Hand Calculation Example: Beta-Binomial Model

Problem Setup

We observe **y = 1 success** in **N = 7 trials**. We use a uniform prior ($a_0 = 1$, $b_0 = 1$).

Step 1: Prior Distribution

Prior: Beta(1, 1) = Uniform(0, 1)

- Prior mean: $E[\theta] = a_0/(a_0+b_0) = 1/(1+1) = 0.5$
- Prior variance: $V[\theta] = (a_0 b_0)/[(a_0+b_0)^2(a_0+b_0+1)] = (1\times1)/[(2)^2(3)] = 1/12 \approx 0.083$

Step 2: Likelihood

Likelihood: $p(y=1|\theta) = C(7,1) \times \theta^1 \times (1-\theta)^6 = 7\theta(1-\theta)^6$

Step 3: Posterior Distribution

Using the conjugacy property:

- Posterior: Beta(a_0 + y, b_0 + N - y) = Beta(1 + 1, 1 + 7 - 1) = Beta(2, 7)

Posterior Statistics:

- **Posterior mean**: $E[\theta|y] = 2/(2+7) = 2/9 \approx 0.222$
- **Posterior mode**: $\text{Mode}[\theta|y] = (2-1)/(2+7-2) = 1/7 \approx 0.143$
- **Posterior variance**: $V[\theta|y] = (2\times7)/[(9)^2(10)] = 14/810 \approx 0.017$

For comparison:

- **MLE**: $\hat{\theta}$_MLE = y/N = 1/7 ≈ 0.143
- **95% Credible Interval**: Use Beta(2,7) quantiles
  - Lower: Beta(2,7).ppf(0.025) ≈ 0.028
  - Upper: Beta(2,7).ppf(0.975) ≈ 0.503

Step 4: Posterior Predictive

For N* = 5 new trials, the posterior predictive is Beta-Binomial(5, 2, 7):

$P(y^* = k \mid y) = C(5,k) \times B(k+2, 5-k+7) / B(2,7)$

Where B(a,b) is the Beta function.

Example calculations:

- $P(y^* = 0 \mid y) \approx 0.318$
- $P(y^* = 1 \mid y) \approx 0.409$
- $P(y^* = 2 \mid y) \approx 0.205$
- $E[y^* \mid y] = N^* \times E[\theta|y] = 5 \times (2/9) \approx 1.11$

Step 5: Model Evidence

$p(y) = C(7,1) \times B(1+1, 7-1+1) / B(1,1) = 7 \times B(2,7) / B(1,1) = 7 \times [\Gamma(2)\Gamma(7)/\Gamma(9)] / [\Gamma(1)\Gamma(1)/\Gamma(2)] = 7 \times [1! \times 6!/8!] / [1] = 7 \times (6!)/(8\times7\times6!) = 7 \times 1/(8\times7) = 1/8 = 0.125$

Sequential Update Example

Starting with uniform prior Beta(1,1):

1. Observe ($N_1=5$, $y_1=2$):
   - Update to Beta(1+2, 1+5-2) = Beta(3, 4)
   - Mean: 3/7 ≈ 0.429
2. Observe ($N_2=3$, $y_2=1$):
   - Update to Beta(3+1, 4+3-1) = Beta(4, 6)
   - Mean: 4/10 = 0.400
3. Observe ($N_3=4$, $y_3=3$):
   - Update to Beta(4+3, 6+4-3) = Beta(7, 7)
   - Mean: 7/14 = 0.500

Final posterior: Beta(7, 7)

Verification: Total data is N=12, y=6 Direct calculation: Beta(1+6, 1+12-6) = Beta(7, 7) ✓

Key Insights

1. **Effect of prior**: With small data (N=7, y=1), the uniform prior pulls the posterior mean (0.222) away from the MLE (0.143) toward 0.5.
2. **Shrinkage**: The posterior mean is a weighted average between the prior mean and the MLE, with weights determined by the "strength" of prior and data.
3. **Uncertainty**: The posterior variance (0.017) is much smaller than the prior variance (0.083), showing how data reduces uncertainty.

4. **Conjugacy benefit**: The Beta-Binomial conjugacy allows all calculations to be done analytically without numerical integration.