Stability with complementarities in decentralized many-to-one matching markets

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Abstract

Complementarities are known to jeopardize stability in many-to-one matching markets. The present work explores a modeling approach tailored for large markets characterized by dynamic allocation with temporary partnerships and search frictions – the prominent example being labour markets – with the hope to obtain less stringent stability requirement. Weaker notions of stability of the process locally in time and asymptotically in market size are defined, according to which equilibrium may be obtained even when a stable equilibrium may not exist on the complete market. A general reduction method is proposed to study stability under general constraint graphs, in particular the category of graphs produced by a sequential search process. Application of this method to a simple search process yields an overall negative results : local stability on simple structures is sufficient, but symetrically local instability can spread to the whole graph. Asymptotic results may be recovered for large markets in which tree graphs are obtained ; it remains extremely challenging to ensure stability in general settings without any assumptions on the production function. Several qualitative properties of such models are explored, notably its (lack of) convergence to a static stable allocation when there exists one and optimality. In the particular case where the usual gross substitutes assumption is satisfied, the model should converge towards the static stable equilibria. This is, however, not the case in general even when the surplus function is submodular and there exists a stable equilibrium. Well known examples of instability are also revisited. Finally, what this approach can and cannot explain highlights several intuitions that might prove fruitful for further studies of stability issues with complementarities.

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Contents

1. Introduction

The problem is introduced and motivated from a purposefuly non-technical perspective, before turning to a litterature review on the subject and finally outlining the approach that was followed in the present work and its main conclusions.

1.1. Motivation

Complementarities are pervasive in economic interactions. In many situations, the objects of choice or the agents with whom we interact are radically heterogeneous in various respects – the relevant outcome is then jointly determined in a possibly complex fashion by the interaction of different types : although several decisions might be distinct, their results are profoundly co-dependant.

If a restaurant is unable to hire any cooks, it will not want to hire any waiters – even less will it hire more waiters to make up for the lack of cook. Classical examples of complementarities are abundant and wide-ranging. Two goods might be worthless one without the other : I have no utility for a roll of film if I do not own a film camera and vice versa. The value of moving to a certain geographical area might depend exclusively on whether or not a significant other is also able to move there. Loosely speaking, we might say that complementarities are characterized by a positive co-dependance in terms of outcome *satisfaction*[1] on the result of two distinct decisions, made by the same agent or not. The labour market is of course an intuitive example as the diversity of skills and production creates many forms of complementarities, from stark non-substitutability – an accountant cannot readily substitute for an architect – to productive interactions – the better the input of your coworkers is, the higher your own contribution to production can be.

Heterogeneity also impacts the structure of interactions. Classical market analysis relies on the assumption that homogeneous goods or services are traded in large quantity, and price adjusts so that quantities traded are at equilibrium – but if those goods or services are different enough, supply and demand break down : quantity adjustment can no longer clear the market and we need new tools to understand the equilibria that may arise. The theory of matching under its various forms (see the litterature review below) has provided a powerful framework for studying assignment problems on bipartite markets[2]. In such settings, agents belonging to two distinct identifiable groups (e.g. buyers and sellers, firms and workers, husbands and wives, donors and receivers) must be matched, either one to one (pair formation), one to many (a firm hires several workers), or even many to many. Heterogeneity can be arbitrarily stark, up to the point of fully idiosyncratic specifications – i.e. the preferences or surplus can be specified as a function of the *identity* of the agents who match. Yet, this is still, strategically speaking *one* market and heterogeneous goods or partnerships can be substituted to one another. Most of the litterature on matching has focused on the concept of *stability* to understand equilibria in bipartite matching markets. Loosely speaking, a matching is stable if no pair[3] could increase

[1] This is notably distinct from co-dependance that comes from a *constraint* e.g. whether I own a certain good does not impact the satisfaction I get from some other good but whether I can buy none, either or both depends on a single budget constraint.

[2] The concept of market can be taken here in a very loose meaning, encompassing all kinds of partnership or exchange situations with or without money transfers

[3] Pair is here to be understand as a mathematical pair of the relevant objects. This will be a usual in the one-to-one framework, while in the many-to-one framework one of the elements of the pair is a set e.g.

the satisfaction of both parties by matching – relative to their current match. This is appealing because it relies on the intuition that no agents will remain in a relatively unsatisfactory match, as they will instead exploit mutually beneficial opportunities.

Heterogeneity and complementarities are, as we have seen, two prominent features of a variety of bipartite matching situations, especially in the many-to-one setup. Yet this is precisely the case in which complementarities are known to jeopardize the existence of a stable equilibrium. Indeed, complementarities are apt to generate cycles of blocking pairs so that no assignment is able to eliminate profitable deviations. For this reason, most of the litterature has focused on variants of the so-called "Gross Substitutes" assumption, which can be (very) loosely expressed as the fact that the desire of an agent to match with another cannot be conditional on the ability to match with a third one.

This clearly precludes any situations that ressembles our restaurant-cook-waiter example previously stated. Yet, this type of configuration seems very realistic and typical of the labour market. From a normative perspective, such situations pose serious problems to market designers trying to optimally conceive assignment procedures. From a descriptive perspective, the behaviour of decentralized matching markets in which complementarities prevent existence of a stable equilibrium is a blind spot of the theory. Some works have tried to recover stability under weaker conditions, but the behaviour of agents in the case of non-existence remains a question mark.

The aim of this work is to consider precisely the unstable cases and find alternative routes to understanding assignment behaviours. More specifically, the motivating example and reference case will be the labour market – and two additional prominent characteristics of real life labour markets will play an important part in specifying the direction of our enquiry : dynamics and frictions. First, labour markets are very poorly described by static matching problems – production and hiring unfold in time and partnerships are hardly ever definitive. Second, such large markets are characterized by frictions insofar as it clearly seems unreasonable to assume that all agents can have knowledge of and access to the whole market ; in practice, this means that not all blocking pairs are susceptible to be formed because agents might not *know* or *be able to* exploit opportunities. Dynamics intuitively give some sense to the (static) idea of instability – there might be improvements, even cycles of improvements, but matchings might still occur temporarily – and frictions weaken, at least locally in time, the likeliness of profitable deviations hence the stability requirement. The two are essentially linked and the aim of the present work is to exploit their structure to explore alternative modeling of the behaviour of unstable markets.

To sum up, the situation we consider is that of a large bipartite many-to-one matching market where transfers are allowed and such that :

- the matching process unfolds dynamically and surplus is produced at all times ;
- matching opportunities are subject to frictions as access to the market is limited and subject to a random meeting search process ;
- the surplus function's inputs are characterized by prominent heterogeneity and its outputs exhibits potentially complex and strong complementarity structures

We explore whether some sort of weaker "local" (in time) stability can be recovered with these ingredients. Intuitively, the unstability result hinges crucially on one shot relationships and

firm/group of workers pairs.

perfect ability to exploit the whole market. The aim of this work is to maintain the essential intuition of the standard (static) stability notion as the driving force behind allocations while showing that a slightly weaker form motivated by frictional dynamics is able to explain the behaviour of markets where non-existence would obtain under usual requirements.

Before turning to the more technical description of our model, its approach and the conclusions we can draw from it, our problem and setup is put into perspective with existing litterature on the subject.

1.2. Litterature review

The modern theory of matching is often traced back to the foundational work of Gale and Shapley (1962) [20] on the so-called "marriage problem". They considered the *Non-transferable utility* (NTU) context with given ordinal preferences on the identity of the match partner and proposed an algorithmic assignment procedure that served both as a proof of existence of a stable equilibirum in the one-to-one case and as a pratical computational approach. Subsequent work by Shapley and Shubik [43] opened the way to *transferable utility* (TU) matching models by considering quasilinear utility and allowing for transfers between matched parties. The NTU litterature has known wide-ranging successes in the field of *market design* as examplified by the works of Alvin Roth (see e.g. [37] [40]) and applications to e.g. school choice, organ donations, or public housing.

The TU Litterature has had a different range of applications both theoretical and empirical – notably the marriage market : see Becker (1973) [7], Dupuy and Galichon (2015) [17], Chiappori (2017) [12], Choo and Siow (2006) [15], Goussé et. al. (2017) [24]) ; hedonic models : see Chiappori et. al. (2010) [14] ; auctions see notably Hatfield and Milgrom (2005) [27] ; and obviously labour markets, see notably Crawford and Knoer (1981) [16], Kelso and Crawford (1982) [28]. It has been shown to be deeply connected with linear programming and the theory optimal transport – see notably Ekeland (2005) [18], Chiappori et. al. (2009) [14]. For a general reference on the theory of optimal transport and related subjects, see for example the monograph by Santambrogio (2015) [42]. For more details on the various connections between optimal transport and economics, see the book by Galichon (2018) [23].

The early works of Gale and Shapley [20] already considered the many-to-one matching problem via the so-called "college admissions" problem, but they restricted themselves to preferences of colleges defined over individual students, thus eliminating *de facto* any complementarities. As was pointed out by Roth (1985) [39], when preferences are taken over *groups* of students (or workers,...), the many-to-one problem is no longer strictly equivalent to the one-to-one case and most importantly stability in general is not guaranteed.

A large subsequent litterature has focused on exploring stability of many-to-one matching markets along several axes. Most of this work focused on the static case by considering a one-shot matching situation. We will focus on the TU case here as it is more relevant to the present work. Kelso and Crawford (1982) [28], building on previous works by Crawford and Knoer (1981) [16], formulated the Gross Substitutes (GS) condition and showed it was sufficient for existence of a stable equilibrium. They also proposed a generalization of the Gale and Shapley deferred acceptance algorithms with transfers that could be interpreted as an approximation of a decentralized allocation process[4]. They exhibited a clear and simple example that failure

[4] Although such process give a sense of dynamics, the spirit of such models remain very much anchored in a static situation : there is no time dimension in the behaviour of the agents, no surplus is reaped until the

of the GS condition could lead to non-existence, and that submodularity was not sufficient for obtaining GS. Roth (1984) [38] extends the model to allow for symmetry between workers and firms. Gul and Stachetti (1999, 2000) [25] [26] expanded on the Gross Substitutes condition by providing alternative formulations ; they reformulated the Kelso and Crawford model as a Walrasian equilibrium problem with discrete commodities and made an important connection with auctions. They also put forward profound methodological relationships between discrete allocation problems, the theory of lattices and matroid theory (see [26]). In the same vein, and still with a setup geared towards multi-unit auctions (but formally equivalent to the many-to-one TU matching problem), Bikhchandani and Ostroy (2002) [8] highlighted connections with linear programming and lattices – the essential tool of this litterature for proving existence relies on showing that the core has a lattice property. Yet all these models crucially rely on the GS substitutes condition and remain silent as to what may happen when it is relaxed. Others examples of strongly connected works focused on auctions include Ausubel and Milgrom (2002) [5] whose proxy ascending package auction model can be recast as a many-to-one matching problem.

The litterature on search and frictional matching models has approached assignment on the labour market from a different perspective and with a different set of tools. The foundational work of Mortensen and Pissarides (see e.g. [35]) initiated a whole litterature on frictional employment from a macroeconomic approach, but with little reference to heterogeneity and the actual structure of matching. The works of Shimer and Smith (see Shimer and Smith 2000 [45] 2001 [46], Shimer 2005 [44] and Smith 2011 [47]) bridged the gap by considering unidimensional heterogeneity (skill) with a common structure of preferences inspired from the Becker marriage model [7]. Their crucial assumption is that matched agents cannot search, hence they are blocked in the relationship until it exogenously breaks hence they might miss other opportuinities. They focused on steady state configurations of the model and explored the geometry of "acceptance sets" (the set of types with whom a given type accepts to match upon encounter). They famously obtain an analogue of the positively assortative matching result of Becker in the case in which there are complementarities of type *accross sides* of the market – i.e the surplus function is supermodular. Subsequent extensions of such models include the addition of explicit search costs (see Atakan (2006) [4]) and richer type spaces (see Goussé et. al. (2017) [24]). It is noteworthy that rich type spaces (discrete or continuous and generally multidimensional with potentially complex structure) seem a natural feature in the spirit of the question we are investigating – complex complementarity structures are intuitively tied to rich heterogeneity – yet it might raise some specific questions. Such setups might jeopardize uniqueness of equilibrium and general properties are harder to investigate – as, for instance, assortativity is no longer a well defined notion. This is not the focus of this work, but one should keep in mind that such issues lurk in the background – for recent works on the question see e.g. Dupuy and Galichon (2015) [17], Chiappori et. al. (2016) [13], or Lindenlaub (2017) [31].

Other approaches to dynamic matching has come from the growing litterature on dynamic bargaining and partnerships. Seminal references on decentralized bargaining in large markets include Rubinstein and Wolinsky (1985) [41], Gale (1987) [21], and Binmore and Herrero (1988) [9]. A key differences in such approaches with respect to the present work is that these models generally consider that partners exit the market upon contracting – which is a prominently unrealistic feature in a labour market. These works are nevertheless strongly related to ours

convergence of the process and all ends after it – in other words, it is really a *one shot* matching situation.

and provide general insights and methodological building blocks for the time dimension of our problem. Subsequent works by Gale and Sabourian (2006) [22], Abreu and Manea (2012a, 2012b) [2] [1] have focused on Markov Perfect Equilibria in dynamic decentralized setups. A recent paper by Elliott and Nava (2018) [19] explores a similar framework for thin markets in which agents have exhaustive knowledge of the market and exhibits conditions for convergence, efficiency, potential delays and mismatches.

The growing litterature on networks, and more specifically bargaining and partnerships on networks, is directly relevant to our modeling approach although it has a different focus. Indeed, as will become apparent below, our approach considers local constraints on matching – that stem from the search process, which de facto generates a network structure by which matching is constrained. Such networks are very regular, hence the focus is much less on general structures than it is in e.g. Abreu and Manea (2012) [2], Manea (2017,2018) [32] [33], Bloch et. al. (2018) [10], Ambrus (2017) [3], yet these works provide interesting insights for our problem.

The last three blocks of litterature mentionned – search, dynamic bargaining and networks – all focus on one-to-one matching and the many-to-one setup might complexify substantially the models, because partnership decisions made at different moment in time interact and modify the gains. Our aim is precisely to bring those dimensions together in such a way that it might actually simplify some problems of instability.

As we have seen, although complementarities are a standard feature of standard consumer theory and textbook examples usually include the case of Leontief preferences or production functions, complementarities of different skills on the labour market and their consequences have often been overlooked - except for complementarities of types *accross* sides of the market leading to positively assortative matching. It seems reasonable to consider that there exists complementarities in the interactions of workers employed by a given firm. An unpublished paper by Kremer and Maskin (1996) [30] considered a simple version of this problem to study segregation by skill, relying on a particular form for the production function where two tasks unequally sensitive to skill must be performed. The macroeconomic and development consequences of such structures have been exhibited in the famous "O-Ring theory" of Michael Kremer (2010) [29]. On the more technical side, a paper by McCann and Trokhimtchouk (2010) [34] gave a general structure to the Kremer/Maskin pair formation problem and provided general results using the theory of optimal transport. Yet, this model explores grouping of workers without considering the heterogeneity of *firms* and restricts to *fixed quotas* – pairs ; although the generation to larger groups is not problematic, the generalization of the model to endogenous group of workers sizes is not self-evident.

Several recent endeavours to go beyond the Gross Substitutes condition in matching model are close to this work in intention but follow a different approach – usually they remain in the static setting and their aim is to find a weaker condition than GS by exploiting the size of the market or some continuity property of preferences. Nguyen, Peivandi and Vohra (2016) [36] exploit the assumption that complementarities are small compared to the size of the market to design mechanisms that satisfy efficiency, no justified envy and asymptotic strategy-proofness. Azevedo and Hatfield (2015) [6] consider a continuous setup in which asymmetry between the two sides of the market and size can help recover stability as long as *one side* of the market has substitutable preferences. Che, Kim and Kojima (2018) [11] only require some convexity and continuity of the demand of firms in the set of available workers ; they consider this in the case of large (infinite) markets and build on it to show that approximately stable matchings exist in large finite economies.

1.3. Approach and main results

The approach of this work lies in between two strands of litterature. On the one hand, it is rooted in the traditional theory of static many-to-one matching in that *stability* is the pivotal concept and intuition to understand equilibrium allocation and payoffs. On the other hand, the ingredients of dynamics and search relates this work to another litterature on dynamic formation of partnership with frictions. The essential difference of those litteratures lies in their vision of the *opportunity cost of matching.* In the former, the opportunity cost of a match is the potential benefit of all the other matches – which were all accessible in the static model – hence the stability requirement. In the latter, the opportunity cost is a form of *waiting cost.* Usually, such models do away completely with the local competition between several possible matches by assuming that no two encounter occur at the same time – hence stability in the usual sense becomes de facto void of meaning ; they recover non-trivial behaviour by assuming either that matches are definitive or that matched agents are unable to search and matches break off exogenously.

By contrast with previously mentionned endeavours to go beyond the GS case, the present work endeavours to modify the stability requirements by adding the elements of search and dynamics into model. The aim is to bridge different litteratures into a somehow more realistic model in order to enquire if what appears to be complications (e.g. frictions) might not in the end simplify the respective problems in each appraoch by softening some constraints.

In that regards the aim of this work is primarily methodological. Although the hope and goal of modeling is, of course, to construct an applicable framework that could then be tested against data, the scope of the present work is much more restricted. We try to outline a new and alternative approach to accomodate the key elements of heterogeneity, complementarities, dynamics and frictions on large matching markets using insights from various strands of litterature, in order to overcome the lack of understanding of the driving forces of actual markets that exhibit such characteristics.

The initial insight of our model was the following : instability relies on precise knowledge of the *whole* market and the possibility of any worker being disputed by any firm ; intuitively, if any two firms do not compete for the allocation of "too many" workers between themselves, i.e if there is some sort of local constraint structure, then we might be able to recover stability locally. This defines a new sort of dynamic or weak stability : although there might not exist a stable equilibrium per se, there exists a constrained stable allocation at any point in time. Such a concept should allow further exploration of the behaviour of the assignement system, include possibly some forms of ergodicity and potential convergence, and its relationship to the static case. Because we expect the model to behave similarly as in the static case (i.e converge to a stable equilibrium) when the GS condition is satisfied, this should allow the interpretation of dynamic/weak stability as a generalization of the usual (static/strong) notion. In that respect, the aim of this work is to preserve the insight of stability as a driving force behind allocations but generalize it consistently to a richer environment that allows dynamics, frictions, and complementarities and then explore whether complementarities are still as much of an issue for existence.

In section 2, we introduce the environment and define a general framework of constrained/local matching, compare it to the usual unconstrained/global setup, and motivate its structure via a dynamic search approach. In section 3, we explore the question of existence of dynamically/weakly stable equilibria of the system using a general reduction theorem and apply this analysis to a particular case of interest : the one-draw-at-a-time model. In section 4, we examine several charateristics and properties of the behaviour of dynamic search models – we notably revisit classical examples of instability in this setup, and investigate the particular case where GS is fulfilled. Section 5 concludes.

Before going any further, we should state that the main result of this work is a strong negative result. Indeed, our separation theorem allows us to reduce stability over the market to stability over simple conflict structures. This results, however, cuts both ways : local instability in small subgraphs can propagate to the whole system. If no assumptions are made on the form of the surplus function, stability cannot in general be ensured *even in the simple 2 by 2 case with two firms and two workers* – so as long as there is the possibility that two firms may be disputing two workers, instability can propagate if we impose no restrictions. This shows that there is no hope of recovering stability by weakening it through a dynamic/local structure. This can, nonetheless, provide a method for obtaining slightly weaker assumptions than GS. Overall, it seems that this framework can have case by case applications and serves efficiently as a generalization of the static case to a dynamic models of frictions when local stability can be ensured (including the GS case). Yet, it falls strikingly short of its initial ambition to provide an alternative all-purpose approach to the behaviour of unstable matching markets. It seems that the best insights from this model actually comes from where it fails : by showing precisely to what extent the standard stability notion can be sustained, it highlights other alternatives paths for further research with the same goal – these prospects are developped in the conclusion.

2. Framework and definitions of essential concepts

2.1. Environment

We begin by defining the modeling environment. Throughout we will refer to a labour market interpretation of the setup to fix ideas, but once again this theoretical approach could be easily transposed to another relevant application.

There are n workers indexed by i and the set of workers is denoted $I := \{1, ..., n\}$. Workers are endowed with a type belonging to an arbitrary space X, we denote $x : I \to X$ the application that associates a worker's type $x(i)$ to its identy i. Similary, there are m firms indexed by $j \in J := \{1, ..., m\}$; a firm's type belongs to type space Y and is given by the application $y : J \to Y$. We will consider throughout the augmented firm set $\tilde{J} := J \cup \{\emptyset\}$, where we use the convention that an unmatched worker is matched to the fictional $\emptyset$ "firm".

The set of finite subsets of X is denoted $\mathcal{X} := \{A \subset X \mid |A| < \infty\}$, where $|\cdot|$ denotes the cardinal of a set. A match between a firm and a set of workers gives rises to a surplus, which is given by the function : $f : Y \times \mathcal{X} \to \mathbb{R}_+$ and is shared between the firm and the workers. Notice that since production depends only on types and not identities, the former as the only relevant productive information – but identities will still play a part as we consider a model with frictions, where random meeting chances are individual. We make the following general assumptions on the production/surplus function :

Assumption 1. *f is such that :*

(i) For any $y \in Y$, $f(y, \emptyset) = 0$

(ii) For any $y \in Y$, for any $A, B \in \mathcal{X}$, $A \subset B \Rightarrow f(y, A) \leq f(y, B)$

The assumption that f is monotone (weakly increasing in the inclusion order) is rather natural and allows to do away with some potential odd effects of undesirability of some workers – which intuitively should flow from effort behaviour rather than productive type and is thus out of the scope of this work.

In practice, although this makes no formal difference, we also assume that the firm is residual claimant on the surplus and pays workers wages. The firm simply seeks to maximize its share of the surplus.

Worker i has utility function given by $u_i : \tilde{J} \times \mathbb{R}_+ \to \mathbb{R}$, such that $u_i(j, w)$ is the utility derived by worker i from working at firm j for wage w. Notice that in general, we might allow worker's preferences to depend on the *identity* and not only the type of the firm. Without loss of generality we can normalize for all i, $u_i(\emptyset, 0) \equiv 0$. We also assume that :

Assumption 2. *For any i and j, we assume that the function $u_i(j, \cdot)$ is strictly increasing and continuous on $\mathbb{R}_+$ (the utility function is continuously strictly increasing in its second argument).*

We assume that a worker always has *at least one* outside option when being offered to match with any given firm – they can always remain unmatched. Hence it is useful to define the relative minimal outside option of a given worker with respect to a given firm, via the minimum acceptable wage for the worker :

Definition 1. *Define $\underline{w}_{ij}$ the minimum possible wage for worker i at firm j by :*

$$\underline{w}_{ij} = \inf_w \{w | u_i(j, w) \geq u_i(\emptyset, 0) = 0\}$$

By the assumptions above it is uniquely characterized for all i, j by :

$$u_i(j, \underline{w}_{ij}) = 0$$

Assuming general preferences can be relevant but it can also unnecessarily complexify some proofs which are easier to do with a simpler setting and easily generalize. Furthermore, in abstract applications it seems there are no compelling reasons to settle for one specific set of assumptions on the shape of utility and workplace preferences. Hence, for those reasons throughout the text we will often assume – usually for clarity of exposition – that preferences are *purely monetary.*

Assumption 3 (Pure monetary preferences). *The utility function will sometime be assumed to be independant of the identity of the hiring firm, hence be purely monetary. In that case, we can assume without loss of generality that for any i and any j :*

$$u_i(j, w) = w$$

A direct consequence is that $\underline{w}_{ij} \equiv 0$.

Of course, the possibility of more general specifications is important and is compatible with all that is presented here – it should certainly be accounted for in more empirical approaches.

Next we define formally the basic concept of a matching under a form suited to our many-to-one problem :

Definition 2. *A* matching *is an assigment of workers to firms :*

$$\alpha : I \to J \cup \{\emptyset\}$$

Where the set of firm identities has been augmented by an empty type to allow for the possibility of workers remaining unmatched. It is clear from this definition that a worker is assigned to at most one firm, however a firm may be assigned more that one worker. The set of workers at firm j under assignment α is $\alpha^{-1}(j)$.

Since we allow for transfers (wages), a matching is not enough to characterize the state of the market.

Definition 3. *An* allocation *is defined by a couple (α, w) where α is a matching and $w \in \mathbb{R}^n_+$ is a wage profile.*

Note that the payoff of the firm is just the surplus from its match minus the wages it pays. We could have defined a more general structure with a weak constraint that the sum of payoffs of a given match does not exceed surplus, but this is formally equivalent since unused surplus will never be left in equilibrium. Hence we have chosen to simplify the setup by focusing only on the payoffs of the workers which are sufficient to fully characterize equilibrium along with the allocation.

In all rigour, with our notations the surplus produced by firm j under matching α writes :

$$f(y(j), x(\alpha^{-1}(j)))$$

but abusing notations slightly for clarity of exposition, we will usually denote it instead as :

$$f(j, \alpha^{-1}(j))$$

i.e the dependance on type is made implicit. We also often use the capital letter F to denote aggregate output as a function of the allocation :

$$F(\alpha) := \sum_j f(j, \alpha^{-1}(j))$$

2.2. Global/Unconstrained Allocations Problems

This section presents the essential (and standard) concepts that characterize the properties of equilibrium allocations in the *global* – equivalently *unconstrained* or *static* setup, where all matches of any firm with any worker is allowed and feasible. Any potential alternative match and surplus share is a possible deviation from equilibrium hence serves as an outside option for realized matches.

The minimal requirements for an allocation are that it is *feasible* and *individually rational* i.e no agent can be forced to be in a situation that is less profitable than being unmatched.

Definition 4. *An allocation (α, w) is feasible (or is said to satisfy aggregate budget balance)*[5] *if it verifies :*

$$\sum_{i \in I} w_i \leq F(\alpha)$$

[5]Although the first is more standard in the litterature, we will prefer the second terminology since feasibility will take on another meaning in our constrained setup.

i.e if wages distributed do not exceed total output.

Definition 5. *An allocation* (α, w) *is said* individually rational *(IR) if :*

- $\forall i,\ w_i \geq \underline{w}_{i\alpha(i)}$
- $\forall j,\ f(j, \alpha^{-1}(j)) - \sum_{i\in\alpha^{-1}(j)} w_i \geq 0$

Individual rationality is defined for an *allocation*, but generally we say that an *assignment* α is individually rational if *there exists* a wage profile w such that the *allocation* (α, w) is individually rational[6].

Next we define the concept of stability, which ensures that there are no profitable deviations from the allocation. We define two versions of stability, depending on whether or not we assume that indifferent individuals in a deviating coalition will participate in it.

Definition 6. *An allocation* (α, w) *is said* (weakly) stable *if :*

$$\forall j,\ \forall C \in \mathcal{X},\ \forall w' \in \mathbb{R}_+^{|C|}\quad f(j, C) - \sum_{i\in C} w'_i \leq f(j, \alpha^{-1}(j)) - \sum_{i\in\alpha^{-1}(j)} w_i$$
$$\text{or } \exists i \in C,\ u_i(j, w'_i) \leq u_i(\alpha(i), w_i)$$

Equivalently, there is no firm-set of workers coalitions and wage profile that can strictly improve the profit/utility of all *of its members.* (α, w) *is also called a core allocation.*

Definition 7. *An allocation* (α, w) *is said* strongly stable *if :*

$$\forall j,\ \forall C \in \mathcal{X},\ \forall w' \in \mathbb{R}_+^{|C|}\quad f(j, C) - \sum_{i\in C} w'_i \leq f(j, \alpha^{-1}(j)) - \sum_{i\in\alpha^{-1}(j)} w_i$$
$$\text{and } \forall i \in C,\ u_i(j, w'_i) \leq u_i(\alpha(i), w_i)$$

or at least one of those inequalities is strict. Equivalently, there is no firm-set of workers coalitions and wage profile that can weakly improve the profit/utility of all of its members and such that the improvement is strict for at least one member. (α, w) *is also called a strict core allocation.*

Similarly we will extend the notion to assignments by saying that α is stable if there exists a wage vector w such that (α, w) is stable. Notice that the Individual Rationality condition is actually just a stability condition with respect to the empty type when it is assumed to always produce zero – it is nonetheless sometimes useful to keep it as a separate notion.

In general, the notion of stability can be microfounded as a relevant equilibrium concept for decentralized allocations by many negociations processes – e.g. bidding wars or some forms of ascending auctions between firms for workers. The strength of the concept, however, crucially lies in that it stops shy of pinpointing actual payoffs (by making extremely specific assumptions about the negociation process) and simply ensures that no renegociation can be profitable for both parties. In that spirit, what matters is that for a given allocation *there exists an non-empty payoff vector* that makes it stable – the particular payoff vector chosen can then depend on any institutional or conjonctural factor that impacts the specific form of the bargaining process.

Next, we define straightforwardly efficiency.

[6]There is no point in making the same distinction for feasibility since clearly any assignement is feasible for some wage profile.

Definition 8. *An assignement* α^* *is said* efficient *or* optimal *if it maximizes total surplus :*

$$\alpha^* \in \arg\max_{\alpha} \sum_{j=1}^{n} f(y(j), x(\alpha^{-1}(j)))$$

It is IR-Efficient if the maximum is taken among individually rational assignments.

Throughout, we will focus on IR-Efficient assignments and allocations – we might even sometimes refer to them as simply efficient, since it does not make sense in our setup to consider assignments which require coercion to be enforced (i.e where at least one agent would prefer remaining unmatched). Note that there always exists a IR-Efficient allocation since we are taking the maximum over a finite and non-empty set – by definition the null assigment $\underline{\alpha}$ such that for all i, $\underline{\alpha} \equiv \emptyset$ is always individually rational with null wage profile.

The following proposition is a general and well-known result on stable allocations.

Proposition 1. *If* (α, w) *is stable, then it is efficient. If* (α, w) *is individually rational and stable, then it is IR-efficient.*

A proof for this result can be found in appendix. This entails that we can restrict the search of stable equilibria to the set of efficient assignments – yet the converse of this result is famously not true in general and there may exist optimal assignments which are not stable.

2.3. Local/Constrained Allocation Problems

Having defined the usual framework, we move to the *constrained* or *local* case. Note that for now everything is taken from a static perspective and we will only later on interpret the static constrained configuration as emerging from a dynamic process.

A *feasibility constraint* is characterized by a subset of worker-firm assignments. It can be viewed as a bipartite graph of undirected links between firms and workers, or equivalently as $m \times n$ matrix A whose coefficients are 0 or 1. The interpretation is that worker i can work at firm j (worker i is attainable or feasible for firm j) if and only if $A_{ij} = 1$. With a slight abuse of notation, for a constraint matrix A we will sometimes conveniently denote $A(j) := \{i | A_{ij} = 1\}$ the set of feasible workers for firm j. By convention we assume that the null matching is always feasible – if we consider the augmented matrix with an added row indexed by $\emptyset$ for unmatched workers we have $A_{i\emptyset} \equiv 1 \ \forall i$. Since we consider only the set of IR matchings, we will usually dispense from augmenting the matrix. Even more so, in the purely monetary utility framework, all matchings are IR which will often further simply matters. We might still adopt different formal conventions depending on context for practical reasons which will be made clear.

The notion of feasibility constraints straightforwardly entails the definition of a feasible matching :

Definition 9. *A matching* α *is said to be feasible under constraint* A *if :*

$$\forall i, j, \ \alpha(i) = j \Rightarrow A_{ij} = 1$$

That is to say if all the worker-firm links in α *are feasible under* A*. For a given constraint matrix* A*, we denote* $\phi(A)$ *the set of feasible matchings under* A*.*

The unconstrained case previously considered corresponds to the case of the complete (bipartite) graph where all links are feasible, i.e $A_{ij} = 1$ for all i, j. The notion of individual rationality is not altered in the constrained framework, but we can redefine stability in exactly the same fashion while only allowing for deviations that are permitted by the constraints.

Definition 10. *An allocation* (α, w) *is said* (weakly) constrained stable *under* A *if :*

$$\begin{aligned} \forall j,\, \forall C \subset A(j),\, \forall w' \in \mathbb{R}_+^{|C|} \;\; & f(j, C) - \sum_{i \in C} w'_i \leq f(j, \alpha^{-1}(j)) - \sum_{i \in \alpha^{-1}(j)} w_i \\ & \text{or } \exists i \in C,\; u_i(j, w'_i) \leq u_i(\alpha(i), w_i) \end{aligned}$$

Equivalently, there is no feasible *firm-set of workers coalitions and wage profile that can strictly improve the profit/utility of all of its members.* (α, w) *is also called a core allocation.*

Definition 11. *An allocation* (α, w) *is said* strongly constrained stable *under* A *if :*

$$\begin{aligned} \forall j,\, \forall C \subset A(j),\, \forall w' \in \mathbb{R}_+^{|C|} \;\; & f(j, C) - \sum_{i \in C} w'_i \leq f(j, \alpha^{-1}(j)) - \sum_{i \in \alpha^{-1}(j)} w_i \\ & \text{and } \forall i \in C,\; u_i(j, w'_i) \leq u_i(\alpha(i), w_i) \end{aligned}$$

or at least one of those inequalities is strict. Equivalently, there is no feasible *firm-set of workers coalitions and wage profile that can weakly improve the profit/utility of all of its members and such that the improvement is strict for at least one member.* (α, w) *is also called a strict core allocation.*

Similarly, efficiency can be restricted to feasible matchings under a given constraint.

Definition 12. *An assignement* α^* *is said* constrained efficient *under* A *if it maximizes the total surplus among* A*-feasible matchings :*

$$\alpha^* \in \arg\max_{\alpha \in \phi(A)} \sum_{j=1}^{n} f(y(j), x(\alpha^{-1}(j)))$$

It is Constrained IR-Efficient if the maximum is taken on the intersection of $\phi(A)$ *with individually rational assignments.*

The same proposition as before holds :

Proposition 2. *If* (α, w) *is constrained stable under* A*, then it is constrained efficient under* A*. If it is in addition individually rational, then it is constrained IR-efficient under* A*.*

The proof is exactly identical to the proof of proposition 1 when we restrict to the class of feasible matchings under a given matrix A.

Our main aim is to explore under which conditions (on the surplus function or the environment) we can ensure existence of a stable equilibrium. Before, we quickly survey and define essential properties of the surplus function and of the environment.

2.4. Properties of the surplus function

We state some relevant properties of set functions that will be useful in characterizing the surplus function. In the simplest case when the surplus is just given by output produced, these properties characterize the production technology. In a general setup where future gains are taken into account, the interpretation might be more intricate. These notions give a proper formalization of, notably, the ideas of complementarity and substitutability.

Throughout the section we consider a set function $f : \mathcal{X} \to \mathbb{R}$ – implicitly it is our production or surplus function, which takes as input a set of workers type and where we ignore the firm index. Later in the text we will sometimes say that the surplus/production function has this or that property without referring to a specific firm : this will mean that we consider that the function has that property for all firm indexes.

The presentation of these definitions and formalism are taken directly from the classical article by Gul and Stacchetti (1999) [25], section 2.

Definition 13. *The set function f is said :*

(i) ***Monotone*** *if for all $A, B \in \mathcal{X}$ such that $A \subset B$: $f(A) \leq f(B)$*

(ii) ***Submodular*** *if for all $A, B \in \mathcal{X}$:*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

(iii) ***Subadditive*** *if for all $A, B \in \mathcal{X}$:*

$$f(A) + f(B) \geq f(A \cup B)$$

(iv) ***Supermodular*** *if $-f$ is submodular*

(v) ***Superadditive*** *if $-f$ is subadditive*

(vi) to have ***Decreasing Marginal Returns*** *if for all $A, B \in \mathcal{X}$ such that $A \subset B$ and all $a \in A$:*

$$u(B) - u(B \setminus \{a\}) \leq u(A) - u(A \setminus \{a\})$$

(vii) to have ***Increasing Marginal Returns*** *if $-f$ has decreasing marginal returns.*

If $f(\emptyset) = 0$ and f is monotone, then f is non-negative – notice that Assumption 1 states precisely that. In that case, it is clear that supermodularity (resp. submodularity) entails superaddivity (resp. subadditivity) but the converse is not true in general. The following equivalence is well known.

Lemma 1. *f is submodular if and only if it has decreasing marginal returns. f is supermodular if and only if it has increasing marginal returns.*

Note that sub- and super-modularity are both very extreme in their own ways, and that a large class of interesting and relevant production functions might be neither – we will give examples of that in section 4. The lemma above show that they give a first possible (although insufficient) idea to approach complementarities : they formalize the idea that the production of a workers is more or less than what they could produce isolated. To specify further what we mean by complementarities, we define the demand correspondance.

Definition 14. *Let f a set function over $\mathcal{X}$ and $K \in \mathcal{X}$ a finite set of available workers. The demand correspondance of f assigns to a wage vector in K a subset of demanded workers. Formally, let $D(w|K)$ the demand for constraint A and wage vector $w \in \mathbb{R}_+^{|K|}$:*

$$D(w|K) := \{C \subset K \,|\, f(C) - \sum_{i \in C} w_i \geq f(B) - \sum_{i \in B} w_i \textit{ for all } B \subset K\}$$

For the purpose of stating the next definition, we need to first define the symetric difference of two sets A and B as :

$$A \Delta B := (A \setminus B) \cup (B \setminus A)$$

We also use the notation introduced by Gul and Stacchetti, for any three sets $A, B, C \in \mathcal{X}$:

$$[A, B, C] := (A \setminus B) \cup C$$

Definition 15. *Consider $f : \mathcal{X} \to \mathbb{R}_+$*

*(GS) f satisfies the **Gross Substitutes** (GS) condition if for any two wage vectors w and w' such that $w' \geq w$, for any feasible workers set $K \in \mathcal{X}$, and any $A \in D(w|K)$, there exist $B \in D(w', K)$ such that $\{a \in A \,|\, w_a = w'_a\} \subset B$*

*(SI) f has the **Single Improvement** (SI) property if for any wage vector w, for any feasible workers set $K \in \mathcal{X}$ and workers set $A \notin D(w|K)$, there exists a workers set B such that $f(A) - \sum_{i \in A} w_i < f(B) - \sum_{i \in B} w_i$, $|A \setminus B| \leq 1$ and $|B \setminus A| \leq 1$*

*(NC) f satisfies the **No Complementarities** (NC) condition if for any wage vector w, for any feasible workers set $K \in \mathcal{X}$ and all bundles $A, B \in D(w|K)$ and $X \subset A$, there exists a bundle $Y \subset B$ such that $[A, X, Y] \in D(w|K)$.*

*(SNC) f satisfies the **Strong No Complementarities** (SNC) condition if for any feasible workers set $K \in \mathcal{X}$ and for all $A, B \subset K$ and $X \subset A$ there exists $Y \subset B$ such that :*

$$f(A) + f(B) \leq f([A, X, Y]) + f([B, Y, X])$$

The first concept (GS) was first defined in Kelso and Crawford (1982) [28] and the other three were defined by Gul and Stacchetti (1999) [25]. The following theorem and lemma, that are also due to Gul and Stacchetti [25], highlights the relation between these definitions.

Theorem 1 (Gul and Stacchetti (1999)). *If f is monotone, then (GS), (SI), and (NC) are equivalent.*

Lemma 2 (Gul and Stacchetti (1999)). *If f is monotone and satisfies (GS), then f and $f(\cdot) - \sum_{i \in \cdot} w_i$ are submodular for any wage vector w.*

These notions, which we will often throughout the text refer to indiscriminately under the term of GS conditions, have then become standard in the litterature – precisely because they guarantee existence. More details on the proof and the intuitions behind this can be found in Gul and Stacchetti [25] and [26]. As the lemma shows, a monotone and (GS) function is also submodular – the converse is in general not true as a famous example of Kelso and Crawford [28] shows. This question and the example will be investigated in more details in section 4.

Our interest in defining these notions is overall negative and mostly as a point of reference : this describes the well behaved case – GS, submodular, with no complementarities. We specifically want to adress cases where these conditions fail. This does not mean that we want to focus on the polar case of supermodularity – instead, we want to obtain an alternative approach when complementarities may arise under various, heterogeneous and possibly unpredictable forms but the environment is characterized by frictions which may or may not prevent the usual exploitation of all the strategic information which usually prevent the existence of stable equilibria.

2.5. From static to dynamic matching : a category of search models

In general, many specific structures for static constrained matching could be found to be practically relevant. The graph structure could flow from some sort of limited information (partial knowledge of the agents on the market) with some pre-determined and possibly random structure ; it could flow from limited accessibility driven by a geographical structure, e.g. if agents are scattered on a two dimensional space, they are only linked to agents of the other group that are close enough to them.

Such cases can, however, be embedded in a larger and more flexible *dynamic* setting with search. This provides a general purpose heuristic to justify how particular graphs structure may arise, and adds new theoretical elements with respect to the standard static model – while the analysis of static constrained models by themselves can generally be tackled with the apparatus that has been developped for the unconstrained case.

We propose a (potentially very wide) category of search models in discrete time, where a constrained matching problem is solved at each time. The idea is simple : at any time, given a previously realized matching, new links are drawn (possibly at a cost) ; some pre-existing links might also be exogenously severed ; the constraint graph is made of the previously realized and unsevered links – i.e we assume that pre-existing relationships, unless exogenously broken, can always be reconducted – and the new potential links just drawn ; a new constrained stable assignment is determined, and the process is iterated. Formally :

- Let α_0 be an initial arbitary matching
- For $t \geq 1$:
  - Firms and/or workers decide whether or not to search at a fixed cost c.
  - Searching firms and workers meet : this is described by the realization of m independant random variable X_j^t for all j which give the set of workers that firm j meets at time t
  - The new constraint matrix A^t is determined by :

$$A_{ij}^t = \begin{cases} 1 & \text{if } \alpha_{t-1}(i) = j \text{ or } i \in X_j^t \\ 0 & \text{otherwise} \end{cases}$$

  - A new constrained IR and stable matching α_t under A^t and according to some generalized surplus function V_t is determined.
  - The process is iterated

Essentially this process amounts to sequential augmentation and reduction of a graph. Matching graphs are made of non-connected stars (with firms at the center connected to workers), so each step amounts to augmenting an initial network of disconnected stars by adding some links from star centers to non-star centers, and then reducing it back to a network of disconnected stars according to some criterion. The equilibrium concept that will be adopted for the decisions on whether or not to participate is that of a standard Nash Equilibrium.

We sometimes denote $X^t := (X_j^t)_{j \in J}$ and $X := (X^t)_{t \geq 1}$ the concatenations of the draws for firms and time. For simplicity we assume that the draws are all independant and that their law is not contingent on the previous matchings realized. We will naturally consider that the draws have different laws for different identities, but usually assume that for a given j, the vabriables X_j^t are iid (i.e the law of the draws does not evolve in time). Notice that these elements allows combined allow potential redundant draws (i.e a firm drawing a worker to which it is already matched), which is not a problem and could be justified heuristically as materializing the possibility for search to lack results – a firm posts a job offer and finds no one or equivalently someone it has alrady hired.

The data of the initial matching α_0, the law of the random draws $\mathcal{L}_X$ and the sequence of generalized surplus functions V defines dynamic matching process. This moves the focus from a single matching to a sequence of matching and allows us to define properties on the process.

Definition 16. *A dynamic matching process* $(\alpha_0, \mathcal{L}_X, V)$ *is said to be :*

- strongly stable *if there exists a stable allocation at each time for any possible draw.*
- almost surely stable *if there exists a stable allocation at each time with probability* 1.
- asymptotically stable *if it almost surely stable when the size of the market (numbers of workers and firms) tends to infinity, for any arbitrary infinite sequence and corresponding limiting distribution of types. If it is sufficient that the number of workers (resp. firms) tend to infinity for a fixed number of firms (resp. workers), then we say that the process is asymptotically stable in workers (resp. firms).*

When there is ambiguity, the stability of a dynamic process will sometimes be referred to as dynamic stability *or* local stability in time *to differentiate it from classical stability.*

Notice that unlike many if not most models in the search litterature, we do not in principle restrict the set of workers than can be met through search to *unemployed* workers. All the market is potentially (randomly) attainable whether already matched or not. This seems to capture an important feature of the labour market, since work contracts are potentially temporary and the fact that a worker is employed would not prevent another firm from making them an offer. But, once again, this model could easily accomodate a specification when search is confined to the set of unemployed workers – or in a weaker form, in which a searching firm is much more likely to meet an unemployed worker.

Another feature that sets this model apart from some of the matching litterature is that surplus is de facto produced at each period – this is prominently a model of *repeated partnership.* This differs from quasi-static models where surplus is reaped only when matches are finalized (whether or not acceptance is deferred throughout the process or any acceptance is definitive). Which, once again seems a reasonable feature of a labour markets : matches are inherently temporary and continuously produce some output.

However, the specification of the dynamic process above crucially specifies that allocation is chosen with respect to a *generalized* surplus function. Indeed, although we assume that there this a "physical" surplus function f which assigns to any firm-set of worker match the output it produces, in general we might want to consider a more complex form for *dynamic* matching – because agents might take into account not only present but future discounted gains. Indeed, there is a strategic dimension to accepting a match which, to the best of our knowledge, is absent as such[7] from all other dynamic search and matching models : a temporarily accepted match entails the certainty that it will be feasible for renewal next period. This element of the model is apt to interact crucially with complementarity structures : a firm might be willing to pay a worker much more than what it contributes to production *right now*, because it expects that the gains from interactions with future hires will outweigh the cost. To use the restaurant example again : a restaurant that is just opening and encounters an exceptional cook might be willing to hire them straight away for a high wage. Although the cook will not produce anything before the restaurant is also able to retain waiters, and although even then having an exceptionnal chef might not make so much of a productive difference before the restaurant hits a critical size, the potential value of retaining the chef is considerable because of *future* gains and guaranteed access. Only a restaurant with overall greater potential value for the chef should be able to hire him.

In general, the dynamic surplus – which we will often refer to as the value function for clarity – could take on many forms. We highlight three possible ways to specify it :

- An obvious candidate is to take the value function equal to the instantaneous surplus function. This corresponds to a fully myopic and/or liquidity constrained setup.
- Another obvious choice is to take the actual value function from the whole dynamic problem. This noticeably complexifies our setup by making the surplus function non-local. Indeed, in theory, if firms anticipate the outcome of the game for future periods, then their expectation depends on the possible realizations of the constraint which itself depends on the *whole* matching today. In practice, this means that firm must take into account into their matching decisions the strategic advantage that other firms have to match with their workers. This requires that we redefine our whole setup to accomodate for this alteration[8]. Yet this does not seem very coherent from a heuristic perspective because it de facto models a contradictory situation where agents are on the one hand subject to search

[7] In search models à la Shimer and Smith, the relationship is de facto renewed with certainty until it exogenously breaks, but since there are no simultaneous encounter and matched workers exit the market the benefit of accessibility while still considering other options is significantly less important. The only opportunity cost lies in between matched to a certain type versus being unmatched and searching. In dynamic matching models à la e.g. Elliott and Nava matches are definitive so this advantadge simply does not exist.

[8] This is feasible in practice although rather inconvenient. Assume that the surplus function has the form $V(j, \alpha)$ where α is an assignment. Aggregate budget balance and individual rationality can be defined exactly as before, but we need to alter stability. First, we need to define the proper (and no longer self-evident) notion of local strategic deviation :

Definition 17. *Consider an allocation α, a firm j, and a set of workers C. The (j, C)-coalitional deviation of α is the allocation α' such that*

$$\forall i \notin C,\ \alpha'(i) = \alpha(i)$$

$$\forall i \in C,\ \alpha'(i) = j$$

A (j, C)-coalitional deviation is said to be feasible under constraint A if $\forall i \in C$, $A_{ij} = 1$.

Stability can then be redefined as :

frictions and unable to exploit the full market structure, yet they somehow have perfect knowledge of all the matches taking place at any given time and can incorporate that into their strategic behaviour. For this reason, going through the complete anticipation process in this search context seems excessive.

- The third and probably most appealing option is an in between, where agents are able to anticipate future gains but cannot strategically exploit the full structure of the market and the game. The value function would then crucially remain *local*, i.e the dynamic expected surplus depends only on the coalition that is or would be formed, but incorporate some form of heuristic for anticipated future surpluses. This heuristic could be more or less refined (we will detail several approaches later on). This is akin to approximate dynamic programming methods for numerical optimization.

Note that although we use a type-based approach to model heterogeneity – surplus-wise, two agents with identical types are strictly identical – we still need idiosyncratic identification $i \in I$, $j \in J$ because of the random meeting process, which introduces different situations and behaviour for otherwise similar types.

This completes the general description of the environment – we will now focus on existence and characterizations of stable dynamic matching processes through analysis of the graphs structures that may arise through the random meeting process. General results will be provided but the main focus of the subsequent sections will be variations on the one-draw-at-a-time model, where the search technology is such that only firms search and each searching firm draws one worker per production period.

3. Existence of stable equilibria in dynamic search models

3.1. A general reduction theorem for constrained allocation problems

Throughout this subsection we consider a constrained matching problem with surplus function f and constraint matrix A. Since stability is defined locally and disconnected subgraphs do not interact strategically, we will assume throughout that the graph defined by the constraint matrix A is connected. Otherwise, the same analysis applies component-wise.

We define the concept of a partial allocation.

Definition 19. *Given a subset of workers $\tilde{I} \subset I$, a partial allocation on $\tilde{I}$ is an allocation $\tilde{\alpha} : \tilde{I} \to J$ of the workers in $\tilde{I}$ that respects the constraint A (or equivalently the constraint given by the subgraph of A defined by the nodes in $\tilde{I}$).*

We define the concept of a localized problem.

Definition 18. *(α, w) is (weakly) stable under A if for any firm j and set of worker C such $C \subset A(j)$, and for any alternative wages w' for the workers in C, denoting α' the (j, C)-coalitional deviation of α, we have :*

$$V(j, \alpha') - \sum_{i \in C} w'_i \leq V(j, \alpha) - \sum_{i \in \alpha^{-1}(j)} w_i$$

$$\textit{or } \exists i \in C,\ u_i(\alpha', w'_i) \leq u_i(\alpha, w_i)$$

Definition 20. *Consider an arbitrary subset of workers $\tilde{I} \subset I$ and a partial allocation $\tilde{\alpha}$ on $\tilde{I}$. We define the* localized allocation problem *under partial allocation $\tilde{\alpha}$ as the allocation problem on $I \setminus \tilde{I}$ under the natural constraint – given by the subgraph of A defined by the nodes in $I \setminus \tilde{I}$ – and with modified surplus function $\tilde{f}$ defined as :*

$$\tilde{f}(j, C) := f(j, C \cup \tilde{\alpha}^{-1}(j))$$

In other words, a localized allocation problem is just the problem that results when some allocations are taken as fixed and we seek to allocate remaining workers while still accounting for the productive input of already allocated workers.

In general, the existence of stable allocations in localized problems does not tell us much about general existence of stable allocations at the market level. Indeed, restricting to a local problem amounts to reducing strategic interactions and potentially suppressing some cycles of blocking coalitions, making existence stability generally easier to attain. This is most stringent in the unconstrained context (when A is the complete bipartite graph) where we can easily find examples such that there does not exist a stable equilibrium but *any* localized allocation problem admits a stable equilibrium.

In the constrained context, however, some specific localized problems do give informations about overall stability. The theorem below gives necessary and sufficient local stability conditions for global stability in a constrained problems.

Definition 21. *Consider a constraint graph with matrix A. We say that a firm j is* pivotal *if there exists subsets $I_1, I_2 \subset I$, $J_1, J_2 \subset J$ such that $I_1 \cap I_2 = \emptyset$, $J_1 \cap J_2 = \{j\}$ and :*

$$\forall i \in I_1,\ \forall k \in J_2 \setminus \{j\},\ A_{ik} = 0$$
$$\forall i \in I_2,\ \forall k \in J_1 \setminus \{j\},\ A_{ik} = 0$$

That is to say a firm is critical if removing it from the graph creates two disconnected components – in other words any paths between a node in $I_1 \cup J_1$ and one in $I_2 \cup J_2$ passes by j. We say that (I_1, J_1), (I_2, J_2) is the partition *of workers and firms according to pivotal firm j.*

We now have all the ingredients to state the essential reduction theorem which sustains our analysis of constrained matching problems.

Theorem 2. *Consider a constrained matching problem with surplus function f and constraint matrix A. Let j be a pivotal firm and $(I_1, J_1), (I_2, J_2)$ the corresponding partition. If for any allocation α_1 on the subgraph defined by (I_1, J_1) there exists a stable allocation in the localized allocation problem under α_1 and for any allocation α_2 on the subgraph defined by (I_2, J_2) there exists a stable allocation in the localized allocation problem under α_2, then there exists a constrained stable allocation under A and f.*

The proof can be found in appendix.

This theorem is in practice a *reduction theorem* – its tells us that we can reduce the issue of stability on the whole constraint graph to stability on smaller subgraphs. The stability of these smaller subgraphs actually *spreads* to the whole graph, hence *local stability conditions* are sufficient to ensure global stability.

Note, however, that the converse is in general not true. It is relatively easy to find examples where a locally unstable situation is made stable by the addition of one links – which is equivalent to saying that whether this link is realized or not impacts local stability, hence there exists an allocation on the other side of the pivotal firm such that local instability obtains. Global constrained stability entails local constrained stability *at the globally stable allocation*, but it does not entail local stability for *any* partial allocation. The following proposition gives a stronger result which is rather self-evident.

Proposition 3. *If there exists a pivotal firm j such that the corresponding partition on one side* never *admits a stable equilibrium for any allocation on the other side, then there does not exist a constrained-stable equilibrium on the market.*

Because stability inherently relies on local deviation, an unstable subgraph is enough to prevent existence – instability spreads to the whole graph just as stability does.

The theorem's main relevance comes from its iterative application. Indeed, we can apply it to specifically chosen pivotal firms such that the partition on one side is clearly stable – we have then effectively reduced the overall stability problem to stability of a subgraph for a certain modification of the surplus function, because once we find a stable allocation on the subgraph we can always propagate back to the whole graph. This process can then be iterated to reduce the problem further to one subgraph.

We define concepts that allow for efficient reduction of the graph structure.

Definition 22. *Let A a constraint graph.*

- *A **worker-leaf** is a worker who is linked to* only *one firm, i.e. i is a worker-leaf if there exists j such that $A_{ij} = 1$ and for all $j' \neq j$, $A_{ij'} = 0$*
- *A **firm-leaf** is a firm who is linked to* only *one worker, i.e. j is a firm-leaf if there exists i such that $A_{ij} = 1$ and for all $i' \neq i$, $A_{i'j} = 0$*
- *A **worker-leaf of order two** is a worker who is linked to two firms only, one of them being a firm-leaf.*

It is clear that we can eliminate iteratively all workers-leaves. Indeed, any firm that is linked to a worker-leaf is pivotal if we consider that one side of the partition is made up only of the worker-leaf. Furthermore there always exists a stable allocation in a problem with two agents, whatever the surplus function. This is intuitive as workers linked to one firm only strategically do not carry weight in general.

Once all worker-leaves have been effectively removed from the graph[9] the same reasoning works for any firm which is connected to a worker-leaf of order two. Indeed, in a situation with two firms and one worker, it is once again straightforward to see that there always exists a stable allocation (this is effectively a trivial one-to-one problem). By the same reasoning, we can thus remove firm-leaves and workers-leaves of order two from the graph. Notice that we cannot have created more worker-leaves in this second step.

We then end up with a graph which is still connected but on which any node is of degree at least two. Stability on this reduced graph for any alteration of the production function that is obtained through exogenous augmentation of the worker set then implies overall stability on

[9]Although we should be careful that they might still have an impact through alteration of the production function, because we want that for any allocations of these workers, i.e for any transformation of the production function that can be obtained by adding a subset of workers exogenously to a given firm, there exists a stable allocation on the subproblem.

the market, and we can interpret this graph as the relevant strategic entity as far as existence of a stable equilibrium is concerned.

Now we can either (or both) study the restrictions on the graph structure and the restrictions on the production function that can ensure this result and guarantee stability. This provides a general method for proving existence that can be tailored to the particular application.

3.2. The myopic one-draw-at-a-time model

3.2.1. Model and graphs structure

We now focus on the one-draw-at-a-time (ODAAT) model, in which each firm draws one worker per period. For convenience, let us modify slightly our notation for the constraint space ; indeed, the constraint matrix at any given period in time is equivalently fully characterized by the data of the matching realized in the previous period and the draw in the current period. Hence for an arbitrary matching and a size-m vector X of elements of I (which gives the identity of the worker drawn by each firm) we redefine as $\phi(\alpha, X)$ the set of matchings that are feasible under the constraint A – previously denoted $\phi(A)$, where A is given by $A_{ij} = 1$ if $\alpha(i) = j$ (assuming no deletion of links) or $X_j = i$ and $A_{ij} = 0$ otherwise.

We focus first on a simple case which we call the *myopic* version of the model, in which allocations problem at every time are independantly considered. This is myopic insofar as there is no strategic lookahead and each equilibrium is defined independantly of future periods. It is more of a methodological distinction since technically a myopic model can accomodate some degree of lookahead and anticipation of future gains – we will come back to this in the next section when discussing dynamics. The easiest way to think about the situation we consider in this section is probably the fully myopic case in which the instantaneous surplus function consider at each period is just the physical production function – i.e only instantaneous gains are considered and shared in the allocation process at any given period. This could obviously be interpreted as an extreme form of bounded rationality (agents account only for present payoffs), but could also be construed as some form of liquidity constraint (firms cannot borrow or transfer liquidity in time and are constrained to pay wages using instantaneous surplus). However, going in such direction begs many difficult modeling question which are not the initial focus of this work and are usually outside of the scope of standard matching models – for instance, in a dynamic matching model where firms are able to borrow, shouldn't we keep track of the capital of firms and specify further their access to liquidity ? We will explore briefly how such questions can be partially adressed through specification of the surplus function, but we largely set aside the issue as a whole, which is left for future works.

We also focus for now on the case of costless search. As far as existence is concerned, adding search costs can only make things easier. Indeed it is clear that (at least in the myopic case) adding search costs only alters the constraint graph structures obtained by removing potential links – i.e less firms draw workers. Thus this only reduces conflicts, hence if there exists a stable (in any acception) allocation for any graph resulting from an costless ODAAT myopic model, then there exists one for the costly case. The same goes for exogenous severance of links, which might be interesting in practice for simulations and applications but does not add any difficulty as far as existence is concerned.

Using the reduction approach outlined in the previous section, we study the properties of general graphs that can be obtained from a general ODAAT process without specifying the law of draw, etc. Such graphs are indeed very regular and iterative applications of the reduction theorem 2 will actually reduce it to *disconnected cycles* as will be shown below.

The proposition belows gives a basic characteristic of the graph in terms of potential conflicts between any two given firms.

Proposition 4. *Any given pair of firms have* at most *two feasible workers in common and any firm can have two feasible workers in common with* at most *one firm.*

This is clearly a drastic limitation of allocation conflicts : the ODAAT process ensures that two firms are never fighting over the allocation of *too many workers at the same time.* This is consistent with an intuition of the labor market as a sequential filling of positions with conflicts rather scarce and limited in scope. Next we define the concept of alternate-draws cycle which is specific to dynamic search-induced graphs. Notice that because the graph is bipartite, any cycle consists of a equal number of firms and workers. For that reason, we call "cycle of order k" a cycle with k firms and k workers linked.

Definition 23. *Alternate-draws cycle Consider a constraint graph obtained from matching α and draw X. An* ***alternate-draw cycle*** *on this graph is a cycle in which every firm drew exactly* one *worker of the two belonging to the cycle to which it is connected. In other words, denote i, j, i' three adjacent nodes on the cycle (respectively worker-firm-worker). Then :*

$$\textit{Either } X_j = i \textit{ or } X_j = i'$$

The following proposition highlights the relevance of this notion.

Proposition 5. *All cycles of a graph obtained via ODAAT are alternate-draws cycle.*

This can be proven (see appendix), as can most of the results in this section, by considering the *oriented* graph obtained from the draw using the convention that a worker drawn is a link from the firm *to* the worker and pre-existing links are bi-directional. Using that convention, a worker has out-degree at most one, hence we can reason recursively starting from two workers with an out-link to a given firm to show that we cannot obtain a cycle. Formally, this oriented graph reasoning is the backbone of all the reasonings that we perform on the structure of draws.

Since all cycles that may arise have this particular structure, we can show that cycles never intersect. Evene more than that, two cycles cannot belong to a connected component.

Proposition 6. *Two cycles cannot be connected – in particular they cannot intersect.*

This means that every connected component of a graph obtained from the ODAAT process is a tree-like structure with at most one cycle branching off into proper tree subgraphs. Now, if we come back to our reduction method and the reduction theorem, it appears quite clearly that stability analysis on ODAAT-induced graph can, first, be performed componentwise and, second, reduces to stability on cycles.

3.2.2. General results

With all these results at hand, we can turn to the main theorem of this section, which gives a general characterization for stability in the ODAAT model. Its proof is a straightforward application of a combination of results previously demonstrated.

Theorem 3. *The ODAAT model is :*

1. *asymptotically stable for any surplus function if and only if the probability that cycles arise is vanishing.*

2. *strongly stable if and only if we can guarantee existence of a stable equilibrium in the cycle case for any sequence of firms and workers and any augmentation of the production function by exogenous coalitions. That is to say that for all* $i_1, ..., i_k \in I$ *and all* $j_1, ..., j_K \in J$*, and for all* $C_1, ..., C_k \subset I$ *such that* $i_1, ..., i_k \notin C_1 \cup ... \cup C_k$*, there exists a stable allocation of* $i_1, ..., i_k$ *constrained by* $\alpha(i_l) \in \{j_l, j_{l+1}\}$ *(using the convention* $j_{k+1} = j_1$*) under the modified surplus function* $\tilde{f}$ *such that* $\tilde{f}(j_l, C) = f(j_l, C \cup C_l)$ *for* $l = 1, ..., k$ *and* $C \subset \{i_{l-1}, i_l\}$*.*

Notice that almost certain stability was left out of the theorem – this is for a compelling modelling reason. Indeed, we can have almost certain stability if for a fixed finite market we have a probability distribution which gives zero mass to any cycles – yet this is in general not reconciliable with our assumption that draws are independant. It can be made so in the case in which all firms draw only from the pool of unemployed workers[10], but this is problematic for several reasons. First, it also erases cycles, making stability essentially a non-issue. Second, it means that employment relationships are somehow rigid – at least this means that an employed worker is locked ; on the firm side we could accomodate for endogenous layoffs. This entails important strategic alterations to the model, because the opportunity cost of forming a partnership is now much higher and straightforwardly dynamic. Our model specification precisely tried to avoid entering into that territory and focuses instead on instaneous stability issues. Hence it seems that forbidding cycles in our framework is not heuristically consistent and asymptotic stability is the best we can hope for.

This theorem can be applied directly to generate interesting corollaries. A natural case of interest is characterized by uniform draws – i.e with n workers, the probability of a given firm to draw any worker irrespective of their identity and type is given by $1/n$. It is clear that in this case the probability of two firms drawing the same worker is null when there are an infinity of workers.

Corollary 1. *The ODAAT model with uniform draws is asymptotically stable.*

This result can be generalized further quite straightforwardly. Indeed, it is clearly sufficient to obtain asymptotic stability that the limiting probability distribution of draws has no atoms with respect to identity – i.e for any i and any j, at the limit when n tends to infinity $\mathbb{P}(X_j = i) = 0$.

Corollary 2. *If the limiting law of draws when the number of workers and firms go to infinity is atomless with respect to identity for every firm, and if the limiting distribution of types is strictly positive, then the ODAAT model is asymptotically stable.*

Note that this does not preclude the existence of atoms in the limiting law of draws with respect to type[11] but since we consider strictly positive distribution of types there cannot be a single agent of any given type. A sufficient and intuitive condition to apply this result is the large class of *anonymous* draws, where for any finite market the law of draws *does not* depend on *identity* and can only depend on types (the law would not be modified by a relabeling of agents).

[10] In that case we even recover strong stability.

[11] We might allow that there exists a type $\tilde{x}$ such that $\mathbb{P}(x(X_j) = \tilde{x}) > 0$ at the limit.

Corollary 3. *The anonymous ODAAT model is asymptotically stable if all types have non-zero mass in the limiting distribution.*

To close this paragraph, let us remark that the methodology outlined here is general and can easily be applied beyond the ODAAT model. However, it is also clear from the analysis conducted here that above one draw per period, substantial conflicts arise which bring us back very quickly into the realm of the gross substitutes condition to guarantee existence – the mere simultaneous 2×3 case can be highly problematic (see section 4.1. below). The main result of this section is thus very negative – if the instability of the cycles setup can spread even in the simple ODAAT model, there is not much hope for recovering strong stability via dynamic search models with friction, and the best we can achieve without heavy restrictions on the surplus function is asymptotic stability. The analysis of the ODAAT model thus leads to the conclusion that such models are of limited interest in general as they will in general fail to capture stability with complementarities.

This outlines several potential directions ahead : if we remain in this framework, we can explore which restrictions guarantee existence in the simplest structures (most notably the 2×2 case), try to explore applications when these conditions are fulfilled, and other qualitative properties such as convergence to the stable equilibrium when there exists one, and the particular case where GS is fulfilled. This will constitute the objective of the next section. Another possibility is to interpret these results as a general failure of this approach and take stock of why it is so in order to explore another direction for an alternative approach of stability and complementarity related issues. For lack of time, this will not be explored in the present work but we believe that this outlines several potent directions which we will come back to in the conclusion.

Before turning to that, we briefly exhibit a natural weakening of the usual GS condition.

3.2.3. The local GS and subadditive cases

It is obvious that if the surplus function satisfies the usual GS condition, then for any constraint there also exists a constrained-stable allocation (which might not be the stable allocation obtained in the complete graph). Indeed, restricting the set of feasible coalitions preserves the stabilizing effect of the GS condition and any of the standard proof of existence (e.g. Kelso and Crawford [28] or Gul and Stacchetti [25]) can be replicated word for word on the restricted set of constrained allocations.

This replication highlights that we can, however, slightly weaken the gross substitutes condition and still obtain the same result. Indeed, in this proof what matters is only that the gross substitutes condition is verified for *feasible* allocations. Using this insight, we define the (weaker) *local gross substitutes* condition.

Definition 24. *Consider a constraint graph A and a surplus function f. We say that f satisfies the local gross substitues condition under A if it satisfies the standard gross substitues condition redefined by a restriction to allocations feasible under A.*

A second trivial replication of the standard proof of existence yields the following intuitive proposition.

Proposition 7. *If the surplus function f satisfies the local gross substitutes condition under A, there exists a stable allocation under A.*

This can be combined with the reduction process to obtain more interesting results, because the local GS condition reduces further using the regularity of the graph structure. Kelso and Crawford [28] considered the case with two firms and two workers, and showed that in this case the gross substitutes condition was equivalent to subbaditivity of the production function. They considered only the case of complete graphs, but if we step into the realm of constrained allocations their proof can straightforwardly be generalized to any situation in which a given firm has access to *only two* workers.

Theorem 4. *Consider the ODAAT model with surplus function f. If for any firm $j \in J$, the function $f(j, \cdot)$ is subadditive in its second argument, then the model is strongly stable.*

The idea of the proof is pretty straightforward. First, notice that if a given function f is subadditive, then its modification $\tilde{f}$ obtained by exogenously adding a set to all values i.e $\tilde{f}(C) = f(C \cup \overline{C})$ is also subbaditive. From the reduction process, we know that the issue of strong stability on an ODAAT-induced graph can be reduced to stability on cycles for any modification of the production function. Then, one of the standard proof when the gross substitutes condition can easily be adapted to the constrained setup – using the fact that subadditivity is sufficient to guarantee GS on a cycle.

It is well known that, in general, that subbaditivity or subdmodularity are *not sufficient* to guarantee GS – we will come back to this when revisiting the classical Kelso-Crawford counter-example. The dynamic ODAAT setup brings a notable weakening of the GS condition into a *local* GS condition for which subadditivity is sufficient.

Before turning to some properties of the model, the following subsection explores the question of dynamics and going beyond the myopic model.

3.3. The full dynamic setup

As we touched on briefly previously, moving beyond the myopic model is not self-evident. As such, dynamics can radically modify the payoff and the structure of the surplus function. For illustrative purposes, let us abstract somehow from the stability issue at least for future dates or equivalently assume that there always exists a unique equilibrium and that it is stable. In order to estimate the dynamic surplus that a matching α can hope to generate, you have to solve a dynamic programming problem of the form :

$$V(\alpha_0, X_0) = \max_{\alpha \in \phi(\alpha_0, X_0)} F(\alpha) + \beta \mathbb{E}[V(\alpha, X_1)]$$

Where α_0 is the initial matching, X_0 the current observed draw and the expectation is taken with respect to next period draw X_1. We have assumed that $\beta \in (0, 1)$ is the common discount factor. This is already a complicated enough problem to solve in practice, but still it does not pinpoint the surplus associated a firm-set of workers partnership. If we denote α^* the maximizer in the above problem, then a rigorous computation of the dynamic surplus v of firm j under allocation α would satisfy :

$$v(j, \alpha_0, X_0) = f(j, (\alpha^*)^{-1}(j)) + \beta \mathbb{E}[v(j, \alpha^*, X_1)]$$

By definition, such a surplus verifies exactly the natural aggregation property :

$$V(\alpha_0, X_0) = \sum_j v(j, \alpha_0, X_0)$$

But notice that it is prominently non-local. To give an intuition, the following reasoning approximates possible strategic reasoning from the point of view of a firm. To anticipate allocations, I need a complete knowledge of the graph. Even though my present situation could be viewed locally through surplus only and thus reduced to knowledge of *my pre-existing* links and the full (!) draw, for future periods a complete anticipation cannot be reduced to productive input. Indeed, hiring a worker, by modifying my constraints for the future draw, can give me an edge in future negociations/allocations – so it might be interesting to hire a worker even if their productive contribution is small. This is of course, not really so simple because there is a feedback loop : if all firms consider the full spectrum of advantages from hiring a given worker this in turn impacts the likeliness of my future allocations and can sometimes reduce the edge having a certain pre-existing link gives me. The point of this is not so much to precisely break down this strategic feedback loop but to highlight that surplus becomes prominently non-local and that computation of expectations requires an extreme level of sophistication.

On the technical side, nothing prevents us from redefining – as we did briefly in a footnote above – our whole setup with a non-local surplus function. Then this would mean that at each period, for a given draw, we should solve the aggregate dynamic programming problem for an arbitrary allocation, compute its separable form with respect to firm identities, and then solve the stable allocation problem with this function as a surplus. This is computationally heavy, but more importantly it relies on questionable assumptions.

As we mentionned previously, this require de facto a perfect knowledge of graphs, draws and the law of future draws for each firm, which is hardly consistent with a search model with frictions and limited access to the market... Another question is the ability of firms to borrow on future gains. In such a dynamic setup, if we insist on step-by-step allocation but take into account future gains, there is an evident borrowing issue. If we go beyond this and consider straight off the full dynamic allocation problem with the corresponding budget constraints over all periods of time and equilibrium profile of wages for all possible paths, we only deepen our issue of rationality and information.

For these reasons, it seems that the true dynamic setup is not really relevant to the problem that we are considering. But if we still want to account for future gains, what can we do to go beyond the truely myopic case ? It seems that a good starting point is to ask what can be done if we insist that the allocation procedure remain local in time and that the surplus function remain local in its inputs (i.e associates to a firm-set of workers a surplus). Take ψ to be this modified surplus function in the (apparently) "myopic" model, then the best we can do is that ψ incorporates *heuristics* for future gains. This leaves a lot of doors open since these heuristics can be more or less refined. As an example, consider a firm that has extremely high computational skills and power, and so is able to solve the full dynamic problem above for any inputs, but just does not know anything about the rest of the market outside of its situation – not the overall allocation of workers on the market nor the production of other firms, perhaps not even the types of the workers on the market – and is unable to acquire information. It might assume an underlying distribution on *all* of these parameters. For instance, assume it knows the size of the markets and has perfect information on its competitors i.e it knows the shape of the production function. Then it can calculate the solution to the dynamic programming problem for all draws of worker types and all possible graphs, then compute an expected value of its dynamic surplus. It might do this with the production function, or incorporate its own heuristics to solve the model.

At the opposite end of the spectrum, the heuristic we choose might be a relatively elementary computation of potential output with limited lookahead – some kind of present-value sum of expectation of the product over the next few period with random additions of workers. This is able to incorporate to various degree uncertainty of meeting certain type of workers, different forms of knowledge of the types present of the market, and the likeliness that a given worker met will be retained by the firm. This can also incorporate some form of liquidity constraint, e.g. through limited lookahead – I can borrow over an estimation of my gains for the next k periods.

This general approach is flexible enough, convenient because it limits computation to the realm of constrained allocation problems that are static (local in time). Most importantly, it makes sense from a modelling perspective on such a market with very little information to work with, possibly limited strategic lookahead, and decentralized allocation – each agent is responsible for the assessment of its value, for which it has limited information and/or computing abilities, then it takes part in some form of bargaining process. We could even consider generally complex heterogeneous structures for surplus heuristics with different levels of sophistication and/or knowledge.

This rather deep methodological argument warrants our taking the myopic framework as a reference case because it still allows a lot of room for consideration of strategic lookahead. Of course, this is not a self-sufficient approach and this is tailored to a specific setup.

4. Properties of the dynamic search model and examples

In the following section, we investigate several aspects of the behaviour of the dynamic search model introduced in the previous sections – focusing on the ODAAT version. The approach is less systematic and subsections focus on different aspects. The first subsection revisits the classical 3×2 example of non-existence exhibited by Kelso and Crawford [28]. We then provide several examples of instability in cycles of order 2 and greater and discuss the qualitative properties of the functions that give rise to these cases. The third subsection shows several results about convergence to a stable equilibrium when there exists one or to an efficient outcome in the general case. The fourth subsection investigates the case where the standard GS assumption is fulfilled (notably from the point of view of convergence).

4.1. Revisiting the Kelso-Crawford example

In their classic 1982 paper, Kelso and Crawford [28] provided a reference example with two firms and three workers, in which there existed no stable equilibrium in the many-to-one problem. One interesting aspect of their example was that the production function they chose was *submodular* yet it failed the gross substitutes condition. Although those two properties, as they showed, are equivalent in the 2×2 case, they are *not* as long as we have three workers.

Let us first restate the example ; it is fully specified by the production function. With such a limited scale setup, we can express it in fully idiosyncratic form without resorting to types. Firms are labeled by $j = 1, 2$, workers by $i = 1, 2, 3$. We consider the myopic setup and use the compact notation $f_{1,1}$ to denote the production of firm 1 with worker 2, $f_{1,12}$ the production of firm 1 with workers 1 and 2, etc. We have obviously $f_{1,\emptyset} = f_{2,\emptyset} = 0$. The productions are given by :

$$\begin{array}{rcl} f_{1,1} = 4 & f_{1,12} = 7.5 & f_{1,123} = 9 \\ f_{1,2} = 4 & f_{1,13} = 7 & \\ f_{1,3} = 4.25 & f_{1,23} = 7 & \end{array}$$

$$\begin{array}{rcl} f_{2,1} = 4.25 & f_{2,12} = 7 & f_{2,123} = 9 \\ f_{2,2} = 4 & f_{2,13} = 7 & \\ f_{2,3} = 4 & f_{2,23} = 7.5 & \end{array}$$

It is easy to see that there are two optimal allocations that are candidate to being stable equilibria – because they are symmetrical, we can focus on one of them e.g. $\alpha^{-1}(1) = \{1\}$ and $\alpha^{-1}(2) = \{2, 3\}$.

We assume that workers have pure monetary preferences hence the minimal acceptable wage of any worker at any firm is simply zero and so the IR constraint is just that wages are non-negative.

It is fairly easy to see that the optimum considered cannot be stable since for a given wage vector w, stability constraints include :

$$\begin{aligned} f_{1,1} - w_1 &\geq f_{1,3} - w_3 \\ f_{1,1} - w_1 &\geq f_{1,12} - w_1 - w_2 \\ f_{2,23} - w_2 - w_3 &\geq f_{2,1} - w_1 \\ f_{2,23} - w_2 - w_3 &\geq f_{2,3} - w_3 \end{aligned}$$

Replacing values and putting the second and fourth inequalities together yields $w_2 = 3.5$, hence the first and third inequalities can be rewritten as :

$$\begin{aligned} w_3 - w_1 &\geq 0.25 \\ w_3 - w_1 &\leq -0.25 \end{aligned}$$

Which is a contradiction, hence there does not exist a wage vector that makes the allocation stable (the core is empty).

What happens if we take the same model but add search frictions of the one-draw-at-a-time form ? It is quite easy to check by hand that the function f satisfies the local gross substitutes condition for any graph obtained via the ODAAT process. We can either check it directly or use the previous proposition to deduce it from the subadditivity of f. This means that the ODAAT process with this production function and any law for the draws is *strongly stable locally in time.*

How does the system then behave ? Once again, with such a small number of agents, it is feasible to check all possible alternatives manually or to perform systematic simulations for all possible sequences of draws. It emerges that the system quickly converge to one of the two (globally) optimal allocations, and then can only alternate between the two. One minor technical detail deserves mention : to compute the dynamics of the system we need to specify a tie-breaking mechanisms, because a situation might arise in which two stable allocations exist and all agents are indifferent between the two. A simple and natural tie-breaking mechanisms consists in randomizing the choice of indifferent workers – i.e workers who at the stable equilibrium are offered exactly the same wage by the two firms. In practice here indifference will only arise for worker 2 who can be assigned to either firm and makes the difference betweeen the two stable equilibria. In practice, we just assume that the worker chooses each firm with probability 1/2 when such a conflict arises.

This means, de facto, that once the system has reached the set of optimal allocations, it behaves as a Markov chain with two states. When the draws follow a uniform distribtuion, it jumps between the two allocations with probability $(1/3) \times (1/2)$ at each period – it can jump if worker 2 is picked by the firm which had only one worker the previous period which happens with probability 1/3, and if the worker chooses to change firm which happens with probability 1/2.

The model does not admit a static stable equilibria and does not converge to any allocation, yet under the ODAAT process it admits a very regular behaviour : it reaches maximal production in finite time, then stays there but is characterized by constant turnover with positive probability.

Although this model is extremely stylized and not relevant for any practical applications[12], it is potent as an illustration of the type of effect that we can expect from the search model that we have presented. Although stability cannot be guaranteed on the complete graph, it exists locally in time and the behaviour of the system can still be approached this way. The set of allocations obtained through local stability in time is always larger than the set of static stable allocations. We can interpret the behaviour of the system – notably the limiting distribution of the Markov chain generated – as a form of weak stability. The same rationale that underpins static stability can help us grasp the properties of the system in this generalized form and rationalize the dynamics of the process with search frictions.

This example also hints at some general properties of the model that will be explored in the following subsections, in terms of optimality and convergence. Although such a behavior seems intuitive, it is still hard to obtain in general because of existence issues and specific sub-equilibria being attained through random draws in more complex situations.

Before turning to those, we present a graphical illustration of one of the simulations performed – see figure 1 below. We assume that random draws are uniform. The left column displays the constraint graph – potential feasible links are depicted by dashed lines ; the right column displayed realized matchings at each period in solid lines. Observe that it takes only three time steps to reach the set of optimal allocations (i.e maximal output) and that subsequently it so happens that we switch between the two optimal allocations in the next two periods.

Let us briefly emphasize that the computation of such dynamics, even in the case where stability (strong, almost sure, or asymptotic) is guaranteed, is far from easy. In the Kelso-Crawford example, the limited number of agents allows for manually checking for stable equilibria, but in general a simple grid search is extremely inefficient. The challenge resides in that if we allow strong complementarities – supermodularity to put it simply – then usual greedy methods will in general fail to converge. The reduction theorem provides in theory a method for obtaining stable equilibrium by reducing the graph to a cycle for a fixed allocation then propagating back to the tree branches – but that means that we must identify clearly cycles and trees.

[12]In particular, the zero outside option assumption entails wage schedules that are not realistic – any worker previously linked to a firm and not drawn by the other can effectively be paid zero, allowing in particular cases the pivotal worker 2 to reap all of the surplus of one of the firms.

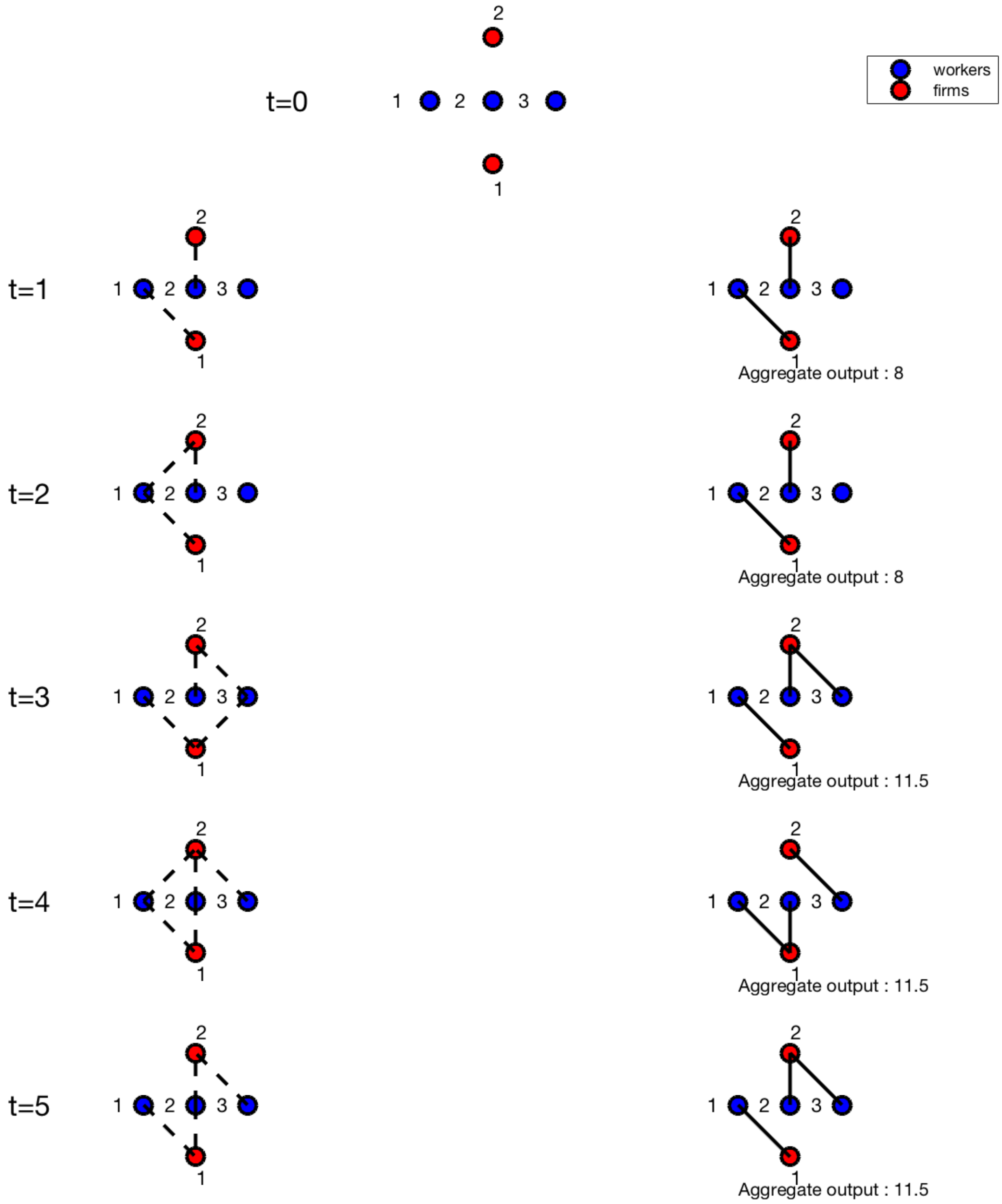

FIGURE 1: A random sequence of matching in the Kelso-Crawford example with uniform ODAAT setup.

With a large number of agents, this might prove computationally heavy. With general graphs structures, even identifying critical firms is in general a NP-complete problem. It should be a little less demanding given the regularity of ODAAT-induced graphs, but might still be far from efficiency.

4.2. Instability in cycles

In this section, we provide some simple and non-pathologic examples of instability in $n \times n$ cycles. As we have seen, the existence of these configurations and possibility that they arise under surplus that exhibit elementary forms of complementary jeopardizes stability even in the very simple ODAAT model. For this reason, it is interesting to gain some intuition on these situations and the shapes of production functions that generate instability.

4.2.1. An elementary 2×2 example

Consider two firms $j = 1, 2$ and two workers $i = 1, 2$ and the complete bipartite graph. Productions are specified, using the same notations as in the Kelso-Crawford example, by :

Workers are assumed to have pure monetary preferences, again for simplicity. If we assume :

$$
\begin{array}{rr}
f_{1,\emptyset} = 0 & f_{2,\emptyset} = 0 \\
f_{1,1} = 10 & f_{2,1} = 100 \\
f_{1,2} = 10 & f_{2,2} = 100 \\
f_{1,12} = 100 + \delta & f_{2,12} = 100 + \epsilon
\end{array}
$$

$$100 > \delta > \epsilon > 10$$

Then there is no stable equilibrium. Clearly the unique candidate optimal assignment is to assign both workers to firm 1, but firm 1 is unable to pay both workers enough to retain them – firm 2 always has a profitable deviations in hiring one of them. We can furthermore exhibit a cycle of blocking coalitions starting from there : the output of firm 1 retaining only one worker drops enough that firm 2 has interest to hire the second worker. But then, because its production is submodular, productivity gains are too small and firm 1 has interest to hire both workers – and we recover the initial allocation.

Clearly, an essential ingredient of the instability here is that one firm has submodular production technology while the other has supermodular technology. Furthermore, scale matters : firm 2 is crucially able to produce more with one worker that half of what firm 1 produces with two.

This example can very well be rationalized : one firm (2) has a task to perform which requires essentially one worker and a second one only marginally improves output. The other (1) has a task to perform which crucially requires two workers. Both tasks, if performed have an approximately equivalent output, with the second producing slightly more than the first but not twice as much.

To show how subtle stability can be in general, consider an alteration of the example in which a third worker is added, which is linked only to firm 1. The production function for firm 2 is unchanged and the production function for firm 1 is augmented symetrically : The underlying idea is the same as before : firm 1 has a two-worker task to perform and would not gain much

$$
\begin{array}{lll}
f_{1,1}=10 & f_{1,12}=100+\delta & f_{1,123}=100+\theta \\
f_{1,2}=10 & f_{1,13}=100+\delta & f_{1,\emptyset}=0 \\
f_{1,3}=10 & f_{2,23}=100+\delta &
\end{array}
$$

from hiring a third worker. If we assume that $100 > \theta > \delta > \epsilon > 10$, then there exists a stable equilibrium where firm 1 hires worker 3 and either worker 1 or 2 and the other firms hires the remaining worker – it is easy to check by hand.

Adding one link has enabled us to restore stability in a graph which could be obtained from the ODAAT process. Notice however, that the production function for firm 1 is now neither supermodular nor submodular, and some restrictions on subsets can be found to be either. More importantly, we find that its restriction to workers 1 and 2 when we take for granted that it will hire worker 3 *is* subadditive, hence by the reduction theorem this explains the stabilizing effect.

Hence it seems that some local subadditivity could be recovered even in a case with complementarities and thus stabilize the process. Nevertheless, it is unclear how general such a method could be. If we allow any random meeting order, strong stability can be extremely hard to guarantee. But this examples hints at the idea that if we consider production functions of a well defined sequentially super- then sub-modular according to scale or type and if the market is large enough, we could recover some asymptotic results.

The essential insight is that supermodularity generates instability but a realistic production funciton is unlikely to be supermodular *everywhere*. Hence if the market is large enough and potential conflicts are scarce enough that they have a vanishing probability to arise (for instance, if firms can grow enough to reach a submodularity threshold before entering in cycle type conflicts), stability and complementarity can be reconciled.

Such a sequentially convex/concave production structure is quite an interesting perspective and seems somewhat realistic. This little example shows how it can help to stabilize markets with complementarities – yet the result is by no means general and remains a mere insight for case-by-case analysis so far. It seems that there would certainly be a path for further research in exploring more specifically complementarity structures of this type and how they can bring about stability.

4.2.2. A general example for odd cycles

Using once again a superadditive production technology, we can come up with an example where this no stable allocation in a 3×3 cycle. The example is symmetrical : each firm produces 1 with one of its two available workers, 2 with the other and 6 with both. Firms are arranged on the cycle so that links that produce 1 and 2 alternate. The setup is illustrated in figure 2 below – the production values have been written directly on the graph for greater clarity. Red dots represent firms, blue dots workers. Dashed lines represen potential links and the numerical values above them are the production values when the firm on the end on that link employs *only* the worker on the other end. Output when a given firm employs the two adjacent workers have been displayed using dotted circles.

This examples appear much more specific, notably because it crucially relies on a structure of symmetry. Casual observations tend to suggest that example of instability in cycles of order strictly greater than two are harder to come by and not as easy to rationalize. For that reason, intuition suggest they are much less likely to occur – but once again, we have not been able

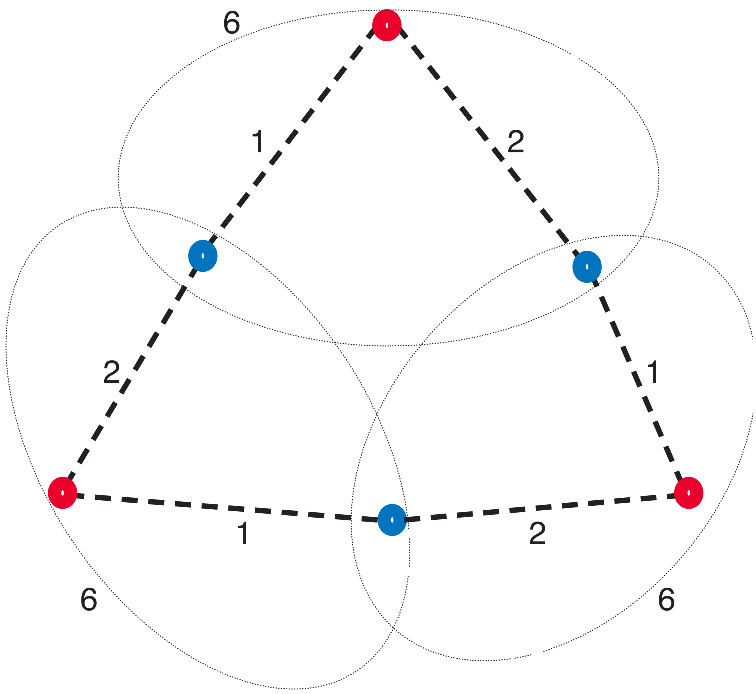

Figure 2: A 3×3 cycle with no stable allocation.

to find a simple and meaningful restriction that guarantees strong stability in general setups, except for subadditivity.

Nevertheless, this 3×3 example is quite useful because it can be extended to any odd cycle by symmetry – it is quite easy to verify directly that the problem of instability and switch between two optimal allocations replicates as long as we have an odd number of firms/workers. To give intuition, we depict in figure 3 below the 5×5 case.

The fact that this example works for any cycle of odd order is sufficient to erase any hope of obtaining strong stability for general surplus functions. However, let us stress again that although no meaningful restriction on the production function has been found, there might be some hope following a similar approach as that outlined for the 2×2 case : we might be able to recover relevant asymptotic results if we identify special cases, limited complementarities, or drastic symmetry requirement, and eliminate their probability of occuring. Thus even though cycles could occur, unstable cycles would have zero probability to occur. This was not the focus of this work, which specifically set as a starting point to not put any restrictions on the surplus function, but this could be the focus of further studies.

4.3. General (lack of) convergence result

It seems natural to ask : if there exists a stable equilibrium on the complete graph, does the ODAAT (or more general search models) converge to that equilibrium ? The answer is, in general, no. Throughout this section, we consider a search process which we assume to be strongly stable

Once again, the main results of this section are negative and probably best illustrated through examples. Before turning to these, we can state some weak results that come for cheap almost

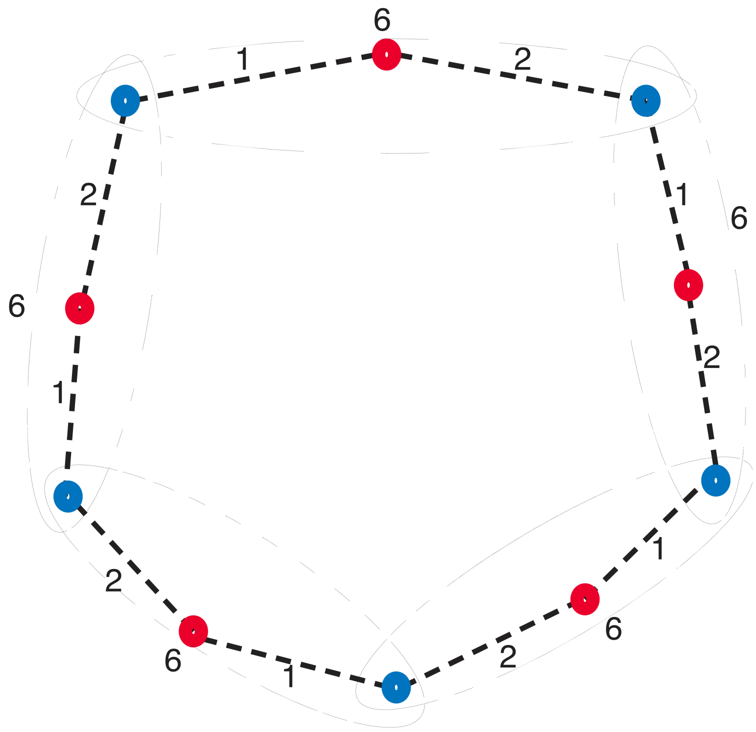

FIGURE 3: A 3×3 cycle with no stable allocation.

by definition of our process.

Proposition 8. *If the system reaches a globally optimal (resp. stable) allocation then it will remain forever in the set of globally optimal (resp. stable) allocations.*

Proposition 9. *Total output (according to the instantaneous surplus chosen) is increasing in time along the path of the search process.*

These results are extremely fragile because they rely on the realization of a random path but do not guarantee its realization. It is clear why the dynamics "stabilize" in the set of globally optimal/stable allocations *if* they reach it, but nothing guarantee that they reach it. In particular proposition 9 gives one reason why : it might be that in some cases we would need to temporarily decrease output to then reach a globally stable/optimal allocation. Such a situation would naturally arise in a case where convex and concave production technologies coexist : in the short run there might be more incentive to allocate workers to concave production technologies, but in the long run it is more efficient to allocate them to convex production technologies – and they would be so directly if all links were feasible to start with.

To illustrate this more clearly, consider once again the 2×2 example presented above, now in dynamic form and in a slightly modified version. Recall that productions are given by :

$$\begin{array}{cc} f_{1,\emptyset} = 0 & f_{2,\emptyset} = 0 \\ f_{1,1} = 10 & f_{2,1} = 100 \\ f_{1,2} = 10 & f_{2,2} = 100 \\ f_{1,12} = 100 + \delta & f_{2,12} = 100 + \epsilon \end{array}$$

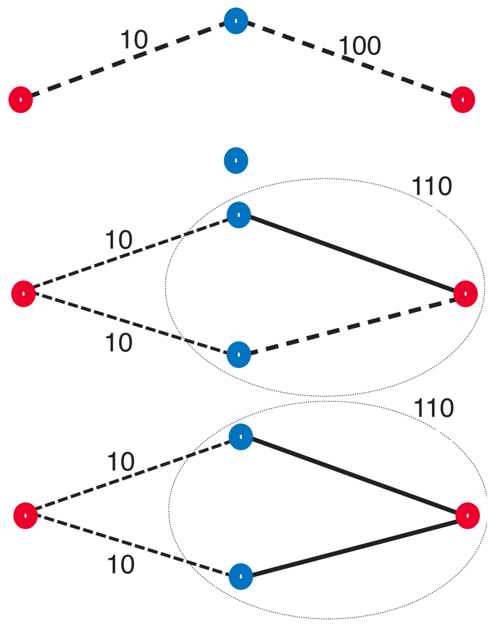

FIGURE 4: Example of a non-convergent allocation process.

Now to ensure that the process is strongly stable we assume $\delta > 100 > \epsilon$. For the sake of illustration take $\delta = 150$ and $\epsilon = 10$. Then consider the following sequence of draws and the resulting assignments (illustrated in figure 4 below) :

- In the first round both firms draw the same worker (dashed lines in the first graph) ;

$\rightarrow$ The disputed worker is assigned to firm 2 (solid line in the second graph) ;

- In the second round, the second firm draws the other worker, and firm 1 draws either worker indifferently (small dashed lines in the second graph)

$\rightarrow$ The first worker remains assigned to firm 2 and the second worker is also assigned to firm 2 (solid lines in the third graph)

$\Rightarrow$ From then on, for whatever draw for firm 1 (small dashed lines in the third graph), both workers remain assigned to firm 2.

Clearly with this sequence the globally stable equilibrium cannot be achieved. Once again, let us stress that this example is somewhat extreme : it hinges on a small market structure with repeated conflicts, which might be extremely unlikely to arise with a large number of heterogeneous firms and even more so with an even larger number of workers (reducing potential conflicts). Yet this example is still relevant for several reasons.

First, such cases cannot in general be ruled out and this is sufficient to conclude that convergence cannot be ensured with no further restrictions. Furthermore it hints at ways to refine the model and seek reasonable restrictions under which such conflicts have vanishing probability to arise.

Second, it is replicable in a large market : just consider a market with only replications of firms 1 and 2, all workers are identical, and production with three or more workers yields no extra output relatively to two workers production. In that context, if the number of worker is not too high (say $n << m/2$) there is a significant probability that *at least some* firms of type 1 will be stuck in a suboptimal regime where they are unable to hire any workers.

Third, although the example apparently hinges on a truely myopic surplus function, it is easy to extend it to the dynamic setup while still so as to retain the failure to converge. Most prominently, this will be due to discounting : if the future is discounted enough, then even if gains are perfectly forecasted we might fail to reach the stable equilibrium. This is a very general distinction to be made : the optimal path is, in general, not the repetition of the optimal static equilibrium as long as there is discounting. This result is quite intuitive and readily extends to imperfect foresight. To illustrate it, let us consider the same example as before, but this time as a dynamic maximization problem – setting aside the issue of stability, although in this particular case they are almost equivalent. Once again, this is a stylization that does not bear any realistic weight but suffices to show how convergence can fail to obtain. Then, consider the exact same sequence of draws exhibited in figure 4, and assume draws are uniform. Working backward from the third step, and assuming that the discount rate is given by β, it is clear that firm 1 will never be awarded any worker as long as :

$$\sum_{k=0}^{\infty}\left[\left(\frac{1}{2}\right)^k\left(10+\frac{1}{2}\frac{\beta^{k+1}}{1-\beta}250\right)\right] < \frac{1}{1-\beta}110$$
$$\Leftrightarrow 20+\frac{2-3\beta}{\beta}250 < \frac{1}{1-\beta}110$$

We can work further backward on previous steps to obtain conditions on β such that it is optimal to assign all workers to firm 2.

Hence because of the sheer time dimension that is added to the problem, the usual static stable equilibria are not guaranteed to remain equilibria in the search process and may not never realized, even from the single point of view of efficiency. This is, on the one hand, a natural consequence of adding time, but it could undermine our interpretation of dynamic stability as a generalization of "usual" stability – or at least substantially alter its perspective. But this interpretation can in large part be rescued through examination of the behaviour of the system and its convergence in the submodular case, which is consistent with static behaviour.

4.4. The gross substitutes and submodular cases

The category of random search processes that we have introduced behaves essentially like a random greedy algorithm – which, although extremely inefficient, should inherit some of the general convergence properties of the non-random version precisely in the gross substitutes and more generally submodular case. Throughout this section we will consider only the true myopic setup – i.e the instantaneous surplus function is just the production function – precisely because we would expect that in the submodular case locally greedy behaviour is equivalent to a more sophisticated lookahead.

For this reason, we would expect that, since the algorithm guarantees constrained efficiency at each step, then in the submodular case – in which we recall that the process is strongly dynamically stable – we could expect that we reach a matching that is unconstrained optimal in finite time. Intuitively, this should be backed by submodularity because we go "in the right direction" for optimality at each step. Such a result is suggested by the simple Kelso and Crawford example in which it easily verifiable that this property holds. It is nevertheless *not* *true* in general.

Indeed, one element counteracts the effect of submodularity in our logic : the history of draws. At each step, efficiency is maximized with respect to the constraint graph which crucially preserves previously formed coalitions. Because of this, the algorithm induced by the search process is only able to compare deviations of one or at most two workers between any two firms – i.e single worker transfers or swaps. Thus, even with submodularity, the process might get trapped with coalitions that are somehow "strong enough" to be immune to small deviations, although they would not be optimal if larger deviations were feasible.

To show this, we must consider an example more complex than the Kelso-Crawford case. Consider the following very stylized setup with 2 firms and 4 workers, where production is specified according to the following two tables. Output is identical for both firms except for the highlighted cases.

$$\begin{array}{llll}
f(1,1)=5 & \mathbf{f(1,12)=8} & f(1,123)=9 & f(1,1234)=9 \\
f(1,2)=5 & \mathbf{f(1,34)=9} & f(1,234)=9 & \\
f(1,3)=5 & f(1,23)=6 & f(1,124)=9 & \\
f(1,4)=5 & f(1,13)=6 & & \\
 & f(1,24)=6 & & \\
 & f(1,14)=6 & &
\end{array}$$

$$\begin{array}{llll}
f(2,1)=5 & \mathbf{f(2,12)=9} & f(2,123)=9 & f(2,1234)=9 \\
f(2,2)=5 & \mathbf{f(2,34)=8} & f(2,234)=9 & \\
f(2,3)=5 & f(2,23)=6 & f(2,124)=9 & \\
f(2,4)=5 & f(2,13)=6 & & \\
 & f(2,24)=6 & & \\
 & f(2,14)=6 & &
\end{array}$$

It is clear from a quick examination of the prodution that the maximum global output is given by :

$$f(1,34)+f(2,12)=18$$

Furthermore, it is clear that the corresponding allocation where firm 1 hires workers 3 and 4 and firm 2 hires workers 1 and 2 is *stable*.

Now, consider the situation in which firm 1 was assigned workers 1 and 2 and firm 2 was assigned workers 3 and 4. This could arise, for example, if two periods have elapsed in which boths firms have encountered a different worker and never conflicted. Such an assignment yields an aggregate output of :

$$f(1,12)+f(2,34)=14$$

Because of the symmetry of the production, any non-redundant couple of draws yield the same outcome, so for the sake of the argument consider that firm 1 has drawn worker 3 and firm 2 has drawn worker 2 – as is depicted in figure 5.

Then it is quite straightforward to see and verify by hand that no firm can hire a worker that the other employs in this situation and a swap is not profitable. Indeed, both cases yield a lower aggregate output ; more specifically if one firm would hire one worker from the other output would decrease from 14 to 11 and if the firms swapped one worker for another aggregate output would decrease to 12. In both cases the decrease is symmetrical and both firms suffer

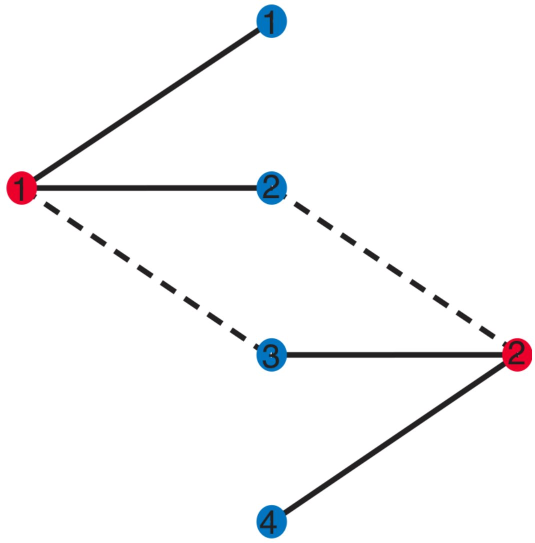

FIGURE 5: A 2×4 counter-example to convergence towards an optimal allocation in the submodular case.

a loss. Hence none of these allocations can be stable. This entails that the only constrained-stable allocation[13] in this context is the status quo allocation previously enforced. Because of the symmetry of the situation for any draws, this means that the market can **never** shift to another allocation and is thus effectively trapped at this suboptimal allocation. Because it considers only one worker deviations or swaps, the process fails to consider the full four workers swap which would be mutually profitable for all agents as it increases aggregate output.

This example shows that submodularity, although it guarantees existence of constrained stable equilibria at any step, is *not enough* to guarantee convergence to a stable equilibrium when there exists one and cannot even ensure that we reach optimality in terms of aggregate output.

Notice that the production function that was used in constructing the example nevertheless crucially fails the gross substitutes assumption. It relies on a complementarity between two workers for each firm along with symmetry between the two firms. To see this for e.g. firm 1, simply consider wage vectors w such that

$$w_1 = w_2 = w_3 = w_4 = 3$$

and w' such that

$$w'_4 = 7$$

and $w'_i = w_i$ for $i = 1, 2, 3$. Then observe that under w, firm 1 demands only workers 3 and 4, but under w' it demands only workers 1 and 2. This clearly contradicts GS because a rise in the cost of worker 4 leads the firm to stop demanding worker 3 for an unchanged wage.

We conjecture that if the surplus is assumed to have no complementarities (or verifies the strong no complementarities condition) such suboptimal traps cannot exist. Essentially, the intuition behind this is that if there are no complementarities, you cannot have a decrease of marginal contribution to production of any worker caused by the removal of one link or the swap of one worker for another – otherwise you create a direct violation of the GS condition.

[13] Which must exist by submodularity, and it is once again easy to check it by hand in this simple context.

This removes the essential ingredient that was used in the example above – essentially with GS we expect that there can be no local maximum however small the deviations we consider are, otherwise the production structure creates joint demands for certain workers. Thus adding links one by one randomly should give us an unconstrained stable equilibrium after some (possibly long) time. If there are no complementarities, there will never be a need to swap and there is actually very limited heterogeneity in workers, marginal increases to production behave essentially as in the standard case in which only the quantity of employed workers matter. Thus the allocation of workers to firm is essentially a simple quantity allocation problem and sequential allocation according to marginal contribution to output is an efficient method.

Conjecture 1. *Consider the ODAAT search models with fully myopic surplus and no search costs. Any such model in which the surplus function satisfies the gross substitutes condition converges towards a globally stable allocation. This remains true if we allow more draws per period.*

Of course, this reasoning is very informal and this result is, so far, just a conjecture. Indeed, although the intuition seems clear, the proof is not self-evident. Because of the structure of the allocation process, wages are allowed to move up and down in time and we cannot view the search process as a replication of any sort of auction or the Kelso-Crawford generalization of the deferred acceptance algorithm. Recovering global unconstrained properties from the sequence of constrained allocations requires a fine analysis of marginal contributions at each step in a possibly very complex fashion with important differences making comparability to the global equilibrium challenging. Hence although the intuition is clear the technical details are complex and we have not yet found a conclusive proof of this conjecture.

It also seems likely that convergence could obtained under much less stringent conditions – and most notably for the submodular case, if we add exogenous separations, i.e random deletion of links. This is akin to an energy/entropy problem in physics, and from this insight we can conjecture as well that if all links have a positive (and possibly high enough) probability of breaking up, then it gets "harder" to be stuck in a local energy minimum and there should exist a random path towards the globally optimal set of allocations and stable allocation if there exists one. The time of convergence could nonetheless in general remain extremely long, as is often the case in physical examples of such conflicts.

5. Conclusion

5.1. Main findings

The present work can be approached from several perspectives which qualify the interpretation that can be given of the results presented here.

The initial aim of this work was to reconcile stability and complementarities in large dynamic matching markets with frictions, in which we expected that although the technical structure of the surplus is characterized by features that usually prevent existence of equilibria, the actual allocation process rendered instability unlikely. The approach was clearly methodological with the intent of building an alternative weaker stability concept for such markets under which we could interpret the behaviour of otherwise unstable situations. The ultimate goal was to find a modeling approach that could rationalize the dynamic behaviour of such markets using the rationale of the usual (static) notion of stability without resorting to restrictions on the shape of the surplus function.

For this purpose, we defined the framework of constrained allocations and a category of search processes characterized by a succession of constrained allocation problems. We defined accordingly a concept of stability locally in time that is a property of the process as well as the surplus – and not just the latter. We focused on one case of particular interest with one draw at a time, which gives rise to very regular graphs structures that are tree-like with at most one cycle per component.

As far as the initial goal is concerned, the main results are negative. Indeed, we developped a reduction method to study stability issues on large graphs which works both ways. If stability can be guaranteed on some well chosen small structures, it spreads out to the whole graph – but if it cannot, instability can spread similarly. Hence there is little hope of recovering stability with no further assumptions.

From this point there are two routes : restrictions on the surplus function and restrictions on the process. Restrictions on the surplus function can provide some weakening of the usual gross substitutes assumption in our setup – most notably, we obtain that submodularity is sufficient for ensuring existence of a stable equilibrium. If we step further into the realm of the gross substitutes assumption, this opens up a new interpretation of our model with respect to the litterature – the emphasis lying no more on the complementarity and stability issues but on the impact of frictions. General questions of convergence arise. Once again, some negative results are obtained. In the submodular case, even when there exists an unconstrained stable equilibrium, we cannot guarantee convergence in general and the process might get stuck in a suboptimal local equilibrium. Examples of such behaviour seem to always involve violations of the gross substitutes condition and we conjecture that convergence can be ensured in the GS case – yet the proof is not as obvious as the underlying intuition and we have yet to provide a satisfactory demonstration of the result.

Restrictions on the process can allow us to recover asymptotical stability by ensuring that the graph obtained is (asymptotically in the number of agents) a tree graph. This is relatively reasonable in large markets, but it dilutes the strategic dimension of the instantaneous assignments. Although this might very well be relevant for some contexts, once again this can be interpreted as a mainly negative results from the point of view of our initial aim. Indeed, stability can be recovered in our model, either asymptotically or under rather mild restrictions on the structure of complementarities, but this still falls short of being a general purpose tool to interpret large market structures with complex complementarities. Hence although there might be some interests in applications of the model on a case-by-case basis, it is in general not able to provide a satisfactory alternative to existing methods and models in the litterature.

By trying to stand as a bridge between two strands of litterature – on static matching and dynamic search models – the model nevertheless highlights key methodological points and it seems that one of the main interests of our results is what they suggest for future research.

5.2. Paths for future research

Trying to preserve semi-static stability in a context where general complementarity structures are allowed seems too demanding – our results show that instability at however small a level can jeopardize the whole system, even when frictions weaken it significantly. Hence the question remains : what *does* happen on such large matching markets when complementarities prevent the existence of a stable equilibrium ?

Essentially there are two ways to approach the strategic dimension and opportunity cost in such a context – either you compare to alternative offers that are simultaneously available (static approach) or you compare to waiting cost against potential future offers (dynamic approach). We tried to bring some elements of search theory into the standard static framework while keeping the static stability notion as a guideline. Perhaps doing the opposite would be fruitful.

In search models *à la* Shimer-Smith, no two encounters arise simultaneously and matched pairs exit the market, so the static notion of stability is completely empty – there remains the opportunity cost of waiting and the computation of steady state equilibrium type acceptance sets as a strategic dimension. But the many-to-one framework crucially modifies such a setup. First, at least one of the parties involved in a match cannot exit the market – otherwise this would not be a many-to-one problem. Hence, the problem of the sequentiality of hirings and the structure of complementarity between hires poses to every firm on the market. This might incidentally allow us to reintroduce the ability to search throughout the market and not only on the subset of unmatched agents – in the one-to-one and unidimensional setup this leads to trivialities in terms of acceptance, but in the many-to-one setup interactions might become more complex.

Furthermore, in a dynamic many-to-one model, the question of payoffs and liquidity constraints and how to handle it technically is not straightforward at all. When a wage is fixed between two parties that agree to match, is it binding over future periods ? Can it be renegociated at any time ? What is properly to be considered the outside option of a worker ? It seems that the structure is that of an intertwined internal renegociation process but a co-dependancy of outside options because firms determine the outside option of workers of other firms if they make offers. Then again, in a dynamic setup with complementarities, firm might want to pay some workers more than their current marginal product to retain their services in expectation of future complementarity effects. What constraints should we put on these ? This opens up an area which is usually left out by matching models : budget constraints. In a static model with a mere division of surplus between matched parties, there is indeed not much point in considering budget constraints' impact. In a many-to-one context where partnerships take place in time and interact, the ability to offer today some of the future (expected) surplus gain might be crucial but it vastly complexifies the problem.

Starting from the same depature and aim as this work, we could conceive an alternative modeling approach that leans more towards the structure of search model. The first step on this paths would without a doubt be a many-to-one extension of the Shimer and Smith model. The essential difficulty is to do so in such a way as to avoid trivialities. If firms always remain on the market, they have no opportunity cost to accepting a match, so why would they refuse ? Do they have limited openings to fill ? Are they somehow bound to a contract ? Do they lose something if they endogenously severe the relationship ? In a more general approach, how do they account for future profits ? Perhaps a good starting point would be to study in depth the problem of a isolated firm facing a sequential hiring problem – which by many aspects is not unlike the standard "secretary problem" in optimal stopping. Another starting point could be to fix an internal wage negociation procedure, for example the Stole/Zweibel [48] process that yields wages robust to renegociations, but make it dependant on outside options determined by endogenous waiting costs (or competing offers) and study how to close the system in a dynamic setting.

Overall, it seems that there are possibly fruitful directions to be explored which let go of the static stability approach and focus on what the relevant *dynamic opportunity cost* to matching should be. This approach is technically and heuristically different although closely related to the one we have tried to follow, and perhaps it could prove more successful in characterizing the behaviour of dynamic matching systems under complementarities.

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A. Proofs for Section 2 (Framework and definitions of essential concepts)

Proposition 1. *If (α, w) is stable, then it is efficient. If (α, w) is individually rational and stable, then it is IR-efficient.*

Proof. Assume that (α, w) is a (weakly) stable allocation. Consider an arbitrary alternative assignment α'. The stability condition entails that for all $j \in J$:

$$f(j, \alpha'^{-1}(j)) - \sum_{i \in \alpha'^{-1}(j)} w_i \leq f(j, \alpha^{-1}(j)) - \sum_{i \in \alpha^{-1}(j)} w_i \tag{A.1}$$

Summing over j and rearranging, this gives :

$$\sum_{j \in J} \left[f(j, \alpha'^{-1}(j)) - f(j, \alpha^{-1}(j)) \right] \leq \sum_{j \in J} \sum_{i \in \alpha^{-1}(j)} w_i - \sum_{j \in J} \sum_{i \in \alpha'^{-1}(j)} w_i \tag{A.2}$$

Define the employed workers sets for both allocations as :

$$\begin{aligned} E :=& \{i \in I \mid \alpha(i) \neq \emptyset\} \\ E' :=& \{i \in I \mid \alpha'(i) \neq \emptyset\} \end{aligned}$$

Next, we use the following lemma :

Lemma 3. *An unemployed worker must have a null wage in any stable allocation.*

The proof is a straightforward application of the feasibility and stability conditions. In particular this means that :

$$\forall i \in I, \; i \notin E \; \Rightarrow \; w_i = 0$$

Notice that, by definintion of E and E' and using the lemma, we can rewrite the right-hand expression in A.2 as :

$$\begin{aligned} \sum_{j \in J} \sum_{i \in \alpha^{-1}(j)} w_i - \sum_{j \in J} \sum_{i \in \alpha'^{-1}(j)} w_i &= \sum_{i \in E'} w_i - \sum_{i \in E} w_i \\ &= - \sum_{i \in E \setminus E'} w_i \leq 0 \end{aligned} \tag{A.3}$$

This directly entails that for an arbitraty matching α', we have :

$$F(\alpha') - F(\alpha) \leq 0 \tag{A.4}$$

Hence α is optimal. In particular, if we restrict to IR matchings (or any particular class of matchings), the exact same proof applies and we conclude that α is optimal among that class of matchings.

□

Proposition 2. *If (α, w) is constrained stable under A, then it is constrained efficient under A. If it is in addition individually rational, then it is constrained IR-efficient under A.*

Proof. The proof is exactly identical to the proof of proposition 1 when we restrict to the class of feasible matchings under a given matrix A. $\square$

B. Proofs for Section 3 (Existence of stable equilibria in dynamic search models)

Theorem 2. *Consider a constrained matching problem with surplus function f and constraint matrix A. Let j be a pivotal firm and $(I_1, J_1), (I_2, J_2)$ the corresponding partition. If for any allocation α_1 on the subgraph defined by (I_1, J_1) there exists a stable allocation in the localized allocation problem under α_1 and for any allocation α_2 on the subgraph defined by (I_2, J_2) there exists a stable allocation in the localized allocation problem under α_2, then there exists a constrained stable allocation under A and f.*

Proof. Let j be a pivotal firm for the constraint matrix A and (I_1, J_1), (I_2, J_2) the corresponding partition of workers and firms. Assume that for any allocation α_1 on the subgraph defined by (I_1, J_1) there exists a stable allocation in the localized allocation problem under α_1 and for any allocation α_2 on the subgraph defined by (I_2, J_2) there exists a stable allocation in the localized allocation problem under α_2. We define the application ψ from the sets of allocations on (I_1, J_1) to itself by the following process :

- Consider an arbitrary allocation α_1 on (I_1, J_1)
- Define an application θ_1 that maps α_1 to an allocation α_2 that is stable in the localized allocation problem under α_1, using the following method :
  - If there exists α_2 such that α_1 is a stable allocation in the localized allocation problem under α_2, choose this $\alpha_2 = \theta_1(\alpha_1)$
  - If not, we assume we have an arbitrary rule for choosing $\alpha_2 = \theta_1(\alpha_1)$ among stable allocations in the localized problem under α_1
- Define an application $\theta_2|_{\alpha_1}$ that maps α_2 to an application α_1' that is stable in the localized allocation problem under α_2, using the following method :
  - If α_1 is stable in the localized problem under α_2, then $\alpha_1' = \alpha_1 = \theta_2|_{\alpha_1}$.
  - If not, we assume we have an arbitrary rule for choosing $\alpha_1' = \theta_2|_{\alpha_1}$ among stable allocations in the localized problem under α_2
- Set $\psi(\alpha_1) = \alpha_1' = \theta_2|_{\alpha_1} \circ \theta_1(\alpha_1)$

Clearly if ψ has a fixed point, then there exists a constrained-stable allocation on (I, J) according to A – it means that there exists two allocations α_1^* and α_2^* on (I_1, J_1) and (I_2, J_2) respectively that are stable for their respective localized allocations problem, hence it is straightforward to see from the definitions that the allocation α^* on (I, J) defined by :

$$\alpha^*(i) = \begin{cases} \alpha_1^*(i) & \text{if } i \in I_1 \\ \alpha_2^*(i) & \text{if } i \in I_2 \end{cases}$$

is constrained stable. In order to prove that ψ has a fixed point, we introduce the notion of cycle :

Definition 25. *Let X a finite space, and $\psi : X \to X$. For $n \in \mathbb{N}^*$, denote by $\psi^{(n)}$ the application defined by iterative applications of ψ, i.e for all $x \in X$ and $n \in \mathbb{N}$, $\psi^{(n+1)}(x) = \psi^{(n)}(\psi(x))$ with $\psi^1 = \psi$. We say that ψ admits a cycle of order n if there exists $x \in X$ such that :*

$$\psi^{(n)}(x) = x$$
$$\forall 0 < k < n,\, \psi^{(k)}(x) \neq x$$

With the convention that $\psi^{(0)}(x) = x$.

Notice that in particular ψ admits a fixed point if and only if it admits a cycle of order 1. We briefly state a useful proposition

Proposition 10. *A cycle can never contain any strict subcycle – where a strict subcycle is defined as a cycle of smaller order nested within the initial cycle.*

> *Proof.* The proof is trivial. Denote $x^{(0)}, ..., x^{(n)}$ the succesive elements of a given cycle of order n. There exists a strict subcycle if there exists k and k' such that $0 < k < k' < n$ and $x^{(k)}, ..., x^{(k')}$ forms a cycle. Clearly any subsequent element of the cycle beyond $x^{(k')}$ must belong to the subcycle, but none of these elements is equal to $x^{(0)}$ by definition, which is a contradiction. □

Corollary 4. *Denote $x^{(0)}, ..., x^{(n)}$ the succesive elements of a given cycle of order n. We have for all $0 < k < k' < n$:*

$$x^{(k)} \neq x^{(k')}$$

The following lemma builds further on this :

Lemma 4. *If ψ has no cycle of order greater or equal to 2, then ψ must have a fixed point.*

> *Proof.* The proof is straightforward by a constructive method. Assume that ψ has no cycles of order more than 2 and let n be the cardinal of X. Choose an arbitrary point $x \in X$. If $\psi(x) = x$, then x is a fixed point of ψ and the proof is over. If $\psi(x) \neq x$, iterate the application of ψ : because ψ has no cycles of order greater than 2, we necessarily have that for all n, $\psi^{(n)}(x) \neq x$. But since there are no cycles of order two, we have furthermore that for any $k \geq 2$, $\psi^{(n+k)}(x) \neq \psi^{(n)}(x)$, hence this means that the iteration of ψ starting from x cannot pass by the same point twice *except* if that point is a fixed point. Assume that there are no fixed point, this means that the iterations of ψ starting from x never pass by the same point twice – i.e the infinite sequence $(\psi^{(n)}(x))_{n \in \mathbb{N}}$ never repeats. This is clearly impossible since the sequence takes its values in a *finite* space. Hence ψ has a fixed point. □

Now, assume that the application ψ defined above has a cycle of order $n \geq 2$. Denote $\alpha_1^{(t)}, \alpha_2^{(t)}$ the sequence of matchings obtained through iterations of ψ, i.e for all t :

$$\alpha_2^{(t)} = \theta_1(\alpha_1^{(t-1)})$$
$$\alpha_1^{(t)} = \psi(\alpha_1^{(t-1)})$$

By assumption we have for $0 < t < t' < n$:

$$\alpha_1^{(0)} = \alpha_1^{(n)}$$
$$\alpha_1^{(t)} \neq \alpha_1^{(0)}$$
$$\alpha_1^{(t')} \neq \alpha_1^{(t)}$$

This entails that, by construction of ψ, we also have :

$$\alpha_2^{(t')} \neq \alpha_2^{(t)}$$

for $0 < t < t' < n$, otherwise the above conditions would be contradicted. From the persepctive of iterations, because we have assumed j to be pivotal, all the relevant information from the previous assignment is contained in the allocation of workers to j on the appropriate side of the market, which we denote for all t by :

$$C_1^{(t)} := (\alpha_1^{(t)})^{-1}(j)$$
$$C_2^{(t)} := (\alpha_2^{(t)})^{-1}(j)$$

By proposition 1, stability entails efficiency, hence we have that for all $t \geq 1$:

$$\alpha_2^{(t)} \in \arg\max_{\alpha} \sum_{k \neq j} f(k, \alpha^{-1}(k)) + f(j, \alpha^{-1}(j) \cup C_1^{(t-1)})$$
$$\alpha_1^{(t)} \in \arg\max_{\alpha} \sum_{k \neq j} f(k, \alpha^{-1}(k)) + f(j, \alpha^{-1}(j) \cup C_2^{(t)})$$

Where the maximum is taken among the sets of stable allocations.This entails that for any alternative sequence of allocations $(\alpha_1^{'(t)}, \alpha_2^{'(t)})$, for all $t \geq 1$:

$$\sum_{k \neq j} f(k, (\alpha_2^{(t)})^{-1}(k)) + f(j, (\alpha_2^{(t)})^{-1}(j) \cup C_1^{(t-1)}) \geq \sum_{k \neq j} f(k, (\alpha_2^{'(t)})^{-1}(k)) + f(j, (\alpha_2^{'(t)})^{-1}(j) \cup C_1^{(t-1)})$$
$$\sum_{k \neq j} f(k, (\alpha_1^{(t)})^{-1}(k)) + f(j, (\alpha_1^{(t)})^{-1}(j) \cup C_1^{(t-1)}) \geq \sum_{k \neq j} f(k, (\alpha_1^{'(t)})^{-1}(k)) + f(j, (\alpha_1^{'(t)})^{-1}(j) \cup C_1^{(t-1)})$$

Let us consider the sequence :

$$\alpha_2^{'(1)} = \alpha_2^{(n)}$$
$$\alpha_1^{'(t)} = \alpha_1^{(t-1)} \text{ for } t \geq 1$$
$$\alpha_2^{'(t)} = \alpha_2^{(t-1)} \text{ for } t \geq 2$$

Notice that with this choice of allocations and with the definition of ψ, all the inequalities above are strict – indeed, ψ was constructed so as to always pick fixed points if they exist, hence if any of the inequalities was an equality, the considered allocation would have been picked, hence we would have for some t : $\alpha_1^{(t)} = \alpha_1^{(t-1)}$ which contradicts the no-subcycle property. Plugging this in the inequality above and simplifying using optimality, we obtain a chain of inequalities :

$$f(j, C_2^{(1)} \cup C_1^0) > f(j, C_2^{(n)} \cup C_1^0)$$
$$f(j, C_1^{(1)} \cup C_2^1) > f(j, C_2^{(1)} \cup C_1^0)$$
$$...$$
$$f(j, C_2^{(n)} \cup C_1^{n-1}) > f(j, C_2^{(n-1)} \cup C_1^{n-1})$$
$$f(j, C_1^{(n)} \cup C_2^n) > f(j, C_2^{(n)} \cup C_1^{n-1})$$

Combining all inequalities, we obtain :

$$f(j, C_1^{(n)} \cup C_2^n) > f(j, C_2^{(n)} \cup C_1^0)$$

But by assumption, $C_1^{(n)} = C_0^{(n)}$, hence this is a contradiction. Intuitively, the existence of cycles of length superior or equal to 2 contradicts the construction of ψ on stability/optimality improving sequences.

We conclude that ψ has no cycles of length greater than 2, hence by the above lemma ψ has a fixed point, thus there exists a constrained-stable equilibrium on the whole market. □

Proposition 3. *If there exists a pivotal firm j such that the corresponding partition on one side* never *admits a stable equilibrium for any allocation on the other side, then there does not exist a constrained-stable equilibrium on the market.*

Proof. The proof is straightforward. Let j be a pivotal firm for the constraint matrix A and (I_1, J_1), (I_2, J_2) the corresponding partition of workers and firms. Assume that there never exists a stable equilibrium in the localized problem on (I_1, J_1) whatever the allocation on (I_2, J_2). Consider any allocation α. Because stability is a local notion, for any global allocation it suffices to restrict attention to the unstable subgraph ; hence we consider $\alpha|_{I_1}$ the restriction of α to I_1. By assumption there exists a coalition in (I_1, J_1) that violates the stability condition for $\alpha|_{I_1}$, hence for α as well, which is sufficient to show that α is unstable. □

Proposition 4. *Any given pair of firms have* at most *two feasible workers in common and any firm can have two feasible workers in common with* at most *one firm.*

Proof. The first part of the proposition is trivial by definition of the ODAAT process : remember that links are either preexisting links or newly drawn links and that each firm has exactly one newly drawn link. Hence a worker connected to two (resp. k) firms must have been drawn by at least one (resp. $k-1$) of them, and a firm connected to two (resp. k) workers must have been already connected to at least one (resp. $k-1$) of them. If two firms have three (resp. k) workers in common, one (resp $k-2$) cannot have been drawn by either which is a contradiction (a worker cannot have been previously linked to two firms). For the second part, assume a firm has two feasible workers in common with another, then as we have just seen that it must have drawn one of them. If there was another firm with which it had to workers in common, this would mean it has drawn two workers which is a contradiction. □

Proposition 5. *All cycles of a graph obtained via ODAAT are alternate-draws cycle.*

Proof. Consider unoriented constraint graph and its alteration as the *oriented* graph obtained from the draw using the convention that a worker drawn is a link from the firm *to* the worker and pre-existing links are bi-directional. Using that convention, a worker has out-degree at most one, and a firm has exactly one unilateral link. Consider a firm j with pre-existing links with two workers i and i'. By definition there is a link from i to j and from i' to j, hence all other links to i, i' *in the undirected graphs* are links *towards* i, i' in the *directed* graph. Let us show that for any firm that is obtained via a path starting from i and *not passing by* j on the undirected graph points towards the worker on that path, and that this worker cannot be i'. The proof is by induction on the path length.

- Consider a firm $j' \neq j$ at distance 1 from i in the undirected graph. It is then linked to i and as we have seen, this link must be directed towards i.
- Assume that a firm j' at distance d from i in the undirected graph points towards the preceding worker on the path and that this worker is never i'. Take a firm j'' at distance two from j' i.e at distance $d+2$ from i. We know that j' cannot be linked to i'. Indeed, assume that j' pointing to $i'' \neq i$ is also linked to i', then because i' is pointing towards j, i' cannot be pointing towards j'. Hence j' has two unilateral links (has picked two workers) which is impossible. Because j' already points to i'', j'' must be pointing towards the worker it has in common with j', which cannot be i' since $i' \notin A(j)$. This shows the claim.

This means no firm reached from following links starting fro i (resp. i') in the undirected graph can be connected to i' (resp. i). In other words, a triplet with a firm and two previously connected workers can never be part of any cycle. This means that any firm in a cycle must have picked one of the workers to which it is connected on the cycles – i.e all cycles are alternate-draws cycles. □

Proposition 6. *Two cycles cannot be connected – in particular they cannot intersect.*

Proof. The proof follow a similar logic as that of the previous proposition. Because we know that cycles are alternate-draws cycles, any path that start from a point of the cycle in the *undirected graph* is a path *towards* that point (and *one way only*) in the *directed graph.* Any two such paths that flow from two nodes of a cycle (whether the two cycles considered are distinct or not) cannot intersect, otherwise it would contradict the fundamental properties of the ODAAT process – either a worker must have two pre-existing links or a firm must have drawn two workers. □

Theorem 3. *The ODAAT model is :*

1. *asymptotically stable for any surplus function if and only if the probability that cycles arise is vanishing.*
2. *strongly stable if and only if we can guarantee existence of a stable equilibrium in the cycle case for any sequence of firms and workers and any augmentation of the production function by exogenous coalitions. That is to say that for all* $i_1, ..., i_k \in I$ *and all* $j_1, ..., j_K \in J$, *and for all* $C_1, ..., C_k \subset I$ *such that* $i_1, ..., i_k \notin C_1 \cup ... \cup C_k$, *there exists a stable allocation of* $i_1, ..., i_k$ *constrained by* $\alpha(i_l) \in \{j_l, j_{l+1}\}$ *(using the convention* $j_{k+1} = j_1$*) under the modified surplus function* $\tilde{f}$ *such that* $\tilde{f}(j_l, C) = f(j_l, C \cup C_l)$ *for* $l = 1, ..., k$ *and* $C \subset \{i_{l-1}, i_l\}$.

Proof. The proof for the theorem is a direct application of theorem 2 and proposition 3. □

Proposition 7. *If the surplus function f satisfies the local gross substitutes condition under A, there exists a stable allocation under A.*

Proof. The proof is a trivial replication of one of the standard proofs of existence (e.g. Kelso and Crawford [28] or Gul and Stacchetti [25]) in which we restrict to feasible coalitions under A. $\square$

Theorem 4. *Consider the ODAAT model with surplus function f. If for any firm $j \in J$, the function $f(j, \cdot)$ is subadditive in its second argument, then the model is strongly stable.*

Proof. The proof is a straightforward application of theorem 2 and proposition 7 combined with two lemmas.

Lemma 5. *Consider a constraint allocation problem in which no firm is connected to more than two workers and every firm's production function is subbaditive. Then the production satisfies the Local Gross Substitutes on the constraint set.*

> *Proof.* The proof is identical to Kelso and Crawford's ([28], section 6) proof in the 2×2 case. We just have to consider the four possible demands of any firm, this allows to separate the space of possible wages in four regions. If the firm has a subadditive production function, it clearly appears that a rise in the wage of one of the worker cannot induce the firm *not* to demand the other if it did for the initial wage. $\square$

Lemma 6. *Consider f as subadditive set function, $\overline{C}$ an arbitrary set, and $\tilde{f}$ the modification of f obtained by exogenously adding $\overline{C}$ to the input of f, i.e $\tilde{f}(C) = f(C \cup \overline{C})$. Then $\tilde{f}$ is also subadditive.*

> *Proof.* This is trivial by definition of subadditivity. $\square$

Remember every component of an ODAAT-induced graph is (at most) one cycle branching off into trees. Then by the second lemma above, it is clear that if the production function is overall subadditive, it is subbaditive on the cycle for any arbitrary allocation on the trees. Hence by the first lemma, it satisfies the local GS condition. This means by proposition 7 that for any allocation on the trees there exists a stable allocation on the cycle. By theorem 2 again, we know that a tree is always stable. Hence using 2 on the whole graph, there exists a constrained-stable allocation. $\square$

C. Proofs for Section 4 (Properties of the dynamic search model and examples)

Proposition 8. *If the system reaches a globally optimal (resp. stable) allocation then it will remain forever in the set of globally optimal (resp. stable) allocations.*

Proof. The proof is trivial : the allocation realized at each step is constrained-stable, hence it is constrained-optimal. If a global optimum was realized at the previous period, it remains feasible next period, hence no suboptimal (globally) matching can be realized. $\square$

Proposition 9. *Total output (according to the instantaneous surplus chosen) is increasing in time along the path of the search process.*

Proof. The proof follows the same logic as that of proposition 8 : the matching realized at each step is constrained-stable, hence constrained-optimal, and the matching realized in the previous period always remains feasible. If output decreased, this would mean that a suboptimal matching has been selected, which is a contradiction. □