Optimal transport coupling in multi-population mean field games

Matching equilibrium displacement and applications to urban planning

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We present a multi-population mean-field game (MFG) model in which the running cost functions of each population are coupled via the potentials of an instantaneous optimal transport problem between their respective densities, with applications to geographical population dynamics and urban planning. The modeling approach is abstract and very stylized ; its applications bear no pretensions of being a realistic nor complete description of observed dynamics. Indeed, our ambition is rather to present conceptual mechanisms susceptible to drive the dynamics of such systems and to explore new tools for analyzing them.

We first approach the model as a system of partial differential equations that are very close to the classical MFG system. We then present a Eulerian variational formulation, as a problem of optimal control of Fokker-Planck equations. Finally, we recast the problem in a third equivalent formulation as an entropy minimization problem in the space of paths measures (Lagrangian approach). The latter is then used to construct an algorithm for practical simulation by performing alternate maximization on the dual problem, which corresponds to Dykstra's algorithm, often referred to as Sinkhorn algorithm or IPFP in optimal transport litterature. It is noteworthy that, although we use variational formulations, the model might not be variational – and is likely not for many interesting cost functions. However, the variational formulations can still be obtained through a fixed point trick.

In applying this model to urban planning problems, we view the populations as e.g. inhabitants and firms, and the instantaneous optimal transport as instantaneous equilibrium on the labour market, while allowing both populations to move. More generally, this framework allows to tackle any situation where you combine an instantaneous matching equilibrium condition and dynamic optimal control – the model provides a Nash equilibrium over a succession of instantaneous equilibria.

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Preliminary note on the text :
This document presents quite an exhaustive review of the work done and the approaches explored during the course of the internship. It is presented it both logical and chronological order.

Contents

1 Introduction

1.1 Motivation and applications

The interaction of many strategic agents is most often constrained by the need to be in some sort of equilibrium. But the outcome of equilibrium and the situation of each agent within it depends on parameters, which can generally be apprehended as locally constant and uncontrolled. However, it might be that, when we switch to a dynamic setting, agents control certain parameters that allows them to change their position within the equilibrium – each agent's impact is marginal, but as all try to achieve a more favourable solution, the equilibrium itself is changed. We studied a modelling approach of such phenomenon, in which there is a constraint that an instantaneous equilibrium between two populations be fulfilled at each time, while allowing players to control certain parameters which dynamically alter the equilibria. Of course, this gives rise to a game structure, each player expecting the others to move in their interest and hence the equilibrium to be altered. Hence there is a second level of equilibrium : a dynamic equilibrium over the movements which give rise to the succession of instantaneous equilibria. We provide a mathematical framework to apprehend such a system, drawing mainly from the theories of mean field games (MFG) and optimal transport (OT).

Our initial motivation came from an urban planning problem, i.e studying the dynamics of cities. Throughout the text, we will continuously refer to this interpretation of the model in order to outline more intuitive insights on the mathematical setting. We believe, however, that the core of the model is much more general and may be adapted to many similarly set problems with two or more populations. In our reference case, the two populations are interpreted as inhabitants and firms, whose density we respectively denote by μ and ν. Initial densities μ_0 and ν_0 are given. Instantaneous equilibrium between the two populations can be understood as an equilibrium on the labour market with a commute cost : every inhabitant/worker is matched to a given firm, so as to minimize commute cost net of wage earned. The wage is set so as to guarantee market equilibrium. Formally, this is modelled via an optimal transport problem, solved at each time t, between given densities μ_t and ν_t, in which we can interpret the Kantorovitch potentials as wage and net cost of commute. The next step is then to allow firms and inhabitants to move : only the initial density is fixed. In this dynamic framework, the inhabitants (resp. firms) will want to minimize their commute cost (resp. wages paid), plus perhaps other criterion – e.g. there might be a congestion effect which entices all agents to avoid crowded areas, or negative externality which drive inhabitants away from firms. This gives rise to a mean-field game with two populations, every agent interacting with others through the densities only, assuming there is an infinity of agents with the same total mass in each population. The main mathematical originality resides in the fact that the Hamilton-Jacobi-Bellman equations of the mean field game are coupled via the potentials obtained from the instantaneous optimal transport (IOT) problem.

The core of the work presented here consisted in finding the right approach to this problem by rewriting it under different forms. We tried to provide a well defined theoretical framework and a suitable algorithmic approach to find numerical solutions. The results presented in the last section exhibit some numerical results corresponding to the framework above, where all agents are averse to congestion (too much individuals in a similar location generates disutility) and there is a negative externality from firms for

inhabitants (e.g. firms generate pollution). The model thus incorporates forces that drive agents both closer to one another (instantaneous commute) and further from one another (congestion and externalities), in a world where motion is costly and subject to random noise. With this setup, we notably observe patterns of centre/suburbs segregation, where one population gradually pushes the other on the outskirts of the city.

1.2 Overview of the model

We present three equivalent formulations of the model, which each have their own specific interest. We assume the game is played on time interval $[0, T]$ and spatial domain Ω. First, the classical MFG forward-backward system of PDEs, augmented by the IOT coupling writes as :

$$\begin{cases} \partial_t u + \sigma_h \Delta u - \frac{|\nabla u|^2}{2} = -f[\mu, \nu] - R_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t v + \sigma_f \Delta v - \frac{|\nabla v|^2}{2\theta} = -g[\mu, \nu] - W_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t \mu - \Delta \mu - \operatorname{div}(\mu \nabla u) = 0 \text{ on } (0, T] \times \Omega \\ \partial_t \nu - \Delta \nu - \operatorname{div}(\nu \frac{\nabla v}{\theta}) = 0 \text{ on } (0, T] \times \Omega \\ (R_t, W_t) = \arg\max_{R,W} \int_\Omega R d\mu_t + \int_\Omega W d\nu_t - \int_\Omega e^{R \oplus W} d\xi \\ u(T, \cdot) = 0 \ ; \ v(T, \cdot) = 0 \\ \mu(0, \cdot) = \mu_0 \ ; \ \nu(0, \cdot) = \nu_0 \end{cases} \tag{MFG}$$

The first two equations are the backward HJB for each population, describing the evolution of the value function of an individual given time and position, for a running cost that is determined by respectively R and W the potentials from IOT, and f and g interaction functionals. The parameter $\theta \geq 1$ in the second population's Hamiltonian captures a higher cost of motion relatively to the first population. The third and fourth equation are standard Kolmogorov-Fokker-Planck forward equations describing the respective evolution of the densities of each population. The fifth equation describes the coupling between the potentials : they achieve the maximum of the dual problem of the regularized OT problem, where ξ is the Gibbs kernel associated to the ground cost. We will come back on this point, but we have chosen to focus on entropic regularization of optimal transport as a way to introduce some noise in the instantaneous equilibrium. If we were to use a standard optimal transport problem, this equation would be replaced by the usual Kantorovitch dual problem. The last two lines merely specify boundary conditions.

This system can be rewritten as a control problem among solutions of certain Fokker-Planck equations, assuming f and g derive from potentials, i.e they are derivatives of certain functionals in the space of measures, in a suitable sense. Ideally, we might want them to derive from the *same* potential – we would then be in the classical setup in which the MFG is said to be *variational* or *potential*, and the problem can be recast in the framework of the calculus of variations. However, in many interesting cases, f and g do not derive from the same potential, hence we will use a fixed point trick to assume separability, i.e that f and g respectively derive from potentials F and G. We then end up with the following problem :

$$\inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \frac{|q|^2}{2\nu} + \int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T F(\mu(t,\cdot)) + G(\nu(t,\cdot))$$
$$\text{Among solutions of :} \tag{EVF}$$
$$\partial_t \mu - \Delta \mu + \text{div}(p) = 0 \text{ on } (0,T] \times \Omega,\ \mu(0,\cdot) = \mu_0$$
$$\partial_t \nu - \Delta \nu + \text{div}(q) = 0 \text{ on } (0,T] \times \Omega,\ \nu(0,\cdot) = \nu_0$$

Where we have denoted :

$$W_2^\epsilon(\mu,\nu)^2 := \inf_{\gamma \in \Pi(\mu,\nu)} \int_\Omega c(x,y) d\gamma(x,y) + \epsilon \int_\Omega d\gamma(log(d\gamma) - 1)$$

the ϵ-regularized Wassertein-2 distance for the ground cost $c(x,y) := |x-y|^2$ (with possibly $\epsilon = 0$ i.e the standard Wasserstein distance). We have used the standard notation $\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(\Omega^2) | \pi_1 \# \gamma = \mu,\ \pi_2 \# \gamma = \nu\}$ for the space of transport plans between μ and ν. This formulation is akin to the Benamou-Brenier [7] approach to optimal transport and the standard formula for potential mean-field games, with the addition of the Wasserstein distance term – which is the anti-derivative of the potentials that appear in the right-hand side of the HJB equations. We will refer to this formulation as the Eulerian Variational Formulation (EVF) throughout the text. It lends itself to numerical simulations via e.g. augmented Lagrangian methods (see for instance [8]), which are theoretically applicable, but in practice too computationally demanding to hope for efficient solving in this case.

Last, our problem can also be recast as an entropy minimization problem in the space of path measures :

$$\inf_{P,Q \in \mathcal{P}(C)} H(P|R_h) + \theta H(Q|R_f) + \int_0^T \epsilon \inf_{\gamma_t \in \Pi(P_t,Q_t)} H(\gamma_t|\chi) + \int_0^T F(P_t) + G(Q_t) \tag{LVF}$$

Where $C = C([0,T],\Omega)$, R_f and R_h are (reversible) Wiener measures, possibly with different variance, and we have denoted $H(p|q)$ the relative entropy of a probability measure $p \in \mathcal{P}(X)$ with respect to a positive measure $r \in \mathcal{M}(X)$, defined as :

$$H(p|r) = \begin{cases} \int_X log(dp/dr) dp \in (-\infty, \infty], & \text{if } p \ll r \\ \infty, & \text{otherwise} \end{cases}$$

We refer to [37] for more detail about the rigorous definition of the relative entropy ; notice that the relative entropy of P, Q on the one hand, and γ_t on the other are not defined on the same spaces, but we use the same notation for convenience. This formulation appeared as the most convenient for numerical simulations, since we can adapt the well-known Sinkhorn algorithm in optimal transport (see e.g. [6], [43], [42], [21]). This actually amounts, in our case, to alternate maximization on the dual problem, i.e Dykstra's algorithm [22]. This approach is very close to the use of Dykstra's algorithm for Bregman divergences developped in [3] and [43].

2 Background and previous work

We introduce very briefly the theoretical background and the previous litterature that this work draws on, along with several notations that we will use throughout the text.

2.1 Mean field games

Mean field games (MFG), introduced in the seminal work of Lasry and Lions [34] [35] [36], provide a powerful framework for understanding strategic interactions between a very large number of agents. All agents are interchangeable from the perspective of each agent, hence interaction is mediated via the distribution (mean field interaction), on which each agent is assumed to have a negligible impact. The solution concept adopted is the classical Nash equilibrium with rational expectations, where every agent best responds to the expected realization of strategies, i.e anticipations are correct. This notion can be further justified, for example by a fictitious play approach – see e.g. Cardaliaguet and Hadikhanloo [13].

A mean field game can be expressed as a forward-backward system of PDEs, comprised a of backward Hamilton-Jacobi-Bellman equation describing the value function of a particular optimizing agent given the position of others and a forward Kolmogorov-Fokker-Planck equation describing the actual motion of the distribution under the optimizing behaviour of agents. The canonical MFG system with quadratic Hamiltonian writes as :

$$\begin{cases} \partial_t u + \Delta u + \frac{1}{2}|\nabla u|^2 = f[\mu_t] \text{ on } [0,T]\times\Omega \\ \partial_t \mu - \Delta \mu + \operatorname{div}(\mu \nabla u) = 0 \text{ on } [0,T]\times\Omega \\ \mu(0,\cdot) = \mu_0, \quad u(T,\cdot) = g[\mu_T] \end{cases} \tag{2.1}$$

We can also write a more general version with non specified cost of motion and induced hamiltonian H for the individual optimal control problem and where we still assume that the diffusion part is uncontrolled (with constant diffusion parameter σ) :

$$\begin{cases} \partial_t u + \sigma\Delta u + H(x, \nabla u(t,x)) = f[\mu_t] \text{ on } [0,T]\times\Omega \\ \partial_t \mu - \sigma\Delta \mu + \operatorname{div}(\mu D_p H(x, \nabla u(t,x))) = 0 \text{ on } [0,T]\times\Omega \\ \mu(0,\cdot) = \mu_0, \quad u(T,\cdot) = g[\mu_T] \end{cases} \tag{2.2}$$

We do not give any more details about the derivation of the mean field game system from individual optimal control problems since we will illustrate it later on in the case of our problem, and give details then about the microfoundations of the approach.

An important special case is variational or potential mean field games, when the system can be recast as a problem of optimal control of a Fokker-Planck equation. For this, we need f and g to derive from potentials, i.e to be the derivatives in some suitable sense of functionnals in the space of probability measures. More specifically, we say that $F : \mathcal{P}_2(\Omega) \to \mathbb{R}$ is a potential of $f : \mathcal{P}_2(\Omega)\times\Omega \to \mathbb{R}$ if :

$$\forall m, m' \in \mathcal{P}_2,\ F(m) - F(m') = \int_0^1 \int_\Omega f(tm + (1-t)m', x) d(m-m')(x) dt \tag{2.3}$$

With this definition, if F is a potential for f and G a potential for g, then the system

(2.2) is equivalent to the problem :

$$\inf_{\substack{\mu,p \\ \partial_t\mu-\sigma\Delta\mu+\mathrm{div}(\mu p)=0 \\ \mu(0,\cdot)=\mu_0}} \int_0^T \int_\Omega H^*(x,p)\mu(t,x)dxdt + \int_0^T F(\mu(t,\cdot))dt + G(\mu_T) \tag{2.4}$$

where H^* denotes the Legendre conjugate of the Hamiltonian. It is possible to obtain another variational form in Lagrangian fashion – we will come back to this point in details. References on this Lagrangian approach can be found in [10] or [9].

For a general introduction to mean field game and applications, one might refer to [31] [30]. For an extensive overview of the topic and a probabilistic approach, one can refer to the two recent monographs by Carmona and Delarue [17] [18]. On variational mean field games, a useful reference is [10] ; on numerical simulations, several approaches have been used, see for example [1] for a finite difference approach, [8] for an augmented lagrangian approach, and [9] for an entropy minimization approach.

2.2 Optimal transport and Wasserstein spaces

The optimal transport problem originated in the work of Gaspard Monge in the 18th century and was subsequently revived by Leonid Kantorovitch in the middle of the 20th century. The problem is the following : given a density μ in a space X – say a sandpile – and a density ν in a space Y – say a hole of same volume –, how to fill the hole with the sand (transport the sand) at the lowest possible cost, where we take as given the cost $c(x,y)$ of moving a grain of sand from location x to y ? Mathematically, the problem writes as :

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{X\times Y} c(x,y)d\gamma(x,y) \tag{2.5}$$

γ represents the *transport plan*, i.e. $\gamma(x,y)$ gives the quantity of mass that is sent (transported) from location x to y. The marginal constraint ensures that the whole density μ is transported to the whole density ν. $\Pi(\mu,\nu)$ denotes the space of transport plans with marginal μ and ν, i.e :

$$\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(X\times Y) \mid \pi_X\#\gamma = \mu,\ \pi_Y\#\gamma = \nu\} \tag{2.6}$$

with π the projection operator, such that $\pi_X\#\gamma$ is the first marginal of γ with $\#$ denoting the usual push forward. That is to say we must have :

$$\int_Y \gamma(\cdot,dy) = \mu(\cdot) \text{ and } \int_X \gamma(dx,\cdot) = \nu(\cdot) \tag{2.7}$$

The previous infimum is uniquely attained under suitable assumptions on the ground cost – in the present text, we will use mainly the quadratic cost $c(x,y) = |x-y|^2$, an important and well-defined particular case first solved by Brenier [12]. The dual of the optimal transport problem, often referred to as Kantorovitch dual problem is particularly important, and will provide a very natural interpretation for our problem later on. It

writes as :

$$\sup_{\substack{\phi,\psi \\ \phi\oplus\psi\leq c}} \int_\Omega \phi d\mu + \int_\Omega \psi d\nu \tag{2.8}$$

Where the dual variables $\phi, \psi : \Omega \to \mathbb{R}$ are called Kantorovitch potentials, and the notation $\phi \oplus \psi$ denotes the function defined on Ω^2 such that $\phi \oplus \psi(x, y) = \phi(x) + \psi(y)$. Such a notation is very handy when dealing with Kantorovitch potentials, for direct sums or tensor products, and we will use it repeatedly throughout the text. More generally for any $f_1, ..., f_n : \Omega \to \mathbb{R}$, we define the functions on Ω^n with values in $\mathbb{R}$:

$$\bigoplus_{i=1}^n f_i : (x_1, ..., x_n) \mapsto \sum_{i=1}^n f_i(x_i) \tag{2.9}$$

$$\bigotimes_{i=1}^n f_i : (x_1, ..., x_n) \mapsto \prod_{i=1}^n f_i(x_i) \tag{2.10}$$

The optimal transport problem defines a distance on the space of probability measures, called the Wasserstein distance, in the case of the quadratic cost :

$$W_2(\mu, \nu) = \inf_{\gamma\in\Pi(\mu,\nu)} \int_{X\times Y} c(x, y) d\gamma(x, y) \tag{2.11}$$

It has many uses – among them, one particularly connected to our problem is the study of gradient flows in the space of probability measures endowed with the Wasserstein distance. It was notably pioneered by Jordan, Kinderlehrer and Otto [33] who showed that the Fokker-Planck equation is the gradient flow of the entropy according to the W_2 distance, and was subsequently applied to e.g. crowd dynamics [41]. A reference on the subject is the book by Ambrosio, Gigli and Savaré [2].

One important property of the Wasserstein distance is that its derivatives (in the sense defined in the preceding paragraph) with respect to μ and ν are the Kantorovitch potentials of the dual problem ϕ and ψ respectively.

The optimal transport problem can also be reformulated as an optimal control of a continuity equation, an approach first proposed by Benamou and Brenier [7]. It amounts to minimizing kinetic energy among continuous curves in the space of probability measures, with prescribed initial an final densities :

$$\inf_{\substack{\rho,v \\ \partial_t\rho_t+\mathrm{div}(v_t\rho_t)=0 \\ \rho_0=\mu,\ \rho_1=\nu}} \int_0^1 \int_\Omega \frac{1}{2}|v_t|^2 d\rho_t \tag{2.12}$$

An important remark is that by passing to the dual problem, we can obtain the primal dual conditions as a forward-backward system of PDEs which is a particular case of mean field game where the running cost is null, the diffusion parameter is zero, and the terminal cost is a hard constraint on the final distribution :

$$\begin{cases} \partial_t u - \frac{1}{2}|\nabla u|^2 = 0 \\ \partial_t \rho_t - \mathrm{div}(\nabla u \rho_t) = 0 \\ \rho_0 = \mu,\ \rho_1 = \nu \end{cases} \tag{2.13}$$

There are deep links between optimal transport, mean field games, gradient flows and more generally analysis in the Wasserstein space. We will notably come back on this point when we exploit the similar nature of mean field game and optimal transport to rewrite both problems as entropy minimization problem, and use this similarity to design an algorithm.

For a broader introduction to optimal transport, or for more details, see for instance the book of Santambrogio [45] or the two books by Villani [46] [47].

2.3 Entropy minimization : Schrödinger problem and regularized optimal transport

Entropy minimization is an essential part of the work presented here, and a crucial tool most notably for building an algorithm. We present the formalism and notations that we will use throughout the text, beginning with the Schrödinger problem. This presentation draws mostly on the survey by Christian Leonard [39], to which one might refer for a more exhaustive review.

The classical Schrödinger problem consists of finding a path measure – i.e a measure on a space of curves – so as to minimize the relative entropy of this path measure with respect to the Wiener measure, and prescribing the first and last marginal. Intuitively, the Schrödinger problem amounts to finding the most likely motion of a density of randomly moving particles during a given time interval and knowing the initial and final densities – the reference case being set by the simple diffusion of particles. Let us introduce some formalism : we denote $\mathcal{P}(Y)$ and $\mathcal{M}_+(Y)$ respectively the space of probability and positive measures on Y. We define the relative entropy of p a given probability measure $p \in \mathcal{P}(Y)$ with respect to r a σ-finite positive measure $r \in \mathcal{M}_+(Y)$ as :

$$H(p|r) := \begin{cases} \int_Y (log(dp/dr) - 1)dp & \text{if } p \ll r \\ \infty & \text{otherwise} \end{cases} \tag{2.14}$$

Where dp/dr denotes the Radon-Nikodym derivative of p with respect to r. Rigorously defining the relative entropy with respect to general unbounded measures generates some technical difficulties which are far beyond the scope of this work. One might refer to [38] [39] [37] for more details and the probabilistic and measure-theoretic background. With a convenient abuse of notation, we will use the same notation for the relative entropy whatever the space Y – and sometimes involving entropy in different spaces in the same expression. We will also denote $E(p)$ the relative entropy of p with respect to the Lebesgue measure. We continue to use the notation Ω for our (real) domain[1], and introduce the space of continuous curves on Ω : $C := C([0,1], \Omega)$. We denote as $R \in \mathcal{M}_+(\Omega)$ the reversible Wiener measure, i.e the law of the Brownian motion with the Lebesgue measure as initial distribution. The *dynamic Schrödinger problem* writes as :

$$\inf_{\substack{P \in \mathcal{P}(C) \\ P_0 = \mu_0,\, P_1 = \mu_1}} H(P|R) \tag{2.15}$$

[1]We state once again that, although we mostly restrain to real domains for the purposes of this document, most results presented hold in much more general settings.

Where μ_0 and μ_1 are given, and we define $\forall t \in [0,1], P_t := e_t \# P$, with $e_t : C \to \Omega$ assigning to a path in C its position at time t – that is to say, P_t gives the distribution of particles at time t. $H(\cdot|R)$ is strictly convex, hence this is a convex minimization problem with a constraint convex set and a solution if it exists is necessarily unique. Next we define the *static Schrödinger problem* :

$$\inf_{\substack{\gamma \in \mathcal{P}(\Omega^2) \\ \gamma_0 = \mu_0,\, \gamma_1 = \mu_1}} H(\pi|R_{01}) \tag{2.16}$$

Where γ_0 and γ_1 are respectively the first and second marginal of the transport plan γ and $R_{01}(dxdy) := R((X_0, X_1) \in dxdy) \propto \exp(-d(x,y)^2/2) vol(dx) vol(dy)$ is the joint law of the initial and final position of the reversible Brownian motion. An important result is that those two problems are equivalent and that we can recover the solution of one from the other. Denote P^* the solution to the dynamic problem and π^* the solution to the static problem. Then P^* disintegrates as :

$$P^*(\cdot) = \int_{\Omega^2} R^{xy} \gamma^*(dxdy) \tag{2.17}$$

and equivalently γ^* can be obtained as :

$$\gamma^* = (e_0, e_1) \# P \tag{2.18}$$

Where $R^{xy} := R(\cdot|X_0 = x, X_1 = y)$ is the Brownian bridge from x to y. It is intuitively apparent that the Schrödinger problem has a very similar flavor to the Monge-Kantorovitch problem, and there indeed exists a relation between the two. First, note that we can express the Monge-Kantorovitch problem in dynamic setting, as an equivalent Lagrangian formulation of the Benamou-Brenier formula, as :

$$\inf_{\substack{Q \in \mathcal{P}(C) \\ e_0 \# Q = \mu_0,\, e_1 \# Q = \mu_1}} \int_C \int_0^1 |\dot{\omega}|^2 dt dQ(\omega) \tag{2.19}$$

There are two key results. First, the dynamic and static Monge-Kantorovitch problems are equivalent – see Léonard (2014) [37]. Second, the static and dynamic Schrödinger problems gamma-converge towards the static and dynamic Monge-Kantorovitch problems respectively as the variance of the reference Wiener measure converges towards zero.

Another way to see this is to start from the *entropic regularization of the optimal transport problem*, which is defined as :

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int c d\gamma - \epsilon \int \gamma(\log(\gamma) - 1) \tag{2.20}$$

This problem conveniently rewrites as :

$$\inf_{\gamma \in \Pi(\mu,\nu)} \epsilon H(\gamma|\xi) \tag{2.21}$$

Where ξ is the Gibbs kernel associated to the ground cost c, i.e $\xi(x,y) := e^{-c(x,y)/\epsilon}$. This formulation is much more general, but in the case of the quadratic cost it is exactly

the static Schrödinger problem, where the variance of the Brownian motion is ϵ. The solution of the regularized problem was shown to converge to the standard solution of the Monge-Kantorovitch problem – see e.g. [21] [6] [42] [20] – and was used to construct a very efficient algorithmic approach, known as Sinkhorn algorithm or Iterative proportional fitting procedure. Indeed, in that case the dual problem writes as :

$$\sup_{\phi,\psi} \int \phi d\mu + \int \psi d\nu - \int e^{\phi \oplus \psi} d\xi \tag{2.22}$$

and the optimal γ is very simply expressed in terms of the dual variables as a scaling of the Gibbs kernel :

$$\gamma = e^{\phi} \xi e^{\psi} \tag{2.23}$$

Since the constraint space is affine, one can then very simply and efficiently obtain the potentials via alternate projections, i.e repeated diagonal scalings of the kernel. Define $a = e^{\phi}$, $b = e^{\psi}$, then the potentials are uniquely determined up to a multiplicative constant by :

$$a = \frac{\mu}{b * \xi}, \; b = \frac{\nu}{a * \xi} \tag{2.24}$$

This approach has been very fruitfully used for numerous applications in, most notably, optimal transport, gradient flows, Wasserstein barycenters, and more recently mean field games – see [20] [6] [43] [9].

2.4 Urban planning and population dynamics

Various strands of academic litterature have tried to tackle urban planning problems from very different perspectives. Given the angle adopted here, we are mostly interested in mathematical modeling approaches in urban planning – hence we do not pretend that our review of litterature on the subject is in any way extensive.

In economics, the first attempts at modeling geographical equilibrium are usually traced back to Hotelling's 1929 model of competition [32]. Another approach was pioneered by Beckmann's continuous transportation model in 1952 [4] (see also [5]). He used a model of minimal flow to study the optimal city structure, which later proved to be tightly connected to the optimal transportation problem (see e.g. the related chapter in the book by Santambrogio [45]). This approach gave rise to many subsequent studies of minimal flow problems in applied mathematics, including notably the modeling of congestion effects – see for instance [16] [15] [11]. Another strand of litterature in geographical and urban economics is represented by the "New economic geography", notably pioneered by the work of Fujita and Krugman, see for instance [27] [24] [26] [25] [28]. Other notable recent endeavours at understanding urban equilibria include the work of Lucas and Rossi-Hansberg [40] [44]. We must also mention an unpublished article by Carlier, Ekeland and Rochet [14], drawing on optimal transportation theory to model optimal location of polluting activites, which was among the initial motivations and a starting point of the present work. Applications of the theory of mean-field games to population dynamics can also be found in the work of Guéant, Lasry and Lions [31] [30].

3 A two population mean field game with optimal transport coupling

3.1 Optimal transport approximation of the labour market

The heuristic of our optimal transport approximation of the labour market is the following. Take two given densities μ and ν representing respectively the positions of inhabitants and firms in a real domain Ω. The commuting cost from position x to y is given by $c(x, y)$, which we will most often assume to be the square euclidean distance. Each inhabitant must decide where to work, so they optimize over commuting cost and wages which they take as given. Notice that the only factor of heterogeneity stems from position – agents are otherwise assumed to be identical. Hence two agents with identical position x will face an identical problem, that is :

$$R(x) = \inf_y c(x, y) - W(y) \tag{3.1}$$

Where we have denoted $W(y)$ wage at position y and $R(x)$ the corresponding (optimal) commute cost net of wage earned, and y is constrained to lie in the support of ν – however, since we will assume later on that position evolves according to a controlled brownian diffusion, mass will effectively always be spread over the whole domain. This gives rise to a transport plan :

$$T(x) = \arg\min_y c(x, y) - W(y) \tag{3.2}$$

We then assume that wages adjust so that equilibrium is reached on the labour market, i.e the mass of inhabitants that work at location x is equal to jobs available at y – for simplicity, we assume that each firm provide one job so as to have a consistent normalization. The equilibrium constraint thus reads :

$$T\#\mu = \nu \tag{3.3}$$

We have implicitly assumed that the optimum above is unique, and hence provides a one-to-one mapping – this might very well not be the case, but the framework is easily adaptable to allow for mass splitting. It is a well known result in optimal transportation theory that this problem is equivalent to the Monge-Kantorovitch optimal transport problem between μ and ν :

$$\inf_{T\#\mu=\nu} \int_\Omega c(x, T(x)) d\mu(x) \tag{3.4}$$

To allow for mass splitting we will rather use Kantorovitch formalism :

$$\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\Omega^2} c(x, y) d\gamma(x, y) \tag{3.5}$$

Then, R and W are simply the potentials in the dual problem :

$$\inf_{\substack{R,W\in C_b(\Omega)\\ R\oplus W\leq c}} \int_\Omega R d\mu + \int_\Omega W d\nu \tag{3.6}$$

Another way to see this is that market equilibrium is efficient in this case, and decentralized minimization is equivalent with minimizing the aggregate transportation cost. It is noteworthy that this is a purely *geographic* labour market equilibrium, which dispenses from any of the usual considerations of e.g. productivity. It is, of course, a very narrow understanding, but it is still informative to our problem ; it could readily be enriched to give a more realistic model, which was not the main focus of this work and is left as a task for future investigations.

For computational convenience as well as for modelling purposes, we will slightly modify the problem above and consider the *entropic regularization* of the OT problem, i.e :

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int c d\gamma + \epsilon \int \gamma(log(\gamma) - 1) \tag{3.7}$$

once again recovering potentials from its dual problem :

$$\inf_{R,W} \int R d\mu + \int W d\nu - \int e^{R \oplus W} e^{-c/\epsilon} \tag{3.8}$$

This amounts to adding noise to the OT coupling, which could be seen as random mistakes in the outcome of the labour market equilibrium – each agent might not exactly choose the optimal firm to work out but one nearby instead. For more detail on the relationship between optimal transport and the so-called Schrödinger problem, see e.g. [29] [19].

3.2 The dynamic optimization problems of inhabitants and firms

We now move from a static perspective, in which positions are given, to a dynamic one in which agents can move. Assume that the dynamics of a given inhabitant starting at position x at time 0 is given by the following controlled diffusion :

$$\begin{aligned} dX_t &= \alpha_t dt + \sqrt{2\sigma_h} dB_t^h \\ X_0 &= x \end{aligned} \tag{3.9}$$

Where B_t^h is a standard Brownian motion, and α_t is a control. We will assume throughout that the diffusion part is not controlled, and that each agents' respective brownian motions are independent. We define similar dynamics for firms, given initial position y :

$$\begin{aligned} dY_t &= \beta_t dt + \sqrt{2\sigma_f} dB_t^f \\ Y_0 &= y \end{aligned} \tag{3.10}$$

Where B_t^f is a standard Brownian motion and similar assumptions were made. Hence we have $\mu_t = \mathcal{L}(X_t)$ and $\nu_t = \mathcal{L}(Y_t)$. Next, we define the control problem for a given inhabitant and firm respectively, taking the evolution of densities as given :

$$\begin{aligned} &\inf_{\alpha} \int_0^T \frac{|\alpha(t, X_t)|^2}{2} + f[\mu_t, \nu_t](t, X_t) + R_t[\mu_t, \nu_t](X_t) dt \\ &\text{Subject to (3.9)} \end{aligned} \tag{3.11}$$

$$\inf_{\beta} \int_0^T \theta \frac{|\beta(t, Y_t)|^2}{2} + g[\mu_t, \nu_t](t, Y_t) + W_t[\mu_t, \nu_t](Y_t) dt \tag{3.12}$$
$$\text{Subject to (3.10)}$$

R and W are the potentials from the instantaneous optimal transport problem defined as above, and f and g are running cost functions. For instance, f and g could incorporate a congestion effect – agents are displeased when their area is too crowded – or externality effects – e.g. negative externality : firms pollute, hence inhabitants dislike being near firms ; positive externalities : when many firms are close they form a cluster which indirectly increases their utility[2]. Furthermore, we have assumed a quadratic cost of motion for inhabitants, and a just added a multiplicative factor θ for firms : the quadratic cost is a natural and convenient framework to work with, but the parameter θ can allow us to consider a relatively heavier cost of motion for firms ($\theta > 1$), which is quite intuitive.

3.3 The MFG system

From the individual optimal control problems (3.11) and (3.12), we look for a Nash equilibrium where the anticipated evolution of densities is self-fulfilling – by deriving the HJB equations for the control problems, we obtain the following forward-backward system of PDEs :

$$\begin{cases} \partial_t u + \sigma_h \Delta u - \frac{|\nabla u|^2}{2} = -f[\mu, \nu] - R_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t v + \sigma_f \Delta v - \frac{|\nabla v|^2}{2\theta} = -g[\mu, \nu] - W_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t \mu - \sigma_h \Delta \mu - \text{div}(\mu \nabla u) = 0 \text{ on } (0, T] \times \Omega \\ \partial_t \nu - \sigma_f \Delta \nu - \text{div}(\nu \frac{\nabla v}{\theta}) = 0 \text{ on } (0, T] \times \Omega \\ (R_t, W_t) = \arg\max_{R,W} \int_\Omega R d\mu_t + \int_\Omega W d\nu_t - \int_\Omega e^{R \oplus W} d\chi \\ u(T, \cdot) = 0 \ ; \ v(T, \cdot) = 0 \\ \mu(0, \cdot) = \mu_0 \ ; \ \nu(0, \cdot) = \nu_0 \end{cases} \tag{3.13}$$

Mean field games with several homogeneous populations are a natural extension of classical mean field games – see for instance chapter 7, section 7.1.1 of the monograph by Carmona and Delarue [17] – and poses no major technical difficulty.

3.4 Infinite horizon and discounting factor

It appears quite natural in our framework to consider the case $T = \infty$ since optimization over a finite horizon when modeling cities does not seem particularly relevant. We then need to add a discounting to the objective function (which we will assume to have a fixed rate) first because it ensures convergence, but also for the purpose of coherent modeling. As a matter of fact, discounting could also be found relevant in the finite horizon case, to take into accounts agents' time preferences. Further along the text, we will move back and forth between the versions with and without discounting. The case without discounting

[2]This last effect is, however, complicated to justify in our model where no consideration of productivity-related issues was taken into account, but it would be a natural extension of the model and could easily be accomodated.

will remain our reference case since it is slightly simpler to tackle when moving to the variational formulations.

Concretely, the addition of discounting at rate ρ_h for inhabitants ρ_f for firms means changing (3.11) and (3.12) respectively to :

$$\inf_{\alpha} \int_0^T e^{-\rho_h t} \left(\frac{|\alpha(t, X_t)|^2}{2} + f[\mu_t, \nu_t](t, X_t) + R_t[\mu_t, \nu_t](X_t) \right) dt \tag{3.14}$$
$$\text{Subject to (3.9)}$$

$$\inf_{\beta} \int_0^T e^{-\rho_f t} \left(\theta \frac{|\beta(t, Y_t)|^2}{2} + g[\mu_t, \nu_t](t, Y_t) + W_t[\mu_t, \nu_t](Y_t) \right) dt \tag{3.15}$$
$$\text{Subject to (3.10)}$$

We then obtain the following modified MFG system :

$$\begin{cases} \partial_t u + \sigma_h \Delta u - \frac{|\nabla u|^2}{2} - \rho u = -f[\mu, \nu] - R_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t v + \sigma_f \Delta v - \frac{|\nabla v|^2}{2\theta} - \rho v = -g[\mu, \nu] - W_t[\mu, \nu] \text{ on } [0, T) \times \Omega \\ \partial_t \mu - \sigma_h \Delta \mu - \operatorname{div}(\mu \nabla u) = 0 \text{ on } (0, T] \times \Omega \\ \partial_t \nu - \sigma_f \Delta \nu - \operatorname{div}(\nu \frac{\nabla v}{\theta}) = 0 \text{ on } (0, T] \times \Omega \\ (R_t, W_t) = \arg\max_{R,W} \int_\Omega R d\mu_t + \int_\Omega W d\nu_t - \int_\Omega e^{R \oplus W} d\chi \\ u(T, \cdot) = 0 \ ; \ v(T, \cdot) = 0 \\ \mu(0, \cdot) = \mu_0 \ ; \ \nu(0, \cdot) = \nu_0 \end{cases} \tag{3.16}$$

Where a zero order term was added in both HJB equations. Note that instead of proving existence for the MFG system, it seems an easier strategy to first prove the equivalence to the variational formulation, and then prove existence in a classical fashion using the direct method in the calculus of variations.

4 A Eulerian variational formulation and a first algorithmic approach

4.1 Eulerian variational formulation

4.1.1 The potential case

First, we assume that f and g derive from the same potential, i.e there exists $\mathcal{F} : \mathcal{P}(\Omega)^2 \to \mathbb{R}$ such that :

$$\begin{aligned} \frac{\delta \mathcal{F}}{\delta \mu}(\mu, \nu) &= f[\mu, \nu] \\ \frac{\delta \mathcal{F}}{\delta \nu}(\mu, \nu) &= g[\mu, \nu] \end{aligned} \tag{4.1}$$

Where we used the previously introduced notation for derivatives in the space of probability measures. In that case our problem can be recast as an optimization problem among solutions of certain Fokker-Planck equations.

$$\begin{aligned}\inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \frac{|q|^2}{2\nu} + \int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T \mathcal{F}(\mu(t,\cdot),\nu(t,\cdot)) \\ \text{Among solutions of :} \\ \partial_t \mu - \sigma_h \Delta \mu + \operatorname{div}(p) = 0 \text{ on } (0,T]\times\Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu - \sigma_f \Delta \nu + \operatorname{div}(q) = 0 \text{ on } (0,T]\times\Omega,\ \nu(0,\cdot) = \nu_0\end{aligned} \tag{4.2}$$

A direct variational approach can be followed to easily yield the equivalence with the MFG system of PDEs, which is standard in mean-field games litterature. Another and perhaps computationally simpler proof consists in showing that the MFG system gives the primal-dual conditions of the problem above. Indeed, if we write the constraints in weak form and rewrite (4.2) as a saddle-point problem (ignoring boundary conditions terms for greater clarity) :

$$\begin{aligned}\inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \frac{|q|^2}{2\nu} + \int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T \mathcal{F}(\mu(t,\cdot),\nu(t,\cdot)) \\ + \sup_{u,v} \int_0^T \int_\Omega \left(\partial_t u + \sigma_h \Delta u\right)\mu + \nabla u \cdot p + \int_0^T \int_\Omega \left(\partial_t v + \sigma_f \Delta v\right)\nu + \nabla v \cdot q\end{aligned} \tag{4.3}$$

Interchanging sup and inf and taking the first order condition in p and q yields :

$$p = -\mu \nabla u \tag{4.4}$$

$$q = -\nu \frac{\nabla v}{\theta} \tag{4.5}$$

Hence we can rewrite the problem as :

$$\begin{aligned}\sup_{u,v} \inf_{\mu,\nu} \int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T \mathcal{F}(\mu(t,\cdot),\nu(t,\cdot)) \\ + \int_0^T \int_\Omega \left(\partial_t u + \sigma_h \Delta u - \frac{|\nabla u|^2}{2}\right)\mu + \int_0^T \int_\Omega \left(\partial_t v + \sigma_f \Delta v - \frac{|\nabla v|^2}{2\theta}\right)\nu\end{aligned} \tag{4.6}$$

Then, first-order conditions in μ and ν directly yield the two HJB equations in the MFG system, which completes the proof.

This formalism also lends itself quite easily to a formal proof of existence using the direct method in the calculus of variations. An easy example of a truly variational case would be to only consider a local congestion effect. For instance, the cost could be of the form log of the sum of density, i.e

$$\begin{aligned}\mathcal{F}(\mu,\nu) = \int_\Omega (\mu+\nu)\left(\log(\mu+\nu) - 1\right) \\ f(\mu,\nu) = g(\mu,\nu) = \log(\mu+\nu)\end{aligned} \tag{4.7}$$

4.1.2 Non-potential case and fixed point workaround

In most interesting cases, however, f and g cannot derive from a same potential – any kind of asymmetrical interactions, for example externalities, where population A has an effect on population B which has no counterpart in a certain reciprocal effect, cannot be tackled in the purely potential case. A possible workaround is to consider a case where the potential $\mathcal{F}$ is *separable*, albeit at the cost of a fixed point argument. In that case we only need f and g to each derive from a potential with respect to one of their variables, which is a much less constraining assumption.

To put it more concretely, let us fix two densities $\tilde{\mu}$ and $\tilde{\nu}$, and assume there exists F and G two functionnals over $\mathcal{P}(\Omega)$ such that :

$$\begin{aligned} \frac{\delta F}{\delta \mu}(\mu|\tilde{\nu}) &= f[\mu, \tilde{\nu}] \\ \frac{\delta G}{\delta \nu}(\nu|\tilde{\mu}) &= g[\tilde{\mu}, \nu] \end{aligned} \tag{4.8}$$

To fix ideas, $\tilde{\mu}$ and $\tilde{\nu}$ could be thought of as fixed anticipation by each population of the other population's behaviour – hence the notation we used to denote the dependence of F and G on $\tilde{\mu}$ and $\tilde{\nu}$ respectively. For simplicity and readability, we will sometimes denote $F(\mu)$ and $G(\nu)$ in the sequel. We then consider the following optimization problem :

$$\begin{aligned} \inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \frac{|q|^2}{2\nu} + \int_0^T W_2^\epsilon(\mu(t,\cdot), \nu(t,\cdot))^2 + \int_0^T F(\mu(t,\cdot)) + G(\nu(t,\cdot)) \\ \text{Among solutions of :} \\ \partial_t \mu - \sigma_h \Delta \mu + \operatorname{div}(p) = 0 \text{ on } (0,T] \times \Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu - \sigma_f \Delta \nu + \operatorname{div}(q) = 0 \text{ on } (0,T] \times \Omega,\ \nu(0,\cdot) = \nu_0 \end{aligned} \tag{4.9}$$

The solution to this problem in μ, ν, which we call $(\bar{\mu}, \bar{\nu})$, gives us a mapping $\Phi : (\tilde{\mu}, \tilde{\nu}) \mapsto (\bar{\mu}, \bar{\nu})$, which under suitable assumptions on F and G admits a fixed point. Such a fixed point is, then, a solution to our original mean-field game system. This could be seen directly by showing that the separability assumption amounts to replacing f and g in the right-hand side of both HJB equations by $f[\mu, \tilde{\nu}]$ and $g[\tilde{\mu}, \nu]$, then applying a fixed point argument there – typically Schauder fixed point, using the regularity of solutions to the HJB equations.

4.1.3 Infinite horizon and discounting

To accomodate the case with discounting in the variational setting, it is useful to start by applying a change of variable in the HJB equations :

$$\begin{aligned} \tilde{u} &= e^{-\rho t} u \\ \tilde{v} &= e^{-\rho t} v \end{aligned} \tag{4.10}$$

Which yield the updated MFG system (in which, for readability, we omit the OT constraint and boundary conditions) :

$$\begin{cases} \partial_t \tilde{u} + \sigma_h \Delta \tilde{u} - e^{\rho t} \frac{|\nabla \tilde{u}|^2}{2} = -e^{-\rho t} f - e^{-\rho t} R \\ \partial_t \tilde{v} + \sigma_f \Delta \tilde{v} - e^{\rho t} \frac{|\nabla \tilde{v}|^2}{2\theta} = -e^{-\rho t} g - e^{-\rho t} W \\ \partial_t \mu - \sigma_h \Delta \mu - e^{\rho t} \operatorname{div}(\mu \nabla \tilde{u}) = 0 \text{ on } (0,T] \times \Omega \\ \partial_t \nu - \sigma_f \Delta \nu - e^{\rho t} \operatorname{div}(\nu \frac{\nabla \tilde{v}}{\theta}) = 0 \text{ on } (0,T] \times \Omega \end{cases} \tag{4.11}$$

Thus by using the same method and taking the appropriate adjoint of the modified Hamiltonian, we obtain the following variational system :

$$\begin{gathered} \inf_{p,q,\mu,\nu} \int_0^T \int_\Omega e^{-\rho t} \frac{|p|^2}{2\mu} + e^{-\rho t} \theta \frac{|q|^2}{2\nu} + \int_0^T e^{-\rho t} W_2^\epsilon(\mu(t,\cdot), \nu(t,\cdot))^2 + \int_0^T e^{-\rho t} \mathcal{F}(\mu(t,\cdot), \nu(t,\cdot)) \\ \text{Among solutions of :} \\ \partial_t \mu - \sigma_h \Delta \mu + \operatorname{div}(p) = 0 \text{ on } (0,T] \times \Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu - \sigma_f \Delta \nu + \operatorname{div}(q) = 0 \text{ on } (0,T] \times \Omega,\ \nu(0,\cdot) = \nu_0 \end{gathered} \tag{4.12}$$

4.2 Augmented Lagrangian algorithm

4.2.1 Dual problem and setup

Our first idea for a numerical approach to finding solutions of this problem was to use an augmented Lagrangian approach, and apply an ADMM-type algorithm in the spirit of previous work by Benamou and Carlier [8], who already applied such methods to mean-field games and gradient flows type problems. For simplicity of exposition, and because this method is better equipped to treat such a case, we are going to consider here only the special case where :

- Motion is deterministic, i.e there is no diffusion ($\sigma_h = \sigma_f = 0$) ;
- the instantaneous optimal transport problem is *not* regularized, i.e $\epsilon = 0$
- there is no running cost appart from commuting costs, i.e $f = g = 0$.
- $\theta = 1$

That is to say, we consider the problem :

$$\begin{gathered} \inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \frac{|q|^2}{2\nu} + \int_0^T W_2(\mu(t,\cdot), \nu(t,\cdot))^2 \\ \text{Among solutions of :} \\ \partial_t \mu + \operatorname{div}(p) = 0 \text{ on } (0,T] \times \Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu + \operatorname{div}(q) = 0 \text{ on } (0,T] \times \Omega,\ \nu(0,\cdot) = \nu_0 \end{gathered} \tag{4.13}$$

There are several reasons why we make such drastic simplifications. First, the augmented lagrangian method is very ill-equipped to deal with a constraint involving a Laplacian, since the augmentation implies that we need to solve, at each iteration of

the algorithm a PDE involving a bi-Laplacian operator (even two in our case). As a consequence, it is better suited to deal with the fully deterministic case. Second, writing out the method is quite notationally heavy, hence for the sake of clarity it is worth sparing a few difficulties, all the more since the generalization with more parameters and non-zero running cost is pretty straightforward. Our intent is then to give, first and foremost, the core idea of the algorithm. Lastly, going through all the computational details in our exposition seemed particulary unnecessary since our final conclusion was that in our case this method, although theoretically valid, demands unreasonable computing power. This assessment of unfeasability prompted us to turn to the third formulation of the model and other algorithmic approaches. Before going into it, we still present the augmented lagrangian approach for the sake of completeness and for potential future uses.

First, we pass to the dual problems by writing (4.13) as a saddle point problem, using the weak form of the continuity equations :

$$\begin{aligned}\inf_{p,q,\mu,\nu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \frac{|q|^2}{2\nu} + \int_0^T \sup_{\substack{R,W \\ R\oplus W\leq c}} \int_\Omega R d\mu_t + \int_\Omega W d\nu_t \\ + \sup_{u,v} \int_0^T \int_\Omega (\partial_t u \mu + \nabla u \cdot p) + \int_0^T \int_\Omega (\partial_t v \nu + \nabla v \cdot q) + \int_\Omega u_0\mu_0 + \int_\Omega v_0\nu_0\end{aligned} \tag{4.14}$$

Where we have also written the Wasserstein distance explicitly. We then formally interchange sup and inf and take the first order conditions in p and q to obtain the following problem :

$$\begin{aligned}\sup_{\substack{u,v \\ R,W \\ u(T,.)=v(T,.)=0 \\ R\oplus W\leq c}} \int_\Omega u_0\mu_0 + \int_\Omega v_0\nu_0 \\ + \inf_{\mu,\nu} \int_0^T \int_\Omega \left(\partial_t u - \frac{|\nabla u|^2}{2} + R\right)\mu + \int_0^T \int_\Omega \left(\partial_t v - \frac{|\nabla v|^2}{2} + W\right)\nu\end{aligned} \tag{4.15}$$

We obtain the dual problem, constrained by the HJB equations i.e the adjoint of the initial continuity equations :

$$\begin{aligned}\sup_{\substack{u,v \\ R,W \\ R\oplus W\leq c \\ \partial_t u - \frac{|\nabla u|^2}{2} + R = 0 \\ \partial_t v - \frac{|\nabla v|^2}{2} + W = 0 \\ u(T,.)=v(T,.)=0}} \int_\Omega u_0\mu_0 + \int_\Omega v_0\nu_0\end{aligned} \tag{4.16}$$

Which we rewrite in a more compact form as :

$$\inf_{X\in E} F(X) + G(\Lambda X) \tag{4.17}$$

Where the set E is e.g. $\{(u,v,R,W) \in H^1([0,T]\times\Omega,\mathbb{R})^4 \mid u(T,\cdot) = v(T,\cdot) = 0\}$ and

we have introduced the notation for the variable :

$$X = (u, v, R, W) \tag{4.18}$$

as well as the operator Λ such that :

$$\Lambda : (u, v, R, W) \mapsto (\partial_t u, \nabla u, \partial_t v, \nabla v, R, W) \tag{4.19}$$

Note that this operator is well chosen since it is injective, which is a crucial condition for applying ADMM. Lastly, we have set the functions F and G such that :

$$\begin{aligned} F : (u, v, R, W) &\mapsto \int u_0 \mu_0 + \int v_0 \nu_0 + \theta(R, W) \\ G : (a, \alpha, b, \beta, \phi, \psi) &\mapsto H(a, \alpha, \phi) + H(b, \beta, \psi) \end{aligned} \tag{4.20}$$

where θ and H are just indicator functions of the constraints :

$$\theta(R, W) = \chi_{\{R \oplus W \leq c\}} := \begin{cases} 0 & \text{if } R \oplus W \leq c \\ \infty & \text{otherwise} \end{cases} \tag{4.21}$$

and

$$H(a, \alpha, \phi) = \chi_{\{a - \frac{\alpha^2}{2} + \phi \leq 0\}} := \begin{cases} 0 & \text{if } a - \frac{\alpha^2}{2} + \phi \leq 0 \\ \infty & \text{otherwise} \end{cases} \tag{4.22}$$

We can now rewrite the problem as a constrained problem :

$$\inf_{\substack{X,Y \\ \Lambda X = Y}} F(X) + G(Y) \tag{4.23}$$

This problem's augmented Lagrangian classically writes as :

$$\mathcal{L}^A(X, Y, \sigma) = F(X) + G(Y) + < \sigma, \Lambda X - Y > + \frac{\tau}{2} ||\Lambda X - Y||^2 \tag{4.24}$$

Where τ is a parameter. If we go back to a detailled form, this gives :

$$
\begin{aligned}
&\mathcal{L}^A(u,v,R,W\ ;\ a,\alpha,b,\beta,\phi,\psi\ ;\ \sigma_a,\sigma_\alpha,\sigma_b,\sigma_\beta,\sigma_\phi,\sigma_\psi)\\
&\quad=\int u_0\mu_0+\int v_0\nu_0+\theta(R,W)\\
&\qquad+H(a,\alpha,\phi)+H(b,\beta,\psi)\\
&\qquad+<\sigma_a,\partial_t u-a>+\frac{\tau}{2}||\partial_u-a||^2\\
&\qquad+<\sigma_\alpha,\nabla u-\alpha>+\frac{\tau}{2}||\nabla u-\alpha||^2\\
&\qquad+<\sigma_\phi,R-\phi>+\frac{\tau}{2}||R-\phi||^2\\
&\qquad+<\sigma_b,\partial_t v-b>+\frac{\tau}{2}||\partial_v-b||^2\\
&\qquad+<\sigma_\beta,\nabla v-\beta>+\frac{\tau}{2}||\nabla v-\beta||^2\\
&\qquad+<\sigma_\psi,W-\psi>+\frac{\tau}{2}||W-\psi||^2
\end{aligned}
\tag{4.25}
$$

4.2.2 Algorithm

The idea behing the alternate direction methods of multipliers is to optimize alternatively in variables X and Y, then update the multipliers accordingly. More details about the algorithm and its convergence can be found e.g. in [23] and [8], see also `http://stanford.edu/~boyd/admm.html`. In compact form, it can be written as :

Algorithm 1 Augmented Lagrangian Algorithm (compact form)

Input : initial pair of primal-dual points $X^{(0)}, Y^{(0)}$; Lagrange multiplier $\sigma^{(0)}$; step τ
for all $k \geq 0$ **do**

1. $X^{(k+1)} := \arg\min_X F(X) + <\sigma^{(k)}, \Lambda X> + \frac{\tau}{2}||\Lambda X - Y^{(k)}||^2$
2. $Y^{(k+1)} := \arg\min_Y G(Y) - <\sigma^{(k)}, Y> + \frac{\tau}{2}||\Lambda X^{(k+1)} - Y||^2$
3. $\sigma^{(k+1)} := \sigma^{(k)} + \tau\left(\Lambda X^{(k+1)} - Y^{(k+1)}\right)$

end for

It is worthwile to go once again into a more detailled presentation. Each these three steps can itself be split into independents substeps, which are each of different nature – and hence require a different approach and technical arsenal. Below, we break down the algorithm into several simpler problems at each steps – note that we incorporate functions θ and H back as constraints.

Steps 1.1. and 1.2 amount to solving two linear elliptic equations in space-time dimension – this can quite easily be done in e.g. Freefem++, as in [8]. Step 1.3. is a constrained linear quadratic problem, which require a totally different set of tools ; we encounter a first very practical problem posed by the algorithm as it might require different softwares or language to be coupled if we are to perform efficiently each of these steps. Steps 2.1. and 2.2. amount to projections on parabolas. Finally, steps 3.1. to 3.6. are the least complicated as they are mere updates on the multipliers.

Algorithm 2 Augmented Lagrangian Algorithm (detailled form)

Input : initial pair of primal-dual points $X^{(0)}, Y^{(0)}$; Lagrange multiplier $\sigma^{(0)}$; step τ
for all $k \geq 0$ **do**

1.1. $u^{(k+1)} := \arg\min_u \int u_0\mu_0 + < \sigma_a^{(k)}, \partial_t u > + \frac{\tau}{2}||\partial_t u - a^{(k)}||^2$
$+ < \sigma_\alpha^{(k)}, \nabla u > + \frac{\tau}{2}||\nabla u - \alpha^{(k)}||^2$

1.2. $v^{(k+1)} := \arg\min_v \int v_0\nu_0 + < \sigma_b^{(k)}, \partial_t v > + \frac{\tau}{2}||\partial_t v - b^{(k)}||^2$
$+ < \sigma_\beta^{(k)}, \nabla v > + \frac{\tau}{2}||\nabla v - \beta^{(k)}||^2$

1.3. $(R^{(k+1)}, W^{(k+1)}) := \arg\min_{\substack{R,W \\ R \oplus W \leq c}} < \sigma_\phi, R > + \frac{\tau}{2}||R - \phi^{(k)}||^2$
$+ < \sigma_\psi, w > + \frac{\tau}{2}||W - \psi^{(k)}||^2$

2.1. $(a^{(k+1)}, \alpha^{(k+1)}, \phi^{(k+1)}) := \arg\min_{\substack{a,\alpha,\phi \\ a - \alpha^2/2 + \phi \leq 0}} - < \sigma_a^{(k)}, a > + \frac{\tau}{2}||\partial_t u^{(k+1)} - a||^2$
$- < \sigma_\alpha^{(k)}, \alpha > + \frac{\tau}{2}||\nabla u^{(k+1)} - \alpha||^2$
$- < \sigma_\phi^{(k)}, \phi > + \frac{\tau}{2}||\nabla R^{(k+1)} - \phi||^2$

2.2. $(b^{(k+1)}, \beta^{(k+1)}, \psi^{(k+1)}) := \arg\min_{\substack{b,\beta,\psi \\ b - \beta^2/2 + \psi \leq 0}} - < \sigma_b^{(k)}, b > + \frac{\tau}{2}||\partial_t v^{(k+1)} - b||^2$
$- < \sigma_\beta^{(k)}, \beta > + \frac{\tau}{2}||\nabla v^{(k+1)} - \beta||^2$
$- < \sigma_\psi^{(k)}, \psi > + \frac{\tau}{2}||\nabla W^{(k+1)} - \psi||^2$

3.1. $\sigma_a^{(k+1)} := \sigma_a^{(k)} + \tau\left(\partial_t u^{(k+1)} - a^{(k+1)}\right)$
...
3.6. $\sigma_\psi^{(k+1)} := \sigma_\psi^{(k)} + \tau\left(W^{(k+1)} - \psi^{(k+1)}\right)$

end for

Also note that the aforementionned problem of bi-Laplacian, which lead us to make the problem deterministic, is quite patent here. If we had kept diffusion in the model, our operator lambda would have had to give $\partial_t u - \Delta u$ and $\partial_t v - \Delta v$ instead of $\partial_t u$ and $\partial_t v$ (in order to incorporate constraints while preserving injectivity). But then, Laplacians would have appeared in the augmented part of the lagrangian, under the square norms, thus requiring us to solve *non-linear* elliptic equations of order two and adding a non-negligible amount of complexity in computing terms.

None of these steps is infeasible independantly – however, when combined, they are likely to require a lot of computing time. For instance, if we use a N points finite elements discretization of space-time, convergence rates should be near $\mathcal{O}(Nlog(N))$ for steps 1.1. and 1.2., $\mathcal{O}(N^2)$ or slightly better for step 1.3., and $\mathcal{O}(N)$ for steps 2.1. and 2.2. (step 3 taking negligible computing time). And remember that this is only *one* iteration of the algorithm and N is *space-time* dimension. If we are to hope for reasonable computing time, we would clearly be restricted to a very coarse discretization – not to mention the case with diffusion, or adding running costs : every added layer complexity just makes practical solvability a more unrealistic goal.

Hence, it appeared that the scope of this algorithmic approach was extremely narrow for our interests, and in the end quite unsatisfactory. This conclusion lead to revisit the whole modelling angle. Especially, it seemed diffusion was essential to modelling and interpretations of our model, so it should be better incorporated in the model. We endeavoured to recast the model in a more fitting framework, and came up with the Lagrangian variational formulation as a solution.

5 A Lagrangian variational formulation and a second algorithmic approach

The idea of applying an entropy minimization approach to mean-field games owes a lot previous work by Benamou, Carlier, Di Marino and Nenna [9], who exploited the equivalence between the static and dynamic formulations of the Schrödinger problem, already well-known in optimal transport, and extended the analysis to the case of mean-field games – the idea of a Eulerian formulation for variational mean-field games was also already present in [10] . This naturally lends itself to a Sinkhorn-type approach for numerical simulations, as developped in [6] or [20]. The generalization of this framework to our two-population framework and the double layer of entropy (instantaneous commute and dynamic transport) also owes a lot to the work of Gabriel Peyré, most notably [43].

5.1 The Lagrangian variational formulation

5.1.1 Entropy minimization approach

There are two key result that this reformulation builds upon :

– First, the formal equivalence between minimizing kinetic energy among solutions of Fokker-Planck equations and minimizing the relative entropy with respect to the disintegration of a reversible Wiener measure, for prescribed initial and final

densities – i.e the static version of Schrödinger's bridge problem. More precisely, if we set two densities μ and ν, and h a positive real we have :

$$\inf_{\substack{\rho,v \\ \partial_t\rho-\frac{1}{2}\Delta\rho+\mathrm{div}(\rho v) \\ \rho(0,\cdot)=\mu\ \rho(h,\cdot)=\nu}} \int_0^h \int_\Omega \frac{|v_t|^2}{2} d\rho_t(x)dt = \inf_{\gamma\in\Pi(\mu,\nu)} H(\gamma|R_{0,h}) - E(\mu) \tag{5.1}$$

where $R_{0,h}$ is the disintegration of the reversible Wiener measure R, i.e :

$$R = \int_{\mathbb{R}^d x \mathbb{R}^d} R(\cdot|x_0, x_h) dR_{0,h}(x_0, x_h) \tag{5.2}$$

and we remind that $E(\mu) = \int \mu(\log(\mu) - 1)$ denotes the entropy of μ with respect to the Lebesgue measure.

– Second is the equivalence between the static and dynamic versions of Schrödinger's problem, which states that optimizing over initial and final position couplings or optimizing over the whole paths yield the same result. More precisely, if we denote :

$$\bar{P} = \underset{\substack{P\in\mathcal{P}(C([0,1],\Omega)) \\ P_0=\mu,\ P_1=\nu}}{\arg\min} H(P|R) \tag{5.3}$$

$$\bar{\pi} = \underset{\substack{\pi\in\mathcal{P}(\Omega^2) \\ \pi_0=\mu,\ \pi_1=\nu}}{\arg\min} H(\pi|R_{0,1}) \tag{5.4}$$

the respective solutions of the dynamic and static problems, we have the following relations, assuming uniqueness[3] of the solutions :

$$\bar{\pi} = \bar{P}_{0,1} \tag{5.5}$$

Fundamental results used here can be found in greater details in e.g. [39], [37], [29], [19]. Note that this framework readily extends to the multi-marginal setup, if we consider couplings between more than two densities in the static setup. An essential result proven in [9] is that these relations extend naturally to the case of mean-field games, that is to say when adding a running cost term to the kinetic energy minimization problem and removing the final density constraint. They use a time-discretization and gamma-convergence to proove that (5.1) and (5.5) extend to :

$$\inf_{\substack{\rho,v \\ \partial_t\rho-\frac{1}{2}\Delta\rho+\mathrm{div}(\rho v) \\ \rho(0,\cdot)=\mu}} \int_0^T \int_\Omega \frac{|v_t|^2}{2} d\rho_t(x)dt + \int_0^T F(\rho_t) = \inf_{\substack{P\in\mathcal{P}(C([0,1],\Omega)) \\ P_0=\mu}} H(P|R) + \int_0^T F(P_t) - E(\mu) \tag{5.6}$$

[3]In the case where solutions are not unique, we have the weaker results that the bridge between initial and final time from the dynamic problem solves the static problem, and that conversely $\int_{\Omega^2} R^{xy}(\cdot)d\bar{\pi}(x,y)$ solves the dynamic problem if $\bar{\pi}$, where R^{xy} denotes the Brownian bridge between x and y, is a solution of the static problem.

Where F is a given running cost functional. As expected, the densities at each time ρ_t in the eulerian problem can be recovered equivalently by taking the marginal of the path measure P at time t, i.e $e_t \# P =: P_t$. Since in our problem, the main interest is finding the *evolution of densities*, the constant term $E(\mu)$ which corrects the value function between the eulerian and lagrangian formulations is of little interest and will from now on be ignored.

Once this result is given, the extension to our problem is pretty straightforward. Indeed, if we start from the first variational formulation we have :

$$\begin{aligned}
&\inf_{\substack{p,q,\mu,\nu \\ \partial_t\mu - \sigma_h\Delta\mu + \mathrm{div}(p)=0 \\ \partial_t\nu - \sigma_f\Delta\nu + \mathrm{div}(q)=0 \\ \mu(0,\cdot)=\mu_0,\, \nu(0,\cdot)=\nu_0}} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta\frac{|q|^2}{2\nu} + \int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T \mathcal{F}(\mu(t,\cdot),\nu(t,\cdot)) \\
&= \inf_{p,\mu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \inf_{q,\nu} \int_0^T \int_\Omega \frac{|q|^2}{2\nu} + \frac{1}{\theta}\int_0^T W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \frac{1}{\theta}\int_0^T \mathcal{F}(\mu(t,\cdot),\nu(t,\cdot)) \\
&= \inf_{p,\mu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \theta \inf_{Q\in\mathcal{P}(C([0,T],\Omega))} H(Q|R^f) + \frac{1}{\theta}\int_0^T W_2^\epsilon(\mu(t,\cdot),Q_t)^2 + \frac{1}{\theta}\int_0^T \mathcal{F}(\mu(t,\cdot),Q_t) \\
&= \inf_Q \theta H(Q|R^f) + \inf_{p,\mu} \int_0^T \int_\Omega \frac{|p|^2}{2\mu} + \int_0^T W_2^\epsilon(\mu(t,\cdot),Q_t)^2 + \int_0^T \mathcal{F}(\mu(t,\cdot),Q_t) \\
&= \inf_Q \theta H(Q|R^f) + \inf_{P\in\mathcal{P}(C([0,T],\Omega))} H(P|R^h) + \int_0^T W_2^\epsilon(P_t,Q_t)^2 + \int_0^T \mathcal{F}(P_t,Q_t)
\end{aligned} \tag{5.7}$$

Where we have denoted R^f and R^h the Wiener measures associated to brownian motions with respective variance σ_f and σ_h. Constraints have been omitted once introduced for better readability, but they should be fairly obvious in context. In the end, if we reorder the above and explicit the regularized Wasserstein distance, we can recast the problem as a two-level entropy-minimization problem with an added running cost :

$$\inf_{\substack{P,Q\in\mathcal{P}(C([0,T],\Omega)) \\ P_0=\mu_0,\, Q_0=\nu_0}} H(P|R^h) + \theta H(Q|R^f) + \int_0^T \epsilon \inf_{\gamma\in\Pi(P_t,Q_t)} H(\gamma|\xi) + \int_0^T \mathcal{F}(P_t,Q_t) \tag{5.8}$$

This is a well-posed convex minimization problem. Due to the nature of the problem, it seems this formulation is the most compact, most fitting of the three. It allows us quite naturally to tackle the dual scale of optimization – instantaneous O.T. and population dynamics – while stressing that they are similar in nature, although acting on different levels, i.e we can take advantage of the specifity of the regularized optimal transport cost as a running cost in the mean-field game. At the same time, this framework incorporates easily the randomness of the model and will prove quite flexible in that respect. Last, although it might seem more complex at first glance, it can be advantageously used to build a very simple and efficient algorithmic approach via alternate minimizations on the dual problem – which we will get too later on.

5.1.2 Dual problem

In this paragraph we exhibit the dual problem to our Lagrangian variational formulation. It is essential here because strong duality applies and the key to our algorithmic apporach will be using alternate maximization on the dual problem. For the purpose of simulations, we will later use the separability assumption introduced previously, but before turning to that aspect, we state its general form.

We begin by rewriting our problem in discrete time. This is done for several reasons ; first, this is what we will use in practice for building an algorithm ; second, using the gamma-convergence result proved in [9], we can safely use it as an approximation for the continuous-time problem ; last, although all of this can be done in continuous time, discrete time avoid having to deal with many technicalities to ensure that all is well-defined, such as semi-group theory and interversion of infimum and integrals. We assume the time interval $[0,T]$ is divided in N equal-length time periods (i.e $N+1$ time steps). Thus, directly interchanging infimum and sum, we get :

$$\inf_{\substack{P,Q\in\mathcal{P}(\Omega^{N+1})\\ P_0=\mu_0,\, Q_0=\nu_0}} \inf_{\substack{(\gamma_k)_{k=0,\dots,N}\\ \forall k,\, \gamma_k\in\Pi(P_k,Q_k)}} H(P|R_N^h) + \theta H(Q|R_N^f) + \epsilon\frac{T}{N}\sum_{k=0}^{N} H(\gamma_t|\xi) + \frac{T}{N}\sum_{k=1}^{N-1}\mathcal{F}(P_t,Q_t) \tag{5.9}$$

Where we denote R_N the discretized Wiener measure over $[0,T]$ with N time periods (superscripts f and h still denote differente variances). We will subsequently stop writing the sets to which P,Q, and γ belong ; for greater concision we also introduce the notation :

$$\tau := \frac{T}{N} \tag{5.10}$$

The conditions on γ can be dualized easily, given P and Q. We introduce the dual variables u_t, v_t for $t\in[0,T]$ and write the saddle-point condition as :

$$\sup_{\substack{(u_k)_{k=0,\dots,N}\\ (v_k)_{k=0,\dots,N}}} \sum_{k=0}^{N}\left(\int_\Omega u_k dP_k + \int_\Omega v_k dQ_k - \int_{\Omega^2} u_k\oplus v_k d\gamma_k\right) \tag{5.11}$$

We could have written it for each time first, but here we directly switched the supremum and the sum as before. Equivalently, we can rewrite it :

$$\sup_{u,v}\int_{\Omega^{N+1}}\left(\bigoplus_{k=0}^{N}u_k\right)dP + \int_{\Omega^{N+1}}\left(\bigoplus_{k=0}^{N}v_k\right)dQ - \sum_{k=0}^{N}\left(\int_{\Omega^2}u_t\oplus v_t d\gamma_t\right) \tag{5.12}$$

To dualize the infimum on P,Q, we begin by splitting the problem into :

- An unconstrained infimum on the marginals μ and ν
- An continuous multi-marginal optimal transport problem with $N+1$ marginal constraints on transport plans P and Q

That is to say we write :

$$\inf_{\substack{P,Q\in\mathcal{P}(\Omega^{N+1})\\ P_0=\mu_0,\, Q_0=\nu_0}} \tag{5.13}$$

as

$$\inf_{\substack{(\mu_k)_{k=0,\dots,N}\\ (\nu_k)_{k=0,\dots,N}}} \inf_{\substack{P,Q\in\mathcal{P}(\Omega^{N+1})\\ \forall k,\, P_k=\mu_k\\ \forall k,\, Q_k=\nu_k}} \tag{5.14}$$

And the initial condition is enforced by introducing, directly into the problem, the functional :

$$\mathcal{F}_0(\mu_0,\nu_0) := \begin{cases} 0 & \text{if } \mu_0=\bar{\mu}_0 \text{ and } \nu_0=\bar{\nu}_0 \\ \infty & \text{otherwise} \end{cases} \tag{5.15}$$

where to avoid confusing notations, we have denoted $\bar{\mu}_0$ and $\bar{\nu}_0$ the given initial densities. The running cost will be discretized similarly, to allow for compact notations, as :

$$\forall k\in\{1,...,N-1\}\;\; \mathcal{F}_k(\mu,\nu) := \mathcal{F}(\mu,\nu) \tag{5.16}$$

$$\mathcal{F}_N := 0 \tag{5.17}$$

This notation allows for a general understanding of the running cost in the discretized setting, which includes the initial condition and the terminal cost – note that the latter is captured by $\mathcal{F}_N$, and that we have assumed it to be zero so far ; an alternative would be to set $\mathcal{F}_k(\mu,\nu) := \mathcal{F}(\mu,\nu)$ to do as if the game kept on being played. We will come back to that when dealing with infinite horizon. Also that the index k, in addition to making for compact notation, will be useful when taking discounting into account, i.e having a depency on time in the running cost.

The constraint on P and Q writes, in saddle point formulation with dual variables ϕ and ψ :

$$\sup_{\substack{(\phi_k)_{k=0,\dots,N}\\ (\psi_k)_{k=0,\dots,N}}} \sum_{k=0}^{N}\left(\int_\Omega \phi_k d\mu_k - \int_\Omega \phi_k dP_k\right) + \sum_{k=0}^{N}\left(\int_\Omega \psi_k d\nu_k - \int_\Omega \psi_k dQ_k\right) \tag{5.18}$$

Which once again we can rewrite equivalently :

$$\sup_{\phi,\psi} \sum_{k=0}^{N}\int_\Omega \phi_k d\mu_k - \int_{\Omega^{N+1}}\left(\bigoplus_{k=0}^{N}\phi_k\right)dP + \sum_{k=0}^{N}\int_\Omega \psi_k d\nu_k - \int_{\Omega^{N+1}}\left(\bigoplus_{k=0}^{N}\psi_k\right)dQ \tag{5.19}$$

Putting it all together, we get :

$$\inf_{\substack{\mu,\nu\\P,Q\\\gamma}} \sup_{\substack{u,v\\\phi,\psi}} H(P|R_N^h) + \theta H(Q|R_N^f) + \epsilon\tau \sum_{k=0}^{N} H(\gamma_k|\xi) + \tau \sum_{k=0}^{N} \mathcal{F}_k(\mu_k,\nu_k)$$
$$+ \int_{\Omega^{N+1}} \left(\bigoplus_{k=0}^{N} u_k\right) dP + \int_{\Omega^{N+1}} \left(\bigoplus_{k=0}^{N} v_k\right) dQ - \sum_{k=0}^{N} \left(\int_{\Omega^2} u_k \oplus v_k d\gamma_k\right)$$
$$+ \sum_{k=0}^{N} \int_{\Omega} \phi_k d\mu_k - \int_{\Omega^{N+1}} \left(\bigoplus_{k=0}^{N} \phi_k\right) dP + \sum_{k=0}^{N} \int_{\Omega} \psi_k d\nu_k - \int_{\Omega^{N+1}} \left(\bigoplus_{k=0}^{N} \psi_k\right) dQ \tag{5.20}$$

This expression simplies greatly when taking first order conditions in P, Q, and γ_k. This allows us to directly express all transport plans as diagonal scalings of the appropriate kernel, using only the dual variables :

$$P = \left(\bigotimes_{k=0}^{N} e^{\phi_k - u_k}\right) R_N^h \tag{5.21}$$
$$Q = \left(\bigotimes_{k=0}^{N} e^{\frac{\psi_k - v_k}{\theta}}\right) R_N^f \tag{5.22}$$
$$\forall k,\ \gamma_k = e^{\frac{u_k \oplus v_k}{\epsilon\tau}} \xi \tag{5.23}$$

Now, we can rewrite our problem as :

$$\begin{aligned}\inf_{\mu,\nu} \sup_{\substack{u,v\\\phi,\psi}} \tau \sum_{k=0}^{N} \left(\mathcal{F}_k(\mu_k,\nu_k) + \int_{\Omega} \frac{\phi_k}{\tau} d\mu_k + \int_{\Omega} \frac{\psi_k}{\tau} d\nu_k\right)\\ - \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\phi_k - u_k}\right) dR_N^h\\ - \theta \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\frac{\psi_k - v_k}{\theta}}\right) dR_N^f\\ - \epsilon\tau \sum_{k=0}^{N} \int_{\Omega^2} e^{\frac{u_k \oplus v_k}{\epsilon\tau}} d\xi\end{aligned} \tag{5.24}$$

Formally interchanging sup and inf, we then directly obtain the dual problem as :

$$\begin{aligned}\sup_{\substack{u,v\\ \phi,\psi}} \; & -\tau \sum_{k=0}^{N} \mathcal{F}_k^*(-\frac{\phi_k}{\tau}, -\frac{\psi_k}{\tau}) \\ & - \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\phi_k - u_k} \right) dR_N^h \\ & - \theta \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\frac{\psi_k - v_k}{\theta}} \right) dR_N^f \\ & - \epsilon\tau \sum_{k=0}^{N} \int_{\Omega^2} e^{\frac{u_k \oplus v_k}{\epsilon\tau}} d\xi \end{aligned} \tag{Dual LVF}$$

Where $\mathcal{F}^*$ denotes the convex conjugate of $\mathcal{F}$.

5.1.3 Infinite horizon and discounting

Infinite horizon poses an interesting problem. A practical limitation of our entropy-minimization approach lies in the close tie between the Wiener measure and kinetic energy, i.e the quadratic cost. For more general moving costs, nothing ensures that we can find an appropriate path measure to translate our optimal control problem into an entropy minimization one. Furthermore, there are a lot of underlying technical problems to make sure that the relative entropy with respect to a certain measure is well defined. This reaches into complex matters that lie between analysis and probability theory, which are far beyond the scope of this text. More details can be found e.g. in [38] [39] [37], or [19], [29].

It does seem, however, that the particular case where the cost is still quadratic but is scaled by a discounting factor (decreasing with time) can be handled relatively easier. Indeed, let us start back from the eulerian variational formulation with discounting :

$$\inf_{p,q,\mu,\nu} \int_0^T \int_\Omega e^{-\rho t} \frac{|p|^2}{2\mu} + e^{-\rho t}\theta \frac{|q|^2}{2\nu} + \int_0^T e^{-\rho t} W_2^\epsilon(\mu(t,\cdot),\nu(t,\cdot))^2 + \int_0^T e^{-\rho t}\mathcal{F}(\mu(t,\cdot),\nu(t,\cdot))$$
$$\text{Among solutions of :}$$
$$\begin{aligned} \partial_t \mu - \sigma_h \Delta \mu + \operatorname{div}(p) = 0 \text{ on } (0,T]\times\Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu - \sigma_f \Delta \nu + \operatorname{div}(q) = 0 \text{ on } (0,T]\times\Omega,\ \nu(0,\cdot) = \nu_0 \end{aligned} \tag{5.25}$$

With the change of variables : $\tilde{p} = e^{-\rho t/2}\frac{p}{\mu}$ and $\tilde{q} = e^{-\rho t/2}\frac{q}{\nu}$, this can be expressed as a classical kinetic energy minimization problem, but the Fokker-Planck equations become :

$$\begin{aligned} \partial_t \mu - \sigma_h \Delta \mu + \operatorname{div}(e^{\rho t/2}\tilde{p}\mu) = 0 \text{ on } (0,T]\times\Omega,\ \mu(0,\cdot) = \mu_0 \\ \partial_t \nu - \sigma_f \Delta \nu + \operatorname{div}(e^{\rho t/2}\tilde{q}\nu) = 0 \text{ on } (0,T]\times\Omega,\ \nu(0,\cdot) = \nu_0 \end{aligned} \tag{5.26}$$

Going back to the mean-field game formulation, and making explicit the drift term, this actually means that we face the following system of PDEs (already presented in (4.11)) :

$$\begin{cases} \partial_t u + \sigma_h \Delta u - e^{\rho t} \frac{|\nabla u|^2}{2} = -e^{-\rho t} f - e^{-\rho t} R \\ \partial_t v + \sigma_f \Delta v - e^{\rho t} \frac{|\nabla v|^2}{2\theta} = -e^{-\rho t} g - e^{-\rho t} W \\ \partial_t \mu - \sigma_h \Delta \mu - e^{\rho t} \operatorname{div}(\mu \nabla u) = 0 \text{ on } (0,T] \times \Omega \\ \partial_t \nu - \sigma_f \Delta \nu - e^{\rho t} \operatorname{div}(\nu \frac{\nabla v}{\theta}) = 0 \text{ on } (0,T] \times \Omega \end{cases} \tag{5.27}$$

Another change of variable, taking $\tilde{\mu} = e^{\rho t}\mu$ and $\tilde{\nu} = e^{\rho t}\nu$, yields :

$$\partial_t \tilde{\mu} - \rho\tilde{\mu} - \sigma_h \Delta \tilde{\mu} - e^{\rho t} \operatorname{div}(\tilde{\mu} \nabla u) = 0 \text{ on } (0,T] \times \Omega \tag{5.28}$$

$$\partial_t \tilde{\nu} - \rho\tilde{\nu} - \sigma_f \Delta \tilde{\nu} - e^{\rho t} \operatorname{div}(\tilde{\nu} \frac{\nabla v}{\theta}) = 0 \text{ on } (0,T] \times \Omega \tag{5.29}$$

These equations are the infinitesimal generators of Brownian motions with killing, which has constant rate ρ, and drifts respectively $-e^{\rho t}\nabla u$ and $-e^{\rho t}\nabla v/\theta$ – this result is obtained by a direct application of Feynman-Kac's theorem. That is to say that, under these changes of variables, discounting can be interpreted as a loss of mass at a constant rate. From the point of view of path measures, it is as if each path had a probability of "dying" at each time – and which paths die or go on is independent of position. Another way of seing it is to see each individual in the population at each time as weighted by its importance in the intertemporal utility function : at time zero everybody has unity weight (all individuals are equally valued), then the weight of the future versions of each individual decrease – note that the rate is the same for everybody. So the utility of a given individual at time t is worth $e^{-\rho t}$ less than the utility of the same individual at time 0.

If we denote $\tilde{R}$ the path measure associated to a Brownian motion with killing rate ρ, then our problem can be written in entropy-minimization form as :

$$\inf_{\substack{P,Q \in \mathcal{P}(C([0,T],\Omega)) \\ P_0=\mu_0,\, Q_0=\nu_0}} H(P|\tilde{R}^h) + \theta H(Q|\tilde{R}^f) + \int_0^T e^{-\rho t} \epsilon \inf_{\gamma \in \Pi(P_t, Q_t)} H(\gamma|\xi) + \int_0^T e^{-\rho t} \mathcal{F}(P_t, Q_t) \tag{5.30}$$

What is really convenient is that, in the discretized version of the model, we will not actually need to make the paths measures explicit – we only need to know its bridges. The killing rate being independant, the probability for a brownian motion with killing of reaching y starting from x during a time interval of length t is given by the probability of being "still alive" times the probability for a standard brownian motion of reaching y starting from x during a time interval of length t. That is to say :

$$\tilde{R}^{xy}_{dt} = e^{-\rho dt} R^{xy}_{dt} \tag{5.31}$$

Where, with a slight abuse of notations, we denoted R^{xy}_{dt} the probability of the process with path measure R of reaching y starting from x during time interval dt. This is the result we will use in practice to introduce discounting in the algorithm.

5.2 Algorithm

The general idea of our algorithmic approach is to generalize the iterative proportional fitting procedure (IPFP), also widely known as Sinkhorn's algorithm in optimal transport (see [6]). In our setting this is more akin to a proximal point algorithm, except we do not have a proper distance but a Bregman divergence – the relative entropy. We aim at computing the projection on an intersection of convex sets. Let us remind that alternate projections converges to the actual projection in the case of affine constraint spaces ; however, in general, if the sets are only convex they do not. The solution is found with Dykstra's algorithm, where projections are corrected at each step by going moving towards the normal direction. A perhaps computationaly simpler way to view it is that Dykstra's algorithm consists in performing alternate maximization on the dual problem. For a detailed presentation of this aproach in the case of Bregman divergences, see for instance [43]. From there, the generalization to several populations is pretty straightforward. The only trick is to integrate the computation of the instantaneous cost of commute within the global problem instead of treating it as a standard running cost – which would entail an embedded algorithm to compute optimal commute given densities, then optimal movement,... which would induce more difficulties and a slower computing time.

5.2.1 The pure transport case

We begin by a first version of the algorithm in the simpler case where running cost $\mathcal{F}$ is null. In that case, the algorithm reduces back to an iterative scaling procedure. This algorithm avoids some qualification constraints problems that might be caused by just applying the generalized algorithm to the case $\mathcal{F} = 0$. It also makes for a clearer presentation of the idea behind the algorithm, and especially of the trick that allows us to compactly compute all transport plans by the same token.

Let us start again from the primal problem :

$$\inf_{\substack{P,Q\in\mathcal{P}(\Omega^{N+1})\\ P_0=\mu_0,\, Q_0=\nu_0}} \inf_{\substack{\gamma_k\in\Pi(P_k,Q_k)\\ k=0,\dots,N}} H(P|R_N^h) + \theta H(Q|R_N^f) + \epsilon\tau \sum_{k=0}^{N} H(\gamma_k|\xi) \tag{5.32}$$

First, notice that we have $2(N+1)+2$ constraints : 2 marginal constraints for instantaneous commute plans γ_k at each time period, and 2 marginal constraints for the initial densities P_0 and Q_0. Also remark all these constraints are affine, so the problem lends itself to a direct use of alternate projection. If we write the problem in saddle point formulation as before, we obtain :

$$\inf_{\substack{P,Q\in\mathcal{P}(\Omega^{N+1})\\ P_0=\mu_0,\, Q_0=\nu_0}} \inf_{\substack{\gamma_k\in\Pi(P_k,Q_k)\\ k=0,\dots,N}} \sup_{\substack{(u_k)_{k=0,\dots,N}\\ (v_k)_{k=0,\dots,N}}} \sup_{\phi_0,\psi_0} H(P|R_N^h) + \theta H(Q|R_N^f) + \epsilon\tau \sum_{k=0}^{N} H(\gamma_k|\xi)$$
$$+ \sum_{k=0}^{N} \left(\int_\Omega u_k dP_k + \int_\Omega v_k dQ_k - \int_{\Omega^2} u_k \oplus v_k d\gamma_k \right)$$
$$+ \int_\Omega \phi_0 d\mu_0 - \int_\Omega \phi_0 dP_0$$
$$+ \int_\Omega \psi_0 d\nu_0 - \int_\Omega \psi_0 dQ_0 \tag{5.33}$$

Computing first order conditions for γ_k yields :

$$\begin{aligned} \epsilon\tau \log(\frac{\gamma_k}{\xi}) - u_k \oplus v_k = 0 \\ \Leftrightarrow \gamma_k = e^{\frac{u_k \oplus v_k}{\epsilon\tau}} \xi \end{aligned} \tag{5.34}$$

Similarly, for P and Q we have

$$P = \exp\left((\phi_0 - u_0) \oplus \bigoplus_{k=1}^{N} -u_k \right) R_N^h \tag{5.35}$$

$$Q = \exp\left(\frac{(\psi_0 - v_0) \oplus \bigoplus_{k=1}^{N} -v_k}{\theta} \right) R_N^f \tag{5.36}$$

We introduce a convenient notation :

$$\begin{aligned} a_k &:= \exp(-u_k) \\ b_k &:= \exp(-\frac{v_k}{\theta}) \\ c_0 &:= \exp(\phi_0) \\ d_0 &:= \exp(\frac{\psi_0}{\theta}) \end{aligned} \tag{5.37}$$

So that the transport plans are given by :

$$P = \left(c_0 \bigotimes_{k=0}^{N} a_k \right) R_N^h \tag{5.38}$$

$$Q = \left(d_0 \bigotimes_{k=0}^{N} b_k \right) R_N^f \tag{5.39}$$

$$\forall k,\ \gamma_k = a_k^{-\frac{1}{\epsilon\tau}} b_k^{-\frac{\theta}{\epsilon\tau}} \xi \tag{5.40}$$

If we just write out the constraints explicitly, we see that we can obtain each dual potential, fulfilling the constraints, given some of the other potentials. For instance, taking the first marginal constraint on γ_k at time $k = 1, ..., N$, we have :

$$\begin{aligned} \pi_1 \# \gamma_k = P_k \Leftrightarrow a_k^{-\frac{1}{\epsilon\tau}} \int_\Omega b_k^{-\frac{\theta}{\epsilon\tau}} d\xi &= a_k \int_{\Omega^N} \left(c_0 \bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j \right) dR_N^h \\ \Leftrightarrow a_k &= \left(\frac{\int_\Omega b_k^{-\frac{\theta}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(c_0 \bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j \right) dR_N^h} \right)^{\frac{\epsilon\tau}{1+\epsilon\tau}} \end{aligned} \tag{5.41}$$

Which give us an expression of a_k as a function of the other a_j – as we would have in a standard standard multi-marginal optimal transport problem – and its counterpart b_k for the other population – as in a classical OT problem. We want to stress how similar this expression is to the classical Sinkhorn algorithm ; this a mere diagonal scaling of a kernel, except that the fixed densities of the OT problem have been replaced by the convolution of the potential corresponding to the other population at the same time (b_k) with the commute ground cost ξ.

Hence we have a very simple expression for obtaining each potential by alternatively projecting on each corresponding constraint, and iterating the procedure. This gives the algorithm below.

This algorithm advantageously shares the property of the standard Sinkhorn algorithm for optimal transport. It is fast and easy to program – since the core of the iterations boils down to matrix multiplications – and could be efficiently parallelized if need be.

5.2.2 Generalized algorithm

With a non-zero running cost, the algorithm gets slightly more involved, since we now need to compute projections according to the Kullback-Leibler divergence – which is just here another way of calling the discretized relative entropy. Let us start back from the dual problem ; note that now we will assume separability of the running cost functional throughout, and we will focus on the case where the problem is not truly variational and we need to use a fixed point trick. This not only allows to dive straight into the more interesting cases, but it is also very practical for concrete numerical simulations, since it

Algorithm 3 Iterative proportionnal fitting algorithm (Pure transport case)

Input : initial potentials $(a_k^{(0)})_{k=0,...,N}$, $(b_k^{(0)})_{k=0,...,N}$, $c_0^{(0)}$, $d_0^{(0)}$
for all $i \geq 0$ **do**

$$c_0^{(i+1)} = \frac{\mu_0}{a_0^{(i)} \int_{\Omega^N} \left(\bigotimes_{k=1}^N a_k^{(i)} \right) dR_N^h}$$

$$d_0^{(i+1)} = \frac{\nu_0}{b_0^{(i)} \int_{\Omega^N} \left(\bigotimes_{k=1}^N b_k^{(i)} \right) dR_N^f}$$

for $k = 0, ..., N$ **do**

$$a_k^{(i+1)} = \left(\frac{\int_\Omega (b_k^{(i)})^{-\frac{\theta}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(c_0^{(i+1)} \bigotimes_{\substack{j=0 \\ j \neq k}}^N a_j^{(i*)} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{1+\epsilon\tau}}$$

$$b_k^{(i+1)} = \left(\frac{\int_\Omega (a_k^{(i+1)})^{-\frac{1}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(d_0^{(i+1)} \bigotimes_{\substack{j=0 \\ j \neq k}}^N b_j^{(i*)} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{\theta+\epsilon\tau}}$$

Where $i* = i+1$ if $j \leq k$, i if $j > k$
end for
end for

Algorithm 4 Iterative proportionnal fitting algorithm (Pure transport case, with discounting)

Input : initial potentials $(a_k^{(0)})_{k=0,...,N}$, $(b_k^{(0)})_{k=0,...,N}$, $c_0^{(0)}$, $d_0^{(0)}$
for all $i \geq 0$ **do**

$$c_0^{(i+1)} = \frac{\mu_0}{a_0^{(i)} \int_{\Omega^N} \left(\bigotimes_{k=1}^N a_k^{(i)} \right) dR_N^h}$$

$$d_0^{(i+1)} = \frac{\nu_0}{b_0^{(i)} \int_{\Omega^N} \left(\bigotimes_{k=1}^N b_k^{(i)} \right) dR_N^f}$$

for $k = 0, ..., N$ **do**

$$a_k^{(i+1)} = \left(\frac{\int_\Omega (b_k^{(i)})^{-\frac{\theta(1+\rho)^{\tau k}}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(c_0^{(i+1)} \bigotimes_{\substack{j=0 \\ j \neq k}}^N a_j^{(i*)} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{(1+\rho)^{\tau k} + \epsilon\tau}}$$

$$b_k^{(i+1)} = \left(\frac{\int_\Omega (a_k^{(i+1)})^{-\frac{(1+\rho)^{\tau k}}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(d_0^{(i+1)} \bigotimes_{\substack{j=0 \\ j \neq k}}^N b_j^{(i*)} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{\theta(1+\rho)^{\tau k} + \epsilon\tau}}$$

Where $i* = i + 1$ if $j \leq k$, i if $j > k$
end for
end for

de facto reduces the complexity of the problems we need to solve at each iteration[4] :

$$\begin{aligned}\sup_{\substack{u,v\\ \phi,\psi}} \; & -\tau\left(\sum_{k=0}^{N} F_k^*(-\frac{\phi_k}{\tau}) + G_k^*(-\frac{\psi_k}{\tau})\right) \\ & - \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\phi_k - u_k}\right) dR_N^h \\ & - \theta \int_{\Omega^{N+1}} \left(\bigotimes_{k=0}^{N} e^{\frac{\psi_k - v_k}{\theta}}\right) dR_N^f \\ & - \epsilon\tau \sum_{k=0}^{N} \int_{\Omega^2} e^{\frac{u_k \oplus v_k}{\epsilon\tau}} d\xi\end{aligned} \tag{Dual LVF}$$

Where we have reused the notations F and G introduced previously and the discretization follows the same logic as previously :

$$\begin{cases} F_0 := \chi_{\mu_0} & \text{for } k = 0 \\ F_k(\mu) := F(\mu) \text{ or } e^{-\rho k} F(\mu) & \text{for } k = 1, ..., N-1 \\ F_N(\mu) := 0 \text{ or } F(\mu) & \text{for } k = N \end{cases} \tag{5.42}$$

$$\begin{cases} G_0 := \chi_{\nu_0} & \text{for } k = 0 \\ G_k(\nu) := G(\nu) \text{ or } e^{-\rho k} G(\nu) & \text{for } k = 1, ..., N-1 \\ G_N(\nu) := 0 \text{ or } G(\nu) & \text{for } k = N \end{cases} \tag{5.43}$$

From these dual variables, we recover the transport plans by computing :

$$P = \left(\bigotimes_{k=0}^{N} e^{\phi_k - u_k}\right) R_N^h \tag{5.44}$$

$$Q = \left(\bigotimes_{k=0}^{N} e^{\frac{\psi_k - v_k}{\theta}}\right) R_N^f \tag{5.45}$$

$$\forall k,\ \gamma_k = e^{\frac{u_k \oplus v_k}{\epsilon\tau}} \xi \tag{5.46}$$

and we introduce a convenient notation similar to that in the previous part :

[4]In the truly variational case, the optimization program which give us potentials ϕ_k and ψ_k (or equivalently c_k and d_k) becomes a joint problem. There is no real objection, and the problem remains tractable although with longer computing time.

$$\begin{aligned} a_k &:= \exp(-u_k) \\ b_k &:= \exp(-\frac{v_k}{\theta}) \\ c_k &:= \exp(\phi_k) \\ d_k &:= \exp(\frac{\psi_k}{\theta}) \end{aligned} \tag{5.47}$$

so that the transport plans are equivalently given by :

$$P = \left(\bigotimes_{k=0}^{N} a_k c_k \right) R_N^h \tag{5.48}$$

$$Q = \left(\bigotimes_{k=0}^{N} b_k d_k \right) R_N^f \tag{5.49}$$

$$\forall k,\ \gamma_k = a_k^{-\frac{1}{\epsilon\tau}} b_k^{-\frac{\theta}{\epsilon\tau}} \xi \tag{5.50}$$

We will switch between the two notations given which is most appropriate in the context. The idea is then to perform alternate minimization on the dual problem above. To illustrate the manipulation, let us fix all potentials at a given iteration except for some ϕ_k. Then, we take :

$$\phi_k := \arg\max_{\phi} -\tau F_k^*(-\frac{\phi}{\tau}) - \int_{\Omega^{N+1}} \left(e^{\phi - u_k} \bigotimes_{\substack{j=0 \\ j\neq k}}^{N} e^{\phi_j - u_j} \right) dR_N^h \tag{5.51}$$

It is very convenient to pass back to a primal form in that sub-problem – in which strong duality clearly applies as well. This yields a projection problem according to the relative entropy. We introduce a compact notation for entropic prox (following that introduced in [9]) :

$$\text{prox}_F^H(\bar{\mu}) := \arg\min_{\mu} H(\mu|\bar{\mu}) + F(\mu) \tag{5.52}$$

Furthermore, we can write more conviently in terms of the dual variable c_k. Putting it all together and skipping a few easy computations, this yields :

$$c_k := \frac{\text{prox}_{\tau F_k}^H \left(a_k \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j c_j \right) dR_N^h \right)}{a_k \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j c_j \right) dR_N^h} \tag{5.53}$$

The kernels in the prox and the in the denominator are the same : they are just the result of integrating the transport plan P in all but one variable and dividing by the potential of interest at that step – in other words, we take N convolutions of the other potentials with the Wiener measure, and the multiplicative factor a_k appears because of the two-levels nature of our problem. This might seem intricate – or at least not very

readable – but the problem here mostly has to do with notations, since the underlying computations are fairly simple.

Similar computations can be done for all potentials ϕ_k/c_k and ψ_k/d_k, to yield a more compact algorithm written directly with the most relevant potentials. Note that the potentials u_k/a_k and v_k/b_k do not involve the running cost functionals and can still be obtained by direct scaling without having to compute proximal operators. Just as before, we just write the corresponding constraint to obtain one potential as a function of the others. For instance, if we fix k, and all potentials except a_k, the constraint $\pi_1 \# \gamma_k = P_k$ writes as :

$$\begin{aligned} a_k^{-\frac{1}{\epsilon\tau}} \int_\Omega b_k^{-\frac{\theta}{\epsilon\tau}} d\xi &= a_k c_k \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j c_j \right) dR_N^h \\ \Leftrightarrow a_k &= \left(\frac{\int_\Omega b_k^{-\frac{\theta}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(c_k \bigotimes_{\substack{j=0 \\ j\neq k}}^{N} a_j c_j \right) dR_N^h} \right)^{\frac{\epsilon\tau}{1+\epsilon\tau}} \end{aligned} \tag{5.54}$$

And similarly for other potentials. Below, we will write two equivalent versions of the algorithm for greater clarity – a version involving the dual form of the optimization program and potentials a_k, b_k, ϕ_k, ψ_k, and a more convenient version for practical use, involving the entropic prox reformulation we just exhibited and the potentials a_k, b_k, c_k, d_k.

A last point must be adressed : the algorithm needs to implement a fixed point strategy. What we have done in practice is to recover densities at each time after each iterations and feed the densities of the previous iterations as a reference to F and G. That is to say, concretely, that F and G are of the form $F(\mu|\tilde{\nu})$ and $G(\nu|\tilde{\mu})$ and we take $\tilde{\nu}$ and $\tilde{\mu}$ to be the densities computed at the previous iterations. We have also tried taking a weighted average of densities at previous iterations, to make it more akin to the standard fictitious play algorithm for mean-field games (as presented for instance in [13]) and having the algorithm fully converge to set expectations before iterating – both gave similar results in practice. There is an interesting remark to be made on the epistemology of our algorithm. Setting aside the question of whether constructive fixed point theorems apply or not, using such a method for simulations seems sensible. We consider strategic agents with a calculating behaviour under rational expectations hypothesis, hence they should be able to perform the same computations as the the economist or mathematician writing the model. This argument – why would the modeller know better or different information than the modelled ? – sustains rational expections. It works both ways : if we are unable to compute a fixed point (say, if constructive arguments do not apply), agents are likely not to be either. In any case, the fictitious play seems a reasonable second best : each agent behaves as if the other's behaviour was given, computes optimal behaviour, then updates its belief. This kind of reasoning is one possible heuristic of the very concept of Nash equilibrium, so it seems that this approach for simulation is warranted, even more than by mathematical considerations by the heuristics of modelling interacting strategic agents. As such, the instabilities or lack of convergence that might be obtained in general settings with this imperfect method might prove more interesting

and relevant for understanding the model than a perfect theoretical result which is out of reach for our means of computation.

6 Numerical results

In this section, we present numerical simulations that were made using the previously described algorithm. With no claim of exhaustivity, we present several stylized cases that were found to be relevant and can give an overall sense of the dynamics of the model – further study of the results could, notably, involve a thorough study of the impact of varying the parameters, that we have not completed for lack of time.

All the results presented here are in dimension one. Although the algorithm readily extends to higher dimensions, it quickly induces purely numerical difficulties. Computing time is still reasonable but grows significantly with the precision of the grid. The main difficulty is dealing with numerical instabilities that appear through iterations producing values too low or too high to be handled. We did make a number of simulations in dimension 2, but we are not yet confident enough of having overcome these technical difficulties to present them.

Throughout this section, we will use normalized sums of gaussian densities (with different mean and variance) as initial population distribution μ_0, ν_0.

6.1 Pure transport case

We first present numerical simulations for the pure transport case – i.e the case with null running cost except for the instantaneous commute. Although this case is of limited interest for applications, it does serve as a benchmark case to test the efficiency of the algorithm. It is also a case in which it is much easier to construct an intuition of what the result should be. If we penalize instaneous transport distance between the densities, they should roughly get "closer" to one another – and would probably converge both to the same stationary distribution. But dynamic motion being costly, they would not instantly superpose, hence they might be some variations in speed of convergence, or one might get closer faster than the other. When the two distributions are the same, there is no incentive to move, so the only movement should come from diffusion (or to counter diffusion) – it is noteworthy that if the diffusion for the two populations have different parameters, it might be extremely costly for agents to maintain the exact same distribution and we would probably end up with *almost* identical distributions. All of this is far from rigorous, but this gives a broad intuition of what we would expect the equilibrium (or equilibria) to look like. And indeed, this intuition is corroborated by the simulations performed with the pure transport algorithm.

First, we present our reference case (see Figure 1), in which we chose what seemed the most intuitive and "realistic" setting in terms of parameters' relationships. In the initial configuration, we assume that firms are concentrated around one location, whereas there are three centers of inhabitants. This assumption could be rationnalized as representing a pre-urban situation in which there exist several small villages and one industrial center. We also assume that moving firms is much more costly than inhabitants, specifically ten times more ($\theta = 10$), which quite naturally induces that in equilibrium inhabitants will

Algorithm 5 Iterative proportionnal fitting algorithm (Generalized case – first expression)

Input : initial potentials $a_k^{(0)}$, $b_k^{(0)}$, $\phi_k^{(0)}$, $\psi_k^{(0)}$ and a priori plans $P^{(0)}$, $Q^{(0)}$.

for all $i \geq 0$ **do**

for $k = 0, ..., N$ **do**

$$a_k^{(i+1)} = \left(\frac{\int_\Omega (b_k^{(i)})^{-\frac{\theta}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(e^{\phi_k^{(i)}} \bigotimes_{\substack{j=0 \\ j\neq k}}^N a_j^{(i*)} e^{\phi_j^{(i)}} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{1+\epsilon\tau}}$$

$$b_k^{(i+1)} = \left(\frac{\int_\Omega (a_k^{(i)})^{-\frac{1}{\epsilon\tau}} d\xi}{\int_{\Omega^N} \left(e^{\psi_k^{(i)}} \bigotimes_{\substack{j=0 \\ j\neq k}}^N b_j^{(i*)} e^{\psi_j^{(i)}} \right) dR_N^f} \right)^{\frac{\epsilon\tau}{\theta+\epsilon\tau}}$$

end for

for $k = 0, ..., N$ **do**

$$\phi_k^{(i+1)} := \arg\max_\phi -\tau F_k^*(-\frac{\phi}{\tau}|Q_k^{(i)}) - \int_{\Omega^{N+1}} \left(a_k^{(i*)} e^\phi \bigotimes_{\substack{j=0 \\ j\neq k}}^N - a_j^{(i*)} e^{\phi_j^{(i*)}} \right) dR_N^h$$

$$\psi_k^{(i+1)} := \arg\max_\psi -\tau G_k^*(-\frac{\psi}{\tau}|P_k^{(i)}) - \int_{\Omega^{N+1}} \left(b_k^{(i*)} e^\psi \bigotimes_{\substack{j=0 \\ j\neq k}}^N b_j^{(i*)} e^{\psi_j^{(i*)}} \right) dR_N^f$$

end for

$P^{(i+1)} = \left(\bigotimes_{k=0}^N a_k c_k \right) R_N^h$
$Q^{(i+1)} = \left(\bigotimes_{k=0}^N b_k d_k \right) R_N^f$

end for
(Where $i* = i+1$ if $j \leq k$, i if $j > k$)

Algorithm 6 Iterative proportionnal fitting algorithm (Generalized case – second expression)

Input : initial potentials $a_k^{(0)}$, $b_k^{(0)}$, $c_k^{(0)}$, $d_k^{(0)}$ and a priori plans $P^{(0)}$, $Q^{(0)}$.

for all $i \geq 0$ **do**

for $k = 0, ..., N$ **do**

$$a_k^{(i+1)} = \left(\frac{\int_\Omega (b_k^{(i)})^{-\frac{\theta}{\epsilon\tau}} d\xi}{c_k^{(i)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N a_j^{(i*)} c_j^{(i)} \right) dR_N^h} \right)^{\frac{\epsilon\tau}{1+\epsilon\tau}}$$

$$b_k^{(i+1)} = \left(\frac{\int_\Omega (a_k^{(i)})^{-\frac{1}{\epsilon\tau}} d\xi}{d_k^{(i)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N b_j^{(i*)} d_j^{(i)} \right) dR_N^f} \right)^{\frac{\epsilon\tau}{\theta+\epsilon\tau}}$$

end for

for $k = 0, ..., N$ **do**

$$c_k^{(i+1)} := \frac{\operatorname{prox}^H_{\tau F_k(\cdot|Q_k^{(i)})} \left(a_k^{(i+1)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N a_j^{(i+1)} c_j^{(i*)} \right) dR_N^h \right)}{a_k^{(i+1)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N a_j^{(i+1)} c_j^{(i*)} \right) dR_N^h}$$

$$d_k^{(i+1)} := \frac{\operatorname{prox}^{\theta H}_{\tau G_k(\cdot|P_k^{(i)})} \left(b_k^{(i+1)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N b_j^{(i+1)} d_j^{(i*)} \right) dR_N^h \right)}{b_k^{(i+1)} \int_{\Omega^N} \left(\bigotimes_{\substack{j=0 \\ j\neq k}}^N b_j^{(i+1)} d_j^{(i*)} \right) dR_N^h}$$

end for

$$P^{(i+1)} = \left(\bigotimes_{k=0}^N a_k^{(i+1)} c_k^{(i+1)} \right) R_N^h$$
$$Q^{(i+1)} = \left(\bigotimes_{k=0}^N b_k^{(i+1)} d_k^{(i+1)} \right) R_N^f$$

end for
(Where $i* = i + 1$ if $j \leq k$, i if $j > k$)

move towards firm faster than the other way around. Accordingly diffusion is much more intense for inhabitants ($\sigma^h = 5$, $\sigma^f = 1$). The latter might also induce faster movement of inhabitants in the early stages since they can "use" the diffusion to get closer to firms, but in later periods it should make it very costly to maintain identical densities – hence it could explain the small but persistant difference in densities at the end. For the entropic regularization parameter, we use $\epsilon = 2$. It induces quite a lot of noise ("mistakes") in the instantaneous matching equilibrium – which is overall a realistic assumption – but it is hard to give a direct and intuitive interpretation of the parameter value. We simulate over five periods of time to give a clear presentation, but adding more periods or a finer time discretization does not make any noticeable differences. We use a 1000 evenly spaced points fixed grid discretization of the segment $[0, 10]$ and assume a periodic domain to avoid having to deal with boundary conditions. There is no discounting ($\rho = 0$) – in most of our simulations, adding discounting seemed merely to have the consequence of slowing the motion down without altering its course, which is at the same time very intuitive (future benefits of getting closer are less powerful so present effort is relatively more costly) and not very interesting in terms of observation[5].

Our simulation clearly exhibits convergence of densities towards one another, as expected. Another salient feature that was consistently observed in all the cases we explored is that, although movement is theoretically continuous it is systematically extremely fast between the first and second time period, however small the time interval – the only obvious exception being cases in which the initial densities largely overlap. A likely interpretation, reinforced by the latter observation, is that given the very strong pull of the Wasserstein distance an initial situation where most of the mass is concentrated on different supports is highly inefficient, hence the benefit to moving closer to one another is considerable in the first period and easily surpasses the cost of rapid motion. Also note the very fast convergence of the algorithm, which reaches the 10^{-10} precision threshold in 131 iterations, taking only a few seconds (see Figure 3)

We then exhibit a more theoretical and less intuitively rationalizable case, for comparison purposes, in which all the parameters are set to 1 – one consequence being that the two populations are completely identical (see Figure 2). It is more than highly likely that there are multiple equilibria in this case – indeed, both populations are totally identical, and the cost-bearing is totally transferable in our problem, hence they could intuitively either both move towards one another, or one could move towards the other indifferently in terms of the total cost. Unfortunately our algorithm, as such, does not allow to compute multiple equilibria so we only obtain one particular case. Hence this might explain the somehow disappointing observation that we again observe inhabitants converging towards firms rather than the other way around – but there is no real reason to rule out this case as an equilibrium either. We still could argue that there is a little more motion of firms towards inhabitants compared to the previous case.

Once again the convergence is very fast, taking under 120 iterations to reach 10^{-10} precision threshold (see Figure 4) – which is, once again a matter of seconds. Iterations obviously take more time as the time or space discretization is made finer, since it amounts to adding dimensions to the matrixes we are manipulating, but we remain in any case in

[5]It could, however, have an important role to play, if we were to try and fit this model or a variation of it over some observed historical dynamics.

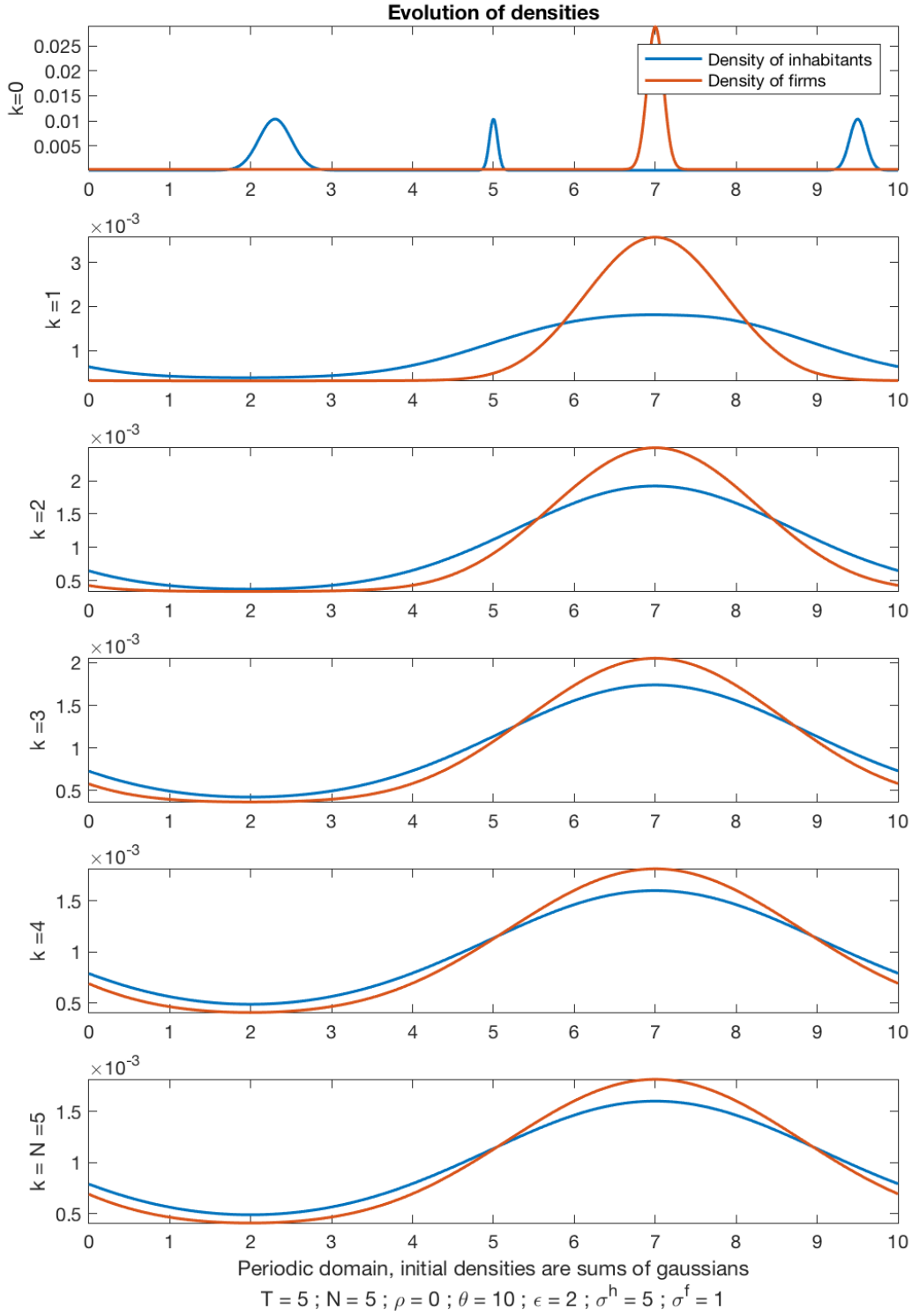

FIGURE 1: Pure transport, reference asymmetrical case

a very tractable ballpark in terms of computation speed.

The main takeway from these simulations is that observations for the pure transport case seem broadly speaking consistent with intuition and expected behaviour. A significant shortcoming of the algorithm is its inability to generate multiple equilibria, as they are very likely to exist – it could be a direction for future work to include by perhaps adding some stochasticity in the algorithm, but we would not yet venture as to how one would go about this.

6.2 Generalized case

We now turn to what we call the generalized case – non-null running cost, allowing us to incorporate congestion and externalities. We will focus on one specific model for the running cost, in which :

$$f[\mu, \nu] = \log(\mu + \nu) + \log(\nu) \tag{6.1}$$
$$g[\mu, \nu] = \log(\mu + \nu) \tag{6.2}$$

We can interpret this cost structure as a case of a symmetric congestion, depending on the sum of densities, and a one way negative externality from firms on inhabitants. Both effects are local and logarithmic. The logarithmic form is particularly practical for simulations since its potential can be written as an entropy, so then the problem really only amounts to one big entropy minimization problem. We also made simulations using quadratic expression instead of logs ; they gave roughly similar results, hence we confine to the presentation of the log version for simplicity. One major limitation of this cost structure is that it is *local*. That is too say, an agent will only be impacted by other agents that are exactly at the same position, which precludes any diffuse effects of e.g. pollution. Although there are no theoretical obstacles to having non-local cost – an interesting choice for negative externality could for example be of the form $f[\mu, \nu](\cdot) = \int |\cdot - y|^2 d\nu(y)$ –, it poses major numerical difficulties, since we then have to solve a non-local optimization problem at every step, which is not as simple and requires significantly more time (at each iteration). A natural extension of our work could inquire about fast methods for solving such problems and trying to integrate that as efficiently as possible within the algorithm. For the local case, the entropic prox problem at each iteration was solved using Newton's method to ensure fast convergence and speed up the algorithm. We more specifically used a damped Newton's method, so as to ensure that the algorithm would not overreach into negative values (which would produce complex numbers and induce numerical instabilities).

We present once again our reference case, but this time in more details and trying to interpret the step by step motion of densities. We take time horizon $T = 5$ and divide this interval into $N = 10$ time steps. Spatial environment is now the segement $[0, 50]$ and space discretization is the same (fixed grid with 1000 evenly spaced points). We now introduce a small discounting with $\rho = 0.4$. Motion is three times costlier for firms than for inhabitants ($\theta = 3$), and the noise of inhabitants' motion has five times higher variance than that of firms ($\sigma^h = 5$, $\sigma^f = 1$). The entropic regularization parameter in

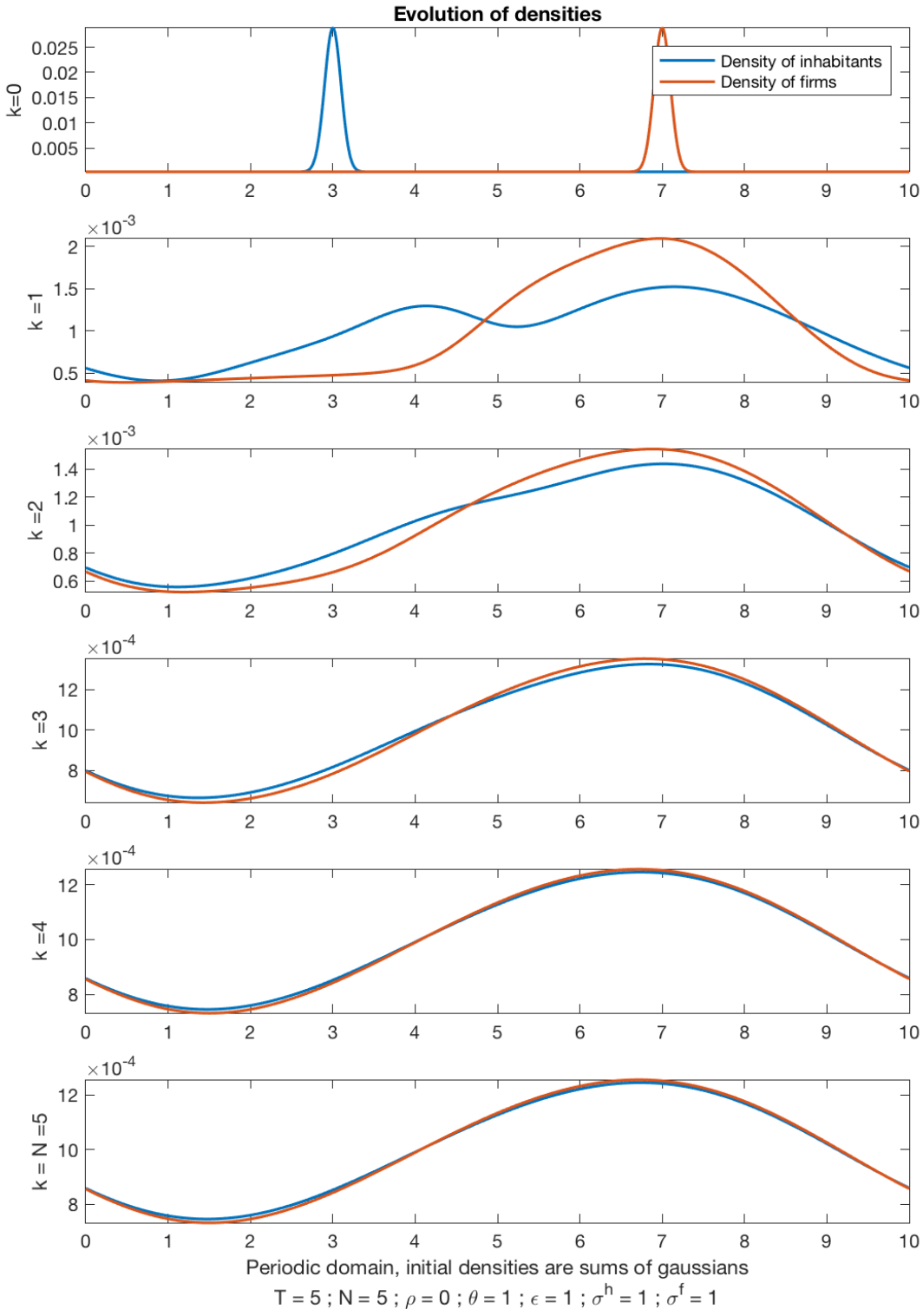

FIGURE 2: Pure transport, symmetrical case

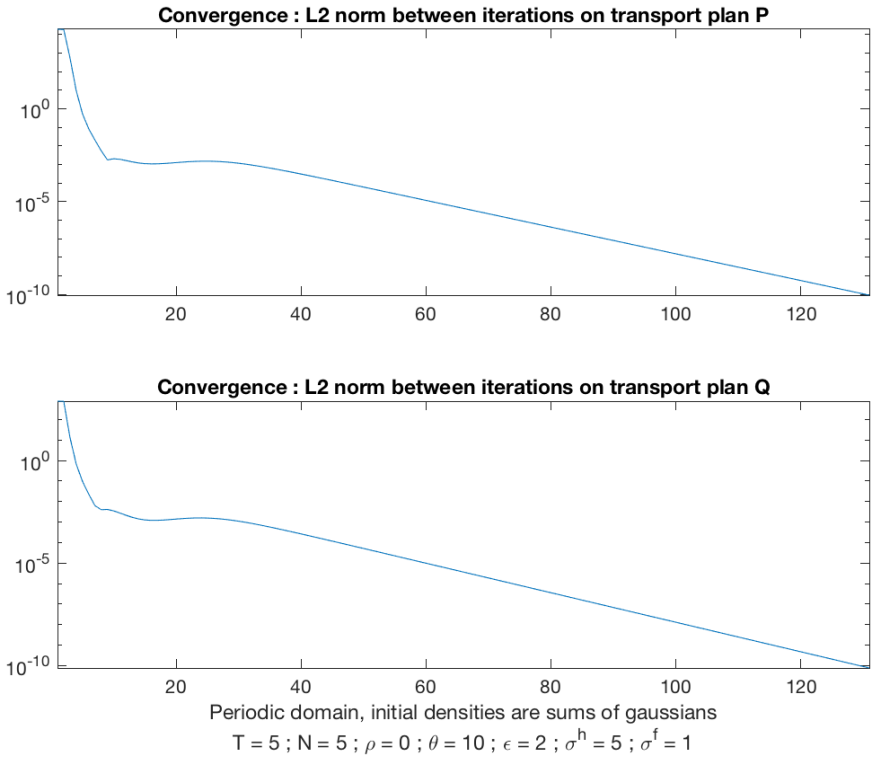

FIGURE 3: Convergence in pure transport, reference asymmetrical case

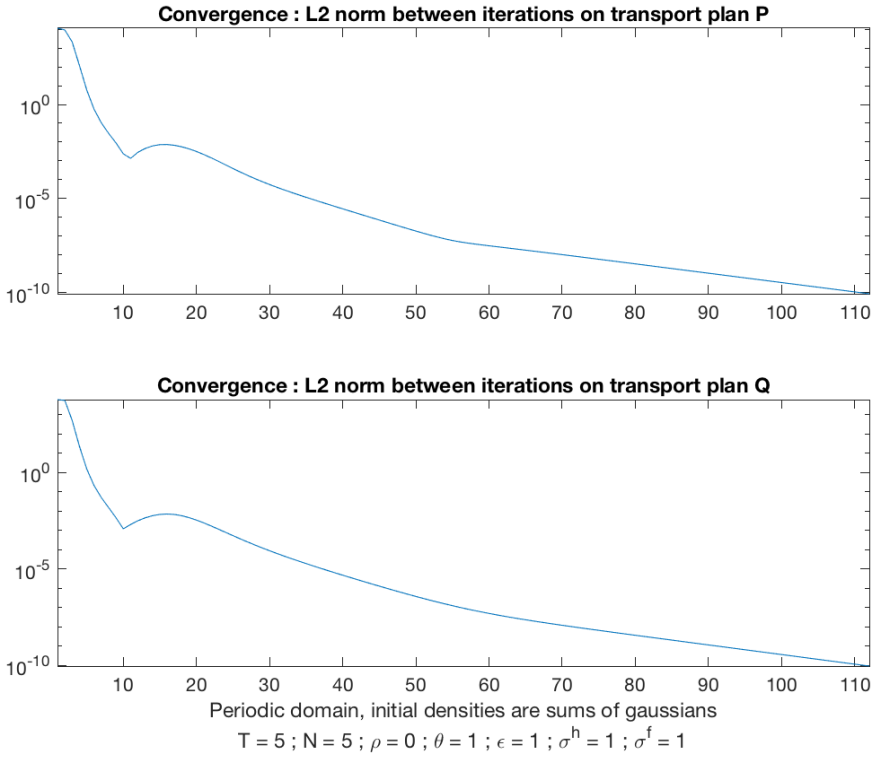

FIGURE 4: Convergence in pure transport, symmetrical case

the instantaneous commute problem is still given by $\epsilon = 2$. Initial densities are, again, normalized sums of gaussians.

This model, although far from realistic, seem to replicate overall coherent feature of urban developpment over the long run, starting from small centers and converging towards one city with a center and a periphery. In between, we clearly see dynamics of one population driving the other away from the center. There is logical tendency to accumulate around so as to minimize commuting distance without suffering from the local externality. Of course, this model is very stylized and the sparsity of its explanatory factors would make it very hard to observe precisely similar historical dynamics. However, this illustration shows the potential of our framework for explaining several salient aspects of long term dynamics with relatively few effects involved.

There is obviously a lot of room for improvement and experiments, notably in the direction of exploring other types of costs, other functional forms, refining effects, perhaps even trying to calibrate to data for very specific contexts ; or adding more dimensions to dynamics, incorporating new factors, such as a more refined market for land and real estate.

Once again, we would like to stress that the algorithm provides fast convergence, even though it is notably slower than in the pure transport. The slowing down has two main reasons : first, and perhaps less importantly, the generalized case involves more potentials, making updates a little costlier ; second, having to compute entropic proxes requires to perform, for each relevant potential ($2(N+1)$ with N time steps discretization), a Newton method. Still, we obtain convergence at the 10^{-10} threshold in under 800 steps, taking only a few minutes. Furthermore, it should be noted that there is certainly a lot of room left for optimizing the code and/or parallelizing it, which would be worthy for more complex applications, finer grids, or higher dimension.

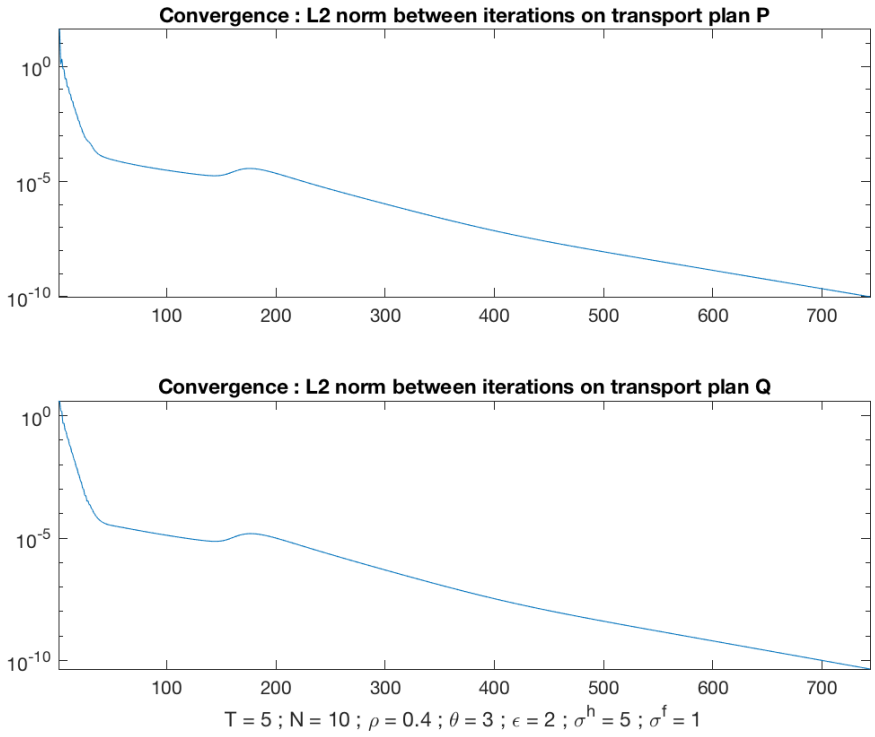

FIGURE 5: Convergence in generalized case

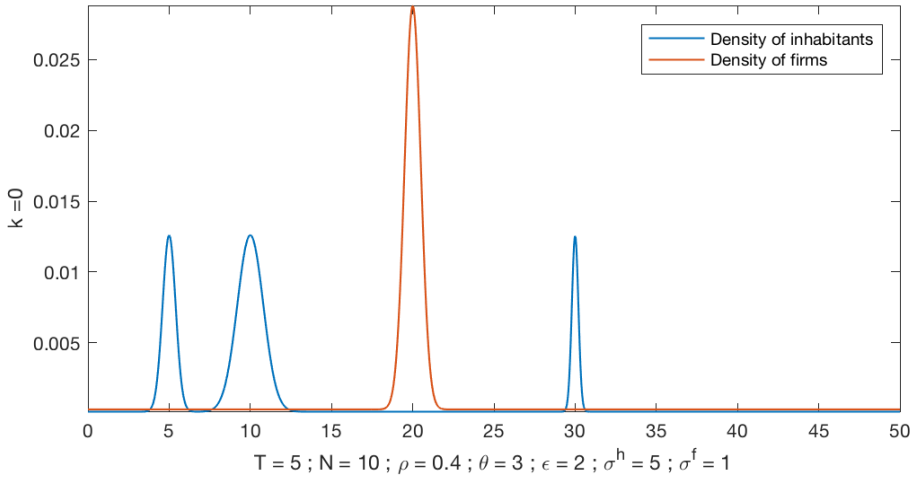

FIGURE 6: Generalized case : Initial densities

In the initial state, although there are technically some firms and inhabitants everywhere, there is one main industrial center and three separate gatherings of inhabitants with distinct supports. This could be interpreted, as a pre-urban state with small towns and a industrial center. We could even go further and justify the existence of this situation by an exogenous new implantation of an industrial center to account for the (temporary) inefficient situation with respect to commuting costs.

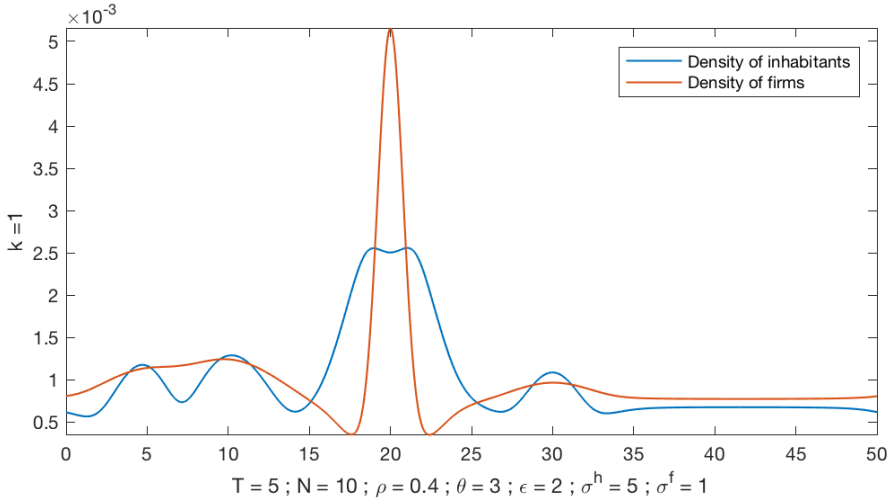

FIGURE 7: Generalized case : Period 1

One period ahead, a significant mass of inhabitants converges around the industrial center and some firms spread out. The initial centers remains but weaken and spread.

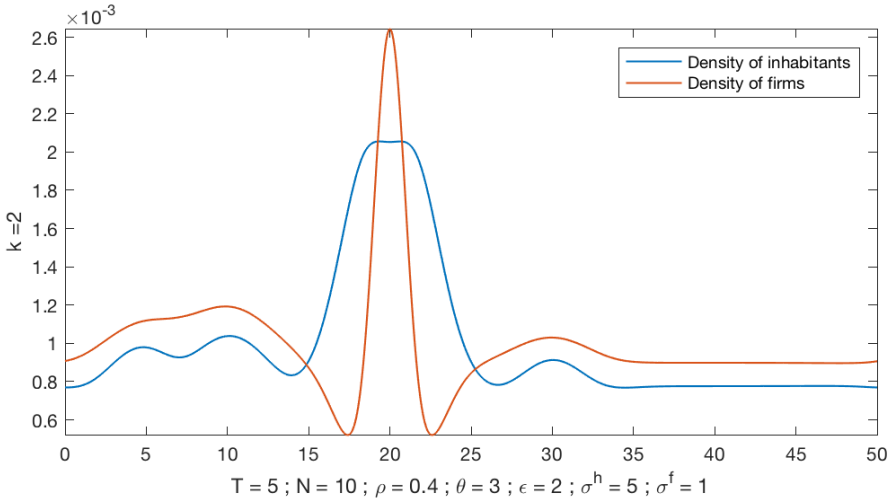

FIGURE 8: Generalized case : Period 3

The city pattern with firms in the center and inhabitants surrouding strengthen. The initial locations remain marked but spread and tend to join into one continuous city. Congestion has dropped outside of the center and the former inhabitants gathering also attract firms.

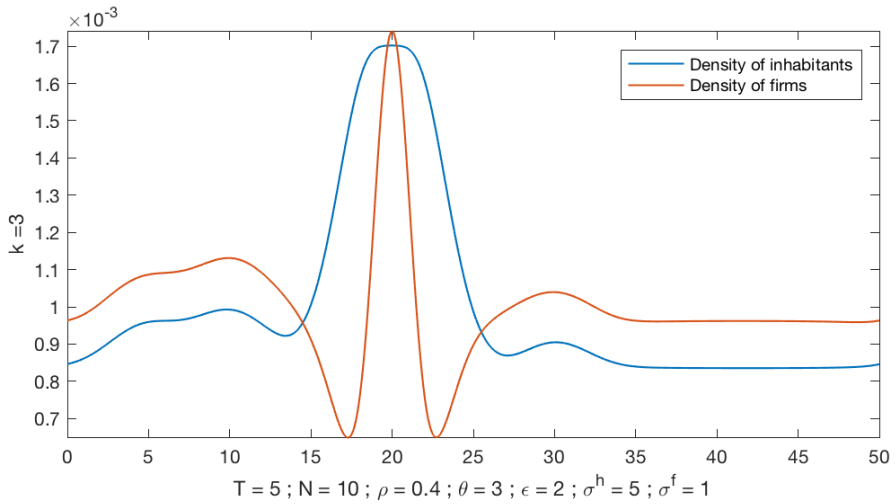

FIGURE 9: Generalized case : Period 4

Inhabitants keeps converging towards the center as firms are driven out by congestion. Outside of the center, the distribution of firms tends to match closely that of remaining inhabitants. The immediate city outskirts where the density of inhabitants is maximal remains a clear low point of firm presence.

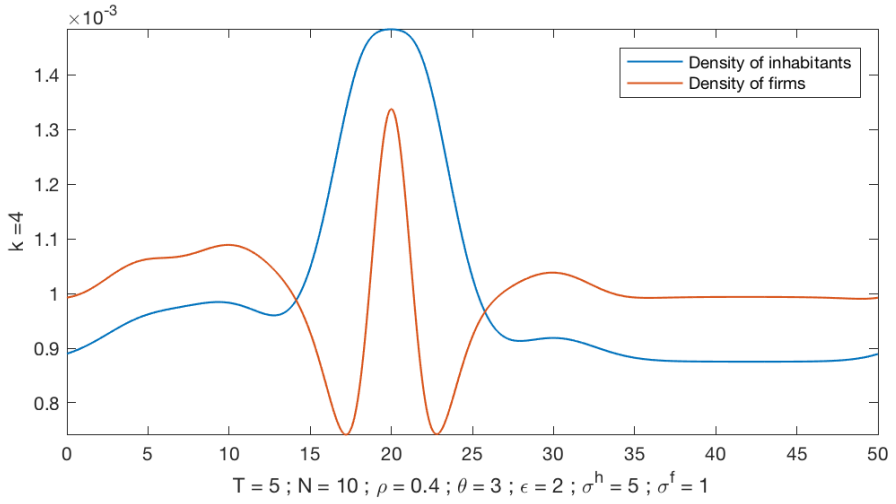

FIGURE 10: Generalized case : Period 5

Lower firm concentration in the city center reduces externality and contributes further to the concentration of inhabitants, creating a feedback effect driving firms towards still well populated suburbs.

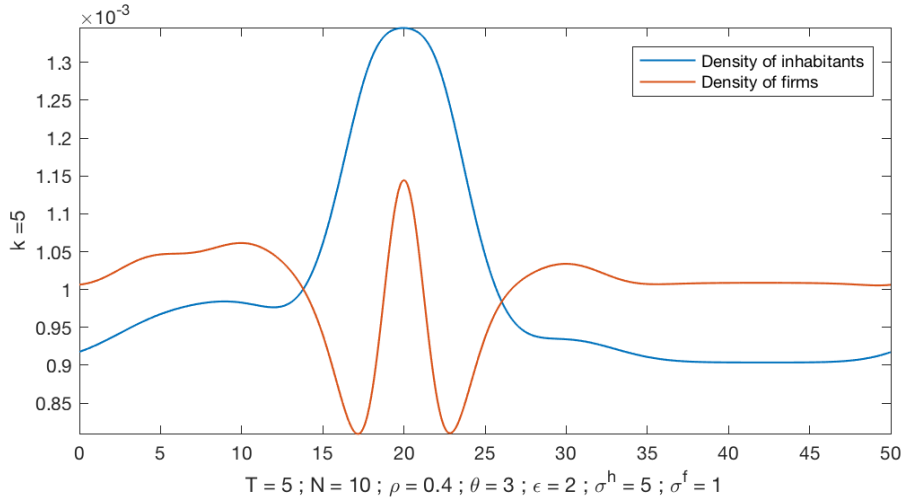

FIGURE 11: Generalized case : Period 6

Crowding out effects of firms in the center further, while outside of the center the distribution tends to smoothen and flatten under the influence of noise.

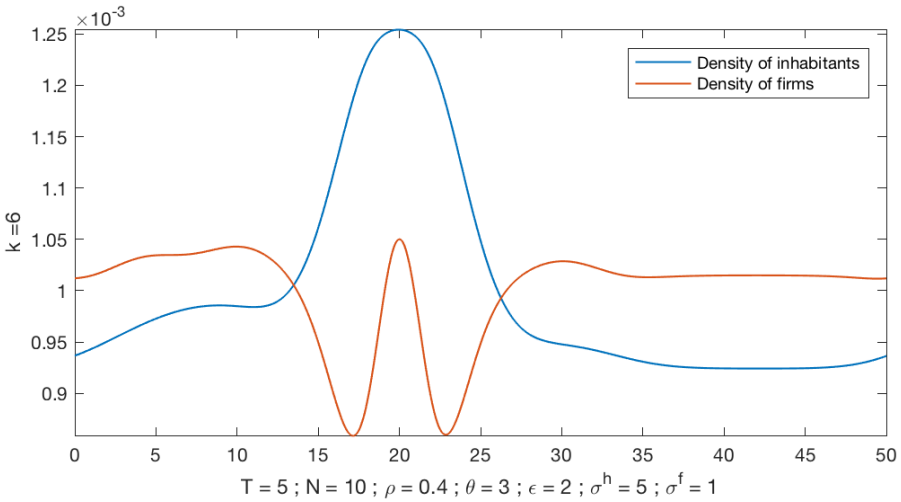

FIGURE 12: Generalized case : Period 7

Although we can still see hints of the initial distribution and a resulting asymmetry, there is now clearly one city comprised of a center and its surrounding (say suburbs). This city ressembles the archetypal European city with inhabitants dwelling in the center and firms around them. Such an pattern now dominates motion of distributions over the next periods, which only deepen what we already observed.

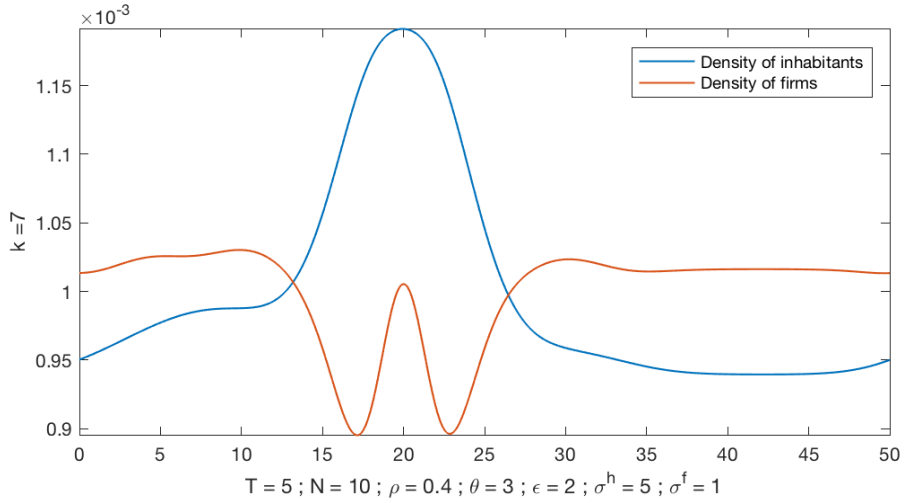

FIGURE 13: Generalized case : Period 8

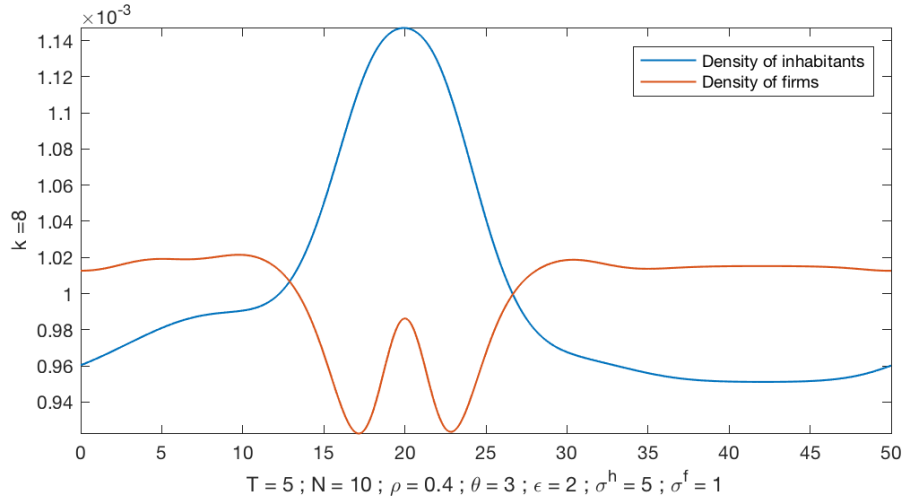

FIGURE 14: Generalized case : Period 9

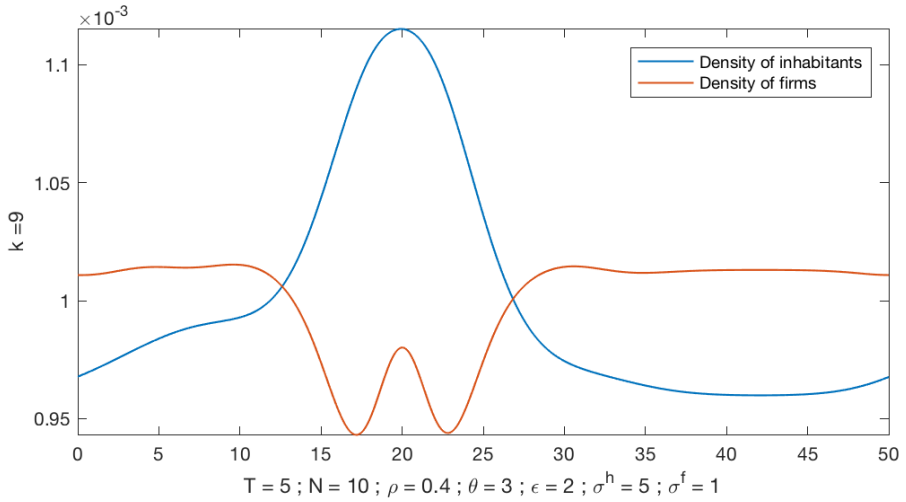

FIGURE 15: Generalized case : Period 10

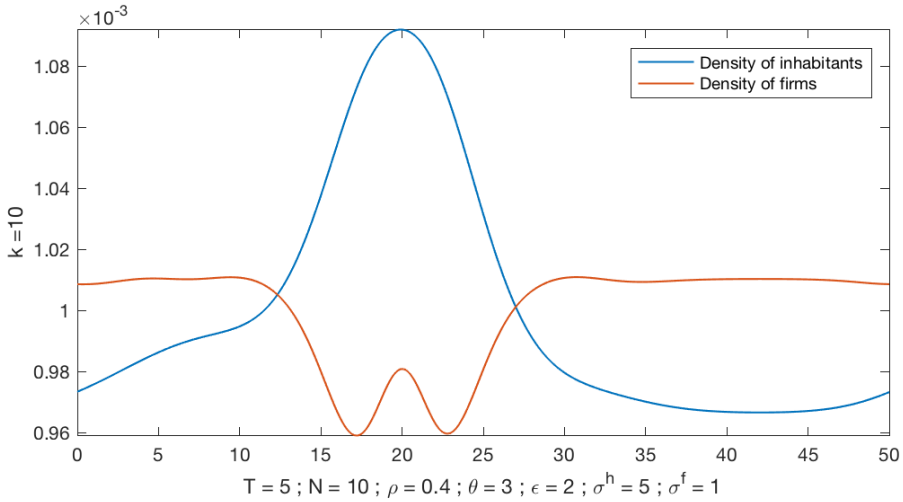

FIGURE 16: Generalized case : Period 11 (final)

In the final period, the dual city is clearer than ever. The center has taken its place in the initial location of firms and is mostly made of inhabitants. Although several firms remain in the city center, most have settled in its surroundings, and seem to be symmetrically distributed around the core. The inhabitants' distribution bears a slight asymmetry inherited from its initial setup, but it tends to even out over time. Due to the effect of diffusion, the whole distribution will also tend to flatten out over time while preserving the relative pattern observed here.

7 Conclusion

Our model provides a very stylized framework for understanding strategic motion of agents within a prescribed instantaneous matching equilibrium. It has the potential to cover a wide variety of situations, and it seems that this could be a rich and potent framework of further studies – with either enrichment from new applications or from new technical elements. Our approach combined several key elements : the understanding of the instantaneous matching equilibrium via an optimal transport problem and its regularization to introduce noise, the framework of multi-population mean field games to understand dynamics in the model and build an equilibrium concept.

We presented the barebone model and explored its elementary properties to find the most suitable approach and demonstrate its practicality by performing simulations. The main result is that the entropy minimization framework is an extremely powerful and versatile tool, which is particularly well equipped to deal with many modeling elements that are essential when tackling economic issues. Our iterative proportional fitting algorithm constitutes an efficient method for performing simulations, which result from its exploitation of the specificities of the model – it might, however, be of use in many similar cases involving optimal transport, mean field games or both.

We started from an intuition that was rooted in urban planning problem and geographical population dynamics, incorporating a labour market. The results of our simulation indeed provided coherent patterns in terms of intuitively expected behaviour of agents in such context, and seem to correspond well overall to long run trends of urbanization observed over the course of the past century or so.

There are many paths for improving and enriching this model. Several are pretty straightforward extensions, e.g. to more than two populations, to different cost functionals. Many involve numerical challenges, like optimization of the algorithm, parallelization, preventing numerical instabilities, efficiently solving for non-local costs, obtaining different equilibria in cases of non-unicity. Other involve theoretical work for fine characterization of the system properties, a thorough study of (non) unicity, influence of the parameters. A large amount of extensions could come from new applications bringing forward their own requirements and their own constraints, or from embedding this setup into richer environments as a building block.

We have mainly endeavoured to lay the groundwork for one modelling approach, but the tools we have used here are far reaching and yet to be exploited in many problems, most notably in economics. We hope that further works will bring forward how potent they can be.

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