南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

考试科目: 线 性 代 数　　开课单位: 数 学 系
考试时长: 120 分钟　　命题教师: 线性代数教学团队

题 号	1	2	3	4	5	6	7
分 值	15 分	20 分	10 分	20 分	20 分	10 分	5 分

本试卷共 (7) 大题, 满分 (100) 分. (考试结束后请将试卷、答题本、草稿纸一起交给监考老师).
This exam paper contains 7 **questions** and the score is 100 in total. (Please hand in your exam paper, answer sheet, and your scrap paper to the proctor when the exam ends).

***Notation**: $C(A)$: the column space of matrix A; $C(A^T)$: the row space of matrix A; $N(A)$: the nullspace of matrix A; $N(A^T)$: the left nullspace of matrix A; O: the zero matrix;* rank A: *the rank of matrix A.*

1. (15 points, 3 points each) Multiple Choice. Only one choice is correct.
(共 15 分, 每小题 3 分) 选择题, 只有一个选项是正确的.

(1) Let A be a 3×4 matrix with rank $A = 2$ and $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, then rank $PA =$

(A) 1.
(B) 2.
(C) 3.
(D) 4.

设 A 为一个秩为 2 的 3×4 矩阵, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, 则 rank $PA =$

(A) 1.
(B) 2.
(C) 3.
(D) 4.

(2) Which of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

(A) $V_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1 \right.$

(B) $V_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 0 \right\}.$

(C) $V_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$

~~(D) V_4~~ $= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0 \right\}.$

下列集合构成 $\mathbb{R}^3$ 的子空间的是?

(A) $V_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1 \right\}.$

(B) $V_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 = 0 \right\}.$

(C) $V_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}.$

(D) $V_4 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mid x_1 \geq 0, x_2 \geq 0 \right\}.$

(3) Let $u, v \in \mathbb{R}^4, \lambda \in \mathbb{R}$, which of the following assertions is **false**?

~~(A)~~ Suppose u, v are nonzero vectors and $u^T v = 0$, then u, v are linearly independent.

(B) $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

~~(C)~~ $\lambda u = 0$, then $\lambda = 0$ or $u = 0$.

(D) $u^T v = 0$, then $u = 0$ or $v = 0$.

设 $u, v \in \mathbb{R}^4, \lambda \in \mathbb{R}$, 以下哪个说法是错误的?

(A) 若 u, v 为满足 $u^T v = 0$ 的两个非零向量, 则 u, v 线性无关.

(B) $||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2)$.

(C) $\lambda u = 0$, 则 $\lambda = 0$ 或 $u = 0$.

(D) $u^T v = 0$, 则 $u = 0$ 或 $v = 0$.

(4) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{2024} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{2025} =$$

(A) $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 4056 & 2031 & 6081 \end{bmatrix}.$

(B) $\begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & 3 \\ 4056 & 2031 & 6081 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2031 & 4056 & 6081 \end{bmatrix}$.

(D) $\begin{bmatrix} 12148 & 5 & 6 \\ 6073 & 2 & 3 \\ 18223 & 8 & 9 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{2024} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{2025} =$$

(A) $\begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 4056 & 2031 & 6081 \end{bmatrix}$.

(B) $\begin{bmatrix} 5 & 4 & 6 \\ 2 & 1 & 3 \\ 4056 & 2031 & 6081 \end{bmatrix}$.

(C) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2031 & 4056 & 6081 \end{bmatrix}$.

(D) $\begin{bmatrix} 12148 & 5 & 6 \\ 6073 & 2 & 3 \\ 18223 & 8 & 9 \end{bmatrix}$.

(5) Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Suppose $Ax = b$ has infinitely many solutions, which of the following assertions must be true?

(A) A is a square matrix.

(B) rank $A = m$.

(C) rank $A = n$.

(D) rank $A < n$.

设 A 为 $m \times n$ 实矩阵, $b \in \mathbb{R}^m$. 如果 $Ax = b$ 有无穷多解, 下列哪个结论一定是正确的?

(A) A 为一个方阵.

(B) rank $A = m$.

(C) rank $A = n$.

(D) rank $A < n$.

2. (20 points, 5 points each) Fill in the blanks.

(共 20 分, 每小题 5 分) 填空题.

(1) Let $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$. Then $A^{-1} =$.

设 $A = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$, 则 $A^{-1} =$ ________.

(2) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 4 & t & 3 \\ 3 & -1 & 1 \end{bmatrix}$ and B be a 3×3 nonzero matrix such that $AB = O$. Then $t =$________.

设 $A = \begin{bmatrix} 1 & 2 & -2 \\ 4 & t & 3 \\ 3 & -1 & 1 \end{bmatrix}$, B 为满足 $AB = O$ 的 3 阶非零矩阵, 则 $t =$________.

(3) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 8 & 7 \\ 5 & 12 & 13 \end{bmatrix}$. A basis of $N(A)$ is ____

设 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 8 & 7 \\ 5 & 12 & 13 \end{bmatrix}$. $N(A)$ 的一个基为 ______.

(4) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 3 \\ 3 & 3 & 5 & 8 \end{bmatrix}$. Then $\dim C(A) =$____

设 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 3 \\ 3 & 3 & 5 & 8 \end{bmatrix}$, 则 $\dim C(A) =$________.

3. (10 points) Find an LU factorization of the matrix

$$\begin{bmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{bmatrix}.$$

求下列矩阵的一个 LU 分解:

$$\begin{bmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{bmatrix}.$$

4. (20 points) Consider the following system of linear equations:

$$(I):\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 = \lambda, \\ x_2 + 2x_3 + 2x_4 + 6x_5 = 3, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 = \mu. \end{cases}$$

(a) For what values of λ and μ does the system (I) have no solution or infinitely many solutions.

(b) Solve for all the solutions of (I) if the system is consistent.

考虑以下线性方程组:

$$(I):\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1, \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 = \lambda, \\ x_2 + 2x_3 + 2x_4 + 6x_5 = 3, \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 = \mu. \end{cases}$$

(a) 当 λ, μ 满足什么条件时, 上述线性方程组 (I) 无解、有无穷多解?

(b) 在方程组 (I) 有解时, 求出其通解.

5. (20 points) Let $V = \mathbb{R}^{2\times 2}$ be the vector space of all 2×2 real matrices. Define a map as follows:

$$T: V \to V,\ T(A) = A + A^T,\ \forall A \in V.$$

(a) Show that T is a linear transformation.

(b) Let **kernel** $T = \{A \in V \mid T(A) = O\}$, where O denotes the 2×2 zero matrix. Show that **kernel** T is a subspace of V and find a basis for **kernel** T.

(c) Find the matrix representation of T with respect to the following ordered basis of V:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) Find all matrices A such that $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

设 $V = \mathbb{R}^{2\times 2}$ 为所有 2×2 实矩阵构成的向量空间. 定义以下映射:

$$T: V \to V,\ T(A) = A + A^T,\ \forall A \in V.$$

(a) 证明: T 为线性变换.

(b) 设 **kernel** $T = \{A \in V \mid T(A) = O\}$, 这里 O 表示 2 阶零矩阵. 证明 **kernel** T 为 V 的一个子空间, 并求 **kernel** T 的一个基向量组.

(c) 求线性变换 T 在如下 V 的有序基下的矩阵表示:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) 求所有满足 $T(A) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ 的矩阵 A .

6. (10 points) In physics, *Hooke's law* states that (within certain limits) there is a linear relationship between the length x of a spring and the force y applied to (or exerted by) the spring. That is, $y = cx + d$, where c is called the **spring constant**. Use the following data to estimate the spring constant (the length is given in inches and the force is give in pounds).

length x	**force** y
3.5	1.0
4.0	2.2
4.5	2.8
5.0	4.3

在物理学中, *Hooke's law* 说的是 (在有限范围内) 作用在弹簧上的力 y 和弹簧的长度 x 存在线性关系. 也就是说, $y = cx + d$, 其中 c 称之为弹簧常数. 请用下列表格提供的数据估计弹簧常数 (长度的单位是英寸, 力的单位是磅).

length x	**force** y
3.5	1.0
4.0	2.2
4.5	2.8
5.0	4.3

7. (5 points) Let A, B be 5×6 real matrices with rank $A = 2$, rank $B = 3$, and rank $(A+B) = 5$. Show that there exist 5×5 invertible matrix P and 6×6 invertible matrix Q such that

$$PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } PBQ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

设 A, B 为 5×6 实矩阵, 且 rank $A = 2$, rank $B = 3$, 以及 rank $(A+B) = 5$. 证明: 存在 5 阶可逆矩阵 P 和 6 阶可逆矩阵 Q 使得

$$PAQ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ 以及 } PBQ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$