一、(1)、$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\ln v\cdot e^{u}\cdot\frac{1}{y}+\frac{e^{u}}{v}=e^{\frac{x}{y}}\left(\frac{1}{y}\ln(x+4y)+\frac{1}{x+4y}\right)$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\ln v\cdot e^{u}\cdot(-\frac{x}{y^2})+\frac{4e^{u}}{v}=e^{\frac{x}{y}}\left(-\frac{x}{y^2}\ln(x+4y)+\frac{4}{x+4y}\right)$

(2)、$\frac{\partial f}{\partial x}=-\frac{x}{\sqrt{1-x^2}}$, $\frac{\partial f}{\partial y}=\frac{y}{\sqrt{y^2-1}}$

$\frac{\partial f}{\partial x}$与$\frac{\partial f}{\partial y}$在p的一个邻域中存在且在p处连续

$\Rightarrow$ f在p处可微

$\Rightarrow \frac{\partial f}{\partial u}(p)=u\cdot Jf(p)=(\frac{\sqrt3}{2},\frac12)\cdot(-\frac{\sqrt3}{3},-\frac{3}{5}\sqrt5)=-\frac12-\frac{3}{10}\sqrt5$

(3)、$S=\int_0^1\sqrt{(2t)^2+(3t^2)^2}\,dt=\int_0^1 t\sqrt{9t^2+4}\,dt=\frac{13\sqrt{13}-8}{27}$

(4)、$r(t)=(a\cos t,b\sin t,0)$

$r'(t)=(-a\sin t,b\cos t,0)$

$r''(t)=(-a\cos t,-b\sin t,0)$

$r'(t)\times r''(t)=(0,0,ab)$

$k(t)=\frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}=\frac{ab}{(a^2\sin^2t+b^2\cos^2t)^{\frac32}}=\frac{ab}{[(a^2-b^2)\sin^2t+b^2]^{\frac32}}$

$k(t)_{min}=\frac{b}{a^2}$, $k(t)_{max}=\frac{a}{b^2}$

二、$\alpha>1$时, $\forall\varepsilon>0$, $\exists\delta=\varepsilon^{\frac{1}{\alpha-1}}$, 使得当$0<\sqrt{x^2+y^2}<\delta$时, $\left|f^{\alpha-1}(x,y)\sin\frac{1}{x^2+y^2}\right|\le\left|f^{\alpha-1}(x,y)\right|=(\max\{|x|,|y|\})^{\alpha-1}<\delta^{\alpha-1}=\varepsilon$

$\Rightarrow\lim\limits_{(x,y)\to(0,0)}f^{\alpha-1}(x,y)\sin\frac{1}{x^2+y^2}=0$

$\alpha\le1$时, 设$g(x)=f^{\alpha-1}(x)\sin\frac{1}{\|x\|^2}$, 取$s_i=(\frac1i,0)$, 则$s_i\to0\ (i\to\infty)$, $g(s_i)=i^{1-\alpha}\sin i^2$,

$\lim\limits_{i\to\infty}g(s_i)$不存在

$\Rightarrow\lim\limits_{x\to0}g(x)$不存在

三、(1)、D

(2)、不可积。对$\forall x_0\in[0,\frac12)\cup(\frac12,1]$, 取$\{s_n\in\mathbb{R}\setminus\mathbb{Q}$且$s_n\ne x_0\}$满足$s_n\to x_0\ (n\to\infty)$, 此时$f(s_n)=0$. 取$\{t_n\in\mathbb{Q}$且$t_n\ne x_0\}$满足$t_n\to x_0\ (n\to\infty)$, 此时$f(t_n)=(t_n-\frac12)^2$, 且有$\lim\limits_{n\to\infty}f(t_n)=(x_0-\frac12)^2>0$, 因此$\lim\limits_{x\to x_0}f(x)$不存在, f在点$x_0$处不连续

$\Rightarrow[0,\frac12)\cup(\frac12,1]\subseteq D(f)$

由于$[0,\frac12)\cup(\frac12,1]$不是零测集, 故$D(f)$不是零测集

$\Rightarrow$ f在[0,1]上不可积

四、证明：C_i为紧致集

$\Rightarrow$从C_i的任一个开覆盖中都能选出有限个开集使其仍为C_i的开覆盖

对C的任一开覆盖，也为C_i的开覆盖，因此能选出有限个开集$\{A_{ij}\}, 1\le j\le n_i$使其能组成$C_i$的开覆盖，故$\{A_{ij}\}, 1\le i\le m, 1\le j\le n_i$这有限个开集为C的开覆盖

$\Rightarrow$C为紧致集 $\square$

五、$f(x)=b\sqrt{\frac{x^2}{a^2}-1}$，$x\in[a,2a]$，$f'(x)=\frac{b}{a}\cdot\frac{x}{\sqrt{x^2-a^2}}$

$$V=\pi\int_a^{2a}f^2(x)dx=b^2\pi\int_a^{2a}\left(\frac{x^2}{a^2}-1\right)dx=\frac{4}{3}ab^2\pi$$

$$S=2\pi\int_a^{2a}f(x)\sqrt{1+(f'(x))^2}dx=\frac{2\pi b}{a^2}\int_a^{2a}\sqrt{(a^2+b^2)x^2-a^4}\,dx$$

$$=2\pi b\sqrt{3a^2+4b^2}-\pi b^2+\frac{\pi a^2 b}{\sqrt{a^2+b^2}}\ln\frac{\sqrt{a^2+b^2}+b}{2\sqrt{a^2+b^2}+\sqrt{3a^2+4b^2}}$$

六、设$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$，则$F(x,y,z)=0$，

在(1,1,2)处的法向量为$\left(\frac{\partial F}{\partial x}(1,1,2),\frac{\partial F}{\partial y}(1,1,2),\frac{\partial F}{\partial z}(1,1,2)\right)=\left(\frac{2}{a^2},\frac{2}{b^2},\frac{4}{c^2}\right)$

$\Rightarrow\left(\frac{2}{a^2},\frac{2}{b^2},\frac{4}{c^2}\right)=k(4,2,3)$

$\Rightarrow\frac{1}{a^2}=2k,\frac{1}{b^2}=k,\frac{4}{c^2}=3k$

由$F(1,1,2)=0$得$\frac{1}{a^2}+\frac{1}{b^2}+\frac{4}{c^2}=1$

$\Rightarrow k=\frac{1}{6}$

$\Rightarrow a=\sqrt{3}, b=\sqrt{6}, c=2\sqrt{2}$

七、
$$J(f\circ g)=J_fJ_g=\begin{pmatrix}\frac{\partial f_1}{\partial x}&\frac{\partial f_1}{\partial y}&\frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x}&\frac{\partial f_2}{\partial y}&\frac{\partial f_3}{\partial z}\end{pmatrix}\begin{pmatrix}\frac{\partial g_1}{\partial u}&\frac{\partial g_1}{\partial v}\\ \frac{\partial g_2}{\partial u}&\frac{\partial g_2}{\partial v}\\ \frac{\partial g_3}{\partial u}&\frac{\partial g_3}{\partial v}\end{pmatrix}=\begin{pmatrix}1&1&1\\ yz&xz&xy\end{pmatrix}\begin{pmatrix}a\cos u\cos v&-a\sin u\sin v\\ a\cos u\sin v&a\sin u\cos v\\ -a\sin u&0\end{pmatrix}$$

$$=\begin{pmatrix}a\cos u(\sin v+\cos v)-a\sin u & a\sin u(\cos v-\sin v)\\ ayz\cos u\cos v+axz\cos u\sin v-axy\sin u & -ayz\sin u\sin v+axz\sin u\cos v\end{pmatrix}$$

$$=\begin{pmatrix}a\cos u(\sin v+\cos v)-a\sin u & a\sin u(\cos v-\sin v)\\ a^3\sin u\sin v\cos v(2\cos^2u-\sin^2u) & a^3\sin^2u\cos u(\cos^2v-\sin^2v)\end{pmatrix}$$

八、证明：(1)、设方向$u=(\cos\theta,\sin\theta)$，$\theta\in[0,2\pi)$，

则$\frac{\partial f}{\partial u}(0,0)=\lim\limits_{t\to0}\frac{f(tu)}{t}=\cos^2\theta\sin\theta$

$\Rightarrow$(0,0)处沿任何方向的方向导数皆存在

(2)、 $$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}$$

取$S_i=(\frac{1}{i},0)$，$t_i=(\frac{1}{i},\frac{1}{i})$，则$S_i\to 0\ (i\to\infty)$，$t_i\to 0\ (i\to\infty)$，

由$f(S_i)=0$，$f(t_i)=\frac{\sqrt{2}}{4}$可得上述极限不存在，故$f$于$(0,0)$处不可微 $\square$