Problem 1. Two reinforced concrete buildings A and B are located in a seismic region. It is estimated that an impending earthquake in the region might be strong (S), moderate (M), or weak (W) with probabilities $P(S) = 0.02$, $P(M) = 0.20$, and $P(W) = 0.78$. The probabilities of failure of each building if these earthquakes occur are 0.20, 0.05, and 0.01, respectively.

(a) Determine the probability of failure of building A if the impending earthquake occurs.

(b) If building A fails, what is the probability that the earthquake was of moderate strength?

(c) Due to similar procedures used in the design and construction of the two buildings, it is estimated that if building A fails the probability that building B will also fail is 0.50, 0.15, and 0.02 for the three types of earthquakes. Determine the probability that both buildings will fail in the impending earthquake.

(d) If building A has failed and building B has survived, what is the probability that the earthquake was not strong?

P.1

(1) $P(AF) = P(S)\cdot P(AFS|S) + P(M)\cdot P(AFM|M) + P(W)\cdot P(AFW|W)$

$= 0.02\times 0.20 + 0.20\times 0.05 + 0.78\times 0.01$

$= 0.0218$

(2) $P(M|AF) = \dfrac{P(AFM)}{P(AF)} = \dfrac{0.20\times 0.05}{0.0218} = 0.45872$

(3) $P(AF/S) = P(BF/S) = 0.2\times 0.02 = 0.004$

$P(AF/M) = P(BF/M) = 0.05\times 0.2 = 0.01$

$P(AF/W) = P(BF/W) = 0.01\times 0.78 = 0.0078$

$P(AS/S) = P(BS/S) = 0.8\times 0.02 = 0.016$

$P(AS/M) = P(BS/M) = 0.95\times 0.2 = 0.19$

$P(AS/W) = P(BS/W) = 0.99\times 0.78 = 0.7722$

These are independent probabilities of failure of A and B.

$P(BFS/AFS) = 0.5$

$P(BFM/AFM) = 0.15$

$P(BFW/AFW) = 0.02$

So

$P(BF/AF) = P(BFS/AFS) * P(AFS) + P(BFM/AFM) * P(AFM) + P(BFW/AFW) * P(AFW)$

$= 0.5\times 0.004 + 0.15\times 0.01 + 0.02\times 0.0078$

$= 0.003656$

(4) $P(BS/AF) = P(BSS/AFS) * P(AFS) + P(BSM/AFM) * P(AFM) + P(BSW/AFW) * P(AFW)$

$= 0.5\times 0.004 + 0.85\times 0.01 + 0.98\times 0.0078$

$= 0.018144$

use Bayes formula:

$P(\bar{S}/(BS/AF)) = \dfrac{(P(BSM/AFM) * P(AFM) + P(BSW/AFW) * P(AFW))}{P(BS/AF)}$

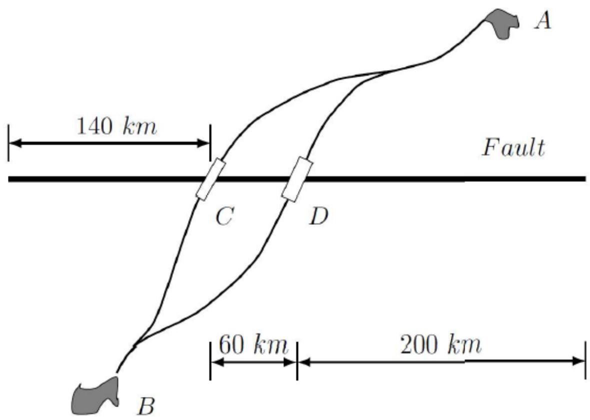
$= \dfrac{(0.0085 + 0.007644)}{0.018144} = \dfrac{0.016144}{0.018144} = 0.88977$

Problem 2. A highway system for transportation between cities A and B includes two parallel bridges C and D that cross a fault line. In the event of an earthquake on the fault, each bridge will fail if the fault rupture crosses it. Other part of the highway are not vulnerable to earthquakes.

(a) Determine the probability of failure of the highway system, i.e., the probability that transportation between cities A and B will be interrupted, if an earthquake caused by a 100 km long rupture occurs on the fault. Assume the rupture is equally likely to occur anywhere along the fault, but not extending beyond its ends.

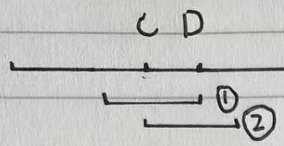
(b) Suppose bridge D is known to have failed due to the earthquake. What is the probability that bridge C has survived?

(c) If transportation between cities A and B is known to be possible after the earthquake, what is the probability that both bridges survived?



P2

(a)



$P_{(F)} = \frac{140-100}{300} = \frac{2}{15}$

(b) $P(DF) = \frac{100}{300} = \frac{1}{3}$

$$P(CS|DF) = \frac{P(CSDF)}{P(DF)} = \frac{\frac{60}{300}}{\frac{1}{3}} = \frac{3}{5}$$

(c) $P(\text{possible}) = 1 - P(F) = 1 - \frac{2}{15} = \frac{13}{15}$

$$P(\text{both survived} \mid \text{possible})$$

$$= \frac{P(\text{both survived} \cdot \text{possible})}{P(\text{possible})}$$

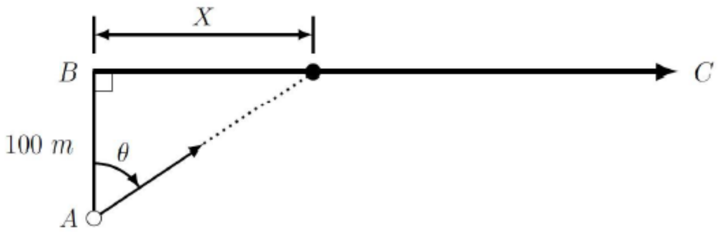
$$= \frac{\frac{100+40}{300}}{\frac{13}{15}} = \frac{7}{13}$$

Problem 3. A bullet is fired from point A towards the semi-infinite line BC at a random angle θ. Assume θ is equally likely to be anywhere between $0°$ and $90°$.

(a) Determine the PDF, $f(x)$, and CDF, $F(x)$, of the coordinate X of the point where the bullet hits on the line.

(b) Sketch the PDF and the CDF. Is it more likely or less likely that the bullet will hit near point B (rather than far from it)?

(c) Determine the probability that X will be less than 50 m.



(a)

$$X = 100tan\theta$$

$$\theta = arctan\frac{X}{100}$$

$$CDF : F(X) = \frac{1}{90}arctan\frac{X}{100}$$

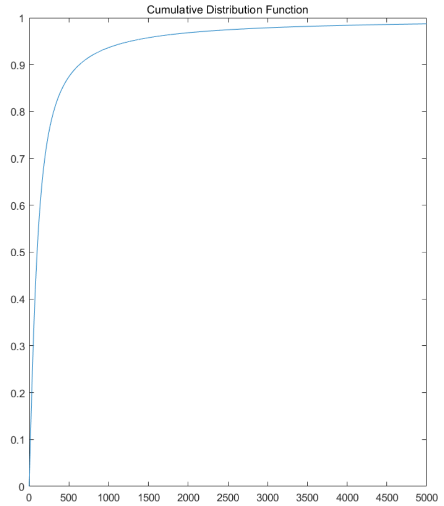
If there is a first derivative of F, its probability density function is:

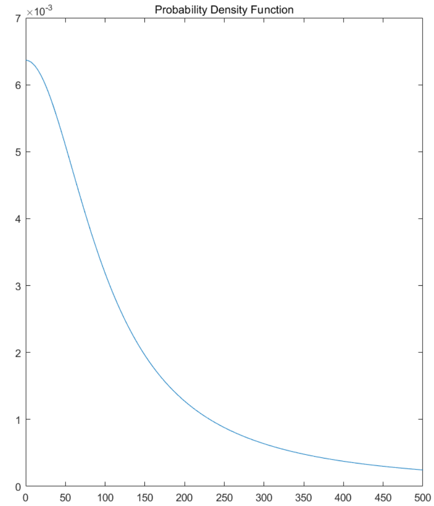
$$f(X) = \frac{dF(X)}{dX}$$

$$PDF : f(X) = 1/(50\pi(X^2/10000+1))$$

(b)

Sketch the CDF and the PDF through MATLAB, the figures are shown below.





The code is shown below.

```
clear;
clc;
syms x;
y=1/90*atand(x/100); %CDF
a=diff(y); %PDF
x = 0:.1:5000;
y=eval(y);
```

```
a=eval(a);

figure;
subplot(1,2,1);
plot(x,y);
title('Cumulative Distribution Function');
xlim([0 5000])

subplot(1,2,2);
plot(x,a);
title('Probability Density Function');
xlim([0 500])
```

As we can see from the figure for PDF, The function is monotonically decreasing, but it is kind of intuitive.

Generate the first derivative with respect to $f(X)$:

$$f^{'}(X) = -x/(250000\pi(x^2/10000+1)^2)$$

$$When X \in [0, 90)$$

$$f^{'}(X) \leq 0$$

Thus, we could draw the conclusion that it is more likely that the bullet will hit near point B.

(c)

$$When X \in [0, 50)$$

$$P(0 \leq X < 50) = F(50) = 0.2952$$

Problem 4. The Rayleigh distribution is used in modeling ocean wave heights. It is described by the probability density function (PDF)

$$\begin{aligned} f_X(x) &= bxe^{-\left(\frac{x}{a}\right)^2} && 0 \leq x \\ &= 0 && \text{elsewhere} \end{aligned}$$

where a and b are constants. Determine:

(a) The constant b in terms of a;

(b) The cumulative distribution function (CDF) of X;

(c) The mean, mode, median, standard deviation, and coefficient of variation values in terms of a.

(d) What is the probability that $X > 2a$?

You may use the following relations for $a > 0$:

$$\int_0^\infty x^{2n} e^{-ax^2} \mathrm{d}x = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x^{2n+1} e^{-ax^2} \mathrm{d}x = \frac{n!}{2a^{n+1}}$$

(a)

$$PDF: f_X(x) = bxe^{-\left(\frac{x}{a}\right)^2}\ 0 \leq x$$

so $CDF: F_X(x) = \int_0^\infty f_X(x)dx = 1$

$$\int_0^\infty bxe^{-\left(\frac{x}{a}\right)^2} dx = 1$$

$b \cdot \frac{2}{a^2} = 1$

$b = \frac{a^2}{2}$

(b)

$CDF: F(x;a) = \int_0^\infty f_X(x)dx = 1 - e^{-(\frac{x}{a})^2}$

(c)

The mean : $\mu(X) = \frac{\sqrt{\pi}}{2}a$

The mode : $\frac{1}{\sqrt{2}}a$

The median : $\sqrt{\frac{ln(4)}{2}}a$

The standard deviation : $std(X) = \frac{\sqrt{4-\pi}}{2}a$

The variance : $var(X) = (1 - \frac{\pi}{4})a^2$

(d)

$CDF: F(x;a) = 1 - e^{-(\frac{x}{a})^2}$

$P(X > 2a) = 1 - P(X \leq 2a) = 1 - F(2a;a) = 1 - (1 - e^{-(\frac{2a}{a})^2}) = \frac{1}{e^4} \approx 0.0183$