Math Camp Notes: Integration

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1 Integration

There are several ways to motivate the notion of an integral. The most common 2 ways you are likely to encounter are the notions of a Riemann integral and a Lebesgue integral. For the purpose of these notes we will focus exclusively on the Riemann integral; which is what you already know from high school maths or introductory calculus courses in college.

2 (Riemann) Integration

Geometrically, the integral of a function f over an interval $[a, b]$ is the area under the curve of the graph of f; more formally, it is defined by the following Riemann sum: let $a = x_0 \leq x_1 \leq x_2 \leq ... \leq x_{n-1} \leq x_n = b$, and define $\Delta_i = x_i - x_{i-1}$. The Riemann sum of f with respect to this partition is

$$\sum_{i=1}^{n} f(x_i)\Delta_i$$

This is nothing but approximating the area under a curve with the sum of areas of n rectangles. Loosely, if this sum approaches some limit (more formally if the limits of the Riemann upper and lower sums are the same) as the size of the partition goes to 0, we call the limiting quantity the integral of f from a to b, written

$$\int_a^b f(x)dx$$

The dx at the end of the integral has an interesting history, but for our purposes just think of it as indicating which variable we are integrating with respect to. This is typically obvious in single-variable integrals, but taking care with variable names becomes important as integrals become more complicated.

[Note, there are many extensions of integrals that would take us too far afield in this course. The Lebesgue integral as mentioned above is a common extension of the Riemann integral which is able to integrate a more general class of functions.]

*These notes are mostly taken from David Thompson's notes from previous math camps.

Note that the integral is a weighted sum (the integral notation is an elongated S, meaning sum). In economics, we encounter integrals most often when dealing with probability distributions. One typical technique for dealing with uncertainty (for example, over future states of the world, or the type of agent you're interacting with) is to posit a probability distribution over the uncertain states. A utility-maximizing agent's expected value will then look like a weighted sum over possible states, so to make an optimal choice he needs to maximize the value of an integral. We'll discuss the maximization aspect of these problems later on, but it's important to be facile with the properties of integrals for first-year coursework.

2.1 Integrability

Is every function integrable? Consider the following example: integrating $f(x) = 1/x^2$ over $[0, 1]$. Let's partition this interval at $1/2, 1/3, 1/4, ...$ Each segment contributes

$$\underbrace{n^2}_{f(x_n)} \underbrace{\frac{1}{n(n+1)}}_{\Delta_n} = \frac{n}{n+1} \geq \frac{1}{2}$$

Thus we have an infinite sum of terms that are all greater than $1/2$, which cannot possibly converge to a limit.

Here's a result that will cover most situations you'll encounter in coursework:

Lemma 2.1. *If f is a bounded on a compact interval $[a, b]$, then f is Riemann integrable if and only if f is continuous almost everywhere on $[a, b]$.*

Intuitively, this says that any continuous functions that doesn't diverge to infinity (i.e. is bounded) can be integrated. In our example above, $f(x)$ approached infinity on $[0, 1]$, so the lemma above did not hold.

Even if your function is unbounded, sometimes a function is still integrable. For example, $\int_0^1 \frac{1}{\sqrt{x}} dx = 2$ even though $1/\sqrt{x}$ is unbounded on $[0, 1]$.

2.2 Properties of integrals

One important property of integrals is that they are linear. That is, if f and g are integrable over $[a, b]$ and $\alpha \in \mathbb{R}$, then:

$$\begin{aligned} \int_a^b (f(x) + g(x))dx &= \int_a^b f(x)dx + \int_a^b g(x)dx \\ \int_a^b \alpha f(x)dx &= \alpha \int_a^b f(x)dx \end{aligned}$$

We can also manipulate the bounds of integrals

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ \int_a^b f(x)dx &= -\int_b^a f(x)dx \end{aligned}$$

2.3 Evaluating Integrals

Linearity is a nice property, but isn't too helpful when faced with a particular integral to evaluate. This section will discuss a few common techniques for evaluating integrals, beginning with one of the most important results in mathematics.

2.3.1 The Fundamental Theorem of Calculus

At its surface, differentiation and integration appear quite different. One deals with tangents and instantaneous rates, whereas the other concerns area and weighted sums. However, the two operations are inverses of each other. Consider the following function:

$$F_0(x) \equiv \int_0^x f(t)dt$$

This function expresses the area under the curve $f(t)$ between 0 and the argument x. Let's differentiate it:

$$\begin{aligned}
F_0'(x) &= \lim_{\Delta \to 0} \frac{F_0(x+\Delta) - F_0(x)}{\Delta} \\
&= \lim_{\Delta \to 0} \frac{\int_0^{x+\Delta} f(t)dt - \int_0^x f(t)dt}{\Delta} \\
&= \lim_{\Delta \to 0} \frac{\int_x^{x+\Delta} f(t)dt}{\Delta} \\
&\approx \frac{\Delta f(x)}{\Delta} \\
&= f(x)
\end{aligned}$$

In the third step we've cancelled the common integral from 0 to x, and in the fourth step we're taking advantage of the fact that $f(t)$ is roughly constant for $t \in [x, x+\Delta]$ (at least for Δ small enough). This argument show heuristically that the derivative of the integral of $f(x)$ is $f(x)$, or that derivatives and integrals are inverses.

A function $F(x)$ such that $F'(x) = f(x)$ is called an **anti-derivative** of $f(x)$, or the indefinite integral of f. Note that $F_0(x)$ is not unique, since for any $c \in \mathbb{R}$ the function $F_0(x) + c$ is also an anti-derivative of $f(x)$.

Now suppose we have an antiderivative $F(x)$ and want to evaluate the integral of f over the interval $[a, b]$. From the above we know $F(x) = F_0(x) + c$ for some constant c. Therefore we have

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^0 f(x)dx + \int_0^b f(x)dx \\
&= -\int_0^a f(x)dx + \int_0^b f(x)dx \\
&= F_0(b) - F_0(a) \\
&= F(b) - F(a)
\end{aligned}$$

This result is one version of the Fundamental Theorem of Calculus:

Theorem 2.1. *(Fundamental Theorem of Calculus) If f is a continuous function on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Thus we have a recipe for calculating integrals: given $f(x)$, find its anti-derivative, and then evaluate the difference of this function between the end points of the integral.

Occasionally we do not specify endpoints for our integrals. In this context, $\int f(x)dx$ means a function whose derivative is $f(x)$. This is called the **indefinite integral** of f. Again note that the indefinite integral is not unique, but is defined up to a constant.

2.3.2 Common Integrals

Using the Fundamental Theorem of Calculus and our knowledge of derivatives, we can quickly evaluate many common integrals. The ones you should know cold for classwork are:

$$\begin{aligned} \int x^a dx &= \frac{1}{a+1}x^{a+1} + C \\ \int e^{ax} dx &= \frac{1}{a}e^{ax} + C \\ \int \frac{1}{x} dx &= \ln|x| + C \end{aligned}$$

2.3.3 Integration By Parts

Recall the product rule: $(fg)' = f'g + g'f$. Assuming integrability, integrating this expression from a to b gives:

$$f(b)g(b) - f(a)g(a) = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx$$

Rearranging gives the formula for **integration by parts**

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

This formula let's you interchange integrals involving f and g' to those involving f' and g, which may be easier to solve. For example

$$\int \underbrace{xe^x}_{f(x)g'(x)} dx = xe^x - \int e^x dx = xe^x - e^x + C$$

2.3.4 Integration by Substitution

Much like integration by parts is the integral rule associated with the product rule for derivatives, integration by substitution is the integral rule associated with the chain rule. Suppose we want to integrate $f(g(x))g'(x)$; note $F(g(x))$ is the antiderivative. Then, using the Fundamental Theorem of Calculus twice we have:

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(u)du\end{aligned}$$

A mnemonic for this process is: let $u = g(x)$, and "differentiate" both sides to get $du = g'(x)dx$. Then substituting and changing the bounds of integration gives the change of variables formula:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Here's an example: calculate $\int_0^t 2xe^{-x^2}$. Notice this is of the form $f(g(x))g'(x)$ where $g(x) = x^2$. Thus, using the change of variables formula we find:

$$\begin{aligned}\int_0^t 2xe^{-x^2}dx &= \int_0^{t^2} e^{-u}du \\ &= 1 - e^{-t^2}\end{aligned}$$

2.4 Differentiating Integral Expressions

Consider the following problem:

$$\frac{d}{dx}\int_{l(x)}^{u(x)} g(t)dt,$$

where the upper and lower bounds of integration, $u(x)$ and $l(x)$, depend on x. Define $f(x) = \int_a^x g(t)dt$. By the fundamental theorem of calculus, $f'(x) = g(x)$. Note:

$$\int_{l(x)}^{u(x)} g(t)dt = f(u(x)) - f(l(x))$$

Thus we can use the chain rule to conlude

$$\begin{aligned}\frac{d}{dx}\int_{l(x)}^{u(x)} g(t)dt &= \frac{d}{dx}\left(f(u(x)) - f(l(x))\right) \\ &= f'(u(x))u'(x) - f'(l(x))l'(x) \\ &= g(u(x))u'(x) - g(l(x))l'(x)\end{aligned}$$

This is a special case of **Leibniz's Rule**, which we'll see when we come to multivariable calculus.

3 Multivariable Integration

The goal of this section is to introduce some of the primary tools needed for calculating integrals involving multivariable functions. You will most typically encounter integrals of this form when taking expected values.

3.1 Leibniz Rule

Let f be a function of two variables and consider the function F defined by

$$F(x) = \int_c^d f(x,t)dt$$

This looks a lot like the setup for the Fundamental Theorem of Calculus, except now the argument is in the function as opposed to the bounds of integration. Given the links, we might expect

$$F'(x) = \int_c^d \frac{\partial f(x,t)}{\partial x} dt,$$

which is indeed the case provided F is "well-behaved". This is a more general case of the well-known Leibniz rule:

Proposition 3.1. *Suppose $f(x,t)$ and $f'_x(x,t)$ are continuous over the rectangle determined by $a \leq x \leq b$, $c \leq t \leq d$. suppose too that $u(x)$ and $v(x)$ are C^1 functions over $[a,b]$, and that the ranges of u and v are contained in $[c,d]$. Then if $F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$,*

$$F'(x) = f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt$$

Proof. Module the technical details of interchanging a limit (the derivative) and an integral, the form of $F'(x)$ follows straight from the chain rule. Define

$$H(x,u,v) = \int_u^v f(x,t)dt$$

Then $f(x) = H(x, u(x), v(x))$, so, by the chain rule:

$$\begin{aligned} F'(x) &= H'_x + H'_u u'(x) + H'_v v'(x) \\ &= \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x} dt + f(x, v(x))v'(x) - f(x, u(x))u'(x) \end{aligned}$$

□

3.2 Multiple Integrals

In probability and game theory it is common to calculate integrals of multivariable functions (particularly when calculating expected values). We will spend some time discussing double integrals in particular, since most of the conceptual challenges with these integrals are introduced when we move from one dimension to 2.

3.2.1 Double Integrals Over Rectangles

Consider a function $f(x,y)$ which we wish to integrate over the rectange $R = [a,b] \times [c,d]$. Geometrically, the integral represents the volume of the region S between the surface of $f(x,y)$ and the rectangle R in the xy-plane, written

$$\int_R f(x,y)dxdy$$

How can we calculate such an integral?

Let t be an arbitrary point in $[a, b]$. The area of the intersection of S and the plane $x = t$ is given by the area under the curve $f(t, y)$:

$$A(t) = \int_c^d f(t, y)dy$$

Now denote by $V(t)$ the volume under f over the rectangle $[a, t] \times [c, d]$. For small amounts Δt, the volume of a slice $V(t + \Delta t) - V(t)$ is approximately $A(t)\Delta t$. Thus we can find the total area by integrating this expression:

$$V = \int_a^b \left(\int_c^d f(t, y)dy \right) dt$$

So a volume calculation can be reduced to calculating two successive integrals in many cases. Here's a more formal result:

Proposition 3.2. *Let f be a continuous function defined over the rectangle $R = [a, b] \times [c, d]$. Then*

$$\int_R f(x, y)dxdy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx = \int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

Note that if f is continuous and R is a rectangle, we can calculate the volume as a double integral in whatever order we choose. The ability to switch orders of integration is particularly nice, since sometimes an integral is more tractable performed in one order vs. another.

Example: Compute $\int_R \left(x^2y + xy^2\right) dxdy$ where $R = [0, 1] \times [-1, 3]$.

$$\begin{aligned}
\int_R \left(x^2y + xy^2\right) dxdy &= \int_0^1 \left(\int_{-1}^3 \left(x^2y + xy^2\right) dy \right) dx \\
&= \int_0^1 \left(\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3 \Big|_{-1}^3 \right) dx \\
&= \int_0^1 \left(4x^2 + \frac{28}{3}x \right) dx \\
&= \frac{4}{3}x^3 + \frac{14}{3}x^2 \Big|_0^1 \\
&= 6
\end{aligned}$$

As an exercise, you can verify that calculating the integral in the opposite order generates the same outcome.

3.2.2 Double Integrals over General Domains

We've seen that integrating over rectangles is relatively straightforward. Sometimes, however, we need to find the volume of a function over a region that isn't a rectangle. For example, when f is a probability distribution, the probability of a particular event occurring is an integral of f over some region A.

Suppose we can define our region A as follows: A consists of all points (x, y) such that $a \leq x \leq b$, and for any x, $u(x) \leq y \leq v(x)$. For instance, we can describe the rectangle $[a, b] \times [c, d]$ in this fashion by taking $u(x) = c$ and $v(x) = d$.

Arguing along the same lines as in the rectangular case, a logical candidate for this volume is:

$$\int_A f(x,y)dxdy = \int_a^b \left(\int_{u(x)}^{v(x)} f(x,y)dy \right) dx$$

That is, a small slice dx will contribute volume of $dx \int_{u(x)}^{v(x)} f(x,y)dy$. This turns out to be the case so long as f, u, and v are continuous.

Proposition 3.3. *Suppose A is a region given by $A = \{(x,y) : a \leq x \leq b, u(x) \leq y \leq v(x)\}$ where u and v are continuous functions and $u(x) \leq v(x)$ for all $x \in [a, b]$. If f is continuous on A, then*

$$\int_A f(x,y)dxdy = \int_a^b \left(\int_{u(x)}^{v(x)} f(x,y)dy \right) dx$$

Moreover, if we can write $A = \{(x,y) : c \leq y \leq d, r(y) \leq x \leq s(y)\}$, then

$$\int_A f(x,y)dxdy = \int_c^d \left(\int_{r(y)}^{s(y)} f(x,y)dx \right) dy$$

This result again says that a double integral can be evaluated as a repeated single integral, and that we may switch the order of integration provided we can express A in the necessary fashion.

Example: Calculate the integral $\int_A (x^2 + y^2)dxdy$ where A is the triangle $\{0 \leq x \leq 1, 0 \leq y \leq x\}$.

$$\begin{aligned}
\int_A (x^2 + y^2)dxdy &= \int_0^1 \left(\int_0^x (x^2 + y^2)dy \right) dx \\
&= \int_0^1 \left(\frac{1}{3}y^3 + xy^2 \Big|_{y=0}^{y=x} \right) dx \\
&= \int_0^1 \frac{4}{3}x^3 dx \\
&= \frac{1}{3}x^4 \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

Notice we can also write $A = \{0 \le y \le 1, y \le x \le 1\}$. Thus

$$\begin{aligned}
\int_A (x^2+y^2)dxdy &= \int_0^1 \left(\int_y^1 (x^2+y^2)dx \right) dy \\
&= \int_0^1 \left(\frac{1}{3}x^3 + xy^2 \Big|_{x=y}^{x=1} \right) dx \\
&= \int_0^1 \left(\frac{1}{3} - \frac{y^3}{3} + y^2 - y^3 \right) dy \\
&= \frac{1}{3}y - \frac{1}{12}y^4 + \frac{1}{3}y^3 - \frac{1}{4}y^4 \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$