Ceren Şahin

I don't always draw a triangle, but when I do it's isosceles.

Chill sis, WolframAlpha's got this.

They proved that such a number existed. I asked them to find it...

GEOMETRİ NOTLARI

ÖNSÖZ

Olimpiyatlara hazırlanmak hayatımda gördüğüm en sıkı çalışma temposu gerektiren bir işti benim için. Özdisiplin ve kararlılıkla sonu görünmeyen bir yolda ilerlemek gibiydi. Bu yolda ilerlerken kendimi yalnız hissettiğimde, motivasyonum düştüğünde, “Acaba ben yetersiz miyim?” gibi sorular kafamı karıştırdığında hep kendime “Bu işten zevk alıyor muyum?” sorusunu sordum. “Matematiğin bu yönünü seviyor muyum?”. Benim için cevabı hep evetti.
Kendime sorduğum bu sorunun cevabını hep evet kılan iki önemli şey vardı: Birincisi, bir soruyu çözünce gelen mutluluk; ikincisi ise soruyu çözmek için arkadaşlarımla ve Özcan Hoca’mla yaptığım beyin fırtınaları. İşte elinizde tuttuğunuz bu kitap bu işi zevk alarak yaptığımın kanıtı. Kitap, Özcan Hoca’nın sayısı bilinmez defterlerinden gelen sorulardan, İftihar Hoca’nın vazgeçilmezi Evan Chen’in geometri kitabından çıkardığım konu anlatımlarından ve kendi çalışmalarım sırasında çözüp beğendiğim uluslararası olimpiyatlarda çıkmış sorulardan meydana geliyor.

Olimpiyat yolculuğumda bana ilham veren şeylerden birisi de Özcan Hoca’nın defterleriydi, ben de olimpiyat yolculuğundakilere bir ilham kaynağı, daha doğrusu bir destek olabilmek için bu kitabı hazırladım. Kendi olimpiyat yolculuğunuzun sonuna ne zaman varacaksınız bilmiyorum ama emin olun yolun sonuna geldiğinizde bir madalya alsanız da almasanız da matematik çalışarak kendinize yaptığınız büyük yatırımın ve verdiğiniz büyük emeklerin getirdiği gururu eminim hep hissedeceksiniz.

Bu kitaptan yardım alacağınız zamanlarda sakın soruların başlığına takılıp soruya karşı bir ön yargı oluşturmayın kafanızda. Ben soruyu çözdüğümde hissettiğim şeyleri temsil etmesi için öznel başlıklar koydum. Siz de aynısını kendi defterinizde yapın :) … Matematik çalışmak üzerine tavsiye verme makamında değilim ama soru üzerine kafa yormadan direkt çözüme bakmayın demeden de geçmek istemem doğrusu. Bu kitabın matematik çalışmalarınızda katkı sağlamasını dilerim. Son olarak, bu yolculukta zorlandığınızda destek almak için Ali Nesin’in *Müstakbel Matematikçiye Öğütler*’ini okumanızı öneririm, ben çok faydasını gördüm zamanında…

TEŞEKKÜR

Matematik olimpiyatlarında benim ve arkadaşlarımın üstünde çok büyük emeği olan, bizim çözdüğümüz her soruyla sevinen, çözemediğimiz her soruda bize doğru ipucunu veren, konuları anlamadığımızda okul bitimine ek dersler koyan, haftanın iki - üç gününü bizimle çalışmaya harcarken geri kalan günlerinde kendini bu alanda hep geliştirmeye çalışan değerli hocam Özcan ASLAN’a, her hafta bize deneme sınavı hazırlayan ve çözümlerini itinayla yapıp anlamadığımızda gerekirse aynı soruyu 5 kez anlatmaktan kaçınmayan, bizde her konunun (özellikle geometrinin) temelinin sağlam olmasını sağlayan kıymetli hocam İftihar HACIOĞLU’na, ileri seviye matematiği bize anlatmak için çaba harcayan tüm hocalarıma, yapamadığım soruları rahatça tartışabildiğim tüm arkadaşlarıma ve bu yolculukta beni hep destekleyen aileme teşekkür ediyorum.

Ceren Şahin
06.02.2022

PREFACE

Studying for the olympiads was a thing that required the hardest study tempo. It was like walking on an endless road with self-discipline and determination. Whenever I felt lonely, not motivated, or whenever I found myself asking questions like "Am I not enough for this thing?", I've always asked the question "Do I enjoy studying maths?". The answer for me was always yes. It had something to do with two main factors: First, the happiness that surrounds you after solving a hard problem, and second, the brainstorming I made with my friends and teachers. So, this book is the very proof that I enjoyed every bit of it. The book is full of questions from Ozcan teacher's numerous notebooks, notes from Evan Chen's geometry book, and questions that I liked and wanted to note down during my self-studies.

One of the things that inspired me in this journey was my teacher's notebooks and I hope this book, my notebook basically, will inspire and support you on your journey as well. I do not know when your olympiad journey will end but I do know that when it does, with or without a medal, you'll always feel the pride in the investment you will have made in yourself by studying maths.

Last but not least, when you're solving the questions in the book do not let the topics of the questions bother you in any way. They are just there because they express how I felt about the questions after solving, or not solving, them. And I suggest you do the same for your own notebook as well :) … I may not be the right person to give advice about maths but I'd feel bad about not advising you this: try to think about the problems before reading the solutions. I hope this notebook helps you in your studies. And I want to finish off by advising one more thing. Whenever you feel down or unmotivated about the whole maths stuff, you might want to take a look at *"Müstakbel Matematikçiye Öğütler"* by Ali Nesin, that has helped me a lot.

ACKNOWLEDGEMENTS

First of all, I would like to thank my teacher Ozcan ASLAN, who contributed a lot to me and my friends in the Mathematics Olympiads, rejoiced with every question we solved, gave us the right clue in every question we could not solve, added additional lessons to the end of the school when we did not understand the subjects, spent two or three days a week working with us, and spent the rest of the week improving himself in this field. I also would like to thank my teacher Iftihar HACIOGLU, who prepared a practice exam for us every week and did not hesitate to explain the same question even five times when we did not understand the solutions and who ensured that the foundation of every subject (especially geometry) was solid. I would like to extend my thanks to all my teachers who spent their time explaining deep maths to us, all my friends with whom I could easily discuss the questions that I could not do, and my family who always supported me on this journey.

Ceren Şahin

06.02.2022

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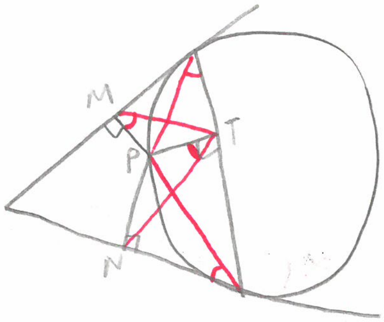
* İçindekiler bölümünde sadece büyük konu anlatımlarının sayfa numaraları vardır. İki konu anlatımı arasında olimpiyatlarda çıkmış birçok soru ve minik teoremler vardır.

ÇEMBER

* $|PT|^2 = |PM| \cdot |PN|$

Kanıt:

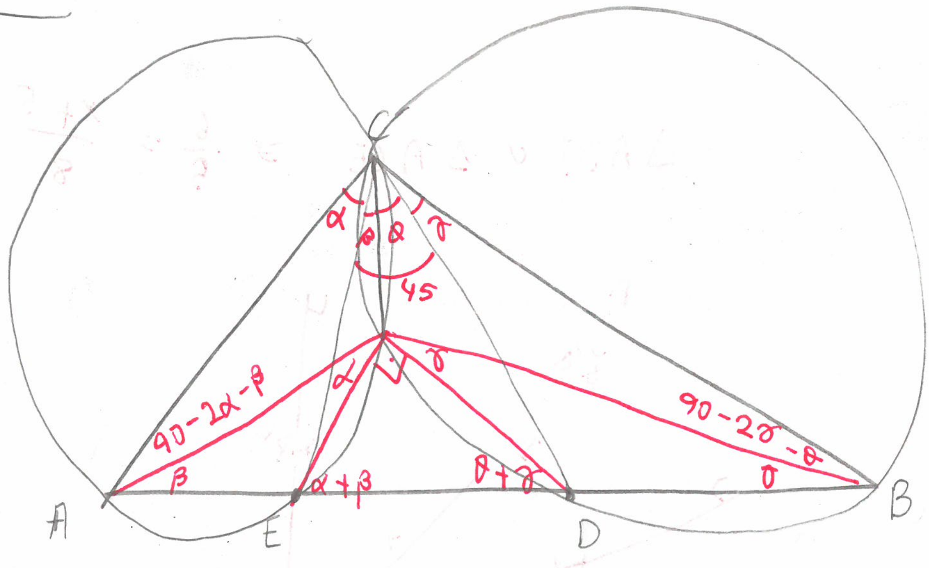


$\triangle PTN \sim \triangle PMT$

olduğundan $\dfrac{PT}{PM} = \dfrac{PN}{PT}$

UMO 13 $\angle C = 90°$ olan bir ABC dik üçgeninin [AB] kenarı üstünde $|AD| = |AC|$ ve $|BE| = |BC|$ olacak şekilde D ve E noktaları seçiliyor. AEC ve BDC üçgenlerinin çevrel çemberlerinin ikinci kez kesiştiği F noktası için $|CF| = 2$ ise $|ED| = ?$

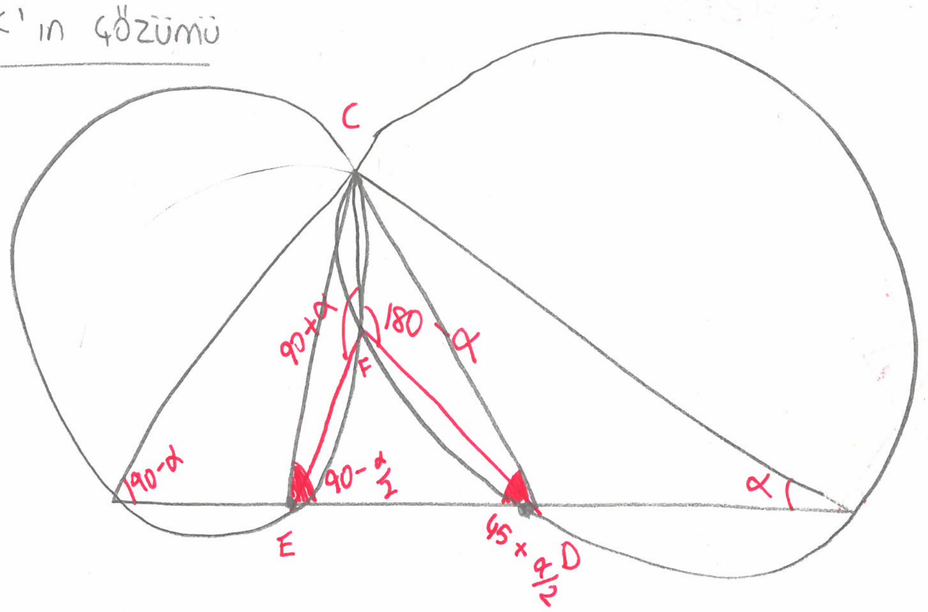
Benim çözümüm



$\angle CAB + \angle CBA = 90° \Rightarrow 90-2\alpha+90-2\gamma = 90 \quad \alpha+\gamma = 45.$

F noktası çok büyük bir olasılıkla $\widehat{ABC}$'nin I'sı ve CED'nin O'su. Bu yüzden $ED = 2\sqrt{2}$.

TÜBİTAK'ın Çözümü

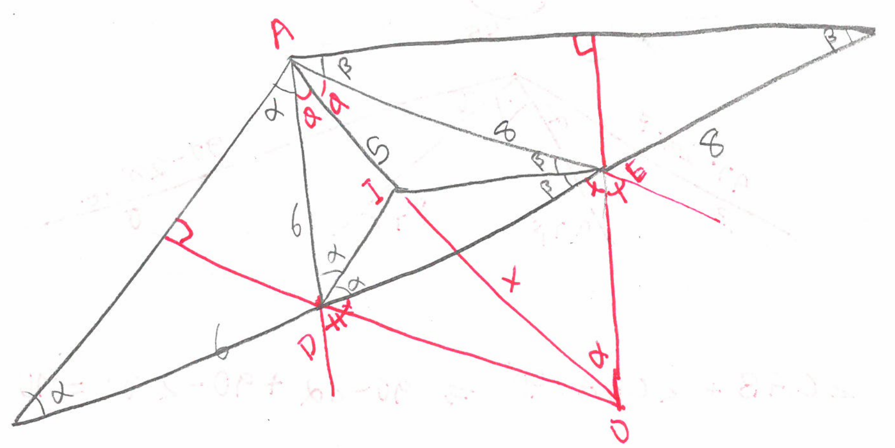


$\angle CFD = 2\angle CED$ ve $\angle CFE = 2 . \angle CDE$ olur.
CED'nin çevrel çemberinin merkezi O için,

$\angle COD = 2 . \angle CED = \angle CFD$ ve $\angle COE = 2 . \angle CDE = \angle CFE$
olur. Yani C, O, F, E ve C, O, D, F çemberdeş. O zaman
O = F olur. O zaman $EF = FD = FC = 2$ $\angle EFD = 90°$
gelmişti. $ED = 2\sqrt{2}$.

VMO 2013

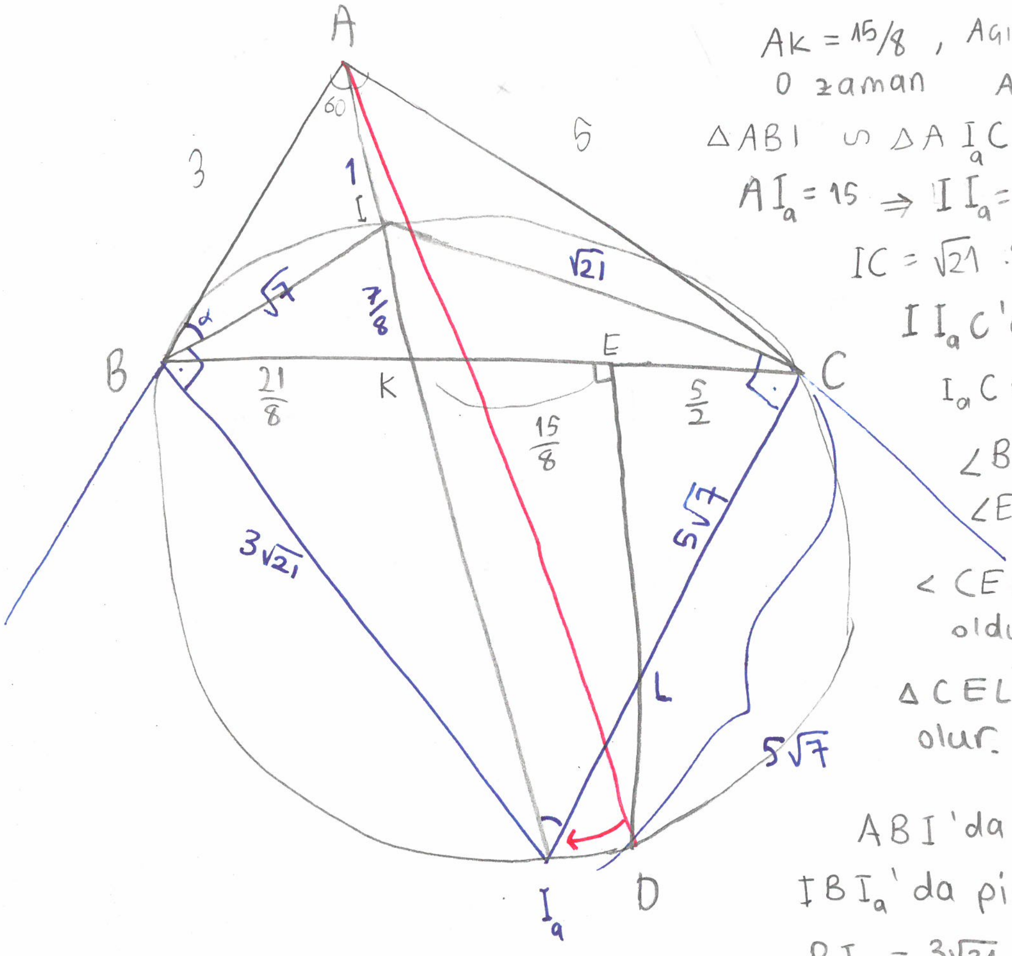
$\triangle ADI \sim \triangle AOE \Rightarrow \frac{6}{5} = \frac{x+5}{8} \Rightarrow x = \frac{23}{5}.$



UMO 15 iç teğet çemberin merkezi I, IBC üçgeninin çevrel çemberinin üstünde I ile farklı tarafta alınan D'den dik indiriliyor. (E'ye) $\frac{BE}{EC} = \frac{9}{5}$ ise $\angle BAD = ?$

$BA = 3$
$AC = 5$
$CB = 7$.

Benim Çözümüm



cos atarsak $\angle A = 120°$

$AK = 15/8$, Açıortay Teo'dan.
O zaman $AI = 1$, $KI = 7/8$

$\triangle ABI \sim \triangle AI_aC$ olduğundan

$AI_a = 15 \Rightarrow II_a = 14$

$IC = \sqrt{21}$ ← AIC'de cos'tan.

II_aC'den pisagordan

$I_aC = 5\sqrt{7}$.

$\angle BII_a = 60 + \alpha$,

$\angle ECL = 60 + \alpha$,

$\angle CEL = \angle IBI_a = 90°$

olduğundan

$\triangle CEL \sim \triangle IBI_a$

olur.

ABI'da cos'tan $BI = \sqrt{7}$

IBI_a'da pisagordan

$BI_a = 3\sqrt{21}$

$\triangle CEL \sim \triangle IBI_a$'dan,

$\frac{\frac{5}{2}}{\sqrt{7}} = \frac{CL}{14} \Rightarrow CL = 5\sqrt{7}$ gelir. $|CL| = |CI_a| = 5\sqrt{7}$ ise

$|LI_a| = 0$. O zaman $D = I_a = L$

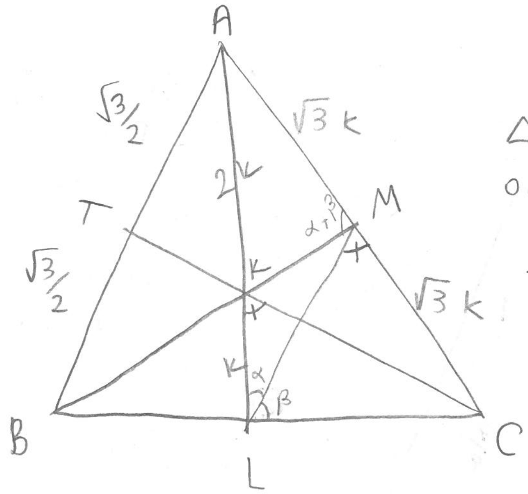
olur. $\boxed{\angle BAD = \angle BAI_a = 60°}$

olur.

UMO 09

ABC üçgeninin AL ve BM kenarortayları K noktasında keşisiyor. C, M, K, L çembersel ve $AB = \sqrt{3}$ ise C'den çıkan kenarortayın uzunluğu nedir

B.Ç.



$AK = 2.KL$,

$\triangle AMK \sim \triangle ALC$ olduğundan

$$\frac{2k}{2.AM} = \frac{AM}{3k}$$

$AM = \sqrt{3}k$.

$ML = AT = \sqrt{3}/2$ olur.

$\triangle ALM \sim \triangle ACK$ olduğundan

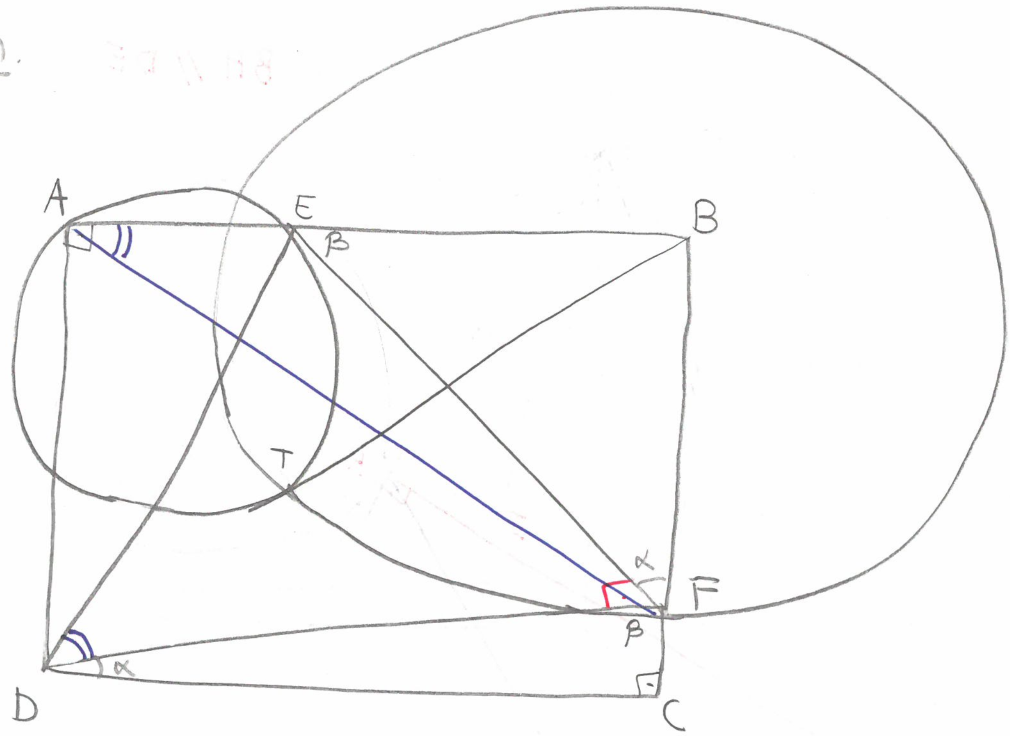
$$\frac{3k}{2\sqrt{3}k} = \frac{3}{2\sqrt{3}} = \frac{ML}{CK} = \frac{\frac{\sqrt{3}}{2}}{CK} = \frac{\sqrt{3}}{2} \Rightarrow CK = 1$$

$CK = 1$ ise $KT = \frac{CK}{2} = \frac{1}{2}$

$TC = 1 + \frac{1}{2} = \frac{3}{2}$

1. Soru: ABCD dikdörtgeninin A köşesinden geçen bir çember [AB] kenarını köşelerden farklı bir E noktasında kesiyor. B'den geçen ve bu çembere teğet olan bir doğrunun değme noktası T olmak üzere, merkezi B olan ve T'den geçen çember de [BC]'yi F noktasında kesiyor. $\angle CDF = \angle BFE$ ise $\angle EDF = \angle CDF$ olduğunu gösterin.

Solution.



$\angle EFD = \angle BAD = 90°$ olduğundan AEFD çemberseldir.

$\angle FDE = \angle FAB = \alpha$ olursa $\triangle ABF \sim \triangle FBE$ benzerliğinden $\frac{BF}{EB} = \frac{AB}{BF} \Leftrightarrow BF^2 = AB \cdot EB$ olur.

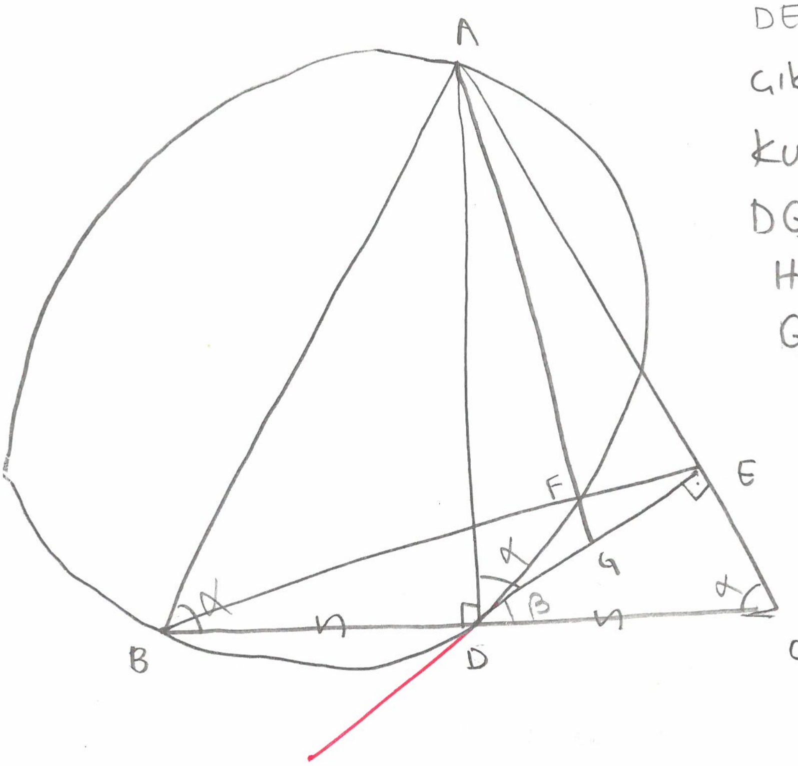
$\overset{\circ}{AET}$'nde kuvvetten $BT^2 = BE \cdot AB$ olur.

$\overset{\circ}{BTF}$'nde B merkez olduğundan $BF^2 = BT^2 = BE \cdot AB$ olur. Benzerlikteki orantı sağlandığından $\triangle ABF \sim \triangle FBE$ olur ve $\angle BAF = \angle EFB = \alpha$ gelir.

3 ARALIK 2011 . ORTAOKUL 2. AŞAMA

2. soru: $AB = AC$ olan bir ABC ikizkenarından BC'nin orta noktası D. D'den AC'ye inilen dikmenin ayağı E. BE doğrusu ABD üçgeninin çevrel çemberini 2. kez F noktasında kesiyor. DE ve AF doğrularının kesişimi G ise $DG = GE$ olduğunu gösterin.

Solution:



Açıları yazıldığında DE doğrusu çembere teğet çıkar.

kuvvetten
$DG^2 = GF \cdot GA$,
Herondan
$GE^2 = GF \cdot GA$,

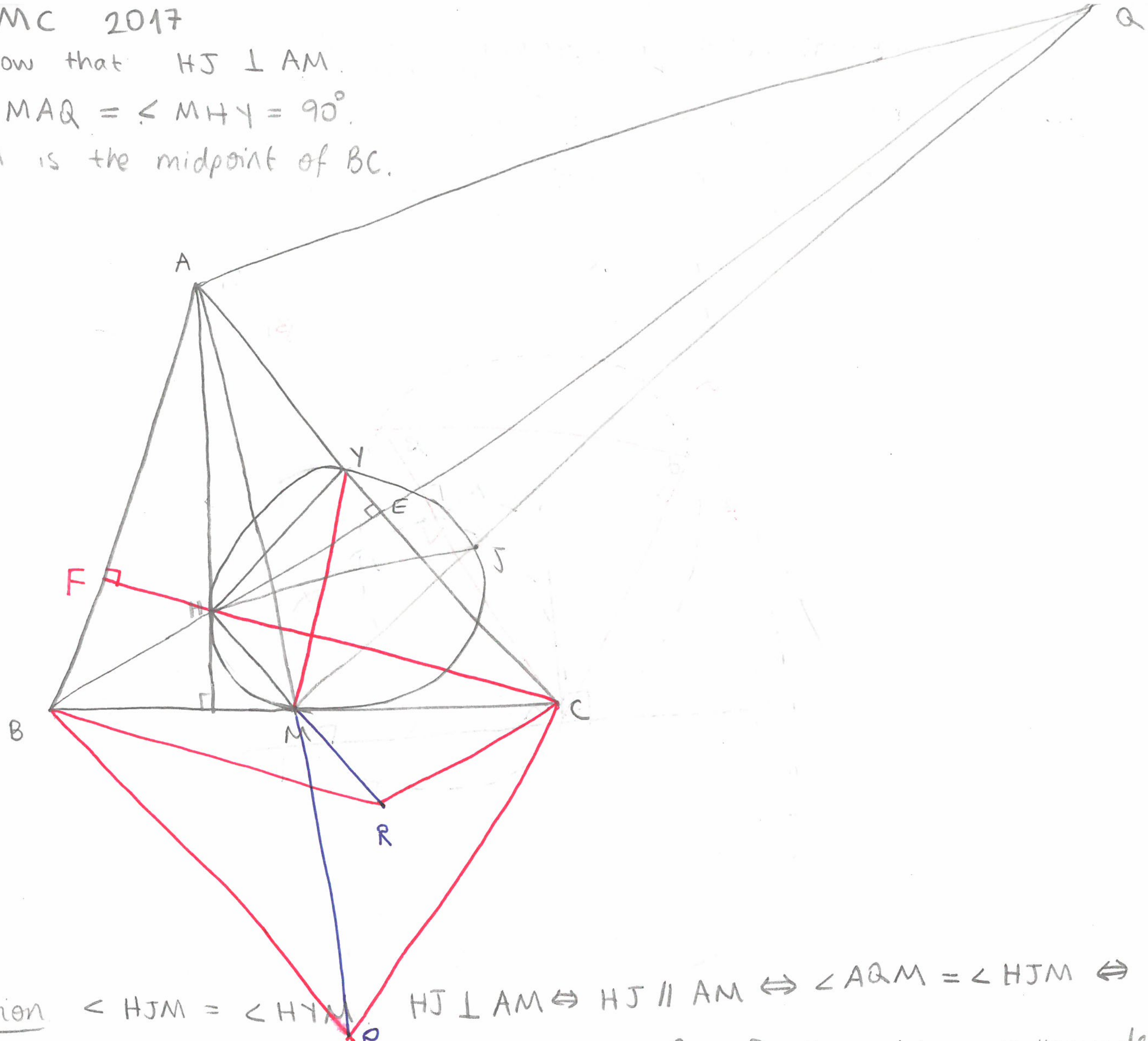
$GE^2 = GD^2$
ve ikisi'de uzunluk olduğu için
$GE = GD$ çıkar.

EMC 2017

Show that $HJ \perp AM$.

$\angle MAQ = \angle MHY = 90^\circ$.

M is the midpoint of BC.



Solution $\angle HJM = \angle HYM$ $\quad HJ \perp AM \Leftrightarrow HJ \parallel AM \Leftrightarrow \angle AQM = \angle HJM \Leftrightarrow$

$\angle AQM = \angle HYM \Leftrightarrow \triangle AQM \sim \triangle HYM$. R ve P, H ve A'nın M üzerinden yansımaları olsun. So, $\angle QAP = \angle YHR = 90^\circ$.

$\triangle AQM \sim \triangle HYM \Leftrightarrow \dfrac{AQ}{HY} = \dfrac{\frac{1}{2}AP}{\frac{1}{2}HR} = \dfrac{AP}{HR} \Leftrightarrow \triangle AQP \sim \triangle HYR \Leftrightarrow \angle QPA = \angle YRH$.

M, midpoint olduğundan, ABPC ve BRCH paralelkenar. So, $\angle ACR = \angle AEB = 90^\circ$.

$\angle YCR = \angle YHR = 90^\circ$ olduğundan YHRC çembersel. So,

$\angle YRH = \angle YCH = \angle YCF = 90^\circ - \angle BAC$. Paralellikten $\angle PBQ = \angle CEQ = 90^\circ$.

So, $\angle PBQ = \angle PAQ = 90^\circ$ olduğundan ABPQ çembersel.

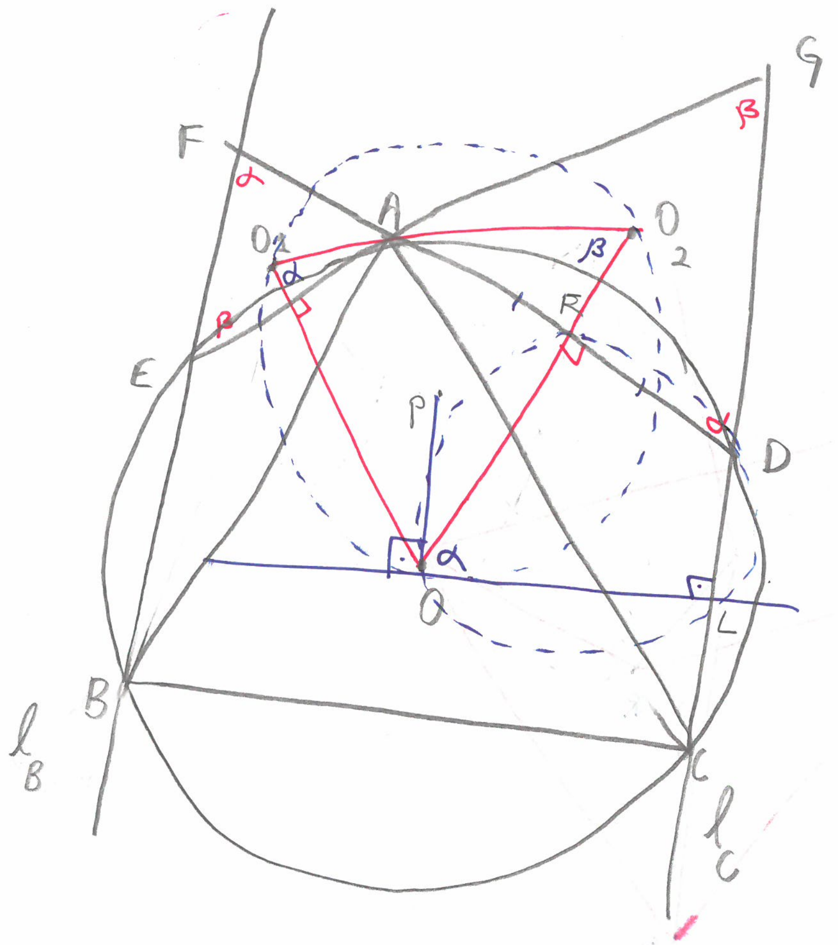
Yani $\angle QPA = \angle QBA = \angle EBA = 90^\circ - \angle BAC$.

$\angle YRH = 90^\circ - \angle BAC = \angle QPA \Rightarrow \angle YRH = \angle QPA$ olduğu kanıtlandı.

EMC 2019 Junior

$l_B \cap l_C$. $O_1 O_2 O$ ← çev. çem. merkezleri üçgeninin çevrel çemberinin merkezi P noktasıdır. $l_B \parallel PO \parallel l_C$ olduğunu gösterin

Çözüm.

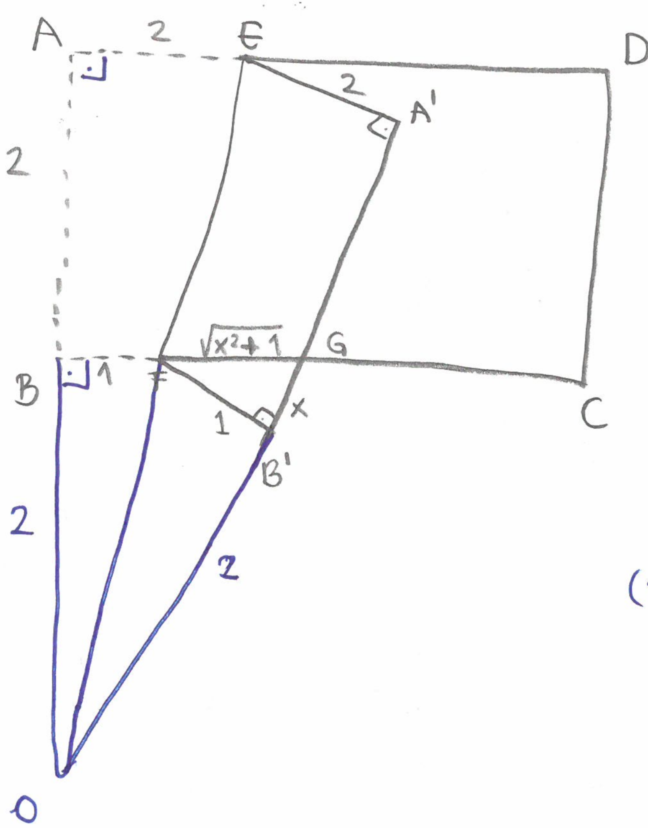


$\Delta EFA \sim \Delta GDA$ olduğundan $O_1 A O_2$ doğrusaldır.

O_2 ve O_1 merkezler olduğundan $\angle O O_2 O_1 = \beta$ ve $\angle O O_1 O_2 = \alpha$ olur. Çemberden $\angle O_2 O L = \alpha = \angle ADG$ O halde OLDR çemberseldir. $OP \perp OL \perp GL$ olduğundan

$OL \parallel l_C \parallel l_B$ $\square$.

Akdeniz 2005



$|B'G| = ?$

ABCD dikdörtgen.

△ OBF ve △ AOE de Thales'ten OB = 2 = OB'

△ OBG de OF açıortay olduğundan

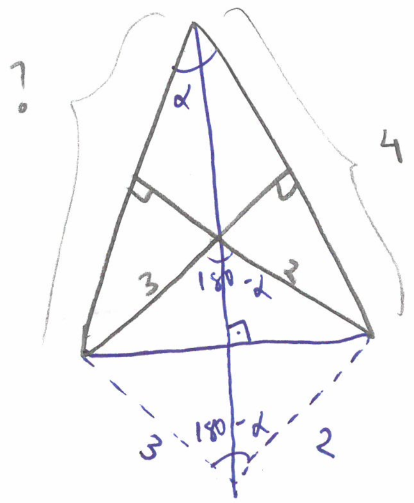
$$\frac{1}{\sqrt{x^2+1}} = \frac{2}{2+x}$$

$$(2+x)^2 = 4x^2+\cancel{4} = x^2+4x+\cancel{4}$$

$$3x^2 = 4x \Rightarrow 3x = 4$$

$$x = 4/3.$$

Güzel Soru



$$?^2 + 2^2 = 3^2 + 4^2$$

$$\Downarrow$$

$$?^2 = 21$$

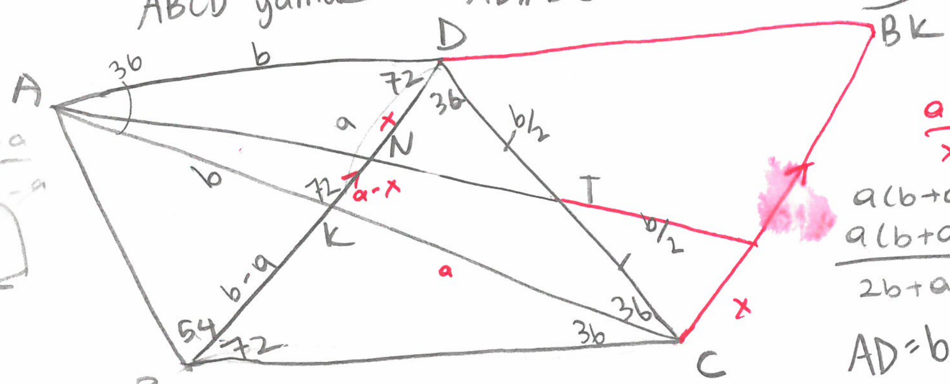
$$? = \sqrt{21}$$

ZOR SORU

AD = BD = CD ve ∠BCD = 72°. DT = TC ise $\frac{ND}{BK} = ?$

ABCD yamuk. AD // BC

$\frac{x}{b-a} = \frac{b-a}{b-a}$ $\boxed{=1}$



$$\frac{a-x}{x} = \frac{b}{b+a}$$

$$a(b+a) = b\cdot x + (b+a)x$$

$$\frac{a(b+a)}{2b+a} = x.$$

AD = b , BC = a olsun.

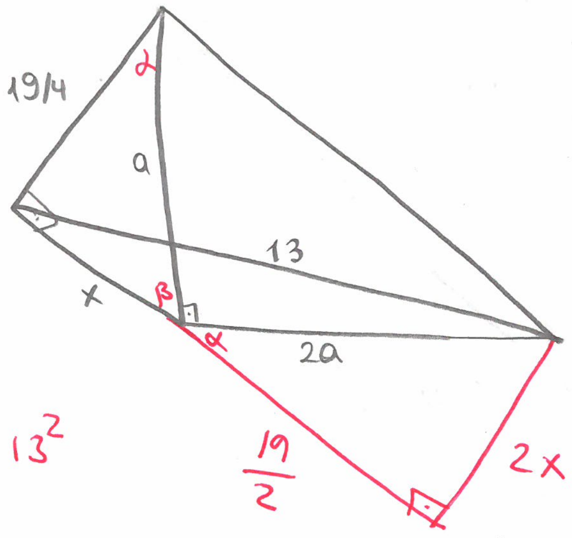
△ADK ∼ △DBC olduğundan BK = b−a, DK = a. △CKB ∼ △AKD $\frac{b-a}{a} = \frac{a}{b}$ $b^2 - ab - a^2 = 0$.

$$\frac{ab+a^2}{2b+a} = \frac{ab+a^2+2(b^2-ab-a^2)}{2b+a}$$

$$= \frac{b(2b+a)-a(2b+a)}{2b+a} = b-a = x \text{ olduğundan}$$

Estetik Soru

$x = ?$



$\left(\frac{19}{2} + x\right)^2 + 4x^2 = 13^2$

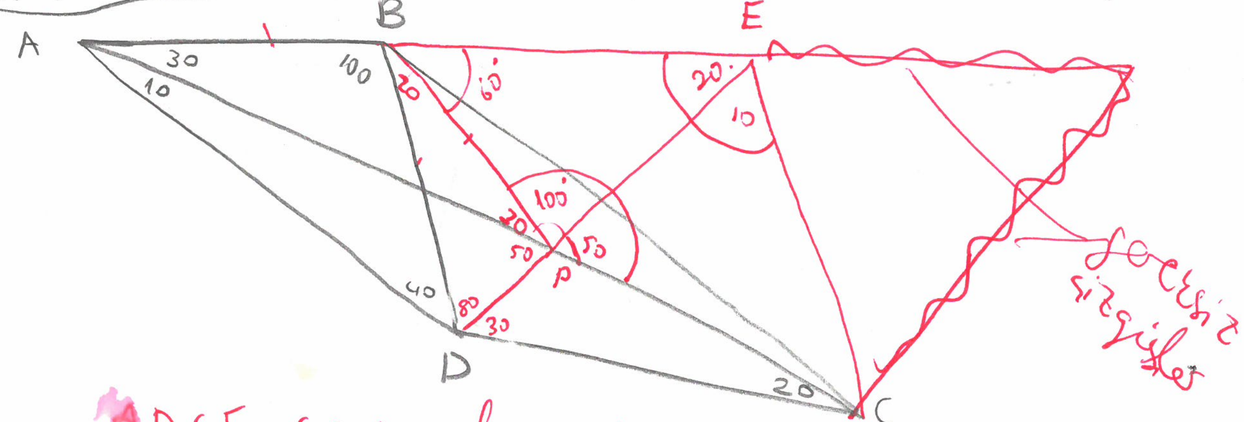
$x = \frac{5}{2}$.

RUMO - 2007

$\angle ADC = 150$. $\angle BAC = 30°$. $\angle ACD = 20°$

Benim çözümüm

$AB = BD$ ise $\angle DBC = ?$



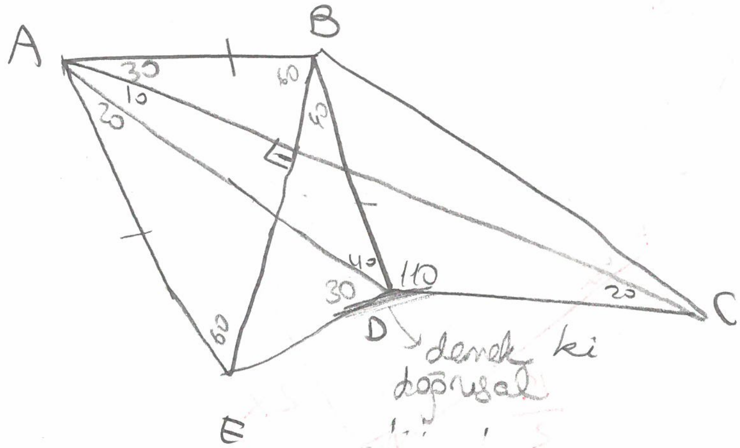
ADCE çembersel çıktı.

$\angle PDC = 30°$ olacak şekilde ayıyırsak B merkezli A, D, P den geçen çember oluşur. Açılarsak $\angle BED = 20° = \angle ACD$ ve $\angle EDC = \angle CAE = 30°$ gelir. Aynı zamanda BECP de çembersel geldi. ~~O halde~~ Çünkü $\angle BPC + \angle BEC = 180°$

O halde $\angle PBC = \angle PEC = 10°$ sonuçta

$\angle DBC = 20 + 10 = 30°$.

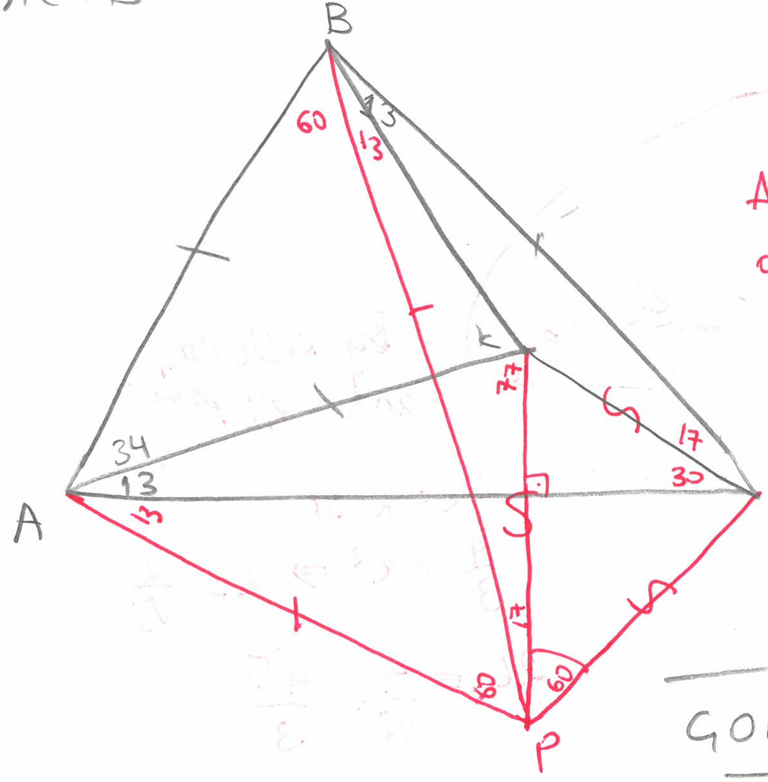
Ömer Gürlü'nün Çözümü



EDC doğrusalmış. $AC \perp BE$ ve $AE = AB$ olduğundan $AECB$ deltoit. $\angle BCA = 20$. $\rightarrow$ $\angle EBC = 70^\circ \Rightarrow \angle DBC = 30^\circ$

$\widehat{ABC} = 86 \longrightarrow \widehat{KCB} = ? = 17$ UAMO 2016

$\widehat{KAC} = 13$



$\triangle APK = \triangle BPC$

old. $PK = PC$

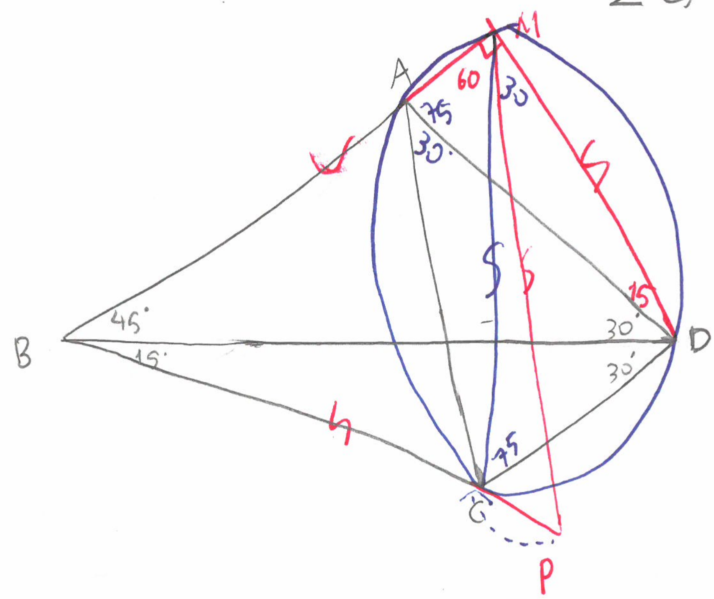
AKCP deltoid.

$KP \perp AC$.

$C < K\widehat{C}B = 17°$

ÇOK İYİ ÇÖZÜM

ÇİLLOP GİBİ ÇÖZÜM $\angle CAD = ?$



Dik üçgeni oluşturalım.

MPB eşkenar üçgenini oluşturalım.

~~MBPD MPB üçgeninin çevrel çemberi üzerindedir.~~

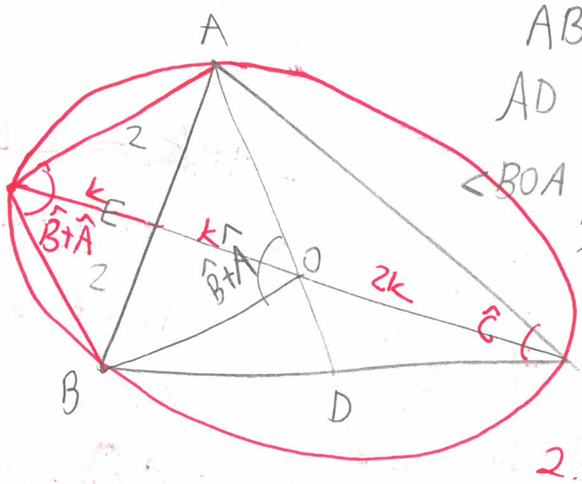
M merkezli çember, B, P, D'den geçer.

$\frac{\angle BMP}{2} = 60 \cdot \frac{1}{2} = 30° = \angle BDP = \angle BDC$ olduğundan P ile C çakışık olur.

$\triangle BCD$ 'de $\angle MCD = 75°$ gelir. $\angle MAD = \angle MCD$ olduğundan,

AMDC çemberseldir. Buradan $\angle CMD = 30° = \angle CAD$.

KEŞKE SORULARI OKUSAYDIM SORUSU



$AB = 4$

AD ve CE kenarortay

$\angle BOA = \angle A + \angle B$ ise $OC = ?$

By Reflection, and by power.

$2.2 = k.3k$

$\frac{4}{3} = k^2 \Rightarrow k = \frac{2}{\sqrt{3}}$.

$2k \Rightarrow \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$.

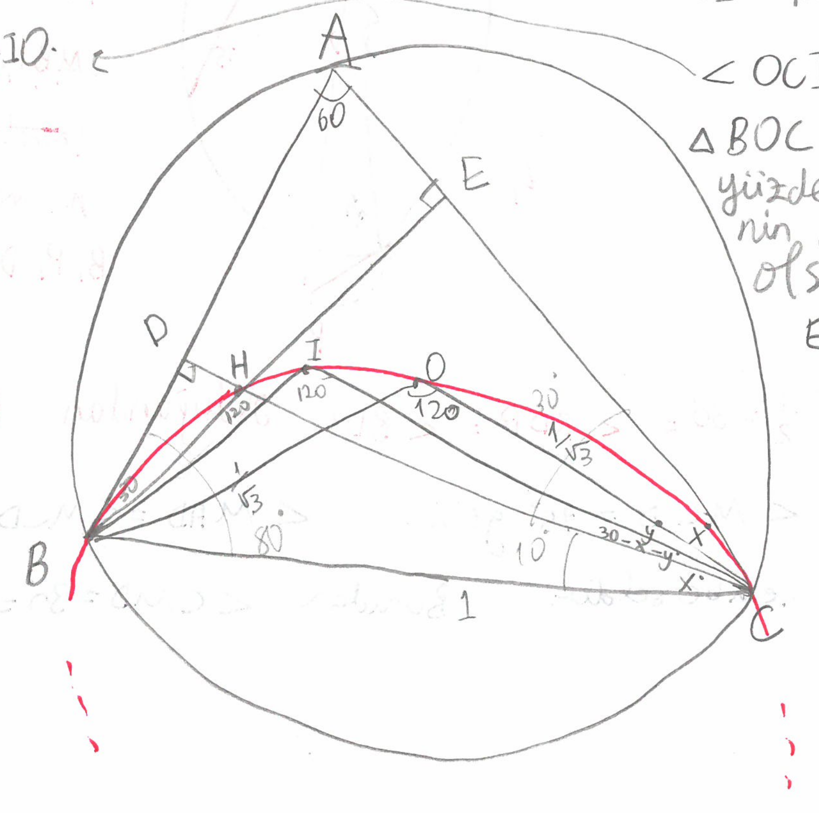
AMC 12 - 2011

ABC üçgeninde $\hat{A} = 60°$. H, I, O, bildiğimiz noktalar. $BC = 1$ $\angle \hat{B} \leq 90°$ ise $BCOIH$ beşgeninin alanı en büyükken $\hat{B} = ? = \underline{80°}$

CI açıortay olduğundan dolayı $HI = IO$.

$\angle OCI = \angle ICH$

ΔBOC tektir. Bu yüzden $BHIO$ dörtgeninin alanı en fazla olsun istiyoruz. En fazla olması için $BH = HI = IO$ olmalı.



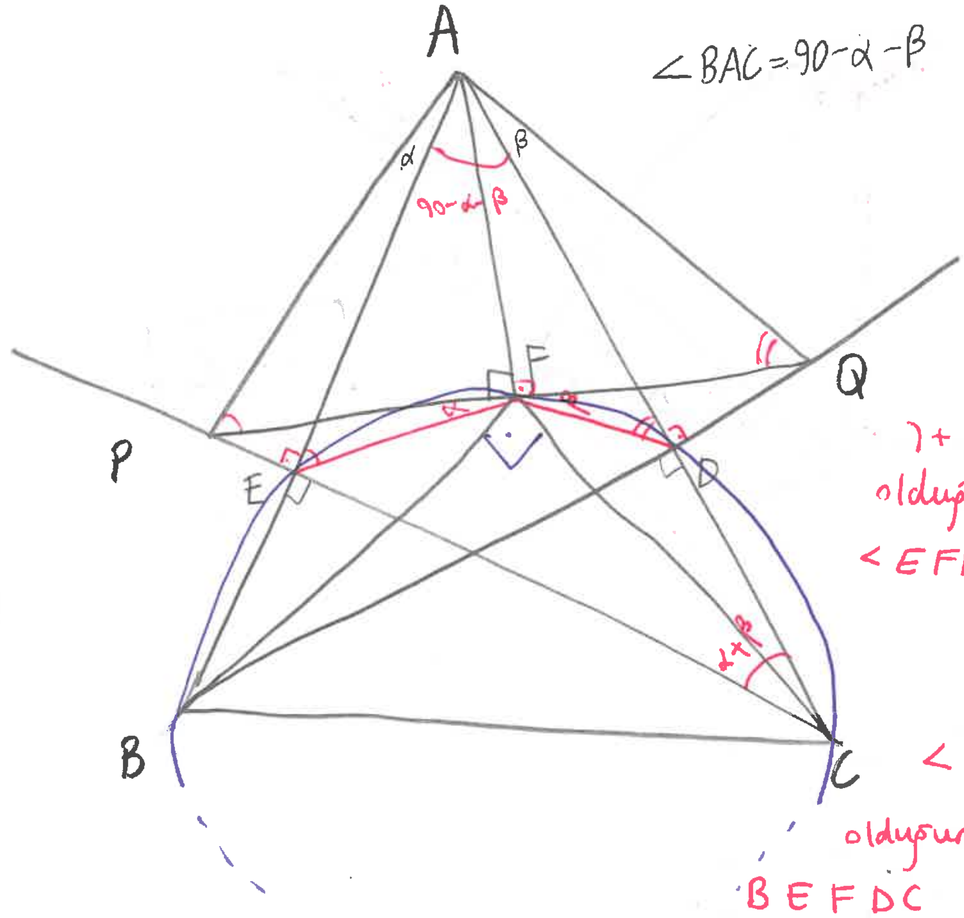
$y = 30 - x - y = x$

$30 - 2x = x$

$x = 10$.

RUMO 2011

$\angle CEA = \angle BDA = \angle AFP = 90°$. Show that $\angle BFC = 90°$



$\angle BAC = 90 - \alpha - \beta$

$) +)) = 90°$

olduğundan

$\angle EFD =) +)) + \angle EAD$

$= 90 + 90 - \alpha - \beta$

$= 180 - \alpha - \beta$.

$\angle ECD = \alpha + \beta$

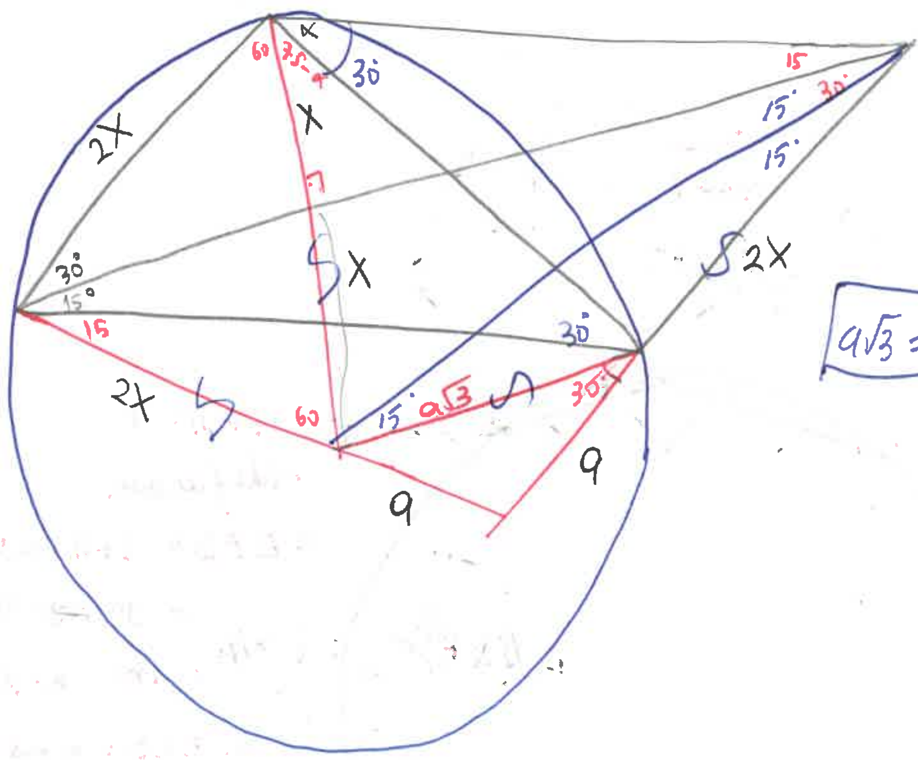
olduğundan

BEFDC çembersel.

$\angle BDC = \angle BFC = 90°$ gelir.

İLAH GİBİ SORU (Allah Çözemez, o derece)

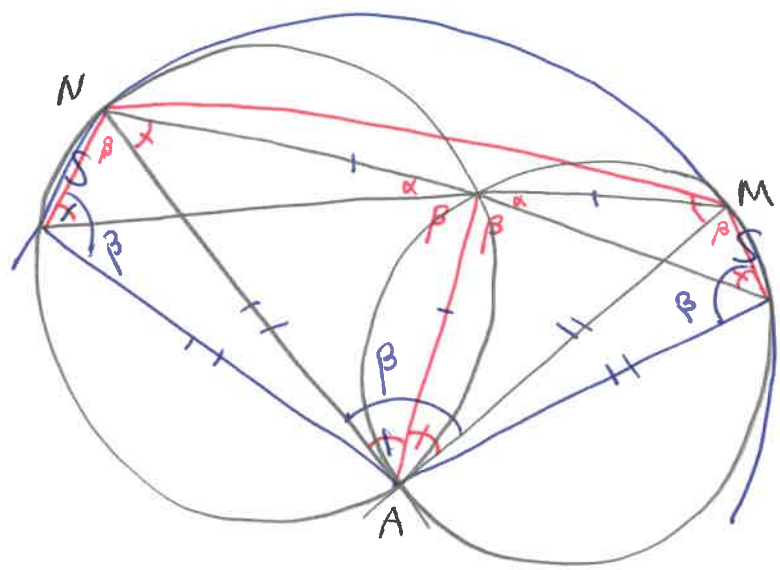
$\alpha = ? = 30^\circ$



$\boxed{a\sqrt{3} = 2x}$!

GÜZEL SORU

Show that $MQ = NP$. (AN and MA are tangents.)



$\gamma + \theta = \beta$

$\triangle NBA = \triangle ABM$ (3 açı + 1 kenar) olduğundan $NB = AB = AM$.

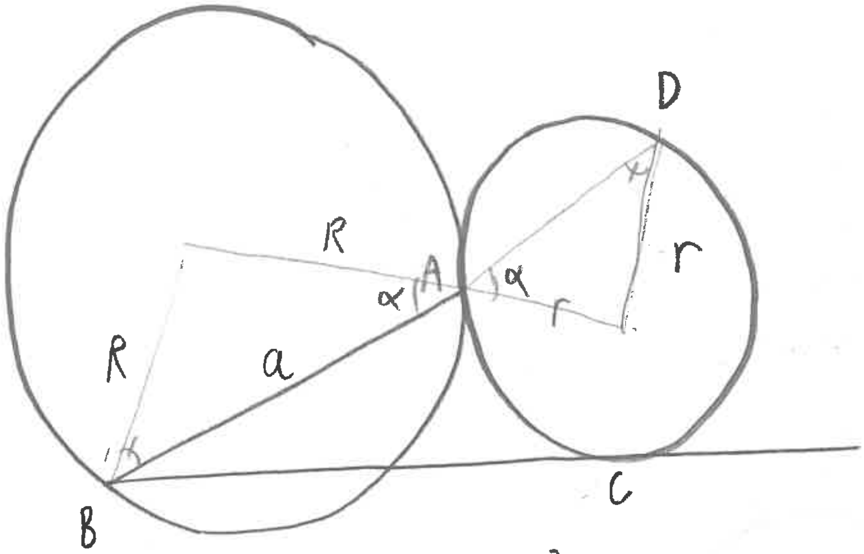
⇓

$\gamma = \gamma = \theta$

NMPQ, A merkezli çem. üzerinde. $\angle NQM = \angle MPN$ olduğundan $\triangle NQA = \triangle MPA$ olduğundan $NQ = MP$. NQPM çembersel ve $NQ = MP$ olduğundan NQPM ikizkenar yamuk. İkizkenar yamuklarda köşegen uzunlukları birbirine eşit. Yani $NP = MQ$

AUMO 1996

$|BC| = ?$



$$\frac{AD}{a} = \frac{r}{R}$$

$$BC^2 = BA \cdot BD = a\left(a + a\frac{r}{R}\right) = a^2\left(1 + \frac{r}{R}\right)$$

$$BC = a\sqrt{1+\frac{r}{R}}$$

Balcan Juniour Team Selection Exam

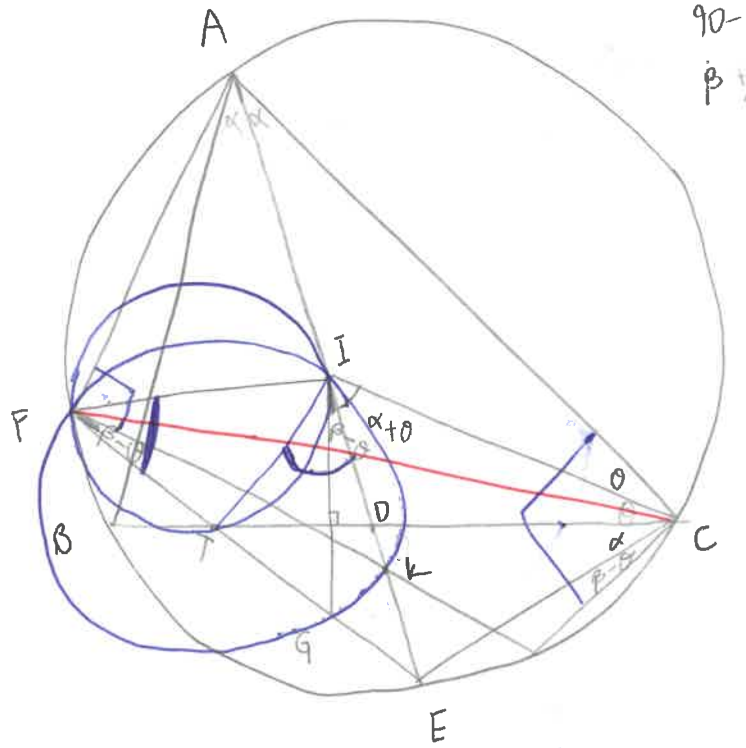
ABC üçgen. $|AB|<|AC|$ I'nın BC'ye göre yansıması I'. Noktalar şekildeki gibi (i) $\frac{AI}{IE}=\frac{ID}{DE}$ (ii) $IA=IF$ olduğunu ispatlayın.

$\beta+\alpha+\theta=90°$

$90-\alpha-2\theta$

$\beta+\cancel{\alpha}+\theta-\cancel{\alpha}-2\theta$

$\beta-\theta$



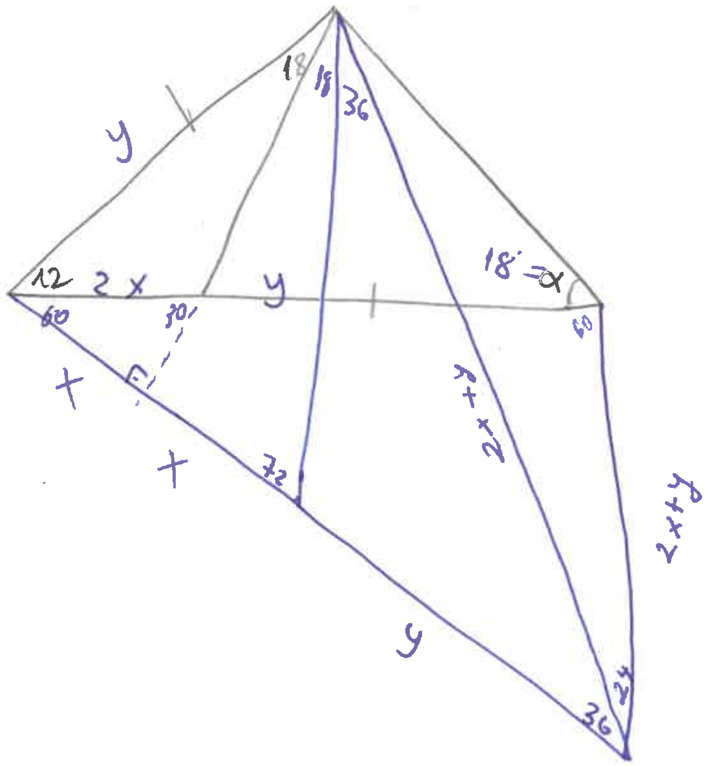
$\frac{TE}{EC}=\frac{EC}{FE} \Rightarrow TE\cdot FE=EC^2=EI^2$

$\frac{TG}{TE}\cdot\frac{ED}{DI}\cdot\frac{1}{1}=1$

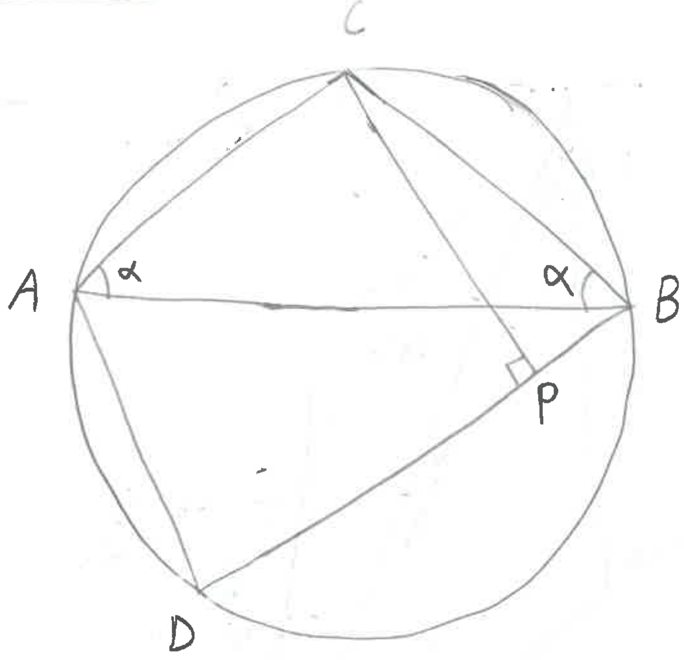
$\frac{TG}{TE}=\frac{ID}{DE}=\frac{AI}{IE}$

17

Güzel Soru



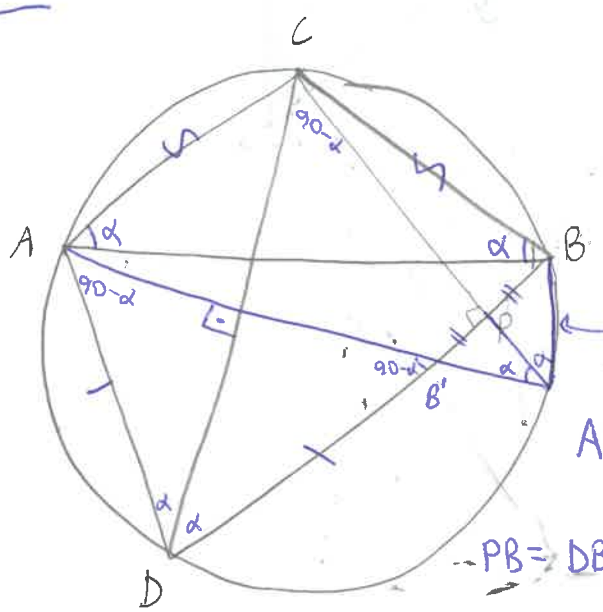
Balcan Junior



$AD + DB = 2DP$

olduğunu gösterin.

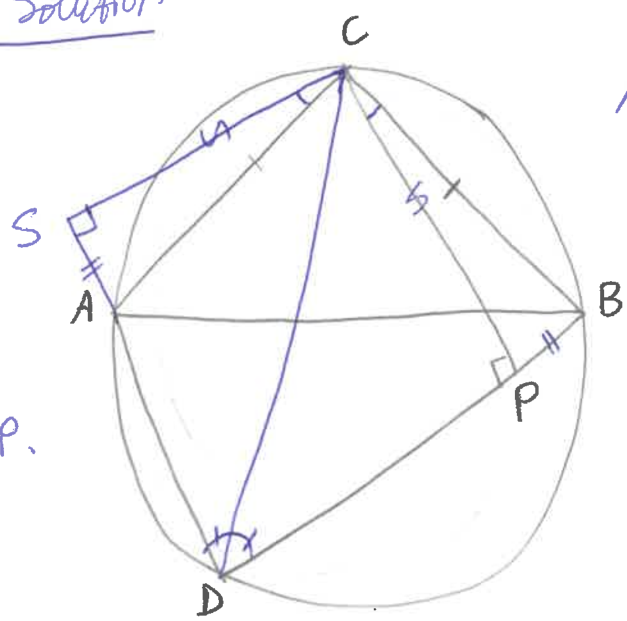
My solution



$B'P = PB$

$$AD + DB = 2DP$$
$$\Updownarrow$$
$$PB = DB - DP = DP - DA = PB'$$

Serdar's Solution



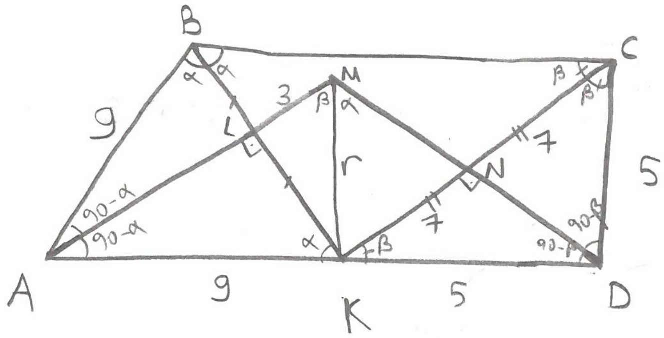
$$AD + DB = (AD + PB) + DP$$
$$= (AD + SA) + DP$$
$$= SD + DP$$
$$= 2DP$$

$SD = DP$.

2004 UMO

$\frac{NM}{KL} = ?$

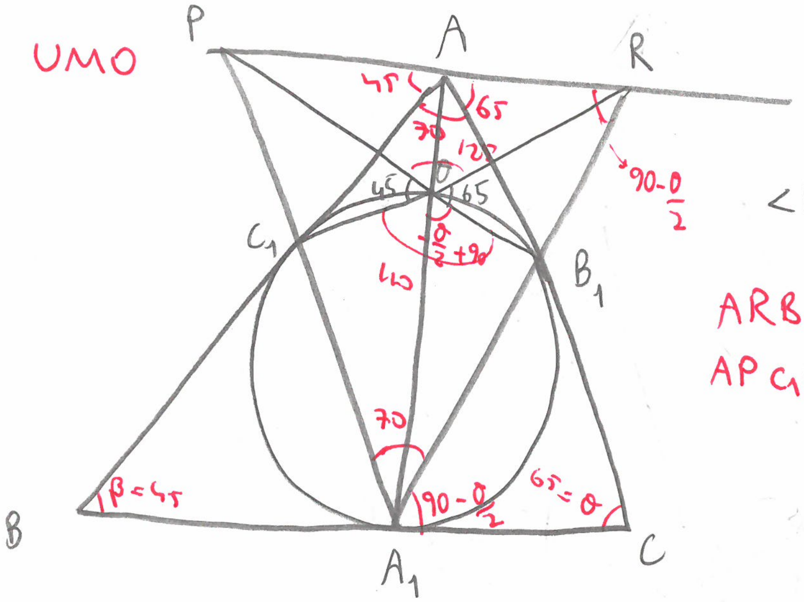
BK, CK, AM, DM içaçıortay.
ABCD yamuk.



$\frac{NM}{7} = \frac{KL}{AL} \wedge \frac{AL}{9} = \frac{7}{r} \Leftrightarrow \frac{NM}{KL} = \frac{7}{AL} = \frac{7r}{7 \cdot 9} = \frac{r}{9}$

$\cos\beta = \frac{3}{r} = \frac{7}{5} \Rightarrow r = \frac{15}{7} \Rightarrow \frac{NM}{KL} = \frac{15}{7 \cdot 9} = \frac{5}{21}$

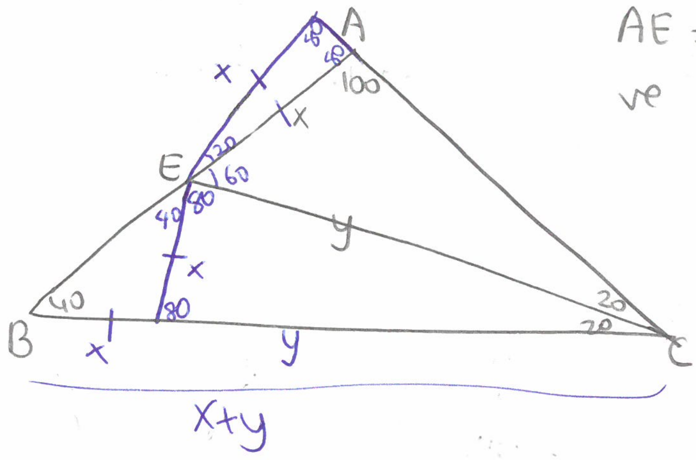
2009 UMO



$\angle PQR = ? = 125$

ARB₁Q ve
APC₁Q çembersel

UMO 2017 Çok Şekil Soru

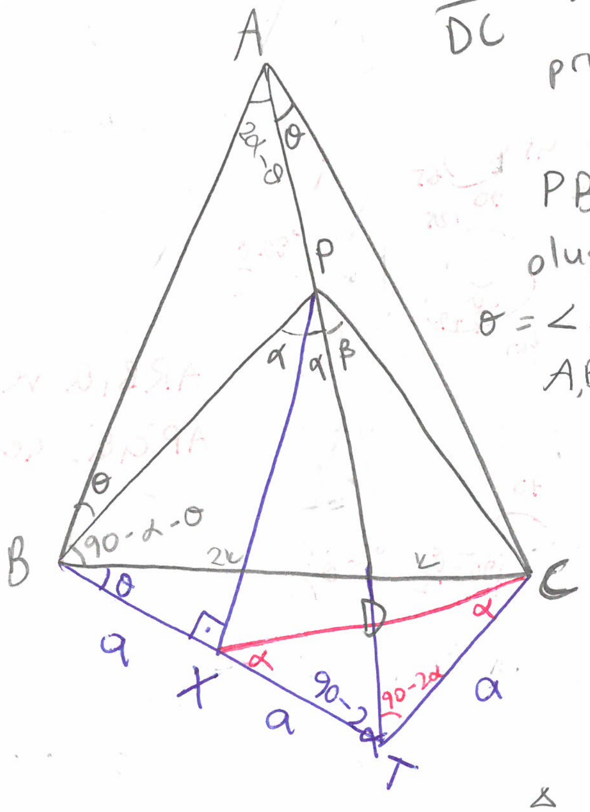


$AE = x$ $EC = y \Rightarrow$ BC'yi x ve y cinsinden bulun.

GÖRME SORUSU

$\frac{BD}{DC} = 2$ $\angle BPD = \angle BAC$

prove $\angle DPC = \angle BAC/2$.



PBT ikizkenar üçgenini oluşturalım.

$\theta = \angle CBT = \angle CAD$ olduğundan A,B,T,C çembersel.

$\angle ATC = 90 - 2\alpha$ gelir.

$[TD]$, $\triangle BTC$'de içaçıortay olduğundan $2.TC = BT = 2a$ olur.

PX dikini indirelim.

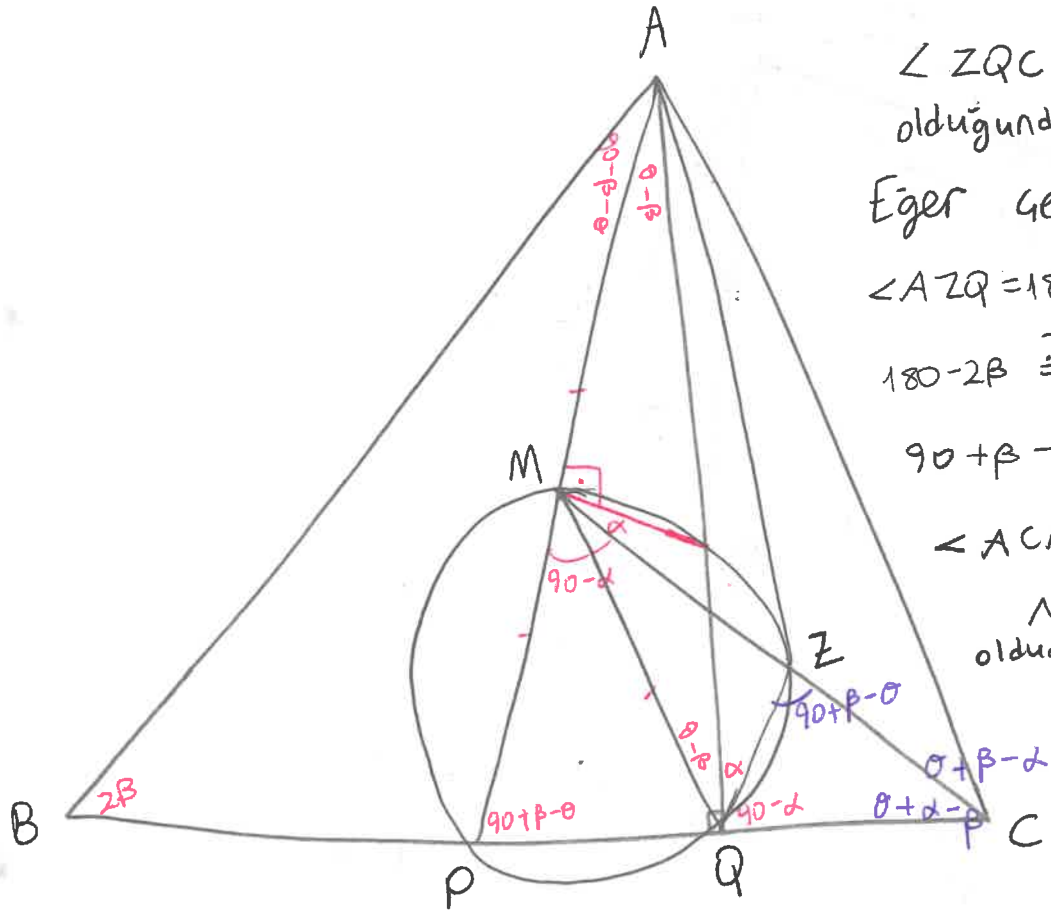
$\triangle XTC$ ikizkenardır ve $\angle XCT = \angle XPT = \alpha$ olduğundan $\angle CPT = \angle CXT = \alpha$ gelir. □

GÜZEL SORU – ZEVKLİ AMA RAHAT –

P noktası $\hat{A}$'nın içaçıortayı. Q, A'dan inilen dikin ayağı. M AP'nin orta noktası. $\triangle PMQ$ nun çev. çemberinin CM'yi kestiği nokta Z olsun. A, Z, Q, B çemberseldir kanıtlayın.

Solution.

$\hat{B}=2\beta$, $\hat{C}=2\theta$ olsun.



$\angle ZQC = \angle ZMP = 90-\alpha$ olduğundan $\angle CZQ = 90+\beta-$

Eğer çemberselse

$\angle AZQ = 180-2\beta$ olmalı.

$180-2\beta \overset{?}{=} 90+\theta-\beta+\angle MZA$

$90+\beta-\alpha \overset{?}{=} \angle MZA$ yani

$\angle ACM \overset{?}{=} \angle MZA$.

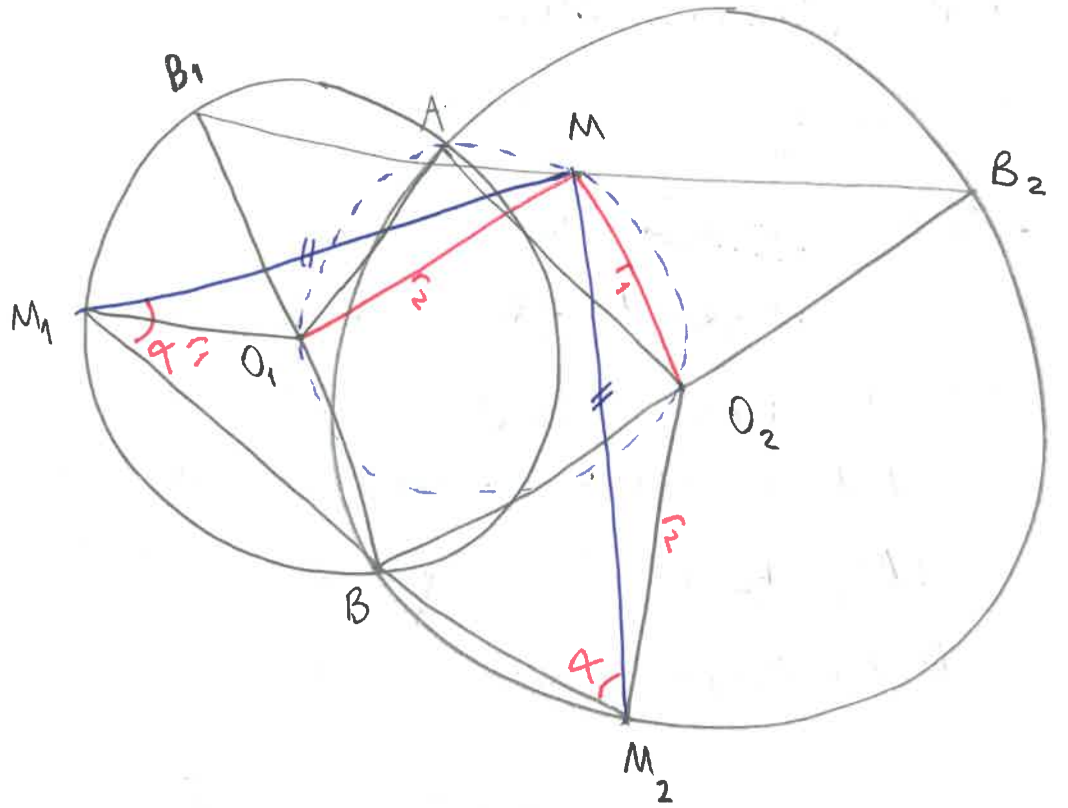
M orta nokta olduğundan $AM=PM=QM$

$\angle ACM = \angle MZA \Leftrightarrow AM^2 = MZ \cdot MC = MQ^2$

$\angle MCQ = \angle ZQM$ olduğundan $MQ^2 = MZ \cdot MC$.

GÖKSEL SORU

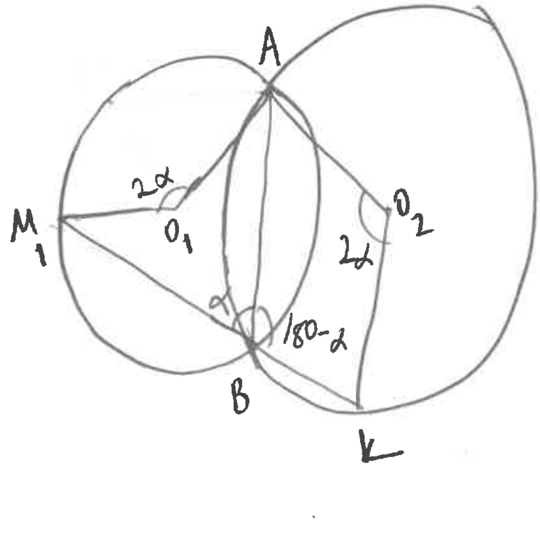
$\angle AO_1M_1 = \angle AO_2M_2$ ise $\angle MM_1B = \angle MM_2B$ old. gösterin.
M, B_1B_2'nin orta noktası.



Solution

M_1BM_2'nin doğrusal old. kanıtlayalım.

M_1BK doğrusal olsun.
$\angle AO_1M_1 = \angle AO_2K$ old. gösterelim.



$B_1M = B_2M \wedge B_1O_1 = O_1B \wedge B_2O_2 = O_2B$ olduğundan

$MO_2 = r_1$, $MO_1 = r_2$ olur.

$\angle O_1AO_2 = \angle O_1BO_2$ (AB yi çekince ikizkenarlardan rahatça görülür.)

$\angle O_1BO_2 = \angle O_1MO_2 \leftarrow$ paralelkenardan gelir.

$\angle O_1MO_2 = \angle O_1AO_2$ olduğundan AMO_2O_1 çembersel.

O zaman $\angle AO_1M = \angle AO_2M$ olur.

Soruda $\angle M_1O_1A = \angle M_2O_2A$ olduğundan

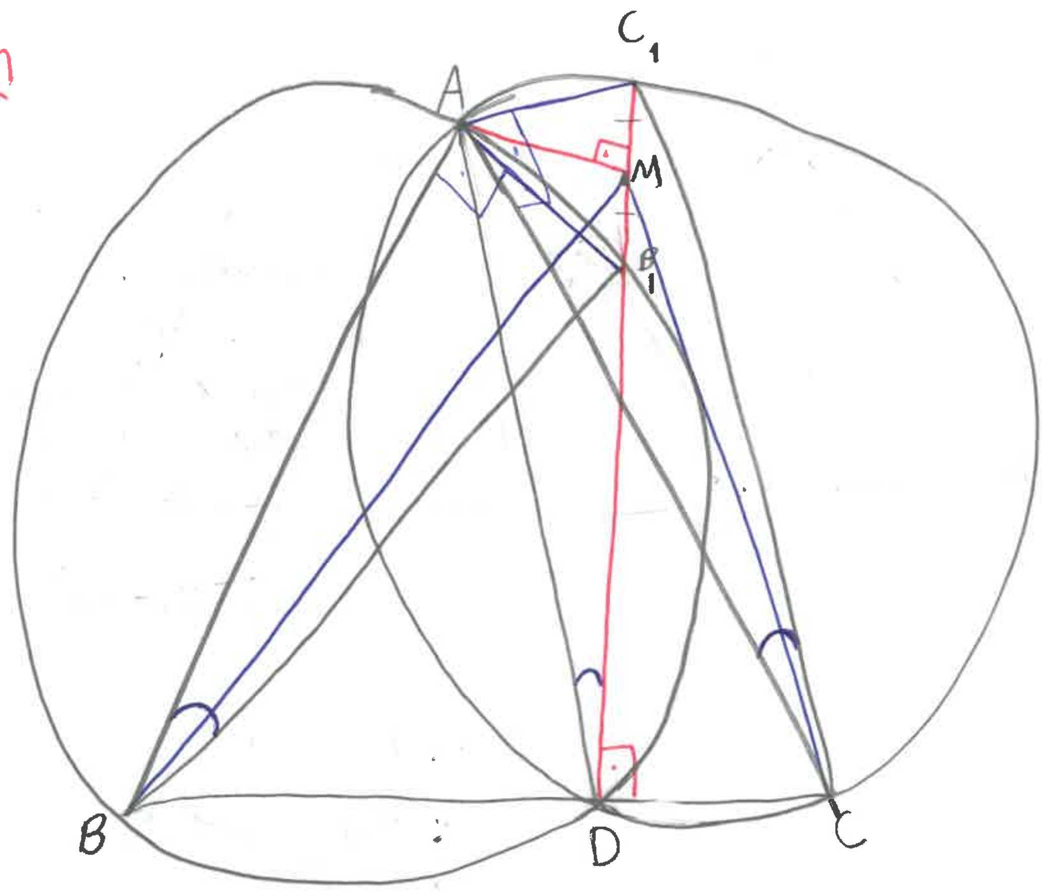
$\angle M_1O_1M = \angle M_2O_2M$ olur. Kenarlar da eşit olduğundan $\triangle M_1O_1M = \triangle M_2O_2M \rightarrow M_1M = M_2M$.

$\triangle M_1MM_2$, M_1BM_2 doğrusal olduğundan ikizkenar üçgendir. $\angle MM_1B = \angle MM_2B$ gelir. $\square$

ŞİRİN KORONALI SORU

$AB = AC$. $D \in [BC]$ $BC > BD > CD > 0$. C_1, C_2 sırayla $\triangle ABD$ ve $\triangle ADC$ 'nin çevrel çemberi. BB_1 ve CC_1 çap. M, $[B_1C_1]$ in orta noktası. $A(MBC)$ sabit olduğunu gösterin

Solution



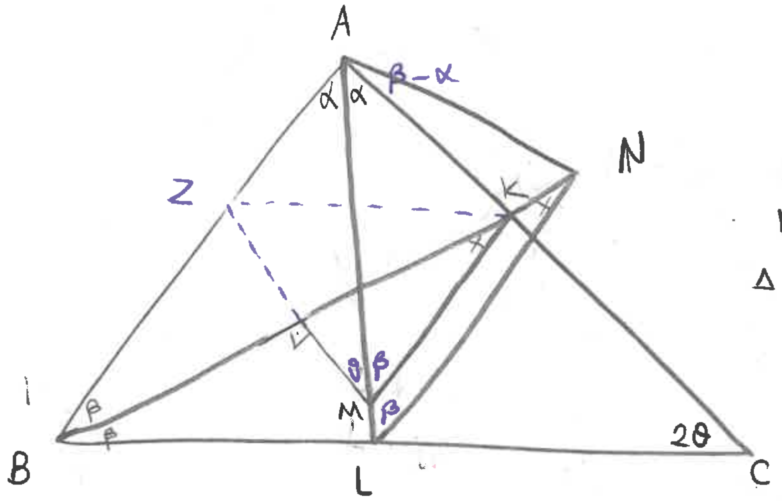
$\triangle ABB_1 = \triangle ACC_1$ (a-a-a ve $AB = AC$ olduğundan)

$AB_1 = AC_1 \Rightarrow AM$ kenar orta dikme

$MD = H_A$ olduğundan $A(BMC) = A(ABC)$

GÖRME SORU

iç Açıortaylar çiziliyor. Bk'nın kenar orta dikmesi Al'yi M de keser. LN // Mk ise NA = NL old. gösterin.



M'yi uzatalım. Z'de kessin. $\angle ZMA = \theta$ gelir.
$\triangle BZK$ ikizkenar geldiğinden
$\triangle ZAK$'de AM iç açıortay
ZM dışaçıortay.
Demek ki $M = I_A$.
$\angle AZK = 2\beta$ olduğundan
$\angle KMA = \beta$..

O zaman, ABMK çembersel.

$\alpha = \angle ZAM = \angle ZKM = x. = \angle BNL = \angle BAL$ olduğundan

ANBL de çembersel. O zaman $\angle ABN = \angle ALN = \beta$ ve

$\angle NBL = \angle NAL = \beta$ Burdan $\triangle ANL$ ikizkenar gelir $AN = NL$.

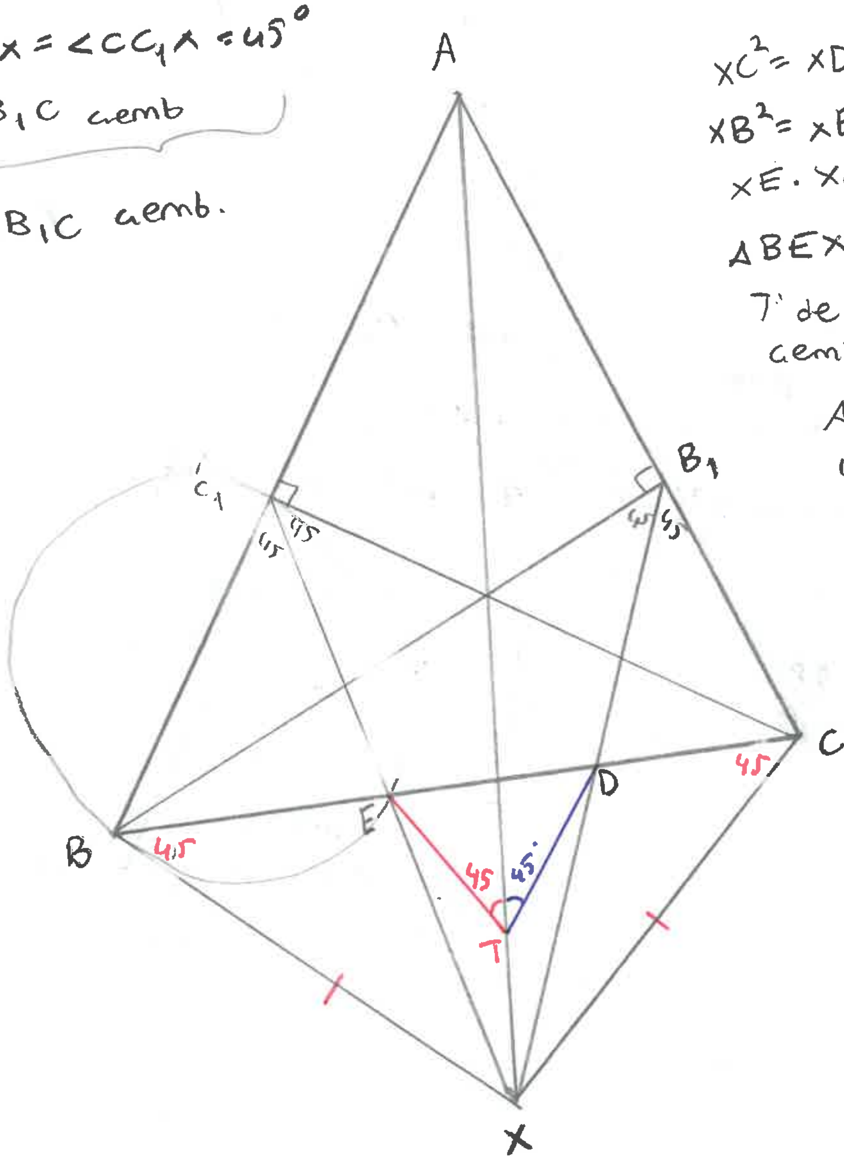
İftihar Hocanın Korona Sınavı (2. Aşama)

2. Soru

$\angle CB_1X = \angle CC_1X = 45^\circ$

$\underbrace{BC_1B_1C \text{ çemb}}$

XBC_1B_1C çemb.



$\left.\begin{array}{l} XC^2 = XD \cdot XB_1 \\ XB^2 = XE \cdot XC_1 \end{array}\right\} \Rightarrow C_1B_1DE$ çemb.

$XE \cdot XC_1 = XD \cdot XB_1$

$\triangle BEX$'in çev. çemberi AX'i T'de kessin. $TXCD$'nin çemb. olduğunu gösterelim.

$ATEC_1$ çemb. gelir.

O zaman

$\underbrace{XE \cdot XC_1} = XT \cdot XA$

$XD \cdot XB_1$

$\downarrow$

Burdan $ATDB_1$ de çemb. gelir.

$\underbrace{\angle ATD} = \angle DB_1C = 45^\circ$

$\angle ATD = \angle DCX$ geldi demek ki $XTDC$ de çemb.

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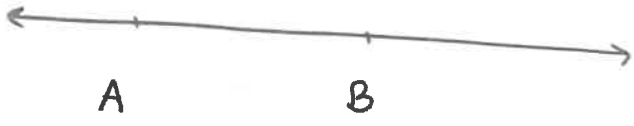
HARMONİK BÖLME



AB doğrusu üzerinde $\frac{TA}{TB} = k, (k \neq 1)$ olan tek bir T noktası vardır.

→ $k<0$ ise T, A ile B arasında

→ $k>0$ ise T, A ile B dışında

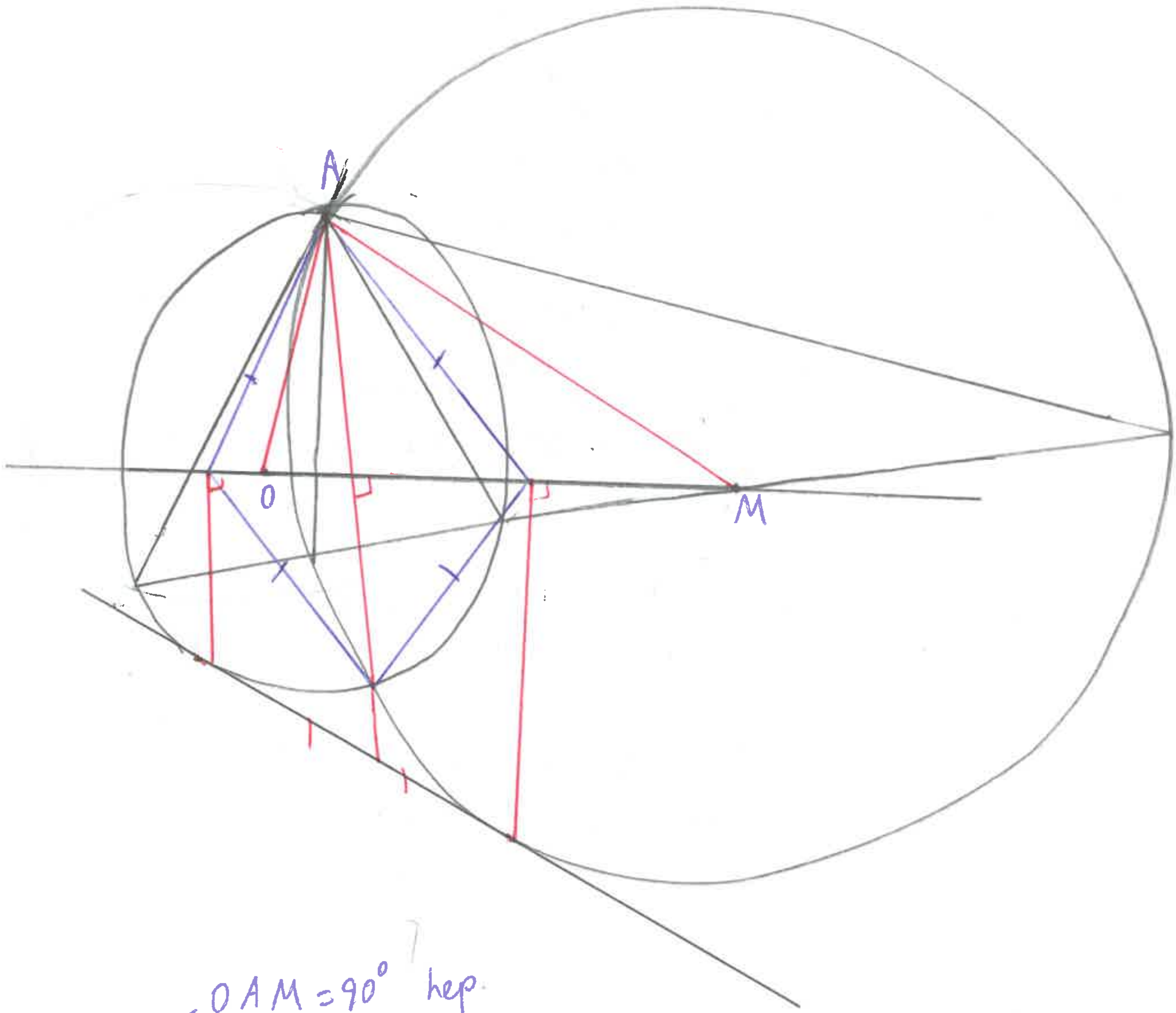
→ $k \to 1$ ise T AB doğrusu üzerinde sonsuza uzaklaşır.

- AB üstünde alınan her C noktasına $\frac{DA}{DB} = -\frac{CA}{CB}$ olacak şekilde bir D noktası karşılık gelir.

$$* \quad \frac{CA}{CB} + \frac{DA}{DB} = 0 \Rightarrow \frac{CA}{CB} \cdot \frac{DB}{DA} = -1$$

* C ve D noktalarına A ve B'nin, karşılık olarak A ve B noktalarına da C ve D'nin harmonik eşlenikleri denir ve (ABCD) diye gösterilir. İlk iki nokta son 2 noktaya göre harmonik eşleniktir denir.

* (ABCD) bölmesinin harmonik olması için gerek ve yeter şart; |AB| 'nin |AC| ve |AD| nin harmonik ortalaması olmasıdır.

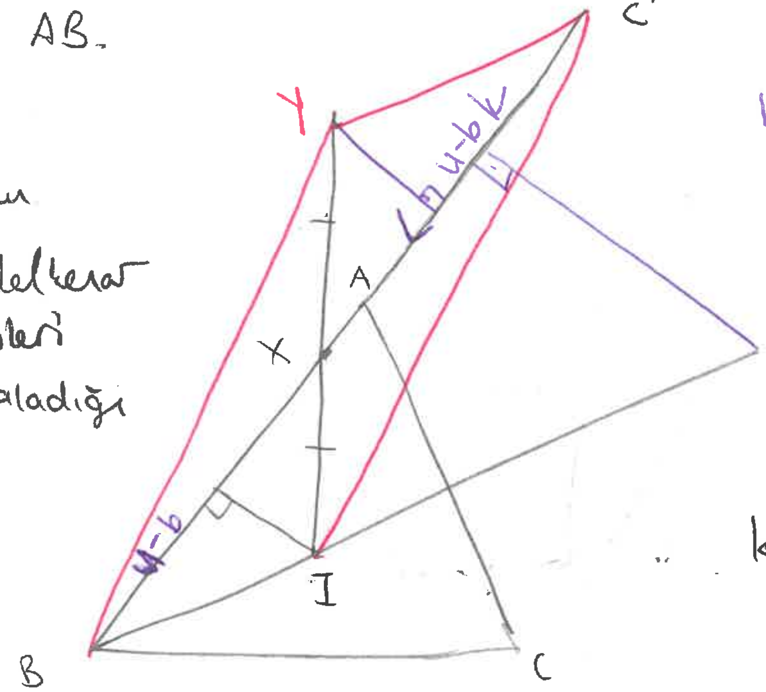


$\angle OAM = 90^\circ$ hep.

St. Petersburg

$2BX = AB+BC$ olan X _ AB üstünde alınsın. Y. I'nın simetrisi

Prove $I_BY \perp AB$.



$AC' = BC$ olsun

$BYC'I$ paralelkenar olur. köşegenleri birbirini ortaladığı için

$kC' = u-b$ old. gösterelim

$BC' = a+c$

$Bk = u$.

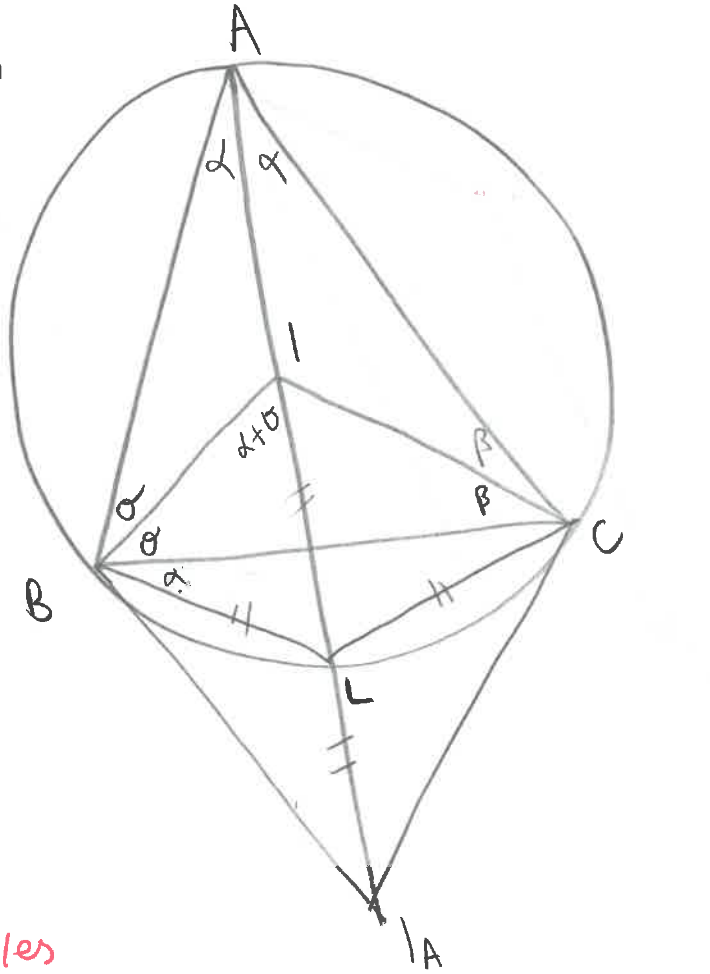
I_B $kC' = a+c-u$

$= u-b$

$kC' = LC' \Rightarrow k = L$.

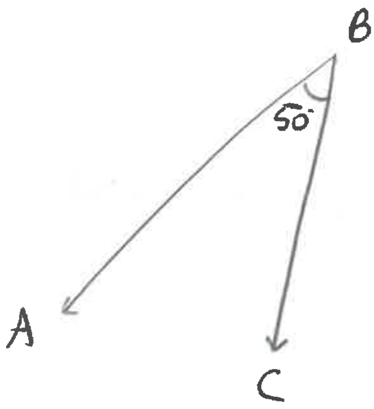
Lemma (The Excenter / Incenter Lemma)

I_A is the reflection of I over L. Then $BICI_A$ is cyclic.



Directed Angles

$\measuredangle ABC > 0$ if A B C appear in clockwise order.

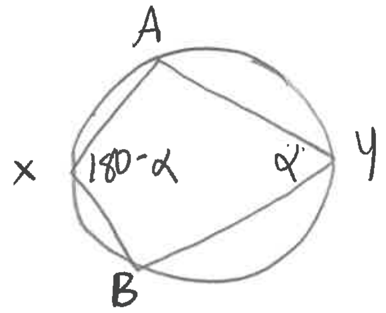


$\measuredangle ABC = 50^\circ$

$\measuredangle CBA = -50^\circ$

Points A, X, B, Y lie on a circle if and only if

$\measuredangle AXB = \measuredangle AYB.$



$\measuredangle AXB = -(180-\alpha)$

$\measuredangle AYB = \alpha$

$-180+\alpha \equiv \alpha \pmod{180}$

Directed Angle Addition: $\measuredangle APB + \measuredangle BPC = \measuredangle APC$.
Triangle Sum: $\measuredangle ABC + \measuredangle BCA + \measuredangle CAB = 0°$.

Such an Easy IMO shortlist Problem.

Let $\triangle ABC$ be acute. The feeds of altitudes are D, E, F accordingly. One of the intersection points of the line EF and the circumcircle is P. The lines BP ve DF meet at point Q. Prove that $AP = AQ$.

Solution

$ABCP$ and $AFDC$ are cyclic. $\measuredangle APQ = \measuredangle PQA$

$\measuredangle APB = \measuredangle ACB = \measuredangle BFD = \measuredangle AFQ$ Thus

$APQF$ are cyclic.

We also know that $BFEC$ is cyclic.

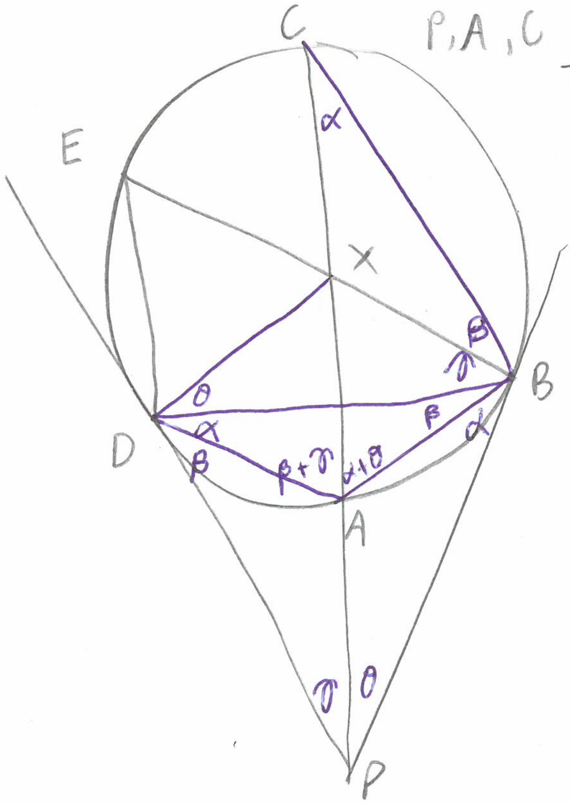
$\measuredangle ACB = \measuredangle ECB = \measuredangle EFA = \measuredangle PFA$.

In $APQF$,

$\measuredangle APQ = \measuredangle APB = \measuredangle ACB = \measuredangle PFA = \measuredangle PQA$. $\square$

EASY JBMO 2011/5

$DE \parallel AC$, PD and PB are tangent P, A, C collinear. Show that $\overline{EB}$ bisects $\overline{AC}$.



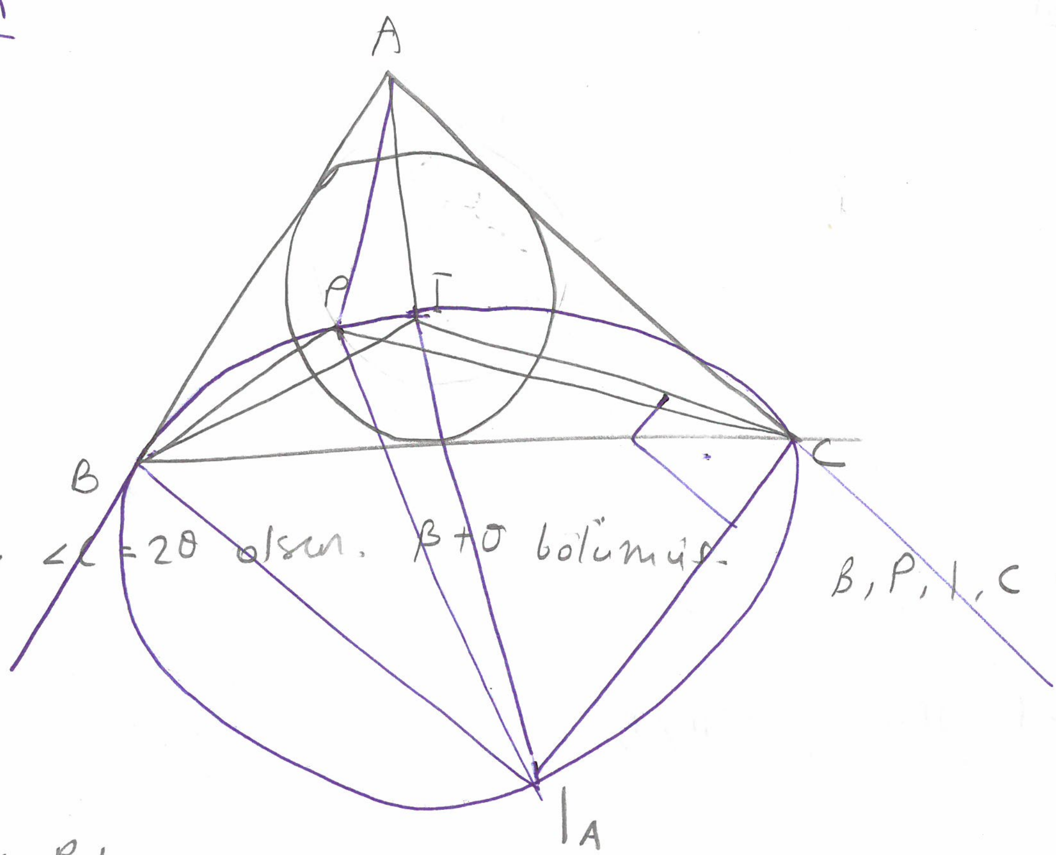
$\triangle CBX \sim \triangle DBA$
$\Downarrow$
$$\frac{CX}{DA} = \frac{XB}{AB}$$
$\triangle DXA \sim \triangle AXB$
$\Downarrow$
$$\frac{XB}{AB} = \frac{AX}{DA}$$

$CX = AX$ $\square$

A Little Easy IMO Problem 2006/1.

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$. Equality holds if and only if $P = I$.

Solution



$\angle B = 2\beta$, $\angle C = 2\theta$ olsun. $\beta + \theta$ bölümüş.

B, P, I, C çembersel

$$AP + PI_A \geq AI + II_A$$

$PI_A \leq II_A$ çünkü II_A çap.

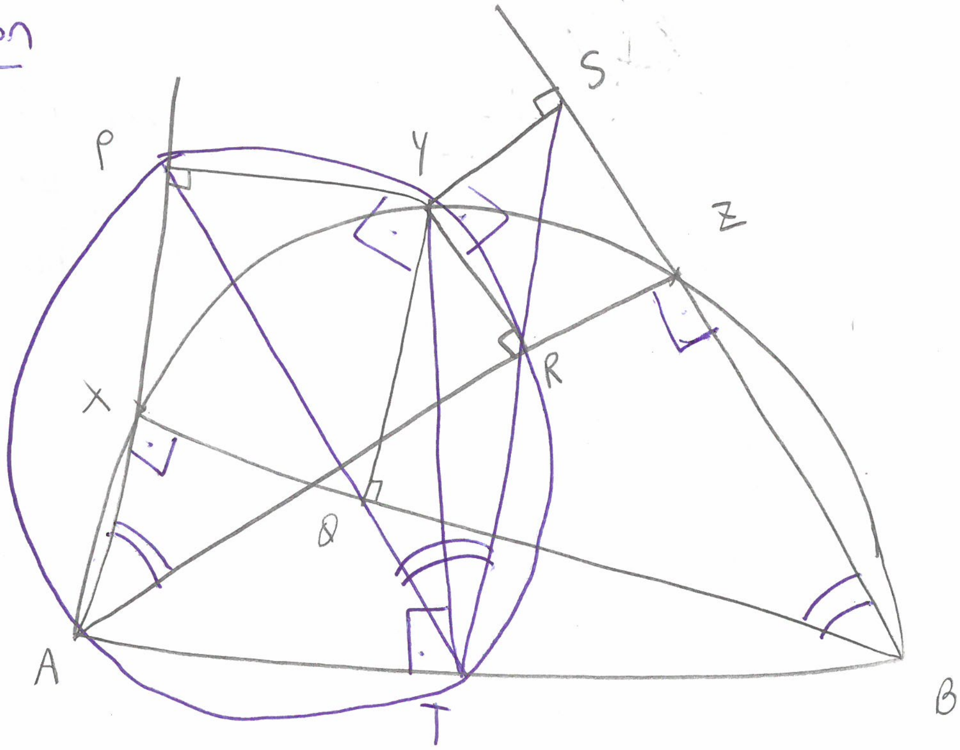
$\Downarrow$

$AP \geq AI$ olmak zorunda. $\square$

NICE USAMO 2010/1

Let AXYZ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by, P, Q, R, S the feet of the altitudes from Y onto lines AX, BX, AZ, BZ, resp. Prove that the acute angle formed by the lines PQ and RS is the half of $\angle XOZ$, where O is the centre of the circle

Solution

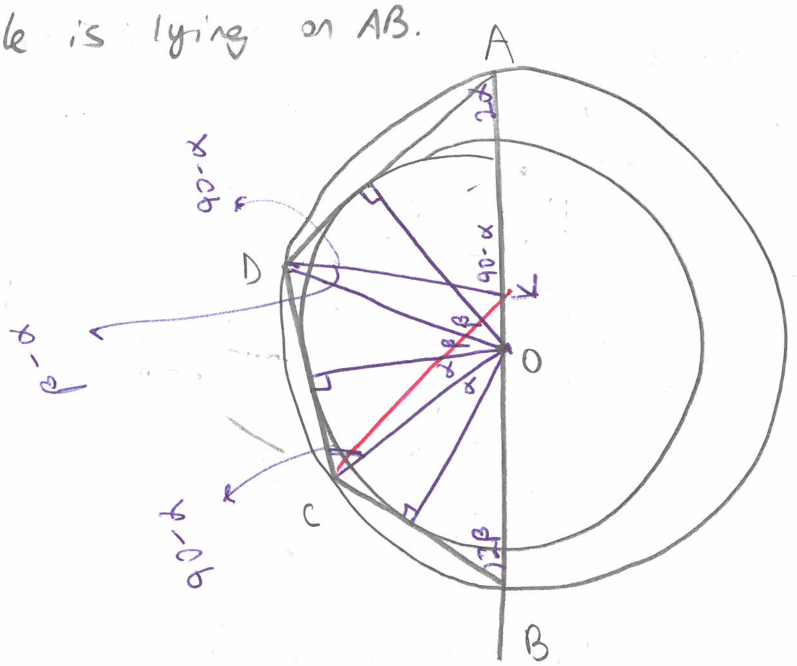


PQ and RS meet on AB because of the Simpson lines in $\triangle ABX$ and $\triangle AZB$. Say T is the point they meet than $YT \perp AB$. AB is diameter so $\angle AXB = \angle AZB = 90^\circ$

Thus $\angle PYQ = \angle SYR = 90^\circ$ Y, R, T, A, P are on the same circle $\square$.

EASY IMO QUESTION ~~1985~~ 1985 / 1

Show that $AB = AD + BC$. The centre of the little circle is lying on AB.



DKOC are cyclic

Thus,

$\angle CKB = 90-\beta$
$= \angle KCB$

Then $AK = AD$

$+\ BK = CB$

$\underline{AB = AD + BC}$. $\square$

Harika Temmuz Sınavında Harika Bir soru

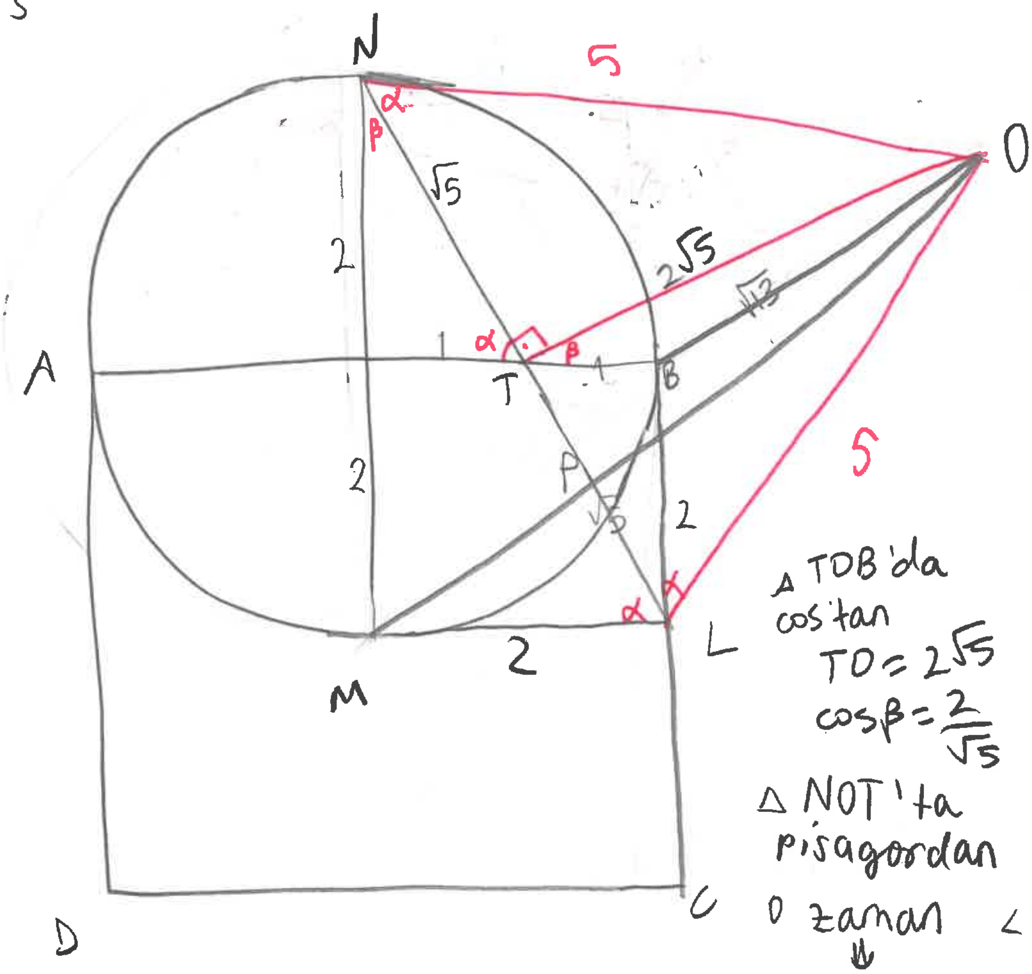
$|AB| = 4$

$|OB| = \sqrt{13}$

$MP \cdot OP = \frac{r}{s}$

$m(r+s) = ?$

N ve L'den geçen çem. merkezi O. AB ve MN ç çemberinin çapları.



$\triangle TOB$'da cos'tan

$TO = 2\sqrt{5}$

$\cos\beta = \frac{2}{\sqrt{5}}$

$\triangle NOT$'ta pisagordan $NO = 5$

O zaman $\angle ONT = \alpha$

$\Downarrow$

$\angle ONM = \alpha + \beta = 90°$

ON teğetmiş.

O halde $OM = \sqrt{41}$, pisagor

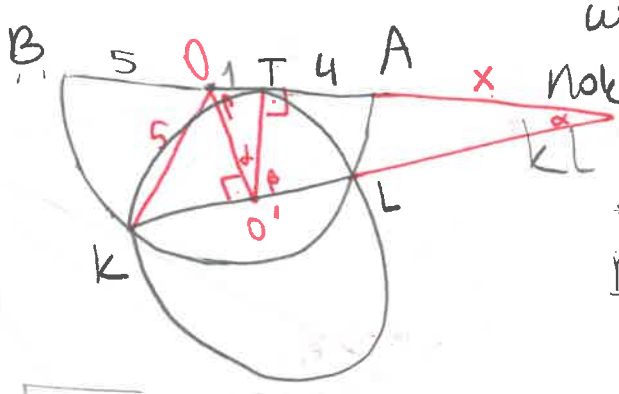
$\angle NLM = \angle NLO = \alpha$ olduğundan

$MP = 2k \quad PO = 5k$

$7k = \sqrt{41} \qquad \frac{2\sqrt{41}}{7} \cdot \frac{5\sqrt{41}}{7} = \frac{10 \cdot 41}{7 \cdot 7} = \frac{410}{49} = \frac{r}{s}$

$r + s = 410 + 49 = 459.$

TEMMUZ SINAVININ HOŞ GEOMETRİ SORUSU



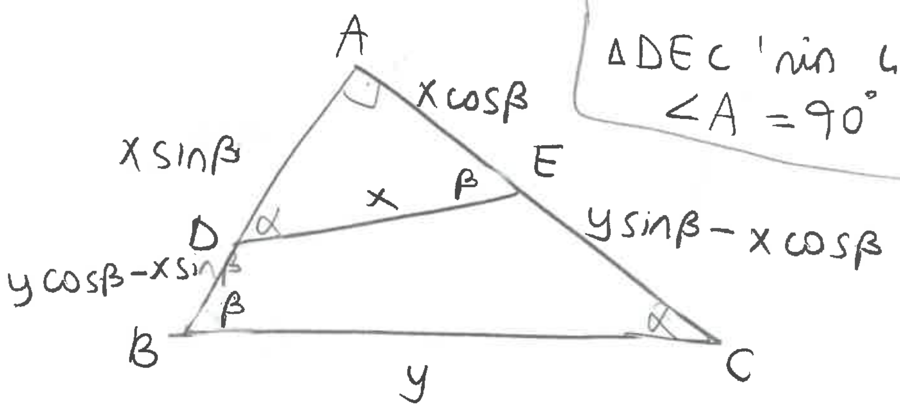
AB çaplı yarım çemberin üstündeki K ve L noktalarının oluşturduğu KL çaplı çember AB'ye T'de teğet.
BT=6, AT=4 ise KL=?

$OO' = \sqrt{5+x}$ $\quad KO' = \sqrt{25-5-x} = \sqrt{20-x} = O'T = \sqrt{4+x}$

$20-x = 4+x \quad x = 8.$

$KL = 2KO' = 2\sqrt{20-x} = 2\sqrt{12} = 4\sqrt{3}.$

YİNE TEMMUZ'DAN ZOR GÖZÜKEN KOLAY SORU



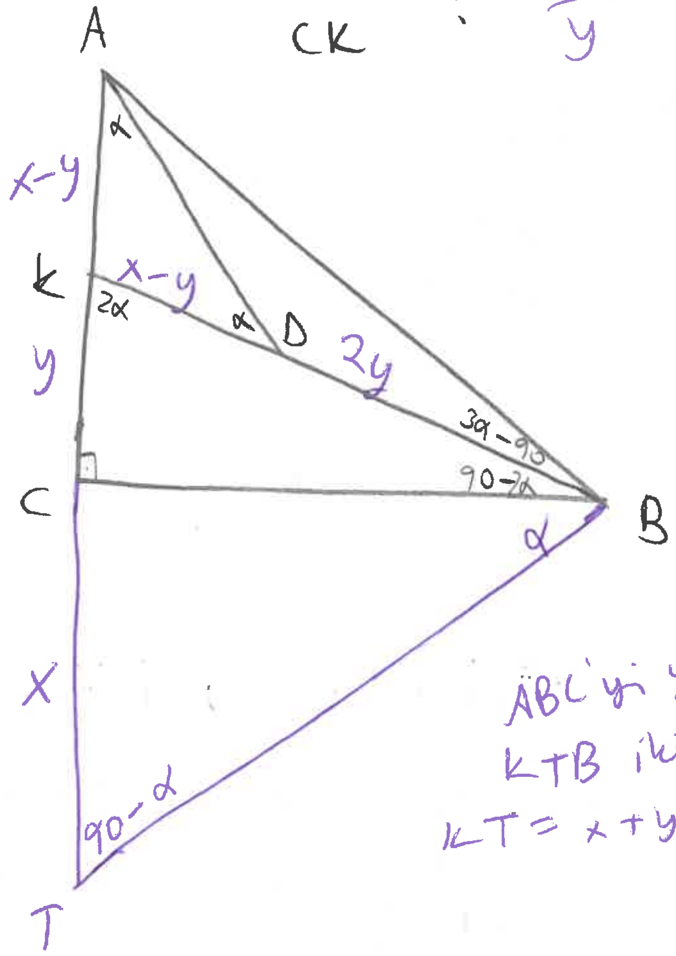
$\triangle DEC$'nin çev. çem. r'si = ?
$\angle A = 90°$ $\angle AED = \angle ABC$.

$$(y\sin\beta - \cancel{x\cos\beta})\cancel{x\sin\beta}\,\frac{1}{\cancel{2}} = \frac{\cancel{x}(y\sin\beta - \cancel{x\cos\beta})\cancel{\sin\beta}\sqrt{x^2+y^2}}{\cancel{2}4R}$$

$DC = \sin\beta\sqrt{x^2+y^2}$ $\quad \boxed{R = \frac{\sqrt{x^2+y^2}}{2}}$

KAYMAK GİBİ BİR ÇÖZÜM
(ALTAN'DAN)

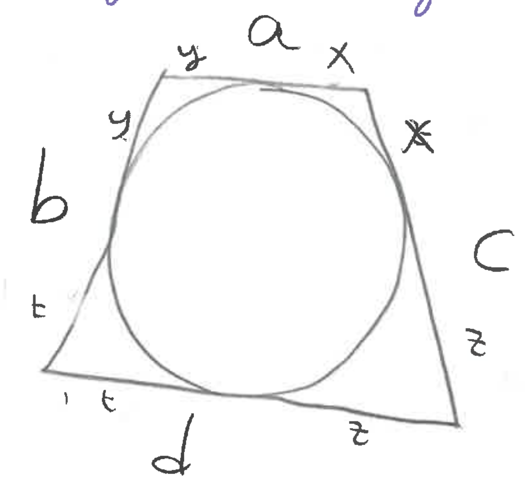
$\frac{BD}{CK} = ? = \frac{2y}{y} = 2.$



ABC'yi yansıt.
KTB ikizkenar geldi!
$KT = x+y = KB \Rightarrow DB = 2y$

Teğetler Dörtgeni

Değerli Bilgi



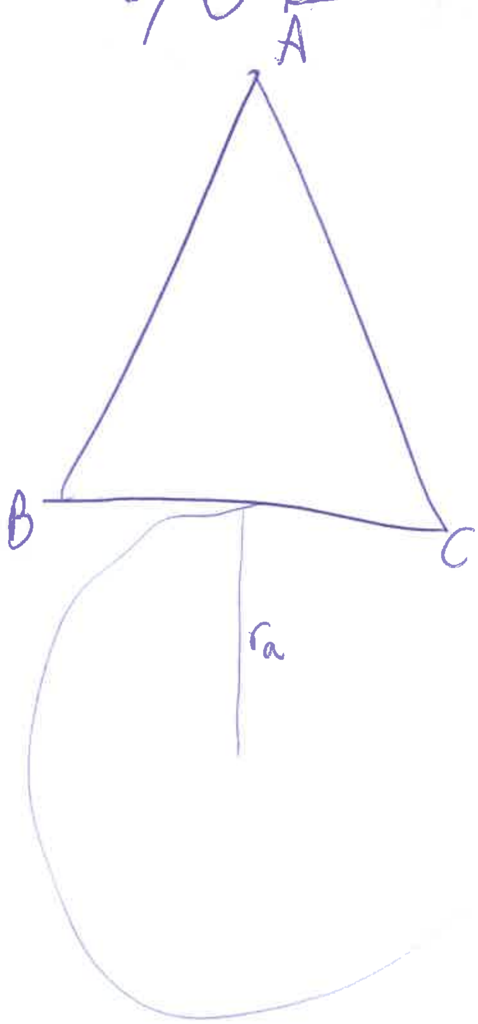
$a+d=b+c$

kanıt çok basit

yaz x, y, z, t diye

$$\underbrace{x+y}_{a}+\underbrace{t+z}_{d}=\underbrace{y+t}_{b}+\underbrace{x+z}_{c}$$

ÇOK ÖNEMLİ BİLGİ



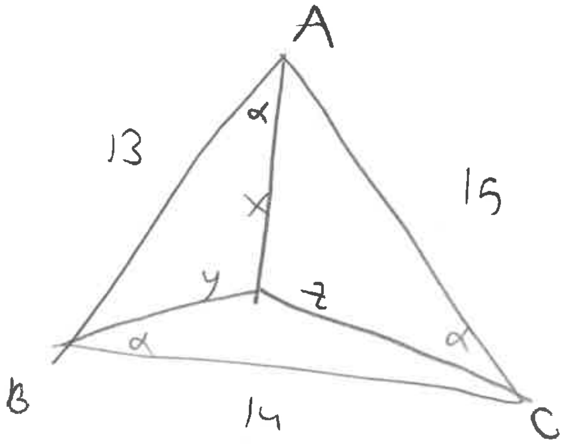
$r_a \to$ A kenarının dış teğet çem. yarıçapı

$$\text{Alan}(ABC)=r_a(u-a)$$
$$=r_b(u-b)$$
$$=r_c(u-c)$$

HOŞ SORU

ABC üçgeninde bir P noktası için $AB=13, BC=14, CA=15$.

$\angle PAB = \angle PBC = \angle PCA$ ise $\tan(PAB)=?$



$$\text{Alan} = \frac{14.12}{2} = 84.$$

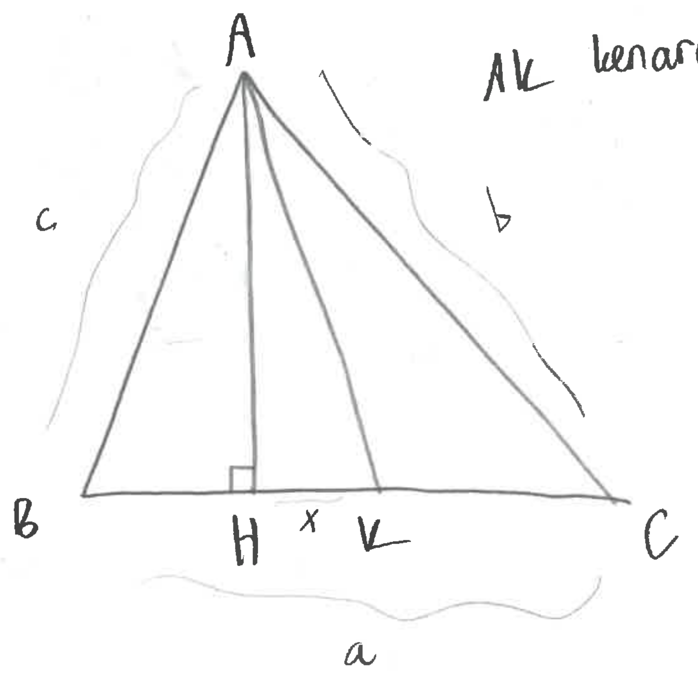
$$\frac{1}{2}\sin\alpha\,(13x+14y+15z) = 84.$$

ikisini oranlarsak $\tan\alpha$ gelir.

$$13^2+\cancel{x^2}-2x.13\cos\alpha=\cancel{y^2}$$
$$14^2+\cancel{y^2}-2y.14\cos\alpha=\cancel{z^2}$$
$$15^2+\cancel{z^2}-2.z.15\cos\alpha=\cancel{x^2}$$

$$13^2+14^2+15^2=2\cos\alpha\,(13x+14y+15z).$$

MİSLER GİBİ TEOREM



AK kenarortay.

$$b^2 - c^2 = 2ax$$

ZOR SORU

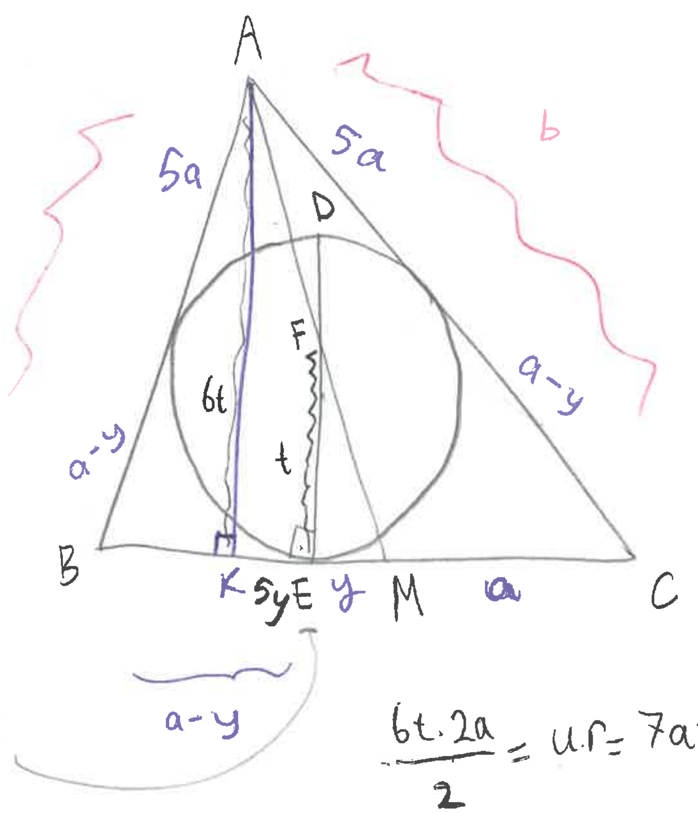
AC > AB.

AM kenarortay

$\frac{DF}{FE} = ?$

AB + AC = 6.BC

GÖZÜM



$b^2 - c^2 = 2.(2a)\ KM$

$(b-c)(b+c) = \cancel{2}(\cancel{2}a) KM$

$3\cancel{b}.(2\cancel{a})$

$3(b-c) = KM = 3.2y = 6y$

$6a - y - 6a + y = 2y$

$\frac{6t.2a}{2} = u.r = 7ar$

$t = \frac{7}{6}r$

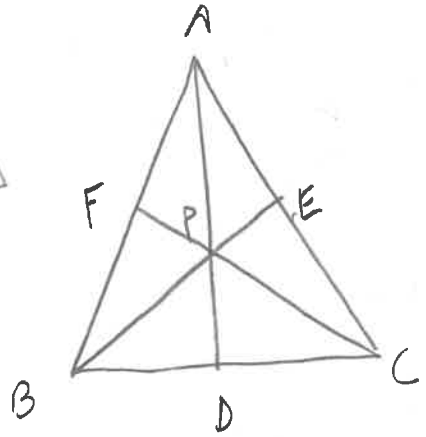
$FD = 2r - t = \frac{5}{6}r$

$\boxed{\frac{DF}{FE} = \frac{5}{7}}$

41

Von Aubel Teoremi

BAK BİL BUNU!



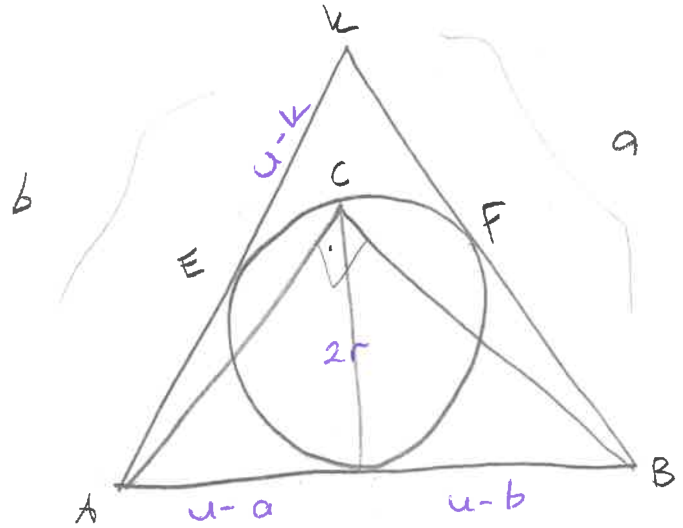
$$\frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=2$$

$$\frac{PD}{AD}+\frac{EP}{BE}+\frac{FP}{CF}=1$$

22 Ağustos'taki Fevkalade Geometrisi

$\angle ACB = 90°$ olduğu veriliyor.

S(ABK) KE'nin kaç katıdır.



$$4r^2=(u-a)(u-b)$$

$$4r^2\cdot u^2(u-k)=u^2(u-k)(u-a)(u-b)$$

$$4S^2\cdot(u-k)=S^2\cdot u$$

$$4(u-k)=u$$

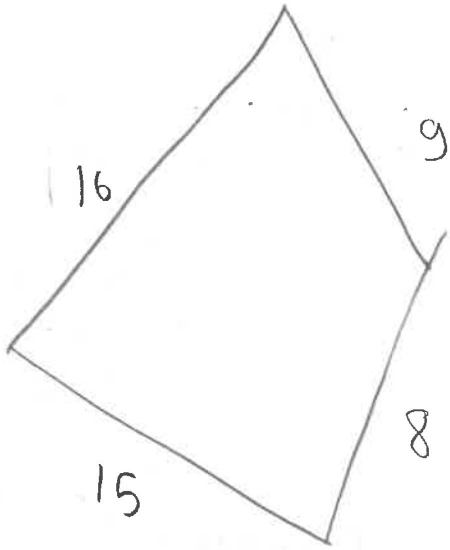
$$3u=4k$$

$$u=\frac{4}{3}k \qquad u-k=\frac{1}{3}k$$

$$\frac{2u}{u-k}=?$$

$$\frac{2u=\frac{8}{3}k}{u-k=\frac{1}{3}k}=8$$

Güzel Soyru



Bu dörtgenin içine çizilebilecek en büyük çemberin yarıçapı kaçtır?

Çözüm

$8+16=9+15$ olduğundan bu bir teğetler dörtgeni.

ÖNEMLİ ! Bir dörtgenin alanının en büyük olması için bunun kirişler dörtgeni olması lazım.

Neden? çünkü $e^2f^2=a^2c^2+b^2d^2-2abcd\cos(A+C)$

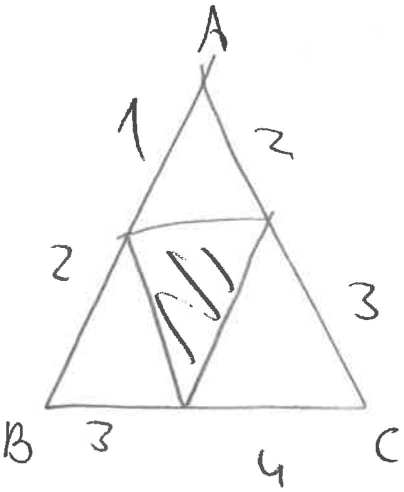
$Alan = ef\cdot\sin\alpha\cdot\frac{1}{2}$ olduğundan alan büyük olması için ef büyük olmalı. ef'nin büyük olması için $2abcd\cos(A+C)$ küçük olmalı. $\cos(A+C)=\cos 180=0$.

Yani dörtgen kirişler dörtgeni

Brahma Gupta'dan $A=\sqrt{24\cdot8\cdot16\cdot15\cdot9}=24r$

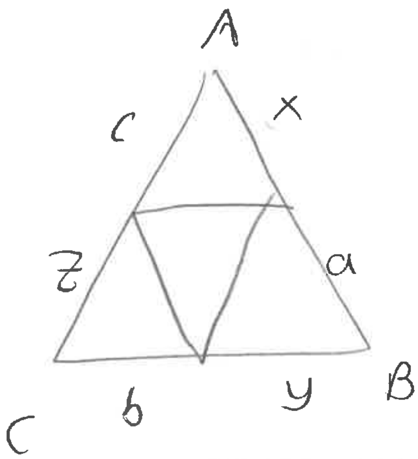
$u=24$

teğet çem. yarıçap



$$\frac{2.4.2+3.3.1}{5.7.3} = \frac{\text{Taralı Alan}}{\text{Tüm Alan}}$$

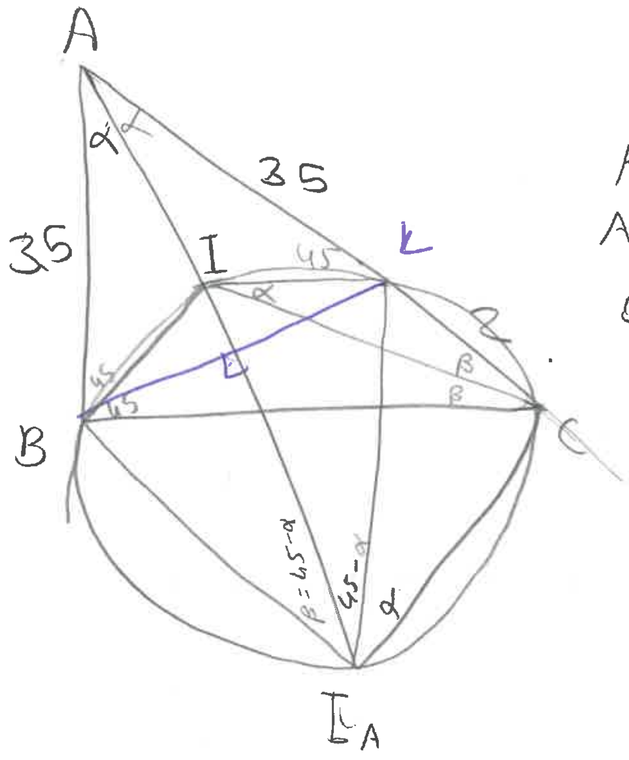
$$= \frac{25}{5.21} = \frac{5}{21}.$$



$$\frac{xyz+abc}{(x+a)(b+y)(c+z)} = \frac{\text{Taralı Alan}}{\text{Tüm Alan}}$$

Benim uzun yoldan çözdüğüm soru

$BC = 12 \quad AB = 35 \quad AC = 37$



Açılarsak

ABI_Ak deltoid gelir.

O zaman

ABk ikizkenar

O halde $Ak = 35$

$kC = 2$ kalır.

ZOR DENEMENİN GÜZEL SORUSU

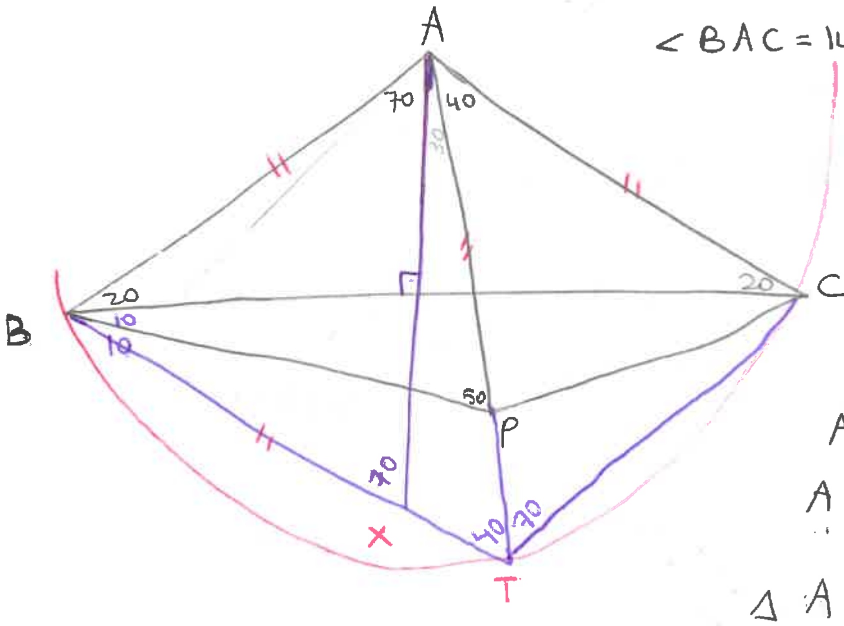
$|AD| = 2 \quad |BC| = 5$ ABCD kirişler dörtgeni

AB ve CD kenarlarının orta noktaları sırasıyla M ve N

$MN \perp CD \quad BD = 11$ ise $|MN| = ?$

GÖZDÜM SORUYU

$\angle BPA = 50°$ $\angle CBP = 10°$
$\angle BAC = 140°$



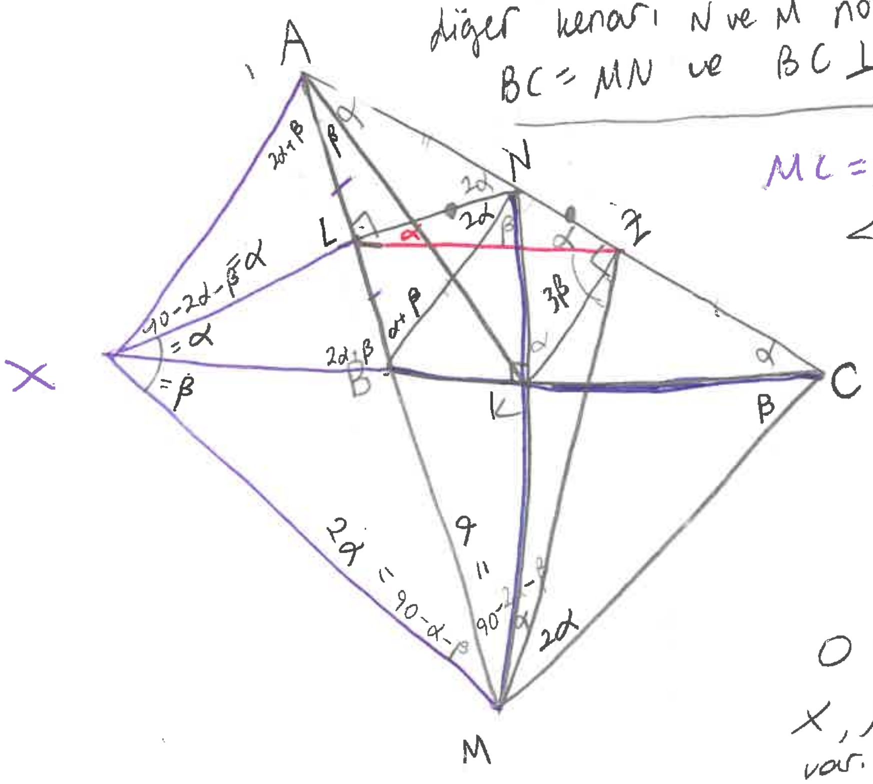
AX'i indir.
AT'yi indir.
$AC \parallel BT$ olur.

$\triangle AXB = \triangle TCA$

$AT = TC$ $AXTC$
ikizkenar yamuk.

ÇOK GÜZEL SORU

AB ve AC nin kenar orta dikmeleri diğer kenarı N ve M noktalarında kesiyor. $BC = MN$ ve $BC \perp MN$ ise $\hat{A} = ?$



$ML = AM$ $\angle NCB = \alpha$, $\angle BCM = \beta$ olsun.
$\angle AXN = \angle AMN = 90 - 2\alpha - \beta$

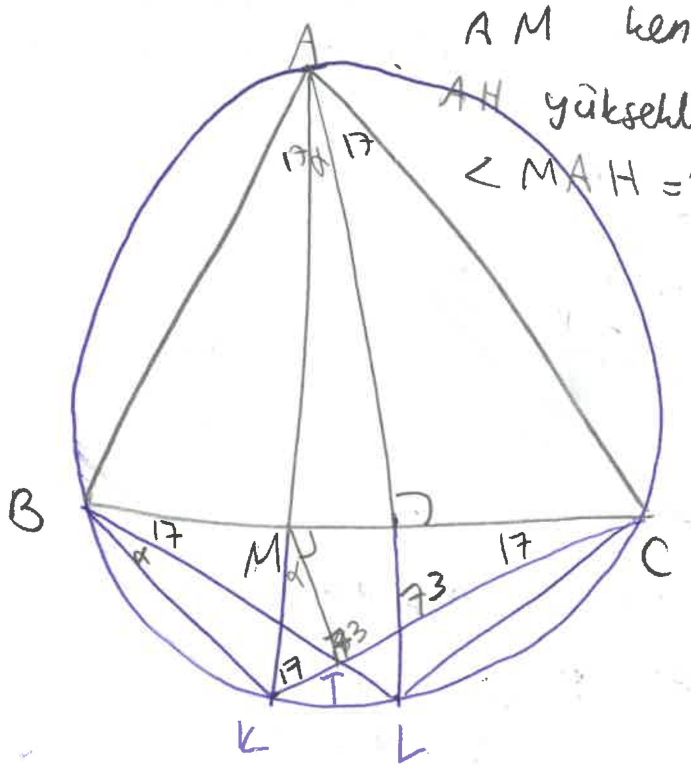
BN'yi çekince BNCM çembersel gelir.

$90 - 2\alpha - \beta = \alpha$
$90 = 3\alpha + \beta$
O halde M merkezli X, A, C'den geçen çember var. $XM = CM$ o zaman
$XK = KC$

$3\beta = 2\alpha + \beta$
$\alpha = \beta$.

$90 = 4\alpha$
$2\alpha = 45$

Çevrel çember gerekli!



AM kenarortay

AH yükseklik ; $\angle BAM = \angle CAH = 17$

$\angle MAH = ?$

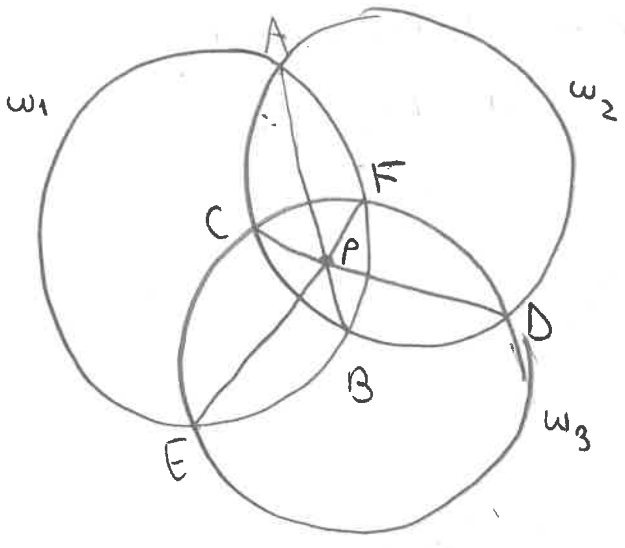
Çevrel çemberi çizelim.

$TM \parallel AL \quad \angle KMT = \alpha$

$\alpha + 17 = 73$

$\alpha = 56$

THEOREM



Show that they intersect at one point.

$Pow_w(P) = OP^2 - r^2$. In w_1 and w_3, EF is the radical axis.

So, $\pm PA \cdot PB = Pow_{w_1}(P) = Pow_{w_2}(P) = \pm PC \cdot PD$

in the same way $Pow_{w_2}(P) = Pow_{w_3}(P)$

$CP \cdot PD = EP \cdot PF$

Thus,

$EP \cdot PF = PA \cdot PB$ $\square$

Euler's Theorem

Let ABC be a triangle. Let R and r denote its circum-radius and inradius, respectively. Let O and I denote its circum-center and incenter. Then $OI^2 = R(R-2r)$

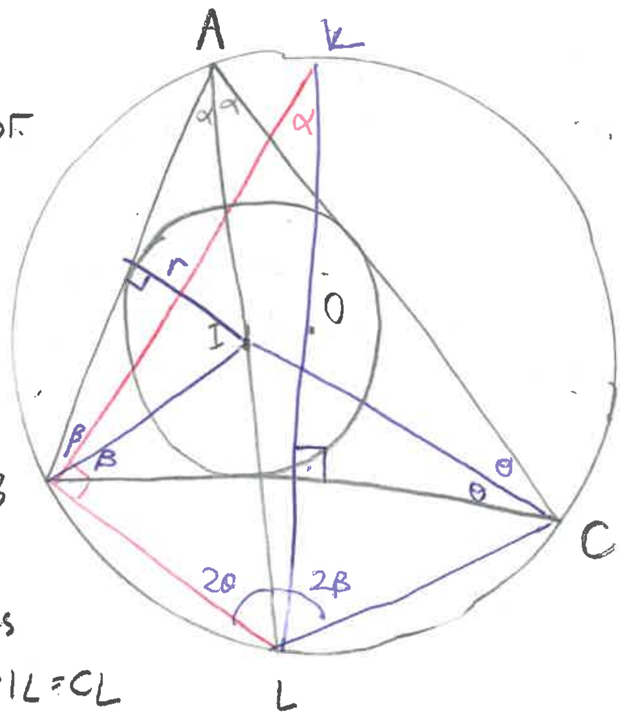
Proof

$$OI^2 = R(R-2r) \Rightarrow \underbrace{R^2 - OI^2}_{} = 2rR = (R-OI)(R+OI) = AI.IL$$

So, we have to prove $2rR = AI.IL \overset{Pow(I)}{\Leftrightarrow} \frac{r}{AI} = \frac{IL}{2R} = \sin\alpha$ (say $\angle BAI = \alpha$).

Let AI intersect the circle at point L.

Let LO intersect " " " " K.



$OL \perp BC$ because AL is the angle bisector

$$\sin\alpha = \frac{BL}{KL} = \frac{BL}{2R}$$

$\angle IBC = \beta \Rightarrow \angle ILC = 2\beta$

and

$\angle ICB = \theta \Rightarrow \angle ILB = 2\theta$

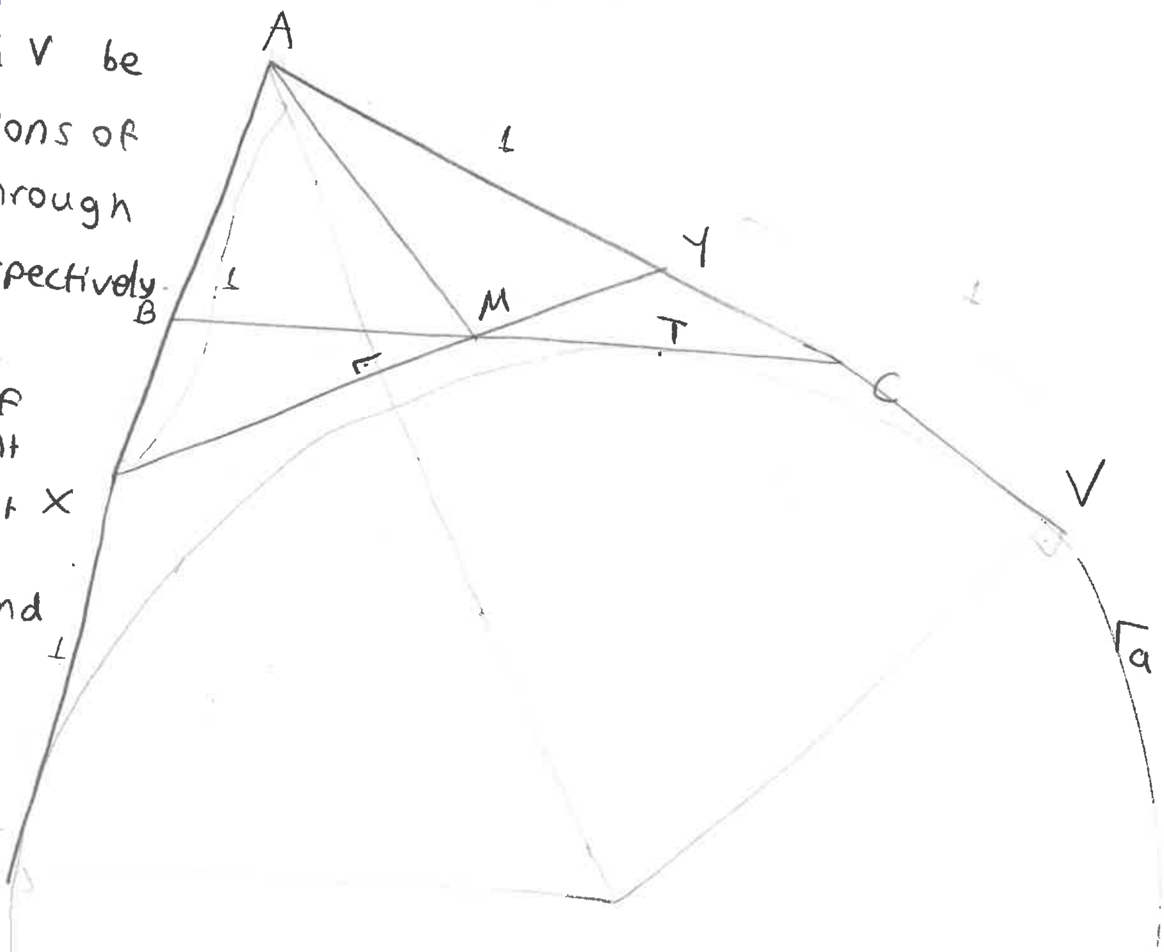
Thus there is a circle centered at L and passes through B, I, C. $BL = IL = CL$

$\sin\alpha = \frac{IL}{2R}$. Also $\sin\alpha = \frac{r}{AI} \Rightarrow \frac{IL}{2R} = \frac{r}{AI} \Leftrightarrow IL.AI = 2R.r$ $\square$

Russian Olympiad 2010

Triangle ABC has perimeter 4. Points X and Y lie on rays AB and AC, respectively such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of either ΔBMA or ΔCMA is 2.

Solution



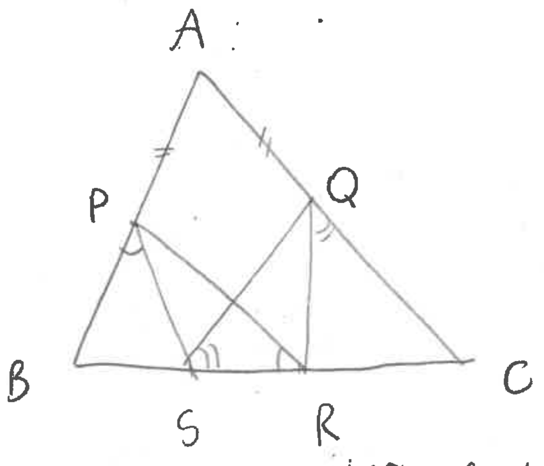
Let U and V be the reflections of point A through X and Y, respectively.
$AU = AV = 2$.
The excircle of A is tangent to BC at point T. If T is between M and C then $G(AMC) \neq 2$ and if $AM = MT$ $G(BMA) = 2$.

Construct a circle of radius zero at A, say ω_0. Then X and Y lie on the radical axis of ω_0 and Γ_a; hence so does M! (Because $XY \perp AI_A$ and M lies on XY.)
Thus $Pow_{\omega_0}(M) = AM^2 = Pow_{\Gamma_A}(M) = MT^2 \Rightarrow MA = MT$
The either-or condition reflects whether T lies on $\overline{BM}$ or $\overline{CM}$

JMO 2012 / 1

Given a triangle ABC and let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$ let S and R be distinct points on segment BC such that S lies between B and R. $\angle BPS = \angle PRS$ and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic

Solution



Let O be the circumcenter of ΔPSR. The circumcirle is tangent to AB at point P bcs $PB^2 = BS.BR$.

Then we have $OP = OQ$ because AO is an angle bisector

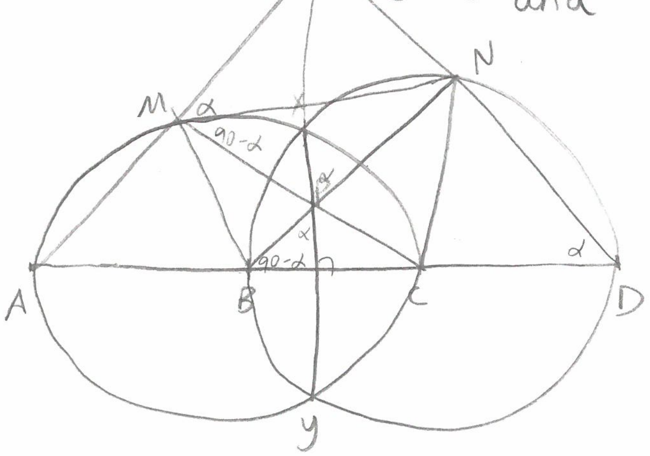
Thus, O is also on that circle $\square$

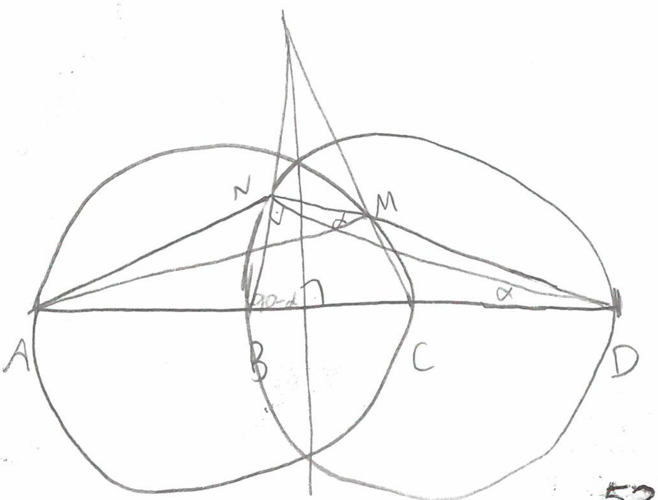
IMO 1995 / 1

Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Solution

Let the lines AM and DN intersect at point Q. Without loss of generality let Q locate above the line AD.
If AM, DN, XY are concurrent then $QM \cdot QA = QX \cdot QY = QN \cdot QD$ hence AMND is cyclic. Let's prove it.
We know that $PM \cdot PC = PX \cdot PY = PN \cdot PB$ thus cyclic. We also know that $\angle AMC = \angle DNB = 90°$.
Let $\angle NDB = \alpha$ then $\angle NBD = 90 - \alpha = \angle PMN$. $\angle AMC = 90°$ so $\angle QMN = \alpha$. Also, because $PY \perp AD$, $\angle BPY = \alpha$. ↳ $\angle QMP = \angle QNP = 90$ so PMQN are cyclic.
Because $\angle BPY = \angle QMN = \alpha$ and PMQN is cyclic then AMND is concyclic and AM, DN, XY are concurrent.

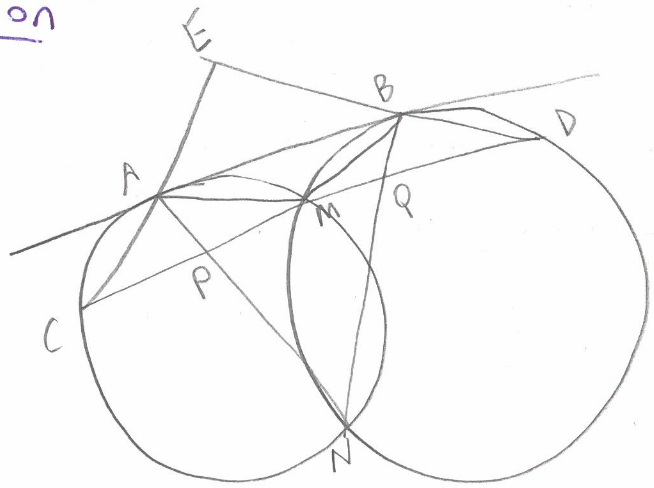




IMO 2000/1

Two circles G_1 and G_2 intersect at 2 points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that EP = EQ

Solution



$\measuredangle ACM = \measuredangle EAB = \measuredangle BAM$ and similarly $\measuredangle EDM = \measuredangle EBA = \measuredangle ABM$.

Thus $\triangle AEB \cong \triangle AMB$ MN bisects AB.

$PQ \parallel AB$ so MN also bisects PQ. $EM \perp AB$ because EAMB is a deltoid. Thus $PE = QE$ $\square$

IMO 2009/2

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circumcircle of $\triangle KLM$. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

Solution

Let w_0 be the circumcircle of the triangle ABC.

$OP = OQ \iff Pow_{w_0}(P) = Pow_{w_0}(Q)$.

$Pow_{w_0}(P) = AP.PC$, $Pow_{w_0}(Q) = AQ.QB$. Let's show that

$AP.PC = AQ.QB \Rightarrow \dfrac{QB}{PC} = \dfrac{AP}{AQ}$.

Bcs L, M and K are midpoints, $MK\,2 = QB$ and $ML\,2 = PC$

So we can write $\dfrac{MK}{ML}$ instead of $\dfrac{QB}{PC}$.

Let $\measuredangle MLK = \alpha$ and $\measuredangle MKL = \beta$. Then $\alpha = \measuredangle MLK = \measuredangle QMK = \measuredangle MQA$.

and similarly $\measuredangle MKL = \measuredangle PML = \measuredangle MPA = \beta$.

From sinus theorem in $\triangle KLM$ we have $\dfrac{MK}{ML} = \dfrac{\sin\alpha}{\sin\beta}$

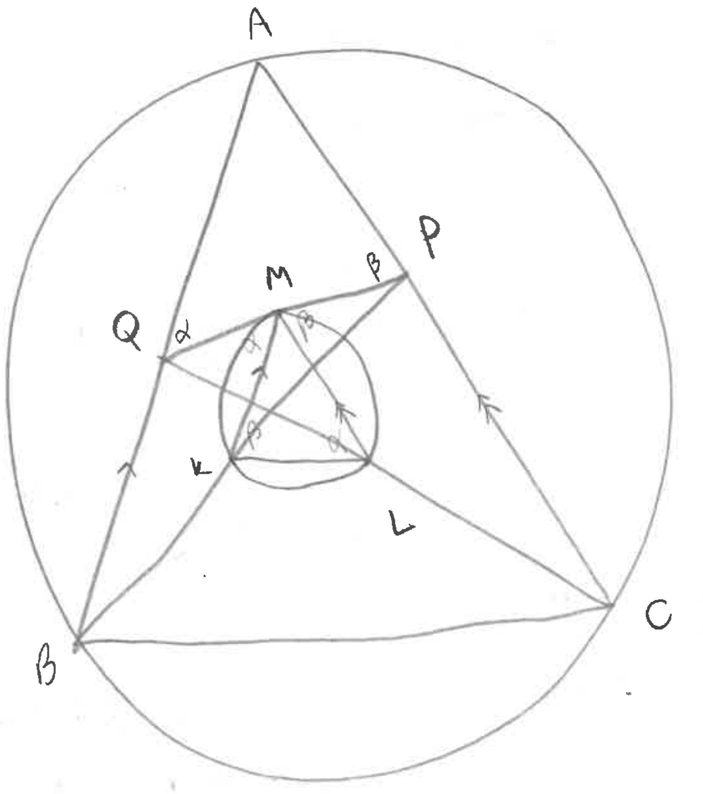
and in $\triangle AQP$ we have $\dfrac{AP}{AQ} = \dfrac{\sin\alpha}{\sin\beta}$

Hence it gives us

$$\frac{\sin\alpha}{\sin\beta} = \frac{AP}{AQ} = \frac{MK}{ML} = \frac{QB}{PC}$$

as desired $\square$

Figure is on the next page.


A
P
M
β
Q
α
K
L
C
B

Problem From Evan Chen 2.36.

Let $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ be the altitudes of a scalene triangle ABC with circumcenter O. Prove that (AOD), (BOE), and (COF) intersect at point X other than O

Solution

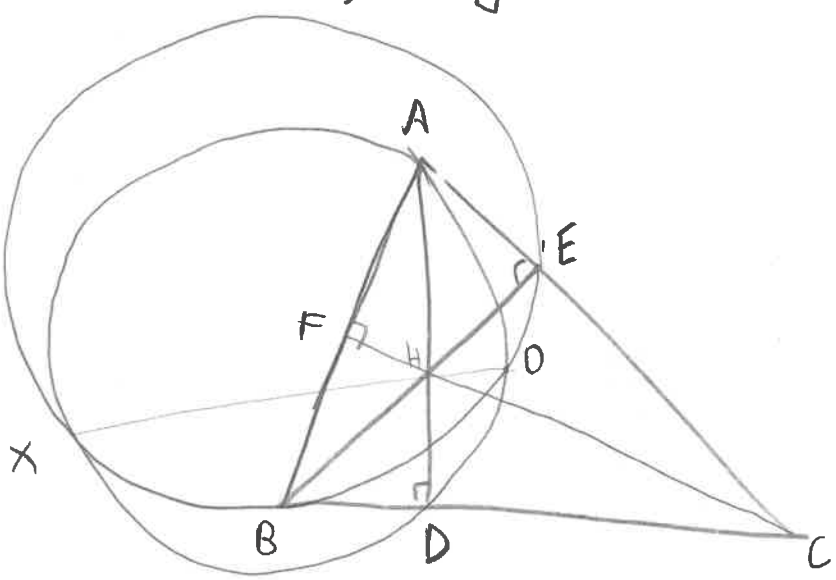
Let X be the intersection point of (AOD) and (BOE).
" H " " " " " the altitudes.
" T " " " " " the line OH with (AOD) other than O. Now, we know that $ABDE$ is a cyclic quadrilateral. Thus, we have

$$BH \cdot HE = AH \cdot HD = OH \cdot HT \Rightarrow BH \cdot HE = OH \cdot HT$$

So T is also on (BOE). So $T = X$. Now, we also know that $CDFA$ is a cyclic quadrilateral. Thus,

$$CH \cdot HF = AH \cdot HD = OH \cdot HX \Rightarrow CH \cdot HF = OH \cdot HX$$

So X is also on (COF). $\square$



Iran TST 2011/1 Holy Radical Axis

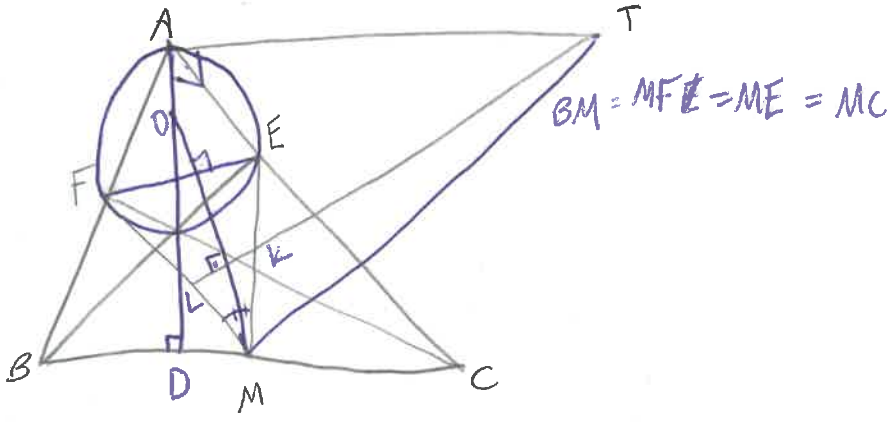
In acute triangle ABC, $\angle B$ is greater than $\angle C$. Let M be the midpoint of $\overline{BC}$ and let E and F be the feet of the altitudes from B and C, respectively. Let K and L be the midpoints of $\overline{ME}$ and $\overline{MF}$, respectively, and let T be on line KL such that $AT \parallel BC$. Prove that $AT = MT$.

Solution $[BE] \cap [FC] = H$ olsun.

$AFHE$ is cyclic. Let D be the feet of the altitude from A. $\angle ADB = 90^\circ = \angle DAT$. Also AH is the diameter of (AFE). So AT is tangent to (AFE). Also $BM = MC = ME = MF$ so the angle bisector of $\angle FME$ is perpendicular to FE and intersects AH at point O, the center of the circle. Also MF and ME are tangent to (AFE). And finally K and L are midpoint so $KL \parallel FE$ thus $KL \perp OM$. Thus the line KL is the radical axis of (AFE) and the circle centered at M with radius zero. So $\mathrm{Pow}_{(AEF)}(T) = \mathrm{Pow}_{(M)}(T)$ as T is on line KL.

$$\downarrow$$

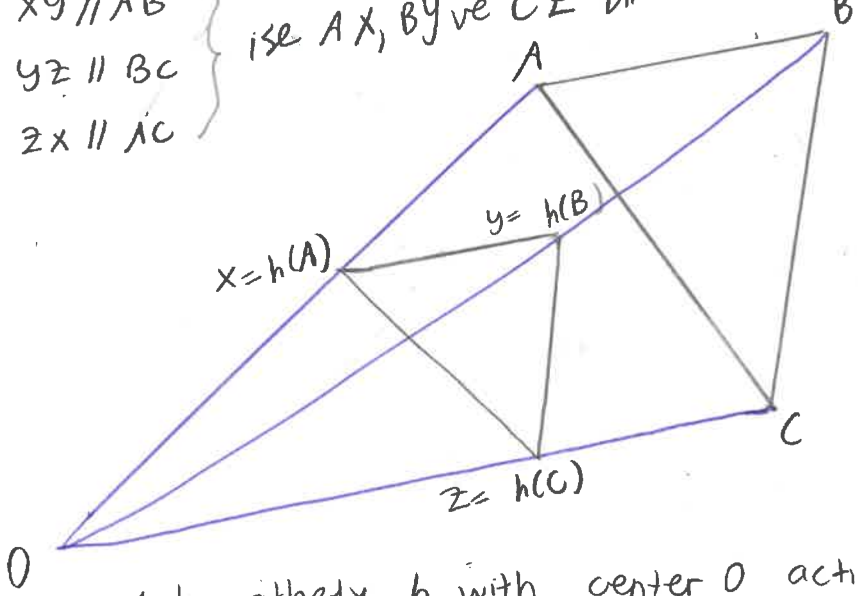
$$AT^2 = MT^2 \Rightarrow AT = MT \text{ as desired.}$$



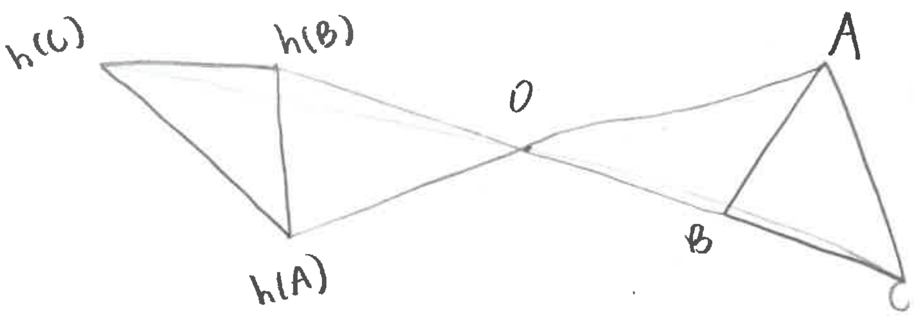
Homothety And the Nine- Nine Point Circle

Benzer üçgenler. ABC ve XYZ üçgen.

$$\left.\begin{array}{l} XY \parallel AB \\ YZ \parallel BC \\ ZX \parallel AC \end{array}\right\}$$ ise AX, BY ve CZ bir O noktasında kesişir.

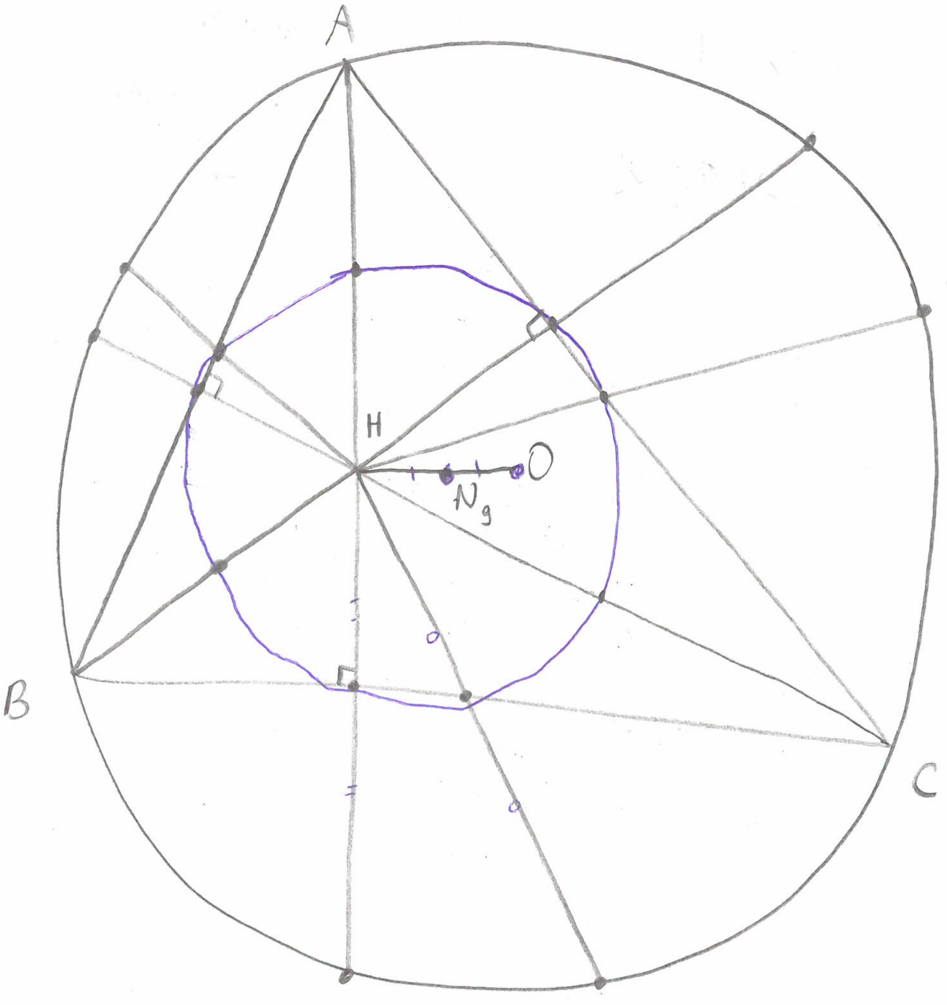


A homothety h with center O acting on ABC



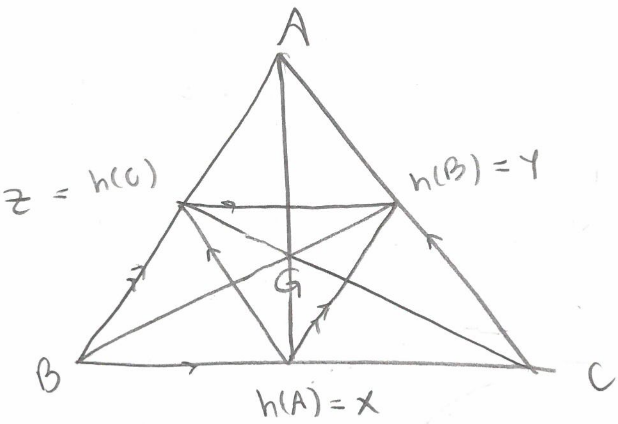
It sends P to another point h(P), multiplying the distance from O by k. The number k is a scale factor. k can be negative.

Nine Point ~~Cent~~ Circle



The reflection of H over the midpoints and over edges are on the circumcircle. Now take homothety h at H and with scale factor $\frac{1}{2}$. This brings all the reflections back onto the sides of the triangle. and center will be N_9 and then radius is the half of R.

PROOF ~~FOR~~ FOR CEVIAN FROM HOMOTHETY



We know that

$2XY = AB$

$2YZ = BC$

$2ZX = AC$

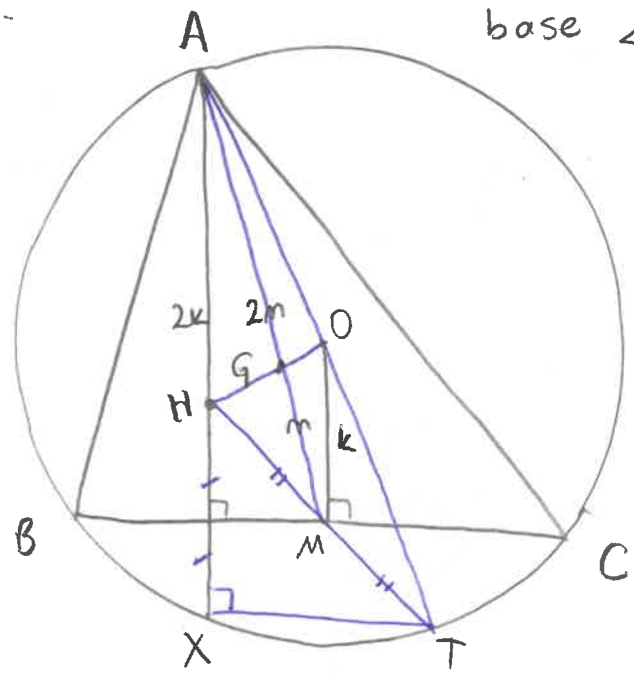
From the parallels we can say that if we take homothety h at G with scale $\frac{1}{2}$ we get ΔXYZ.

Thus $\left(\frac{BG}{GY} = 2\right)$.cyc $\square$

ELEGANT PROOF FOR EULER LINE

In a triangle H, G, O are collinear and $HG/GO = 2$

Proof



From the intermediate base $\angle AXT = 90^\circ$ so A, O, T are collinear.

$\triangle AHT \sim \triangle OMT$ so $AH = 2OM$ Now say $T = \overline{AM} \cap \overline{OH}$

$\triangle AHT \sim \triangle MOT$

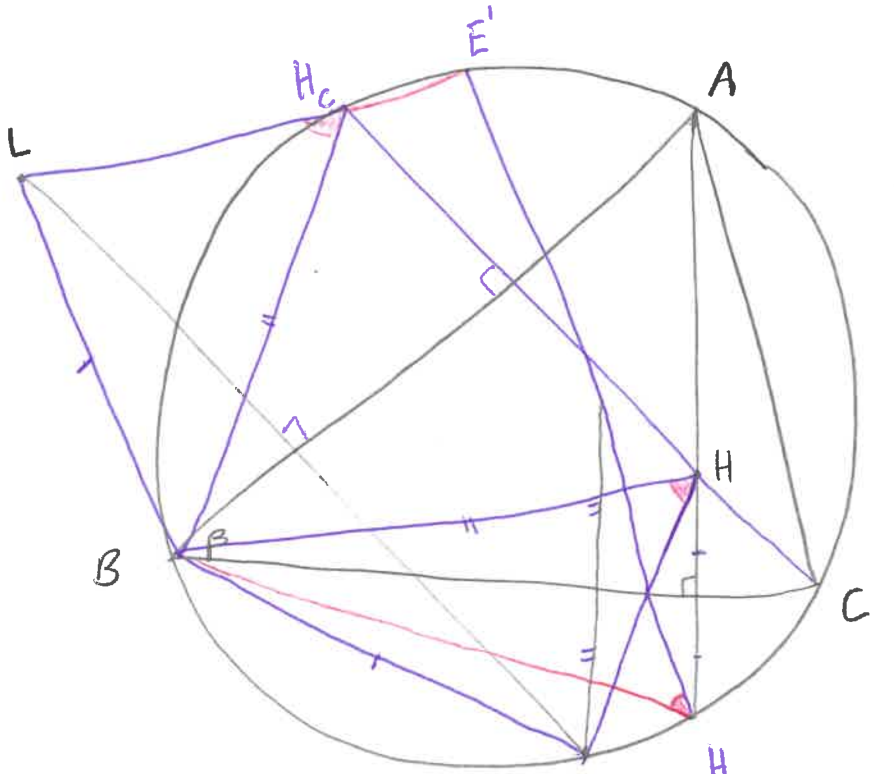
so $AT : TM = 2$ and $MC = MB$ thus $T = G$ and $\frac{HG}{GO} = 2$.

□

AN IMPRESSIVE QUESTION FROM THE VERY FIRST EGMO 2012/7

Let ABC be an acute-angled triangle with circumcircle Γ and orthocenter H. Let K be a point of Γ on the other side of $\overline{BC}$ from A. Let L be the reflection of K across $\overline{AB}$, and let M be the reflection of K across $\overline{BC}$. Let E be the second point of intersection of Γ with the circumcircle of triangle BLM. Show that the lines KH, EM and BC are concurrent

Solution:



Obviously, MH_A, HK, BC are concurrent. Let H_AM intersect Γ at E' other than H_A. Lets show $BLE'M$ is concyclic.

$\angle LBM = \angle LBA + \angle ABM = \angle ABK + \angle ABM = 2.(\angle ABM + \angle MBK)$
$= 2.\angle ABC = 2\beta$.

$\angle H_CE'H_A = \angle H_CCH_A = 180 - 2.\angle BAC = 180 - 2\beta$.

Now all we have to show is $\angle LE'M = 180 - 2\beta = \angle H_CE'M$ which means L, H_C, E' are collinear. Let's show this.

If we draw $\overline{BH}$ we see that

$\triangle BLH_c = \triangle BKH$ from symmetry.

Thus $\angle LH_cB = \angle KHB$.

Say $T = E'H_A \cap KH$ Then $\triangle THB = \triangle TH_AB$ thus

$$\measuredangle LH_cB = -\measuredangle KHB = -\measuredangle THB = -\measuredangle TH_AB = \measuredangle E'H_AB = \measuredangle E'H_cB$$

So L, H_c, E' are collinear. $\square$

Shortlist 2000 / G3

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E and F on sides BC, CA and AB respectively such that $OD + DH = OE + EH = OF + FH$ and the lines AD, BE, CF are concurrent.

Solution

Define D as follows $D = BC \cap OH_A$ and define E and F similarly. Then

$$OD + DH = OD + DH_A = R$$
$$OE + EH = OE + EH_B = R$$
$$OF + FH = OF + FH_C = R$$

So all we have to show is AD, BE and CF are concurrent.
To prove that we'll use Ceva theorem.

$\measuredangle DBH_A = 90 - \hat{C}$, $\measuredangle DH_AB = \measuredangle OH_AB = 90 - \measuredangle BAH_A = \hat{B}$.
$\measuredangle DH_AC = \hat{C}$, $\measuredangle BCH_A = 90 - \hat{B}$

$$\left.\begin{aligned} \frac{BD}{\sin B} &= \frac{DH_A}{\sin(90-C)} \\ \frac{\sin C}{DC} &= \frac{\sin(90-B)}{DH_A} \end{aligned}\right\} \frac{BD}{DC} = \frac{\cos B . \sin B}{\cos C . \sin C}$$

Similarly, $\frac{EC}{EA} = \frac{\cos C . \sin C}{\cos A . \sin A}$ and $\frac{FA}{FB} = \frac{\cos A . \sin A}{\cos B \sin B}$

So we have

$$\frac{BD}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$$ which concurrengous point from Ceva theorem.

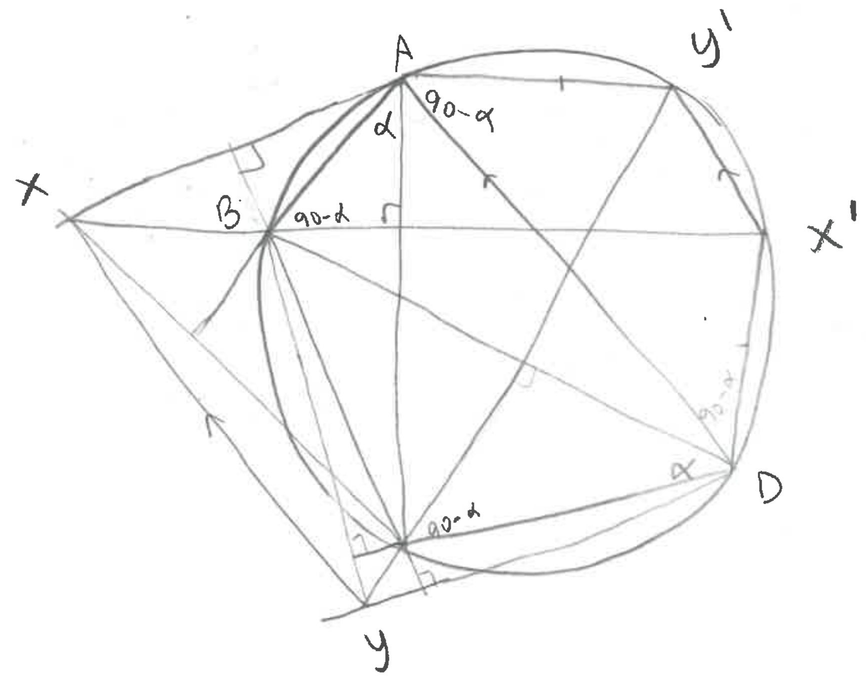
Problem 3.17.

In cyclic quadrilateral $ABCD$ points X and Y are the orthocentres of ΔABC and ΔBCD. Show that $AXYD$ is parallelogram.

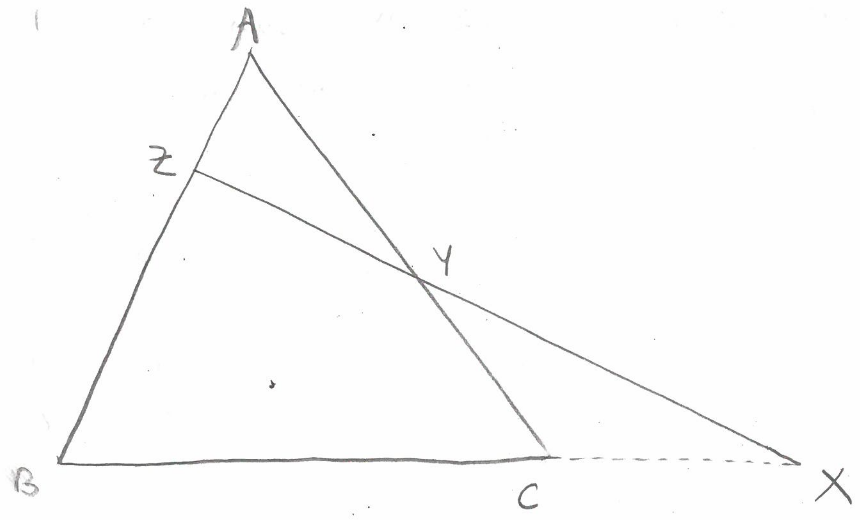
Solution

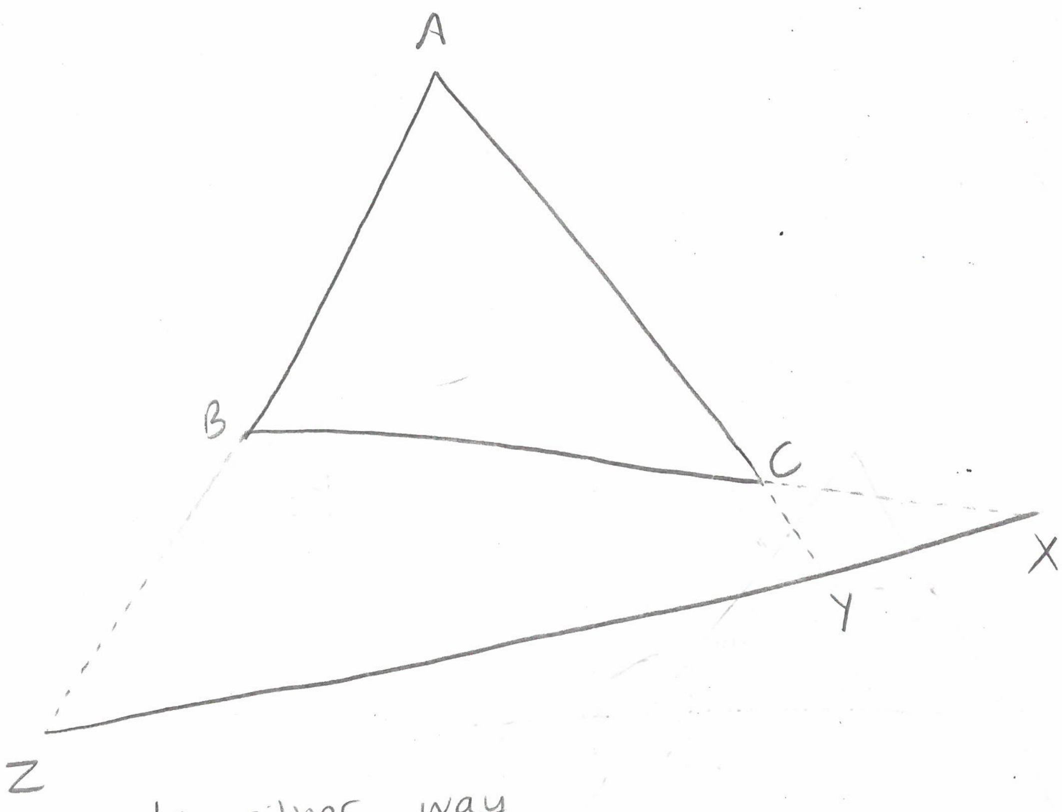
ΔABC an ΔBCD might be acute angled or not So X and Y might be inside the circle or outside the circle. But we know for sure that their reflection over AC and BD resp. are on the circle say X' and Y'

$XA \parallel YD$. This is clear in either way. Now acılarsak we see $\measuredangle Y'AD = \measuredangle ADX'$ so $ADX'Y'$ is a trapezoid. thus $AD \parallel X'Y'$ and because X' and Y' are reflections we know $XY \parallel X'Y'$ so $AD \parallel XY$ Thus $ADYX$ is a parallelogram.



THE REAL MENALAUS





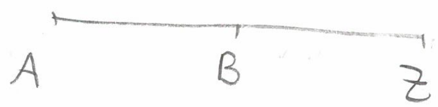
In either way

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

Definition: Directed Length

Given collinear points A, Z, B we say that the ratio $\frac{AZ}{ZB}$ is positive if Z lies between A and B, and negative otherwise.

$$\frac{AZ}{ZB} > 0$$

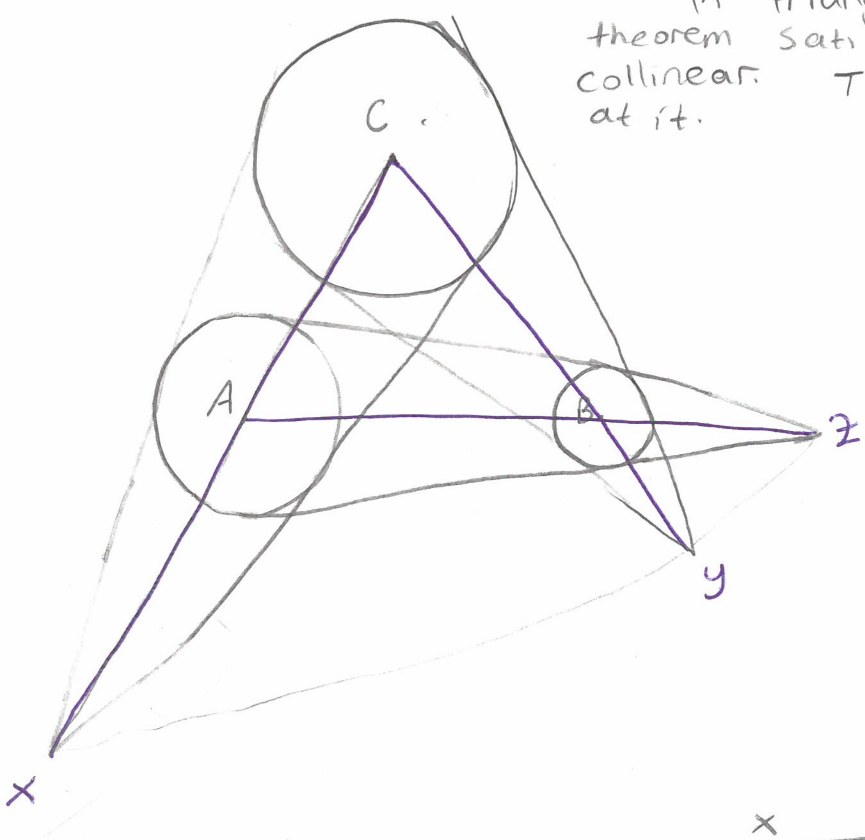


$$\frac{AZ}{ZB} < 0.$$

Monge's Theorem

Consider disjoint circles $\omega_1, \omega_2, \omega_3$ in the plane, no two congruent. For each pair of circles the intersection of their common external tangents are collinear.

Proof.



In triangle ABC in menalaus theorem satisfies then x, y, z are collinear. Then lets have a look at it.

$$\frac{By}{yC} \cdot \frac{CX}{XA} \cdot \frac{AZ}{ZB} = ?$$

$$\frac{By}{yC} = -\frac{r_b}{r_c}$$

$$\frac{CX}{XA} = -\frac{r_c}{r_a}$$

$$\frac{AZ}{ZB} = -\frac{r_a}{r_b}$$

$\times$

$$\frac{By}{yC} \frac{CX}{XA} \frac{AZ}{ZB} = -1$$ as desired □

Çok Çok Basit EGMO Sorusu

2013/1

The side BC of the triangle ABC is extended beyond C to D so that CD = BC. The side CA is extended beyond A to E so that AE = 2 CA. Prove that if AD = BE then the triangle ABC is right-angled.

Çözüm

Let DA intersect EB at point M.

In triangle EBD it is easy to see that A is the centroid so $ME = MB = \frac{EB}{2} = \frac{AD}{2} = AM$

Thus $\measuredangle EAB = 90^\circ = \measuredangle BAC$ $\square$

APMO 2004/2

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

Solution

Without loss of generality assume B and C lies on the same side of the line $\overline{OH}$.
Let X, Y, Z be the feet of the altitudes from A, B, C to the line $\overline{OH}$ respectively. Let M be the midpoint of BC. Then AM ∩ OH = G, the centroid from Euler line. Let T be the feet of the altitude from M to $\overline{OH}$. Then BYZC is a right trapezium.
Also $\frac{AH}{MT} = \frac{AG}{GM} = 2$.

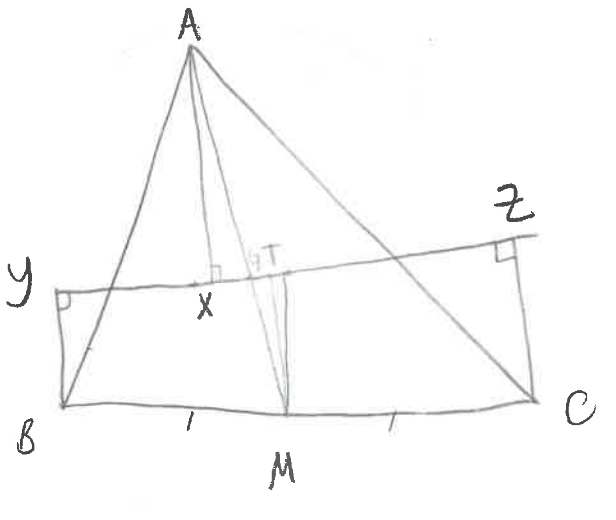
Thus,

$$\frac{BY + CZ}{2} = MT = \frac{AH}{2}$$

$$\Downarrow$$

$$BY + CZ = AH$$

Thus, $A(AOH) = A(BOH) + A(COH)$



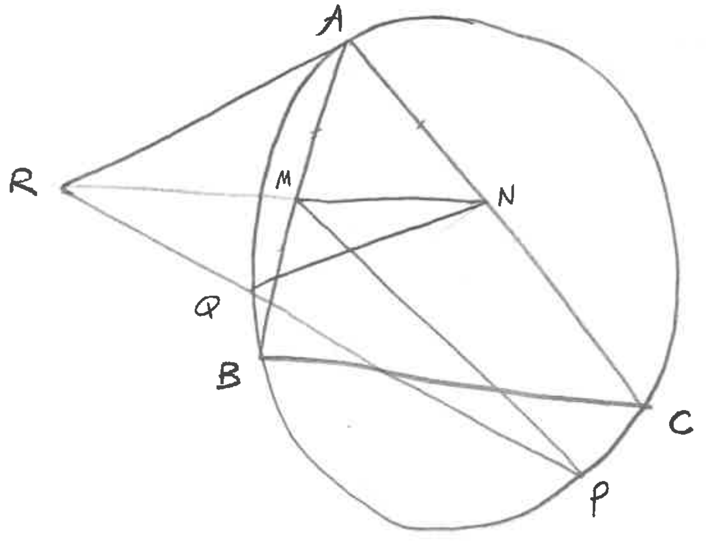
GÖRMEM 3 SAAT ALDI AMA HER ŞEYİ ZATEN BULMUŞUM SORUSU

USA TSTST 2011/4

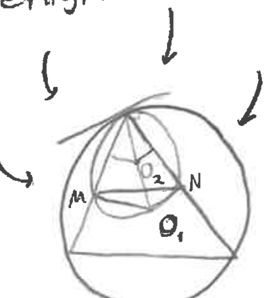
Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocentre and circumcentre respectively. Let M and N be the midpoints of sides AB and AC respectively. Rays MH and NH meet ω at P and Q " . Lines MN and PQ meet at R. Prove that $\overline{OA} \perp \overline{RA}$.

Solution

Let HM and HN intersect (ABC) at M' and N' other than P and Q respectively. Then $\measuredangle QPM' = \measuredangle QN'M' = \measuredangle QNM$. So, $MNPQ$ is cyclic. Also taking homothety h at A with scale factor $\frac{1}{2}$ gives us the fact that (AMN) is tangent to (ABC). Finally R is the radical centre of circles (ABC), (AMN), $(MNPQ)$ Thus RA is tangent to (ABC) which proves the statement.



Note: Homoteti yerine direkt benzerlikten de gelir (AMN) nin (ABC)'ye teğetliği.



M, N orta noktalar

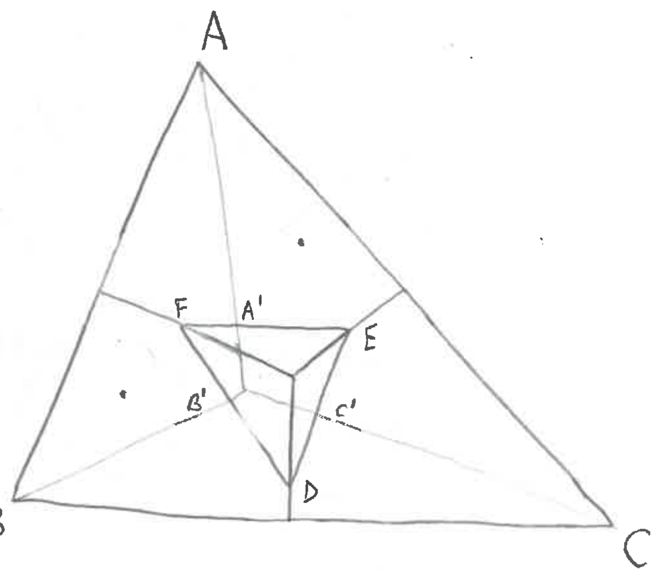
VAY ANASINI SAYIN SEYİRCİLER

1997 USAMO Problem 2

$\triangle ABC$ is a triangle Take points D, E, F on the perpendicular bisectors of BC, CA, AB resp. Show that the lines through A, B, C perpendecilar to EF, FD, DE respectively are concurrent.

Solution 1

If the statement is true than Carnot theorem holds.



$AF = BF$
$AE = EC$
$BD = CD$

$$FA'^2 - A'E^2 + EC'^2 - C'D^2 + DB' - B'F^2 = AF^2 - AA'^2 - AE^2 + AA'$$
$$+ EC^2 - CC' - DC^2 + CC' + BD^2 - BB'^2 - BF^2 + BB'^2 =$$
$$AF^2 - AE^2 + EC^2 - DC^2 + BD^2 - BF^2 = 0 \quad \square$$

Solution 2

Construct circles centered at F, E, D with radii BF, AE, DC respectively. Then the lines AA', BB', CC' are radical axeses so they must concur at the radical centre $\square$

! If D, E, F are collinear then the lines inscribed in the question are parallel and never concur

USAMO 1998/2

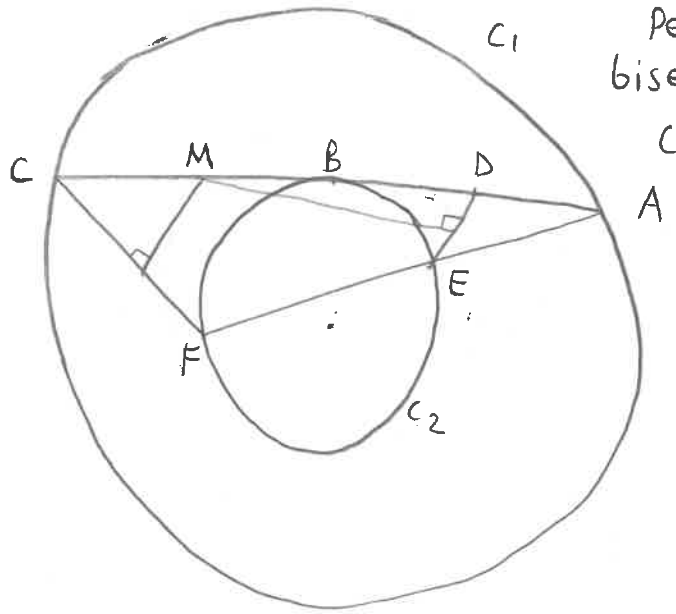
Let C_1 and C_2 be concentric circles with C_2 in the interior of C_1. From a point A on C_1 one draws ...

(şekile göre $\frac{AM}{MC} = ?$)

Diagram

BD = DA

Perpendicular bisectors of DE and CF intersect at M.



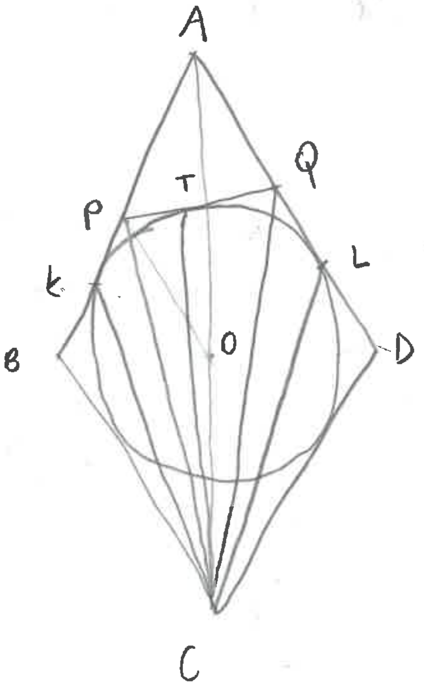
Solution

$DA \cdot CA = AB^2 = AE \cdot AF \Rightarrow CFED$ cyclic. M is the centre because of the perp. bisectors. Then $CM = MD$ so

$\frac{AM}{MC} = \frac{5}{3}$ □

ABCD is a fixed rhombus. Segment PQ is tangent to the inscribed circle of ABCD, where P is on side AB, Q is on side AD. Show that, when segment PQ is moving, the area of $\triangle CPQ$ is a constant.

Solution



$$[COP] = [OPA]$$

$$[OTP] = [OKP]$$

$$[OTQ] = [OQL]$$

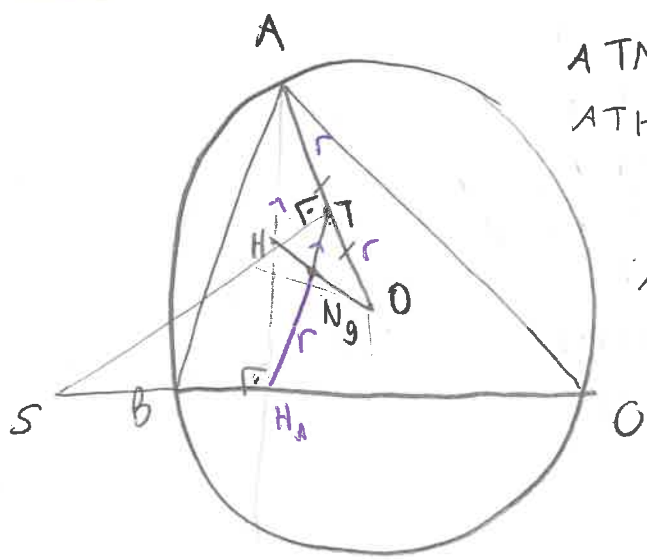
$$+ \quad [COQ] = [AOQ]$$

$$[CPQ] = [AKOL]$$

Belarus TST 2019

Let O be the circumcenter and H be the orthocentre of an acute-angled triangle ABC. Point T is the midpoint of the segment AO. The perpendicular bisector of AO intersects the line BC at point S. Prove that the circumcircle of the triangle AST bisects the segment OH.

Solution 1 by ME!



ATN_9H_A ikizkenar yamuk

ATH_AS çemb.

$\Downarrow$

ATN_9H_AS çemb.

Shortlist 2001/G1 Çok iyi çözüm

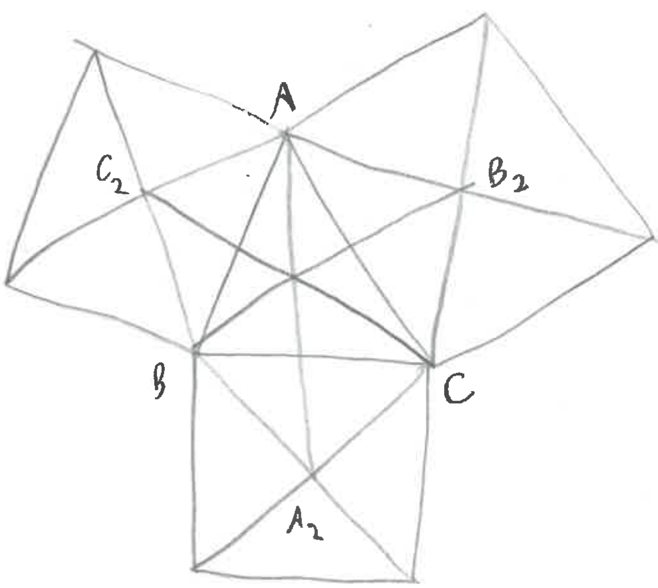
Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the 2 remaining vertices of the square is on side AB and the other is on AC. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1, CC_1 are concurrent.

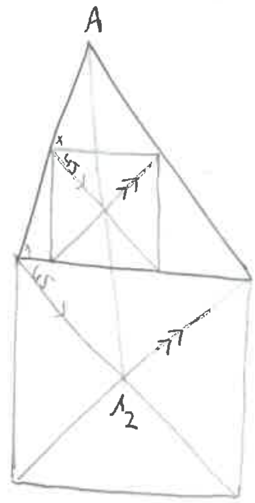
Amazing Solution (Fikir benden işlem kitaptan)

Take homothety h at A on the square so that the edge is $\overline{BC}$ and the new square is suited outside of the triangle. Let A_2 be the center of the bigger square then A, A_1, A_2 are collinear. Do the same on B and C. Now we have to prove AA_2, BB_2 and CC_2 are concurrent. Now see if Ceva theorem Holds $\triangle ABC$.

$$\frac{\sin\angle ACC_2}{C_2A} = \frac{\sin(45+\hat{A})}{C_2C}, \qquad \frac{C_2B}{\sin\angle BCC_2} = \frac{C_2C}{\sin(45+\hat{B})} \quad \Rightarrow \quad \frac{\sin\angle ACC_2}{\sin\angle BCC_2} = \frac{\sin(45+\hat{A})}{\sin(45+\hat{B})}$$

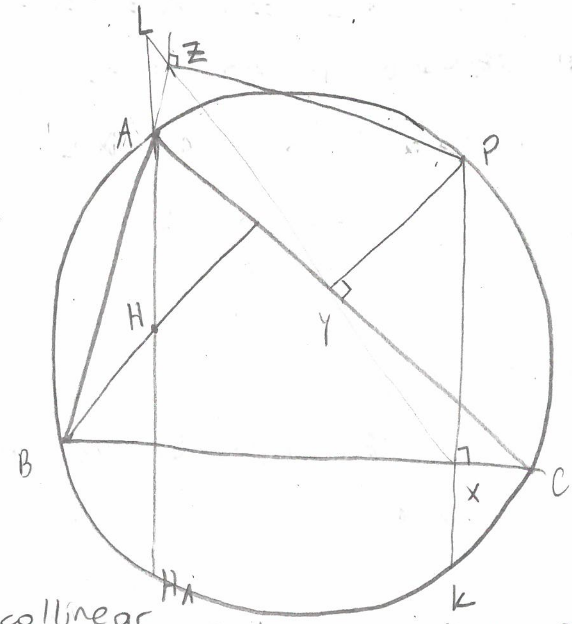
In the same way we get





$$\frac{\sin(45+\hat{A})}{\sin(45+\hat{B})} \cdot \frac{\sin(45+\hat{B})}{\sin(45+\hat{C})} \cdot \frac{\sin(45+\hat{C})}{\sin(45+\hat{A})} = 1$$ as desired. This proves the concurrency $\square$

SIMSON LINE
AND SOME PROPOSITIONS



X, Y, Z are collinear and the line passing through them is called Simson Line.

Proof: $\measuredangle ZYP = \measuredangle ZAP = \measuredangle BCP = \measuredangle XCP = \measuredangle XYP$ □

Proposition 1. LAKX is a parallelogram.

Proof: obviously $LH \parallel XK$. Also $\measuredangle AKP = \measuredangle ACP = \measuredangle YXP$ so $AK \parallel LX$. □

Position 2. Let K' be the orthocenter of $\triangle PBC$. Then $AHK'P$ is a parallelogram.

Proof. HH_AKK' is a trapezoid and $KH_A = K'H$

Also $AH_A \parallel PK$ so AH_AKP is also a trapezoid and $AP = H_AK = HK'$. HK' is the reflection of H_AK over BC so $AP \parallel HK'$ and $AP = HK'$. □

Proposition 3. $LHXP$ is a parallelogram.

Proof: $LH \parallel PX$. If we can show $LH = PX$ then proof ends.

$$LH = LA + AH \underset{\uparrow}{=} PK' + LA \overset{?}{=} PX = PK' + XK'$$

$(AH = PK'$ from proposition 3)

$$PK' + LA \overset{?}{=} PK' + XK' \Leftrightarrow LA \overset{?}{=} XK'$$

$$\Downarrow$$

$$LK' = AX \Leftrightarrow LAXK' \text{ is a parallelogram.}$$

we know $LA \parallel XK'$

Lets show that $LK' = AX$.

From proposition 1 $AKXL$ is a parallelogram also $KX = XK''$ Thus $\triangle AKX = \triangle LXK'$ then $AX = LK'$ $\square$

Lemma 4.9. The Diameter of the incircle

Let ABC be a triangle whose incircle is tangent to BC at point D. If DE is a diameter of the incircle and ray AE meets $\overline{BC}$ at X, then $BD = CX$ and X is the tangency point of the A-excircle to $\overline{BC}$.

Proof: Let the parallel line passing from E to BC intersect $\overline{AB}$ and $\overline{AC}$ at points B' and C' respectively.

Then $AI \perp B'C'$ and $\triangle AB'C' \sim \triangle ABC$ so,

$$\frac{AE}{AX} = \frac{AC'}{AC} = \frac{AI}{AI_A} = \frac{r}{R} \quad \text{thus } IE \parallel I_AX \text{ and}$$

$$\frac{IE}{I_AX} = \frac{r}{R} \quad \text{also } IE = r \text{ so } I_AX = R \text{ Thus } X \text{ is the tangency}$$

point. Then it is easy to check that $BD = CX$ by using s. $\square$

Lemma 4.10. (Diameter of the Excircle

In the notation of Lemma 4.9 suppose $\overline{XY}$ is a diameter of the A excircle. Show that D lies on $\overline{AY}$.

Proof: Let AD intersect A excircle at Y' (the furthest intersection point). If $\frac{AD}{AY'} = \frac{AE}{AX}$ Then $DE \parallel Y'X$

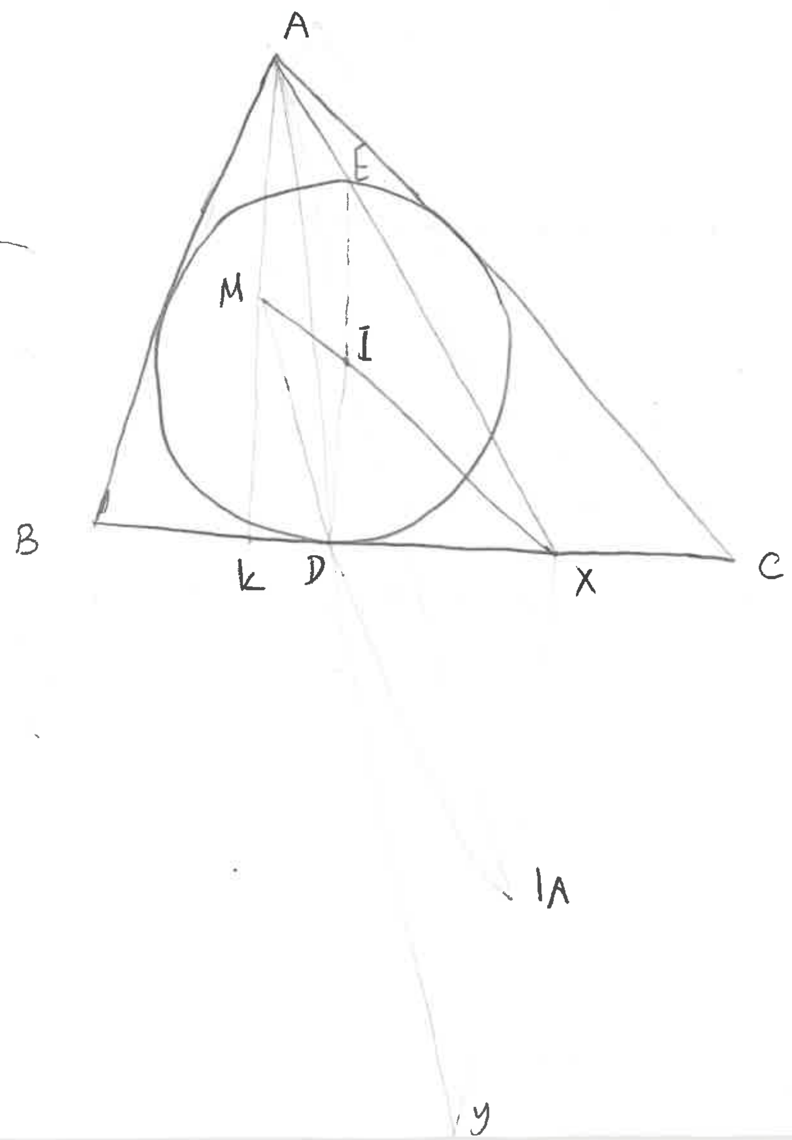
DE contains I so $Y'X$ contains I_A. And,

$\triangle AB'C' \sim \triangle ABC$ so $\frac{AB'}{AB} = \frac{AD}{AY'} = \frac{AC'}{AC} = \frac{AE}{AX}$

so $Y'X$ contains I_A thus $Y = Y'$ $\square$

Midpoints of Altitudes

Show that X, I, M and I_A, D, M are collinear.
The diagram is above.



let ray XI meet AK at point M_1

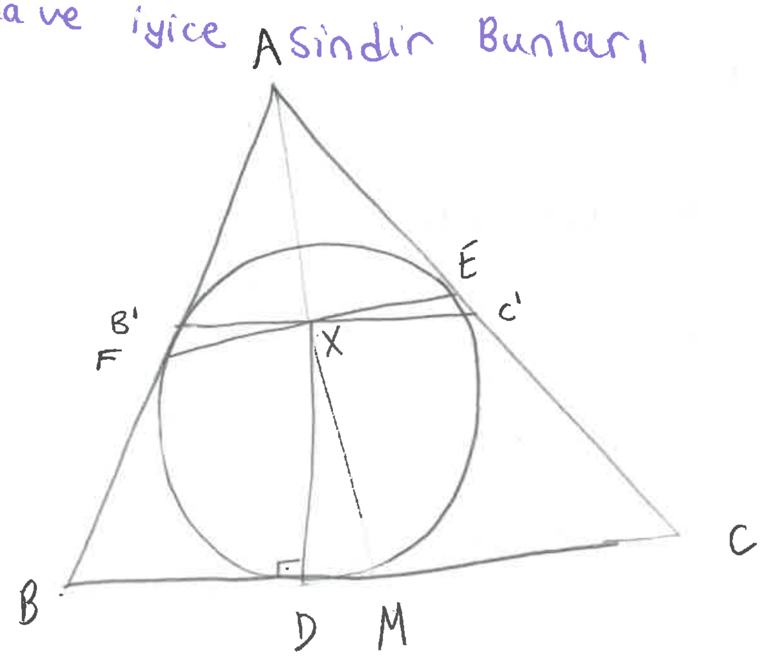
Then, $\dfrac{XD}{XK} = \dfrac{XE}{XA} = \dfrac{r}{M_1A} = \dfrac{r}{M_1K}$

thus $M_1A = M_1K \Rightarrow M = M_1$.

$\triangle AKD \sim \triangle AXD$ so $I_A\, D\, M$ must be collinear. □

More Incircle and Excenter

Hatırla ve iyice Asindir Bunları



1. X is on $\overline{EF}$ such that $XD \perp BC$ then AX bisects BC.

<u>Proof:</u> $B'C' \parallel BC$ $X \in [B'C']$ olsun. I, XD üstünde $IX \perp B'C' \Rightarrow B'X = XC'$ Burdan $MB = MC$

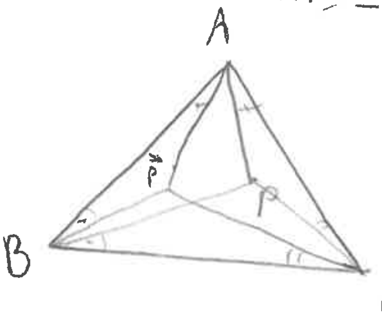
2. I is on $(AB'C')$ $\angle IFX = \angle IB'X = \angle IAC'$ □

bcs. $B'FIX$ is cyclic.

ISOGONAL CONJUGATE

Let P be any point not collinear with any of the sides. There exists a unique point P^* satisfying the relations:

$\measuredangle BAP = \measuredangle P^*AC$, $\measuredangle CBP = \measuredangle P^*BA$, $\measuredangle ACP = \measuredangle P^*CB$

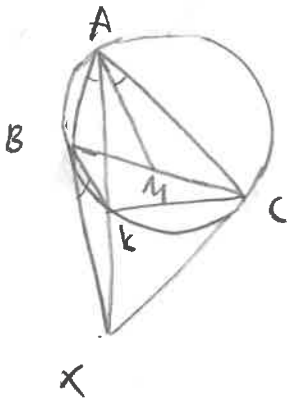


P^* is basically the reflection of AP ~~with~~ over the angle bisector of A.

Symedians

__Theorem.__ Let X be the intersection of the tangents to (ABC) at B and C. Then line AX is a symmedian. Let M be the intersection of the isogonal of AX on $\overline{BC}$; then $BM = MC$.

__Proof__

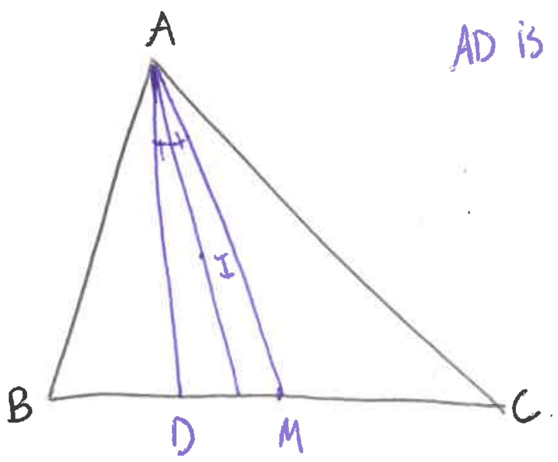


$$\frac{BX}{Xk} = \frac{\sin C}{\sin \alpha} = \frac{AM}{MC}$$

$$\frac{XC}{Xk} = \frac{\sin B}{\sin \measuredangle BAM} = \frac{AM}{BM}$$

$XB = XC$ so $\dfrac{AM}{MC} = \dfrac{AM}{BM} \Rightarrow MC = MB$ $\square$

1.

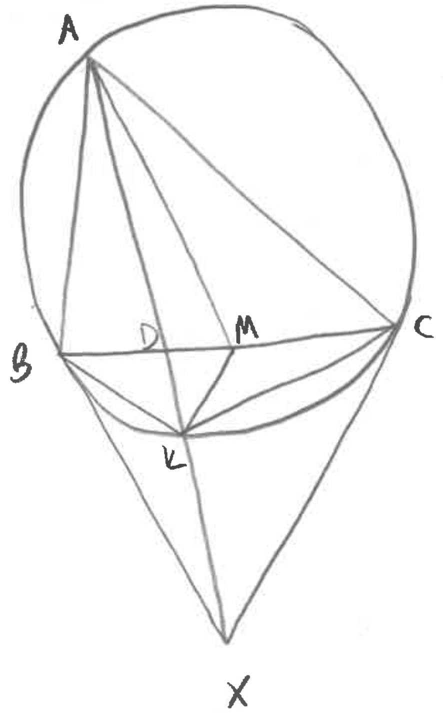


AD is the A-symmedian

Properties of Symmedian

In the diagram below,

M is midpoint.



1. AX is A-symmedian of $\triangle ABC$.

Proof: we have to show that $\angle BAX = \angle MAC$.

$$\frac{BK}{AB} = \frac{XK}{BX} = \frac{XK}{CX} = \frac{AC}{KC} = \frac{AM}{}$$

$$\frac{AB}{BK} \stackrel{?}{=} \frac{AM}{MC} = \frac{AM}{BM}$$

$$\frac{MC}{AM} = \frac{AC}{KC}$$

2. $\overline{KA}$ is a K-symmedian of $\triangle KBC$

Proof: Know we have 1st property:

Statement is true if

$$\triangle ACK \sim \triangle BMK$$

$$\Longleftrightarrow$$

$$\frac{AK}{AC} = \frac{BK}{BM} = \frac{BK}{CM} \Longleftrightarrow \frac{AK}{BK} = \frac{AC}{CM}$$

$$\frac{AK}{BC} = \frac{\sin\angle ACK}{\sin\angle BAK} = \frac{\sin\angle AMC}{\sin\angle MAC} = \frac{AC}{CM} \quad \square.$$

3. $\triangle ABK$ and $\triangle AMC$ are directly similar.

4. $$\frac{BD}{DC} = \left(\frac{AB}{AC}\right)^2$$

Proof: $$\frac{BK}{BA} = \frac{XK}{BX} = \frac{XK}{CX} = \frac{CK}{AC} \Rightarrow \frac{BK}{CK} = \frac{AB}{AC}$$

$$\frac{BD}{DC} = \frac{\sin BXD}{\sin DXC}.$$

$$\left.\begin{aligned} \frac{\sin BXD}{\sin\alpha} &= \frac{BK}{XK} \\ \frac{\sin\alpha}{\sin KBC} &= \frac{BK}{KC} \\ \frac{\sin KCX}{\sin KXC} &= \frac{KX}{CK} \end{aligned}\right\} \frac{\sin BXD}{\sin KXC} = \frac{BK^2}{CK^2} = \left(\frac{AB}{AC}\right)^2 \quad \square$$

5. $\overline{BC}$ is the B-symmedian of $\triangle BAK$, and the C-symmedian of $\triangle CAK$.

<u>Proof:</u> From 1 and 2 we know $\triangle ABM \sim \triangle AKC$.

so $\frac{AB}{BM} = \frac{AK}{KC} \Leftrightarrow \frac{AB}{\frac{BC}{2}} = \frac{2KT}{KC} \Rightarrow \frac{AB}{BC} = \frac{KT}{TC}$

thus $\triangle ABC \sim \triangle KTC$ thus $\angle TCK = \angle BCA$

then this gives the C-symmedian. say $\angle BAK = \alpha$

$\angle KAC = \beta$ then $\angle KCB = \angle TCA = \alpha$ $\angle CTX = \angle CBX$
$= \alpha + \beta$

Thus $BTCX$ is cyclic. so $\angle BTX = \angle BCX = \alpha + \beta$

then $\angle ABT = \beta$ also $\angle KAC = \angle CBK = \beta$

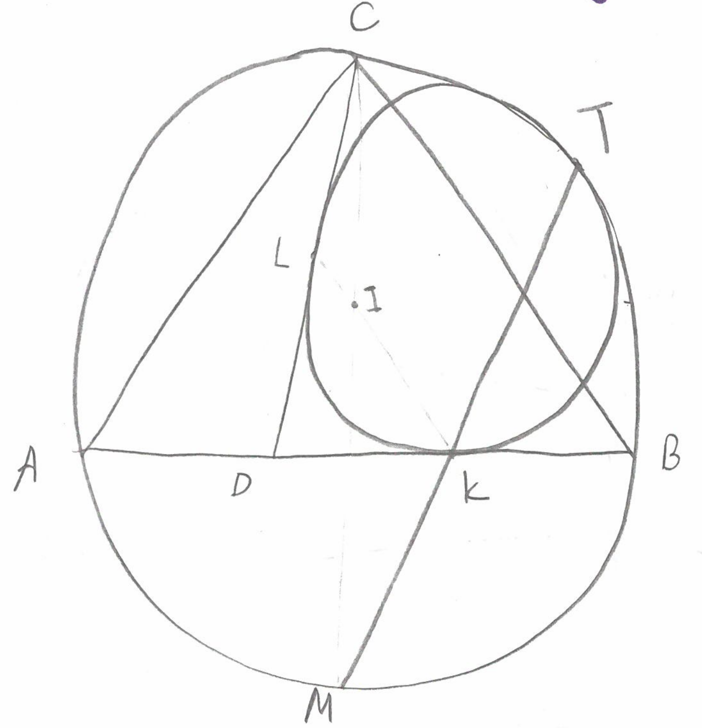
so BC is B-symedian of $\triangle ABK$.

(Note: T is the midpoint of AK)

6. (BXC) passes through the midpoint of AK

Proof: It is proved above.

Circles Inscribed in Segments



Now just consider the points $TAMBK$. Then $\overset{\frown}{AM} = \overset{\frown}{MB}$. Proof is very easy.

Now consider the whole diagram.

1. C, L, I, T are concyclic.

<u>Proof:</u> $\angle MTB = \angle KBM \Rightarrow \triangle MBT \sim \triangle MKB$.

$\measuredangle TLI = \measuredangle TLK = \measuredangle TKB = -\measuredangle BKM = -\measuredangle MBT = -\measuredangle MCT$

$= -\measuredangle ICT = \measuredangle TCI$ □

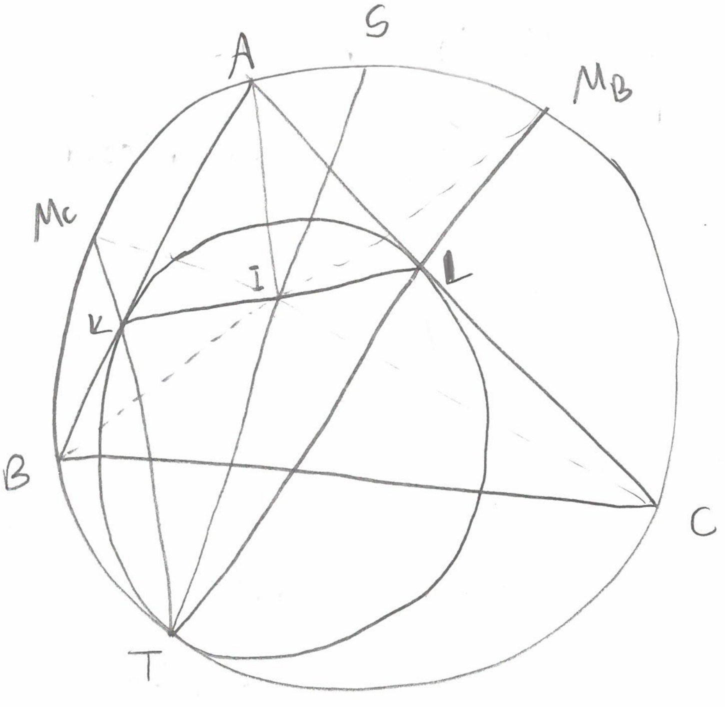
2. I is the incircle of $\triangle ABC$

<u>Proof:</u> $\triangle MKI \sim \triangle MIT \Rightarrow MI^2 = MK.MT = MB^2 = MA^2$

$MI = MA = MB$. $\measuredangle CMB = \measuredangle CAB = 2 . \measuredangle IAB$ also

$\measuredangle ACM = \measuredangle MCB$ □

Mixtilinear Incircles



a) The midpoint I of $\overline{KL}$ is the incenter of $\triangle ABC$

b) Lines TK and TL bisects the arcs AB and AC resp.

c) Line TI passes through the midpoint of arc $\widehat{BC}$ not containing T.

d) $\angle BAT = \angle CAE$. E is the tangency point of A-excircle to BC.

e) $\angle BTA = \angle CTD$. D is the " " " the incircle ABC to BC.

f) Quadrilaterals BKIT and CLIT are concyclic.

a, b, c, f kanıtını yaptım.

BAMBASIT HONG KONG 1998

Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90°$ and let H and K be the feet of the altitudes from Q to lines PR and PS. Prove that $\overline{HK}$ bisects $\overline{QS}$

~~Sora~~

Solution.

$PKHQ$ is cyclic.

$\measuredangle HKQ = \measuredangle HPQ = \measuredangle RPQ = \measuredangle RSQ$.

Let the lines HK and SR intersect at point T

Then $\measuredangle HKQ = \measuredangle TKQ = \measuredangle TSQ$ Thus

$STQK$ is cyclic also $\measuredangle TSK = \measuredangle SKQ = 90$

so $STQK$ is a cyclic rectangle. Thus its diagonals bisects each other which means

$\overline{KH} = \overline{KT}$ bisects $\overline{SQ}$ $\square$

EASY BUSY THIS IS CRAZY
USA TST 2014 SORU

Let ABC be an acute triangle and let X be a variable interior point on arc $\widehat{BC}$ (minor one). Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection point of line PQ and the perpendicular from B to $\overline{AC}$. Let ℓ be the line passing through P parallel to $\overline{XR}$. Prove that while X varies along minor arc $\widehat{BC}$, the line ℓ always passes through a fixed point.

Solution

Let $\overline{RB}$ intersect ℓ at T, AC at L and (ABC) at Y. Then $BY \parallel XP$, $\ell \parallel RX$ so, $RXPT$ is a parallelogram.

also $\measuredangle RYX = \measuredangle BYX = \measuredangle BCX = \measuredangle QPX = \measuredangle RPX$

So $RXPY$ is cyclic and also we know that $RY \parallel XP$ so $RXYP$ is a trapezoid. So,

$\measuredangle PTY = \measuredangle XRY = \measuredangle RYP = \measuredangle TYP \Rightarrow TL = LY$ thus T is the orthocenter of $\triangle ABC$. So every ℓ passes through the orthocenter.

BASIT JAPANIESE OLYMPIAD

Triangle ABC is inscribed in circle Γ. A circle with center O is drawn, tangent to side BC at point P, and internally tangent to the arc $\widehat{BC}$ of Γ not containing A at point Q. Show that if $\angle BAO = \angle CAO$ the $\angle PAO = \angle QAO$.

Solution

AO is the angle bisector of $\triangle ABC$. Let AO meet (ABC) at point L. Let K be a point on Γ such that $KL \perp BC$. Then BLC is an isosceles triangle an L is be midpoint of the arc $\widehat{BC}$ containing A. We know that Q, P, K are collinear. Thus O_1 the center of Γ, $O_1 = KL \cap QO$.

Then $\angle QAO = \angle QAL = \angle QKL = \angle QKO_1 = \angle QPO$

Then $AQOP$ is cyclic

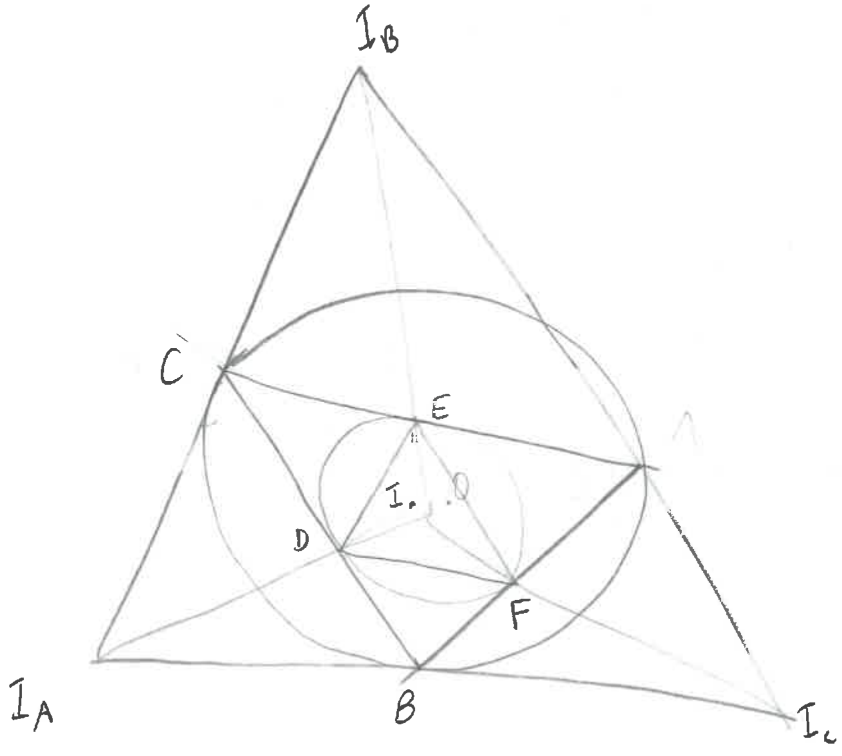
$\angle QKO_1 = \angle KQO_1 = \angle PQO = PAO$

$\Downarrow$

$\angle QAO = \angle PAO$ $\square$

Vietnam TST 2003/2

Let ABC be a scalene triangle with circumcenter O and incircle I. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C ~~pespectively~~ respectively. The incircle of $\triangle ABC$ touches the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F resp. Prove that lines A_0D, B_0E, C_0F and OI are concurrent. A_0, B_0, C_0 are the midpoints of $\overline{AH}, \overline{BK}, \overline{CL}$ resp.



$\triangle DEF$ $\triangle I_A I_B I_C$ homoteti alırsan

I_AD, I_BE I_CF noktadaş

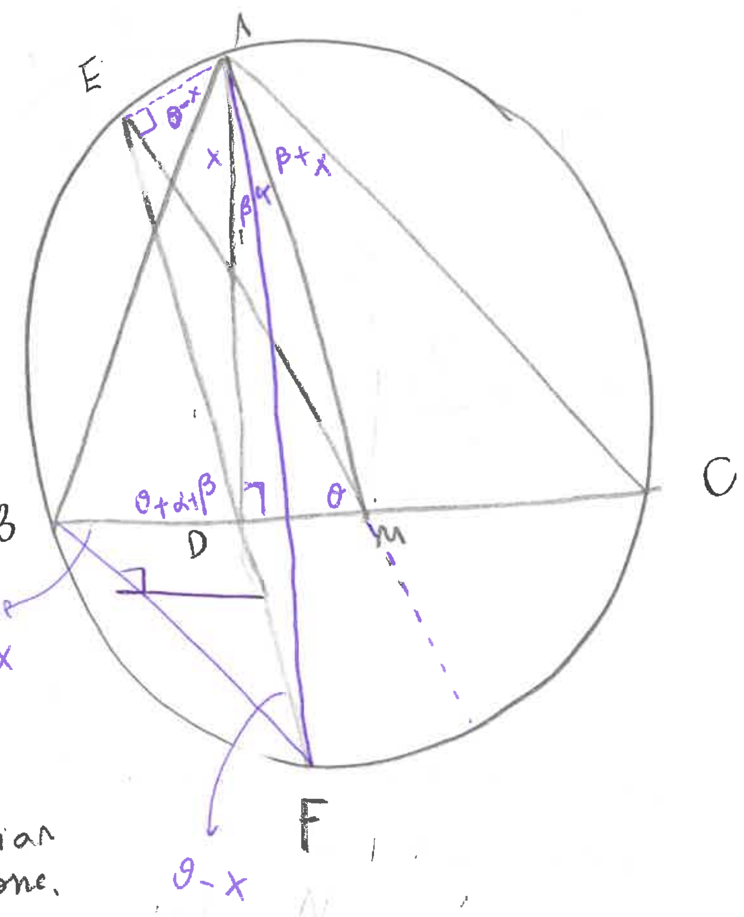
O, $I_AI_BI_C$'de 9 point circle'ın merkezi

I, " orthocenter

APMO 2012/4

Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of $\overline{BC}$, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of $\triangle ABC$ and the ray MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

Solution



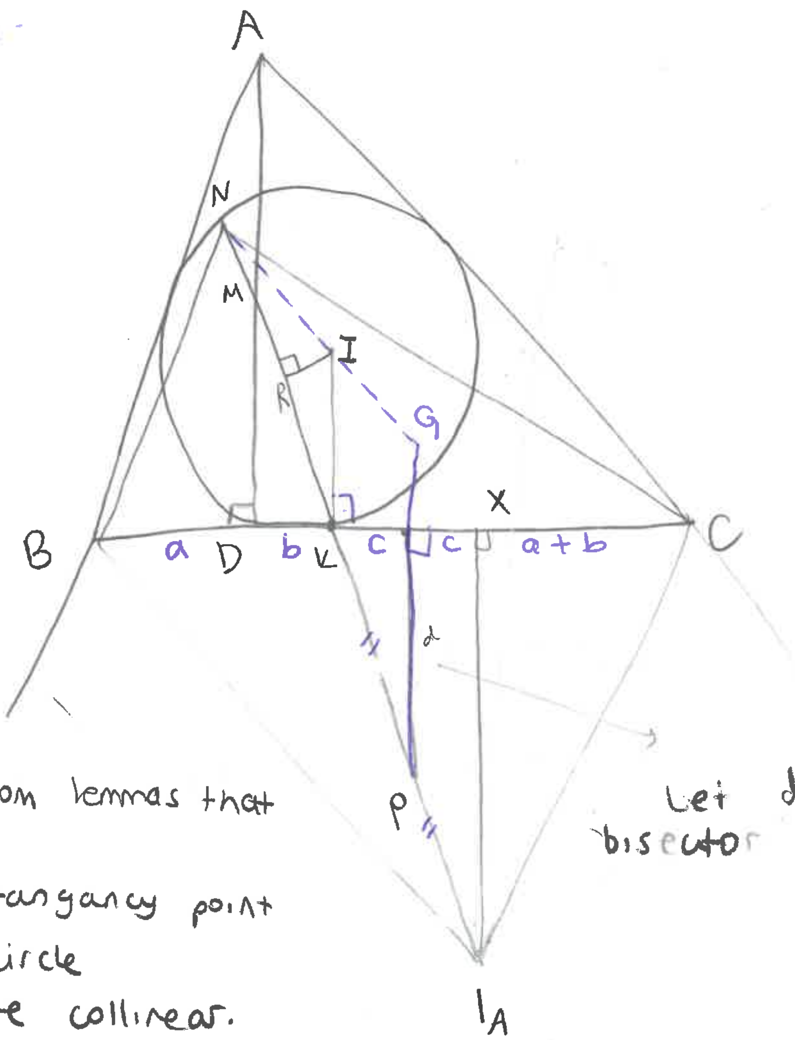
If we can show AF is A-symmedian then we are done.

Açılayınca bariz. $\square$

Shortlist 2002 / G7

The inc. Ω of the acute $\triangle ABC$ is tangent to $\overline{BC}$ at point K. Let $\overline{AD}$ be the altitude, and let M be the midpoint of $\overline{AD}$. If N is the common point of the circle Ω and the line KM, then prove that the inc. Ω and the circ. of $\triangle BCN$ are tangent to each other at the point N.

Solution



Let d be the perp. bisector of $\overline{BC}$.

we know from lemmas that
$BK = XC$
X is the tangancy point of A-excircle
$M\,K\,I_A$ are collinear.

Now,
$\sphericalangle IRI_A = 90^\circ$ so $IRBI_AC$ is cyclic.

$$RK \cdot KI_A = BK \cdot KC = \frac{NK}{2} \cdot KI_A = \frac{NK}{2} \cdot 2PK = NK \cdot PK$$

Thus $NBPC$ is cyclic. Becaus $d \perp BC$ then the center of $(NBPC)$ must lie on d. Let $G = NI \cap d$
Then
$$\angle INK = \angle IKN = \angle GPN = \angle GNP \text{ so } GN = GP$$
and $BG = GC$ so G must be the center of $(NBPC)$. I, G, N are collinear so they are tangent at N

HOLY BIC! SORUSU

IRAN TST 2009/9

Let ABC be a triangle with the incenter I and contact triangle DEF. Let M be the foot of the perpendicular from D to $\overline{EF}$ and let P be the midpoint of $\overline{DM}$. If H is the orthocenter of $\triangle BIC$ prove that $\overline{PH}$ bisects $\overline{EF}$.

Solution

Lemma 1. In a triangle ABC let D, E, F be the tangency point of incircle to sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ resp. Let K be the intersection point of BI and EF. Then $CK \perp BI$.

Proof. Let $CK \perp BI$. Now let's show F, E, K are collinear.
$\measuredangle KCI = \frac{\measuredangle A}{2} = \measuredangle FEI$ and $\measuredangle IKC = \measuredangle IEC$ so $IEKC$ is cyclic. Then let KE intersect $\overline{AB}$ at P.
so $\measuredangle KCI = \measuredangle KEI = \measuredangle PEI = \measuredangle FEI$. Thus $F = P$. $\square$

Now $K = BI \cap FE$ and $L = CI \cap FE$. We have, from lemma 1, $BL \perp CI$ and $CK \perp BI$. So $H = BL \cap CK$.
Now, I, E, K, C, D and I, L, F, B, D are cyclic. Thus,
$\measuredangle IKD = \measuredangle ICD = \measuredangle ECI = \measuredangle EKI$
$\measuredangle ILD = \measuredangle IBD = \measuredangle FBI = \measuredangle FLI = \measuredangle KLI$. Thus
In triangle DKL, I is the incenter. Also we have $\measuredangle BLC = \measuredangle BKC = 90$ so $HLIK$ is cyclic and D, I, H are collinear. Thus H is the D-excircle center.

Lemma 2. Midpoints of altitudes.
Proof. 7 yaprak geride
D, M, I_A collinear. $\square$

From lemma 2 we know that the intersection of MH and KL is the tangency point of incircle in $\triangle KLD$. Let's call it T. Now we have $IT \perp KL = IT \perp EF$.

I is the circumcenter of $\triangle DEF$ so T must be the midpoint of EF (bcs $TI \perp EF$) $\square$

BİR AZ LİNEER CEBİR

matrix = rectangular array of numbers.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Determinant of matrix A, denoted det A or $|A|$, is a special value associated with the matrix A

We define only the determinant of a 2x2 matrix and a 3x3 matrix. We have

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

for a 2x2 matrix. For a 3x3 matrix we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & c_3 \\ a_2 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

or equivalently

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

In the definition, the 2x2 sub-matrices are called <u>minors</u>.

Proposition 1. (Swapping Rows or Columns)

Let A be a matrix and B be a matrix formed by swapping either a pair of rows or a pair of columns in A. Then $\det A = -\det B$. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$

Proposition 2 (Factoring) For any real number k, we have

$$k \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix}$$

Similar statements hold for the other rows and columns.

Theorem 3. (Elementary Row Operations) For any real number k, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+kb_1 & a_2+kb_2 & a_3+kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example

$$\begin{vmatrix} \frac{1}{2}\left(p+a+c-\frac{ac}{p}\right) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{c}-\frac{p}{ca}\right) & 1 \\ \frac{1}{2}\left(p+a+b-\frac{ab}{p}\right) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{b}-\frac{p}{ba}\right) & 1 \\ \frac{1}{2}\left(p+a+b+c\right) & \frac{1}{2}\left(\frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) & 1 \end{vmatrix}$$

$$= \frac{1}{4}\begin{vmatrix} p+a+c-\frac{ac}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{c}-\frac{p}{ca} & 1 \\ p+a+b-\frac{ab}{p} & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}-\frac{p}{ba} & 1 \\ p+a+b+c & \frac{1}{p}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1 \end{vmatrix}$$

$=$

$$\frac{1}{4}\begin{vmatrix} -b-\frac{ac}{p} & -\frac{1}{b}-\frac{p}{ca} & 1 \\ -c-\frac{ab}{p} & -\frac{1}{c}-\frac{p}{ba} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=$$

$$\frac{1}{4}\begin{vmatrix} b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ca} & 1 \\ c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ba} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$=$$

$$\frac{1}{4}\left[0\begin{vmatrix} \frac{1}{b}+\frac{p}{ca} & 1 \\ \frac{1}{c}+\frac{p}{ba} & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 1 & b+\frac{ac}{p} \\ 1 & c+\frac{ab}{p} \end{vmatrix} + 1\begin{vmatrix} b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ca} \\ c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ba} \end{vmatrix}\right]$$

$$=$$

$$\frac{1}{4}\begin{vmatrix} b+\frac{ac}{p} & \frac{1}{b}+\frac{p}{ca} \\ c+\frac{ab}{p} & \frac{1}{c}+\frac{p}{ba} \end{vmatrix}$$

$$=$$

$$\frac{1}{4}\left[\left(b+\frac{ac}{p}\right)\left(\frac{1}{c}+\frac{p}{ba}\right)-\left(c+\frac{ab}{p}\right)\left(\frac{1}{b}+\frac{p}{ca}\right)\right]=0.$$

Cramer's Rule

Consider a system of equations

$$a_x \, x + a_y \, y + a_z \, z = a$$

$$b_x \, . \, x + b_y \, y + b_z \, z = b$$

$$c_x \, x + c_y \, . \, y + c_z \, z = c$$

The solution for x is

$$x = \begin{vmatrix} a & a_y & a_z \\ b & b_y & b_z \\ c & c_y & c_z \end{vmatrix} \div \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Proof.

$$\begin{vmatrix} a & a_y & a_z \\ b & b_y & b_z \\ c & c_y & c_z \end{vmatrix} = \begin{vmatrix} a_x \, x + a_y . y + a_z \, z & a_y & a_z \\ b_x \, x + b_y \, y + b_z \, z & b_y & b_z \\ c_x \, x + c_y . y + c_z . z & c_y & c_z \end{vmatrix}$$

$$= \begin{vmatrix} a_x . x & a_y & a_z \\ b_x . x & b_y & b_z \\ c_x . x & c_y & c_z \end{vmatrix} = x . \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad \square$$

GEOMETRİYE DEVAM

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

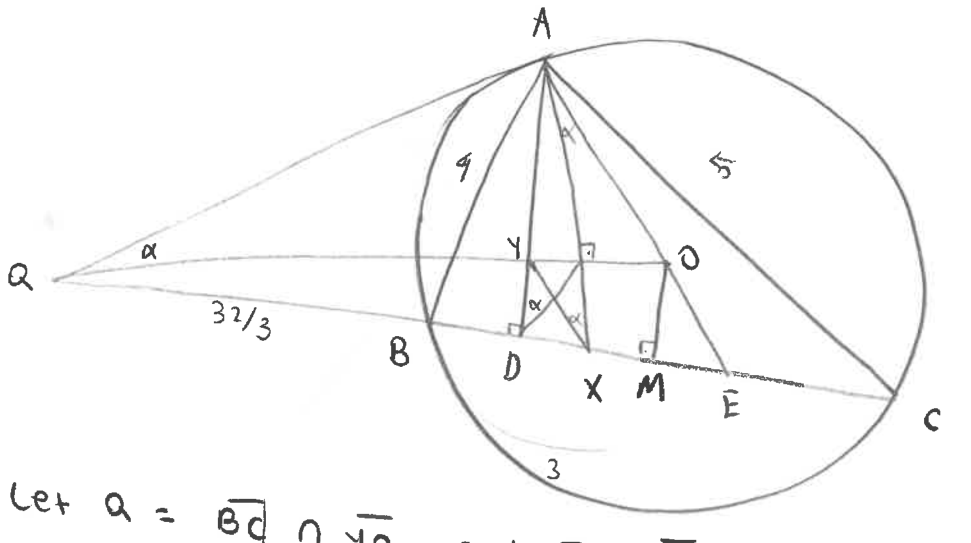
$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

KOORDİNAT DÜZLEMİNDEN SORU

Harvard-MIT Math Tournament 2014

Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, $BC = 6$. Let D be the foot of the altitude from A to $\overline{BC}$ and E the intersection of lines AO and BC. Suppose that X is on $\overline{BC}$ between D and E s.t. there is a point Y on $\overline{AD}$ satisfying $\overline{XY} \parallel \overline{AO}$ and $\overline{YO} \perp \overline{AX}$. Determine the length of BX.

My solution



Let $Q = \overline{BC} \cap \overline{YO}$ and $T = \overline{AX} \cap \overline{YO}$ and $\measuredangle XAE = \alpha$. Then

$\alpha = \measuredangle AXY = \measuredangle TXY = \measuredangle TDY = \measuredangle TDA = \measuredangle TQA$

Thus $\measuredangle QAT = 90 - \alpha \Rightarrow \measuredangle QAO = 90$ so QA is tangent to (ABC).

Now from the given lengths we can find $AD = \frac{5}{4}\sqrt{7}$

$BD = \frac{9}{4}$ $\quad DM = \frac{3}{4}$ $\quad ME = \frac{3}{31}$ $\quad EC = \frac{90}{31}$ $\quad R = \frac{8}{\sqrt{7}}$

$$\frac{DE}{AD} = \frac{21}{31\sqrt{7}} = \frac{AD}{QD}$$

$$\downarrow$$

$$QB = \frac{32}{3}$$

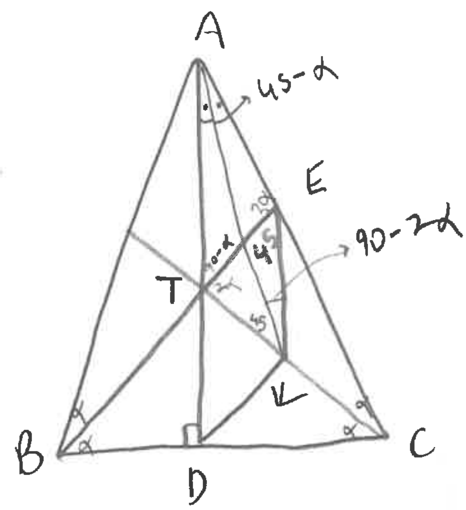
$\frac{64}{7} = AO^2 = QT \cdot QO = QX \cdot QM$

$= \left(\frac{32}{3} + BX\right)\left(\frac{32}{3} + 3\right) \Rightarrow BX = \frac{96}{41}$ $\square$

IMO 2009/4 NEFİS ÇÖZDÜM TRİGO'DAN

Let ABC be a triangle with AB = AC. The angle bisector of $\angle CAB$ and $\angle ABC$ meet the sides BC and AC at D and E resp. Let K be the incenter of $\triangle ADC$. Suppose that $\angle BEK = 45°$. Find all the possible values of $\angle CAB$.

My Solution



ATKE de ceva'dan $\sin 2\alpha \cdot \sin(90-2\alpha) \sin 3\alpha \cdot \cancel{\sin(45-\alpha)}$
$=$
$\cancel{\sin(45-\alpha)} \sin 45 \sin 45 \sin(90-\alpha)$

$\sin 2\alpha \cos 2\alpha \sin 3\alpha = \frac{1}{2}\cos\alpha$
$\Downarrow$
$\sin 4\alpha \sin 3\alpha = \cos\alpha \Rightarrow \cos\alpha - \cos 7\alpha = 2\cos\alpha$

$\cos(180-\alpha) = \cos 7\alpha$

1) $180 - \alpha = 7\alpha \Rightarrow \angle BAC = 90$

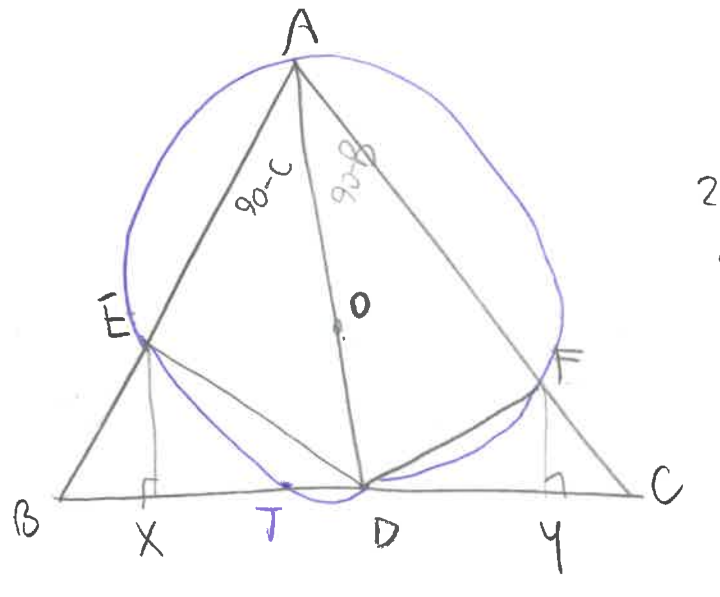
2) $180 + \alpha = 7\alpha \Rightarrow \angle BAC = 60$.

CGMO 2004/6 Chinese Girl Math Olympiad

Mantığı HariKA

Let ABC be an acute triangle with O as its circumcenter. Line AO intersect BC at D. Points E and F are on $\overline{AB}$ and $\overline{AC}$ resp. s.t. A, E, D, F are concyclic. Prove that the length of the projection of line segment EF on side $\overline{BC}$ does not depend on E and F.

Solution



(AED)'nin T BC'yi 2. kesim noktası olsun,

$$XY = BC - (BX + YC)$$

$$BX + YC = BE\cos B + FC\cos C.$$

$$BE = \frac{BT \cdot BD}{BA} \qquad CF = \frac{CD \cdot CT}{CA}$$

sin teodan

$$\frac{BD}{DC} = \frac{c\cos C}{b\cos B}$$

$BT = u$, $CT = v$ diyelim
$u + v = a = BC$

$$BX + YC = \frac{BD\cos B}{c}u + \frac{CD\cdot\cos C}{b}v$$

$$= \cos B\cos C\left(\underbrace{\frac{BD}{c\cos C}}u + \underbrace{\frac{CD}{b\cos B}}v\right) = \cos B\cos C\,(u+v)\frac{BD}{c\cos C}$$

$$= \cos B \cdot BD \cdot \frac{a}{c}$$

E ve F tek bir unique T belirler

XY T'ye bağlı olmadığından

E ve F'ye de bağlı değildir

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\sin\alpha\cdot\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Bunları ezberlemek zorundayım! MUST

TRIGONOMETRİNİN GÖZÜNÜ SEVEYİM!

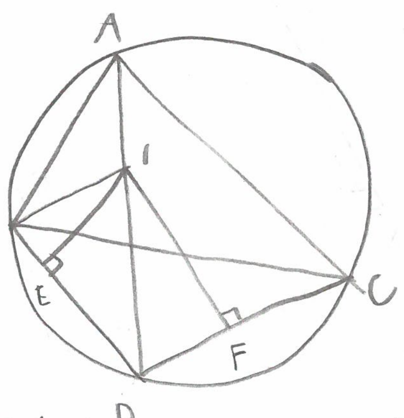
Iran Olympiad 1999

You know I. AI meet (ABC) at D. Denote the feet of the perpendiculars from I to lines BD and CD by E and F, resp. If $IE + IF = \frac{1}{2}AD$, calculate $\angle BAC$.

Solution Ben çözdüm.

We have, from $\triangle ABD$,

(I) $\frac{AD}{R} = \frac{AD}{BD} = \frac{\sin(B+\frac{A}{2})}{\sin\frac{A}{2}}$



We know there is a circle centered at D passing through B, I, C. Let R be the radius of it. $DI = DB = DC = R$

$IE = R\sin C$
$IF = R.\sin B$
$\Big\}\ R(\sin C + \sin B) = \frac{1}{2}AD$ (II)

Now, we have from (I) and (II),

$$\sin C + \sin B = \frac{\sin(B+\frac{A}{2})}{2\sin\frac{A}{2}}$$

$\Updownarrow$

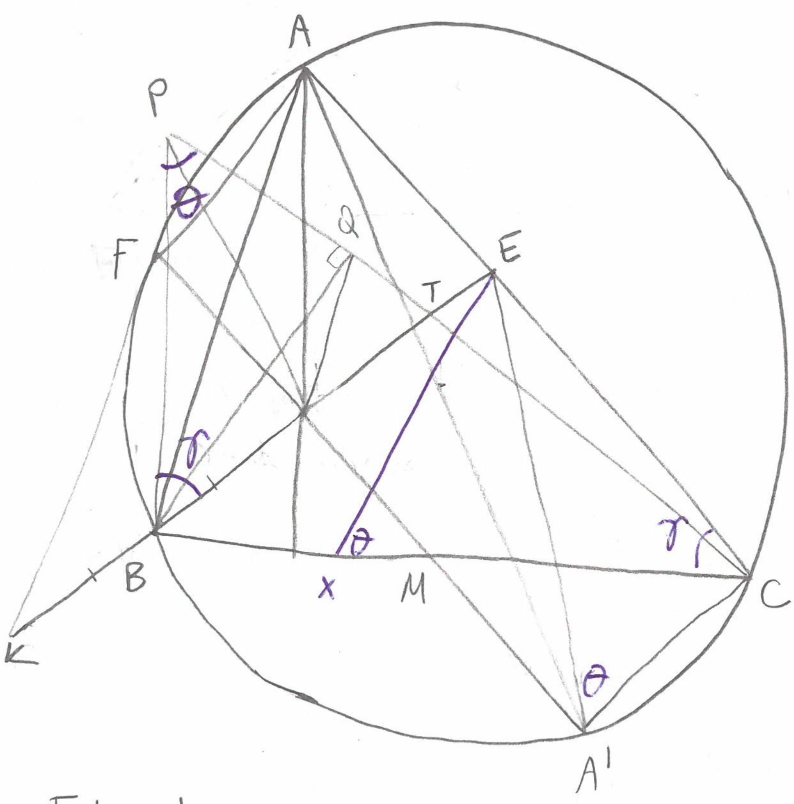
$$\frac{2\sin(90-\frac{A}{2})\cos\left(\frac{B-C}{2}\right)}{\cos\frac{A}{2}} = \frac{\sin(B+\frac{A}{2})}{\sin A} \Leftrightarrow \sin A = \frac{1}{2} \Leftrightarrow A = 30^\circ \vee A = 150^\circ.$$

$\cos\left(\frac{B-C}{2}\right)$ $\downarrow$ $\sin(B+\frac{A}{2})$

TST Serbia 2018

Let H be the orthocenter of ABC, $AB \neq BC$, and let F be a point on circumcircle of ABC such that $\angle AFH = 90°$. K is the symmetric point of H wrt B. Let P be a point such that $\angle PHB = \angle PBC = 90°$, and Q is the foot of B to CP. Prove that HQ is tangent to the circumcircle of FHK.

Solution.



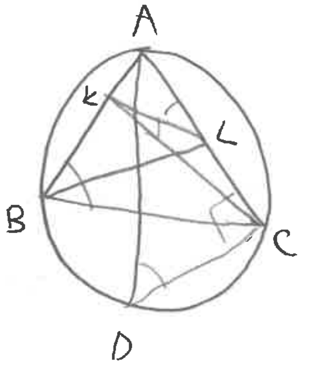
Kuvvetten $FKA'E$ çemberseldir. Dolayısıyla $QH \parallel EA'$ old. kanıtlamak yeterlidir. $\angle ACB = \gamma$ olsun. $\rightarrow$ $\angle PBH = \gamma$ olur. $\angle CA'E = \theta$

$$\tan\theta = \frac{EC}{CA'} = \frac{EC}{BH} = \frac{BC\cos\gamma}{BP\cos\gamma} = \frac{BC}{BP} = \tan\angle CPB$$

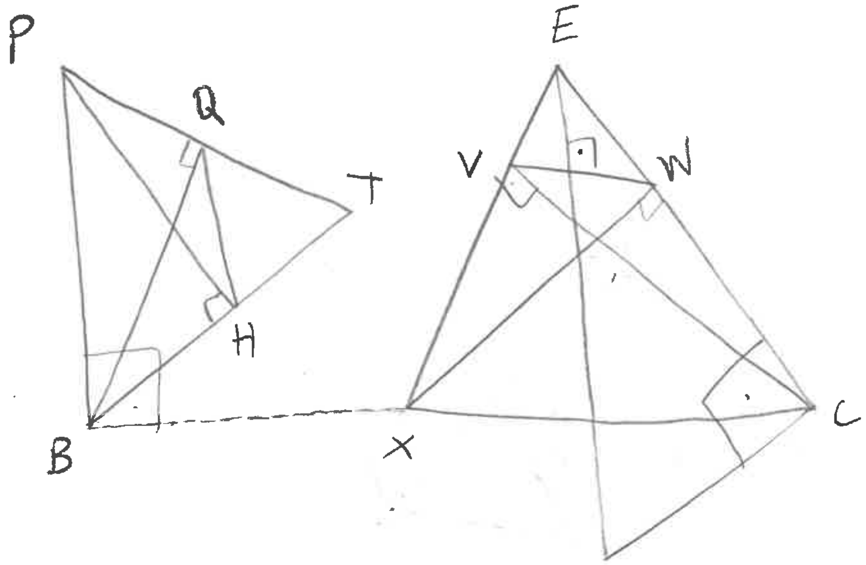
$BC \cap (ECA') = X$ olsun. $\angle EA'C = \angle EXC = \theta$. $\rightarrow \triangle EXC \sim \triangle TPB$

$\triangle EXC$'yi $90°$ döndürüp homoteti uygularsak $\triangle PBT$'yi elde ederiz ve $PB \parallel XC$ olur.

Lemma 1.



$KL \perp AD$



üçgenler benzer olduğundan

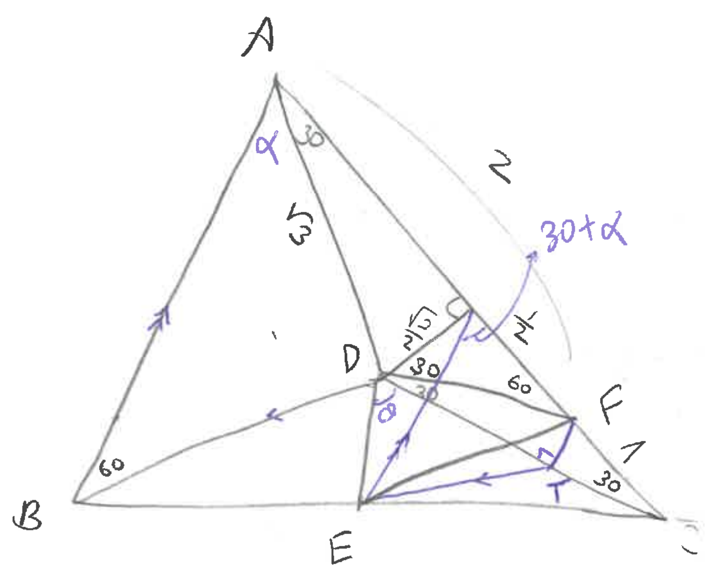
EXC'yi 90° saat yönünde döndürürsek

$\underbrace{QH \parallel VW}_{\text{benzerlikten dolayı}} \underbrace{\parallel EA'}_{\text{Lemma 1'den dolayı}}$ olur. $\rightarrow$ $QH \parallel EA'$ □

CGMO 2007/5

Point D lies inside triangle ABC such that $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Point E is the midpoint of segment $\overline{BC}$. Point F lies on segment $\overline{AC}$ with $AF = 2FC$. Prove that $DE \perp EF$.

Solution



$\theta \stackrel{?}{=} \alpha$

$\triangle ABD$'de

$$\frac{BD}{\sin\alpha} = \frac{\sqrt{3}}{\sin 60} = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$$

$$BD = 2\sin\alpha \Rightarrow ET = \sin\alpha$$

$$\angle BDE = 120 + \alpha - \theta = \angle DET$$

$\triangle DET$'de

$$\frac{\sin\alpha}{\sin\theta} = \frac{DT}{\sin(120+\alpha-\theta)} = \frac{\sqrt{3}}{2\sin(60+\theta-\alpha)}$$

$$\underbrace{2 \cdot \sin(60+\theta-\alpha)\sin\alpha} = \sin 60 \sin\theta \cdot 2$$

$$\cos(60+\theta-2\alpha) - \cos(60+\theta) = \cos(60-\theta) - \cos(60+\theta)$$

$$\Downarrow$$

$$60+\theta-2\alpha = 60-\theta \quad \vee \quad \underbrace{60+\theta-2\alpha = \theta-60}$$

$$\alpha = 60 \Rightarrow \angle BDC = 180$$

$$\Downarrow$$

$$D \in BC$$

D doesn't lie inside $\triangle ABC$.

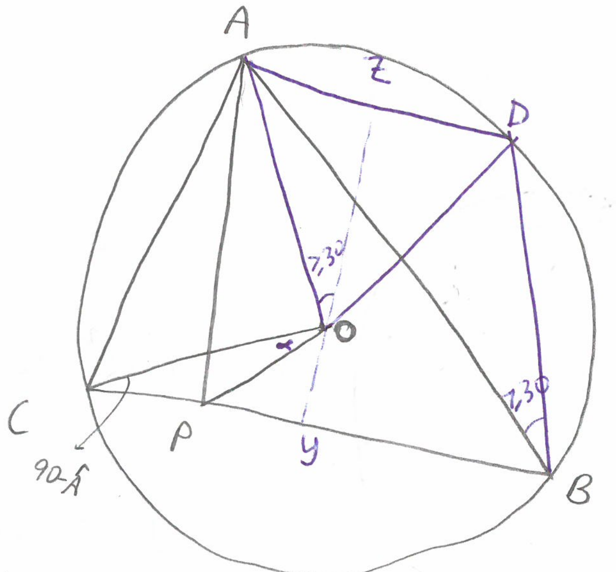
$$\Downarrow$$

$$\theta = \alpha \quad \square$$

IMO 2001/1

Consider an acute triangle ΔABC. Let P be the foot of the altitude of triangle ΔABC issuing from the vertex A, and O be the circumcenter of the triangle ΔABC. Assume that $\angle C \geq \angle B + 30^\circ$. Prove that $\angle A + \angle COP < 90^\circ$

Solution as bright as a diamond. (from AOPS)



It suffices to prove $OP > CP$.

Take D on circumcircle s.t. $AD \parallel BC$. Then $\angle ABD \geqslant 30$. Thus $\angle AOD \geq 60$. Let Z and Y be the midpoints of AD and BC. Then $AZ = PY \geqslant \frac{R}{2}$ ($O = Y$ is impossible because the A would be 90°.)

Now $CP = CY - PY < R - PY \leq \frac{R}{2} \Rightarrow CP < \frac{R}{2}$

$OP > PY \geqslant \frac{R}{2} \Rightarrow OP > \frac{R}{2}$

Then $OP > CP$ as desired

Solution 2 (yarısı benden yarısı AOPS'tan)

We have to prove $OP > CP$

Say $CP < OP$. Then $2CP > CP + OY > CO = R$ $\quad CP = b\cos C$

$2b\cos C > R = \frac{b}{\sin B \cdot 2} \Rightarrow 4\sin B \cos C > 1$

But,

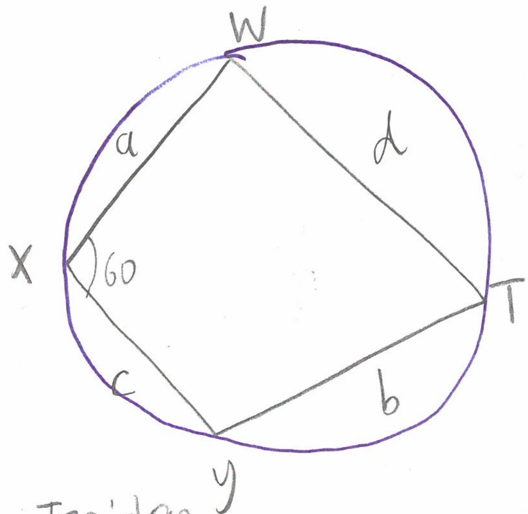
$4\sin B\cos C \leq 4\sin B\cos(30+B) \leq 2(\sin(2B+30) - \sin 30) \leq 2(1-\frac{1}{2}) = 1$ #

IMO 2001/6

Let $a > b > c > d$ be positive integers such that $ac+bd = (b+d+a-c)(b+d-a+c)$.
Prove that $ab+cd$ is not a prime.

Solution (hem de geodan)

$$ac+bd = (b+d)^2 - (a-c)^2$$
$$\Downarrow$$
$$a^2 - ac + c^2 = b^2 + bd + d^2$$



$\angle WXY = 60°$ alınırsa

$WY^2 = a^2+c^2-ac = b^2+d^2+bd$

$= b^2+d^2-2bd\cos 120$

old. $\angle WTY = 120°$ gelir.

Yani $XWTY$ cemb.

Ptolemy Teo'dan

$$WY^2 = \frac{(ab+cd)(ad+bc)}{ac+bd}.$$

If $ab+cd=p$ is a prime

then $p \cdot \frac{ad+bc}{ac+bd} = WY^2$ cannot be an integer because $ad+bc < ac+bd < p$. But $WY^2 = a^2-ac+c^2$ is clearly an integer. Thus $ab+cd$ cannot be a prime.

COMPLEX NUMBERS

$z = a + bi$, $a,b \in \mathbb{R}$ $\quad i^2 = -1.$

$\forall z \in \mathbb{C}$ şöyle yazılabilir:

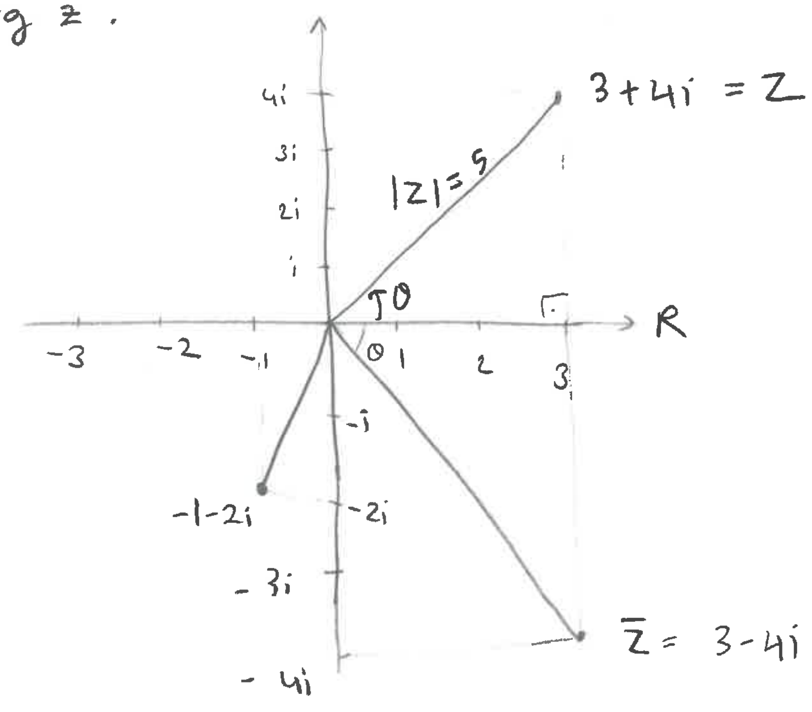
$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$r \geq 0, r \in \mathbb{R}$, $\theta \in \mathbb{R}$

The magnitude of $z = a + bi = re^{i\theta}$, denoted $|z|$ is equal to r, or equivalently,

$$|z| = \sqrt{a^2+b^2} = r$$

The number θ is called the argument of z, denoted arg z.

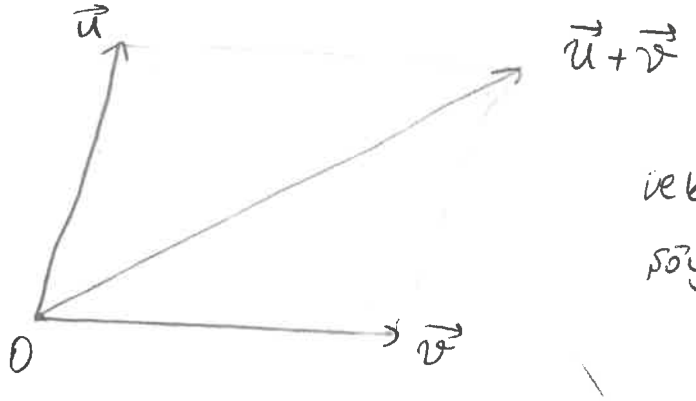


The complex conjugate of z (or just conjugate) is the number

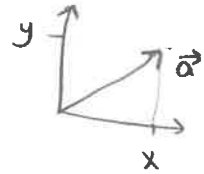
$$\bar{z} = a - bi = re^{-i\theta}$$

$\rightarrow |z|^2 = a^2 + b^2 = (a-ib)(a+ib) = z\bar{z}$

DOT PRODUCT and VECTORS



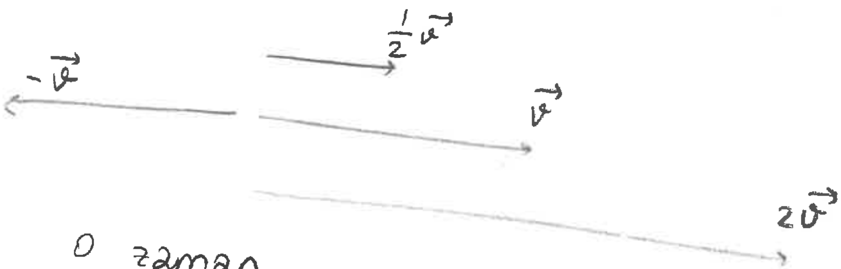
vektörü koordinatlarıyla şöyle gösterebiliriz $\langle x, y \rangle$



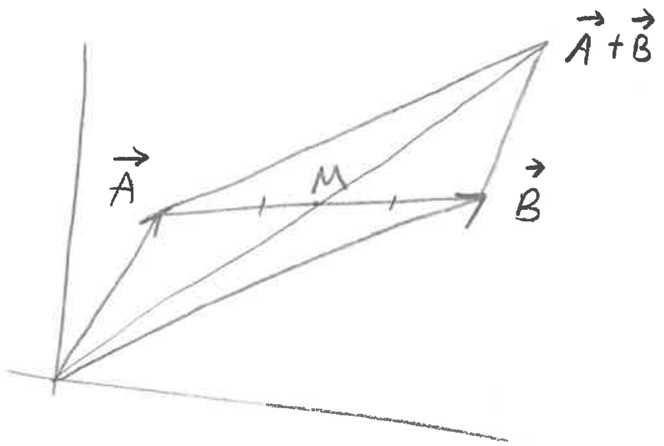
Toplama.

$\langle x, y \rangle + \langle a, b \rangle = \langle x+a, y+b \rangle$

Böylece bir paralel kenar oluşur.



O zaman,



AB 'nin orta noktası M ise $M = \frac{1}{2}(\vec{A}+\vec{B})$

Dot product.

$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos\theta$ θ is the angle between $\vec{v}$ and $\vec{w}$

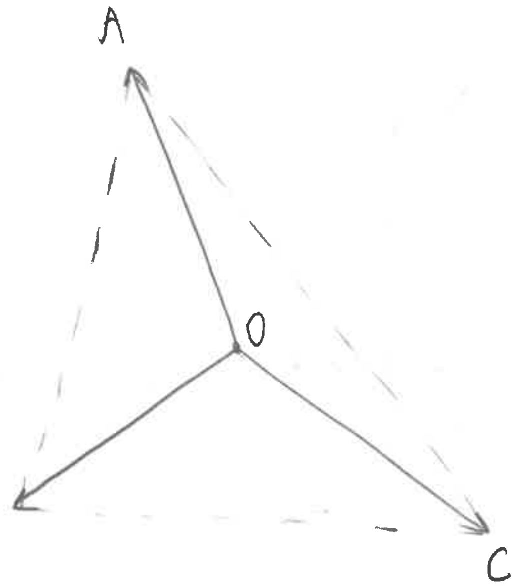
$\langle a, b \rangle \cdot \langle x, y \rangle = ax + by$ ← gördüğün gibi çarpımın sonucu bir vektör değil.

$|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

$\vec{v} \perp \vec{w} \iff \vec{v} \cdot \vec{w} = 0$

Consider a triangle ABC with circumcenter O.
If we set O as the zero vector $\vec{0}$ then let's see what we have.

$$|\vec{A}| = |\vec{B}| = |\vec{C}| = R$$



$$\vec{A}.\vec{B} = R^2 . \cos 2C = R^2(1-2\sin^2 C) = R^2\left(1-2\left(\frac{c}{2R}\right)^2\right)$$

$$= R^2 - \frac{1}{2}c^2$$ similarly,

$$\vec{B}\cdot\vec{C} = R^2 - \frac{1}{2}a^2$$

$$\vec{C}.\vec{A} = R^2 - \frac{1}{2}b^2$$

Lets compute OH! ($\vec{H} = \vec{A} + \vec{B} + \vec{C}$, is to be proved later)

$$OH^2 = |\vec{OH}|^2 = \vec{OH}.\vec{OH} = \vec{H}.\vec{H} = (\vec{A}+\vec{B}+\vec{C})(\vec{A}+\vec{B}+\vec{C})$$

$$= \vec{A}.\vec{A} + \vec{B}\cdot\vec{B} + \vec{C}\cdot\vec{C} + 2(\vec{A}.\vec{B} + \vec{B}.\vec{C} + \vec{C}\cdot\vec{A})$$

$$= 3R^2 + 2\left(3R^2 - \frac{1}{2}(a^2+b^2+c^2)\right)$$

$$= 9R^2 - (a^2+b^2+c^2)$$!

Back to complex numbers,

$z_1 = r_1 . e^{i\theta_1}$

$z_2 = r_2 . e^{i\theta_2}$

$z_1 . z_2 = r_1 . r_2 . e^{i(\theta_1+\theta_2)}$

so

$|z_1||z_2| = |z_1 z_2|$ and

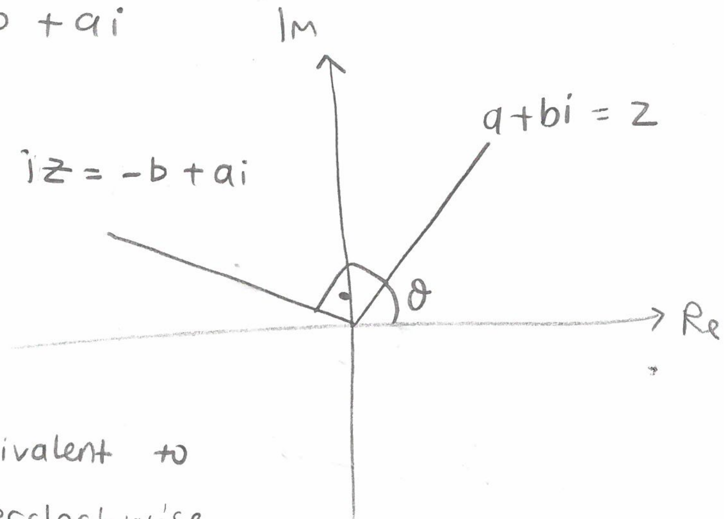
$\arg z_1 z_2 = \arg z_1 + \arg z_2$

z 'leri mod 360'ta aldığımızdan aslında

$\arg z_1 z_2 \equiv \arg z_1 + \arg z_2 \pmod{360}$

$iz = i.(a+bi) = -b + ai$

$= r . 1\, e^{i(\theta+90)}$

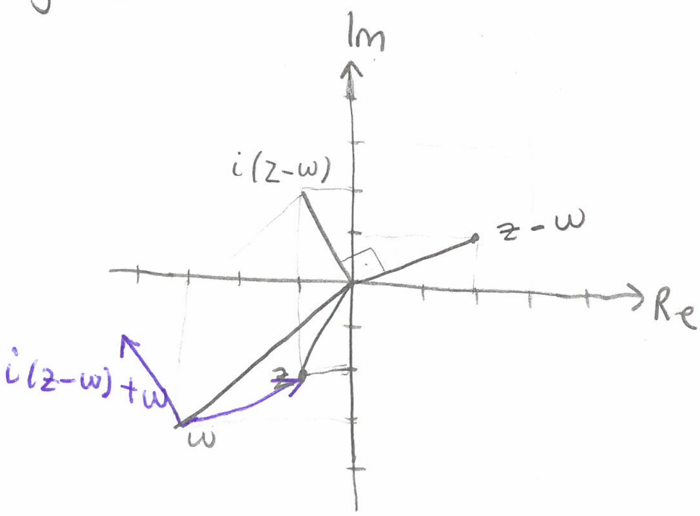


$r = |i| = 1$

$\arg i = 90$

Multiplying by i is equivalent to rotating by 90° counterclockwise around the origin.

Peki z 'yi w'nin etrafında döndürmek için ne yapacağız?



Önce w 'yı çıkar yani 0'ya getir vektörü sonra döndür sonra geri ekle.

$z \rightarrow i(z-w)+w$

We can generalize further to any complex number other than i. For any complex number w and nonzero α the map

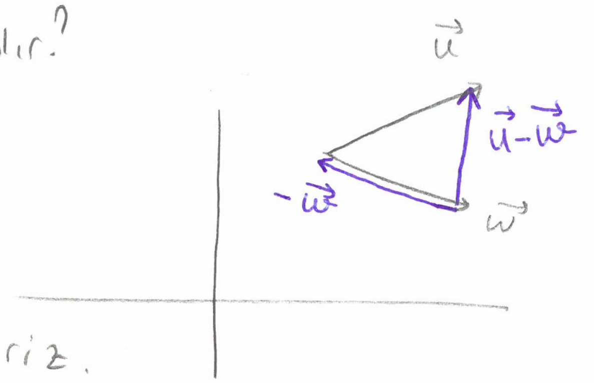
$$z \mapsto \alpha(z-w)+w$$

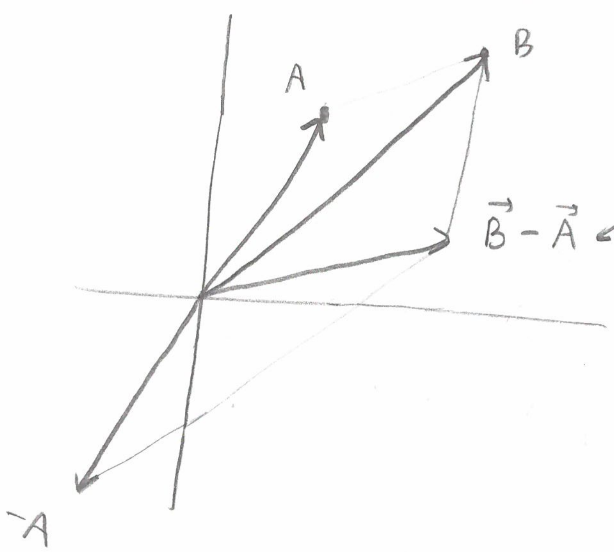
is a spiral similarity. That means it is a map that rotates by $\arg\alpha$ and dilates by $|\alpha|$; it is a composition of a rotation and a homothety.

$\vec{u}-\vec{w}$ nasıl yapılır?

$\vec{u}-\vec{w}=\vec{u}+(-\vec{w})$

uc uca ekleme yönteminden $\vec{w}$'nin okunu ters çevirebiliriz.



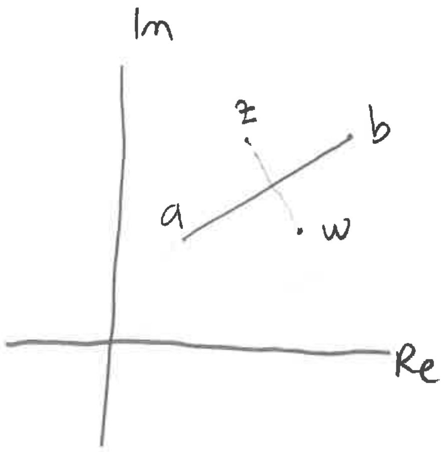


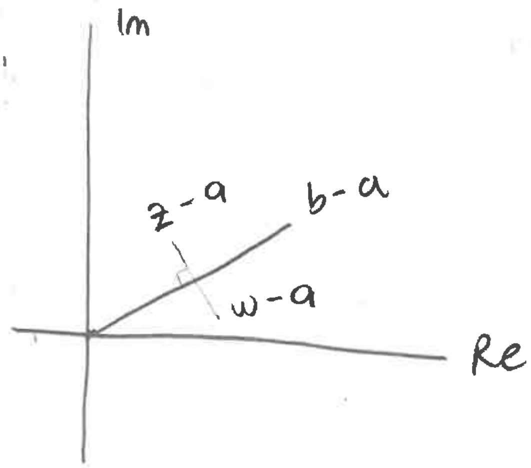
Böylece 0'ya gelmiş oluyoruz.

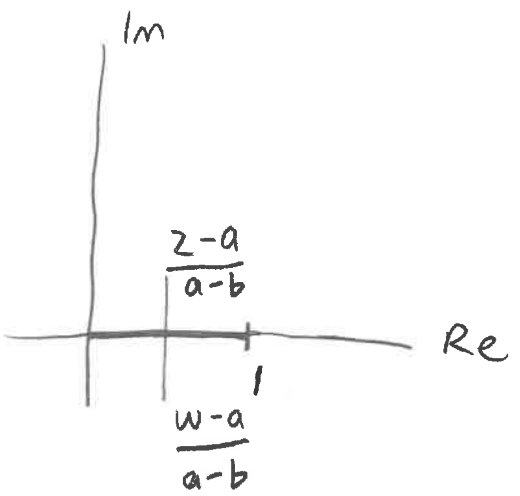
Lemma 6.2 Complex Reflection.

Let W be the reflection of z over a given $\overline{AB}$. Then

$$w = \frac{(a-b)\bar{z} + \bar{a}b - a\bar{b}}{\bar{a} - \bar{b}}$$







$$z \mapsto \frac{z-a}{b-a}$$

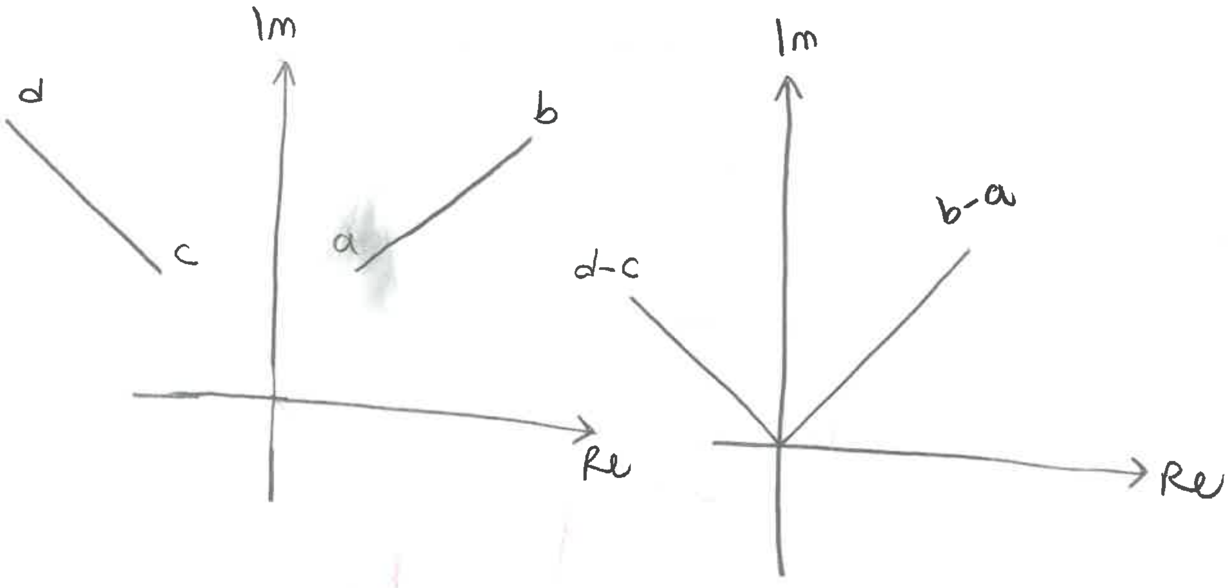
$$w \mapsto \frac{w-a}{b-a}$$

These two are conjugates right now!

$$\frac{w-a}{b-a} = \overline{\left(\frac{z-a}{b-a}\right)} = \frac{\bar{z}-\bar{a}}{\bar{b}-\bar{a}}$$

With enough calculation one can have the desired result.

Perpendicularity.



$$\arg \frac{z}{w} = \arg z \cdot w^{-1} = \arg z + \arg w^{-1}$$

$$= \arg z - \arg w$$

If $\arg(d-c) - \arg(b-a) = 90$ then $AB \perp DC$.

$$\arg(d-c) - \arg(b-a) = 90 \iff \arg \frac{d-c}{b-a} = 90.$$

Which means $\frac{d-c}{b-a}$ is pure imaginary. (without real part)

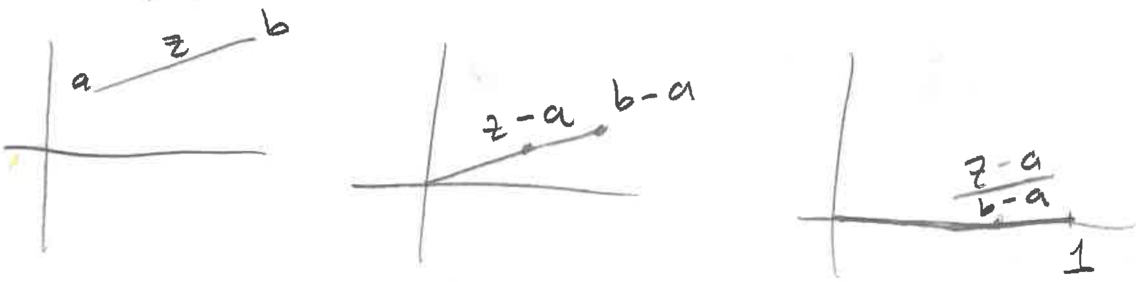
Perpendicularity Criterion. The complex numbers a, b, c, d have the property $\overline{AB} \perp \overline{CD}$ if and only if

$$\frac{d-c}{a-b} + \overline{\left(\frac{d-c}{a+b}\right)} = 0.$$

~~Corrot~~ Collinearity Criterion

Complex numbers z, a, b are collinear if and only if

$$\frac{z-a}{z-b} = \overline{\left(\frac{z-a}{z-b}\right)}$$



Shoelace Formula

$A = (x_1, y_1)$ $B = (x_2, y_2)$ $C = (x_3, y_3)$

then the are of the triangle ABC is,

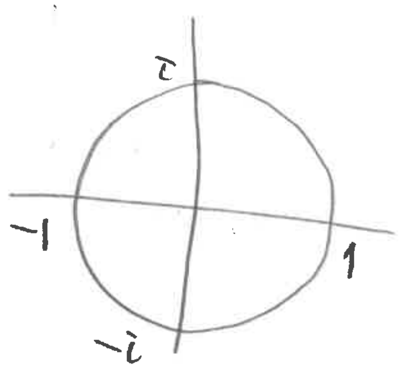
$$\frac{1}{2}\left|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right|$$

Complex Shoelace Formula

$a, b, c \in \mathbb{C}$. The signed are of triangle ABC is

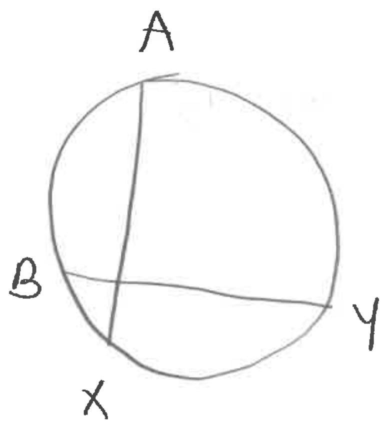
$$\frac{i}{4}\left|\begin{matrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{matrix}\right|$$

$$\left.\begin{aligned} a &= a_x + a_y \cdot i \\ b &= b_x + b_y \cdot i \\ c &= c_x + c_y i \end{aligned}\right\}$$ yazıp normal shoelace formulünü uygularsak gelir.



Unit circle

$|z| = 1$ olan z'lerden oluşur.



$AX \perp BY \iff ax + by = 0.$

$a, b, x, y \in \mathbb{C}.$

Proof.

$$\frac{a-x}{b-y} + \overline{\left(\frac{a-x}{b-y}\right)} = 0.$$

$|z|^2 = z \cdot \bar{z}$ old. $1 = z.\bar{z}$ (unit circle'da)

$\frac{1}{z} = \bar{z}.$

$$\frac{a-x}{b-y} + \frac{\bar{a}-\bar{x}}{\bar{b}-\bar{y}} = 0.$$

$$\frac{a-x}{b-y} + \frac{\frac{1}{a} - \frac{1}{x}}{\frac{1}{b} - \frac{1}{y}} = 0. \Rightarrow \left(\frac{a-x}{b-y}\right)\underbrace{\left(1 + \frac{ax}{by}\right)}_{0} = 0.$$

$by + ax = 0.$

Lemma 6.11. (Complex Foot)

If a and b, $a \neq b$, are on the unit circle and z is an arbitrary complex number, then the foot z to AB is given by

$$\frac{1}{2}(a+b+z-ab\bar{z})$$

Proof. Putting $\bar{a} = \frac{1}{a}$, $\bar{b} = \frac{1}{b}$ in the formula of finding the foot of the perpendicular from z to AB.

Euler Line - complex

1) The circumcenter is $o = 0$.

2) The centroid is $g = \frac{1}{3}(a+b+c)$

3) The orthocentre is $h = a+b+c$ (If ABC is not an equilateral triangle)

Proof 3. Since $AH \perp BC$ and $BH \perp AC$ and $CH \perp AB$.

$$\frac{a-h}{b-c} + \frac{\bar{a}-\bar{h}}{\bar{b}-\bar{c}} = 0$$

$$\frac{h-a}{b-c} + \frac{\bar{h}-\frac{1}{a}}{\frac{1}{b}-\frac{1}{c}} = \frac{h-a}{b-c} - bc\,\frac{\bar{h}-\frac{1}{a}}{b-c} = 0.$$

$$bc\left(\bar{h}-\frac{1}{a}\right) = h-a$$

$$abc\bar{h} - bc = ha - a^2$$

$$abc\bar{h} - ha = bc - a^2 \quad \text{Similarly,}$$

$$abc\bar{h} - hb = ac - b^2$$

$$abc\bar{h} - hc = ab - c^2$$

$$h(b-a) = (b-a)(a+b+c) \Rightarrow h = a+b+c$$

If $a \neq b$.

Since $h = 3g$ it follows that O, G, H are collinear and $OH = 3OG$, this establishes the Euler line.

Complex nine-point circle

If a, b, c lie on the unit circle, and H is the orthocentre of $\triangle ABC$, the point $n_g = \frac{1}{2}(a+b+c)$ is a distance of $\frac{1}{2}$ from the midpoint of $\overline{BC}$, the midpoint of $\overline{AH}$, and the foot from A to $\overline{BC}$.

Proof.

Remember $|a|=|b|=|c|=1$ as they are on the unit circle.

First lets check the distance to the midpoint of BC.

$$\left|n_g - \frac{b+c}{2}\right| = \left|\frac{a}{2}\right| = \frac{1}{2}|a| = \frac{1}{2}.$$

Now, the distance to the midpoint of $\overline{AH}$.

$$\left|n_g - \frac{a+h}{2}\right| = \left|n_g - \frac{a+(a+b+c)}{2}\right| = \left|-\frac{a}{2}\right| = \frac{1}{2}|a| = \frac{1}{2}.$$

Now, the distance to the foot from A to $\overline{BC}$.

$$\left|n_g - \frac{1}{2}\left(b+c+a-bc.\bar{a}\right)\right| = \left|n_g - \frac{1}{2}\left(b+c+a-\frac{bc}{a}\right)\right|$$

$$= \left|\frac{1}{2}\frac{bc}{a}\right| = \frac{1}{2}\frac{|b||c|}{|a|} = \frac{1}{2} \qquad \square$$

Concyclic Complex Numbers

Let a, b, c, d be distinct complex numbers, not all collinear. Then A, B, C, D are concyclic if and only if

$$\frac{b-a}{c-a} \div \frac{b-d}{c-d}$$

is a real number.

Proof

$$\arg\left(\frac{b-a}{c-a}\right) - \arg\left(\frac{b-d}{c-d}\right) = 0.$$ olmalı.

gerisini henüz getiremedim.

Complex similarity

Two triangle ABC and XYZ are similar if and only if

$$0 = \begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix}$$

Proof.

$$\begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix} = \begin{vmatrix} a-c & x-z & 1 \\ b-c & y-z & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a-c & x-z \\ b-c & y-z \end{vmatrix} = 0$$

$$\Longleftrightarrow \frac{a-c}{a-b} = \frac{x-z}{x-y}$$ ki zaten

üçgende şöyle bir mapleme yapabiliriz

$a-c \longmapsto AC$ $\quad x-z \longmapsto XZ$

$a-b \longmapsto AB$ $\quad x-y \longmapsto XY$

$$\frac{AC}{AB} = \frac{XZ}{XY}$$ ise

zaten benzer olurlar.

Complex Intersection

If lines AB and CD are not parallel then their intersection is given by

$$\frac{(\bar{a}b - a\bar{b})(c-d) - (a-b)(\bar{c}d - c\bar{d})}{(\bar{a}-\bar{b})(c-d) - (a-b)(\bar{c}-\bar{d})}$$

Proof.

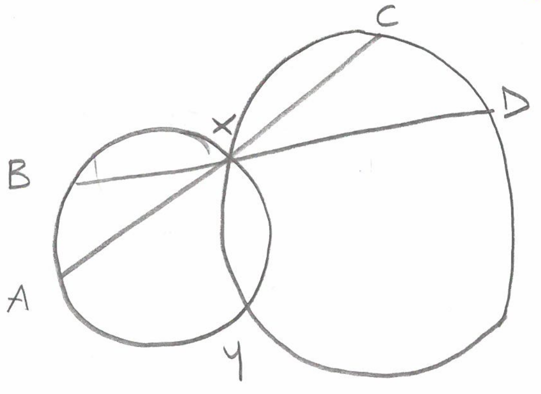
Solve this:

$$0 = \begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = \begin{vmatrix} z & \bar{z} & 1 \\ c & \bar{c} & 1 \\ d & \bar{d} & 1 \end{vmatrix}$$

Burdan $\frac{z-b}{a-b} = \overline{\left(\frac{z-b}{a-b}\right)}$ ve $\frac{z-d}{c-d} = \overline{\left(\frac{z-d}{c-d}\right)}$ geliyor.

z demek ki kesişim. Bir de z'yi çekip alırsan yukarıdaki ifade gelecek.

Intersection of 2 Circles



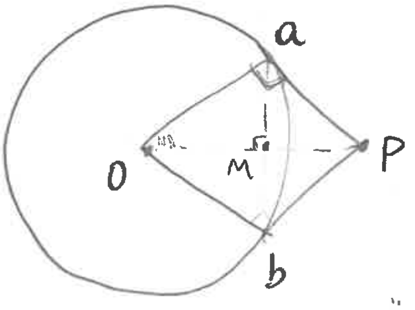
$$y = \frac{ad - bc}{a+d-b-c}$$

Complex Tangent Intersection

Let A and B be points on the unit circle with $a+b \neq 0$. Then

$$\frac{2ab}{a+b} = \frac{2}{\bar{a}+\bar{b}}$$

is the intersection point of the tangents at A and B.



OM · OP = 1 (easy to show from similar triangles)

$OM \cdot OP = 1 \Rightarrow |m| \cdot |p| = 1$

$m \cdot p = \bar{m}\, p = 1$ Çünkü $\arg m = \arg p = 0$.

dolayısıyla $\vec{m} \cdot \vec{p} = |m||p| \cos 0 = 1$.

$m = \bar{m}$ olduğu da OP'yi Re seçersek görülür.

$$p = \frac{1}{\bar{m}} = \frac{2}{\bar{a}+\bar{b}} = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b} \quad \square$$

Complex Incenter

Given ABC on the unit circle, it is possible to pick complex numbers u, v, w such that

(a) $a = u^2 \quad b = v^2 \quad c = w^2$ and

(b) the midpoints of arc $\widehat{BC}$ not containing A is $-vw$; the analogous midpoints opposite B and C are $-wu$ and $-uv$

In this case incenter is given by
$-(uv + vw + wu)$.

Complex Circumcenter

The circumcircle of a triangle XYZ is given by the quotient

$$\begin{vmatrix} x & x\bar{x} & 1 \\ y & y\bar{y} & 1 \\ z & z\bar{z} & 1 \end{vmatrix} \div \begin{vmatrix} x & \bar{x} & 1 \\ y & \bar{y} & 1 \\ z & \bar{z} & 1 \end{vmatrix}$$

In particular if $z = 0$

$$\frac{xy(\bar{x} - \bar{y})}{\bar{x}y - \bar{y}x}$$

MÜKEMMEL ÇÖZÜMLÜ

SORU USA TSTST 2013/1

Let ABC be a triangle and D, E, F be the midpoints of the arcs BC, CA, AB on the circumcircle. Line l_a passes through the feet of the perpendiculars from A to $\overline{DB}$ and $\overline{DC}$. Line m_a passes through the feet of the perpendiculars from D to $\overline{AB}$ and $\overline{AC}$. Let A_1 denote the intersection of l_a and m_a. Define points B_1 and C_1 similarly. Prove that $A_1B_1C_1 \triangle DEF$.

Solution

Line l_a is the simson line from D onto $\triangle ABC$ and m_a is the simson line from A onto $\triangle DBC$. Let H and H_A be the orthocentres of the triangles $\triangle ABC$ and $\triangle DBC$. We know that l_a and m_a bisect AH_A and DH. Say M_1 and M_2 are the midpoints of AH_A and DH. Then

$$m_1 = \frac{(b+c+d)+a}{2} = \frac{(a+b+c)+d}{2} = m_2.$$

So $A_1 = M_1 = M_2$. Similarly B_1 and C_1. Thus taking homothety h from H over DEF takes $\triangle DEF$ onto $\triangle A_1B_1C_1$.

2nd Solution

Tamamen Complexle. ABC'yi unit circle al önce dikmeleri tanımla. Sonra fazlalıklardan kurtulmanı sağlayacak bir mapleme yap.

$T: \alpha \mapsto 2\alpha - (x^2+y^2+z^2)\,xy$ gibi mesela

Sonra kesişimi bul. Sadece 1 tanesinin mesela A_1. Map'i geri al. Bu arada D, E, F'lere $-xy$, $-yz$, $-zx$ gelecek şekilde al a, b, c'yi,

$a_1 = \frac{1}{2}(a+b+c+d)$ gelecek yani

HD'nin orta noktası. Soru bitti burdan.

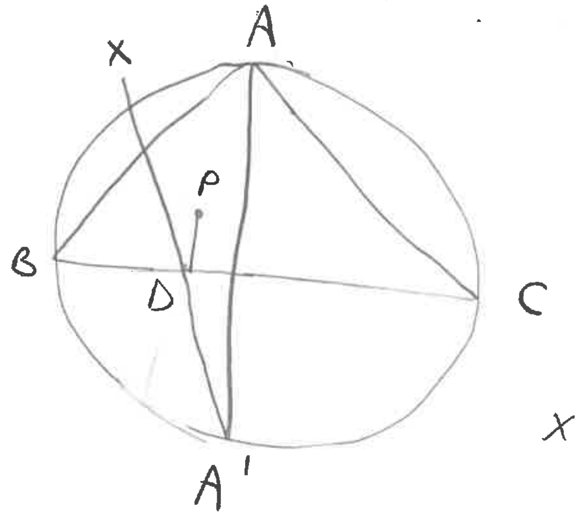
Tamamen cebir yani.

Chinese TST 2011

Let ABC be a triangle, and let A', B', C' be points on its circumcircle, diametrically opposite A, B, C, respectively. Let P be any point inside $\triangle ABC$ and let D, E, F be the foot of the altitudes from P onto $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. Let X, Y, Z denote the reflections of A', B', C' over D, E, F respectively. Show that triangles XYZ and ABC are similar to each other.

Solution Obvious to see that the solution is from complex numbers.

Let (ABC) be the unit circle.



$a' = -a$

$x = (h + p - bc\bar{p})$

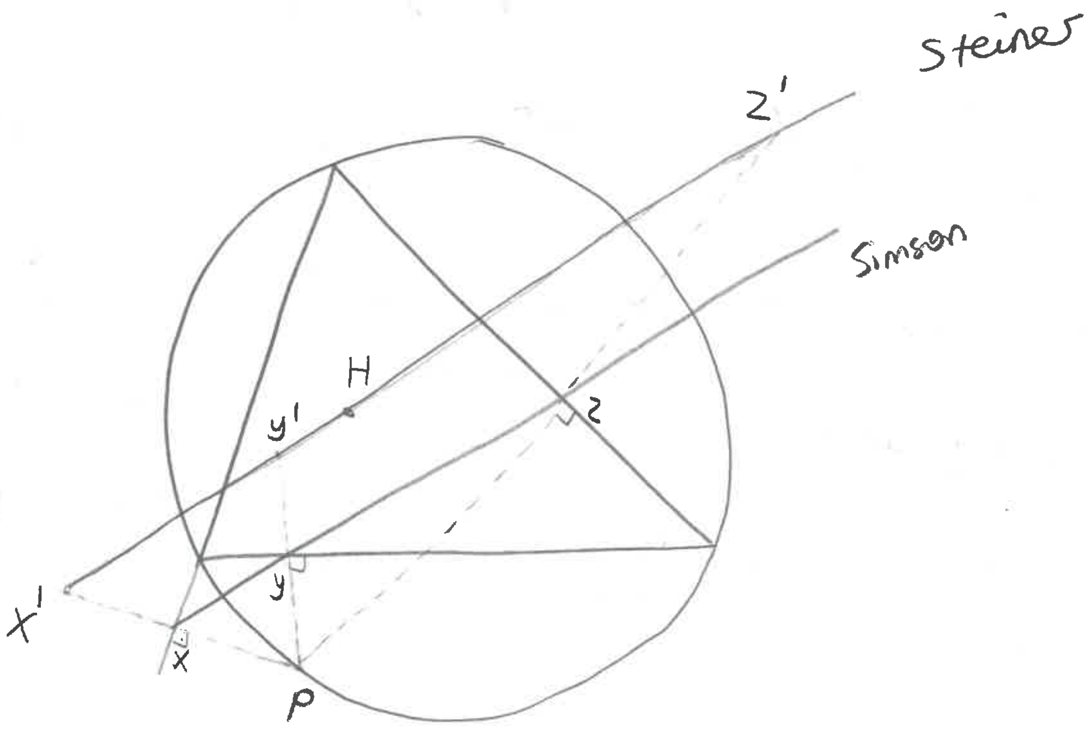
$d = \frac{1}{2}(b + c + p - bc\bar{p})$

$\triangle ABC \sim \triangle A'B'C'$ old. O'a eşit.

$$\frac{x + a'}{2} = \frac{x - a}{2} = d = \frac{1}{2}(b + c + p - bc\bar{p})$$

$$\begin{vmatrix} a & h+p-bc\bar{p} & 1 \\ b & h+p-ca\bar{p} & 1 \\ c & h+p-ab\bar{p} & 1 \end{vmatrix} = -abc\bar{p} \begin{vmatrix} a & \frac{1}{a} & 1 \\ b & \frac{1}{b} & 1 \\ c & \frac{1}{c} & 1 \end{vmatrix}$$

$$= -abc\bar{p} \begin{vmatrix} a & \bar{a} & 1 \\ b & \bar{b} & 1 \\ c & \bar{c} & 1 \end{vmatrix} = 0. \Rightarrow \triangle XYZ \sim \triangle ABC.$$



x', y', z' ; P'nin sırayla x, y, z noktalarına göre simetriği olsun. X' y' Z' de doğrusaldır ve H'tan geçer.

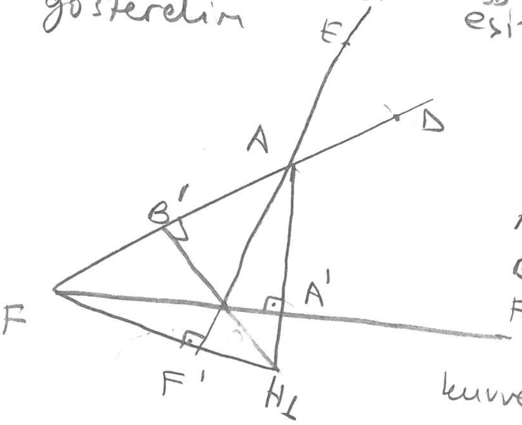
Kanıt. Simson line zaten doğrusal. P'den 2 oranında homoteti uygularsak $X \mapsto X'$ $y \mapsto y'$ $z \mapsto z'$ olur. Doğrusal olur. Simson doğrusunun PH'ı ortaladığını zaten biliyoruz.

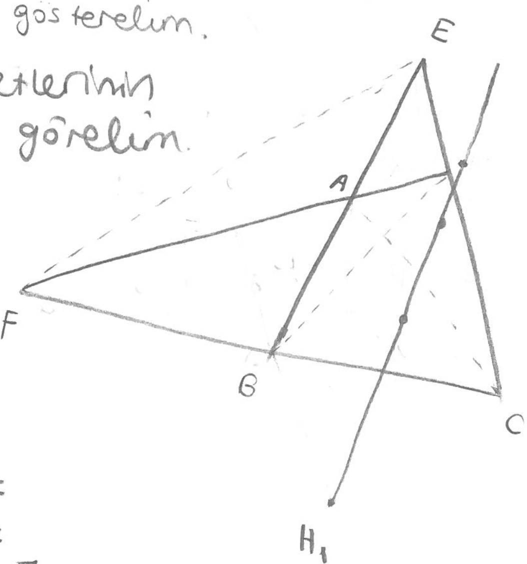
(*) • 4 doğrunun oluşturduğu üçgenlerin diklik merkezleri doğrudaştır.

• Bu doğru köşegenlerin orta noktalarını birleştiren doğruya diktir.

İspat. $[XY]$ çaplı çemberi ω_{XY} ile gösterelim.

H_1'in ω_{AC}, ω_{EF}, ω_{BD}'ye göre kuvvetlerinin eşit olduğunu görelim.

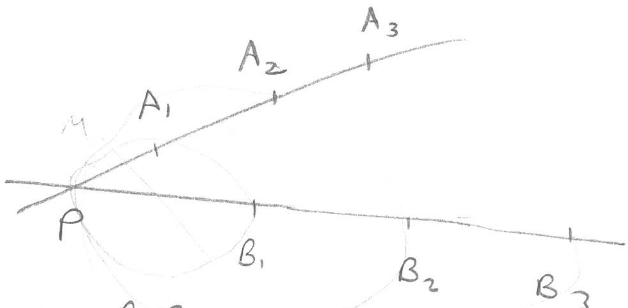




$A' \in \omega_{AC}$
$B' \in \omega_{BD}$
$F' \in \omega_{FE}$

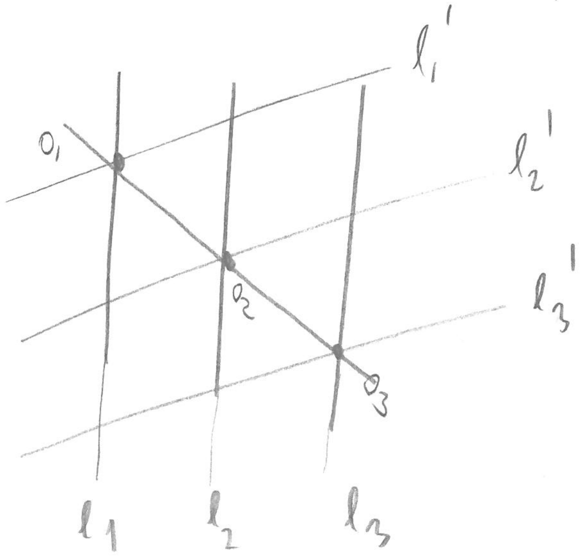
kuvvetler: $HA'.AH = BH.B'H = F'H.HF$ ∎

Lemma.



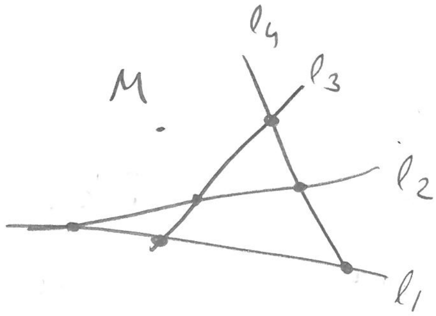
$A_1A_2 : A_1A_3 = B_1B_2 : B_1B_3$ ise (PA_iB_i) çemberleri P'den farklı bir noktada daha kesişir.

İspat. $[PA_i]$'nin orta dikmesi l_i, $[PB_i]$'nin orta dikmesi l_i' olsun.



P'nin $O_1O_2O_3$ doğrusuna göre simetriği M olsun. $M \in (PA_iB_i)$ olur.

(*) ın ispatı

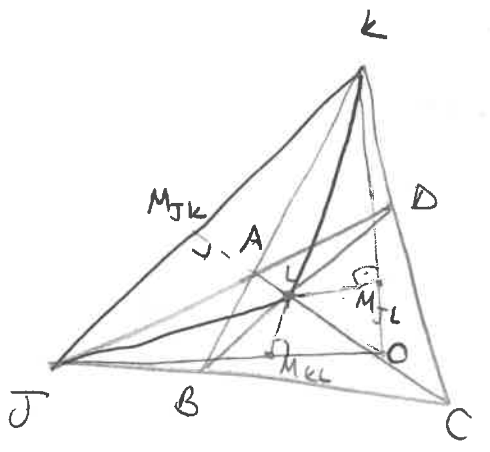


M miquel noktası.

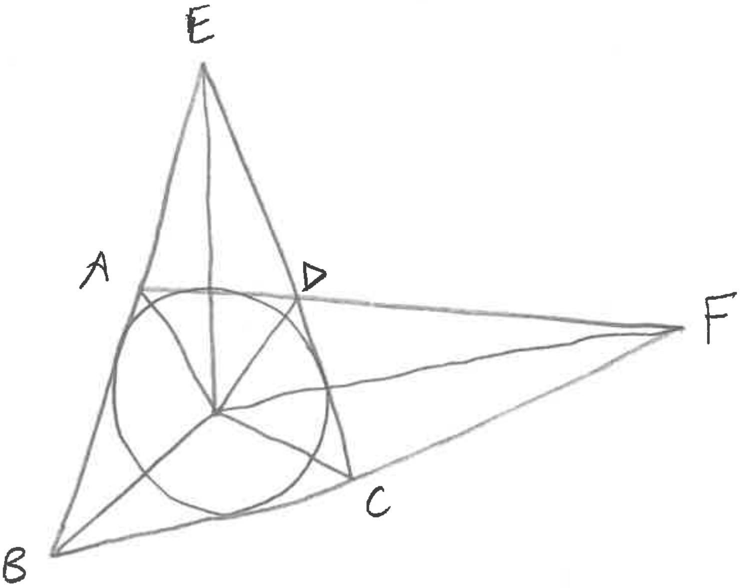
M'nin l_i 'ye göre simetriği M_i olsun.
M_1, M_2, M_3, M_4 doğrusal.
(Simson'a homotetiden → Steiner doğrusundan)

$\Rightarrow$ H_1, H_2, H_3, H_4 bu doğru üstünde
(yine Steiner'den) $\boxed{S}$

A, B, C, D çemberdeş (O merkezli) olduğunda.



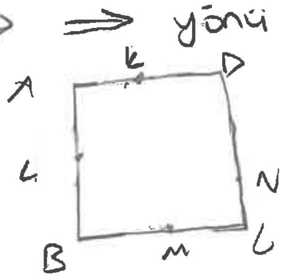
- ΔJKL'nin diklik merkezi O'dur.
- ΔJKL'de diklik ayakları M_{JK}, M_{KL}, M_{LJ} olur.



- Açıortaylar noktadaştır $\Leftrightarrow$ iç teğet çember vardır.

$AB + DC = AD + BC \Leftrightarrow$ iç teğet çem. var $\Leftrightarrow AE + CF = AF + CE$

$\Rightarrow$ yönü ispatı.



K, L, M, N nok. öyle alalım ki $AK = AL$, $BL = BM$, $CM = CN$, $DK = DN$ olsun.

- $L\hat{K}A = 90 - \frac{A}{2}$, $\angle NKD = 90 - \frac{D}{2}$, ...

$\Rightarrow \angle LKA = \frac{A+D}{2}$, $\angle LMA = \frac{B+C}{2}$

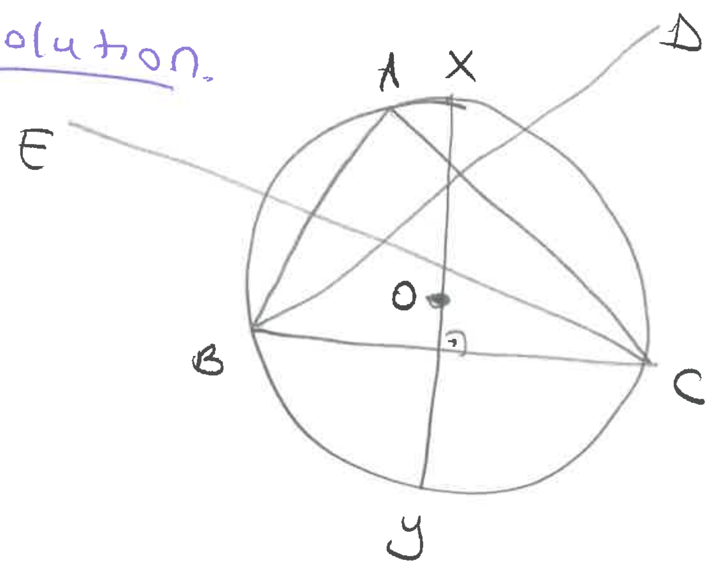
o zaman K, L, N, M çemb.

(KLMN) nin merkezi I olsun. IA, IB, IC, ID açıortay. Ama (KLMN)'nin iç teğet olmayanı da vardır. Çünkü K, L, M, N'yi tek türlü seçmedik.

Online Math Open Fall 2013

Let ABC be triangle with $AB = 13$, $AC = 25$, and $\tan A = \frac{3}{4}$. Denote the reflections of B, C across $\overline{AC}, \overline{AB}$ by D, E. Let O be the circumcenter of $\triangle ABC$. Let P be a point such that $\triangle BPO \sim \triangle PEO$ and let X and Y be the midpoints of the major and minor arcs $\widehat{BC}$ of (ABC). Find $PX \cdot PY$.

Solution.



$x + y = 0$.

Benzerlikten
$$\begin{vmatrix} d & p & 1 \\ p & e & 1 \\ 0 & 0 & 1 \end{vmatrix} = de - p^2 = 0 \Rightarrow de = p^2$$

$x, y, B, C \in (ABC)$ ve $XY \perp BC \Rightarrow xy + bc = 0$.

$$|(p-x)(p-y)|^2 = (p-x)(\bar{p}-\bar{x})(p-y)(\bar{p}-\bar{y})$$
$$= (p^2 - (x+y)p + xy)(\bar{p}^2 - (\bar{x}+\bar{y})\bar{p} + \bar{x}\bar{y})$$
$$= (p^2 - bc)(\bar{p}^2 - \bar{b}\bar{c})$$
$$= |de - bc|^2$$

130

$$d = c + a - \frac{ac}{b}$$

$$e = b + a - \frac{ab}{c}$$

$$|de - bc| = \left|2a^2 - a^2\frac{c}{b} - a^2\frac{b}{c}\right|$$

$$= \left|-\frac{a^2}{bc}(b-c)^2\right|$$

$$= \left|-\frac{a^2}{bc}\right| \left|(b-c)^2\right|$$

$$= BC^2$$

BC de ΔABC'de cos todan $\sqrt{274}$ gelir.

$PX \cdot PY = \sqrt{274}$

GÖZÜM BİR MUAZZAM

BİR MUAZZAM ANLATAMAM

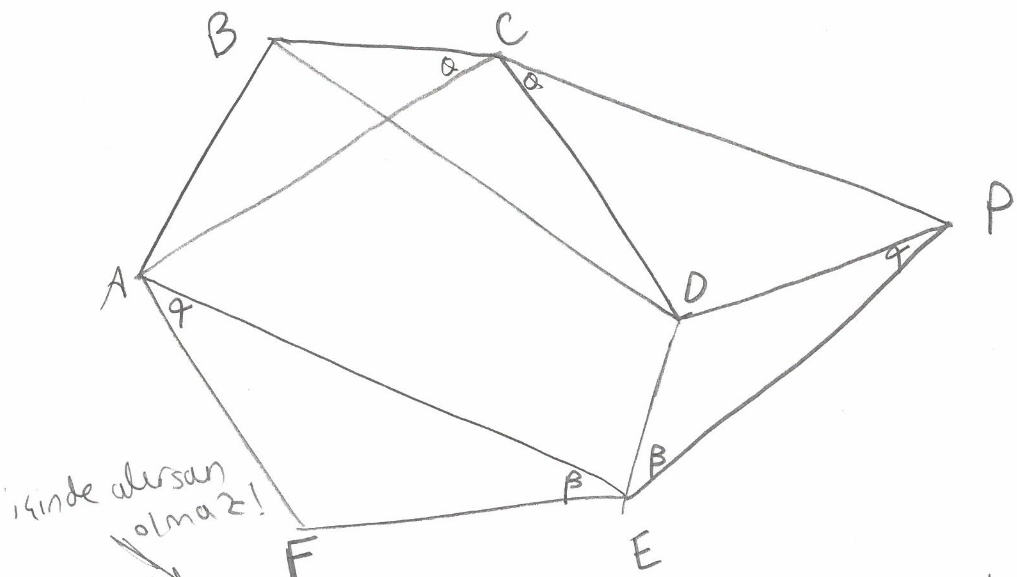
YANİ!

IMO 1998 Shortlist G6

ABCDEF is a convex hexagon. with $\angle B + \angle D + \angle F = 360°$ and $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Show that

$$\frac{BC \cdot DF \cdot EA}{EF \cdot AC \cdot BD} = 1.$$

Solution



Altıgenin dışında öyle bir P noktası alalım ki $\triangle AFE \sim \triangle PDE$ olsun. O zaman,

$$\frac{FE}{DE} = \frac{AE}{EP} \text{ ve } \angle FED = \angle AEP \text{ old. } \triangle FED \sim \triangle AEP$$

gelir. $\angle B + \angle D + \angle F = 360 = \angle D + \angle PDE + \angle CDP$

$= \angle D + \angle F + \angle CDP \Rightarrow \angle B = \angle CDP$ gelir.

$$\frac{AB \cdot CD \cdot EF}{BC \cdot DE \cdot FA} = \frac{AB \cdot CD \cdot DE}{BC \cdot DE \cdot DP} = 1 \Rightarrow \frac{AB}{BC} = \frac{DP}{DC} \text{ old.}$$

$\triangle ABC \sim \triangle PDC$ gelir.

$\frac{BC}{CD} = \frac{AC}{CP}$ geldi benzerlikten ve $\angle BCD = \angle ACP$ old

$\triangle BCD \sim \triangle ACP$ gelir.

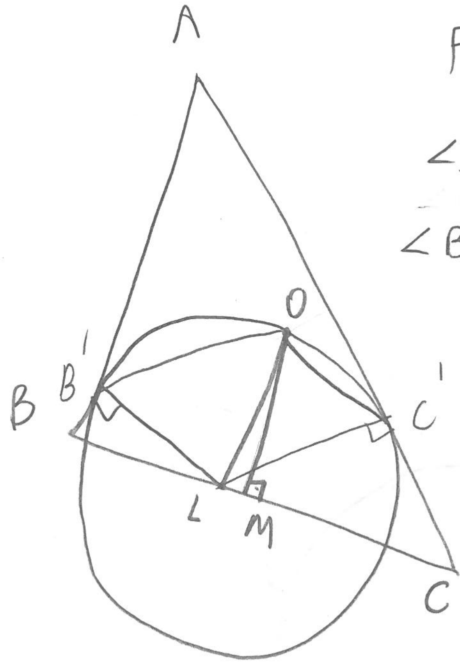
$$\frac{BC}{BD} = \frac{AC}{AP} \quad , \quad \frac{DF}{EF} = \frac{AP}{AE}$$

$$\frac{BC}{BD} \cdot \frac{DE}{EF} \cdot \frac{AE}{AC} = \frac{AC}{AP} \cdot \frac{AP}{AE} \cdot \frac{AE}{AC} = 1$$

ISL 2011 / G1 VAY ANASINI KOLAYMIŞ BE!

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to $\overline{AB}$ at B' and $\overline{AC}$ at C'. Suppose also that the circumcenter O of triangle ABC lies on the shorter arc B'C' of ω. Prove that the circumcircle of ABC and ω meet at 2 points.

Solution



First lets show $60 > \hat{A}$.

$\angle A = \alpha$ olsun.

$\angle B'OC' = 90 + \frac{\alpha}{2} > \angle BOC = 2\alpha$

$\Downarrow$

$180 > 3\alpha \Rightarrow 60 > \alpha$.

Now,

$OL \geq OM = R\cos(\alpha) > R\cos 60 = \frac{R}{2}$

$OL > \frac{R}{2}$ ise bitti ∎

SUPER ÇÖZÜMLÜ SORU

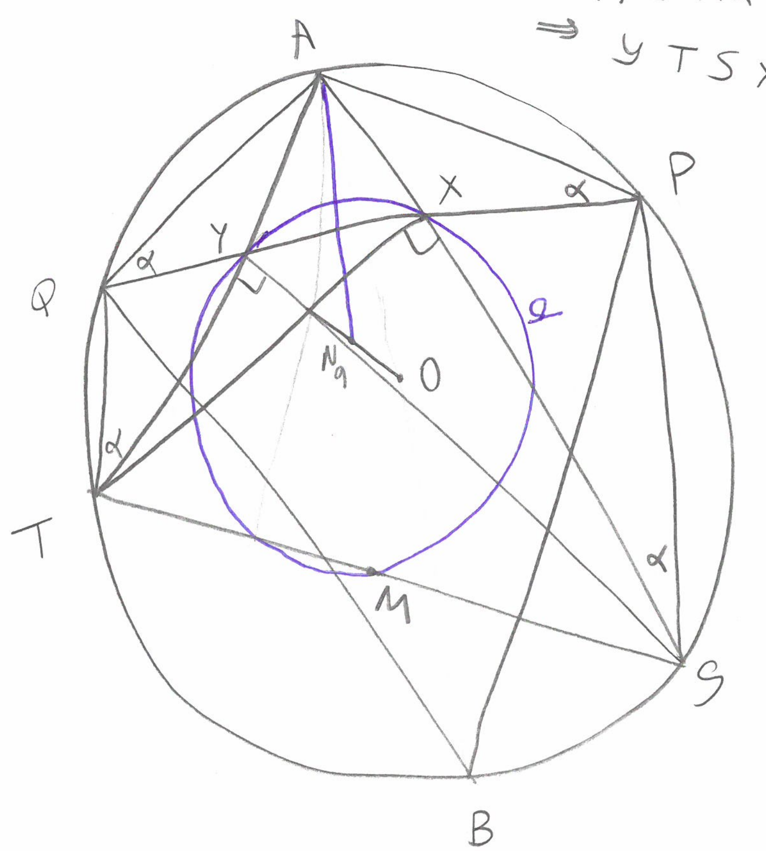
USAMO 2015 / 2

Quadrilateral $APBQ$ is inscribed in circle ω with $\angle P = \angle Q = 90^\circ$ and $AP = AQ < BP$. Let X be a variable point on segment $\overline{PQ}$. Line AX meets ω again at S (other than A) Point T lies on arc AQB of ω such that $\overline{XT}$ is perpendicular to $\overline{AX}$. Let M denote the midpoint of chord $\overline{ST}$. As X varies on segment $\overline{PQ}$, show that M moves along a circle.

Solution

$AY . AT = AQ^2 = AP^2 = AX . AS$

$\Rightarrow$ $YTSX$ çemb.



$\text{Pow}_{\Omega}(A) = |AN_9| - \left(\frac{R}{2}\right)^2 = AX . \frac{AS}{2} = \frac{AP^2}{2}$ old.

$|AN_9|$ sabit. N_9, A merkezli $|AN_9|$ yarıçaplı çemberde geziniyor. G, ağırlık merkezi olsun $\triangle AST$'de.

G merkezli $-\frac{1}{2}$ oranlı homotetiden

$A \mapsto M$, $N_g \mapsto N$ ($NN_g = NO$ olacak nokta)

dur.

$AO \cap MN = Z$ olsun. $N_gN = NO$ ve $MN \parallel AN_g$

old. $AZ = ZO$ gelir.

$\frac{AG}{GM} = 2$ olduğundan $AN_g \cdot \frac{1}{2} = MN = NZ$

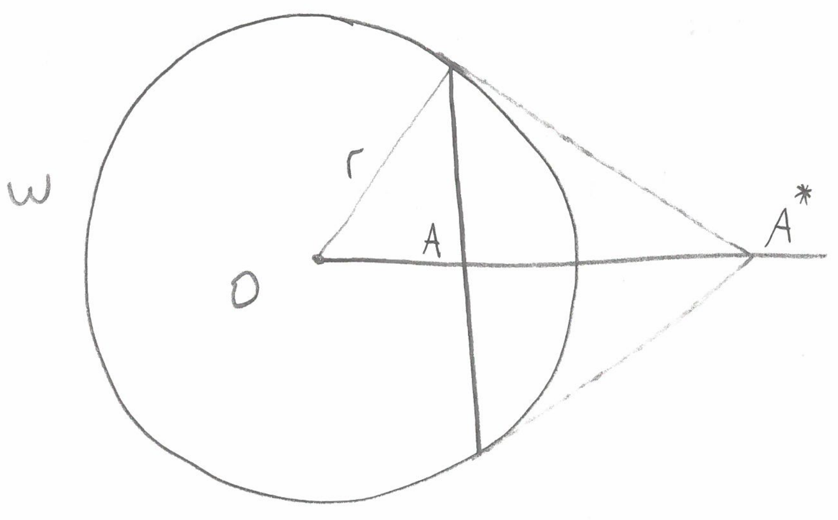
gelir. $$AN_g = MN + NZ = MZ$$

$AN_g = MZ = \frac{AP^2}{2}$ old. sabit M, Z merkezli

çember etrafında döner.

INVERSION

The idea is to view every line as a circle with infinite radius. We add a special point P_∞ to the plane, which every ordinary line passes through (and no circle passes through). This is called the point at infinity.



We say an inversion about w is a map which sends any point A to the point A^* lying on a ray OA such that $OA \cdot OA^* = r^2$.

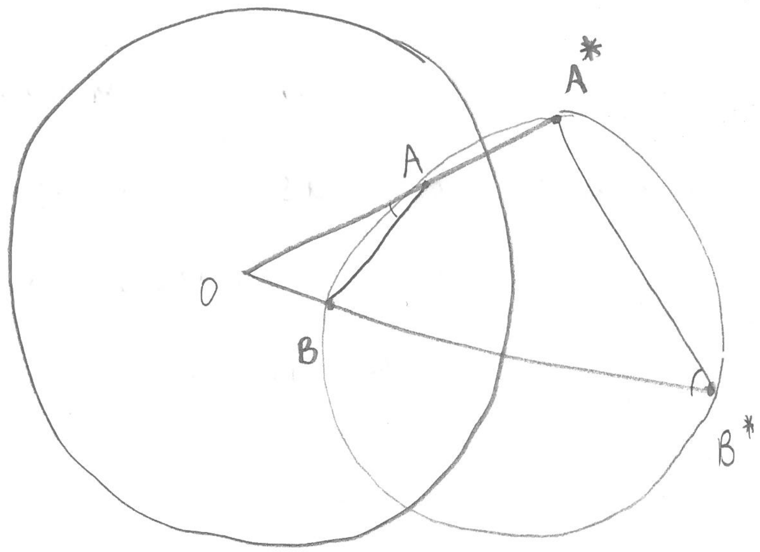
- The center O of the circle is sent to P_∞.
- The point P_∞ is sent to O.

$A = O \Rightarrow \quad OA \cdot OA^* = 0 \cdot \infty = r^2$ doğru.

$$\boxed{\frac{r^2}{\infty} = 0, \quad \frac{r^2}{0} = \infty}$$

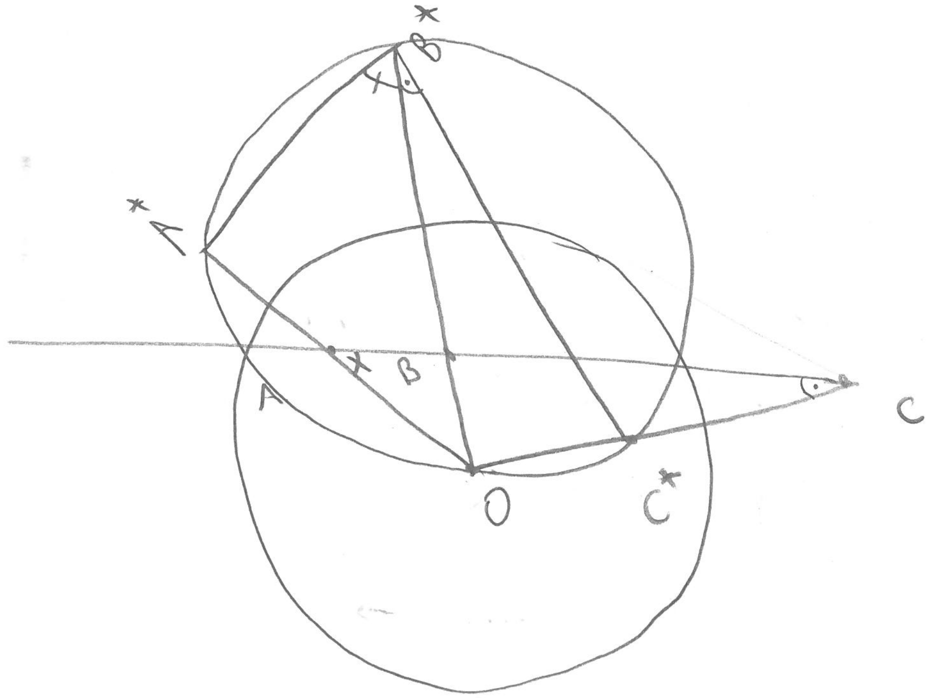
- $A A^* B^* B$ is cyclic.

$$\measuredangle OAB = -\measuredangle OB^*A^*$$



- A line passing through O inverts to itself.

- A line not passing throug O inverts to circle passing through O.

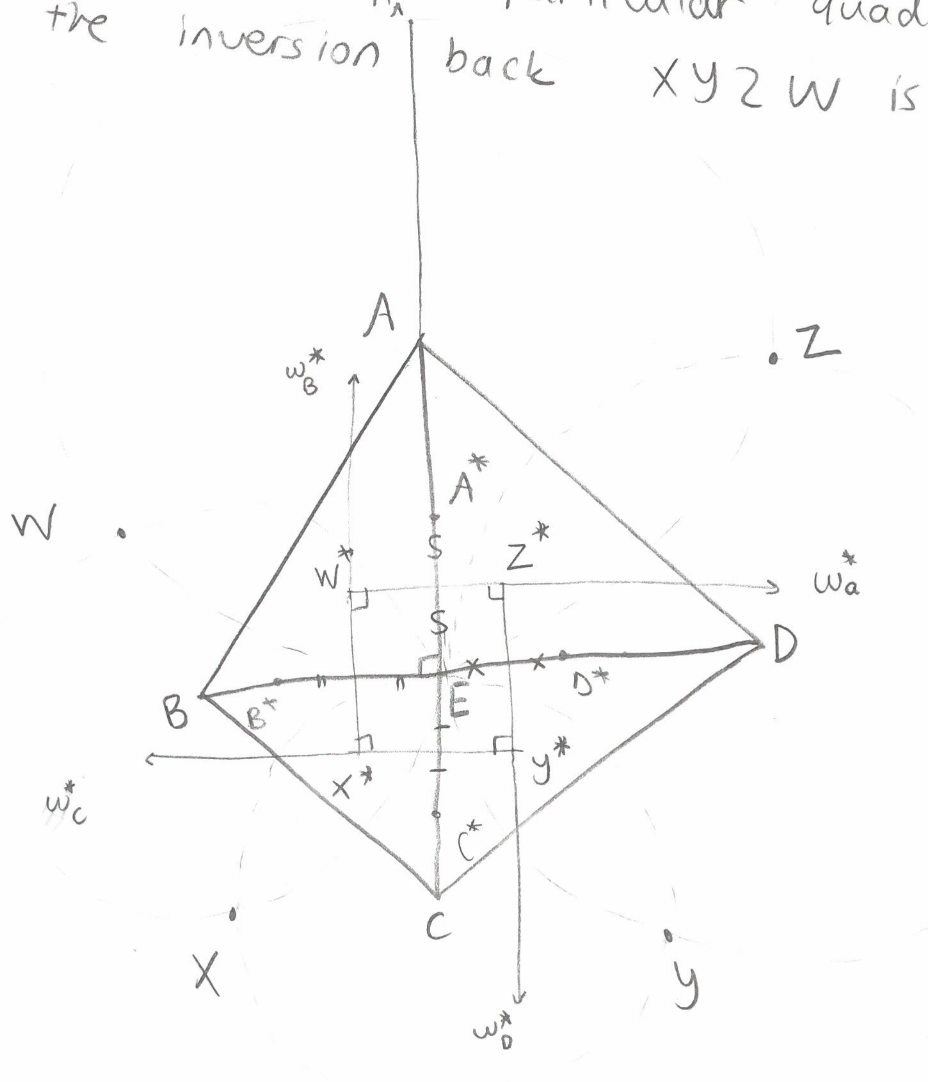


USAMO 1993/2

Let $ABCD$ be a quadrilateral whose diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular and intersect at E. Prove that the reflections of E across $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ are concyclic.

Solution. Observe that $EA = WA = ZA$. Thus construct the circle ω_A with radius AE. Construct $\omega_B, \omega_C, \omega_D$ similarly. Then W is the other intersection of ω_A and ω_B. Similarly X, Y, Z ...

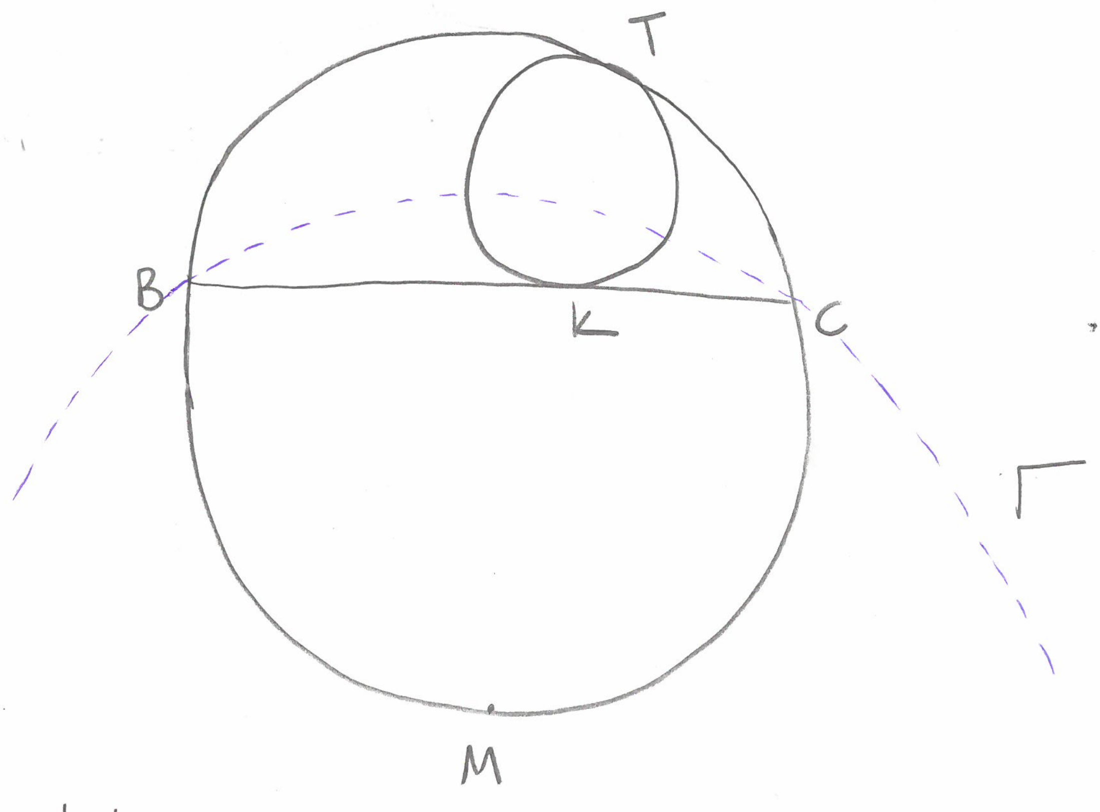
Let's make an inversion centered at E with radius 1. Then $W^* Z^* Y^* X^*$ becomes a rectangle which is in particular quadrilateral. Taking the inversion back $XYZW$ is quadrilateral.



BEAUTIFUL PROOF FOR LEMMA 4.33.

Let $\overline{BC}$ be a chord of a circle Ω. Let ω be a circle tangent to chord $\overline{BC}$ at K and internally tangent to Ω at T. Then ray TK passes through the midpoint M of the arc $\widehat{BC}$ not containing T. Moreover, MC^2 is the power of M with respect to ω.

Proof.



Let's take an inversion with radius $MB = MC$ centered at M. Then Ω turns into a line passing through BC. Because B and C are on Γ so they won't change their place. Now we claim that ω just gets sent to itself.

Because BC and Ω trade places w^* is also a circle tangent to both. Also the centers of w^* and w are collinear with M. This is enough to fore $w^* = w$.

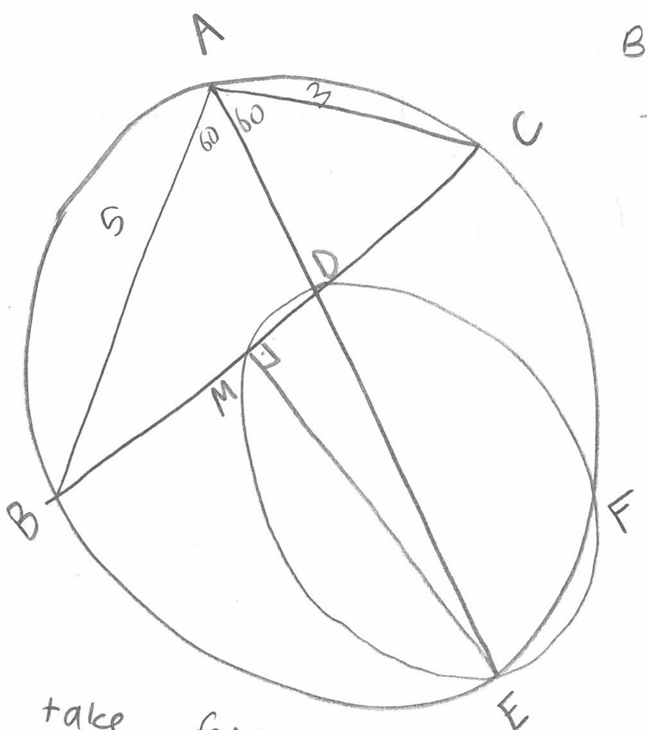
Now $T^* = K$ because k is the point of w^* and BC so, k^* is the tangency point of $(MB^*C^*) = \Omega$ and $(w^*)^* = w$. And T satisfies this property of k^*. Thus $T = k^*$. So T, k, M are collinear and due to Γ $MB^2 = MK \cdot MT$ ■

HOLY INVERSION!

2012 AIME II Problem 15

Triangle ABC is inscribed in circle with $AB = 5$, $BC = 7$ and $AC = 3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$. $(m,n) = 1$. Find $m+n$.

Solution.



Basic calculation resuts:

$AD = \frac{15}{8}$

$DC = \frac{21}{8}$

$DB = \frac{35}{8}$

$DE = \frac{49}{8}$

Observe that

$AD \cdot AE = 5 \cdot 3 = 15$.

Now, if we take force-overlaid inversion about A, D and E trades place so γ stays still. Now M (midpoint of BC) is on γ. And as a resut of inversion $\omega \leftrightarrow BC$ Thus $M^* \in \omega$ and because γ stays still $M^* \in \gamma$ thus $M^* = F$. Now $AM \cdot AM^* = AM \cdot AF = 15$

$AM = \frac{\sqrt{19}}{2} \Rightarrow AF^2 = \frac{900}{19} \Rightarrow m+n = 919$.

The Inversion Distance formula

Let A and B be points other than O and consider an inversion about O with radius r.

Then,
$$A^*B^* = \frac{r^2}{OA.OB} \cdot AB$$

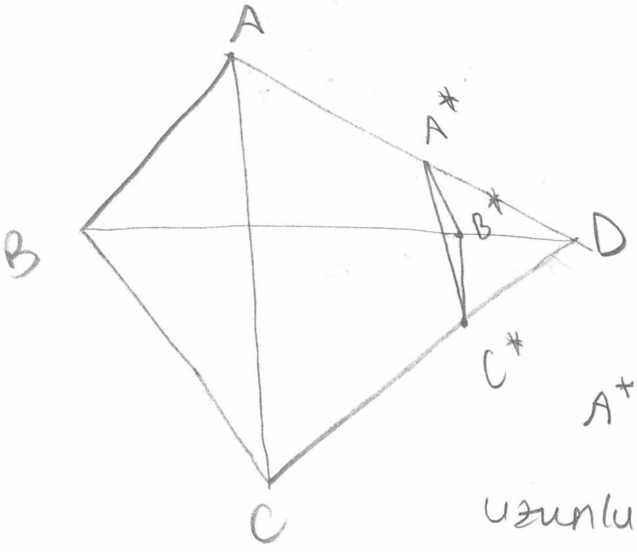
Equivalently,
$$AB = \frac{r^2}{OA^* \cdot OB^*} \cdot A^*B^*$$

FEVKALADE PTOLEMY KANITI.

A, B, C, D 3'ü doğrusal olmayan noktalar olsun.

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$

Kanıt.



D'den inversion yapalım. r yarıçaplı. ABCD çembersel değilse $A^*B^*C^*$ üçgen oluşturur.

$$A^*B^* + B^*C^* > A^*C^*$$

uzunluk formulunden yerine yazarsak gelir. Eşitlik durumu ABCD çemberselken $A^*B^* + B^*C^* = A^*C^*$ olur.

Chinese Olympiad 2006.

Açılardan rahat gelen ama evirtimden çok estetik olan soru.

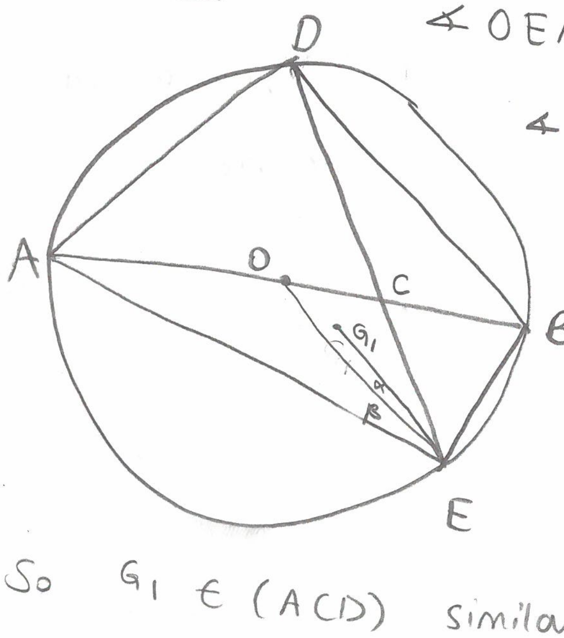
Let ADBE be a quadrilateral inscribed in a circle with diameter $\overline{AB}$ whose diagonals meet at C. Let γ be the circumcircle of $\triangle BOD$, where O is the midpoint of $\overline{AB}$. Let F be on γ such that $\overline{OF}$ is a diameter of γ and let ray FC meet γ again at G. Prove that A, O, G, E are concyclic.

My Solution

Lemma. $G_1 = (BOD) \cap (AOE)$ olsun.

$G_1 \in (CAD)$ ve $G_1 \in (CBE)$

Kanıt.



$\measuredangle OEA = \beta$, $\measuredangle G_1EO = \alpha$ olsun.

$\measuredangle G_1OB = \measuredangle G_1DB = \alpha + \beta$.

$\measuredangle OEA = \measuredangle EAO = \measuredangle EDB = \beta$

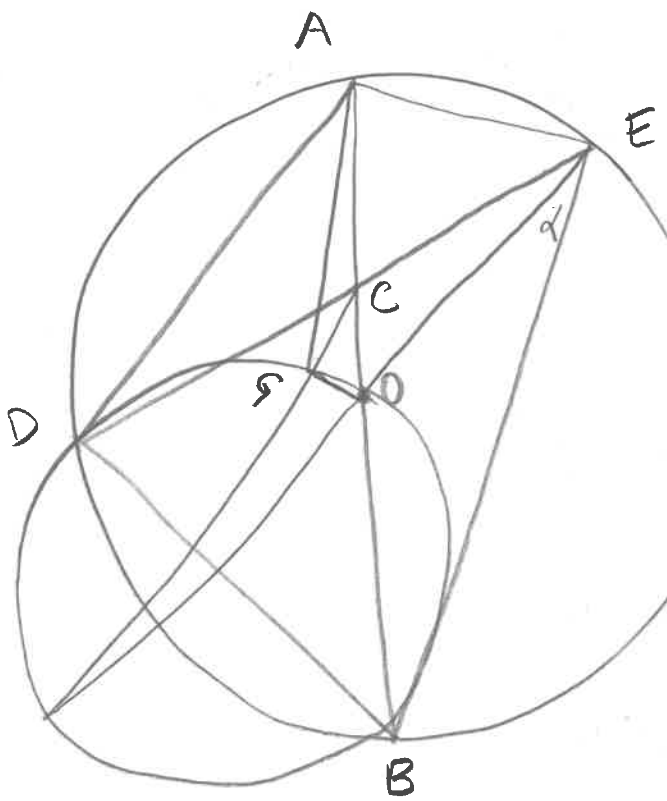
So $\measuredangle G_1DE = \measuredangle G_1DC = \alpha$

Also

$\measuredangle G_1EO = \measuredangle G_1AO = \alpha$

$\measuredangle G_1AC = \measuredangle G_1DC = \alpha$

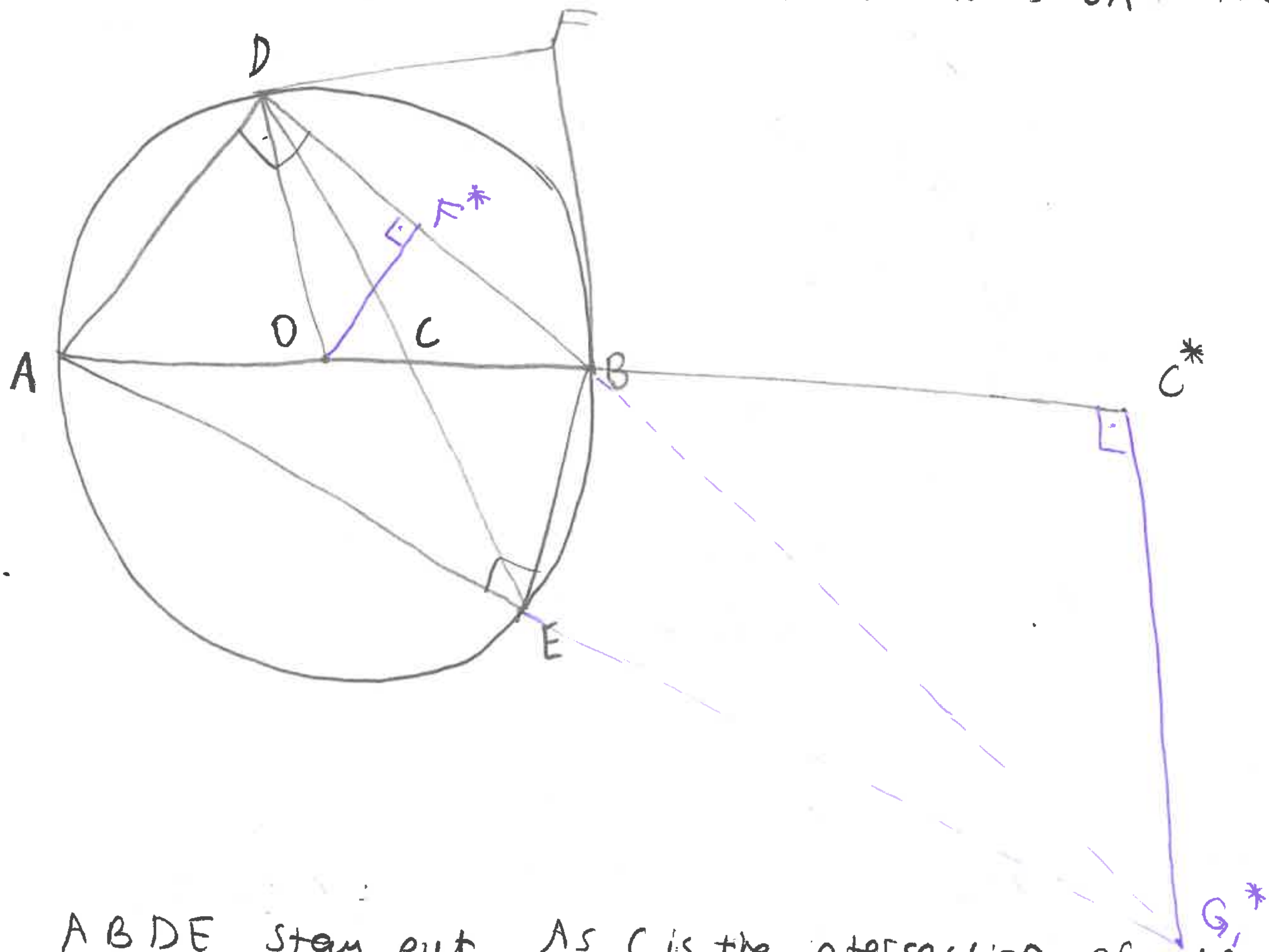
So $G_1 \in (ACD)$ similarly $G_1 \in (CBE)$ $\square$



$\angle OEB = \alpha$ olsun. $G = (OBD) \cap (AOE)$ olsun. Lemmadan $G \in (ACD)$ ve $G \in (EBC)$
$\angle OEB = \angle OBE = \angle ABE = \angle ADE = \angle AGC = \alpha$.
$\angle AEO = 90 - \alpha$ old. $\angle AGO = 90 + \alpha$. $\angle AGC = \alpha$ old.
$\angle CGO = 90$. GC'nin (BOD)'yi kestiği yere F' dersek $\angle OGF' = 90°$ olacağından $OF' = OF$. $\square$

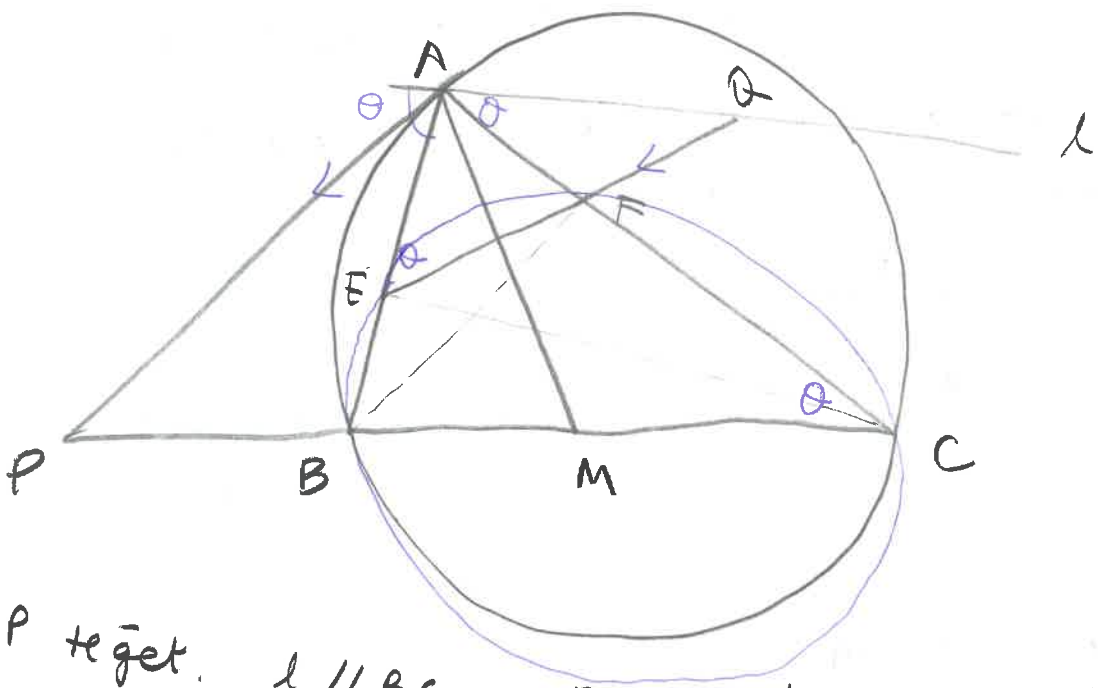
Solution . From Inversion by Evan Chen

$G_1 = (BOD) \cap (AOE)$ olarak tanımlayalım.
G_1, C, F, 'nin doğrusal olduğunu gösterelim.
Take inversion about O with radius OA. Then



$ABDE$ stay put. As C is the intersection of AB and DE, C^* is on AB and C^*DOE is cyclic. In (BOD) F^* is the midpoint of DB because $\angle ODF = \angle OBF = 90°$. So FD and FB are tangent to $(ADBE)$. G is the intersection of (FDB) and (AOE) So G^* is the intersection of $(FDB)^* = DB$ and $(AOE)^* = AE$. Now G_1, C, F are collinear if and only if O, G_1^*, C^*, F^* are concyclic. As $G_1^*D \perp AD$ and $BE \perp AG_1^*$ and O is the midpoint of AB; $(DOEC^*)$ is the nine-point circle of $\triangle ABG_1^*$. Thus $AC^* \perp G_1^*C^*$ and that gives $OF^*C^*G_1^*$ is cyclic. $\square$

ÇOK GÜZEL ÇÖZÜM



AP teğet. $\ell \parallel BC$. E ve F diklik ayakları. M orta nokta. $AM \perp PQ$ old. gösterin

Çözüm. Şekil üstünde. M merkezli BM yarıçaplı çemberi çizelim. E ve F'den de geçer $BM = r$ dersek

$$AQ^2 = QF \cdot QE = QM^2 - r^2$$

$$AP^2 = PM^2 - r^2$$

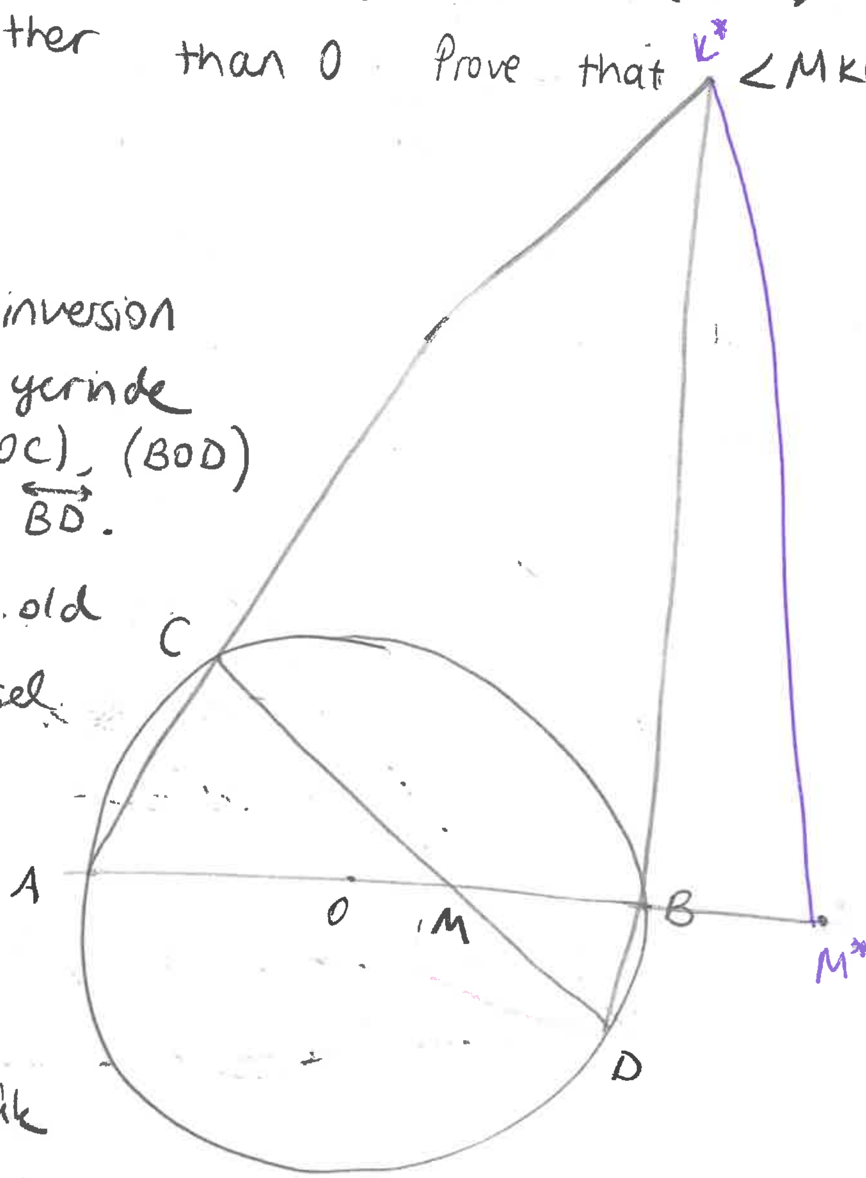
$$AP^2 + QM^2 = PM^2 - r^2 + AQ^2 + r^2 = AQ^2 + PM^2$$

$$AP^2 + QM^2 = AQ^2 + PM^2 \Rightarrow AM \perp PQ \quad \square$$

IRAN OLYMPIAD 1996

BEN HARİKAYIM YAA ♡

Consider a cemicircle with diameter $\overline{AB}$. A line intersects line AB at M and the semicircle at C and D such that $MC > MD$ and $MB < MA$. Suppose (AOC) and (BOD) meet at point K other than O. Prove that $\angle MKO = 90°$



Solution

O merkezli AO yarıçaplı inversion sonucu ABCD yerinde kalır. $K \in (AOC), (BOD)$ old $K^* \in \overleftrightarrow{AC}, \overleftrightarrow{BD}$.

C, M, D doğrusal old

O, C, M^*, D çembersel

$\measuredangle MKO = 90$

$\Longleftrightarrow$

$\angle OM^*K^* = 90$

$\triangle AK^*B$'de

AD, BC yükseklik

O AB'nin orta noktası

o zaman (COD) 9-nokta çemberi. $M^* \in (COD)$ ve $M^* \in AB$ old. K^*M^* diğer yükseklik olmak zorundadır.

$K^*M^* \perp AB \Rightarrow \angle OM^*K^* = 90$

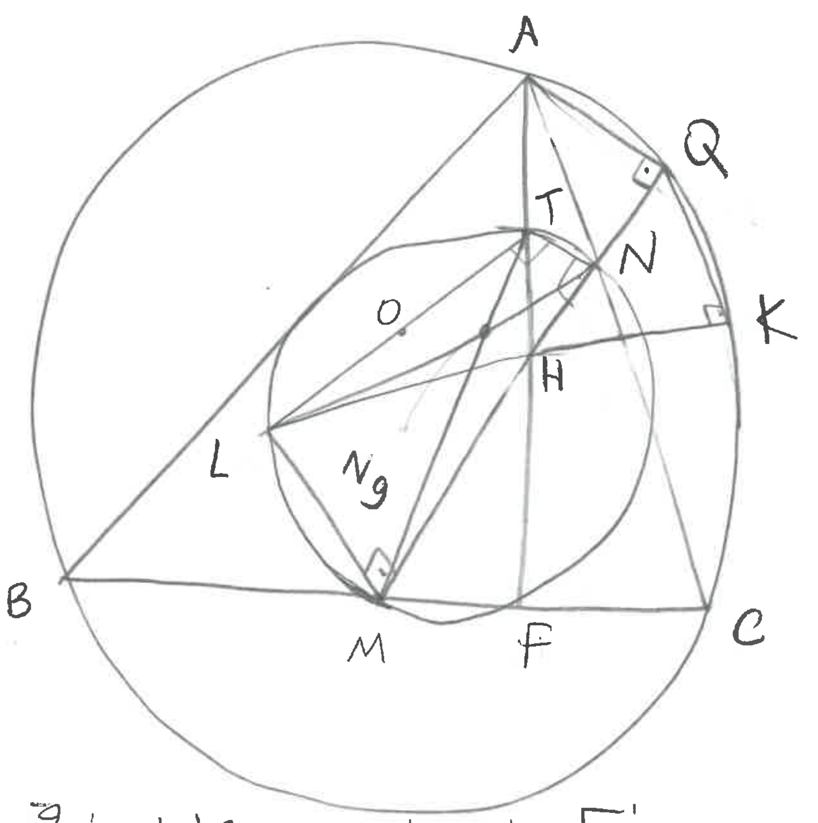
$\square$

INVERSION'DAN ACAYİP BASİTLEŞEN 2015 IMO SORUSU BAK DAHA 5 SENE ÖNCESİ

Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter and F the foot of the altitude from A. Let M be the midpoint of $\overline{BC}$. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$.

Assume that the circumcircles of triangles KQH and FKM are tangent to each other.

Solution



H merkezli 9-nokta çemberini Γ'ye geçiren negatif inversion sonucu $L = K^*$ $M = Q^*$ $F = A^*$

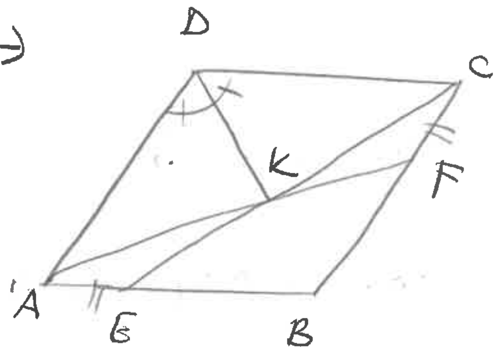
$\underbrace{(K^* F^* M^*)}_{(LAQ)}$'nin $\underbrace{Q^* K^*}_{ML}$'ye teğet olduğunu göstercem.

149.

MT is a diameter. But also $\angle QML = \angle NML = 9$
So $LMNT$ is a rectangle and O is on LT because $HN_g = N_gO$ and $N_g = TM \cap NL$.

$LM \parallel TN \parallel AQ$ so! $LO \perp AQ$ thus AQL is isosceles triangle and the circumcenter O_1 of (AQL) lies on the line LO and $ML \perp LO$ thus ML is tangent to (AQL) $\square$

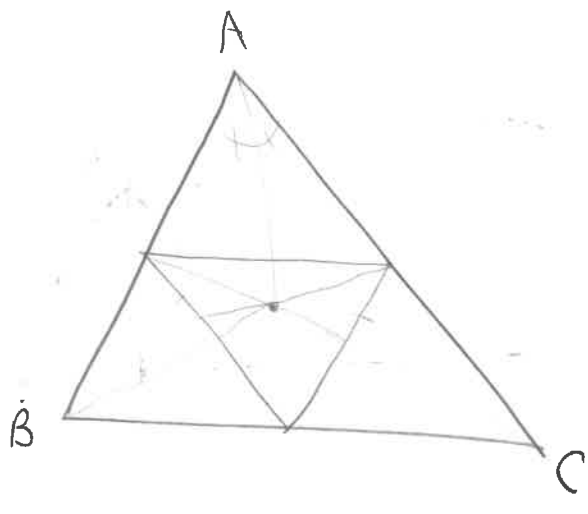
HAP BILGILER

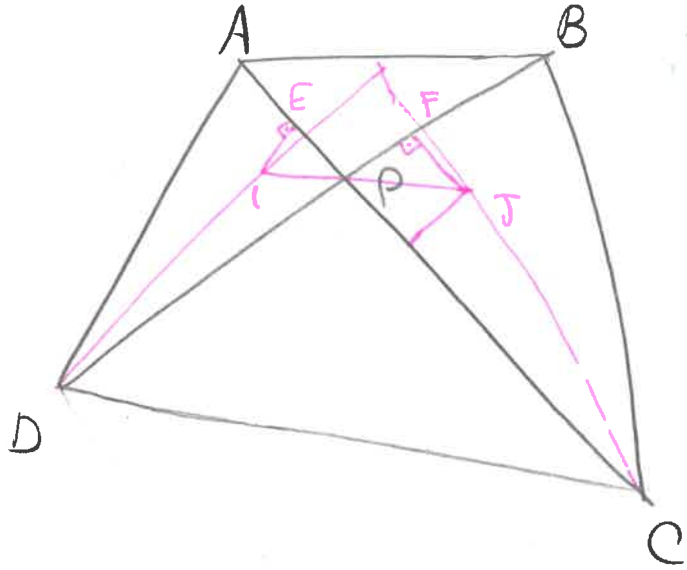
→ 

FC = AE ⇒ DK açıortay

(ABCD paralelkenar)

→ Medyan üçgenin Nagel noktası ABC üçgeninin iç teğet çem. merkezidir.





$AC+AD=BC+BD$ ise $\angle ADB$, $\angle ACB$, $\angle APB$ 'nin iç açıortaylarının çakıştığını ispatlayın

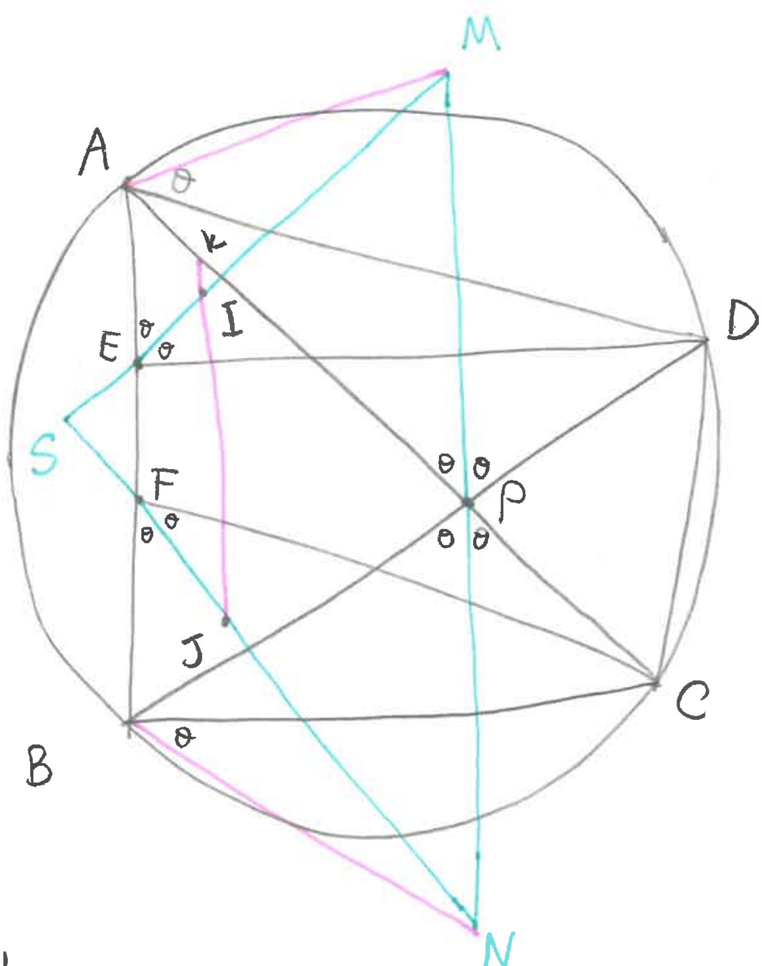
I, J $\triangle ADP$ ve $\triangle BCP$ 'nin iç teğ. çem mer. olsun
$AC+AD = BC+BD$ $\Rightarrow$ $AE = BF$ geliyor. işlemler sonucu. $\triangle DAP$'nin dış teğ çem. AP'yi K'de kessin
$\triangle BCP$'nin " " " BP'yi L'de " .
$PK = AE = BF = PL$ old. dış teğ çemberler kesişir □

PASLANMIŞIM SORUSU
(China MO 2018)

ABCD is a cyclic quadrilateral whose diagonals intersect at P. The circumcircle of $\triangle APD$ meets segment AB at points A and E. The circumcircle of $\triangle BPC$ meets segment AB at points B and F. Let I and J be incenters of $\triangle ADE$ and $\triangle BCF$, respectively. Segments IJ and AC meet at K. Prove that the points A, I, K, E are cyclic.

(MAEPD)
ve
(FBNCP)

MP, NP
acıortaylar olur.



$\triangle PAM \sim \triangle PBN$

ve

$\triangle PAB \sim \triangle SMN$

$NJ = NB$

$MI = MA$

$$\frac{MI}{NJ} = \frac{MA}{NB} = \frac{PA}{PB} = \frac{SM}{SN} \Rightarrow JI \parallel MN$$

Bu da sonucu bitirir.