Lecture Notes - Correspondences

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Contents

This lecture first introduces the concepts of correspondences and their continuity, and then discuss two important results, Kakutani's fixed point theorem and Berge's theorem of maximum. You may refer to FMEA Chapter 14.1, 14.2, and 14.4.

1 Definitions

Definition 1.1. *A* ***correspondence*** *F from X to Y is a set-valued function that associates every element in X a subset of Y*[1]*.*

$$\begin{aligned} F :& X \rightrightarrows Y \\ & x \mapsto F(x) \end{aligned}$$

The set X is called the ***domain*** *of the correspondence F, and Y is called the* ***codomain*** *of F. $F(x)$ is called the* ***image*** *of point $x \in X$.*

You may consider the concept of correspondence as a generalization of functions, in the sense that $F(x)$ is a set in Y instead of an element in Y. Clearly, a single-valued correspondence $F : X \rightrightarrows Y$ can be viewed as a function from X to Y.

Listed below are some terminologies that we use to describe the properties of correspondences.

*The present lecture notes were largely based on math camp materials from Palaash Bhargava, Paul Koh, and Xuan Li. All errors in this document are mine. If you find a typo or an error, please send me an email at cesar.barilla@columbia.edu.

[1] By the definition above, a correspondence is a relation, and so this terminology is in fact a redundant one.

Definition 1.2. *A correspondence $F : X \rightrightarrows Y$ is said to be XXX-valued at $x_0 \in X$ iff $F(x_0)$ is a XXX set. If F is XXX-valued at all $x_0 \in X$, we say F is XXX-valued.*

These "XXX" can be

1. *non-empty*
2. *single (singleton)*
3. *open*
4. *closed*
5. *compact*
6. *convex*

Notice that the 3 - 5 above requires Y to be a metric space (Y, d_Y), and 6 requires Y to be a (real) vector space $(Y, +, \cdot)$.

1.1 Upper Hemi-continuity

Similar to functions, it is possible to talk about continuity of a correspondence if its domain and codomain are both metric spaces. However, there are two distinct notions of continuity for correspondences, known as *upper hemi-continuity* and *lower hemi-continuity*, and they capture different aspects of continuity of a correspondence. Let's first look at upper hemi-continuity.

Definition 1.3. *Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be* ***upper hemi-continuous (uhc) at*** *$x_0 \in X$ iff $\forall$ open set U in (Y, d_Y) s.t. $F(x_0) \subset U$, $\exists$ $\delta > 0$ s.t. $F\left(B_\delta(x_0)\right) \subset U$.*

The correspondence $F : X \rightrightarrows Y$ is said to be ***upper hemi-continuous (uhc)*** *iff it is upper hemi-continuous at x_0 for all $x_0 \in X$.*

The definition requires that whenever the open set U covers the entire image of the point x_0, then it must also entirely cover all nearby images. What is not allowed by uhc at x_0 is sudden appearance of large chunk of image when x deviates from x_0.

For example, consider the correspondence $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as[2]

$$F_1(x) := \begin{cases} \{0\}, \text{ if } x \le 0 \\ [-1, 1], \text{ if } x > 0 \end{cases}$$

Clearly F_1 fails to be uhc at 0, because if we let $U := \big(-1/2, 1/2\big)$, whenever x moves away a little from 0 to the right, the image $F_1(x)$ becomes $[-1, 1]$, which is not covered by U. The problem of this correspondence at 0 is that many new points suddenly appear when x deviate from 0 to the right, and this is a violation of uhc. Therefore, uhc can be intuitively interpreted as "no sudden appearance of large chunk of image when deviating from a point".

Consider a slightly different correspondence $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_2(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ [-1, 1], \text{ if } x \ge 0 \end{cases}$$

[2] In $\mathbb{R}^n$, we use the Euclidean metric d_2 by default.

The image of F_2 at 0 is $[-1,1]$, and so there is no sudden appearance of image when deviating from 0. Therefore, F_2 is uhc at 0. Clearly, F_2 is also uhc at all other points in $\mathbb{R}$, and so F_2 is uhc.

Uhc does not allow sudden appearance of image when deviating from a point, but it allows "smooth changes" in the image when deviating from a point, if the correspondence is closed-valued at this point. For example, consider the correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_3(x) := [x, x+1]$$

for any $x \in \mathbb{R}$. Under F_3, the image $F_3(x) = [x, x+1]$ changes "smoothly" when x changes, and it can be shown that F_3 is uhc.

Claim 1.4. *The correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined above is uhc.*

Proof. Take any $x_0 \in \mathbb{R}$. WTS: F_3 is uhc at x_0.

Take any open set $U \supset [x_0, x_0+1]$. WTS: $\exists\, \delta > 0$ s.t. $U \supset F(x)$ for any $x \in (x_0 - \delta, x_0 + \delta)$.

Because x_0 and $x_0 + 1$ are in the open set U, they are interior points of U, and so $\exists\, \delta > 0$ s.t.

$$(x_0 - \delta, x_0 + \delta) \subset U$$
$$(x_0 + 1 - \delta, x_0 + 1 + \delta) \subset U$$

Therefore, we have $(x_0 - \delta, x_0 + 1 + \delta) \subset U$.

For any $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$F(x) = [x, x+1] \subset (x_0 - \delta, x_0 + 1 + \delta) \subset U$$

□

However, when the correspondence is not closed-valued, then even smooth changes in the image may violate uhc. For example, consider a slightly different correspondence $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_4(x) := (x, x+1)$$

It can be shown that it is not uhc at any point in $\mathbb{R}$. To see this, for each $x_0 \in \mathbb{R}$, let $U := F_4(x_0) := (x_0, x_0 + 1)$, and U cannot cover $F(x)$ as long as $x \neq x_0$.

In applications, however, we almost always work with closed-valued correspondences, in which case uhc allows smooth changes, but does not allow sudden appearance of image.

For single-valued correspondences, uhc is equivalent to continuity of functions.

Proposition 1.5. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a single-valued correspondence $F : X \rightrightarrows Y$. Define $f : X \to Y$ as $f(x) := y$ s.t. $y \in F(x)$. Then F is uhc at $x_0 \in X$ iff f is continuous at x_0.*

This proof is straightforward, and is left as an exercise.

For compact-valued correspondences, there is a *sequential definition of uhc*, which is formulated in the following proposition[3].

Proposition 1.6. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a correspondence $F : X \rightrightarrows Y$, and let $x_0 \in X$. Then the following two statements are equivalent:*

(1) F is compact-valued at x_0, and F is uhc at x_0.

(2) For any sequence (x_n) in X convergent to x_0, any sequence (y_n) s.t. $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, there exists a subsequence (y_{n_k}) convergent to some $y_0 \in F(x_0)$.

[3]This is the definition of uhc in the book by SLP, who only study compact-valued correspondences.

Proof. (1) $\Rightarrow$ (2):

Take any sequence (x_n) in X convergent to x_0, any sequence (y_n) s.t. $y_n \in F(x_n)$ for each $n \in \mathbb{N}$.

WTS: There exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

For each $k \in \mathbb{N}$, consider the set

$$U_k := \bigcup_{y \in F(x_0)} B_{1/k}(y)$$

Because arbitrary union of opens is still open, we know that U_k is an open set. By construction $U_k \supset F(x_0)$, and so by uhc of F at x_0, there exists $\delta_k > 0$ s.t. $F\left(B_{\delta_k}(x_0)\right) \subset U_k$. Because $x_n \to x_0$, there exists N_k s.t. $x_n \in B_{\delta_k}(x_0)$, and thus $y_n \in U_k$ for any $n > N_k$.

Therefore, we can find a subsequence $\left(y_{n_k}\right)$ s.t. $y_{n_k} \in U_k$ for each $k \in \mathbb{N}$. By construction of U_k, for each k, there exists $z_k \in F(x_0)$ s.t. $d_Y\left(y_{n_k}, z_k\right) < 1/k$. Because F is compact-valued at x_0, we know that $F(x_0)$ is compact in (Y, d_Y). So there exists a subsequence $\left(z_{k_l}\right)$ convergent to some $y_0 \in F(x_0)$. So we have $d_Y\left(z_{k_l}, y_0\right) \to 0$, and

$$\begin{aligned} 0 \le d_Y\left(y_{n_{k_l}}, y_0\right) &\le d_Y\left(y_{n_{k_l}}, z_{k_l}\right) + d_Y\left(z_{k_l}, y_0\right) \\ &< \frac{1}{k_l} + d_Y\left(z_{k_l}, y_0\right) \to 0 + 0 = 0 \end{aligned}$$

Therefore, we have $d_Y\left(y_{n_{k_l}}, y_0\right) \to 0$, which means $y_{n_{k_l}} \to y_0$. Therefore, we have found a subsequence of (y_n) that converges to some point in $F(x_0)$.

(1) $\Leftarrow$ (2):

(a) WTS: F is compact-valued at x_0.

Take any sequence (y_n) in $F(x_0)$. WTS: There exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

Let $x_n = x_0$ for all $n \in \mathbb{N}$. Then we have $x_n \to x_0$ and $y_n \in F(x_n)$ for each $n \in \mathbb{N}$. By assumption, there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

(b) WTS: F is uhc at x_0.

Suppose that F is not uhc at x_0. Then $\exists\, U$ open in (X, d_X) s.t. $U \supset F(x_0)$, but $\forall\, \delta > 0$ we have $U \not\supset F\left(B_\delta(x_0)\right)$. Then for any $n \in \mathbb{N}$, we have $U \not\supset F\left(B_{1/n}(x_0)\right)$, i.e. there exists $x_n \in B_{1/n}(x_0)$ and $y_n \in F(x_n)$ s.t. $y_n \notin U$. Because $x_n \to x_0$, by assumption there exists a subsequence $\left(y_{n_k}\right)$ convergent to some $y_0 \in F(x_0)$.

Because $\left(y_{n_k}\right)$ is in $Y \backslash U$, which is closed in (Y, d_Y), we have $y_0 \in Y \backslash U$, and so $y_0 \notin F(x_0)$. Contradiction. $\square$

Without compact-valuedness, uhc alone does not imply property (2) in the proposition above. For example, consider $F_5 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_5(x) = (0, 1)$$

for any $x \in \mathbb{R}$. Clearly, F_5 is uhc everywhere, but it does not satisfy property (2) at any $x_0 \in \mathbb{R}$, since compact-valuedness is necessary for property (2).

1.2 Closed Graph Property

There is a concept, called *closed graph property*, that is closely related to uhc.

Definition 1.7. *Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to have* ***closed graph property (cgp) at*** *$x_0 \in X$ iff $\forall$ sequence (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$, we have $y_0 \in F(x_0)$.*

The correspondence $F : X \rightrightarrows Y$ is said to have ***closed graph property (cgp)*** *iff it has closed graph property at x_0 for all $x_0 \in X$.*

Clearly, cgp implies closed-valuedness.

Claim 1.8. *Let (X, d_X) and (Y, d_Y) be metric spaces. If the correspondence $F : X \rightrightarrows Y$ is cgp at $x_0 \in X$, then it is closed-valued at x_0.*

Proof. Take any sequence (y_n) in $F(x_0)$ convergent to $y_0 \in Y$. WTS: $y_0 \in F(x_0)$.

Let $x_n = x_0$ for all $n \in \mathbb{N}$, then we have $x_n \to x_0$, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$. By cgp, we have $y_0 \in F(x_0)$. □

The **graph**[4] of a correspondence $F : X \rightrightarrows Y$ is defined as

$$Gr(F) := \left\{(x, y) \in X \times Y : y \in F(x)\right\}$$

For a correspondence $F : X \rightrightarrows Y$, where (X, d_X) and (Y, d_Y) are metric spaces, the name of the property "closed graph property" comes from the fact that F has cgp (everywhere in X) iff its graph is closed in $(X \times Y, d_{X\times Y})$, where the metric for the product space is defined as

$$d_{X\times Y}\left((x, y), (x', y')\right) := \sqrt{\left[d_X(x, x')\right]^2 + \left[d_Y(y, y')\right]^2}$$

for any $(x, y), (x', y') \in X \times Y$.

It can be shown that $d_{X\times Y}$ as defined above is a valid metric for $X \times Y$. Also, we can show that $(x_n, y_n) \to (x_0, y_0)$ in $(X \times Y, d_{X\times Y})$ iff $x_n \to x_0$ in (X, d_X) and $y_n \to y_0$ in (Y, d_Y), and this is left as an exercise.

Claim 1.9. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then a correspondence $F : X \rightrightarrows Y$ has cgp iff $Gr(F)$ is closed in $(X \times Y, d_{X\times Y})$.*

Proof. $\Rightarrow$:

Take any $\left((x_n, y_n)\right)$ in $Gr(F)$ that is convergent to $(x_0, y_0) \in X \times Y$. WTS: $(x_0, y_0) \in Gr(F)$.

Because $(x_n, y_n) \to (x_0, y_0)$, we have $x_n \to x_0$ and $y_n \to y_0$. Because $(x_n, y_n) \in Gr(F)$ for all n, we have $y_n \in F(x_n)$ for all n. Because F has cgp, we know that F has cgp at x_0, and so $y_0 \in F(x_0)$, which implies $(x_0, y_0) \in Gr(F)$.

$\Leftarrow$:

Take any $x_0 \in X$. WTS: F has cgp at x_0.

Take any (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$. WTS: $y_0 \in F(x_0)$.

Because $x_n \to x_0$ and $y_n \to y_0$, we have $(x_n, y_n) \to (x_0, y_0)$ in $(X \times Y, d_{X\times Y})$. Because $y_n \in F(x_n)$ for each n, we have $(x_n, y_n) \in Gr(F)$ for each n. Because $Gr(F)$ is closed in $(X \times Y, d_{X\times Y})$, we have $(x_0, y_0) \in Gr(F)$. □

Closed graph property is closely related to uhc, and their relation is formulated by the following two propositions.

[4]This is in fact a redundant definition since $Gr(F) = F$, if we view F as a relation from $X \times Y$.

Proposition 1.10. *Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ is uhc at $x_0 \in X$, and is closed-valued at x_0, then F has cgp at x_0.*

Proof. Take any sequence (x_n) in X convergent to x_0, $y_n \in F(x_n)$ for each $n \in \mathbb{N}$, and $y_n \to y_0 \in Y$.

WTS: $y_0 \in F(x_0)$.

Suppose $y_0 \notin F(x_0)$, i.e. $y_0 \in Y \backslash F(x_0)$. Because F is closed-valued at x_0, $Y \backslash F(x_0)$ is open in (Y, d_Y), and so $\exists\ \varepsilon > 0$ s.t. $B_{2\varepsilon}(y_0) \subset Y \backslash F(x_0)$. And the "closed ball"

$$\bar{B}_\varepsilon(y_0) := \{y \in Y : d_Y(y, y_0) \leq \varepsilon\}$$

is contained in $B_{2\varepsilon}(y_0)$ and therefore in $Y \backslash F(x_0)$, and therefore $F(x_0) \subset Y \backslash \bar{B}_\varepsilon(y_0)$. It can be shown that a closed ball is a closed set (exercise), and $F(x_0)$ is covered by the open set $Y \backslash \bar{B}_\varepsilon(y_0)$. By uhc of F at x_0, $\exists\ \delta > 0$ s.t. $F(B_\delta(x_0)) \subset Y \backslash \bar{B}_\varepsilon(y_0)$.

Because $x_n \to x_0$ and $y_n \to y_0$, there exists $\hat{n}$ s.t. $x_{\hat{n}} \in B_\delta(x_0)$ and $y_{\hat{n}} \in \bar{B}_\varepsilon(y_0)$. However, because $F(B_\delta(x_0)) \subset Y \backslash \bar{B}_\varepsilon(y_0)$, we have $y_{\hat{n}} \in F(x_{\hat{n}}) \subset F(B_\delta(x_0)) \subset Y \backslash \bar{B}_\varepsilon(y_0)$, which contradicts $y_{\hat{n}} \in \bar{B}_\varepsilon(y_0)$. □

The result above states that uhc implies cgp if we have closed-valuedness. Without closed-valuedness, this implication does not hold since a uhc correspondence may not have closed-valuedness. For example, consider F_5 as previously defined. Clearly, F_5 is uhc everywhere, but it does not have cgp anywhere, since closed-valuedness is necessary for cgp.

A correspondence $F : X \rightrightarrows Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is said to be **locally bounded at** x_0 iff $\exists\ \delta > 0$ and a compact set K in (Y, d_Y) s.t. $F(B_\delta(x_0)) \subset K$.

The next proposition works in the other direction.

Proposition 1.11. *Let (X, d_X) and (Y, d_Y) be metric spaces. If a correspondence $F : X \rightrightarrows Y$ has cgp at $x_0 \in X$, and F is locally bounded at x_0, then F is uhc at x_0.*

The proof of this proposition is similar to the proof of Proposition 1.6, part (b) of the direction "(1) $\Leftarrow$ (2)".

Proof. Suppose that F is not uhc at x_0. Then $\exists U$ open in (Y, d_Y) s.t. $F(x_0) \subset U$, but $\forall \delta > 0$ we have $F(B_\delta(x_0)) \not\subset U$. Then for any $n \in \mathbb{N}$, we have $F(B_{1/n}(x_0)) \not\subset U$, i.e. there exists $x_n \in B_{1/n}(x_0)$ and $y_n \in F(x_n)$ s.t. $y_n \notin U$. By assumption there exists $\hat{\delta} > 0$ and compact set K in (Y, d_Y) s.t. $F(B_{\hat{\delta}}(x_0)) \subset K$. By construction, we have $x_n \to x_0$, and so $\exists N$ s.t. $x_n \in B_{\hat{\delta}}(x_0)$ and so $y_n \in K$ for any $n > N$.

By sequential compactness of K, there exists a subsequence (y_{n_k}) of $(y_n)_{n>N}$ convergent to some $y_0 \in K$. Because the subsequence $(y_{n_k}) \subset Y \backslash U$, which is closed, we have $y_0 \in Y \backslash U$. However, because F has cgp at x_0, and $x_{n_k} \to x_0$, $y_{n_k} \in F(x_{n_k})$, $y_{n_k} \to y_0$, we have $y_0 \in F(x_0) \subset U$. Contradiction.

□

The result above states that cgp implies uhc if we have local boundedness. Without local boundedness, cgp does not imply uhc. For example, consider $F_6 : \mathbb{R} \rightrightarrows [0, 1)$ defined as

$$F_6(x) = \begin{cases} \{e^x\}, & x < 0 \\ \{0\}, & x \geq 0 \end{cases}$$

which is clearly not uhc at 0. However, F_6 has cgp at 0. Notice that 1 is not in the codomain, and so when x_n converges to 0 from the negative real line, $y_n \in F(x_n)$ does not converge. This is not a violation of the proposition above, because F_6 is not locally bounded at 0. Notice again that 1 is not in the codomain, and so we cannot find a compact set K in $\big([0,1), d_2\big)$ to bound all images of points nearby 0.

Another example is $F_7 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_6(x) = \begin{cases} \{1/x\}, \, x \neq 0 \\ \{0\}, \, x = 0 \end{cases}$$

As a consequence of the two propositions above, under closed-valuedness and local boundedness, uhc and cgp are equivalent.

1.3 Lower Hemi-continuity

Now let's define lower hemi-continuity.

Definition 1.12. *Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be* ***lower hemi-continuous (lhc) at*** *$x_0 \in X$ iff $\forall$ open set U in (Y, d_Y) s.t. $F(x_0) \cap U \neq \emptyset$, $\exists$ $\delta > 0$ s.t. $F(x) \cap U \neq \emptyset$ for any $x \in B_\delta(x_0)$.*

The correspondence $F : X \rightrightarrows Y$ is said to be ***lower hemi-continuous (lhc)*** *iff it is lower hemi-continuous at x_0 for all $x_0 \in X$.*

The definition requires that whenever the open set U covers a part of the image of the point x_0, then it must also cover a part of all nearby images. What is not allowed by lhc at x_0 is sudden disappearance of large chunk of image when x deviates from x_0.

For example, consider the correspondence $F_2 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_2(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ [-1, 1], \text{ if } x \geq 0 \end{cases}$$

as previously defined. Clearly F_2 fails to be lhc at 0, because if we let $U := \big(1/2, 3/2\big)$, whenever x moves away a little from 0 to the left, the image $F_2(x)$ becomes $\{0\}$, which does not share an intersection with U. The problem of this correspondence at 0 is that many points suddenly disappear when x deviate from 0 to the left, and this is a violation lhc. Therefore, lhc can be intuitively interpreted as "no sudden disappearance of large chunk of image when deviating from a point".

Consider the slightly different correspondence $F_1 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_1(x) := \begin{cases} \{0\}, \text{ if } x \leq 0 \\ [-1, 1], \text{ if } x > 0 \end{cases}$$

as previously defined. The image of F_1 at 0 is $\{0\}$, and so there is no sudden disappearance of image when deviating from 0. Therefore, F_1 is lhc at 0. Clearly, F_1 is also lhc at all other points in $\mathbb{R}$, and so F_1 is lhc.

Lhc does not allow sudden disappearance of image when deviating from a point, but it allows "smooth changes" in the image when deviating from a point. For example, consider the correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$

$$F_3(x) := [x, x+1]$$

for any $x \in \mathbb{R}$ as previously defined. Under F_3, the image $F_3(x) = [x, x+1]$ changes "smoothly" when x changes, and it can be shown that F_3 is lhc.

Claim 1.13. *The correspondence $F_3 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined above is lhc.*

Proof. Take any $x_0 \in \mathbb{R}$. WTS: F_3 is lhc at x_0.

Take any open set U s.t. $[x_0, x_0 + 1] \cap U \neq \emptyset$.

WTS: $\exists\ \delta > 0$ s.t. $[x, x + 1] \cap U \neq \emptyset$ for any $x \in (x_0 - \delta, x_0 + \delta)$.

Let $\hat{x} \in [x_0, x_0 + 1] \cap U$. Because U is open, there exists $\delta > 0$ s.t. $(\hat{x} - \delta, \hat{x} + \delta) \subset U$.

Take any $x \in (x_0 - \delta, x_0 + \delta)$. By construction, we have $x - x_0 \in (-\delta, \delta)$, and so

$$\hat{x} + (x - x_0) \in (\hat{x} - \delta, \hat{x} + \delta) \subset U$$

Because $\hat{x} \in [x_0, x_0 + 1]$, we have

$$\begin{aligned} \hat{x} + (x - x_0) &\in \big[x_0 + (x - x_0), x_0 + (x - x_0) + 1\big] \\ &= [x, x + 1] \end{aligned}$$

Therefore, we have $\hat{x} + (x - x_0) \in [x, x + 1] \cap U$, and so $[x, x + 1] \cap U \neq \emptyset$. $\square$

Lhc allows smooth changes in the image, regardless of whether the correspondence is closed valued. If we consider a slightly different correspondence $F_4 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as

$$F_4(x) := (x, x + 1)$$

for any $x \in \mathbb{R}$, a slightly modification of the proof above can show that F_4 is also lhc.

For single-valued correspondences, lhc is equivalent to continuity of functions.

Proposition 1.14. *Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a single-valued correspondence $F : X \rightrightarrows Y$. Define $f : X \to Y$ as $f(x) := y$ s.t. $y \in F(x)$. Then F is lhc at $x_0 \in X$ iff f is continuous at x_0.*

This proof is straightforward, and is left as an exercise.

The following proposition provides the *sequential definition of lhc.*

Proposition 1.15. *Let (X, d_X) and (Y, d_Y) be metric spaces. A correspondence $F : X \rightrightarrows Y$ is lhc at $x_0 \in X$, iff for any $y_0 \in F(x_0)$ and sequence (x_n) in X convergent to x_0, there exists $N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0.*

In the proposition above, we start to construct the sequence (y_n) starting from $n = N + 1$, because $F(x_n)$ may be empty for small n's.

Proof. $\Rightarrow$:

Take any $y_0 \in F(x_0)$ and sequence (x_n) in X convergent to x_0.

WTS: $\exists\ N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0.

For each $k \in \mathbb{N}$, we have $y_0 \in F(x_0) \cap B_{1/k}(y_0)$, and so $F(x_0) \cap B_{1/k}(y_0) \neq \emptyset$. By lhc, $\exists\ \delta_k > 0$ s.t. for any $x \in B_{\delta_k}(x_0)$, we have $F(x) \cap B_{1/k}(y_0) \neq \emptyset$.

Because $x_n \to x$, $\exists\ N \in \mathbb{N}$ s.t. $x_n \in B_{\delta_1}(x_0)$ for any $n > N$.

For each $n > N$, arbitrarily take

$$y_n \in \bigcap_{k \in \left\{k' \in \mathbb{N} : x_n \in B_{\delta_{k'}}(x_0)\right\}} \Big[F(x_n) \cap B_{1/k}(y_0)\Big]$$

This is possible because $F(x_n) \cap B_{1/k}(y_0) \neq \emptyset$ whenever $x_n \in B_{\delta_k}(x_0)$.

Now I want to show that $(y_n)_{n>N}$ converges to y_0.

Take any $\varepsilon > 0$. $\exists\, K$ s.t. $1/k < \varepsilon$ for any $k > K$. Because $x_n \to x_0$, $\exists\, \hat{N} > N$ s.t. $x_n \in B_{\delta_K}(x_0)$ for any $n > \hat{N}$. Therefore for any $n > \hat{N}$, we have $x_n \in B_{\delta_K}(x_0)$, which implies $y_n \in B_{1/K}(y_0)$, which in turn implies $y_n \in B_\varepsilon(y_0)$.

$\Leftarrow$:

Suppose, by contradiction, that $\exists$ open set U in (Y, d_Y) s.t. $F(x_0) \cap U \neq \emptyset$, but $\forall\, \delta > 0$, $\exists\, x \in B_\delta(x_0)$ s.t. $F(x) \cap U = \emptyset$. This implies that for any $n \in \mathbb{N}$, $\exists\, x_n \in B_{1/n}(x_0)$ s.t. $F(x_n) \cap U = \emptyset$, i.e. $F(x_n) \subset Y \backslash U$.

By construction, we have $x_n \to x_0$. Arbitrarily take $y_0 \in F(x_0) \cap U$, and by assumption there exists $N \in \mathbb{N}$ and $y_n \in F(x_n)$ for any $n > N$ s.t. the sequence $(y_n)_{n>N}$ converges to y_0. Because $y_n \in F(x_n) \subset Y \backslash U$ for any $n > N$, and $Y \backslash U$ is closed in (Y, d_Y) since U is open, we have $y_0 \in Y \backslash U$. This contradicts the construction of y_0. $\square$

As we have discussed, uhc for closed-valued correspondences means no sudden appearance of image when deviating from a point, while uhc means no sudden disappearance of image when deviating from a point. Therefore, we might expect a relation between F being uhc and F^c being lhc. In fact, we have one direction, but not the other.

For a correspondence $F : X \rightrightarrows Y$, let's define its complement $F^c : X \rightrightarrows Y$ as

$$F^c(x) := Y \backslash F(x)$$

for any $x \in X$. Again, this is a redundant definition if we realize that F is essentially a subset of $X \times Y$.

Proposition 1.16. *Let (X, d_X) and (Y, d_Y) be metric spaces, and consider a correspondence $F : X \rightrightarrows Y$. If F^c is uhc at $x_0 \in X$, then F is lhc at x_0.*

The proof is left as an exercise.

However, F^c being lhc does not imply F being uhc, even if we further assume F to be compact-valued. For example, consider the correspondence $F_7 : \mathbb{R} \rightrightarrows \mathbb{R}$ defined as:

$$F_8(x) := \begin{cases} \{0\}, \text{ if } x < 0 \\ \{1\}, \text{ if } x \geq 0 \end{cases}$$

Clearly F is compact-valued, and $F(x)$ is not uhc at 0. However, F^c is lhc at all $x_0 \in \mathbb{R}$.

Finally, a correspondence is said to be continuous iff it is both uhc and lhc.

Definition 1.17. *Let (X, d_X) and (Y, d_Y) be metric spaces. The correspondence $F : X \rightrightarrows Y$ is said to be **continuous at** $x_0 \in X$ iff F is both uhc and lhc at x_0. The correspondence F is said to be **continuous** iff F is continuous at x_0 for all $x_0 \in X$.*

2 Kakutani's Fixed Point Theorem

Definition 2.1. *A correspondence F from X to X itself is called a **self-correspondence**.*

*For a self-correspondence $F : X \rightrightarrows X$, a point $x^* \in X$ is called a **fixed point** of F iff $x^* \in F(x^*)$.*

When the self-correspondence F is single-valued, clearly $x^* \in X$ is a fixed point of F iff $F(x^*) = \{x^*\}$, which is consistent with notion of fixed points for functions. Therefore, the definition above can be considered as a generalization of the notion of fixed points to correspondences.

Theorem 2.2 (Kakutani's Fixed Point)**.** *Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, compact-valued, convex-valued, and uhc, then there exists a fixed point $x^* \in X$ of F.*

In the theorem above, compactness is w.r.t. the metric space $(\mathbb{R}^n, d_2)$, and convexity is w.r.t. the vector space $(\mathbb{R}^n, +, \cdot)$ over $\mathbb{R}$, where $+$ and $\cdot$ are the usually defined vector addition and scalar multiplication for real vectors.

If F is single-valued, then nonempty-valuedness, compact-valuedness, and convex-valuedness of F holds trivially, and uhc reduces to the continuity of functions. So the theorem above reduces to Brouwer's fixed point theorem. Therefore, Kakutani's fixed point theorem should be viewed as a generalization of Brouwer's fixed point theorem.

Because the codomain X of F is compact in the theorem, compact-valuedness is equivalent to closed-valuedness, and so we can replace the compact-valuedness assumption by closed-valuedness.

Again because the codomain X is compact, (compact-valuedness + uhc) is equivalent to cgp. To see this, the direction "$\Rightarrow$" is given by Proposition 1.10, and the other direction "$\Leftarrow$" is given by Proposition 1.11, since local boundedness holds trivially. Therefore we have the following corollary.

Corollary 2.3. *Let X be a nonempty, compact, and convex set in $\mathbb{R}^n$. If the self-correspondence $F : X \rightrightarrows X$ is nonempty-valued, convex-valued, and has cgp, then there exists a fixed point $x^* \in X$ of F.*

Kakutani's fixed point theorem plays the central role in the proof of the existence of Walrasian equilibrium in general equilibrium theory, and also in the proof of the existence of Nash equilibrium in non-cooperative game theory.

3 Berge's Theorem of Maximum

Theorem 3.1. *[Berge's Theorem of Maximum] Let (X, d_X) and (A, d_A) be metric spaces. Let $f : X \times A \to \mathbb{R}$ be a continuous function w.r.t. the metric $d_{X\times A}$. Let $\alpha_0 \in A$, and suppose that the correspondence $D : A \rightrightarrows X$ is nonempty-valued at α_0, compact-valued at α_0, and continuous at α_0.*

Define the correspondence $X^* : A \rightrightarrows X$ as

$$X^*(\alpha) := \arg\max_{x\in X} \left\{ f(x, \alpha) : x \in D(\alpha) \right\}$$

for any $\alpha \in A$.

Let $\hat{A} := \left\{ \alpha \in A : X^*(\alpha) \neq \emptyset \right\}$, and define the function $f^* : \hat{A} \to \mathbb{R}$ as

$$f^*(\alpha) = \max_{x\in X} \left\{ f(x, \alpha) : x \in D(\alpha) \right\}$$

Then X^* is nonempty-valued at α_0, compact-valued at α_0, and uhc at α_0, and f^* is continuous at α_0.

In the theorem above, the maximization problem we are looking at is a parameterized problem

$$\max_{x\in X} f(x, \alpha) \text{ s.t. } x \in D(\alpha)$$

where both the objective function f and the constraint set D depend on the parameter α. The theorem states that if the objective function f is continuous, and the constraint set D is nonempty- and compact-valued, and is both uhc and lhc in the parameter α, then the set of maximizers X^* is compact and uhc in α, and the maximum value f^* is also continuous in α.

Proof. Let's prove the theorem in three steps:

Step 1: X^* is nonempty-valued at α_0

Because $f : X \times A \to \mathbb{R}$ is continuous w.r.t. $d_{X\times A}$, clearly the function $f_{\alpha_0} : X \to \mathbb{R}$ defined as

$$f_{\alpha_0}(x) = f(x, \alpha_0), \text{ for any } x \in X$$

is continuous w.r.t. d_X (exercise). Because $D(\alpha_0)$ is nonempty and compact by assumption, Weierstrass theorem implies that

$$X^*(\alpha_0) := \arg\max_{x\in X} \left\{ f_{\alpha_0}(x) : x \in D(\alpha_0) \right\}$$

is nonempty. So we know that X^* is nonempty-valued at α_0.

Step 2: X^* is compact-valued at α_0 and uhc at α_0

Let's show this using Proposition 1.6.

Take any sequence (α_n) in A convergent to α_0, any sequence (x_n) s.t. $x_n \in X^*(\alpha_n)$ for each $n \in \mathbb{N}$.

WTS: $\exists$ subsequence $\left(x_{n_k}\right)$ convergent to some $x_0 \in X^*(\alpha_0)$.

Because $x_n \in X^*(\alpha_n) \subset D(\alpha_n)$ for each n, and because D is compact valued at α_0 and uhc at α_0, by Proposition 1.6, $\exists$ subsequence $\left(x_{n_k}\right)$ convergent to some $x_0 \in D(\alpha_0)$.

Take the x_0 found this way, and it is sufficient to show that $x_0 \in X^*(\alpha_0)$, i.e. $f(x_0, \alpha_0) \geq f(z, \alpha_0)$ for any $z \in D(\alpha_0)$.

Take any $z \in D(\alpha_0)$. WTS: $f(x_0, \alpha_0) \geq f(z, \alpha_0)$

Because D is lhc at α_0 and $\alpha_{n_k} \to \alpha_0$, by sequential definition of lhc (Proposition 1.15), there exists $K \in \mathbb{N}$ and $z_k \in D\left(\alpha_{n_k}\right)$ for each $k > K$, s.t. $z_k \to z$.

Because $x_{n_k} \to x_0$, $\alpha_{n_k} \to \alpha_0$, we have $\left(x_{n_k}, \alpha_{n_k}\right) \to (x_0, \alpha_0)$ in $(X \times A, d_{X\times A})$. Because f is continuous w.r.t. $d_{X\times A}$, we have $f\left(x_{n_k}, \alpha_{n_k}\right) \to f(x_0, \alpha_0)$.

Because $z_k \to x_0$, $\alpha_{n_k} \to \alpha_0$, we have $\left(z_k, \alpha_{n_k}\right) \to (x_0, \alpha_0)$ in $(X \times A, d_{X\times A})$. Because f is continuous w.r.t. $d_{X\times A}$, we have $f\left(z_k, \alpha_{n_k}\right) \to f(z, \alpha_0)$.

For each k, we have $f\left(x_{n_k}, \alpha_{n_k}\right) \geq f\left(z_k, \alpha_{n_k}\right)$ because $x_{n_k} \in X^*\left(\alpha_{n_k}\right)$. Therefore we have $f(x_0, \alpha_0) \geq f(z, \alpha_0)$.

Step 3: f^* is continuous at α_0

By (1), we have $\alpha_0 \in \hat{A}$, i.e. α_0 is in the domain of the function f^*. Therefore it makes sense to talk about continuity of f^* at α_0.

Let's show the continuity of f^* using the sequential definition of continuous functions.

Take any sequence (α_n) in $\hat{A}$ convergent to α_0. WTS: $f^*(\alpha_n) \to f^*(\alpha_0)$.

Suppose $f^*(\alpha_n) \nrightarrow f^*(\alpha_0)$. Then there exist $\hat{\varepsilon} > 0$ s.t. for any $N \in \mathbb{N}$ there exists $\hat{n} > N$ s.t. $\left|f^*(\alpha_{\hat{n}}) - f^*(\alpha_0)\right| \geq \hat{\varepsilon}$. Then we can find a subsequence $\left(\alpha_{n_k}\right)$ s.t. $\left|f^*\left(\alpha_{n_k}\right) - f^*(\alpha_0)\right| \geq \hat{\varepsilon}$ for each k.

For each k, because $\alpha_{n_k} \in \hat{A}$, the set $X^*\left(\alpha_{n_k}\right)$ is nonempty. Arbitrarily take some $x_k \in X^*\left(\alpha_{n_k}\right)$. Because X^* is compact-valued at α_0 and uhc at α_0, by Proposition 1.6, there exists a subsequence $\left(x_{k_l}\right)$ convergent to some point $x_0 \in X^*(\alpha_0)$. Because $\alpha_{n_{k_l}} \to \alpha_0$, $x_{k_l} \to x_0$, we have $\left(x_{k_l}, \alpha_{n_{k_l}}\right) \to (x_0, \alpha_0)$, and so $f\left(x_{k_l}, \alpha_{n_{k_l}}\right) \to f(x_0, \alpha_0)$. Because $x_{k_l} \in X^*\left(\alpha_{n_{k_l}}\right)$ and $x_0 \in X^*(\alpha_0)$, we have $f\left(x_{k_l}, \alpha_{n_{k_l}}\right) = f^*\left(\alpha_{n_{k_l}}\right)$ and $f(x_0, \alpha_0) = f^*(\alpha_0)$, and therefore

$$f^*\left(\alpha_{n_{k_l}}\right) \to f^*(\alpha_0)$$

However, we have $\left|f^*\left(\alpha_{n_k}\right) - f^*(\alpha_0)\right| \geq \hat{\varepsilon}$ for each k, by construction of the subsequence $\left(\alpha_{n_k}\right)$. Contradiction. $\square$

By Theorem of Maximum, we can only conclude that the set of maximizers X^* is uhc in the parameter α. In fact, X^* may easily fail to be lhc, even when the objective function f and the constraint correspondence D are continuous in the parameter α. For example, consider the following problem:

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \alpha x_1 + x_2 \text{ s.t. } p_1x_1 + p_2x_2 \leq m$$

where the parameters $\alpha > 0, p_1, p_2, m > 0$. Clearly, the objective function $f : \mathbb{R}^2_+ \times \mathbb{R}_+$ defined as

$$f(x, \alpha, p_1, p_2, m) := \alpha x_1 + x_2$$

is continuous. The constraint correspondence $D : \mathbb{R}_+ \times \mathbb{R}^2_{++} \times \mathbb{R}_{++} := S \rightrightarrows \mathbb{R}^2_+$ defined as

$$D(\alpha, p_1, p_2, m) := \left\{x \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 \leq m\right\}$$

is nonempty- and compact-valued, and continuous at all $(\alpha, p_1, p_2, m) \in S$. Therefore the assumptions of the Theorem of Maximum are satisfied. However, it is not difficult to see that the set of maximizers $X^* : \mathbb{R}_+ \rightrightarrows \mathbb{R}^2_+$ is

$$X^*(\alpha, p_1, p_2, m) := \begin{cases} \left\{\left(0, \frac{m}{p_2}\right)\right\}, \text{ if } p_1 > \alpha p_2 \\ \left\{x \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 = m\right\}, \text{ if } p_1 = \alpha p_2 \\ \left\{\left(\frac{m}{p_1}, 0\right)\right\}, \text{ if } 0 < p_1 < \alpha p_2 \end{cases}$$

which is clearly uhc but not lhc at the point (α, p_1, p_2, m) where $p_1 = \alpha p_2$.

If in addition to the assumptions in the Theorem 3.1, D is convex-valued and f is strictly concave in x, then X^* is single-valued (see Proposition 2.3 in Lecture 5). That is, there is a unique maximizer for any α that satisfies the relevant conditions. In this case, we can think of X^* as a continuous function x^*, such that $X^*(\alpha) = \{x^*(\alpha)\}$. Moreover, we have the following result if the parameter space A and the space of the choice variable X are both Eulidean spaces:

Lemma 3.2. ** Let A be a set in $(\mathbb{R}^l, d_2)$, X be a set in $(\mathbb{R}^m, d_2)$. Let $f : X \times A \to \mathbb{R}$ be a continuous function w.r.t. the metric $d_{X\times A}$ and $f(x, \alpha_0)$ is strictly quasi-concave in x for some $\alpha_0 \in A$. Let $D : A \rightrightarrows X$ be non-empty at α_0, compact-valued at α_0, convex-valued at α_0, and continuous at α_0. Then for any $\varepsilon > 0$, there exists $\delta_{\alpha_0} > 0$ (which might depend on α_0) s.t. for $x \in D(\alpha_0)$,*

$$|f^*(\alpha_0) - f(x, \alpha_0)| < \delta_{\alpha_0}$$

implies

$$d_2(x^*(\alpha_0), x) < \varepsilon.$$

If the conditions on D and f hold for $\forall\alpha \in A$, and A is compact, then for any $\varepsilon > 0$, there exists $\delta > 0$ (independent of α) s.t. for any $\alpha \in A$, $x \in D(\alpha)$,

$$|f^*(\alpha) - f(x, \alpha)| < \delta \Rightarrow d_2(x^*(\alpha), x) < \varepsilon.$$

This lemma says when we consider a maximization problem in a real space, if in addition to the conditions on the objective function f and the constraint-set correspondence D in Theorem 3.1, we also have f is strictly quasi-concave in x, and D is convex-valued, then for given $\alpha \in A$, as long as the value of the objective function evaluated at $x \in D(\alpha)$ is close enough to that evaluated at the maximizer $x^*(\alpha)$, then the point x can get arbitrarily close to the maximizer.

Now we have the following theorem about the maximizers of a convergent sequence of parametrized functions on a parameterized constraint set.

Theorem 3.3. ** Assume A, X, D satisfy the relevant conditions for $\forall \alpha \in A$ as in the Lemma above. Let $\{f_n\}$ be a sequence of continuous functions $f_n : X \times A \to \mathbb{R}$. Assume that for each n and $\alpha \in A$, $f_n(\cdot, \alpha)$ is strictly concave. Assume $f : X \times A \to \mathbb{R}$ is also strictly concave in x and continuous. Let $f_n \to f$ uniformly[5]. Let*

$$f_n^*(\alpha) = \max_{x \in D(\alpha)} f_n(x, \alpha), n = 1, 2, ...$$

$$f^*(\alpha) = \max_{x \in D(\alpha)} f(x, \alpha)$$

*Then $f_n^* \to f^*$ **pointwise**. If A is compact, then $f_n^* \to f^*$ **uniformly**.*

This theorem states that when we consider maximization problems in a real space, under certain conditions, the value function of objective function f_n under certain constraints (common to n) gets close to the value function of f under the same constraint, if (f_n) as a sequence of functions gets close to f.

[5] $f_n \to f$ uniformly iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\sup_{\alpha \in A, x \in D(\alpha)} |f_n(x, \alpha) - f(x, \alpha)| < \varepsilon$. That is, uniform convergence of functions is convergence under the uniform metric defined for the space of functions: $d(f, g) := \sup_{x \in D} |f(x) - g(x)|$.