SDU Summer School

Deep Learning

Summer 2022

Welcome to the Summer School

Linear Algebra

- Fundamental Building Blocks
- Vectors
- Matrices
- Square Matrices
- System of Linear Equations
- Matrices as Linear Transformation

Scalar and Vector

Scalar

- Single number
- Normally: $x \in \mathbb{R}$ or $x \in \mathbb{N}$

Vector

- An array of numbers
  - Arranged in order
  - Each no. identified by an index
- $\boldsymbol{x} = \begin{bmatrix} x_1 \\ \dots \\ x_d \end{bmatrix}$ and $\boldsymbol{x}^T = [x_1, \dots, x_d]$, $\boldsymbol{x} \in \mathbb{R}^d$
- We think of vectors as points in space
  - Each element gives coordinate along an axis

Matrix

- 2-D array of numbers
- Each element identified by two indices
- Denoted by bold typeface $\boldsymbol{A}$
- Elements indicated as $A_{m,n}$

$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

- $A_{i:}$ is ith row of $\boldsymbol{A}$, $A_{:j}$ is jth column
- $\boldsymbol{A}$ has m rows and n columns, then $A \in \mathbb{R}^{m \times n}$

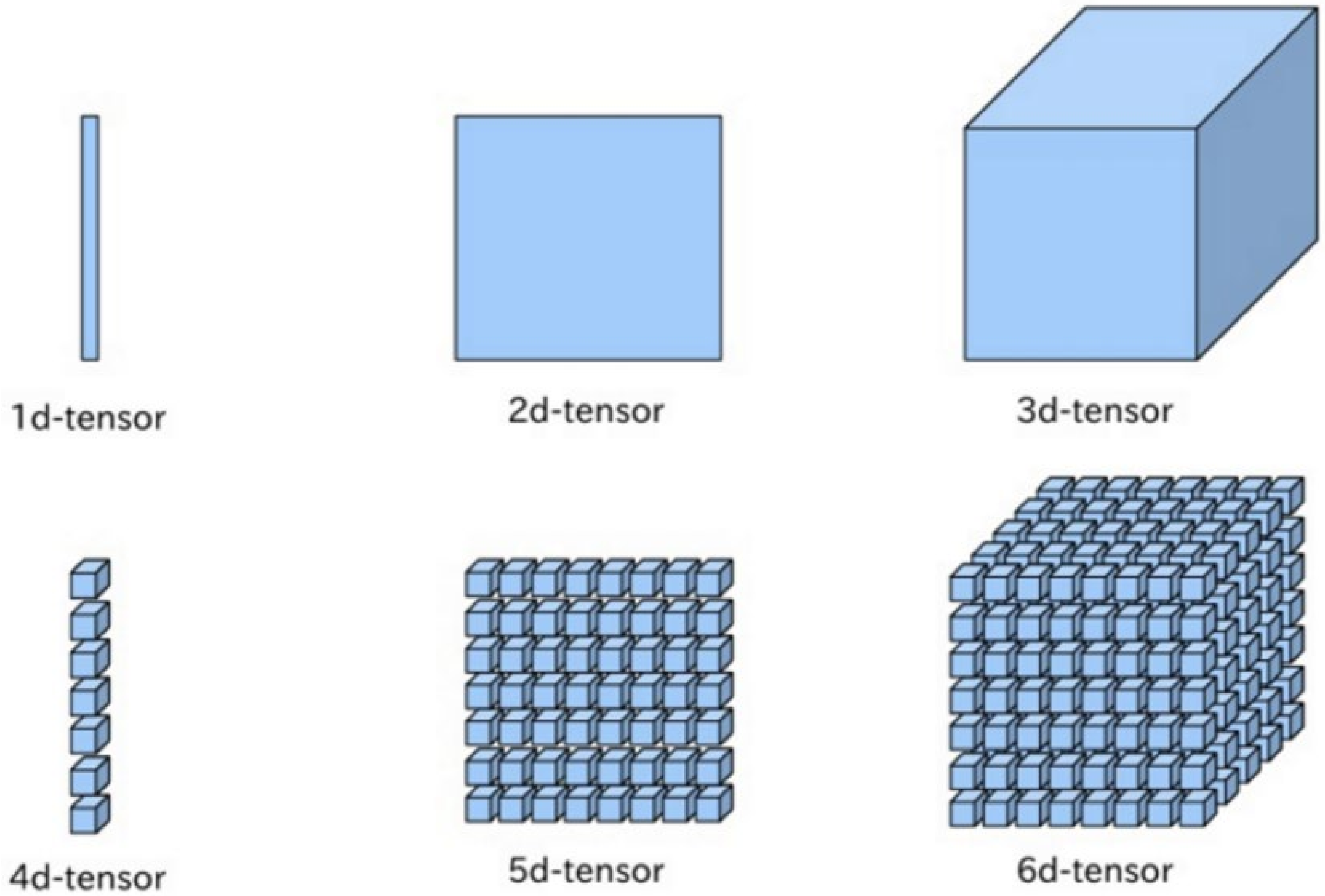
Tensor

- Sometimes need an array with more than two axes
- An array arranged on a regular grid with variable number of axes is referred to as a **Tensor**

- We denote a tensor with bold non-italic typeface: $\mathbf{A}$ in comparison to a normal Matrix $\boldsymbol{A}$
- I try to keep the notation consistent, but double-check ☺
- An Element (i, j, k) of tensor denoted by $\mathrm{A}_{i,j,k}$

- Example: RGB image is a 3D Tensor (height x width x color channel)

Shape of Tensors



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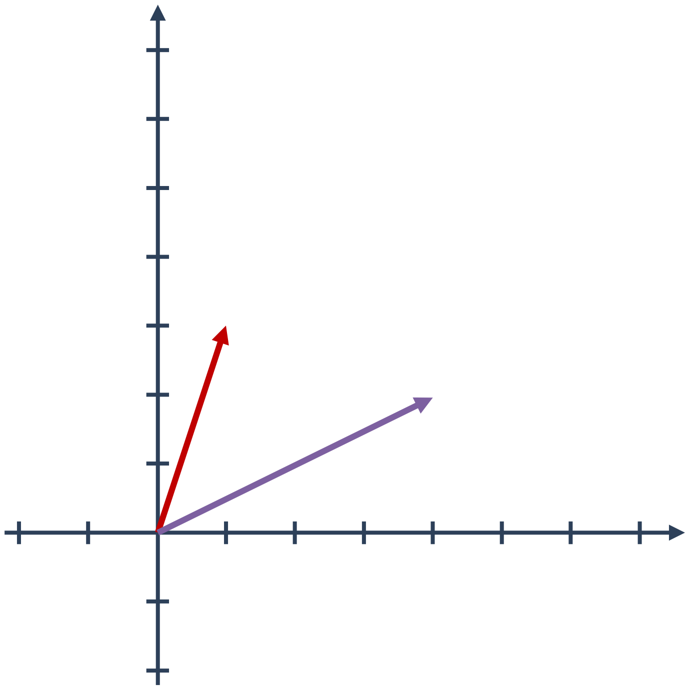
Vector Interpretation

- Vectors can are often depicted as arrows in Euclidean space.

- Let assume two Vectors:

$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$



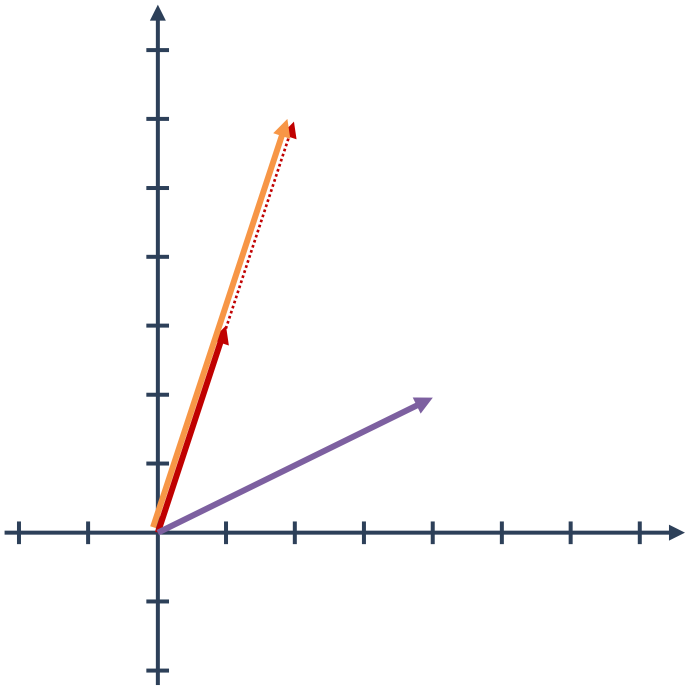
Vector Operations: Multiply with a Scalar

- Vectors can be multiplied with a scalar by elementwise multiplying the entries:

$$\lambda \cdot x = \lambda \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ \vdots \\ \lambda \cdot x_d \end{pmatrix}$$

- Example:

$$2 \cdot \boldsymbol{a} = 2 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 \\ 2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$



Vector Operations: Addition

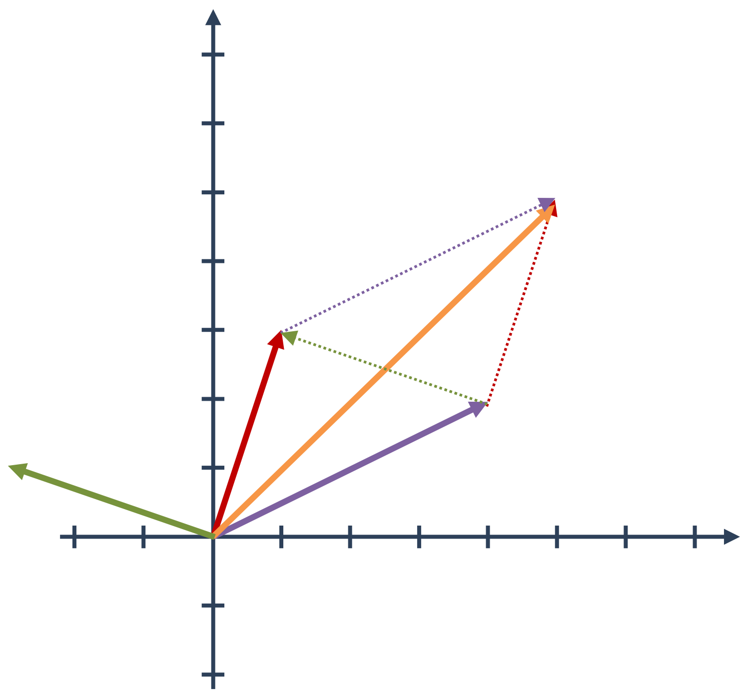
- Vectors can be added by element-wise adding the components:

$$x + y = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_1 + y_1 \end{pmatrix}$$

- Examples:

$$\boldsymbol{a} + \boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$\boldsymbol{a} - \boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$



Vector Operations: The Dot Product

- The scalar product of two vectors is the sum of the element-wise multiplication of the entries:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} = x_1 y_1 + \cdots + x_d y_d = \sum_{i=1}^{d} x_i y_i = \boldsymbol{x}^T \boldsymbol{y}$$

- Note: The result is a real number, not a vector
  - For this reason, the dot product is sometimes called the scalar product (or inner product).

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 1 \cdot 4 + 3 \cdot 2 = 10$$

Properties of the Scalarproduct

- The dot product obeys many of the laws that hold for ordinary products of real numbers.
- Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be vectors, λ is a scalar.

- Then:
  1. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  2. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
  3. $(\lambda \boldsymbol{a}) \cdot \boldsymbol{b} = \lambda(\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot (\lambda \boldsymbol{b})$
  4. $0 \cdot a = 0$

Vector Operations: Addition

- The length of a vector is normally calculated as the square root of the summed squares of the entries:

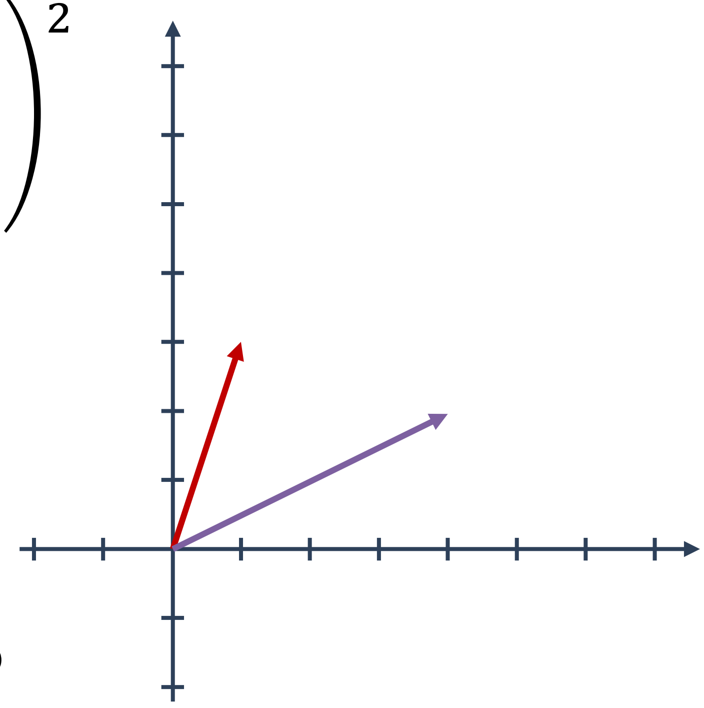
$$|x| = \left|\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}\right| = \sqrt{x_1^2 + \cdots + x_d^2} = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$$

- Relation to the dot-product:

$$x \cdot x = |x|^2$$

- Example:

$$|\boldsymbol{a}| = \left|\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right| = \sqrt{1^2 + 3^2} = \sqrt{10} \approx 3.16$$



L^p Norms for Vectors

- What we have seen is just a member of a family of norms:

$$L^p = \left(\sum_i |x_i|^p \right)^{1/p}$$

- L^2 Norm
  - Most commonly used
  - If nothing is explicitly stated, the Euclidean norm is used
- L^1 Norm
  - Useful when 0 and non-zero have to be distinguished (since L^2 increases slowly near origin, e.g., $0.1^2 = 0.01$)
- L^∞ Norm
  - $||\boldsymbol{x}||_\infty = \max |x_i|$
  - Called max norm

Angle between Vectors

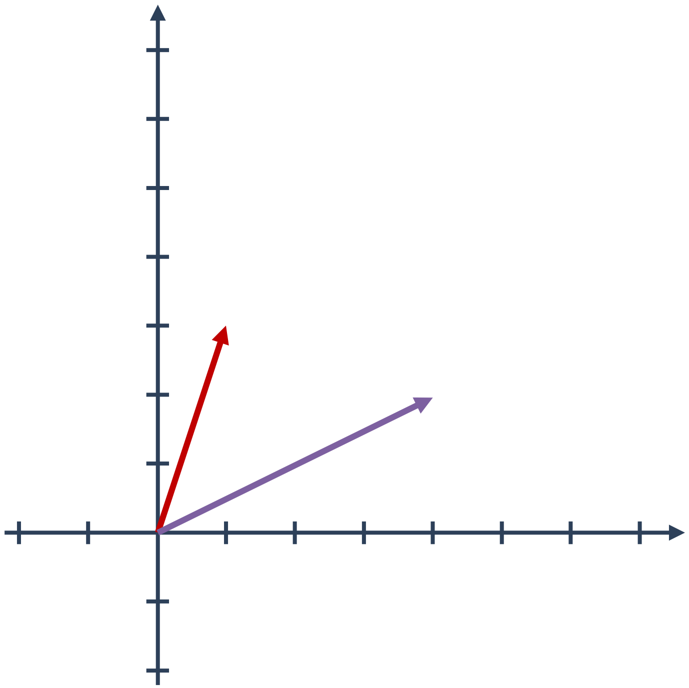
- Dot product of two vectors can be written in terms of their L^2 norms and angle θ between them

$$\boldsymbol{x} \cdot \boldsymbol{y} = |\boldsymbol{x}||\boldsymbol{y}| \cdot \cos\theta$$

- Example:

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|a| \cdot |b|}\right) =$$

$$= \cos^{-1}\left(\frac{10}{\sqrt{10} \cdot \sqrt{20}}\right) = 45°$$



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Names around Matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- **Order of matrix**: It represent the number of rows and number columns of a matrix. For above matrix, the order is 3×3
- **Square matrix**: If a matrix has same number of row and columns then it is called square matrix.
- Special Shapes:
  - Row Matrix: If a matrix only has one row: a transposed Vector
  - Column Matrix: Similarly, if a matrix only has one column: a Vector
  - 1×1 Matrix: A scalar

Matrix Operations: Addition

- Two Matrices can be added by simply element-wise add the entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

- Example:

$$\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1+3 & 2+4 \\ 5+2 & 6+5 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 7 & 11 \end{pmatrix}$$

  - The matrices necessarily need to be of the same order (or dimensionality)

Matrix Operations: Multiplication with a Scalar

- A Matrix can be multiplied by a scalar by multiplying each element of the matrix with the scalar:

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda \cdot a & \lambda \cdot b \\ \lambda \cdot c & \lambda \cdot d \end{pmatrix}$$

- Example:

$$2 \cdot \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 & 2 \cdot 4 \\ 2 \cdot 2 & 2 \cdot 5 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 4 & 10 \end{pmatrix}$$

Matrix Operations: Multiplication

- Two multiply two matrices, we perform the "Dot Product" between rows of the first matrix and columns of the second matrix:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C}$$

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

  - Two matrices can only be multiplied if number of columns in the first matrix is same as the number of rows of the second matrix.
  - If matrix $\boldsymbol{A}$ is of order $m \times n$ and matrix $\boldsymbol{B}$ is of order $r \times s$ the $\boldsymbol{A} \cdot \boldsymbol{B}$ will be possible if $n = r$.
  - The resulting matrix will be of order $m \times s$.

Matrix Operations: Multiplication

- Two multiply two matrices, we perform the “Dot Product” between rows of the first matrix and columns of the second matrix:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C}$$

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{21} \\ c_{21} & c_{22} \end{pmatrix}$$

Matrix Product Properties

- Distributivity over addition:

$$\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$$

- Associativity:

$$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$$

- Not commutative: $\boldsymbol{AB} = \boldsymbol{BA}$ is not always true

- Dot product between vectors is commutative:

$$\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{x}$$

- Transpose of a matrix product has a simple form:

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$$

Transpose of a Matrix

- Mirror image across principal diagonal

$$\boldsymbol{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix}, A^T = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

- Vectors are then either a Matrix with a single column or a single row; For convenience often written in a line: $\boldsymbol{x}^T = [x_1, x_2, x_3]$

- Scalar is a Matrix with just one element, thus:

$$x^T = x$$

Norm of a Matrix

- Frobenius norm

$$\|\boldsymbol{A}\| = \left(\sum_{i,j} A_{i,j}^2\right)^{1/2}$$

- It is analogous to L^2 norm of a vector

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Square Matrices

- Remember, square matrices have the same number of rows as columns

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}$$

- Some operations are special to square matrices and are not defined for arbitrary matrices

Identity Matrix

- An identity or unit matrix of size d is square matrix of order $d \times d$ where all the diagonal elements are '1' and all the other elements are '0'.
- It is denoted by $\boldsymbol{I}$.

$$\boldsymbol{I}_1 = (1) \quad \boldsymbol{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Important Property ($\boldsymbol{A}$ being a square matrix):

$$\boldsymbol{A} \cdot \boldsymbol{I} = \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A}$$

Inverse of a Matrix

- For a square matrix, the inverse matrix will be such that the product of the original matrix and the inverse matrix will produce an identity matrix.

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$$

- Note: Not all square matrices have inverses. A square matrix which has an inverse is called invertible or nonsingular matrix and a square matrix which does not has an inverse is called noninvertible or singular matrix.
- A square matrix is singular if and only if its **determinant** is 0

Square Matrices: Determinant of a Matrix

- Determinant of a square matrix det(A) is a mapping to a scalar
- It is equal to the product of all eigenvalues of the matrix
- Measures how much multiplication by the matrix expands or contracts space

Square Matrices: How To Calculate a Determinant

- For a 1x1 matrix:

$$\det(a_{11}) = a_{11}$$

- For a 2x2 matrix:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For a 3x3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} =$$
$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

(There exists a rather simple way to do this with arbitrarily large matrices, but we omit this in this course).

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System of Linear Equations

- A system of linear equations (or linear system) is a collection of two or more linear equations involving the same set of variables.
- Example:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

- We see, that this matrix formulation is equivalent:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Solution set of linear equations

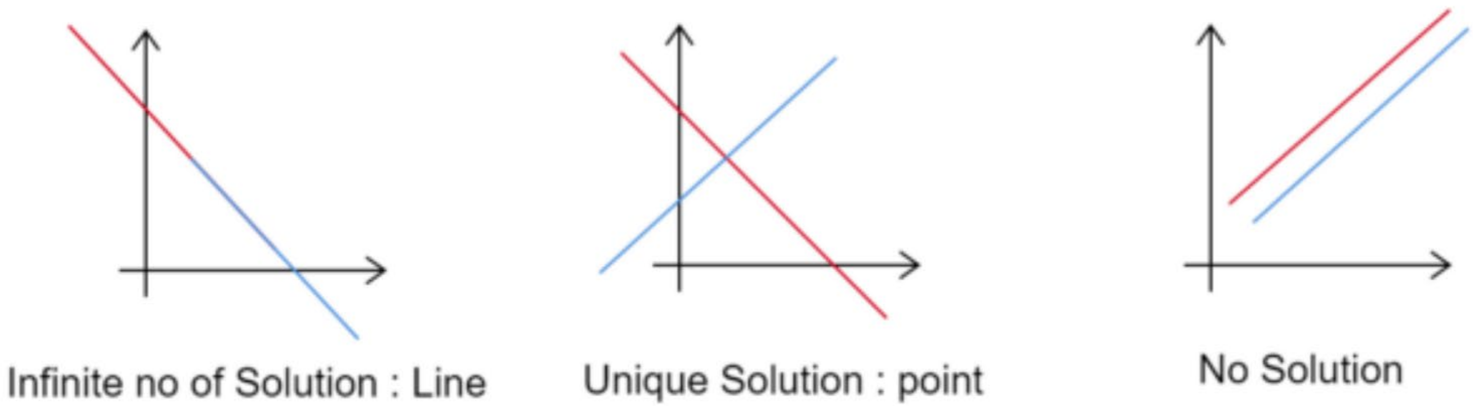
- We are often in the situation that we have given $\boldsymbol{A}$ and $\boldsymbol{b}$ and are looking for x satisfying our system of linear equations:

$$\boldsymbol{Ax} = \boldsymbol{b}$$

- Every linear system may have only one of three possible number of solutions:
  - The system has a single unique solution.
  - The system has infinitely many solutions.
  - The system has no solution.

Geometrical Representation

- For a system of two variables (x and y), each linear equation determines a line on the xy-plane.
- The solution set is the intersection of these lines, and is hence either a line, a single point or don't have any common point.



Consistency and Linear Independence

- **Consistency:** A linear system is said to be consistent if it has at least one solution and is said to be inconsistent if it has no solution.

- **Linear Independence**: A linear system is said to be independent if none of the equations can be written as a linear combination of others.
  - For example, equations $x + y = 2$ and $2x + 2y = 4$ are not linearly independent as the 2nd equation can be obtained by multiplying 2 with the 1st equation.

Rank of a Matrix

- Rank of Matrix: The maximum number of linearly independent rows of a matrix is called the row rank, and the maximum number of linearly independent columns is called the column rank of the matrix.

- For any matrix $\boldsymbol{A}$

row rank of A = column rank of A = rank of A

Linear Equations: Closed-Form Solutions

1. Matrix Formulation: Ax=b
Solution: $x=A^{-1}b$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

2. Gaussian Elimination followed by back-substitution

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

L_2-$3L_1 \rightarrow L_2$ $\qquad$ L_3-$2L_1 \rightarrow L_3$ $\qquad$ -$L_2/4 \rightarrow L_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Special Matrices & Vectors

- **Unit Vector**
  - A vector with unit norm

$$||\boldsymbol{x}||_2 = 1$$

- **Orthogonal Vectors**
  - A vector x and a vector y are orthogonal to each other if

$$\boldsymbol{x}^T \boldsymbol{y} = 0$$

  - Vectors are at 90 degrees to each other

- **Orthogonal Matrix**
  - A square matrix columns and rows are orthogonal unit vectors

$$\boldsymbol{A}^{-1} = \boldsymbol{A}^T$$

Special Matrices & Vectors

- **Diagonal Matrix**
  - Mostly zeros, with non-zero entries in diagonal
  - $diag(\boldsymbol{v})$ is a square diagonal matrix with diagonal elements given by entries of vector $\boldsymbol{v}$
  - Multiplying $diag(\boldsymbol{v})$ by vector $\boldsymbol{x}$ only needs to scale each element x_i by v_i

- **Symmetric Matrix**
  - Is equal to its transpose:

$$\boldsymbol{A} = \boldsymbol{A}^T$$

  - E.g., a distance matrix is symmetric with $A_{ij} = A_{ji}$

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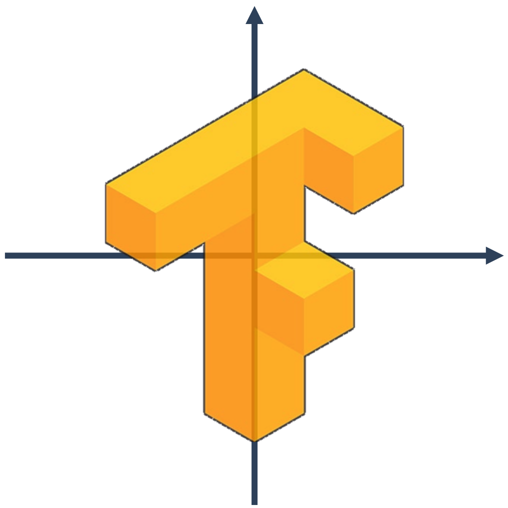
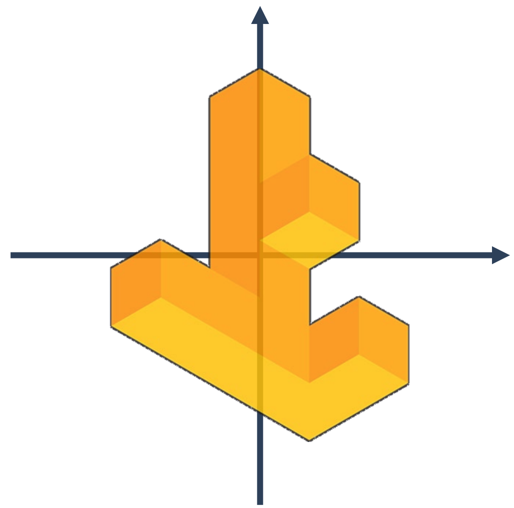
Matrices as Linear Transformation

- At the end of the day, a Matrix defines a linear transformation
- A Matrix $A \in \mathbb{R}^{n \times m}$ projects points from $\mathbb{R}^n$ to their image in $\mathbb{R}^m$
- This is later exactly what we will do with the hidden layers
- Let's have a look at some examples for $\mathbb{R}^{2 \times 2}$ since we can visualize them

Examples: Mirror

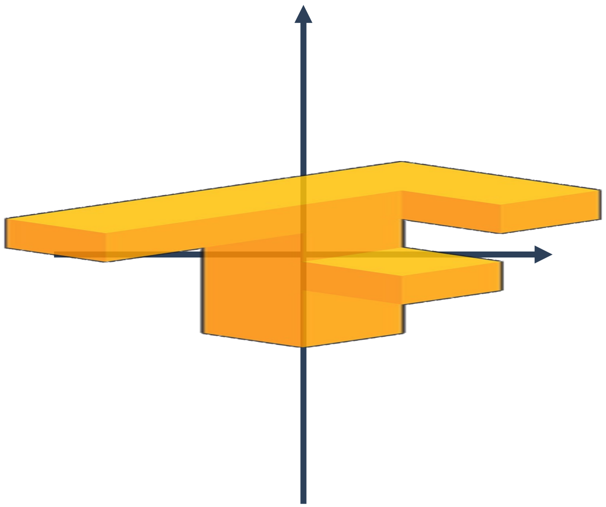




- Reflection on the x-Axis:

$$Mx = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Examples: Stretching

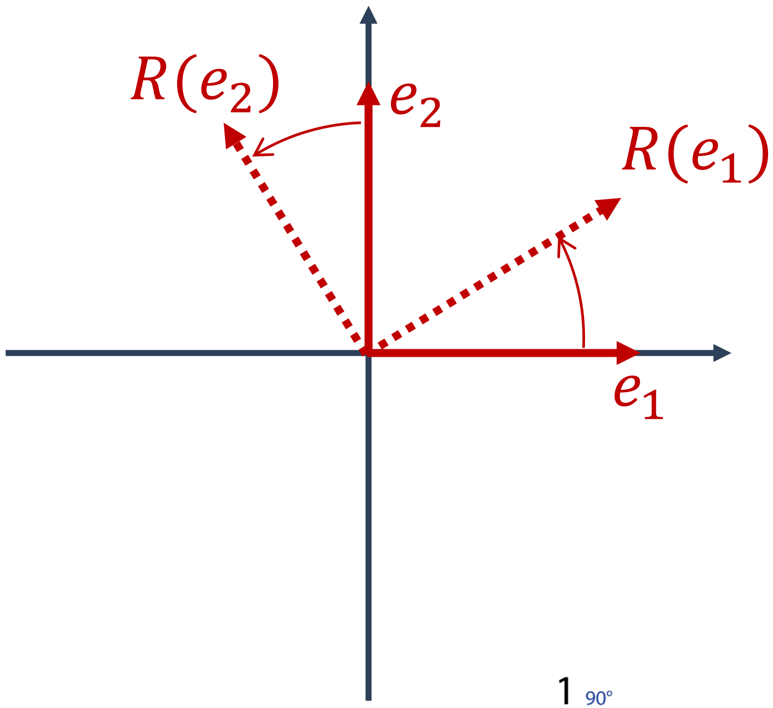


- Reflection on the x-Axis:

$$Sx = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 0.5 \cdot y \end{bmatrix}$$

Example: Rotation

- We are looking for the image of the standard basis e_1, e_2



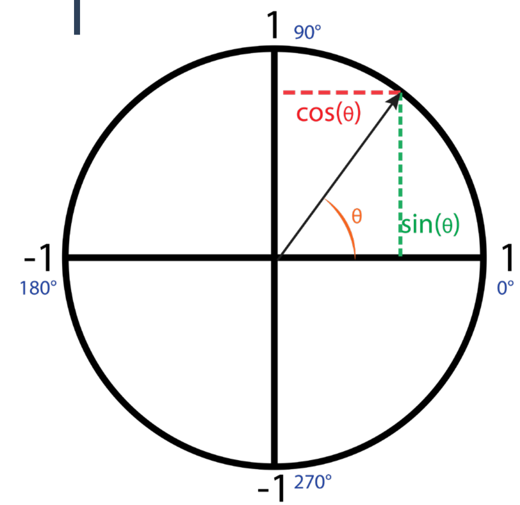
- We need the following images:

$$R(e_1) = R\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

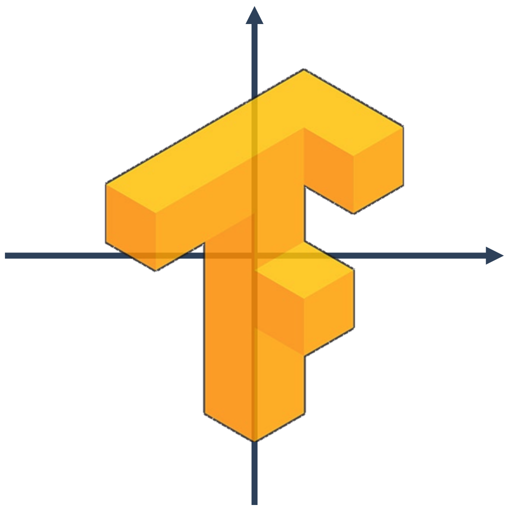
$$R(e_2) = R\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

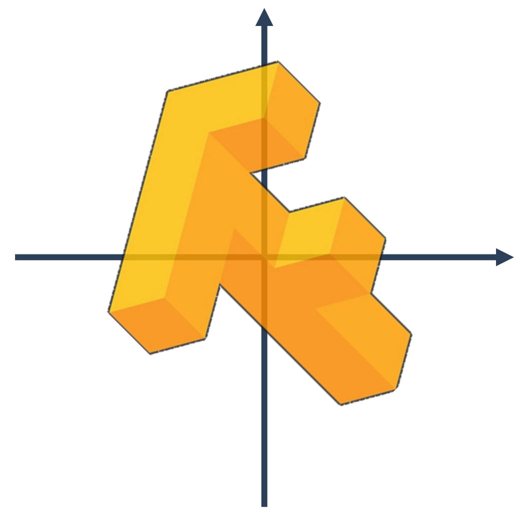
- Together for a rotation of 45 degree:

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$$



Examples: Rotation

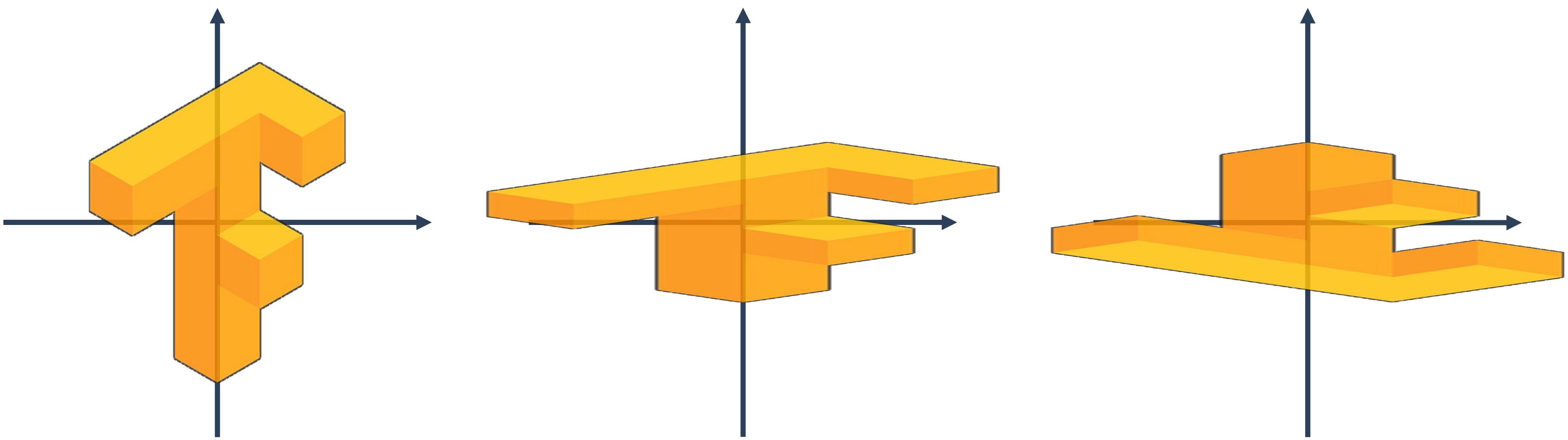




- Rotation on the x-Axis:

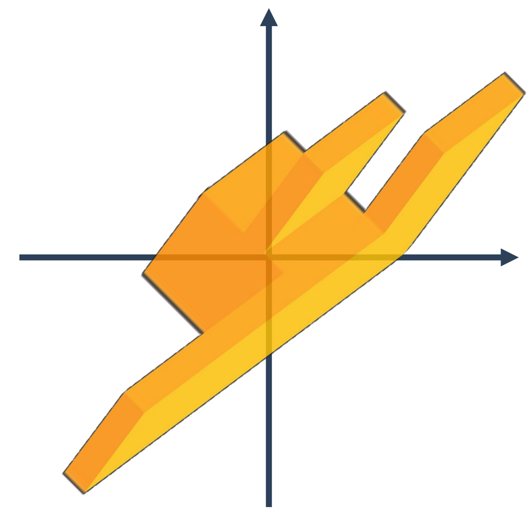
$$Rx = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{bmatrix}$$

Example: Combinations



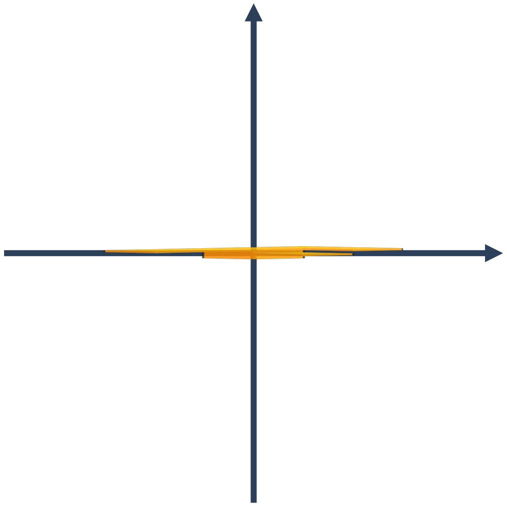
- Simply combine the transformations

$$M = RMS = \begin{bmatrix} 2\frac{1}{\sqrt{2}} & -2\frac{1}{\sqrt{2}} \\ -\frac{1}{2}\frac{1}{\sqrt{2}} & -\frac{1}{2}\frac{1}{\sqrt{2}} \end{bmatrix}$$



On the invertibility of the projections

- It becomes clear that not every transformation is invertible.



- For instance $Mx = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x$
  - Every point (x, y) gets projected onto the same value x
  - Impossible to "reconstruct" y
- Non square matrices project points into higher/lower dimensional spaces (as the example above does)