MAT 157: Analysis I

Summary. We are in the process of assembling a mathematical description of $\mathbb{R}$, the set of real numbers. So far, we have observed that the real numbers form a **field**: There are two operations, addition ($+$) and multiplication ($\cdot$), linked by the distributive law. They are associative, commutative, and have neutral elements (0 and 1, respectively); each number a has an additive inverse ($-a$); if $a \neq 0$ then it has a multiplicative inverse (a^{-1}). Moreover, $\mathbb{R}$ has an **order** relation ($<$) that is compatible with the addition and multiplication, making it an ordered field.

Two more properties will be needed to fully characterize $\mathbb{R}$: The **Archimedean** property describes how the natural numbers lie inside $\mathbb{R}$; specifically, for every real number x we can find a natural number $n > x$. Finally, **completeness** expresses that the real line s a continuum.

But first, we will spend some time with the natural numbers ($\mathbb{N}$), the integers ($\mathbb{Z}$), and the rational numbers ($\mathbb{Q}$) We have characterized the set of natural numbers $\mathbb{N}$ by the properties that

1. 1 is in $\mathbb{N}$;
2. each n in $\mathbb{N}$ has a successor (denoted $n+1$) in $\mathbb{N}$;
3. 1 is not the successor of any natural number;
4. $n+1 = m+1$ only if $m = n$;
5. the principle of **induction**.

From these properties, addition, multiplication, and the order on $\mathbb{N}$ can be constructed recursively (see Spivak, Problem 2.24). Furthermore, the integers can be constructed as differences $m - n$ of natural numbers, and the rationals as quotients p/q. In this course, we will not consider these constructions in detail (the issue is that the representations of a number as a difference or as a quotient are not unique: $m - n = (m+k) - (n+k)$ and $p/q = (k \cdot p)/(k \cdot q)$ for every natural number k). However, we will discuss how to construct $\mathbb{R}$ from $\mathbb{Q}$.

Assignments:

Read Sections 1 and 2.

Hand-in (due Thursday, September 21):

1. *Completing the square.*

   (a) Find the smallest possible value of $2x^2 - 5x + 1$.

   (b) Find all values of a such that $x^2 + axy + y^2 > 0$ whenever x and y are not both zero.

   Explain your method!

2. *Arithmetic-geometric mean inequality (Problem 1.7).* We will prove very soon that for every real number $a \geq 0$ there is a unique real number $x \geq 0$ with $x^2 = a$. This number is denoted by $\sqrt{a}$.

   Prove that if $0 < a < b$ then
$$a < \sqrt{ab} < \frac{a+b}{2} < b\,.$$

3. On the set C of real 2-vectors $z = (a, b)$, define the following two operations:

   addition: $(a, b) + (c, d) := (a + c, b + d)$
   multiplication: $(a, b) \cdot (c, d) := (a \cdot c - b \cdot d, a \cdot d + b \cdot c)$.

   (a) Prove that this defines a field by verifying Properties (P1)-(P9).
   (Try to do as little algebra as possible by appealing to known properties of the real numbers. Start with associativity, commutativity, and the distributive law, then identify the zero and the additive inverse. For multiplication, verify that $\mathbf{1} := (1, 0)$ is the neutral element, and $(a, b)^{-1} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$.)

   (b) Find an element z in C such that $z^2 = -\mathbf{1}$.

   (c) Is there a subset $\mathcal{P}$ in C that makes C into an ordered field? Why not?

4. *Triangle inequality (Problem 1.14).* Define the **absolute value** of a real number by

$$|a| = \begin{cases} a\,, & \text{if } a \geq 0\,, \\ -a\,, & \text{if } a \leq 0\,. \end{cases}$$

   Show that

   (a) $|a| = \sqrt{a^2}$;

   (b) $|a| = |-a|$;

   (c) $|a| \leq b$ if and only if $-b \leq a \leq b$. In particular, $-|a| \leq a \leq |a|$.

   Use this to prove that $|a + b| \leq |a| + |b|$ for all real numbers a, b.

5. Prove directly from the basic properties of $\mathbb{N}$ that that every natural number n is either even ($n = 2k$) or odd ($n = 2k - 1$).

6. *Towers of Hanoi (Problem 2.26).* There is a puzzle consisting of three pegs, with n disks of decreasing diameter stacked on Peg 1 (see the figure on p. 34). A disk may be moved from one peg to another, provided it is not placed on top of a smaller disk.

   Prove that the entire stack can be moved from Peg 1 to Peg 3 in $2^n - 1$ moves, and that this cannot be done in fewer than $2^n - 1$ moves. (Please describe your strategy precisely, and use induction.)