Lecture Notes - Static Optimization

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Contents

1 General Setup

1.1 Definition

Definition 1.1. *Let f be a function from X to the poset $(Y, \leq)$, and let $D \subset X$. A* ***maximization problem*** *takes the form*

$$\max_{x \in X} f(x) \;\; s.t. \;\; x \in D$$

where f is called the ***objective function****, x is called the* ***choice variable****, and D is called the* ***constraint set*** *or* ***feasible set****. A point $x \in X$ is said to be* ***feasible*** *iff $x \in D$.*

*The present lecture notes were largely based on math camp materials from Palaash Bhargava, Paul Koh, and Xuan Li. All errors in this document are mine. If you find a typo or an error, please send me an email at cesar.barilla@columbia.edu.

The set of ***maximizers****, or* ***maximum points****, of this problem is defined as*

$$\arg\max_{x\in X}\left\{f\left(x\right):x\in D\right\}:=\left\{x^{*}\in D:f\left(x^{*}\right)\geq f\left(x\right)\ \forall\ x\in D\right\}$$

If the set of maximizers is nonempty, then this problem is said to ***have a solution****. In this case, we define the* ***maximum****, or the* ***maximum value****, of this problem as* $f\left(x^{*}\right)$*, where* x^{*} *is an arbitrary maximizer, and denote it as* $\max_{x\in X}\left\{f\left(x\right):x\in D\right\}$.

The maximum does not need to exist in general, i.e the set of maximizers can be empty. Consider for example the function :

$$\begin{aligned} f:(0,1)&\to(0,1)\\ x&\mapsto x\end{aligned}$$

This function does not have a maximum because for every point $x\in(0,1)$ I can find a point $x'\in(0,1)$ such that $f(x')>f(x)$ (by moving arbitrarily close to 1). In other words, the set $f((0,1))$ does not have a maximum.

Notice that although the set of maximizers can be non-singleton when nonempty, the maximum does not depend on the selection of x^{*} from the set of maximizers. This follows directly from the anti-symmetry property of the partial order on Y. Therefore, if it exists, the maximum is a well defined (unique) element of Y :

$$\max_{x\in D}f(x)\in Y.$$

The set of maximizers, by contrast, is a *subset* of D in general :

$$\operatorname*{argmax}_{x\in D}f(x)\subset D$$

If $(Y,\leq)$ has the least upper bound property, then we know that the following supremum always exist :

$$\sup_{x\in D}f(x).$$

This will notably be the case when $Y=\mathbb{R}$, which will be our main case of of interest. Then, the question of whether a maximum exists can be interpreted as whether this supremum is *attained* by a point in D. There exists a smallest upper bound to $f(x)$, but is it feasible to attain this value from a point in D ? The previous example already hints at the fact that both the properties of the set D and the function f will jointly determine the answer to this question – this is because we are essentially studying the set $f(D)$.

We can define minimization problem analogously. In fact, we can always transform a minimization problem into a maximization problem by reversing the order $\leq$ on the codomain, and therefore it is without loss to only study maximization problems. In most applications, of course, the codomain of the objective function f is the totally ordered set $(\mathbb{R},\leq)$. In this case, we can transform a minimization problem of function f to a maximization problem of $-f$. Throughout these notes we will focus on maximization, but all our results are directly applicable to minimization using the following transformation :

$$\max_{x\in D}f(x)=-\min_{x\in X}(-f)(x)$$
$$\operatorname*{argmax}_{x\in D}f(x)=\operatorname*{argmin}_{x\in X}(-f)(x)$$

Observe that so far, we have not specified what the set X (the domain of the function) is allowed to be. In general, this could be anything, but different set structures will lead to different techniques. For instance, X could be a finite set, a finite or infinite dimensional vector space or a subset of one, a subset of a metric space, etc. The structure of the space will naturally determine the tools that we can use to study the problem. We introduce some usual terminology to categorize different common types of optimization problems :

- "Unconstrained optimization" refers to an optimization problem where D is an open set of a metric space.
- "Optimization under equality constraint" refers to an optimization problem where D is of the form $D = \{x \in X, g(x) = 0\}$, where $g : X \to Z$, and both X, Z are metric spaces. If Z is a vector space of dimension p (typically $Z = \mathbb{R}^p$), we say that p is the "number of constraints".
- "Optimization under inequality constraint" refers to an optimization problem where D is of the form $D = \{x \in X, \forall i \in I, g_i(x) \leq 0\}$, where $g_i : X \to \mathbb{R}$ for all $i \in I$. If $|I|$ is finite, we refer to it again as the number of constraints.
- We will sometimes talk about mixed constraints when combining equality and inequality constraints.
- We say that $x_0 \in D$ is a *global maximum* if it is a solution to the optimization problem, i.e $f(x_0) \geq f(x)$ for all $x \in D$.
- We say that $x_0 \in D$ is a *local maximum* if there exists a neigborhood of x_0 in D (notice that this requires X to be a metric space to allow us to talk about distance) such that $f(x_0) \geq f(x)$ for all x in this neighborhood.

There are generally three main kind of questions that we can investigate when looking at an optimization problem :

1. Does there exist (at least) a solution ? In other words, can we find a maximizer ? Is this maximizer unique ? This is usually the first question in the logical order : we cannot say much if there is no maximum and our problem is not well defined. Existence questions are usually approached using Analysis tools.
2. What are the properties of the solution ? How to characterize the maximizers and the maximum ? What can we say about them ? This is where we usually start to get more economics-relevant results that allows us to characterize some behavior and its properties of interest. To answer these questions, we often use the tools of differential calculus.
3. Can we *explicitly* identify the solution or an approximation of it ? In simple cases, this will actually be answered with the same tools as the previous question. In general, we might need to construct algorithms or numerical methods to approximate complex solutions that we cannot find explicitly.

In those lecture notes, we will talk mostly about the first two questions, although in many cases we will look at problems simple enough that the second and third questions are actually answered together.

1.2 General Properties

The following proposition relates a maximization problem on a bigger set to a maximization problem over a subset (this can allow us, in particular, to relate local maxima to global maxima).

Proposition 1.2. ***(Variant 1)*** *Let f be a function from X to the poset $(Y, \leq)$, and let $E \subset D \subset X$. Suppose that $\forall\ x \in D$, $\exists\ \hat{x} \in E$ s.t. $f(\hat{x}) \geq f(x)$. Consider the following two problems:*

$$\max_{x \in X} f(x) \ \ s.t. \ x \in D$$

and

$$\max_{x \in X} f(x) \ \ s.t. \ x \in E$$

The maximizers in the two problems have the following relation

$$\arg\max_{x \in X} \left\{ f(x) : x \in E \right\} = \left(\arg\max_{x \in X} \left\{ f(x) : x \in D \right\} \right) \cap E$$

and if one of the two problems has a solution, then the other also has a solution. Furthermore, when the two problems have a solution, they have the same maximum.

(Variant 2)

Let f be a function from X to the totally ordered set $(Y, \leq)$. Let $D \subset X$, and x_0 be some arbitrary element of D, and define $E := \left\{ x \in D : f(x) \geq f(x_0) \right\}$. Then we have

$$\arg\max_{x \in X} \left\{ f(x) : x \in E \right\} = \arg\max_{x \in X} \left\{ f(x) : x \in D \right\}$$

and the two problems have the same maximum if they have a solution.

Intuitively, Variant 1 of Prop.1.2 says that when we choose $x \in D$ to maximize $f(x)$, we can instead focus only on $E \subset D$ without loss of optimality, if for any alternative $x \in D$ we can find an alternative $\hat{x} \in E$ that is weakly better than x[1].Variant 2 says that if the alternative x_0 is feasible, then we can ignore all alternatives strictly worse than x_0 without loss of optimality[2].

Proposition 1.3. *Let f be a function from X to the poset $(Y, \leq)$, and let $D \subset X$. Let $\{D_\alpha\}_{\alpha \in A}$ be a family of subsets of D s.t.*

$$\bigcup_{\alpha \in A} D_\alpha = D$$

For each $\alpha \in A$, let

$$X_\alpha^* := \arg\max_{x \in X} \left\{ f(x) : x \in D_\alpha \right\}$$

Suppose that $X_\alpha^* \neq \emptyset$ for any $\alpha \in A$. Then

$$\begin{aligned} &\arg\max_{x \in X} \left\{ f(x) : x \in D \right\} \\ &= \arg\max_{x \in X} \left\{ f(x) : x \in \bigcup_{\alpha \in A} X_\alpha^* \right\} \end{aligned}$$

[1] The proposition above has an important application in mechanism design. When the principal chooses from the space of all mechanisms to maximize some objective function, the *Revelation Principle* states that any allocation that can be implemented by some mechanism can also be implemented by a direct truthful mechanism. Therefore, the proposition above implies that it is without loss of optimality for the principal to focus only on the space of direct truthful mechanism, which is a much smaller space compared to the space of all mechanisms. In this way, the maximization problem of the principal is greatly simplified.

[2] Notice that Variant 2 requires that the codomain Y is a totally ordered set, and the result does not hold if the order $\leq$ defined on the codomain is not complete.

Intuitively, this proposition says that when we optimize f over the set D, we can partition D into pieces and optimize in each piece. Then we can collect the maximizers over each piece and compare them.

1.3 Examples

1. **Choice problem.** Let X a set of possible alternatives for a decision-maker – i.e the decision maker has to choose *one* element in X. Assume that the preferences of the decision maker are represented by an objective function $f : X \to \mathbb{R}$ such that $f(x') \geq f(x)$ if and only if x' is a better alternative than x, i.e x' is preferred to x. In the case of a consumption problem, this would often be referred to as a utility function, but in general this could be anything that captures the preferences of the decision-maker[3]. Then the decision problem can be viewed as a maximization problem : the best alternative x^* would be one that dominate all others i.e $f(x^*) \geq f(x)$ for all x. In other words, we can view model the decision problem as :

$$\max_{x \in X} f(x)$$

   Where the chosen alternative would be anything in $\operatorname{argmax}_{x \in D} f(x)$.

   We can make several modelling remarks here. First, this is only *one model* for the decision problem. We can approach it from a normative perspective : if we assume that f is the right objective function (i.e we take for granted that it represents the true preferences of the decision maker), then solving the problem will tell us what the decision maker should choose under those preferences. The validity of the answer is obviously contingent on having the right objective function. We might also try to have a descriptive approach and try to predict what a decision-maker would choose. Then the objective function might not be a representation of the true preferences if the decision process is imperfect (e.g. noisy), but it would have to be a representation of the criterion that the decision maker uses. When trying to have a descriptive approach, the model we choose to predict behavior should ultimately be comparible (and compared) to actual relevant decisions to assess its accuracy and predictive power. It is also very important to found the structure and assumptions of our approach on general principle. A model is never universal, it is only one self-consistent approximation that may or may not be applicable to represent more complex phenomena.

2. **Best response.** Consider a game with two players that simultaneously choose an action. Player 1 chooses an action from a set A and player two chooses from a set B. The resulting payoff for player 1 when they choose $a \in A$ and the other chooses $b \in B$ is given by $u_1(a, b)$; this is the payoff function or utility function, which is a function from $A \times B$ to $\mathbb{R}$. We might wonder, if player 1 knew what player 2 is going to play, what would be their best choice (i.e what would they like to play) ? This defines the best response problem. Fix some $b \in B$, we want to find $a^* \in A$ such that $u_1(a^*, b) \geq u_1(a, b)$ for all $a \in A$. This is a maximization problem in a, which has b has a fixed parameter, therefore it defines a mapping from a given b to the set of maximizers for that b :

$$b \mapsto \operatorname*{argmax}_{a \in A} u_1(a, b)$$

[3]It is not obvious (and it is not true in general) that arbitrary preferences of the decision maker can be represented by a *function*. It is an important topic in decision theory to find explicit axiomatic characterizations of fundamental preferences (represented as abstract relations on the set of alternatives) such that we can indeed represent preferences as a utility function. Intuitively, you can see that representing preferences as a utility function requires everything to be comparible (completeness) since all real numbers are comparible and preferences to be transitive (since we have transitivity on $\mathbb{R}$).

This is known as Player 1's best response correspondence, and this plays a crucial role in defining the concept of Nash Equilibrium of a game.

3. **Least Squares.** Consider a problem where we want to assess the effect of some variable x (e.g. years of education) on some variable y (e.g. wage). Assume that for some external reason, we know that the relationship between those two variables can be well approximated by a linear function $y = \beta x$ for some $\beta \in \mathbb{R}$, and we are interested in estimating the value of that parameter β – in the education/wage example, this will tell us approximately how much more an individual can expect to earn for an additional year of education. Furthermore assume that we have observed data $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ giving us actual realized pairs (x, y). One way to approach our estimation problem is to try choose the β that minimizes the prediction error of the linear specification. If we fix a β, then for observation i the predicted value will be $\hat{y}_i = \beta x_i$, hence the estimation error will be $y_i - \hat{y}_i = y_i - \beta x_i$. For technical reasons, it is a good idea to focus on the square of the error (measuring the magnitude of approximation) $|y_i - \beta x_i|$. Therefore the total approximation error over all observations for a given β will be given by :

$$\sum_{i=1}^{n} |y_i - \beta x_i|^2$$

One estimation procedure that can be shown to be effective if we have good reasons to use a linear approximation is to try to minimize the sum of squared errors in β. This leads to the following minimization problem :

$$\min_{\beta \in \mathbb{R}} \sum_{i=1}^{n} |y_i - \beta x_i|^2$$

This is only one possible statistical model and one possible approach at solving it, but it is quite a general idea that in statistics and econometrics, we can often express our problem as minimizing some loss function that captures how well our estimation fits the observed data.

4. **Pareto-Optimal Allocation.** Assume there are n agents among which we need to split some ressource whose total amount is normalized to 1 (think about cutting a cake). Assume that agent i gets utility $u_i(x_i)$ from a share $x_i \in [0, 1]$ of the cake. We can ask : what is the split of the cake that maximizes total utility ? In which case this will lead to the maximization problem :

$$\max_{(x_1,\dots,x_n)\in[0,1]^n} \sum_{i=1}^{n} u_i(x_i)$$
$$\text{s.t. } \sum_{i=1}^{n} x_i \leq 1$$

The inequality constraint $\sum_{i=1}^{n} x_i \leq 1$ is a *ressource* constraint : we cannot distribute more than the total size of the cake. We can generalize this problem to different objectives over resulting utilities. For example, we might add relative weights (λ_i) to the utilities of each agent, giving a new maximization problem :

$$\max_{(x_1,\dots,x_n)\in[0,1]^n} \sum_{i=1}^{n} \lambda_i u_i(x_i)$$
$$\text{s.t. } \sum_{i=1}^{n} x_i \leq 1$$

We might care only about, say, the utility of the worse-off individual under a given split of the cake. Given $(x_1, ..., x_n)$ a repartition, the worse-off individual has utility $\min_{1\leq i\leq n} u_i(x_i)$. In that case, we would try to maximize :

$$\max_{(x_1,\dots,x_n)\in[0,1]^n} \min_{1\leq i\leq n} u_i(x_i)$$
$$\text{s.t. } \sum_{i=1}^{n} x_i \leq 1$$

There are many more examples of optimal allocation problems under various constraints, and this has many applications in e.g. public goods or common resources problem.

2 Existence of Maximizers

The first issue about maximization problems is the existence of maximizers. We have already seen results that provide existence results in our Real Analysis lecture. Indeed, remember that Weierstrass theorem in Lecture 1 states that a continuous real-valued function on a compact set must achieve its maximum/minimum. Let's rewrite it as the proposition below.

Proposition 2.1. *Let $f : X \to \mathbb{R}$, $D \subset X$ nonempty, and consider the maximization problem*

$$\max_{x\in X} f(x) \ \ s.t. \ x \in D$$

If there exists a metric d defined on the set D s.t. (D, d) is a compact metric space, and the function $f|_D$, i.e. f restricted in D, is continuous w.r.t. the metric d, then

$$\arg\max_{x\in X} \left\{ f(x) : x \in D \right\} \neq \emptyset$$

i.e. the maximization problem has a solution.

In the proposition above, we use the usually defined order $\leq$ and the Euclidean distance d_2 for the codomain $\mathbb{R}$. Function f restricted in D is a new function $f|_D : D \to \mathbb{R}$ defined as $f|_D(x) = f(x)$ for any $x \in D$[4].

Observe that compactness and continuity together play an essential role : because the image of a compact set by a continuous function is a compact set, and because compact subsets of $\mathbb{R}$ have a maximum, the function attains its maximum. Those conditions are sufficient, but in general not necessary. It is however, a very frequent theme in optimization method to show existence that compactness and continuity play a part together. Although there is no absolutely universal method, those notions and techniques can be applied and adapted to many particular cases.

[4] We distinguish between f and $f|_D$ because continuity is not defined without the metric d, while a metric is not necassary on $X\backslash D$.

Weierstrass theorem provides a sufficient condition for a maximization problem to have a solution. However, sometimes we cannot directly apply Weierstrass theorem to argue that a maximization problem has a solution. For example, consider the following maximization problem:

$$\max_{(x_1,x_2)\in\mathbb{R}^2_{++}} \ln x_1 + \ln x_2 \text{ s.t. } x_1 + x_2 = 3$$

Notice that the constraint set D in this problem is

$$D := \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 + x_2 = 3\right\}$$

which is not compact under the Euclidean distance d_2, since it is not closed in $\left(\mathbb{R}^2, d_2\right)$. Therefore, we cannot directly apply Weierstrass theorem, although the objective function $\ln x_1 + \ln x_2$ is continuous.

However, we can transform this problem to another problem to which Weierstrass theorem applies, using Proposition 1.2. Because $(1,2) \in D$, and $f(1,2) = \ln 2$, we can define

$$E := \left\{(x_1, x_2) \in \mathbb{R}^2_{++} : x_1 + x_2 = 3,\ \ln x_1 + \ln x_2 \geq \ln 2\right\}$$

By Proposition 1.2, the problem

$$\max_{(x_1,x_2)\in\mathbb{R}^2_{++}} \ln x_1 + \ln x_2 \text{ s.t. } (x_1, x_2) \in E$$

has the same set of maximizers as the original problem. It can be shown that E is compact under d_2, and so we can invoke Weierstrass theorem to argue that the maximization problem over E has a solution, and therefore, the original problem over D also has a solution. It's important to remember that how the problem is written matters, and in general the choice of the underlying space and the distance will influence which techniques we can use. This is true in finite dimension and even more so in infinite dimension, and highlights the importance of having our general toolbox of real analysis at hand to analyze those problems.

The next issue is about the uniqueness of the maximizer. One of the main techniques to get uniqueness is to use concavity (for a minimum, convexity for a maximum) of functions to relate local and global properties. Concavity captures the idea that "mixtures improve the objective function" (the image of the convex combination is higher than the convex combination of the images), therefore strict concavity will imply uniqueness, because given two maxima we can take a convex combination of them and do strictly better. In general, we get a more general result by only requiring quasi-concavity. Formally, we have the following result :

Proposition 2.2. *Let X be a set in real vector space $(V, +, \cdot)$, and let $f : X \to \mathbb{R}$. If $D \subset X$ is a convex set in V and $f|_D$ is a strictly quasi-concave function, then* $\arg\max_{x\in X} \left\{f(x) : x \in D\right\}$ *contains at most one point, i.e. the maximization problem has a unique maximizer if it exists.*

Proof. Suppose $x^*, x^{**} \in \arg\max_{x\in X} \left\{f(x) : x \in D\right\}$ and $x^* \neq x^{**}$. By strict quasi-concavity of f, we have

$$f\left(x^*\right) = f\left(x^{**}\right) < f\left(\frac{1}{2}x^* + \frac{1}{2}x^{**}\right)$$

Because D is a convex set, we know that $\frac{1}{2}x^* + \frac{1}{2}x^{**} \in D$, and this contradicts the assumption that $x^*, x^{**} \in \arg\max_{x\in X} \left\{f(x) : x \in D\right\}$. □

In the proposition above, if we replace strict quasi-concavity by quasi-concavity, then we don't have this uniqueness result. Instead we have the following result.

Proposition 2.3. *Let X be a set in real vector space $(V,+,\cdot)$, and let $f : X \to \mathbb{R}$. If $D \subset X$ is a convex set in V and $f|_D$ is a quasi-concave function, then* $\arg\max_{x\in X}\left\{f(x) : x \in D\right\}$ *is a convex set in V.*

Proof. Suppose $x^*, x^{**} \in \arg\max_{x\in X}\left\{f(x) : x \in D\right\}$. Then $\forall\lambda \in [0,1]$, $f(\lambda x^* + (1-\lambda)x^{**}) \geq \min\left\{f(x^*), f(x^{**})\right\}$ because f is quasi-concave. As the value of maximum $f(x^*)$ is unique if exists, we have $f(\lambda x^* + (1-\lambda)x^{**}) = f(x^*)$ and therefore $\lambda x^* + (1-\lambda)x^{**} \in \arg\max_{x\in X}\left\{f(x) : x \in D\right\}$ □

It's important to remember that there is no universal method to prove either existence or uniqueness of minimizers. The notions of compactness, continuity, concavity provide useful tools because they capture fundamental properties that will imply those results. The spirit of those results can be adapted and extended to ad hoc cases, or conversely non-existence or non-uniqueness can be proven by finding e.g. appropriate violations of compactness, continuity or concavity respectively. Those results are general guides that cover a lot of cases, but it's important to understand their logic to be able to replicate and adapt those results to problems of interest.

3 Unconstrained Optimization on $\mathbb{R}^n$

From now on, we focus on maximization problems for real-valued functions defined on a subset of $\mathbb{R}^n$. In this paragraph, we consider unconstrained optimization problem where the set over which the optimization problem is defined is an open set in $\mathbb{R}^n$. What makes such problems special (and easier to deal with) is that in an open set, we can always compare a point to all the points in a ball around it. Since all points are interior points, the conditions for being a local maximizer will have the same form for all points : we only need to check that the point achieves the highest value compared to a ball around it (which is symmetric). If we had non-interior points (boundary points) in the set, this would introduce an asymmetry. We will deal with that in the next paragraphs in the form of (in)equality constraints.

First, we consider single variable functions for simplicity, and then generalize it to multivariable functions. The next theorem provides the *necessary first order condition* and the *necessary second order condition* for an interior maximizer. The result does not require that D is open but restricts the attention to interior maximizers which is equivalent (we ignore maximizers on the boundary).

Theorem 3.1. *Let X be a set in $\mathbb{R}$, and $D \subset X$. Let $f : X \to \mathbb{R}$, and consider the problem*

$$\max_{x\in X} f(x) \;\; s.t. \;\; x \in D$$

and let $x^* \in int(D)$ be a maximizer of the problem.

(1) If f is differentiable at x^*, then $f'(x^*) = 0$.

(2) If f is differentiable in an open ball around x^*, and is twice differentiable at x^*, then $f''(x^*) \leq 0$.

In the theorem above, x^* is required to be an interior point of the constraint set D w.r.t. the whole real line $(\mathbb{R}, d_2)$, instead of (X, d_2). Assuming x^* to be an interior point of D implies that x^* is also an interior point of the domain X, and so we are able to talk about the derivative of f at x^*. In (2), we require $f'(x)$ to exist in some open ball around x^*, and so we are able to talk about $f''(x^*)$. This is why openness plays such a crucial role : we can consider derivatives at every point in the interior, which gives us information about local variations of the function, and if a set is open any maximizer has to be an interior point by definition of an open set.

Observe that necessary conditions equivalently characterize *local* maximizers. Since any global maximum has to be a local maximum, we can consider the set of local maximizers as the set of *candidate points* to be the maximum. Those conditions, however, are only necessary and not sufficient : being a local maximizer (a candidate global maximizer) does not imply being a global maximizer in general.

Now let's generalize this result to multivariate functions.

Theorem 3.2. *Let X be a set in $\mathbb{R}^n$, and $D \subset X$. Let $f : X \to \mathbb{R}$, and consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; x \in D$$

and let $x^* \in int(D)$ be a maximizer of the problem.

(1) If f is differentiable at x^*, then $\nabla f(x^*) = 0$.

(2) If f is differentiable in an open ball around x^*, and is twice differentiable at x^*, then $H_f(x^*)$ is negative semi-definite.

In the theorem above, again x^* is required to be an interior point of the constraint set D w.r.t. the whole Euclidean space $(\mathbb{R}^n, d_2)$, instead of (X, d_2). Assuming x^* to be an interior point of D implies that x^* is also an interior point of the domain X, and so we can talk about total/partial derivatives of f at x^*.

To maximize f, in practice we usually take partials of f and set them equal to 0, and then solve for the maximizers. Setting all partials equal to 0 is called the **(necessary) first order condition (FOC)** of the maximization problem. The theorem above implies that FOC is necessary for interior maximizers at which all partials exist.

Note that those results only hold *if f is differentiable.* If f is not differentiable, the previous theorem is simply not applicable and does not give us any information. In particular, this **does not** say that a non-differentiable function has or does not have a maximum, or where it might or might not be – it does not say anything at all.

Another way to interpret the theorem above is to look at Taylor approximations. At the first order, we have :

$$f(x+h) = f(x) + \nabla f(x) \cdot h + o(||h||)$$

when $||h|| \to 0$. Hence, if $f(x+h) \geq f(x)$ for all h (which is a consequence of x being a maximizer), this must mean that $\nabla f(x) \cdot h \leq 0$ for all h small enough. In particular, taking $h = t\nabla f(x)$ for $t > 0$ small enough we have :

$$\nabla f(x) \cdot (t\nabla f(x)) = t||\nabla f(x)||^2 \leq 0$$

Which is only possible if $\nabla f(x) = 0$.

We can similarly interpret the second-order condition : if x is a candidate maximizer, i.e $\nabla f(x) = 0$, then we have as $||h|| \to 0$:

$$f(x+h) = f(x) + \nabla f(x) \cdot h + \frac{1}{2} h^T \mathcal{H}_f(x) h + o(||h||^2) = f(x) + \frac{1}{2} h^T \mathcal{H}_f(x) h + o(||h||^2)$$

hence when h small :

$$f(x+h) - f(x) \approx \frac{1}{2} h^T \mathcal{H}_f(x) h$$

so $f(x+h) - f(x)$ is of the sign of $h^T \mathcal{H}_f(x) h$: if $\mathcal{H}_f(x)$ is negative semi-definite (f is locally concave at x), $f(x+h) - f(x) \leq 0$ for all h small enough, i.e $f(x') \leq f(x)$ for x' in some small ball around x, in other words x is a local maximum.

Those first and second order approximations highlight that the first order condition corresponds to being a *critical point* which is a local extremum, but the second order term determines whether it is a local minimum or maximum.

There are two things to be careful about when using FOC. First, FOC is not a sufficient condition for an x^* to be a maximizer, i.e. an x^* at which all partials are 0 may or may not be a maximizer. Second, FOC is only necessary for interior maximizers at which all partials exist; if a maximizer x^* is on the boundary of D then FOC may or may not hold at x^*; if some partials do not exist at a maximizer x^*, then it doesn't even make sense to talk about FOC at x^*.

In practice, we solve for all solutions to FOC, and consider them as "type 1" candidates for maximizers. If D is open, this is all we have to consider. If D is not open, we also collect all points on the boundary of D and all points at which some partials do not exist, and consider them as "type 2" candidates. Then we combine the two types of candidates and examine them carefully. It is possible that the problem does not have a solution at all, in which case no candidate is a maximizer. However, if we know that the problem has a maximizer, possibly by Weierstrass theorem, then we know that it must be among the candidates we have found (Proposition 1.3). Then the maximizers are exactly those candidates that give us the highest value among all candidates.

Negative semi-definite $H_f(x^*)$ is sometimes called the **necessary second order condition (necessary SOC)** of the maximization problem. The theorem above states that necessary SOC is necessary for interior maximizers at which f is twice differentiable, and so it may help us to rule out some solutions to FOC but are not maximizers of the problem.

Negative definite $H_f(x^*)$ is sometimes called the **locally sufficient second order condition (locally sufficient SOC)** of the maximization problem, because when f is C^2 in some open ball around x^*, a negative definite $H_f(x^*)$ is sufficient for x^* to be a *strict local maximizer*, in the sense that $\exists\ \delta > 0$ s.t. $f(x^*) > f(x)$ for any $x \in B_\delta(x^*) \setminus \{x^*\}$. See FMEA Theorem 3.2.1, 2.3.2, and 1.8.1 for a proof. Clearly, a negative definite $H_f(x^*)$ is not sufficient for x^* being a (global) maximizer, since $H_f(x^*)$ only gives us local properties of the function f.

Now let's state several sufficient conditions for x^* being a (global) maximizer.

Theorem 3.3. *Let X be a convex set in $\mathbb{R}^n$, and $D \subset X$. Let $f : X \to \mathbb{R}$ be a concave function, and consider the problem*

$$\max_{x\in X} f(x) \ \ s.t.\ x \in D$$

If f is differentiable at $x^* \in int(X) \cap D$, and $\nabla f(x^*) = 0$, then x^* is a maximizer of the problem.

In the theorem above, the objective function f is assumed to be concave, which is a global property.

Proof. WTS: $f(x^*) \geq f(x)$ for any $x \in D$.

Let's show an even stronger statement: $f(x^*) \geq f(x)$ for any $x \in X$.

Suppose $\exists\ \hat{x} \in X$ s.t. $f(\hat{x}) > f(x^*)$. Clearly, we have $\hat{x} \neq x^*$. By concavity of f, we know that

$$f\left(\lambda\hat{x} + (1-\lambda)x^*\right) \geq \lambda f(\hat{x}) + (1-\lambda) f\left(x^*\right)$$

for any $\lambda \in [0,1]$. Let $z := \hat{x} - x^*$, and for any $\lambda \in (0,1]$, we have

$$\begin{aligned}\frac{f(x^*+\lambda z) - f(x^*)}{\lambda} &= \frac{f\left(\lambda\hat{x} + (1-\lambda)x^*\right) - f(x^*)}{\lambda} \\ &\geq \frac{\lambda f(\hat{x}) + (1-\lambda) f(x^*) - f(x^*)}{\lambda} \\ &= f(\hat{x}) - f\left(x^*\right) > 0\end{aligned}$$

Because f is differentiable at x^* and $\nabla f(x^*) = 0$, we have

$$\lim_{\lambda \to 0} \frac{f(x^* + \lambda z) - f(x^*)}{\lambda} = \frac{d}{d\lambda} f\left(x^* + \lambda z\right)\Big|_{\lambda=0} = \nabla f\left(x^*\right) \cdot z = 0$$

which contradicts

$$\frac{f(x^* + \lambda z) - f(x^*)}{\lambda} \geq f(\hat{x}) - f\left(x^*\right) > 0$$

for all $\lambda \in (0, 1]$. □

If we replace the concavity assumption in the theorem above by quasi-concavity, the sufficiency result does not hold. For example, consider quasi-concave function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^3$. Clearly $0 \in int(\mathbb{R})$ and $f'(0) = 0$, but 0 is not a maximizer on $D = \mathbb{R}$.

However, if we further assume that the function is C^2 in some open ball around x^*, and that $H_f(x^*)$ is negative definite, and we can restore the sufficiency.

Theorem 3.4. *Let X be a convex set in $\mathbb{R}^n$, and $D \subset X$. Let $f : X \to \mathbb{R}$ be a quasi-concave function, and consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; x \in D$$

Suppose that
(1) f is differentiable at $x^* \in int(X) \cap D$, $\nabla f(x^*) = 0$, and
(2) f is C^2 in some open ball around x^*, and $H_f(x^*)$ is negative definite.
Then x^* is a maximizer of the problem.

4 Optimization under Equality Constraints in $\mathbb{R}^n$

In the previous section, we saw how to deal with interior points : this provided us with a method to deal with open sets and some non-open sets by considering "individually" all boundary and non-differentiability points as candidates. In general, this second might not be practical of feasible if there is, in a sense, "a lot" of boundary points.

We now consider the particular case of **equality constraints**, where all admissible points are boundary points so we cannot use the interior characterization directly. However, due to the particular structure of equality constraints, we can use the ideas of the unconstrained approach to rewrite the problem in a more convenient way. This will lead us to introducing the **Lagrangian Formalism**, that we will then extend to deal with both equality and inequality constraints (Kuhn-Tucker Theorem, in the next paragraph).

We consider the following problem. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^k$, with $g(x) = (g_1(x), ..., g_k(x))$ and $c = (c_1, ..., c_k) \in \mathbb{R}^k$. The set $\{x \in \mathbb{R}^n, g(x) = c\}$ is called *a level set* of g, and is pinned down by the choice of the constant c. We consider the problem of optimizing f on a level set of g :

$$\max_{x \in \{x, g(x) = c\}} f(x)$$

which we rewrite equivalently in the constrained form :

$$\max_{x \in \mathbb{R}^n} f(x)$$
$$\text{s.t. } g(x) = c$$

Where $g(x) = c$ explicitly rewrites as $g_i(x) = c_i$ for all i :

$$\begin{cases} g_1(x) & = c_1 \\ \vdots & \\ g_k(x) & = c_k \end{cases}$$

Existence questions are quite similar in spirit to the general case. For example, if we can prove that the level set $\{x, g(x) = c\} \subset \mathbb{R}^n$ is compact in $\mathbb{R}^n$ and f is continuous, we will get existence immediately.

Observe that if we can rewrite the level set as a parametrized region, where the parameter belongs to an open set, we can rewrite the whole problem as an unconstrained problem. Typically, if there exists $x : \mathbb{R} \to \mathbb{R}^n$ such that :

$$\{x, g(x) = x\} = \{x(t), t \in \mathbb{R}\}$$

Then we can rewrite :

$$\max_{x, g(x)=c} f(x) = \max_{t \in \mathbb{R}} f(x(t))$$

which is an unconstrained optimization problem. This can sometimes be done directly (we sometimes refer to this as "substitution") and allows to use the tools of the previous paragraph directly.

Example 4.1. *If $g : \mathbb{R}^2 \to \mathbb{R}$, with $g(x, y) = x + y$, then :*

$$\{(x, y), g(x, y) = 0\} = \{(t, -t), t \in \mathbb{R}\}$$

Sometimes, finding a parametrization is neither easy nor convenient, but thankfully we can develop tools to deal with equality constraints directly when the functions f and g are differentiable. This is the object of the next theorem, stated for one constraint.

Theorem 4.2. *Let $f, g : D \subset \mathbb{R}^n \to \mathbb{R}$ and $x^* \in int(D)$. If x^* is a local extremum of f under the constraint $g = c$, if f is differentiable at x^*, g is differentiable in a neigborhood of x^* and if $\nabla g(x^*) = 0$, then there exists $\lambda \in \mathbb{R}$ such that :*

$$\nabla f(x^*) = \lambda g(x^*)$$

λ is called the Lagrange multiplier associated to the constraint.

i.e at an extremum, the gradient of the objective function must be *colinear* to the gradient of the constraint : this is the constrained equivalent of the first order condition in an unconstrained problem. This result can be given a geometric interpretation and follows from linear algebra considerations which we will make more precise when talking about several constraints.

We now introduce Lagrangian notations. Given the problem of maximizing f under the constraint $g = c$, define the *Lagrangian of the problem* as the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that :

$$\mathcal{L}(x, \lambda) = f(x) - \lambda(g(x) - c)$$

The previous theorem rewrites as follows : if x^* is an extremum of f under the constraint $g = c$, then there exists λ such that (x^*, λ) is a critical point of $\mathcal{L}$, i.e :

$$\begin{aligned}\nabla\mathcal{L}(x^*, \lambda) = 0 &\Leftrightarrow \begin{cases} \frac{\partial\mathcal{L}}{\partial x}(x^*, \lambda) = 0 \\ \frac{\partial\mathcal{L}}{\partial\lambda}(x^*, \lambda) = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \nabla f(x^*) - \lambda\nabla g(x^*) = 0 \\ g(x^*) - c = 0 \end{cases}\end{aligned}$$

The theorem (and the Lagrangian notation) extend to more than one constraint.

Theorem 4.3. *Let $f, g_1, ..., g_k : D \subset \mathbb{R}^n \to \mathbb{R}$ and $c = (c_1, ..., c_k) \in \mathbb{R}^k$. If $x^* \in int(D)$ is a local extremum of f under the constraints $g_i = c_i$ for all i and if*

(i) f is differentiable at x^*

(ii) g is C^1 in a neighborhood of x^*

(iii) the family $(\nabla g_1(x^*), ..., \nabla g_k(x^*))$ is independent

then there exists $(\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$ such that :

$$\nabla f(x^*) = \sum_{i=1}^{k} \lambda_i \nabla g_i(x^*)$$

The Lagrangian with several constraints is defined as :

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{k} \lambda_i (g_i(x) - c_i) = f(x) - \lambda \cdot (g(x) - c)$$

If x^* is an extremum of f under the constraint $g = c$, then there exists $\lambda \in \mathbb{R}^k$ such that (x^*, λ) is a critical point of $\mathcal{L}$, i.e :

$$\begin{aligned}\nabla\mathcal{L}(x^*, \lambda) = 0 &\Leftrightarrow \begin{cases} \frac{\partial\mathcal{L}}{\partial x}(x^*, \lambda) = 0 \\ \frac{\partial\mathcal{L}}{\partial\lambda}(x^*, \lambda) = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \nabla f(x^*) - \lambda \cdot \nabla g(x^*) = 0 \\ g_i(x^*) - c_i = 0 \quad \forall i \end{cases}\end{aligned}$$

This theorem has a fundamental geometric interpretation that relies on the notion of tangent hyperplanes. Essentially, we can see with one constraint that the gradient of f at an optimum has to be orthogonal to the line tangent to the space $\{g(x) = c\}$ at that point. Essentially this captures the idea that "otherwise, we could move a little bit while staying in the constraint space and improve f". This tangency condition captures the idea that locally, we cannot improve on the value of f without exiting the space. A classical result on the tangent line is that the gradient of g at x is orthogonal to the tangent line to the space $\{g = c\}$ at x. If both $\nabla f(x^*)$ and $\nabla g(x^*)$ are orthogonal to the same line, they must be colinear, giving the result.

In general, the proof of the theorem is a consequence of Farkas lemma which is an important result in linear algebra :

Lemma 4.4 (Farkas Lemma)**.** *Let V a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Let $a \in V$ and $(a_1, ..., a_k) \in V^k$. The following statements are equivalent :*

(i) For all $x \in V$, if $\langle a_i, x\rangle \leq 0$ for all $i = 1, ..., k$, then $\langle a, x\rangle \leq 0$

(ii) There exists $(\lambda_1, ..., \lambda_k) \in \mathbb{R}_+^k$ such that $a = \sum_{i=1}^k \lambda_i a_i$.

Farkas lemma can be interpreted in terms of linear transformations and combined with result about differentiation (recall that derivatives correspond to local linear approximations) to prove the following proposition :

Proposition 4.5. *Let x^* a local extremum of f under $g = c$. If f is differentiable at x^*, g is C^1 in a neighborhood of x^* and $g'(x^*)$ (understood as the linear map from $\mathbb{R}^n$ to $\mathbb{R}^k$ corresponding to the matrix) is surjective, then for every $h \in \mathbb{R}^n$:*

$$g'(x^*)h = 0 \ \Rightarrow \ \nabla f(x^*) = 0$$

Observe that $h \mapsto g'(x^*)h$ surjective corresponds exactly to the condition that the family $(\nabla g_1(x^*), ..., \nabla g_k(x^*))$ is independent (no two constraints are colinear at the optimum, i.e there is no "redundant" constraint). The theorem follows from the proposition or Farkas lemma.

Using the Lagrangian formulation, we can further give sufficient second order conditions to be a local maximum.

Theorem 4.6. *Let $f, g_1, ..., g_k : D \subset \mathbb{R}^n \to \mathbb{R}$ and $c = (c_1, ..., c_k) \in \mathbb{R}^k$. If $x^* \in int(D)$ is a local maximum of f under the constraints $g_i = c_i$ for all i and :*

(i) f is twice differentiable at x^*

(ii) g is C^2 in a neighborhood of x^*

(iii) the family $(\nabla g_1(x^*), ..., \nabla g_k(x^*))$ is independent

then there exists $(\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$ such that :

$$\nabla_x \mathcal{L}(x^*, \lambda) = \nabla f(x^*) - \sum_{i=1}^k \lambda_i \nabla g_i(x^*) = 0$$
$$h^T \mathcal{H}_{xx} \mathcal{L}(x^*, \lambda) h \leq 0 \text{ for all } h \in ker(g'(x^*))$$

The last condition equivalently means that the Lagrangian, interpreted as a function of x, is locally concave in the constraint space. If this function is further globally concave, then the maximum is unique (the proof is similar to the previous unconstrained case). We will see those results again in more detail and more generality in the next section when incorporating inequality constraints.

5 Inequality Contstraints : Kuhn-Tucker Theorem

We now wish to extend the results of the previous section to inequality constraints. Inequality constraints are more complex in nature because they might not bind : if we have a constraint $g(x) \leq c$, and it turns out that at the optimum $g(x) < c$, then x is essentially an interior point and it is "as if" the constraint was not there locally. On the other if we saturate the constraint, i.e $g(x) = c$ at the optimum, then we need a machinery similar to that we just introduced to deal with an extra equality constraint – given the admissible directions of increase are locally reduced.

This section discusses the Kuhn-Tucker Theorem, which is a crucial result for constrained optimization. You may refer to FMEA Chapter 3.3 - 3.10.

First, let's define the problem we study in this section, and introduces the concept of *constraint qualification (CQ)*.

Definition 5.1. *Let X be an open set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions. Consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; g(x) \geq 0 \;\; and \;\; h(x) = 0$$

For a feasible point $\hat{x} \in X$, the inequality constraint $g_j(x) \geq 0$ is said to be ***binding at*** $\hat{x}$ *iff $g_j(\hat{x}) = 0$.*

We say that the ***constraint qualification (CQ) holds at*** $\hat{x}$ *iff the derivatives of all binding constraints*

$$\left\{\nabla g_j(\hat{x})\right\}_{\{j : g_j \, binding \; at \; \hat{x}\}} \cup \left\{\nabla h_l(\hat{x})\right\}_{l=1}^m$$

in $\mathbb{R}^n$ are linearly independent; otherwise we say that the ***constraint qualification (CQ) fails at*** $\hat{x}$.

As stated in the definition above, we study the problem

$$\max_{x \in X} f(x) \;\; \text{s.t.} \;\; x \in D$$

where the constraint set D is described by a set of k weak inequalities and a set of m equalities:

$$D := \left\{x \in X : g(x) \geq 0 \text{ and } h(x) = 0\right\}$$

In the problem, we require the domain X to be open, and the inequalities to be weak[5].

In practice, we often define the **Lagrangian function** of the maximization problem as

$$\begin{aligned}\mathcal{L}(x, \lambda, \mu) &= f(x) + \lambda^T g(x) + \mu^T h(x) \\ &= f(x) + \sum_{j=1}^k \lambda_j g_j(x) + \sum_{l=1}^m \mu_l h_l(x)\end{aligned}$$

and λ_j's and μ_l's are called the **Lagrangian multipliers**.

Now let's state Kuhn-Tucker theorem.

Theorem 5.2 (Kuhn-Tucker)**.** *Let X be an open set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions. Consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; g(x) \geq 0 \;\; and \;\; h(x) = 0$$

[5] If we have strict inequality in some of the constraints, then those constraints should be considered as a part of the definition of the open set X. If X is not open by the nature of the problem, then we should consider the "closed boundary" of X as a weak inequality constraint. For example, if the problem is

$$\max_{(x_1, x_2) \in \mathbb{R}^2_+} x_1 x_2$$

s.t.

$$\begin{cases} x_2 > 1 \\ x_1 + x_2 \leq 4 \end{cases}$$

then we should rewrite it as

$$\max_{(x_1, x_2) \in \mathbb{R} \times (1, +\infty)} x_1 x_2$$

s.t.

$$\begin{cases} x_1 \geq 0 \\ 4 - x_1 - x_2 \geq 0 \end{cases}$$

If x^* is a maximizer of the problem above, and CQ holds at x^*, then there exists a unique $(\lambda, \mu) \in \mathbb{R}_+^k \times \mathbb{R}^m$ s.t. the following two conditions hold:

(1) ***First order condition (FOC):***

$$\nabla f\left(x^*\right) + \lambda^T g'\left(x^*\right) + \mu^T h'\left(x^*\right) = 0$$

(2) ***Complementary slackness condition (CSC):***

$$h_l(x^*) = 0$$

for each $l \in \{1, \ldots, m\}$.

$$\lambda_j \geq 0,\ g_j\left(x^*\right) \geq 0,\ \textit{and}\ \lambda_j g_j\left(x^*\right) = 0$$

for each $j \in \{1, \ldots, k\}$.

Clearly, when there is no constraint ($k = m = 0$), the problem in the theorem above becomes $\max_{x \in X} f(x)$, and the FOC reduces to $\nabla f(x^*) = 0$, which is the necessary condition we saw in the previous section for an interior maximizer x^* at which f is differentiable. Under the assumptions of this theorem, x^* is automatically an interior point of X because X is open, and f is differentiable at x^* because f is differentiable everywhere.

The FOC in the theorem above is essentially

$$\nabla f\left(x^*\right) + \sum_{j=1}^{k} \lambda_j \nabla g_j\left(x^*\right) + \sum_{l=1}^{m} \mu_l \nabla h_l\left(x^*\right) = 0$$

or equivalently, for each $i \in \{1, \ldots, n\}$

$$\frac{\partial f}{\partial x_i}\left(x^*\right) + \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i}\left(x^*\right) + \sum_{l=1}^{m} \mu_l \frac{\partial h_l}{\partial x_i}\left(x^*\right) = 0$$

which is essentially setting the partials of the Lagrangian $\mathcal{L}(x, \lambda, \mu)$ w.r.t. x_i, $i = 1, 2, ..., n$ to zero:

$$\left.\frac{d}{dx}\mathcal{L}\left(x, \lambda, \mu\right)\right|_{x=x^*} = 0$$

or equivalently, for each $i \in \{1, \ldots, n\}$

$$\frac{\partial \mathcal{L}}{\partial x_i}\left(x^*, \lambda, \mu\right) = 0$$

Simply put, Kuhn-Tucker theorem states that if x^* is a maximizer and satisfies CQ, then there exist λ and μ s.t. (x^*, λ, μ) satisfies FOC + CSC.

In practice, we often write down the following system of conditions

$$\begin{cases} x \in X \\ \frac{\partial f}{\partial x_i}(x) + \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i}(x) + \sum_{l=1}^{m} \mu_l \frac{\partial h_l}{\partial x_i}(x) = 0,\ \forall\ i = 1, \ldots n \\ h_l(x) = 0,\ \forall\ l = 1, \ldots m \\ \lambda_j \geq 0,\ g_j(x) \geq 0, \text{ and } \lambda_j g_j(x) = 0,\ \forall\ j = 1, \ldots, k \end{cases}$$

which is sometimes known as the **Kuhn-Tucker condition**. Then we can solve for all solutions (x, λ, μ) to this system. Kuhn-Tucker theorem states that if x^* is a maximizer and satisfies CQ, x^*

must be a part of some solution (x, λ, μ) to the K-T condition, and therefore, we must be able to find this x^* by solving for all solutions to the K-T condition.

Notice that the theorem only works for maximizer x^*'s at which CQ holds. If CQ fails at x^*, then there may not exist (λ, μ) s.t. (x^*, λ, μ) satisfies FOC and CSC, even if x^* is a maximizer of the problem. Therefore, we may never be able to find such maximizers by solving the K-T condition. An example is given below.

Example 5.3. *Consider the problem*

$$\max_{(x_1,x_2)\in\mathbb{R}^2} -x_2$$

s.t.

$$x_1^2 - x_2^3 = 0$$

Because $x_2^3 = x_1^2 \geq 0$*, and so* $x_2 \geq 0$*, and clearly the unique maximizer of this problem is* $\left(x_1^*, x_2^*\right) = (0,0)$*. However, if we write down the Lagrangian*

$$L\left(x_1, x_2, \lambda\right) = -x_2 + \lambda\left(x_1^2 - x_2^3\right)$$

and consider the FOC

$$\begin{cases} \frac{\partial L}{\partial x_1}\left(x_1, x_2, \lambda\right) = 2\lambda x_1 = 0 \\ \frac{\partial L}{\partial x_2}\left(x_1, x_2, \lambda\right) = -1 - 3\lambda x_2^2 = 0 \end{cases}$$

clearly there exists no $\lambda \in \mathbb{R}$ *s.t.* $(0,0,\lambda)$ *satisfies the FOC above. Therefore, we will never find the correct maximizer* $\left(x_1^*, x_2^*\right) = (0,0)$ *by solving the FOC.*

This is not a violation of Kuhn-Tucker theorem because CQ fails at $(0,0)$*, and so K-T theorem is silent about the maximizer* $\left(x_1^*, x_2^*\right) = (0,0)$*. To see why CQ fails at* $(0,0)$*, let* $h\left(x_1, x_2\right) := x_1^2 - x_2^3$ *and we have* $\nabla h\left(0,0\right) = (0,0)$*, which is a not linearly independent when considered as a set of only one vector.*

The right way to use K-T theorem to find the maximizers of a problem is the following:

First, we collect all x's that appears in some solution (x, λ, μ) to K-T condition, and consider them as "type 1" candidates for the maximizers. Then we collect all x's at which CQ fails, and consider them as "type 2" candidates. Then we combine the two types of candidates and examine them carefully. It is possible that the problem does not have a solution at all, in which case no candidate is a maximizer. However, if we know that the problem has a maximizer, possibly by Weierstrass theorem, then we know that it must be among the candidates we have found. Then the maximizers are exactly those candidates that give us the highest value among all candidates.

Our procedure above involves solving the K-T condition

$$\begin{cases} x \in X \\ \frac{\partial f}{\partial x_i}(x) + \sum_{j=1}^{k} \lambda_j \frac{\partial g_j}{\partial x_i}(x) + \sum_{l=1}^{m} \mu_l \frac{\partial h_l}{\partial x_i}(x) = 0, \ \forall\ i = 1, \ldots n \\ h_l(x) = 0, \ \forall\ l = 1, \ldots m \\ \lambda_j \geq 0, \ g_j(x) \geq 0, \text{ and } \lambda_j g_j(x) = 0, \ \forall\ j = 1, \ldots, k \end{cases}$$

and one may wonder how we can solve this system in practice. This problem is difficult in general, and we can only analytically solve this system when the functions take very simple forms. Notice that the third line implies that either $g_j(x) = 0$ or $g_j(x) > 0$,which implies $\lambda_j = 0$, and so we have two cases to discuss for each j. In total, we have 2^k cases to discuss. In each case, we have $n + m + k$ equations and k weak inequalities for $n + m + k$ unknowns. Also, the number 2^k of cases we need to discuss increases very fast as k increases, and this is another difficulty for this kind of problems in general.

In many economic applications, we might be able to use our economic intuitions to guess which constraints are binding at optimum and which are not. You can verify your guess that the constraint $g_j(x) \geq 0$ is binding at optimum by showing that there is no solution to the K-T condition with $\lambda_j = 0$.

5.1 Applying Kuhn-Tucker Theorem: an Example

Sometimes, K-T theorem cannot be directly applied to solve a problem, and we need to analyze the problem carefully, and try to transform the original problem into another to which K-T theorem applies. Now let's carefully analyze a specific maximization problem as an example.

Consider the problem

$$\max_{(x_1,x_2)\in\mathbb{R}_+^2} x_1^\alpha x_2^{1-\alpha}$$

s.t.

$$p_1x_1 + p_2x_2 \leq m$$

where $\alpha \in (0,1)$, $p_1, p_2 \in \mathbb{R}_{++}$, and $m \in \mathbb{R}_+$ are parameters.

The first issue we should consider is the existence of maximizers. Because power function is continuous, and the objective function $x_1^\alpha x_2^{1-\alpha}$ is a product of two power functions, and so it is continuous. If we can also show that the constraint set

$$D(p,m) := \left\{(x_1,x_2) \in \mathbb{R}_+^2 : p_1x_1 + p_2x_2 \leq m\right\}$$

is a nonempty compact set, then by Weierstrass theorem the problem must have a solution. Clearly we have $(0,0) \in D$, and so we only need to show compactness

Claim 5.4. *The constraint set $D(p,m)$ is compact in $\left(\mathbb{R}^2, d_2\right)$, for any $p \in \mathbb{R}_{++}^2$ and $m \in \mathbb{R}_+$.*

Proof. By Heine-Borel, it is sufficient to show that D is closed and bounded in $\left(\mathbb{R}^2, d_2\right)$.

(1) Closedness

Take any sequence (x^n) in D s.t. $x^n \to x^0 \in \mathbb{R}^2$. WTS: $x^0 \in D$ (Recall that what we use here is the sequential definition of closed sets.)

Because $x^n \to x^0$, we have $x_1^n \to x_1^0$, and $x_2^n \to x_2^0$. Because for each n, we have $x^n \in D$, and so $x_1^n \geq 0$ and $x_2^n \geq 0$, and so $x_1^0 \geq 0$ and $x_2^0 \geq 0$ (weak inequality is preserved under limit), and therefore $x^0 \in \mathbb{R}_+^2$. Because $p_1x_1 + p_2x_2$ is a continuous function in x, we have $p_1x_1^n + p_2x_2^n \to p_1x_1^0 + p_2x_2^0$. Because $p_1x_1^n + p_2x_2^n \leq m$ for each n, we have $p_1x_1^0 + p_2x_2^0 \leq m$. As a result, we have $x^0 \in D$.

(2) Boundedness

Take any $x \in D$, we have $p_1x_1 \leq m$ and $p_2x_2 \leq m$, and so

$$d_2(x,0) = \sqrt{x_1^2 + x_2^2} \leq \sqrt{\left(\frac{m}{p_1}\right)^2 + \left(\frac{m}{p_2}\right)^2} = m\sqrt{p_1^{-2} + p_2^{-2}}$$

Let $r := m\sqrt{p_1^{-2} + p_2^{-2}} + 1$, and we have $D \subset B_r(0)$. □

In fact, this claim is also true in $\mathbb{R}^n$, i.e. $D(p,m) := \left\{\mathbf{x} \in \mathbb{R}_+^n : p \cdot \mathbf{x} \leq m\right\}$ is compact in $(\mathbb{R}^n, d_2)$ for any $p \in \mathbb{R}_{++}^n$ and $m \in \mathbb{R}_+$.

According to the claim above, we know that the problem always has a solution by Weierstrass theorem.

The difficulty to apply K-T theorem is that the objective function has the domain $\mathbb{R}^2_+$ by its nature, because the power function z^α is only defined on $\mathbb{R}_+$ for $\alpha \in (0,1)$ in general. Clearly the domain $\mathbb{R}^2_+$ is not open in $\mathbb{R}^2$, and so it does not satisfy the assumption of K-T theorem. Although it is possible to smoothly extend the objective function $x_1^\alpha x_2^{1-\alpha}$ to $\mathbb{R}^2$, we still cannot apply K-T theorem because the objective function is not differentiable on the two axes. Therefore, we need to approach the problem in another way.

If $m = 0$, then the only feasible point is $(x_1, x_2) = (0,0)$, and the problem becomes trivial: it has a unique maximizer $(0,0)$, and the maximum is 0.

If $m > 0$, consider the feasible point $\hat{x} = \left(m/(2p_1), m/(2p_2)\right) \in \mathbb{R}^2_{++}$. At this point, the objective takes a strictly positive value. However, whenever $x_1 = 0$ or $x_2 = 0$, the objective takes the value 0. Therefore, there cannot be any maximizer on the two axes, and it is without loss of optimality to focus on the domain $\mathbb{R}^2_{++}$. Consider the new problem

$$\max_{(x_1,x_2)\in\mathbb{R}^2_{++}} x_1^\alpha x_2^{1-\alpha}$$

s.t.

$$p_1 x_1 + p_2 x_2 \le m$$

This new problem has the same set of maximizers as the original problem, and so we can solve this new problem instead. To see this, take any maximizer x^* of the original problem, and then we have $f(x^*) \ge f(\hat{x}) > 0$, and so $x^* \in \mathbb{R}^2_{++}$, and so it is a maximizer of the new problem. On the other hand, take any maximizer x^* of the new problem. Because $f(x^*) \ge f(\hat{x}) > 0$, and $f(x) = 0$ for any $x \in \mathbb{R}^2_+ \backslash \mathbb{R}^2_{++}$, we know that x^* is a maximizer of the original problem.

In this new problem, the domain $\mathbb{R}^2_{++}$ is an open set in $\mathbb{R}^2$. Also, the objective function $x_1^\alpha x_2^{1-\alpha}$ is C^1 on the entire domain $\mathbb{R}^2_{++}$, and so K-T theorem applies. Define $g : \mathbb{R}^2_{++} \to \mathbb{R}$ as

$$g(x) := m - p_1 x_1 - p_2 x_2$$

and $x \in \mathbb{R}^2_{++}$, we have $\nabla g(x) = (-p_1, -p_2) \neq 0$, which is linearly independent when considered as a set of only one vector. Therefore, CQ holds at all feasible points.

Write down the Lagrangian

$$L(x_1, x_2, \lambda) = x_1^\alpha x_2^{1-\alpha} + \lambda(m - p_1 x_1 - p_2 x_2)$$

and then the K-T condition

$$\begin{cases} x \in \mathbb{R}^2_{++} \\ \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \\ (1-\alpha) x_1^\alpha x_2^{-\alpha} - \lambda p_2 = 0 \\ \lambda \ge 0,\ m - p_1 x_1 - p_2 x_2 \ge 0, \text{ and } \lambda(m - p_1 x_1 - p_2 x_2) = 0 \end{cases}$$

By the two FOCs, we have $\lambda > 0$, and so by CSC we have $m - p_1 x_1 - p_2 x_2 = 0$. Also, comparing the two FOCs gives us

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^\alpha x_2^{-\alpha}} = \frac{\lambda p_1}{\lambda p_2}$$

i.e.

$$\frac{p_1 x_1}{p_2 x_2} = \frac{\alpha}{1-\alpha}$$

and so we have

$$(x_1, x_2, \lambda) = \left(\frac{\alpha m}{p_1}, \frac{(1-\alpha) m}{p_2}, \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}\right)$$

as the unique solution to the K-T condition. So $(x_1, x_2) = \left(\alpha m/p_1, (1-\alpha)\, m/p_2\right)$ is the unique type 1 candidate in this problem.

Because CQ holds at all feasible point, there is no type 2 candidate at all. Because the problem has a solution by Weierstrass, we know that the unique type 1 candidate

$$(x_1, x_2) = \left(\frac{\alpha m}{p_1}, \frac{(1-\alpha)\, m}{p_2}\right)$$

must be the unique maximizer of the problem.

This conclusion also applies to the case $m = 0$, and so we can unify the two cases. Therefore, for any $\alpha \in (0,1)$, $p_1, p_2 \in \mathbb{R}_{++}$, and $m \in \mathbb{R}_+$, the problem has a unique maximizer $\left(x_1^*, x_2^*\right) = \left(\alpha m/p_1, (1-\alpha)\, m/p_2\right)$.

5.2 Sufficient Conditions

The K-T theorem we have studied provides a condition that is necessary for maximizers at which CQ holds, and it is by no means a sufficient condition. However, the theorem below provides a sufficient condition for an x^* being a maximizer of a constrained maximization problem.

Theorem 5.5. *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions. Consider the problem*

$$\max_{x \in X} f(x) \;\; \text{s.t.} \;\; g(x) \geq 0 \text{ and } h(x) = 0$$

If x^* is feasible, and there exists $(\lambda, \mu) \in \mathbb{R}_+^k \times \mathbb{R}^m$ s.t. the following three conditions hold
(1) FOC:

$$\nabla f\left(x^*\right) + \lambda^T g'\left(x^*\right) + \mu^T h'\left(x^*\right) = 0$$

(2) CSC:

$$\lambda_j \geq 0, \; g_j\left(x^*\right) \geq 0, \text{ and } \lambda_j g_j\left(x^*\right) = 0$$

for each $j \in \{1, \ldots, k\}$, and
(3) The Lagrangian $L_{\lambda,\mu} : X \to \mathbb{R}$ defined as

$$L_{\lambda,\mu}(x) := f(x) + \lambda^T g(x) + \mu^T h(x)$$

is a concave function,
then x^* is a maximizer of this problem.

In the theorem above, condition (1) and (2) are FOC and CSC in the K-T theorem. The additional requirement (3) requires the Lagrangian function to be concave in x. According to this theorem, when we solve the K-T condition for type 1 candidates, if we happen to find a solution $\left(\hat{x}, \hat{\lambda}, \hat{\mu}\right)$ to K-T condition s.t. under this $\left(\hat{\lambda}, \hat{\mu}\right)$ the Lagrangian is a concave function in x, then we can immediately conclude that $\hat{x}$ is a maximizer of the problem. But keep in mind that there might be other maximizers, since the theorem is silent about uniqueness.

As a special case of the theorem above, when there is no constraint at all, i.e. $k = m = 0$, the FOC reduces to $\nabla f(x^*) = 0$, and the concavity of the Lagrangian reduces to the concavity of the objective function f. This is consistent with Theorem 3.3.

The next theorem provides yet another sufficient condition for an x^* being a maximizer.

Theorem 5.6. *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions, and f is concave. Consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; g(x) \geq 0 \text{ and } h(x) = 0$$

If x^* is feasible, and there exists $(\lambda, \mu) \in \mathbb{R}^k_+ \times \mathbb{R}^m$ s.t. the following three conditions hold
(1) FOC:

$$\nabla f\left(x^*\right) + \lambda^T g'\left(x^*\right) + \mu^T h'\left(x^*\right) = 0$$

(2) CSC:

$$\lambda_j \geq 0, \; g_j\left(x^*\right) \geq 0, \text{ and } \lambda_j g_j\left(x^*\right) = 0$$

for each $j \in \{1, \ldots, k\}$, and
(3) $\lambda_j g_j$ is quasi-concave for each $j = 1, \ldots, k$, and $\mu_l h_l$ is quasi-concave for each $l = 1, \ldots, m$,
then x^* is a maximizer of this problem.

This theorem requires the objective f to be concave, g_j to be quasi-concave if $\lambda_j > 0$, h_l to be quasi-concave (quasi-convex) if $\mu_l > 0$ ($\mu_l < 0$). There is no restriction on g_j (h_l) if λ_j (μ_l) is zero.

We often deal with objective functions which are quasiconcave, rather than concave. The following result gives conditions under which the Kuhn-Tucker conditions are sufficient for a maximum, when f is quasiconcave:

Theorem 5.7. *Let X be an open and convex set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions, and f is quasi-concave. Consider the problem*

$$\max_{x \in X} f(x) \;\; s.t. \;\; g(x) \geq 0 \text{ and } h(x) = 0$$

If x^* is feasible, and there exists $(\lambda, \mu) \in \mathbb{R}^k_+ \times \mathbb{R}^m$ s.t. the following three conditions hold
(1) FOC:

$$\nabla f\left(x^*\right) + \lambda^T g'\left(x^*\right) + \mu^T h'\left(x^*\right) = 0$$

(2) CSC:

$$\lambda_j \geq 0, \; g_j\left(x^*\right) \geq 0, \text{ and } \lambda_j g_j\left(x^*\right) = 0$$

for each $j \in \{1, \ldots, k\}$, and
(3) $\nabla f(x^*) \neq 0$, $\lambda_j g_j$ is quasi-concave for each $j = 1, \ldots, k$, and $\mu_l h_l$ is quasi-concave for each $l = 1, \ldots, m$,
then x^* is a maximizer of this problem.

5.3 Comparative Statics

Let's consider the parameterized optimization problem $P(\alpha)$:

$$\max_{x \in X} f(x, \alpha) \;\; \text{s.t.} \;\; g(x, \alpha) \geq 0 \text{ and } h(x, \alpha) = 0$$

where the parameter α is taken from some set A. For each α, if the problem $P(\alpha)$ has a solution, then we can calculate the maximum value of the problem $P(\alpha)$, and define it as $f^*(\alpha)$. Then it might be interesting to study how the value function $f^*(\alpha)$ changes as the parameter α changes.

Theorem 5.8 (Envelope)**.** *Let X be an open set in $\mathbb{R}^n$, and A be an open set of parameters in $\mathbb{R}^s$. Let $f : X \times A \to \mathbb{R}$, $g : X \times A \to \mathbb{R}^k$, and $h : X \times A \to \mathbb{R}^m$ be C^1 functions. For each parameter $\alpha \in A$, define the problem $P(\alpha)$ as*

$$\max_{x \in X} f(x, \alpha) \;\; s.t. \;\; g(x, \alpha) \geq 0 \text{ and } h(x, \alpha) = 0$$

Let $\hat{A} := \left\{\alpha \in A : \arg\max P(\alpha) \neq \emptyset\right\}$, and define the value function $f^* : \hat{A} \to \mathbb{R}$ as

$$f^*(\alpha) := \max_{x \in X} \left\{f(x, \alpha) : g(x, \alpha) \geq 0 \text{ and } h(x, \alpha) = 0\right\}$$

For parameter $\alpha^* \in A$, suppose:
(1) In the problem $P(\alpha^*)$, there is a unique maximizer x^*, and CQ holds at x^*.
(2) There exists $\varepsilon > 0$ and $r > 0$ s.t. $\forall\, \alpha \in B_\varepsilon(\alpha^*)$, $\left(\arg\max P(\alpha)\right) \cap B_r(x^*) \neq \emptyset$.
Then the value function f^* is differentiable at α^*, and

$$\begin{aligned} f^{*\prime}(\alpha^*) &= \frac{d}{d\alpha} L\left(x^*, \lambda^*, \mu^*, \alpha\right)\Big|_{\alpha=\alpha^*} \\ &= \frac{d}{d\alpha} f\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*} + \lambda^{*T} \frac{d}{d\alpha} g\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*} + \mu^{*T} \frac{d}{d\alpha} h\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*} \end{aligned}$$

where λ^* and μ^* are the unique Lagrangian multipliers found by K-T theorem for the problem $P(\alpha^*)$.

In the theorem above, condition (1) guarantees that K-T theorem applies to the problem $P(\alpha^*)$, and so we can find a unique λ^* and μ^* s.t. (x^*, λ^*, μ^*) satisfies FOC and CSC. Condition (2) implies that $f^*(\alpha)$ is well-defined for any $\alpha \in B_\varepsilon(\alpha^*)$, and so we can talk about differentiability of f^* at α^*.

The proof of this theorem is not straightforward. However, let's provide a heuristic "proof", assuming away some technical aspects of the problem. Assume that for each $\alpha \in B_\varepsilon(\alpha^*)$, we can find $x(\alpha) \in \arg\max P(\alpha)$ s.t. $x(\alpha)$ is differentiable at α^*. Also, assume that for each $\alpha \in B_\varepsilon(\alpha^*)$, in the problem $P(\alpha)$, CQ holds at $x(\alpha)$. By K-T theorem, there exists $\lambda(\alpha)$ and $\mu(\alpha)$ s.t. $\left(x(\alpha), \lambda(\alpha), \mu(\alpha)\right)$ satisfies FOC and CSC for the problem $P(\alpha)$. Assume that $\lambda(\alpha)$ and $\mu(\alpha)$ are differentiable at α^*.

By definition of f^*, we have $f^*(\alpha) = f\left(x(\alpha), \alpha\right)$ for any $\alpha \in B_\varepsilon(\alpha^*)$, and therefore

$$\begin{aligned} f^{*\prime}(\alpha^*) &= \frac{d}{d\alpha} f\left(x(\alpha), \alpha\right)\Big|_{\alpha=\alpha^*} \\ &= \frac{d}{d\alpha}\left[f\left(x(\alpha), \alpha\right) + \lambda(\alpha)^T g\left(x(\alpha), \alpha\right) + \mu(\alpha)^T h\left(x(\alpha), \alpha\right)\right]\Big|_{\alpha=\alpha^*} \end{aligned}$$

The second equality is because CSC implies that $\lambda(\alpha)^T g\left(x(\alpha), \alpha\right)$ and $\mu(\alpha)^T h\left(x(\alpha), \alpha\right)$ are constantly 0 for any $\alpha \in B_\varepsilon(\alpha^*)$. Then by chain rule, we have

$$\begin{aligned} f^{*\prime}(\alpha^*) = {} & \frac{d}{dx} f\left(x, \alpha^*\right)\Big|_{x=x^*} \cdot x'(\alpha^*) + \frac{d}{d\alpha} f\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*} \\ & + \lambda^{*T}\left(\frac{d}{dx} g\left(x, \alpha^*\right)\Big|_{x=x^*} \cdot x'(\alpha^*) + \frac{d}{d\alpha} g\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*}\right) \\ & + \mu^{*T}\left(\frac{d}{dx} h\left(x, \alpha^*\right)\Big|_{x=x^*} \cdot x'(\alpha^*) + \frac{d}{d\alpha} h\left(x^*, \alpha\right)\Big|_{\alpha=\alpha^*}\right) \\ & + g\left(x^*, \alpha^*\right)^T \cdot \lambda'(\alpha^*) + h\left(x^*, \alpha^*\right)^T \cdot \mu'(\alpha^*) \end{aligned}$$

By feasibility, the last term $h\left(x^*,\alpha^*\right)^T\cdot\mu'\left(\alpha^*\right)=0\cdot\mu'\left(\alpha^*\right)=0$. In the second last term, if $g_j\left(x^*,\alpha^*\right)=g_j\big(x\left(\alpha^*\right),\alpha^*\big)>0$, we have $g_j\big(x\left(\alpha\right),\alpha\big)>0$ when α is in some open ball around α^*. By CSC, we have $\lambda_j\left(\alpha\right)=0$ when α is in this open ball around α^*, and so $\lambda_j'\left(\alpha^*\right)=0$. Therefore, the second last term $g\left(x^*,\alpha^*\right)^T\cdot\lambda'\left(\alpha^*\right)=0$. Therefore, we have

$$
\begin{aligned}
& f^{*\prime}\left(\alpha^*\right) \\
&= \left(\frac{d}{dx}f\left(x,\alpha^*\right)\bigg|_{x=x^*}+\lambda^{*T}\frac{d}{dx}g\left(x,\alpha^*\right)\bigg|_{x=x^*}+\mu^{*T}\frac{d}{dx}h\left(x,\alpha^*\right)\bigg|_{x=x^*}\right)\cdot x'\left(\alpha^*\right) \\
&+ \frac{d}{d\alpha}f\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*}+\lambda^{*T}\frac{d}{d\alpha}g\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*}+\mu^{*T}\frac{d}{d\alpha}h\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*}
\end{aligned}
$$

By FOC, we have

$$\frac{d}{dx}f\left(x,\alpha^*\right)\bigg|_{x=x^*}+\lambda^{*T}\frac{d}{dx}g\left(x,\alpha^*\right)\bigg|_{x=x^*}+\mu^{*T}\frac{d}{dx}h\left(x,\alpha^*\right)\bigg|_{x=x^*}=0$$

and therefore we have

$$
\begin{aligned}
f^{*\prime}\left(\alpha^*\right) &= \frac{d}{d\alpha}f\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*}+\lambda^{*T}\frac{d}{d\alpha}g\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*}+\mu^{*T}\frac{d}{d\alpha}h\left(x^*,\alpha\right)\bigg|_{\alpha=\alpha^*} \\
&= \frac{d}{d\alpha}L\left(x^*,\lambda^*,\mu^*,\alpha\right)\bigg|_{\alpha=\alpha^*}
\end{aligned}
$$

which is the envelope result we want to show[6].

With the envelope theorem, we do not need to derive the value function $f^*(\alpha)$ explicitly to analyze how it responds to changes to the parameter α. The derivative of the Lagrangian are usually simpler, because in many cases the constraints are linear in the parameters (e.g. the budget constraint is linear in endowment and prices).

Now let's verify the equation in the envelope theorem for the eample given in Section 4.4. The solution to the Kuhn-Tucker conditions is given as

$$\big(x_1^*(p_1,p_2,m),x_2^*(p_1,p_2,m),\lambda^*(p_1,p_2,m)\big)=\left(\frac{\alpha m}{p_1},\frac{\left(1-\alpha\right)m}{p_2},\frac{\alpha^\alpha\left(1-\alpha\right)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}\right)$$

The value function of the maximization problem is

$$v(p_1,p_2,m)=\left(x_1^*\right)^\alpha\left(x_2^*\right)^{1-\alpha}=\frac{m\alpha^\alpha(1-\alpha)^{1-\alpha}}{p_1^\alpha p_2^{1-\alpha}}$$

Taking its first order derivative w.r.t. (p_1,p_2,m) we have:

$$\frac{\partial v}{\partial p_1}=-\frac{m\alpha^{1+\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha+1}p_2^{1-\alpha}}=-\lambda^*x_1^*$$

$$\frac{\partial v}{\partial p_2}=-\frac{m\alpha^{\alpha}(1-\alpha)^{2-\alpha}}{p_1^{\alpha}p_2^{2-\alpha}}=-\lambda^*x_2^*$$

[6] In the heuristic calculation above, we assumed that $x\left(\alpha\right)$, $\lambda\left(\alpha\right)$, and $\mu\left(\alpha\right)$ are all differentiable at α^*. However, the result is still true even when this assumption does not hold.

$$\frac{\partial v}{\partial m} = \frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{p_1^{\alpha}p_2^{1-\alpha}} = \lambda^*$$

From the envelope theorem, we have:
$\frac{\partial \mathcal{L}}{\partial p_1}(x_1^*, x_2^*, \lambda^*, p_1, p_2, m) = -\lambda^* x_1^*$,
$\frac{\partial \mathcal{L}}{\partial p_2}(x_1^*, x_2^*, \lambda^*, p_1, p_2, m) = -\lambda^* x_2^*$,
$\frac{\partial \mathcal{L}}{\partial m}(x_1^*, x_2^*, \lambda^*, p_1, p_2, m) = \lambda^*$.

5.4 Interpretation of Lagrangian Multipliers*

We can use envelope theorem to obtain an interpretation of the Lagrangian multipliers.

Let X be an open set in $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}^k$, and $h : X \to \mathbb{R}^m$ be C^1 functions. Consider the parameterized problem $P(a, b)$

$$\max_{x \in X} f(x)$$

s.t.

$$\begin{cases} g(x) + a \geq 0 \\ h(x) + b = 0 \end{cases}$$

where $(a, b) \in \mathbb{R}^k \times \mathbb{R}^m$ are parameters. If the problem $P(a, b)$ has a solution, define $f^*(a, b)$ as the maximum value of the problem $P(a, b)$.

When we move (a, b) around $(a^*, b^*) = (0, 0)$, we are considering perturbations around the original problem $P(0, 0)$

$$\max_{x \in X} f(x)$$

s.t.

$$\begin{cases} g(x) \geq 0 \\ h(x) = 0 \end{cases}$$

A small positive a_j can be viewed as a slight relaxation of the constraint $g_j(x) \geq 0$, which might make the feasible set slightly larger, which in turn might make the maximum value slightly higher. We are interested in how such a slight relaxation of the constraint $g_j(x) \geq 0$ will affect the maximum value, i.e. we are interested in the partial derivative $\frac{\partial f^*}{\partial a_j}(0, 0)$.

If in the original problem $P(0, 0)$ there is a unique maximizer x^*, CQ holds at x^*, and $\exists\, \varepsilon > 0$ and $r > 0$ s.t. $\forall\ (a, b) \in B_{\varepsilon}(0, 0)$, $\exists\ x \in \big(\arg\max P(a, b)\big) \cap B_r(x^*)$, then we can invoke the envelope theorem at $(a^*, b^*) = (0, 0)$, and we have

$$\begin{aligned}
&\frac{d}{d(a,b)} f^*(a, b)\Big|_{(a,b)=(0,0)} \\
&= \frac{d}{d(a,b)} f\left(x^*\right)\Big|_{(a,b)=(0,0)} + \lambda^{*T} \frac{d}{d(a,b)} \Big(g\left(x^*\right) + a\Big)\Big|_{(a,b)=(0,0)} \\
&+ \mu^{*T} \frac{d}{d(a,b)} \Big(h\left(x^*\right) + b\Big)\Big|_{(a,b)=(0,0)} \\
&= 0 + \lambda^{*T} \cdot \big[I_k | 0_{k \times m}\big] + \mu^{*T} \cdot \big[0_{m \times k} | I_m\big] \\
&= (\lambda_1, \ldots, \lambda_k, \mu_1, \ldots \mu_m)
\end{aligned}$$

Therefore, we have

$$\frac{\partial f^*}{\partial a_j}(0, 0) = \lambda_j$$

for each $j = 1, \ldots, k$, and

$$\frac{\partial f^*}{\partial b_l}(0,0) = \mu_l$$

for each $l = 1, \ldots, m$.

Therefore, the Lagrangian multiplier λ_j corresponding to the inequality constraint $g_j(x) \geq 0$ measures the marginal increase in the maximum value under a marginal relaxation of the constraint $g_j(x) \geq 0$. As a consequence, λ_j is sometimes called the **shadow price of the constraint** $g_j(x) \geq 0$.

In a firm's maximization problem, the objective function is usually the firms profit function, and a constraint $g_j(x) \geq 0$ usually represents the requirement that total usage of some resource (labor/capital/electricity/...) is weakly less than the total amount of this resource available to the firm. Then λ_j can be called the **shadow price of this resource** (labor/capital/electricity/...), and by envelope theorem, it measures the marginal increase in profit by marginally increasing the total amount of this resource available to the firm. In other words, λ_j is the price the firm is willing to pay for an additional unit of this resource.

By K-T theorem, the Lagrangian multiplier λ_j corresponding to the weak inequality constraint $g_j(x) \geq 0$ is required to be nonnegative. This is consistent with our interpretation of λ_j as the marginal gain by slightly relaxing the constraint $g_j(x) \geq 0$, because a relaxation of a constraint never decreases the maximum value. Also, CSC in K-T theorem states that if the constraint $g_j(x) \geq 0$ is not binding at optimum, i.e. $g_j(x^*) > 0$, where x^* is the unique maximizer, then we must have $\lambda_j = 0$, i.e. we will gain nothing by slightly relaxing the constraint. On the other hand, if there is a strictly positive marginal gain by slightly relaxing the constraint $g_j(x) \geq 0$, i.e. $\lambda_j > 0$, then there is no reason not to fully exploit the constraint in the optimization, i.e. the constraint must be binding at optimum.

Notice that CSC only requires at least one of λ_j and $g_j(x)$ is zero, and in fact they could be both zero. In other words, it is possible for some constraint to be binding, while slightly relaxing this constraint does not increase the maximum. For example, consider the problem

$$\max_{x \in \mathbb{R}} -x^2 \text{ s.t. } x \geq 0$$

in which case both the Lagrangian multiplier is 0 and the constraint is binding.

The Lagrangian multiplier μ_l corresponding to the equality constraint $h_l(x) = 0$ measures the marginal change in the maximum value under a marginal perturbation of the equality constraint $h_l(x) = 0$. By its nature, it may be positive or negative, which is consistent with the assumption on μ_l in K-T theorem.

6 A Brief Introduction to Dynamic Programming

Consider the infinite horizon inequality-constrained maximization problem (one-sector optimal growth problem):

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t.\ & c_t + k_{t+1} \leq f(k_t) \\ & c_t, k_{t+1} \geq 0, t = 0, 1, ... \\ & k_0 > 0 \ \ given. \end{aligned}$$

where U and f are strictly increasing. Here the choice variables are a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$. Since k_0 is a parameter, we can find the value function $v : \mathbb{R}_+ \to \mathbb{R}$ which gives the maximized value of the object function given k_0:

$$\begin{aligned} v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t U(c_t) \\ s.t.\ & c_t + k_{t+1} \leq f(k_t) \\ & c_t, k_{t+1} \geq 0, t = 0, 1, ... \\ & k_0 > 0 \ \ given. \end{aligned}$$

Since the problem is time-independent, $v(k_1)$ would be the maximized value of the object function if the program starts in period $t+1$. $\beta v(k_1)$ is this value discounted at period $t = 0$. So we can write the problem in $t = 0$ as

$$\begin{aligned} & \max_{c_0, k_1} U(c_0) + \beta v(k_1) \\ s.t.\ \ & c_0 + k_1 \leq f(k_0) \\ & c_0 \geq 0, k_1 \geq 0 \\ & k_0 > 0 \ \ given. \end{aligned}$$

By definition of v we substitute out c_0:

$$v(k_0) = \max_{0 \leq k1 \leq f(k_0)} U\big(f(k_0) - k_1\big) + \beta v(k_1)$$

Since subscript does not matter, we can write

$$v(k) = \max_{0 \leq y \leq f(k)} U\big(f(k) - y\big) + \beta v(y)$$

Now the unknown is not a variable but the function v. We call this a **functional equation**.

Theorem 6.1 (Blackwell's sufficient condition for contraction)**.** *Let $X \subset \mathbb{R}^k$ and $B(X)$ a real vector space of bounded functions $f : X \to \mathbb{R}$, with norm defined as $||f|| = \sup_{x \in X} |f(x)|$. Let $T : B(X) \to B(X)$ be an operator satisfying*

(1) (monotonicity) If $f, g \in B(X)$ and $f(x) \leq g(x)$ for $\forall x \in X$, then $(Tf)(x) \leq (Tg)(x)$ for $\forall x \in X$.

(2) (discounting) There exists some $\beta \in (0, 1)$ s.t.

$$(T(f + a))(x) \leq (Tf)(x) + \beta a, for\ \forall f \in B(X), a \geq 0, x \in X,$$

where $(f + a)$ is defined as $(f + a)(x) = f(x) + a$.

Then T is a contraction with modulus β.

In dynamic programming, Blackwell's sufficient conditions are often easy to verify. In the problem above, we define an operator T as

$$(Tv)(k) = \max_{0 \leq y \leq f(k)} U\big(f(k) - y\big) + \beta v(y)$$

We want to find a fixed point of operator T, i.e. a function v s.t. $Tv = v$. We can verify that T is a contraction using Blackwell, and by contraction mapping theorem we know that such a fixed point exists. To find it we iteratively apply T, starting with an arbitrary function v_0 until convergence (under certain specified convergence criteria).