Problem Set 3
MA Math Camp 2021
Due Date : Monday September 6th, 2021

Answers should be typed and submitted in PDF format on Gradescope (see the course website for details). Be sure to answer every question thoroughly, and try to write complete and rigorous yet concise proofs. You can contact me if you have *specific* questions about the problem set, or if you think you have spotted a typo or mistake.

1. State if and where the following function are differentiable, and compute their derivative :
   a) $f : x \mapsto \frac{1}{1+x^2}$ defined over $\mathbb{R}$
   b) $f : x \mapsto \sqrt{x^2-1}$ defined over $(1, \infty)$
   c) $f : x \mapsto a^x$ defined over $\mathbb{R}$
   d) $f : (x, y) \mapsto cos(x)sin(y)$ over $\mathbb{R}^2$

2. a) Verify that Schwarz theorem (symmetry of the second order derivatives) holds for the following C^2 functions :
      i. $f(x, y) := x\exp(xy)$
      ii. $f(x, y) := ln(x^2 + y^2 + 1)$
      iii. $f(x, y) := (y + 2)tan(x)$

   b) Prove that the following function is not C^2 at 0 :

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

   (Hint : assume that it is C^2 at $(0, 0)$ and find a violation of Schwarz theorem)

3. Let $F : \mathbb{R} \to \mathbb{R}$ a C^1 function. Show that the following function is continuous on R^2 :

$$f(x,y) := \begin{cases} \frac{F(x)-F(y)}{x-y} & \text{if } x \neq y \\ F'(x) & \text{if } x = y \end{cases}$$

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ a differentiable function. Differentiate the functions : $u(x) = f(x, -x)$, $g(x, y) = f(y, x)$.

5. For the following functions from a given interval I to $\mathbb{R}$, compute $\sup_{x\in I} f(x)$, $\inf_{x\in I} f(x)$, state if these are attained and at which point(s) :
   a) $f(x) = x(1 - x)$ on $I = [0, 1]$
   b) $f(x) = 1 - e^{-x}$ on $I = \mathbb{R}^+$
   c) $f(x) = 3x^4 - 4x^3 + 6x^2 - 12x + 1$ on $I = \mathbb{R}$
   d) $f(x) = \frac{1}{\sqrt{x^2-x+1}}$ on $I = [0, 1]$

6. Find the maximum and minimum of $f(x, y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$ using the Kuhn-Tucker method. Using the substitution $y^2 = 1 - x^2$ solve the same problem as a single variable unconstrained problem. Do you get the same results? Why or why not?

7. A consumer's utility maximization problem is

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}_{++}\times\mathbb{R}_{+}} \quad & \alpha \ln x + y \\ \text{s.t. } & px + qy \leq m \\ & y \geq 0 \end{aligned}$$

where, $\alpha > 0$, $p > 0$, $q > 0$, $m > 0$ are parameters.

a) Argue that the budget constraint must hold with equality.

b) Write the Lagrangian. State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

c) Are there any admissible points where the constraint qualification fails?

d) Solve for the maximizer (x^*, y^*).

e) Find the value function $v(p, q, m)$. What does the Envelope Theorem tell you about the derivative of $v(p, q, m)$ with respect to q?

8. A firm produces two outputs, x and y, using a single input z. The price of x has been normalized to 1; the price of y is p. The firm's program is

$$\begin{aligned} \max_{(x,y)\in\mathbb{R}^2} \quad & x + py \\ \text{s.t. } & x^2 + y^2 \leq z \\ & x \geq 1, \ y \geq 0 \end{aligned}$$

$p > 0$ and $z > 0$ are parameters.

a) Write the Lagrangian.

b) State the Kuhn-Tucker necessary conditions for a maximum. Are these conditions sufficient for a maximum?

c) Are there any admissible points where the constraint qualification fails? Can any of these points be a solution to the program?

d) Solve for the maximizer (x^*, y^*).

e) Find the value function, $f^*(p, z)$.

f) What does the Envelope Theorem tell you about the derivative of $f(p, z)$ with respect to z?