Expanding CertiGraph: Dijkstra, Prim, and Kruskal

Anshuman Mohan, Wei Xiang Leow, Aquinas Hobor

NUS Programming Language and Verification Seminar
December 15, 2020

Verified Software Toolchain

Certifying Graph-Manipulating C Programs via Localizations within Data Structures

SHENGYI WANG, National University of Singapore, Singapore
QINXIANG CAO, Shanghai Jiao Tong University, China
ANSHUMAN MOHAN, National University of Singapore, Singapore
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VST + CompCert + CertiGraph

A Coq library to verify executable code
against realistic specifications
expressed with mathematical graphs

This Work

Verified
Software
Toolchain

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In doing so, we:

- Test existing features [Dijk labels edges]
- Expand into undirectedness [Prim, Krus]
- Make nontrivial calls to verified methods [Krus calls UF]

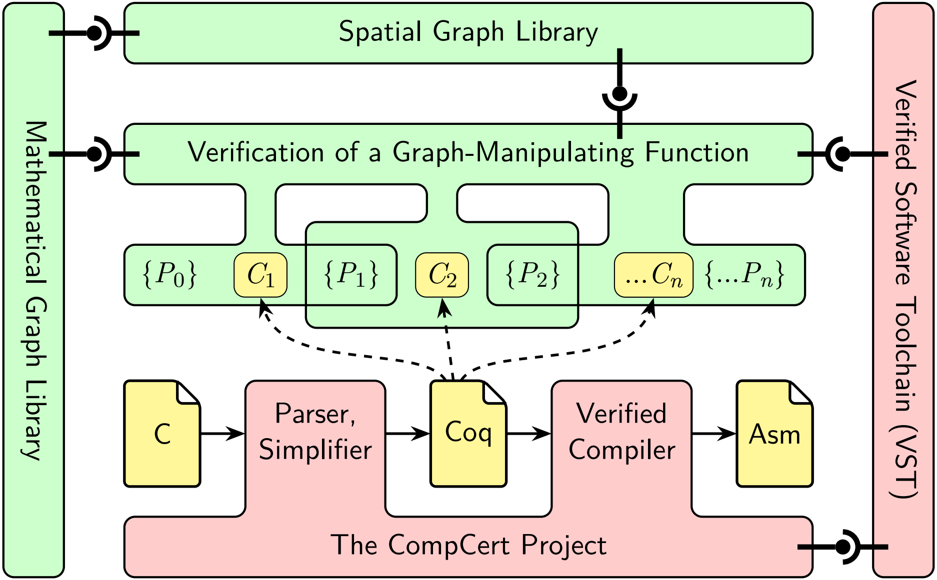
Challenges

Using CompCert C, which
  is executable and realistic
  but has real-world complications

Aiming for full functional correctness

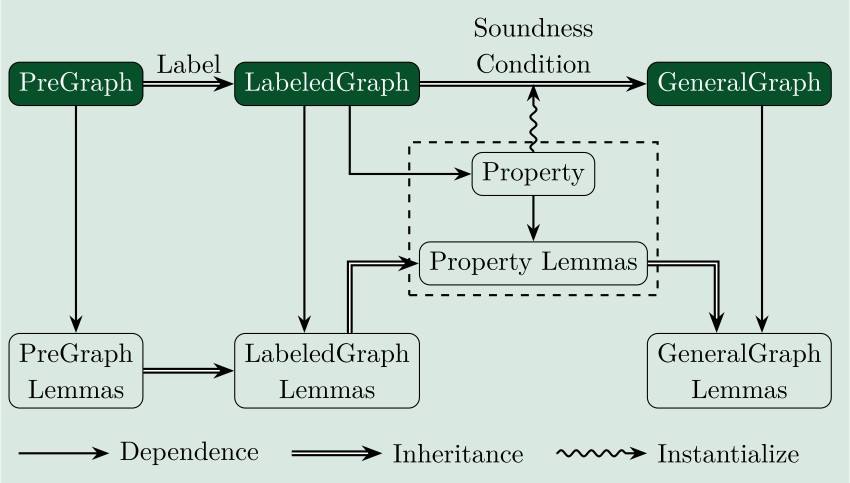
Maintaining modularity and reuse

Workflow



Spatial Graph Library
Mathematical Graph Library
Verification of a Graph-Manipulating Function
Verified Software Toolchain (VST)
$\{P_0\}$
C_1
$\{P_1\}$
C_2
$\{P_2\}$
$\ldots C_n$
$\{\ldots P_n\}$
C
Parser, Simplifier
Coq
Verified Compiler
Asm
The CompCert Project

Math Graph Architecture



Instantiating DijkGraph

A PreGraph is a hextuple (`VType`, `EType`, `vvalid`, `evalid`, `src`, `dst`)

$$\textbf{Dijk\_PG}(\gamma) \stackrel{\text{def}}{=} \begin{array}{l} \texttt{VType := Z} \\ \texttt{EType := VType * VType} \\ \texttt{src := fst} \\ \texttt{dst := snd} \\ \forall v.\ \texttt{vvalid}(\gamma, v) \Leftrightarrow 0 \leqslant v < \texttt{size} \\ \forall s, d.\ \texttt{evalid}(\gamma, (s, d)) \Leftrightarrow \texttt{vvalid}(\gamma, s) \wedge \texttt{vvalid}(\gamma, d) \end{array}$$

A LabeledGraph is a quadruple (PreGraph, `VL`, `EL`, `GL`)

$$\mathbf{Dijk\_LG(}\gamma\mathbf{)} \stackrel{\text{def}}{=} \text{Dijk\_PG as shown}$$

```
VL := list EL
EL := Z
GL := unit
```

A GeneralGraph adds arbitrary soundness conditions

$$
\begin{aligned}
\textbf{DijkGraph}(\gamma) \stackrel{\text{def}}{=}\ & \text{Dijk\_LG as shown, and} \\
& \mathit{FiniteGraph}(\gamma) \land \\
& \forall i, j.\ \texttt{vvalid}(\gamma, i) \land \texttt{vvalid}(\gamma, j) \Rightarrow \\
& \qquad i = j \Rightarrow \texttt{elabel}(\gamma, (i, j)) = 0 \land \\
& \qquad i \neq j \Rightarrow 0 \leqslant \texttt{elabel}(\gamma, (i, j)) \leqslant \lfloor \texttt{MAX/size} \rfloor \land \\
& \ldots
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& \ldots
\end{aligned}
$$

Representing DijkGraph in Memory

$$\mathsf{list\_rep}(\gamma, i) \stackrel{\text{def}}{=} \texttt{data\_at array graph2mat}(\gamma)[i] \ \texttt{list\_addr}(\gamma, i)$$

$$\mathsf{graph\_rep}(\gamma) \stackrel{\text{def}}{=} \mathop{\circledast}_{\texttt{vvalid}(\gamma, v)} v \mapsto \mathsf{list\_rep}(\gamma, v)$$

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Relies on restrictions placed at the Math level

Code and Specification

```
#define IFTY INT_MAX - INT_MAX/size
void dijkstra (int graph[size][size], int src,
                         int *dist, int *prev) {
```

$\{\mathsf{DijkGraph}(\gamma)\}$

Code and Specification

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#define IFTY INT_MAX - INT_MAX/size
void dijkstra (int graph[size][size], int src,
                         int *dist, int *prev) {
```

$\{\mathsf{DijkGraph}(\gamma)\}$

```
 int pq[size];
 int i, j, u, cost;
 for (i = 0; i < size; i++)
 {  dist[i] = inf; prev[i] = inf; pq[i] = inf;  }
 dist[src] = 0; pq[src] = 0; prev[src] = src;
```

Code and Specification

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#define IFTY INT_MAX - INT_MAX/size
void dijkstra (int graph[size][size], int src,
                        int *dist, int *prev) {
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$\{\mathsf{DijkGraph}(\gamma)\}$

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```

$\{\mathsf{DijkGraph}(\gamma) \wedge \mathit{dijk\_correct}(\gamma, \mathit{src}, \mathit{prev}, \mathit{dist}, \mathit{priq})\}$

```
 // big while loop
```

Code and Specification

$\{\mathsf{DijkGraph}(\gamma) \wedge \mathit{dijk\_correct}(\gamma, src, prev, dist, priq)\}$

Code and Specification

$\{\mathsf{DijkGraph}(\gamma) \wedge dijk\_correct(\gamma, src, prev, dist, priq)\}$

```
while (!pq_emp(pq)) {
 u = popMin(pq);
 for (i = 0; i < size; i++) {
  cost = graph[u][i];
   if (cost < inf) {
    if (dist[i] > dist[u] + cost) {
     dist[i] = dist[u] + cost; prev[i] = u; pq[i] = dist[i];
}}}}
```

Code and Specification

$\{\mathsf{DijkGraph}(\gamma) \land \mathit{dijk\_correct}(\gamma, \mathit{src}, \mathit{prev}, \mathit{dist}, \mathit{priq})\}$

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```

$$\left\{\begin{array}{l}\mathsf{DijkGraph}(\gamma) \land \forall \mathit{dst} \in \mathit{priq}.\ \mathit{priq}[\mathit{dst}] = \texttt{inf} \land \\ \ \mathit{dijk\_correct}(\gamma, \mathit{src}, \mathit{prev}, \mathit{dist}, \mathit{priq})\end{array}\right\}$$

```
return;
}
```

$$
\begin{aligned}
& \mathit{dijk\_correct}(\gamma, \mathit{src}, \mathit{prev}, \mathit{dist}, \mathit{priq}) \stackrel{\text{def}}{=} \\
& \quad \forall \mathit{dst}.\ \mathit{dst} \in \mathit{popped}(\mathit{priq}) \Rightarrow \\
& \qquad\qquad\qquad \exists \mathit{path}.\ \mathit{path\_correct}(\gamma, \mathit{prev}, \mathit{dist}, \mathit{path}) \wedge \\
& \qquad\qquad\qquad \mathit{path\_glob\_optimal}(\gamma, \mathit{dist}, \mathit{path}) \wedge \\
& \qquad\qquad\qquad \mathit{path\_entirely\_in\_popped}(\gamma, \mathit{prev}, \mathit{path}) \wedge \\
& \qquad\quad \mathit{priq}[\mathit{dst}] < \texttt{inf} \Rightarrow \\
& \qquad\qquad\qquad \texttt{let}\ m\ \texttt{:=}\ \mathit{prev}[\mathit{dst}]\ \texttt{in}\ m \in \mathit{popped}(\mathit{priq}) \wedge \\
& \qquad\qquad\qquad \forall m' \in \mathit{popped}(\mathit{priq}).\ \mathit{cost}(\mathit{path2m}+ :: (m, \mathit{dst})) \leqslant \\
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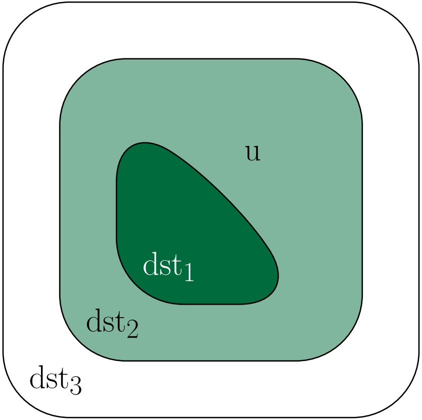
Key Transformation: Growing the Subgraph



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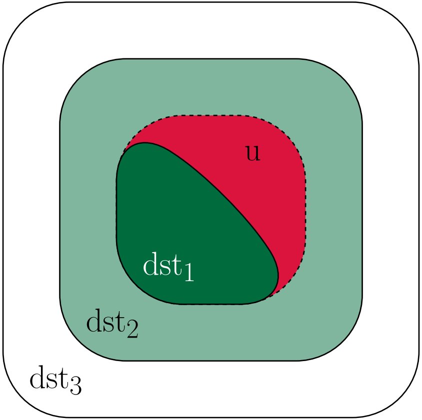

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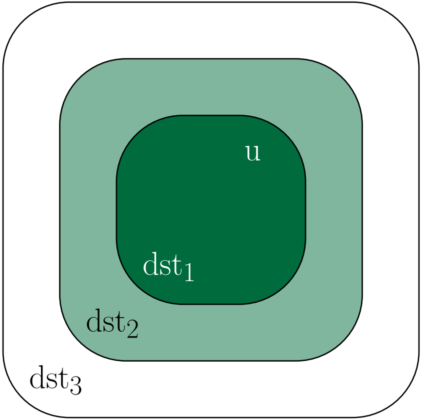

Overflow Strikes Again

The longest optimal path has `size-1` links

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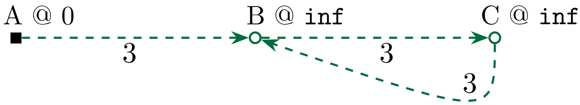

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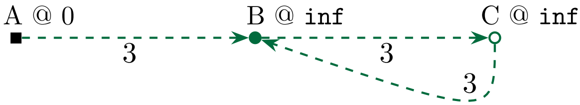

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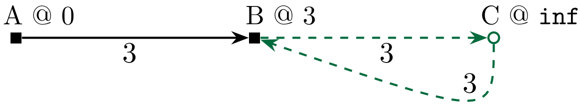

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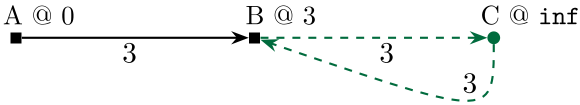

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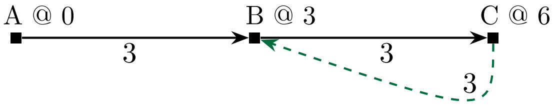

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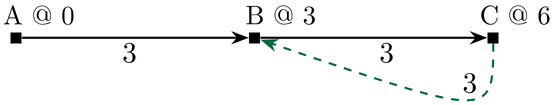

if $3 > 9$ then relax C $\rightsquigarrow$ B

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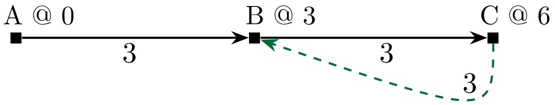

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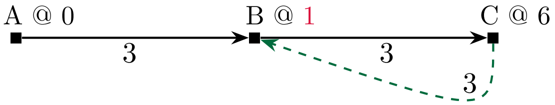

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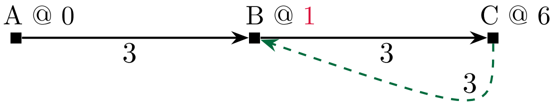

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One solution: Conservatively set upper bound to $\lfloor$`MAX/size`$\rfloor$

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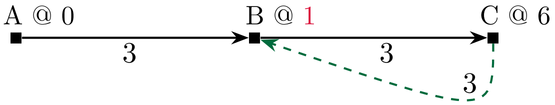

if $3 > 1$ then relax C $\rightsquigarrow$ B

One solution: Conservatively set upper bound to $\lfloor$`MAX/size`$\rfloor$

Max path cost is then $\lfloor$`MAX/size`$\rfloor$ `* (size-1) = MAX -` $\lfloor$`MAX/size`$\rfloor$

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There are other ways to fix this!

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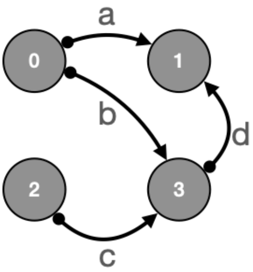
Sadly, intuition supports `inf = MAX`

Extending to get undirectedness

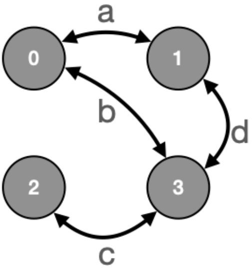
Consider the AdjMat representation of a directed graph:



	0	1	2	3
0	0	a	inf	b
1	inf	0	inf	inf
2	inf	inf	0	c
3	inf	d	inf	0

Extending to get undirectedness

Versus that of an undirected graph:



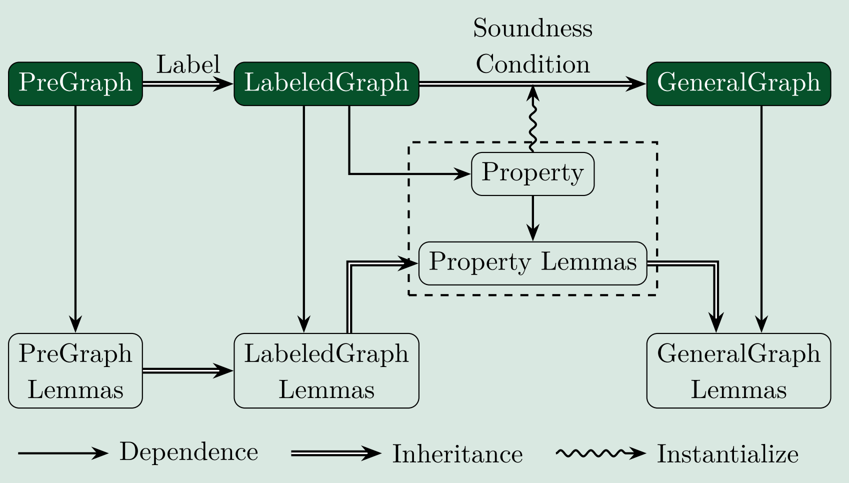
	0	1	2	3
0	0	a	inf	b
1	a	0	inf	d
2	inf	inf	0	c
3	b	d	c	0

Extending to get undirectedness

Thanks to our model, this is remarkably easy to model:

```
Class SoundUAdjMat (g: UAdjMatLG) := {
  sadjmat: @SoundAdjMat size inf g;
  undirec: forall e, evalid g e -> src g e <= dst g e;
}.
```

Recall: Math Graph Architecture



A note on modularity