**Solution for Problem Set 3**
**MA Math Camp 2021**

1. State if and where the following function are differentiable, and compute their derivative :
   a) $f : x \mapsto \frac{1}{1+x^2}$ defined over $\mathbb{R}$
   b) $f : x \mapsto \sqrt{x^2-1}$ defined over $[1, \infty)$
   c) $f : x \mapsto a^x$ defined over $\mathbb{R}$, with $a > 0$
   d) $f : (x, y) \mapsto \cos(x)\sin(y)$ over $\mathbb{R}^2$

   **Solution :**

   a) $f'(x) = \frac{-2x}{(1+x^2)^2}$ differentiable over $\mathbb{R}$
   b) $f'(x) = \frac{x}{\sqrt{x^2-1}}$ differentiable over $(1, \infty)$
   c) Observe that $f(x) = \exp(x\ln(a))$, hence $f'(x) = \ln(a)\exp(x\ln(a)) = \ln(a)a^x$ for $x \in \mathbb{R}$
   d) $\nabla f(x) = (-\sin(x)\sin(y), \cos(x)\cos(y))$ for $x, y \in \mathbb{R}^2$

2. a) Verify that Schwarz theorem (symmetry of the second order derivatives) holds for the following $C^2$ functions :
      i. $f(x, y) := x\exp(xy)$
      ii. $f(x, y) := ln(x^2 + y^2 + 1)$

   **Solution :**

   i. Let $f(x, y) := x\exp(xy)$, we have :

$$\nabla f(x, y) = \exp(xy)\begin{pmatrix} xy+1 \\ x^2 \end{pmatrix}$$

   and :

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \exp(xy)(x + x(xy+1)) = \exp(xy)(2x + yx^2) = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

   ii. Let $f(x, y) := ln(x^2 + y^2 + 1)$, we have :

$$\nabla f(x, y) = \frac{2}{1 + x^2 + y^2}\begin{pmatrix} x \\ y \end{pmatrix}$$

   and :

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{-2xy}{(1 + x^2 + y^2)^2} = \frac{\partial^2 f}{\partial x \partial y}(x, y)$$

b) The function is clearly $C^2$ over $\mathbb{R}^2 \setminus \{(0,0)\}$, furthermore its gradient then is :

$$\nabla f(x,y) = \frac{1}{(x^2+x^2)^2}\begin{pmatrix} y^3(x^2+y^2) - 2x^2y^3 \\ 3xy^2(x^2+y^2) - 2xy^4 \end{pmatrix} = \frac{1}{(x^2+x^2)^2}\begin{pmatrix} y^5 - x^2y^3 \\ 3xy^2 + xy^4 \end{pmatrix}$$

We can verify easily that $f$ is $C^1$ over $\mathbb{R}^2$ and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$. Let's show that the cross partial derivatives exist at $(0,0)$ but do not coincide (hence if $f$ was $C^2$ over $\mathbb{R}^2$, Schwarz theorem would be violated) :

$$\frac{1}{h}\left(\frac{\partial f}{\partial x}(0,h) - \underbrace{\frac{\partial f}{\partial x}(0,0)}_{=0}\right) = \frac{1}{h}\frac{1}{h^4}(h^3(h^2+0^2) - 2h^3 0^2) = 1 \xrightarrow[h\to 0]{} 1 = \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

$$\frac{1}{h}\left(\frac{\partial f}{\partial y}(h,0) - \underbrace{\frac{\partial f}{\partial y}(0,0)}_{=0}\right) = \frac{1}{h}\frac{1}{h^4}(3h^3 0^2 + h0^4) = 0 \xrightarrow[h\to 0]{} 0 = \frac{\partial^2 f}{\partial x \partial 1}(0,0)$$

3. Let $F : \mathbb{R}^2 \to \mathbb{R}$ a $C^1$ function. Show that the following function is continuous on $R^2$ :

$$f(x,y) := \begin{cases} \frac{F(x)-F(y)}{x-y} & \text{if } x \neq y \\ F'(x) & \text{if } x = y \end{cases}$$

**Solution :** Since $F$ is continuous, the function $f$ is continuous at every point $(x,y)$ with $x \neq y$. To show that it is continuous at any point $(a,a) \in \mathbb{R}^2$, we first restate the mean value theorem : for any $x, y \in \mathbb{R}$, there exists $c$ between $x$ and $y$ such that :

$$F(x) - F(y) = F'(c)(x-y)$$

Let $\epsilon > 0$. Since $F'$ is continuous at $a$, there exists $\delta > 0$ such that if $|t-a| < \delta$, $|F'(t) - F'(a)| < \epsilon$. Now consider $(x,y)$ close enough to $(a,a)$ in the sense that $|x-a| < \delta$ and $|y-a| < \delta$ (NB : this is equivalent to taking $(x,y)$ in the ball of radius $\delta$ around $(a,a)$ according to the sup norm). Take $c$ between $x$ and $y$ as in the statement of the mean value theorem. Since $c$ is between $x$ and $y$, directly $|c-a| < \delta$ (using the triangular inequality and that $c = \lambda x + (1-\lambda)y$ for some $\lambda \in [0,1]$). Now consider two cases :

- If $x = y$, then $|f(x,y) - f(a,a)| = |F'(x) - F'(a)| < \epsilon$ since $|x-a| < \delta$
- If $x \neq y$, then $|f(x,y) - f(a,a)| = |F'(c) - F'(a)| < \epsilon$ since $|c-a| < \delta$

This shows that $f$ is continuous at $(a,a)$. **Remark :** A proof that does not use the continuity of $F'$ cannot be complete since if $F'$ is not contiuous, $f$ is not continuous along the diagonal $\{x = y\}$ hence it cannot be continuous on $\mathbb{R}^2$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ a differentiable function. Differentiate the functions : $u(x) = f(x,-x)$, $g(x,y) = f(y,x)$.

**Solution :** We have :

$$\begin{aligned} u'(x) &= \frac{\partial f}{\partial x}(x,-x) - \frac{\partial f}{\partial y}(x,-x) \\ \frac{\partial g}{\partial x}(x,y) &= \frac{\partial f}{\partial y}(y,x) \\ \frac{\partial g}{\partial y}(x,y) &= \frac{\partial f}{\partial x}(y,x) \end{aligned}$$

5. For the following functions from a given interval $I$ to $\mathbb{R}$, compute $\sup_{x\in I} f(x)$, $\inf_{x\in I} f(x)$, state if these are attained and at which point(s) :

   a) $f(x) = x(1-x)$ on $I = [0,1]$

   **Solution :** $f$ is continuous and $I$ is compact, hence $f$ attains it supremum and its infimum. Observe that $f(x) \geq 0$ for all $x \in [0,1]$ and $f(0) = f(1) = 0$, hence $\min_I f = 0$. This implies that the maximum must be attained on $(0,1)$, hence the interior condition $f'(x) = 0$ must be verified at that point. We can directly compute that the derivative only cancels out at a single point $x = 1/2$, therefore this has to be the maximum and we have $\max_I f = f(1/2) = 1/4$.

   b) $f(x) = 1 - e^{-x}$ on $I = \mathbb{R}^+$

   **Solution :** $f$ is continuous but $I$ is not bounded so we cannot a priori conclude on the existence of extrema. However, we can observe that $1 \geq e^{-x} > 0$ hence $f$ is bounded. Since $f(0) = 0$, we have $\min_I f = 0$. Observing that $\lim_{+\infty} f = 1$ ensures that $\sup_I f = 1$ – but the supremum is not attained.

   c) $f(x) = 3x^4 - 4x^3 + 6x^2 - 12x + 1$ on $I = \mathbb{R}$

   **Solution :** Again, since the domain is not bounded we cannot a priori conclude about the existence of extrema even though $f$ is continuous. Considering the limit as $x \to \infty$, we see that $\sup_I f = \infty$. Computing the derivative of $f$ yields $f'(x) = 12(x-1)(x^2+1)$. It cancels out at a single point, $x_0 = 1$. Furthermore $f''(x) = 12(3x^2 - 2x + 1) > 0$ for any $x \in \mathbb{R}$, hence $f$ is convex and $f$ attains a local minimum at $x_0$. Since $f'$ is increasing and zero at $x_0 = 1$, $f$ is decreasing on $(-\infty, 1]$ and increasing on $[1, +\infty)$. Therefore $f$ is bounded below and attains its infimum at $x_0$ : $\min_I f = f(1) = -6$.

   d) $f(x) = \frac{1}{\sqrt{x^2-x+1}}$ on $I = [0,1]$

   **Solution :** We can verify that the denominator is strictly positive over $I$ (e.g by observing that $x^2 - x + 1 = (x^2-1) + x$), hence $f$ is continuous over $I$. Over $x \in I$, $3/4 \leq x^2 - x + 1 \leq 1$, hence since the function $x \mapsto 1/\sqrt{x}$ is decreasing, we have $\min_I f = f(1) = 1$, $\max_I f = f(1/2) = \sqrt{4/3}$.

6. Find the maximum and minimum of $f(x,y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$ using the Kuhn-Tucker method. Using the substitution $y^2 = 1 - x^2$ solve the same problem as a single variable unconstrained problem. Do you get the same results? Why or why not?

   **Solution :** As the object $f$ is continuous and the unit circle is compact, by Weierstrass this program has global minimum and maximum.

$$\begin{array}{ll} \max_{(x,y)\in\mathbb{R}^2} & x^2 - y^2 \\ \text{s.t.} & x^2 + y^2 = 1 \end{array}$$

Lagrangian: $\mathcal{L}(x, y, \lambda) = x^2 - y^2 + \lambda\left(1 - x^2 - y^2\right)$

FOC: $\begin{cases} \frac{\partial\mathcal{L}}{\partial x} = 2(1-\lambda)x = 0 \\ \frac{\partial\mathcal{L}}{\partial y} = -2(\lambda+1)y = 0 \end{cases}$

Solve the two equations along with the constraint, we have either $x = 0, y = \pm 1$ or $x = \pm 1, y = 0$, which are candidates for optimizers. Notice that $-1 = f(0,1) = f(0,-1) < f(1,0) = f(-1,0) = 1$ and that $x^2 - y^2 \leq x^2 \leq x^2 + y^2 = 1$ and $x^2 - y^2 \geq -y^2 \geq -\left(x^2 + y^2\right) = -1$, we know the maximum is 1 and the minimum is $-1$.

If we substitute $y^2 = 1 - x^2$ into the objective function, we get the unconstrained problem

$$\max_{x\in\mathbb{R}} 2x^2 - 1$$

which has no solution. The reason for the difference is that we have not imposed the constraint that $1 - x^2 \geq 0$, but this is necessary since $1 - x^2$ must equal $y^2$ for some real number $y$.

7. A consumer's utility maximization problem is

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}_{++}\times\mathbb{R}_+} \quad & \alpha \ln x + y \\ \text{s.t.} \quad & px + qy \leq m \\ & y \geq 0 \end{aligned}$$

where, $\alpha > 0$, $p > 0$, $q > 0$, $m > 0$ are parameters.

a) Argue that the budget constraint must hold with equality.

**Solution :** Suppose not, and $m - px - qy = c > 0$ at an optimum. Then we could increase $y$ by $c/q$, which still satisfies all the constraints and obtain a strictly higher value of the objective function, a contradiction.

b) Write the Lagrangian. State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

**Solution :** The Lagrangian is given by :

$$\mathcal{L}(x, y, \lambda, \mu) = \alpha \ln(x) + y + \lambda(m - px - qy) + \mu y$$

Assuming the budget constraint holds with equality, the necessary conditions are :

$$\begin{aligned} \frac{\alpha}{x} - \lambda = 0 \\ 1 + \mu - \lambda q = 0 \\ y \geq 0, \mu \geq 0, y\mu = 0 \end{aligned}$$

$$px + qy = m$$

Since the objective function is concave and the constraints are linear, these conditions are also sufficient for a maximum.

c) Are there any admissible points where the constraint qualification fails?

**Solution :** Denote $h(x, y) := m - px - qy$ and $g(x, y) = y$. We have :

$$(\nabla g(x, y), \nabla h(x, y)) = \left( \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Those vectors are linearly independent since $p > 0$, hence there are no points at which the constraint qualification fails – observe that this holds whether or not the positivity constraint on $y$ is active : if it isn't, then the family composed of only the vector $(p, q)$ is independent.

d) Solve for the maximizer $(x^*, y^*)$.

**Solution :** First consider the case $y = 0$. Then we must have $x = m/p$, which implies $\alpha/m = \lambda$. Plugging this into the second FOC yields :

$$1 + \mu = \frac{\alpha q}{m} \;\Leftrightarrow\; \mu = \frac{\alpha q}{m} - 1$$

For this to be consistent (i.e for $(m/p, 0)$ to be a candidate point), we need $\frac{\alpha q}{m} - 1 \geq 0$.

Next consider the case $y > 0$, then $\lambda q = 1$ and $\alpha/x = p/q$, hence $x = \frac{\alpha q}{p}$. Plugging this into the budget constraint, we get :

$$\begin{aligned} px + qy = m \;&\Leftrightarrow\; \alpha q + qy = m \\ &\Leftrightarrow\; y = \frac{m}{q} - \alpha \end{aligned}$$

This must be nonnegative for this to be a consistent solution, so this is only possible if $\frac{m}{q} - \alpha \geq 0$.

Summing up, the solution is given by :

$$\begin{cases} x = \dfrac{m}{p}, y = 0 & \text{if } \alpha q \geq m \\ x = \dfrac{\alpha q}{p}, y = \dfrac{m}{q} - \alpha & \text{if } \alpha q \leq m \end{cases}$$

e) Find the value function $v\,(p, q, m)$. What does the Envelope Theorem tell you about the derivative of $v\,(p, q, m)$ with respect to $q$?

**Solution :** Substituting the solution into the objective function gives :

$$v(p, q, m) = \begin{cases} \alpha \ln\left(\dfrac{m}{p}\right) & \text{if } \alpha q \geq m \\ \alpha \ln\left(\dfrac{\alpha q}{p}\right) + \dfrac{m}{p} - \alpha & \text{if } \alpha q \leq m \end{cases}$$

The envelope theorem tells us that the derivative of the value function with respect to $q$ is equal to the derivative of the Lagrangian with respect to $q$, evaluated at the optimum point. Verify it by first differentiating $v(p,q,m)$ in $q$ directly :

$$\frac{\partial v}{\partial q}(p,q,m) = \begin{cases} 0 & \text{if } \alpha q \geq m \\ \dfrac{\alpha}{q} - \dfrac{m}{q^2} & \text{if } \alpha q \leq m \end{cases}$$

Now observe that the derivative of the Lagrangian with respect to $q$ is :

$$\frac{\partial \mathcal{L}}{\partial q}(x,y,\lambda,\mu|p,q,m) = -\lambda y$$

At the optimum we have, when $\alpha q \geq m$, $y = 0$, which is consistent with the previous derivation. When $\alpha q \leq m$, $\lambda = 1/q$ and $y = m/q - \alpha$, which also gives the same expression as before.

8. A firm produces two outputs, $x$ and $y$, using a single input $z$. The price of $x$ has been normalized to 1; the price of $y$ is $p$. The firm's program is

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2} \ & x + py \\ \text{s.t. } & x^2 + y^2 \leq z \\ & x \geq 1, \ y \geq 0 \end{aligned}$$

$p > 0$ and $z > 0$ are parameters.

a) Write the Lagrangian.

**Solution :** The Lagrangian is given by :

$$\mathcal{L}(x,y,\lambda,\mu,\nu) = x + py + \lambda(z - y^2 - x^2) + \mu y + \nu x$$

b) State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

**Solution :** First order necessary conditions are given by :

$$\begin{aligned} 1 - 2\lambda + \nu &= 0 \\ p - 2\lambda y + \mu &= 0 \\ y^2 + x^2 &\leq z \\ x &\geq 0 \\ y &\geq 0 \\ \lambda(z - y^2 - x^2) &= 0 \\ \nu x &= 0 \\ \mu y &= 0 \\ \lambda, \mu, \nu &\geq 0 \end{aligned}$$

Since the objective function is linear (therefore concave), and the constraints are convex, these conditions are sufficient for a maximum.

c) Are there any admissible points where the constraint qualification fails? Can any of these points be a solution to the program?

**Solution :** First consider points at which only the first constraint binds, i.e $x^2 + y^2 = z$, but $x, y > 0$. The CQ will fail only if the gradient $\begin{pmatrix} 2x \\ 2y \end{pmatrix}$ is equal to zero, which cannot be the case since $x, y > 0$. Consider next the case where the first constraint binds, $y = 0$, but $x > 0$. The gradient associated to the second constraint is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which is linearly indendent from the gradient of the first constraint $\begin{pmatrix} 2x \\ 2y \end{pmatrix}$, hence the CQ holds. The case $x = 0$ but $y > 0$ is symmetric. Lastly, it can never be the case that all three constraint bind since $z > 0$ and if the first constraint does not bind the CQ is directly always verified : the gradients associated to the other two constraints $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are always linearly independent. We conclude that there are no points at which the CQ fails.

d) Solve for the maximizer $(x^*, y^*)$.

**Solution :** First suppose that only the first constraint binds. Then $\nu = \mu = 0$. Dividing the second FOC by the first yields $px = y$ hence $p^2x^2 = y^2$. We substitute in the constraint to get $x^2 + p^2x^2 = z$, which then gives :

$$x = \sqrt{\frac{z}{1+p^2}}$$
$$y = p\sqrt{\frac{z}{1+p^2}}$$

This is our first solution candidate. Since this solution has $x$ and $y$ strictly positive, we can never have a maximum at which both of the nonnegativity constraints bind : $x = y = 0$ yields a strictly lower value than the previous point. Now suppose $\nu > 0$ but $\mu = 0$. Then $x = 0$ so from the first constraint $1 + \nu = 0$, which cannot be satistifed for any $\nu > 0$. The same goes for the symmetric case $\nu = 0$ and $\mu > 0$. Hence the only candidate is the one we previously found. This must be the maximum since the Kuhn-Tucker conditions are sufficient here.

e) Find the value function, $f^*(p, z)$.

**Solution :** Substituting the optimal $(x^*, y^*)$ into the objective function yields :

$$f^*(p,z) = p^2\sqrt{\frac{z}{1+p^2}} + \sqrt{\frac{z}{1+p^2}} = \sqrt{z(1+p^2)}$$

f) What does the Envelope Theorem tell you about the derivative of $f(p, z)$ with respect to $z$?

**Solution :** Applying the Envelope Theorem yields :

$$\begin{aligned} \frac{\partial f^*}{\partial z}(p,z) &= \left[\frac{\partial \mathcal{L}}{\partial z}(x,y,p,z)\right]_{x=x^*(p,z),y=y^*(p,z)} \\ &= \lambda^*(p,z) \end{aligned}$$

We can verify by plugging in the value of the multiplier that this coincides with directly differentiating the expression from the previous question in $z$.