Columbia MA Math Camp

Introduction to Propositional Logic

Vinayak Iyer

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Outline of Today's Class

- Introduction to Propositional Logic
- Introduction to Set Theory
- Overview of how to write proofs

But first, more important stuff :

- How to approach classes
- Rubinstein Interview 1
- Rubinstein Interview 2

Before we start...

- This is meant to prepare a basic foundation of mathematics that will help you in your classes
    - Not meant to teach you advanced mathematics
- Your goal should be to understand the concepts well. The more you use this time to really grasp concepts, the easier it will be in your classes
- I will try and go slow, but please let me know if I am going too fast.
- Please interrupt me and ask questions! You will benefit and so will others!

Why do we need to learn Propositional Logic?

- In economics, we rely on a lot of proofs. You will encounter proofs in most of your classes.
- So it is useful to go through some foundations of logic which will help you in your class

Propositions

Definition

A **proposition** is a statement that can be either **True (T)** or **False (F)**

Examples :

- *"The earth is flat"* - **False**
- *"March has 31 days"* - **True**
- *"You should maintain a work-life balance"* - **Not a proposition**

Notation : Lower case letters are often used to represent propositions.

Example :

- p : *"The earth is flat"*
- q : *"March has 31 days"*

Logical Operators (Connectives)

Definition

A **connective** is a symbol that combine propositions. Propositions separated by connectives make a **compund proposition**.

There are 3 basic connectives :

1. **Conjunction :** "p and q" and is denoted by $\wedge$
   - *"The Earth is flat and March has 31 days"*
2. **Disjunction :** "p or q" and denoted by $\vee$
   - *"The Earth is flat or March has 31 days"*
     Note : The meaning of $\vee$ is *inclusive* in the sense that the either one statement is true or both can be true
3. **Negation :** "$\neg p$" is the negation of the statement p

Truth Values

- We now must determine the ***truth values*** of compound statements. (What were compound statements?)
- Note that the truth or falsity of a compound statement depends on the truth or falsity of the individual statements.
- We do this through a ***truth table*** which lists all possible combinations of the truth value of a compound statement based on the truth value of the individual components.

$p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$\neg p$

p	$\neg p$
T	F
F	T

Examples of Evaluating Compound Propositions

Example

Suppose p, q, r are 3 statements. Suppose p and r are true while q is false. Evaluate whether or not the following statements are true.

- $\neg(p \wedge q)$:- Since p is True and q is False $\implies$ $p \wedge q$ is False $\implies$ $\neg(p \wedge q)$ is **True**
- $(\neg p) \wedge (\neg q)$:- $(\neg p)$ is False and $(\neg q)$ is True $\implies$ $(\neg p) \wedge (\neg q)$ is **False**
- $p \vee ((\neg q) \wedge r)$:- (?)

Filling out a Truth Table

In the previous slide I specified the truth values of the individual statements. If I had not specified the truth values, then we would have to fill out the truth table based on all possible combinations of p, q and r. So let's fill it out for $p \vee ((\neg q) \wedge r)$.

p	q	r	$p \vee \neg q \wedge r$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Conditional Propositions

Definition

The conditional operator ***if*** is used in a compound statement of the form *"if p then q"* and is denoted by $p \to q$. It is also called the logical implication sometimes.

- p is called the **antecedent or hypothesis**
- q is called the **consequent or conclusion**
- $p \to q$ is also read as *"p implies q"*

Example :

- p : *"You got an A"*
- q : *"I give you a dollar"*
- $p \to q$: *"If you get an A, then I will give you a dollar"*

Truth Table for $\rightarrow$

Like the other logical operators, the **logical implication** ($\rightarrow$) also has a truth table.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- The last two are by definition. To understand why, let us go back to the previous example.
- $p \rightarrow q$:- *"If you get an A, then I will give you a dollar"*
  - Statement is true if I keep my promise and false if I don't...
  - If you don't get an A, then irrespective of whether or not I give you a dollar, I haven't broken my promise!

Example

Example

Suppose p, q and r are statements such that p, r are true and q is false. Evaluate the following compound statements.

1. $\neg(p \to q)$:- From the truth table, since p is true and q is false $\implies$ $p \to q$ is False and the negation of a False statement is True. Hence $\neg(p \to q)$ is **True**
2. $(p \wedge q) \to r$:- $(p \wedge q)$ is False. This implies that $(p \wedge q) \to r$ is a **True** statement.
3. $(p \to q) \to r$:- (?)

Necessary and Sufficient Conditions

- You will come across these terms a LOT in economics!
- Consider the logical statement $p \to q$
  - p is said to be a **sufficient condition** for q
    - If the statement *"If you get an A, then I will give you a dollar" is true*, then knowing that you got an A is sufficient to know that you also received a dollar.
  - q is said to be a **necessary condition** for p
    - Similarly, for the statement above to be true, then getting a dollar is necessarily true if you get an A.
- Of course p being sufficient for q does not mean it is necessary and vice versa.

More examples of Necessary vs Sufficient Conditions

1. **Consider the statement *"If it rains, then it is cloudy"***
   - Then knowing that it is raining is **sufficient** to know that it is cloudy. (But being cloudy is **not sufficient** for rain!)
   - It is **necessary** for it to be cloudy for rain to occur. In other words, if it is not cloudy, then it cannot rain! (It is **not necessary** for it to rain for it be cloudy!)
2. How does this relate to economics? Most of you must have heard the phrases **"First Order Condition" and "Second Order Condition"** right?
   - The FOC when maximizing a function $f(x)$ is given by $f'(x) = 0$. For x to be an interior maximum, it is **necessary** that $f'(x) = 0$ (but not sufficient!)
   - The SOC sufficient condition is generally given by the statement that if x is such that $f'(x) = 0$ and $f''(x) < 0$, then x is a maximizer. But this is not necessary! - Find the maxima of $f(x) = -x^2$

Some statements can be both necessary and sufficient

- Consider the conditional statement given by :
  - *"A number being 2,4,6 or 8* $\rightarrow$ *it is an even number*
    - As per the discussion previously, knowing a number being 2,4,6 or 8 is **sufficient** to know that it is even. But it is **not necessary** that number be 2,4,6 or 8 for it to be even
- Now consider the alternate statement :
  - *"A number being 2,4,6 or 8* $\rightarrow$ *it is a positive even number* $<$*10*
    - Now a number being 2,4,6 or 8 is **both necessary and sufficient** for it to be a positive even number < 10
    - In this case, we can actually write *"A number being 2,4,6 or 8* $\leftrightarrow$ *it is an even number* $<$*10*

Definition

We write $p \leftrightarrow q$ or p if and only if q when p is **both necessary and sufficient** for each other.

What does the truth table look like? When is $p \leftrightarrow q$ true?

Converse and Contrapositive

Definition

For a conditional statement $p \to q$

- The **converse** is defined as $q \to p$
- The **contrapositive** is defined as $(\neg q) \to (\neg p)$

Properties :

1. If $p \to q$ is true, that **does not** mean that $q \to p$ is true!
   1.1 When p is False and q is True, then $p \to q$ is True but $q \to p$ is False
2. The contrapositive $(\neg q) \to (\neg p)$ is **equivalent** to $p \to q$. (Why?)

Question : **Does $p \to q$ imply $\neg p \to \neg q$?**

Example by Rubinstein!

Q6. My paper has just been rejected. What should I do?

I have a lot of experience with the mental state you must be in, so I have three pieces of advice:

(a) Don't read the referee reports. They are likely to depress you. Even if they are potentially useful, you are not in a state of mind to benefit from them.

(b) Find comfort in my motto: "A paper that has not been rejected should not be published." But beware of the faulty logic in assuming that "every paper that has been rejected should be published."

Figure 1: Words of Wisdom from Rubinstein

- What is the faulty logic? $p \to q$ DOES NOT IMPLY $\neg A \implies \neg B$
- Contrapositive is that "*A paper that has been published must have been rejected*"

Examples of Converse and Contrapositive Statements

Consider the statement *"If a paper that has been published, then it must have been rejected"*

- **Converse** : *If a paper has been rejected, then it must be published*
- **Contrapositive** : *If a paper has not been rejected, then it must not be published*

Note : The contrapositive has the same truth value as the original statement.

Logical Equivalence

Definition

Two statements p and q are **logically equivalent** denoted by $p \leftrightarrow q$ when the truth values in **all** rows in the truth table are the same.

For example, show that the statements $(p \rightarrow q)$ and $(\neg p \vee q)$ are logically equivalent

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

DeMorgan's Law

Fact

Consider 2 propositions p and q. Then the statements $\neg(p \vee q)$ *and* $(\neg p \wedge \neg q)$ *are logically equivalent i.e.*

$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

Proof.

p	q	$\neg(p \vee q)$	$(\neg p \wedge \neg q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

$\square$

Definition

A **tautology** is a formula that is always true i.e. for every combination of truth values of its components, the compound statement is true.

Example

Show that $(p \to q) \vee (q \to p)$ is a tautology.

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \vee (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Final Example

Example

Suppose x is a real number. Consider the statement :

$$\text{If } x^2 = 4, \text{ then } x = 2$$

Construct the **converse** and **contrapositive** of the statement. Determine the truth or falsity of the original, converse and the contrapositive using your knowledge of algebra. In the converse statement, identify the **necessary and sufficient statements**.

Proof.

The converse of the statement is "If $x = 2$, then $x^2 = 4$". The contrapositive is "If $x \neq 2$, then $x^2 \neq 4$".

Truth value : The original statement is false since $(-2)^2 = 4$. The contrapositive has the same truth value as the original, hence is also false, while the converse is true i.e. If $x = 2$, then $x^2 = 4$.

Knowing that $x = 2$ is sufficient to know that $x^2 = 4$, while x^2 being equal to 4 is necessary for x being equal to 2. $\square$