Columbia MA Math Camp

Set Theory

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Motivation of Study Set Theory

- Set theory is one of the fundamental building blocks of mathematics
- Many important concepts later such as relations, functions and sequences are defined using the language of sets
- You will encounter a lot of this during your micro course (and math methods of course)

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Small Digression on Proofs

- We went through some basics of logic where we talked about implications like $p \to q$.
- In set theory, you will frequently encounter questions which give you some information and then ask you to show that $p \to q$.
- The way to show it is to assume that p is true and then use the information to show that q must be true
- Sometimes it is easier to prove the contrapositive i.e. try to prove that $\neg q \to \neg p$.
    - Assume that $\neg q$ is true and then proceed to show that $\neg p$ must be true.

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Common Notation

- $\in$: **"in"**; e.g. $x \in \mathbb{N}$ means x is a **natural number**.
- $\forall$: **"for all"**; e.g. $\forall x \in \mathbb{N}$ means for all natural numbers x.
- $\exists$: **"there exists"**; e.g. $\forall x \in \mathbb{N}, \exists y \in \mathbb{Z}$ such that $x + y = 0$
- $!$: **"unique"**; typically used in conjunction with $\exists$
- $\Rightarrow$: **"implies"**; e.g. $A \Rightarrow B$ means A implies B.
  - We have been using $\rightarrow$. We will switch to $\implies$ from now as this is more commonly used.

Basic Concepts

- A set is a collection of objects and each individual object is called an **element**
- Lowercase letters are used for elements and Capital letters for sets.
- The notation $x \in X$ means that the object x **is an element** of the set X.
- A set is typically written in curly brackets $\{1, 2, 3\}$
    - The order of the elements listed does not matter
- For more complicated sets we use **"set-builder"** notation, e.g.

$$\{x \in \mathbb{N} | x^2 < 100\}$$

- The item before the vertical line defines the domain of our search.
- In the example above, we are searching for natural numbers which satisfy the requirement to the right of the vertical line

Common Sets

- **Common sets:**
  - $\mathbb{N}$: natural numbers $\{0, 1, 2, ...\}$
  - $\mathbb{Z}$: integers $\{..., -2, -1, 0, 1, 2, ...\}$
  - $\mathbb{Q}$: rational numbers; all numbers of form $\frac{p}{q}$ with $p, q \in \mathbb{Z}, q \neq 0$
  - $\mathbb{R}$: real numbers; most of econ happens here
- We do allow a set to contain **no element** at all, and we call it the **empty set** denoted by $\emptyset$
  - The empty set $\emptyset$ is a subset of every set. (Why?)

Comparing sets - Inclusion

- A is a **subset** of B if every element of A is an element of B; write $A \subseteq B$ or $B \supseteq A$
  - In other words, $x \in A \implies x \in B \; \forall x$
- Two sets are **equal** if they contain exactly the same elements. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- A set A is a **proper subset** of B if $A \subseteq B$ and $A \neq B$. This is sometimes written $A \subset B$ or $A \subsetneq B$
  - Not all sets are comparable. Give me an example?
- Set inclusion is **transitive**: $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.
- For finite sets, the **cardinality** of a set $|A|$ is the number of elements of A

How to prove Set Inclusion is Transitive?

Lemma 1.1

Set inclusion is transitive i.e. $A \subseteq B$ *and* $B \subseteq C$ *implies* $A \subseteq C$

Proof.

We want to show $A \subseteq C$. By definition, this means that we need to show for any $x \in A$ it must be that $x \in C$. Take any $x \in A$. By definition of $A \subseteq B$ and because $x \in A$, we have that $x \in B$. Again by the definition of $B \subseteq C$, we have $x \in C$. $\square$

Constructing new sets

- The **union** of A and B, $A \cup B$ is the collection of elements in A or B (or both)

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

- The **intersection** of A and B, $A \cap B$ is the collection of elements that belong to both A and B

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

  If $A \cap B = \emptyset$, then we say that A and B are disjoint.

- The **difference** of A and B, $A \setminus B$ or $A - B$, is the collection of elements in A and not in B.

$$A\ B := \{x : x \in A : x \notin A\}$$

Can take unions and intersections of big collections of sets, typically indexed by an **index set**. Most common index set is $\mathbb{N}$, e.g.

$$\bigcup_{i \in \{0,1,2,\ldots\}} A_i$$

Some properties - I

Lemma 1.2

$A \cup B = B$ iff $A \subseteq B$ (Note that this is a equivalence because of the iff)

Proof.

We show subset containment both ways i.e. both the $\implies$ and the $\impliedby$.

Let's first prove the $\impliedby$ side.

- Suppose $A \subseteq B$. WTS that $A \cup B = B$. This means that we need to show both $A \cup B \subseteq B$ and $B \subseteq A \cup B$. Let us first show $A \cup B \subseteq B$.
- Take any $x \in A \cup B$. WTS $x \in B$. Because $x \in A \cup B$, then, by definition $\cup$, either $x \in A$ or $x \in B$
- If $x \in A$, then by definition of $x \subseteq B$ we have $x \in B$. So either case, we have $x \in B$. Thus $A \cup B \subseteq B$ is proved. Proving $B \subseteq A \cup B$ is left as an exercise.

Now let us prove the other direction i.e. $\implies$

- Given $A \cup B = B$, WTS $A \subseteq B$.
- Take any $x \in A$, WTS that $x \in B$. By definition of $A \cup B$ and $x \in A$, we have $x \in A \cup B$. Because $A \subseteq B = B$, we have $x \in B$. Thus $\implies$ is proved.

Some properties - II

Lemma 1.3

Intersection is distributive with respect to the union:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof.

We show subset containment both ways.

- Let $x \in A \cap (B \cup C)$.
- By definition, $x \in A$ and $x \in B \cup C$, so $x \in B$ or $x \in C$.
- Thus $x \in A \cap B$ or $A \cap B$, so $x \in (A \cap B) \cup (A \cap C)$.

Now the other direction

- Let $x \in (A \cap B) \cup (A \cap C)$.
- By definition, $x \in A \cap B$ or $x \in A \cap C$, so $x \in A$ and $x \in B$ or $x \in C$.
- Thus $x \in A \cap (B \cup C)$

□

Proof of General Statement of Distributive Property

Note that we don't need to limit the proof before to finite unions. We can prove it for an infinite unions.

Lemma 1.4
Prove that the intersection is distributive wrt to the union i.e.

$$B \cap \left(\bigcup_{i \in \mathbb{N}} A_i \right) = \bigcup_{i \in \mathbb{N}} (B \cap A_i)$$

Proof : To do this, we need to show that both the $\subseteq$ and $\supseteq$ to show that both sets are equal.

$\subseteq$: Take any $x \in B \cap \left(\bigcup_{i \in \mathbb{N}} A_i \right)$. By definition of $\cap$, this implies that $x \in B$ and $x \in \bigcup_{i \in \mathbb{N}} A_i$. By definition of $\bigcup_{i \in \mathbb{N}} A_i$, $\exists i \in \mathbb{N}$, such that $x \in A_i$. So we have for that i, $x \in B \cap A_i$. Therefore $x \in \bigcup_{i \in \mathbb{N}} (B \cap A_i)$

Proof Continued...

$\supseteq$: Take any $x \in \bigcup_{i \in \mathbb{N}} (B \cap A_i)$. By definition, $\exists i \in \mathbb{N}$ such that $x \in B \cap A_i$.

Therefore we have that $x \in B$ and $x \in A_i$. We can conclude then that $x \in \bigcup A_i$ and therefore $x \in B \cap \left(\bigcup_{i \in \mathbb{N}} A_i \right)$

Complements and DeMorgan's Laws

We normally think of sets living in some larger space Ω. The **complement** of a set A, A^c, is the collection of elements not in A.

$$A^c := \Omega \backslash A$$

Further, the complement of the complement of a set is the set itself.

$$(A^c)^c = A$$

Complements play nicely with unions and intersections. The following are DeMorgan's Laws for 2 sets A and B.

$$\begin{aligned}(A \cap B)^c &= A^c \cup B^c \\ (A \cup B)^c &= A^c \cap B^c\end{aligned}$$

General statement of DeMorgan's Laws

Lemma 1.5

Let A_i be a collection of sets. We have that :

(a) $\left(\bigcup_{i\in\mathbb{N}} A_i\right)^c = \bigcap_{i\in\mathbb{N}} A_i^c$

(b) $\left(\bigcap_{i\in\mathbb{N}} A_i\right)^c = \bigcup_{i\in\mathbb{N}} A_i^c$

Proof : We'll prove only the first statement. As usual we need to prove both $\subseteq$ and $\supseteq$.

$\subseteq$: Take any $x \in \left(\bigcup_{i\in\mathbb{N}} A_i\right)^c = \Omega \backslash \left(\bigcup_{i\in\mathbb{N}} A_i\right)$. This implies $x \in \Omega$ and $x \notin \left(\bigcup_{i\in\mathbb{N}} A_i\right)^c$.

Therefore $x \notin A_i \ \forall i \in \mathbb{N}$ which implies that $x \in A_i^c \ \forall i$. Thus, $\bigcap_{i\in\mathbb{N}} A_i^c$.

Proof Continued...

$\supseteq$: Take any $x \in \bigcap_{i \in \mathbb{N}} A_i^c$.

By definition, we have that $x \in A_i^c \; \forall i \in \mathbb{N}$. This implies $x \in \Omega$ and $x \notin A_i$ for any $i \in \mathbb{N}$. So we have $x \in \Omega$ and $x \notin \bigcup\limits_{i \in \mathbb{N}} A_i$ which implies that $x \in \left(\bigcup\limits_{i \in \mathbb{N}} A_i \right)^c$

Cartesian Products

The **Cartesian product** is our last common method for generating new sets: take all pairs from A and B:

$$A \times B \equiv \{(a, b) | a \in A, b \in B\}$$

Typically work in $\mathbb{R}^n$:

$$\mathbb{R}^n \equiv \{(a_1, ..., a_n) | a_i \in \mathbb{R} \, \forall i = 1, ..., n\}$$

Note order matters: $(2, 1) \neq (1, 2)$

(a, b) is known as an ordered pair where the element $a \in A$ and $b \in B$

So in other words $A \times B \neq B \times A$. So the order in which you take the Cartesian product matters (unlike in union or intersection!)

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Functions

A **function** $f : X \to Y$ from a set X to a set Y is a rule that assigns **exactly one** element $y = f(x)$ in Y to each x in X.

Terminology:

- The **domain** of f is X; the **codomain** of f is Y
- The **image** or **range** of f is the collection of values f takes :

$$f(X) \equiv \{f(x)|x \in X\} \subseteq Y$$

- A function f is **one-to-one**, or **injective**, if $f(a) \neq f(a')$ whenever $a \neq a'$
- A function f is **onto**, or **surjective**, if the range of f is Y.
- A function f is **bijective** if it is one-to-one and onto

Functions (cont.)

We're frequently interested in inverting equations like $f(x) = 0$.

- If a function is injective, it admits an **inverse function** $f^{-1} : range(f) \to X$ defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

- For any function, we can define the **inverse image** of a set $S \subseteq Y$:

$$f^{-1}(S) = \{x \in X | f(x) \in S\}$$

- Example: $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.
    - Not injective because $f(1) = f(-1)$; inverse function does not exist
    - But, inverse image is defined: $f^{-1}(\{1\}) = \{-1, 1\}$

Relations

A **relation** R from a set A to a set B is a subset of $A \times B$. If $(a, b) \in R$, we say a and b are **in relation**, sometimes written aRb.

- A relation describes pairs of elements that are (surprise, surprise) related in some fashion
- To give a sense of how general this is, note every function is a relation: $aRb \Leftrightarrow f(a) = b$. (Formal definition of functions in terms of relations?)
- What is the difference between functions and relations? 2 things...

Here are a few examples. We'll take $A = B = \mathbb{R}$.

- The "$=$" relation: In this case, $R = \{(x, x), x \in \mathbb{R}\}$.
- The "$<$" relation: here, $R = \{(x, y) | x < y\}$.
- The "$\leq$" relation: here, $R = \{(x, y) | x \leq y\}$.

Relations could also be really abstract. Take P to be the set of professors and S to be the set of students. We can define the *advising* relation R if student s is advised by professor p and denote it by pRs

Relations (cont.)

A relation from A to itself is called a **binary relation**. Binary relations can satisfy a number of properties, such as:

- A relation is **transitive** if xRy and yRz implies xRz for all $x, y, z \in A$
- A relation is **symmetric** if xRy implies yRx for all $x, y \in A$
- A relation is **complete** if xRy or yRx for all $x, y \in A$

Relevance for econ? Preferences in consumer theory

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Motivation

- Econ isn't math, but it doesn't hurt to be formal when we can
- You'll see a lot of proofs in coursework and papers - even if you're not writing proofs day-to-day, important to know how to read and evaluate them

Proof Methods

Most proofs we will encounter are conditional statements of the form “if A then B”. Notation/terminology:

- A implies B
- $A \Rightarrow B$

Three general approaches to proving such statements

- **Direct proof:** Assume A, show that B holds
- **Proof by contrapositive:** Assume $\neg B$, show $\neg A$
- **Proof by contradiction:** Assume A and $\neg B$ and derive a logical contradiction

Direct Proofs

Example: if n and m are even integers, $n + m$ is even

- Since n and m are even, $n = 2k$ and $m = 2j$ for some integers j, k
- Therefore $n + m = 2k + 2j = 2(k + j)$ is even

Proofs by contraposition

An **implication** is equivalent to its **contrapositive**: $A \Rightarrow B$ if and only if $\neg B \Rightarrow \neg A$

- How would you disprove the statement $A \Rightarrow B$? What about $\neg B \Rightarrow \neg A$?
- In set-theoretic language, $A \subseteq B \Leftrightarrow B^c \subseteq A^c$

Example: if n^2 is even, then n is even

- Assume n is odd
- Then $n = 2k + 1$ for some integer k
- Then $n^2 = 4k^2 + 4k + 1$ is odd

Proof by contradiction

Proof by contradictions are powerful but can be weird

- Like with contraposition, we assume $\neg B$. We also assume A, so we have more "ammo" than we do with direct/contrapositive proofs
- The end of the proof is less clear - we need to show that A and $\neg B$ produce a logical inconsistency

Example: If x is rational and y is irrational, then $x + y$ is irrational

- Assume $x + y$ is rational. Then $x + y = \frac{p}{q}$ for some integers p, q.
- Since x is rational, $x = \frac{a}{b}$ for some integers a, b.
- Therefore $y = \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{qb}$ is rational, a contradiction

Generally best to use direct or contrapositive proofs (many "contradiction" proofs are contrapositives in disguise)

Common types of proofs

"If and only if": $A \Leftrightarrow B$

- Approach: Two separate proofs! Show $A \Rightarrow B$ and $B \Rightarrow A$

"For all" : (e.g. "for all $x \in X$, property P holds")

- Approach: take an arbitrary element of X and show that P holds

"There exists": (e.g. "there exists an $x \in X$ such that property P holds")

- Approach: constructive or nonconstructive. Example : Intermediate Value Theorem.

Uniqueness proofs (e.g. "there exists a unique x satisfying P")

- Typically take two elements x_1 and x_2 satisfying P and show they must be the same

General guidance

- Always have a roadmap in mind. Be clear about what you need to prove, and how you're planning to do it
- Understand your actors: what objects are you working with, and what do we know about them (either from assumptions or previous results)
    - Be comfortable with definitions
- If you're stuck, try a different approach (e.g. contradiction)
- The conditions of a theorem are clues - if you haven't used one yet you're likely missing something
- Proofs are rarely presented the way they're derived (this is true of almost all endeavors, academic or otherwise)