μ_n = exp(f_n)
f_n = w^T x_n = w_0 + w_1 * age_n
w | κ ~ N(0, κ²I)
κ ~ N_+(0, 1) (half-normal)
```

Poisson Distribution

The probability mass function:

```
P(y_n | μ_n) = (μ_n^y_n * e^(-μ_n)) / y_n!
```

Log pmf:

```
log P(y_n | μ_n) = y_n * log(μ_n) - μ_n - log(y_n!)
```

Half-Normal Distribution

For κ ~ N_+(0, 1):

```
p(κ) = √(2/π) * exp(-κ²/2)  for κ ≥ 0
     = 0                     for κ < 0
```

Log pdf:

```
log p(κ) = log(√(2/π)) - κ²/2 = 0.5*log(2/π) - κ²/2
```

2. Joint Distribution

The log joint distribution:

```
log p(y, w, κ) = log p(y | w) + log p(w | κ) + log p(κ)
```

Log Likelihood

```
log p(y | w) = Σ_n log P(y_n | μ_n)
             = Σ_n [y_n * log(μ_n) - μ_n - log(y_n!)]
             = Σ_n [y_n * w^T x_n - exp(w^T x_n) - log(y_n!)]
```

Log Prior on Weights

```
log p(w | κ) = log N(w | 0, κ²I)
             = -D/2 * log(2π) - D * log(κ) - ||w||²/(2κ²)
             = -1 * log(2π) - 2 * log(κ) - (w_0² + w_1²)/(2κ²)
```

Log Hyperprior

```
log p(κ) = 0.5 * log(2/π) - κ²/2
```

Full Log Joint

```
log p(y, w, κ) = Σ_n [y_n * (w_0 + w_1 * x_n) - exp(w_0 + w_1 * x_n) - log(y_n!)]
               - log(2π) - 2 * log(κ) - (w_0² + w_1²)/(2κ²)
               + 0.5 * log(2/π) - κ²/2
```

3. Small Numerical Example

Let's work through a tiny example with 3 data points:

Data

```
age = [35, 40, 45]
x = [-1, 0, 1]  # standardized
deaths = [3, 4, 5]
```

Initial Parameters

```
w = [1.0, 0.5]  # w_0 = 1.0, w_1 = 0.5
κ = 1.0
```

Step 1: Compute f_n = w^T x_n

```
f_0 = 1.0 + 0.5 * (-1) = 0.5
f_1 = 1.0 + 0.5 * (0) = 1.0
f_2 = 1.0 + 0.5 * (1) = 1.5
```

Step 2: Compute μ_n = exp(f_n)

```
μ_0 = exp(0.5) = 1.649
μ_1 = exp(1.0) = 2.718
μ_2 = exp(1.5) = 4.482
```

Step 3: Compute Log Likelihood

```
log L = Σ_n [y_n * log(μ_n) - μ_n - log(y_n!)]

n=0: 3 * log(1.649) - 1.649 - log(3!) = 3 * 0.5 - 1.649 - 1.792 = -1.941
n=1: 4 * log(2.718) - 2.718 - log(4!) = 4 * 1.0 - 2.718 - 3.178 = -1.896
n=2: 5 * log(4.482) - 4.482 - log(5!) = 5 * 1.5 - 4.482 - 4.787 = -1.769

Total: -5.606
```

Step 4: Compute Log Prior

```
log p(w | κ) = -log(2π) - 2 * log(1) - (1² + 0.5²)/(2 * 1²)
             = -1.838 - 0 - 0.625
             = -2.463
```

Step 5: Compute Log Hyperprior

```
log p(κ) = 0.5 * log(2/π) - 1²/2
         = -0.226 - 0.5
         = -0.726
```

Step 6: Total Log Joint

```
log p(y, w, κ) = -5.606 - 2.463 - 0.726 = -8.795
```

4. Metropolis-Hastings Algorithm

Proposal Distribution

Use a multivariate normal proposal:

```
θ' ~ N(θ, σ²I)
```

where θ = [w_0, w_1, κ]

Accept/Reject Step

```
α = min(1, p(θ') / p(θ))
```

Accept θ' with probability α.

Example MH Step

Current: θ = [1.0, 0.5, 1.0] Propose: θ' = [1.1, 0.4, 0.9]

1. Compute log p(θ) = -8.795
2. Compute log p(θ') using same steps
3. Compute acceptance ratio
4. Accept/reject based on uniform random draw

5. Posterior Predictive Distribution

After obtaining posterior samples {w^(s), κ^(s)} for s = 1,...,S:

For new age x*:

```
f*^(s) = w_0^(s) + w_1^(s) * x*
μ*^(s) = exp(f*^(s))
y*^(s) ~ Poisson(μ*^(s))
```

Summary Statistics

```
E[y* | y] ≈ (1/S) Σ_s y*^(s)
Var[y* | y] ≈ (1/S) Σ_s (y*^(s))² - (E[y* | y])²
```

6. Example: Prediction at Age 75

Suppose we have posterior samples:

```
w^(1) = [2.0, 0.3], κ^(1) = 0.8
w^(2) = [1.8, 0.4], κ^(2) = 0.9
w^(3) = [2.1, 0.2], κ^(3) = 0.7
```

For age 75 (standardized x* = 2.5):

Sample 1:

```
f*^(1) = 2.0 + 0.3 * 2.5 = 2.75
μ*^(1) = exp(2.75) = 15.64
y*^(1) ~ Poisson(15.64) → e.g., 17
```

Sample 2:

```
f*^(2) = 1.8 + 0.4 * 2.5 = 2.8
μ*^(2) = exp(2.8) = 16.44
y*^(2) ~ Poisson(16.44) → e.g., 14
```

Sample 3:

```
f*^(3) = 2.1 + 0.2 * 2.5 = 2.6
μ*^(3) = exp(2.6) = 13.46
y*^(3) ~ Poisson(13.46) → e.g., 15
```

Posterior Predictive Summary:

```
E[y* | y] ≈ (17 + 14 + 15)/3 = 15.33
SD[y* | y] ≈ 1.53
```