Problem 1. A random variable X has the mean $\mu = 100$ and standard deviation $\sigma = 30$.

(a) Compute the probabilities for the events $\{X > 130\}$ and $\{X > 160\}$ and list them for the following distributions: uniform, gamma, normal, lognormal.

(b) The probability of which event is more sensitive to the choice of the distribution? Try to explain why this is the case.

(a)

- Uniform Distribution

The mean of the distribution is:

$E(X) = \frac{1}{2}(b + a) = 100$

The variance is:

$V(X) = \sigma^2(X) = \frac{1}{12}(b - a)^2 = 30^2$

We get that:

$a = 100 - 30\sqrt{3}$

$b = 100 + 30\sqrt{3}$

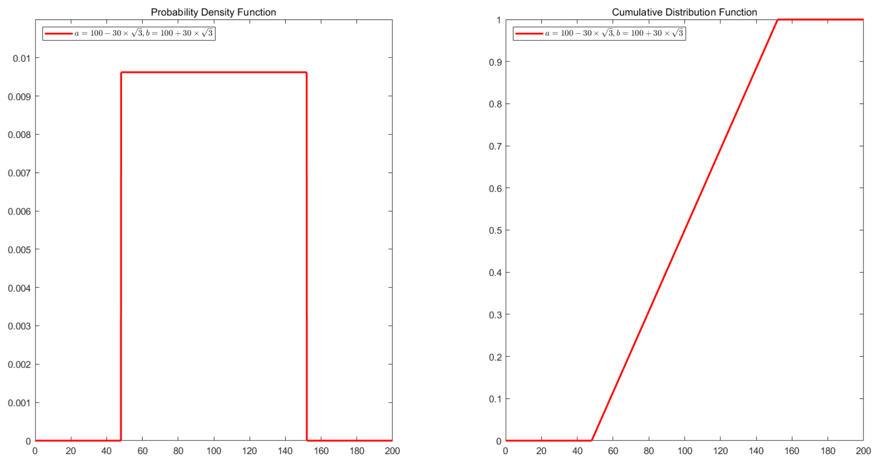
MATLAB code:

```
clc;
clear;
pd1 = makedist('Uniform','lower',100-30*sqrt(3),'upper',100+30*sqrt(3));
x = 0:.1:200;
pdf1 = pdf(pd1,x);
cdf1 = cdf(pd1,x);

figure;
subplot(1,2,1);
plot(x,pdf1,'r','LineWidth',2);
legend({'$a = 100-30\times \sqrt{3}, b = 100+30\times
\sqrt{3}$'},'Location','northwest','Interpreter','LaTex');
title('Probability Density Function');
ylim([0 0.011])

subplot(1,2,2);
plot(x,cdf1,'r','LineWidth',2);
legend({'$a = 100-30\times \sqrt{3}, b = 100+30\times
\sqrt{3}$'},'Location','northwest','Interpreter','LaTex');
title('Cumulative Distribution Function');

1-cdf(pd1,130)
1-cdf(pd1,160)
```



$P(X > 130) = 0.2113$

$P(X > 160) = 0$

- Gamma Distribution

Shape parameter a and scale parameter b

The mean of the distribution is:

$E(X) = ab = 100$

The variance is:

$V(X) = \sigma^2(X) = ab^2 = 30^2$

We get that:

$a = \frac{100}{9}$

$b = 9$

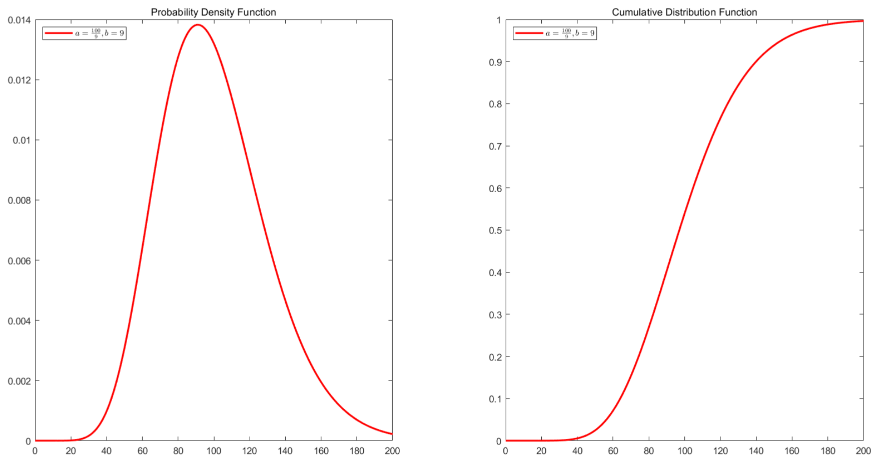
MATLAB code:

```
clc;
clear;
x = 0:.1:200;
pdf1 = gampdf(x,100/9,9);
cdf1 = gamcdf(x,100/9,9);

figure;
subplot(1,2,1);
plot(x,pdf1,'r','LineWidth',2);
legend('$a = \frac{100}{9}, b = 9$','Location','northwest','Interpreter','LaTex');
title('Probability Density Function');

subplot(1,2,2);
plot(x,cdf1,'r','LineWidth',2);
legend('$a = \frac{100}{9}, b = 9$','Location','northwest','Interpreter','LaTex');
title('Cumulative Distribution Function');

1-gamcdf(130,100/9,9)
1-gamcdf(160,100/9,9)
```



$P(X > 130) = 0.1557$

$P(X > 160) = 0.0363$

- Normal Distribution

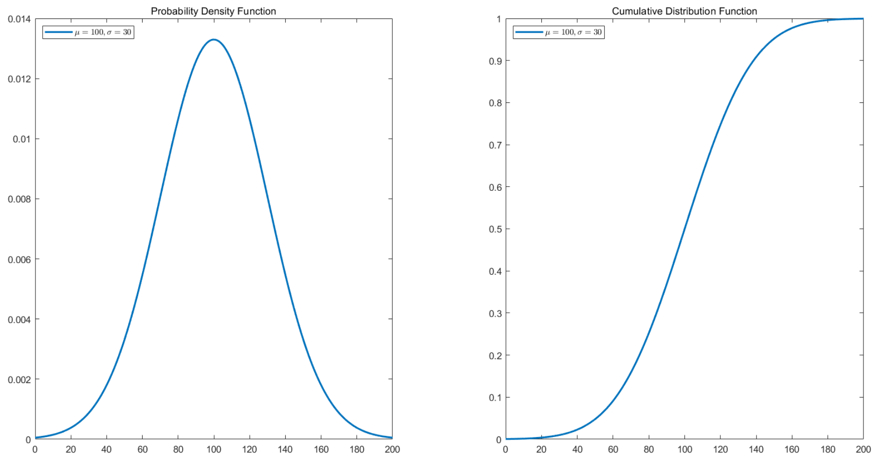
MATLAB code:

```
clc;
clear;
mu = 100;
sigma = 30;
pd = makedist('Normal','mu',mu,'sigma',sigma);
x = 0:.1:200;
pdf_normal = pdf(pd,x);
cdf_normal = cdf(pd,x);

figure;
subplot(1,2,1);
plot(x,pdf_normal,'LineWidth',2);
legend('$\mu = 100, \sigma = 30$','Location','northwest','Interpreter','LaTex');
title('Probability Density Function');

subplot(1,2,2);
plot(x,cdf_normal,'LineWidth',2);
legend('$\mu = 100, \sigma = 30$','Location','northwest','Interpreter','LaTex');
title('Cumulative Distribution Function');

1-cdf(pd,130)
1-cdf(pd,160)
```



$P(X > 130) = 0.1587$

$P(X > 160) = 0.0228$

- Log-normal Distribution

Given $\mu_X = 100, \sigma_X^2 = 30^2$

In order to produce a distribution with desired mean μ_X and variance σ_X^2, one uses

$$\mu = ln(\frac{\mu_X^2}{\sqrt{\mu_X^2+\sigma_X^2}}) = ln(\frac{100^2}{\sqrt{100^2+30^2}}) \approx 4.5621$$

$$\sigma = \sqrt{ln(1+\frac{\sigma_X^2}{\mu_X^2})} = \sqrt{ln(1+\frac{30^2}{100^2})} \approx 0.2936$$

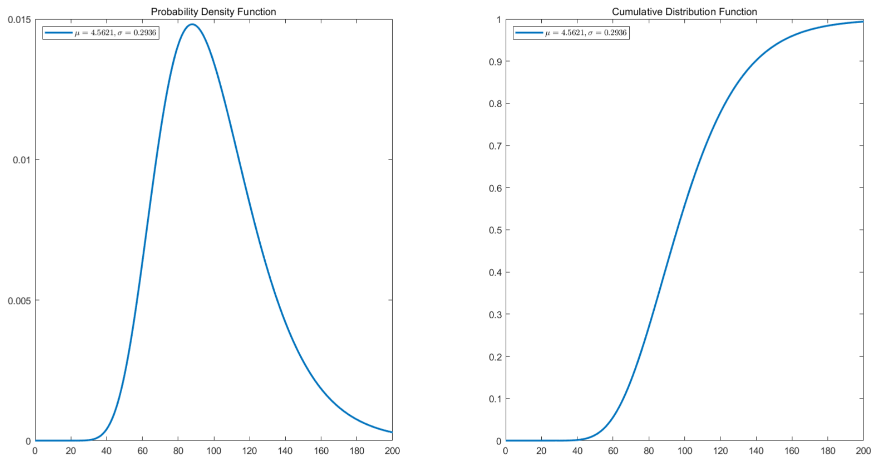
MATLAB code:

```
clc;
clear;
mu = log(100)-1/2*log(1+9/100);
sigma = sqrt(log(1+9/100));
pd = makedist('Lognormal','mu',mu,'sigma',sigma);
x = 0:.1:200;
pdf_lognormal = pdf(pd,x);
cdf_lognormal = cdf(pd,x);

figure;
subplot(1,2,1);
plot(x,pdf_lognormal,'LineWidth',2);
legend('$\mu = 4.5621, \sigma = 0.2936$','Location','northwest','Interpreter','LaTex');
title('Probability Density Function');

subplot(1,2,2);
plot(x,cdf_lognormal,'LineWidth',2);
legend('$\mu = 4.5621, \sigma = 0.2936$','Location','northwest','Interpreter','LaTex');
title('Cumulative Distribution Function');

1-cdf(pd,130)
1-cdf(pd,160)
```



$P(X > 130) = 0.1491$

$P(X > 160) = 0.0402$

(b)

Table 1: Results of (a)

	$P(X > 130)$	$P(X > 160)$
Uniform Distribution	0.2113	0
Gamma Distribution	0.1557	0.0363
Normal Distribution	0.1587	0.0228
Log-normal Distribution	0.1491	0.0402

We compute the variance of each event through NumPy:

```
import numpy as np
arr1 = [0.2113,0.1557,0.1587,0.1491]
arr2 = [0,0.0363,0.0228,0.0402]
print('The variance of list1:',np.var(arr1))
print('The variance of list2:',np.var(arr2))
```

```
The variance of list1: 0.0006169799999999995
The variance of list2: 0.000247111875
```

As we know, the larger the variance is, the more sensitive the event is to the choice of the distribution.

So we conclude that the event {X > 130} is more sensitive to the choice of the distribution.

The reason for this is that for most distributions, when X is around 130, the absolute value of the rate of change(or the first order derivative) is much larger that which is around 160. That is why the event {X > 130} behaves more sensitively than the event {X > 160}.

Problem 2. The seismic fragility of a building, denoted $g(x)$, is defined as the conditional probability of failure of the building for a given peak ground acceleration x, i.e.,

$$g(x) = P(\text{Failure} \mid \text{peak ground acceleration} = x)$$

For a particular class of buildings, the fragility function is given by

$$g(x) = \begin{cases} 0 & x < 0.1 \\ 2.5(x - 0.1) & 0.1 < x < 0.5 \\ 1 & 0.5 < x \end{cases}$$

where x is measured in units of gravity acceleration. Suppose the peak ground acceleration of an earthquake has the exponential distribution with mean 0.05 units of gravity acceleration. determine:

(a) The probability of failure of the building during an earthquake.

(b) If the building is known to have failed, what is the probability density function of the peak ground acceleration of the earthquake that caused the failure.

Note: you may use the integral $\int x \exp(-ax)\, dx = -\left(\frac{1+ax}{a^2}\right) \exp(-ax)$.

(a)

MATLAB code:

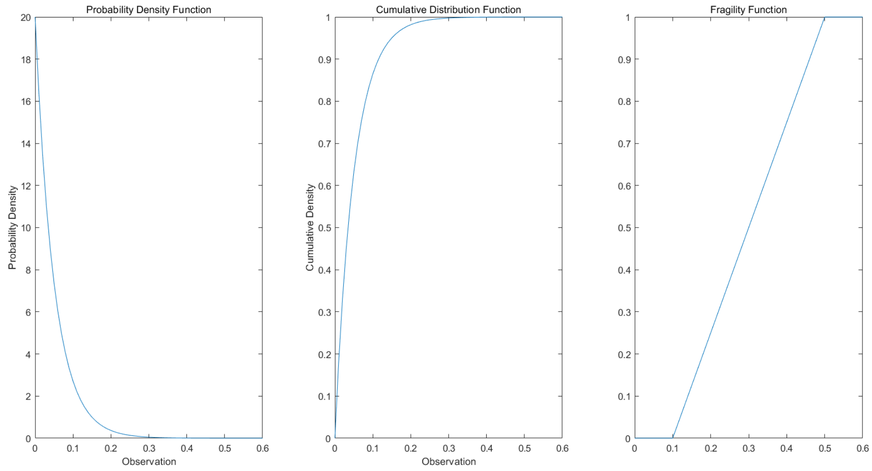
```
clc;
clear;
x = 0:0.01:1;
y1 = exppdf(x,0.05);
y2 = expcdf(x,0.05);

figure;
subplot(1,3,1);
plot(x,y1);
xlabel('Observation')
ylabel('Probability Density')
title('Probability Density Function');
xlim([0 0.6])

subplot(1,3,2);
plot(x,y2);
xlabel('Observation')
ylabel('Cumulative Density')
title('Cumulative Distribution Function');
xlim([0 0.6])

subplot(1,3,3);
t1=0:.01:0.1;
v1 = 0*t1;
t2 = 0.1:01:0.5;
v2 = 2.5*(t2 - 0.1);
t3 = 0.5:.01:1
v3 = 0*t3 + 1;
```

```
t = [t1 t2 t3];
v = [v1 v2 v3];
plot(t,v);
title('Fragility Function');
xlim([0 0.6])b
```



We know $x \sim Exp(0.05)$

so $PDF: f(x;0.05) = \begin{cases} 20e^{-20x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$CDF: F(x;0.05) = \begin{cases} 1 - e^{-20x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

Thus,

$P(Failure|PGA < 0.1) = 0$

$P(Failure|PGA > 0.5) = 1 \times \int_{0.5}^{\infty} d(1 - e^{-20x}) = \int_{0.5}^{\infty} 20e^{-20x} = 4,\ 54 \times 10^{-5}$

$P(Failure|0.1 < PGA < 0.5) = \int_{0.1}^{0.5} 2.5(x - 0.1)d(1 - e^{-20x}) = \int_{0.1}^{0.5} 2.5(x - 0.1) \cdot 20e^{-20x}dx = 0.0169$

$P(Failure) = P(Failure|PGA < 0.1) + P(Failure|PGA > 0.5) + P(Failure|0.1 < PGA < 0.5) = 0.0169$

(b)

PDF :

$$P(PGA = x|Failure) = \frac{1}{P(Failure)} \cdot P(Failure|PGA = x) \cdot f(x;0.05) = \begin{cases} 0 & x < 0.1 \\ \frac{1}{0.0169} \cdot 2.5(x - 0.1) \cdot 20e^{-20x} & 0.1 < x < 0.5 \\ \frac{1}{0.0169} \cdot 20e^{-20x} & x > 0.5 \end{cases}$$

Problem 3. Cracks in the weld of a structural member have random lengths A with the PDF

$$f_A(a) = \lambda \exp(-\lambda a) \quad a \geq 0$$

where $\lambda = 10\,mm^{-1}$. An X-ray device is used to detect the welds. The probability that a crack will be detected depends on its length and is given by

$$\begin{aligned} P(\text{crack will be detected} \mid A = a) &= 25a^2 \quad 0 \leq a \leq 0.2\,mm \\ &= 1 \quad\quad a \geq 0.2\,mm \end{aligned}$$

(a) Determine the PDF of the length of a crack that has been detected.

(b) Determine the PDF of the length of a crack that has escaped detection.

(c) Plot and compare the above two PDF's together with the PDF of the crack length before detection.

You may use the relation

$$\int x^2 \exp(ax)\,dx = \frac{\exp(ax)}{a^3}\left(a^2x^2 - 2ax + 2\right)$$

Matlab code:

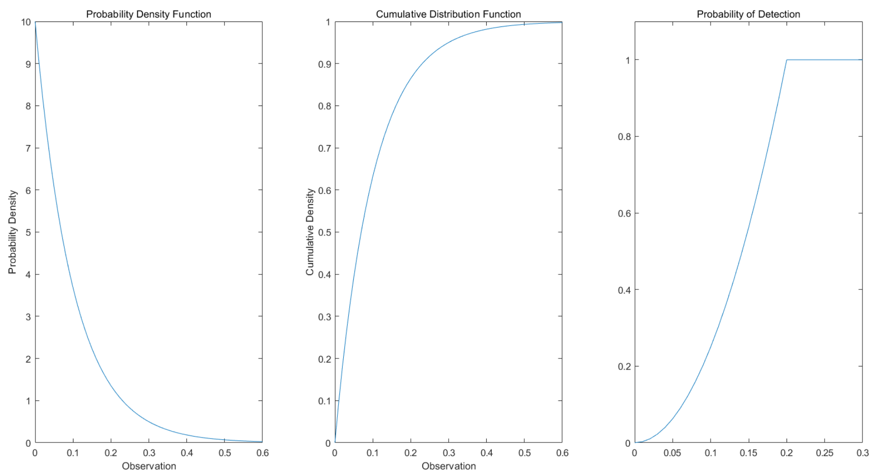
```
clc;
clear;
x = 0:0.01:1;
y1 = exppdf(x,0.1);
y2 = expcdf(x,0.1);

figure;
subplot(1,3,1);
plot(x,y1);
xlabel('Observation')
ylabel('Probability Density')
title('Probability Density Function');
```

```
xlim([0 0.6])

subplot(1,3,2);
plot(x,y2);
xlabel('Observation')
ylabel('Cumulative Density')
title('Cumulative Distribution Function');
xlim([0 0.6])

subplot(1,3,3);
t1=0:.01:0.2;
v1 = 25.*t1.^2;
t2 = 0.2:.01:0.5;
v2 = t2*0 + 1;
t = [t1 t2];
v = [v1 v2];
plot(t,v);
title('Probability of Detection');
xlim([0 0.3])
ylim([0 1.1])
```



PDF of length A is:

$f_A(a) = 10e^{-10a}$

CDF of length A is:

$F_A(a) = 1 - e^{-10a}$

So, as for length A,

$P(0 \leq a \leq 0.2mm) = 0.8647$

$P(a \geq 0.2mm) = 0.1353$

Thus, $P(crackwillbedetected|a \geq 0.2mm) = 0.1353 \times 1 = 0.1353$

$P(crackwillbedetected|0 \leq a \leq 0.2mm) = \int_0^{0.2} 25a^2 d(1 - e^{-10a}) = \int_0^{0.2} 25a^2 \cdot 10e^{-10a} da = 0.1617$

$P(crackwillbedetected) = P(crackwillbedetected|a \geq 0.2mm) + P(crackwillbedetected|0 \leq a \leq 0.2mm) = 0.1353 + 0.1617 = 0.2970$

(a)

PDF :

$$P(A = a|crackwillbedetected) = \frac{1}{P(crackwillbedetected)} \cdot P(crackwillbedetected|A = a) \cdot f_A(a) = \begin{cases} \frac{1}{0.297} \cdot 25a^2 \cdot 10e^{-10a} & 0 \leq a \leq 0.2mn \\ \frac{1}{0.297} \cdot 1 \cdot 10e^{-10a} & a \geq 0.2mm \end{cases}$$

(b)

PDF :

$$P(A = a|crackwillescapedetection) = \frac{1}{P(crackwillescapedetection)} \cdot P(crackwillescapedetection|A = a) \cdot f_A(a) = \frac{1}{1 - P(crackwillbedet}$$

(c)

MATLAB code:

```
clc;
clear;
x = 0:0.001:1+0.001;
y1 = exppdf(x,0.1);

figure;
plot(x,y1,'r','LineWidth',1);
hold on;

t1=0:.001:0.2;
```

```
v1 = 1/0.279.*(10.*exp(-10.*t1)).*25.*t1.^2;
t2 = 0.2:.001:1;
v2 = 1/0.279.*(10.*exp(-10.*t2));
t = [t1 t2];
v = [v1 v2];
plot(t,v,'b','LineWidth',1);
hold on;

a1=0:.001:0.2;
b1 = 1/0.703.*(10.*exp(-10.*a1)).*(1-25.*a1.^2);
a2 = 0.2:.001:1;
b2 = a2.*0;
a = [a1 a2];
b = [b1 b2];
plot(a,b,'gr','LineWidth',1);
legend('PDF of the crack length before detection','PDF of the length of a crack that has been detected','PDF of the length of a crack that has escaped detection','Location','northeast')
xlim([0 1])
ylim([0 16])
hold off;
```