Diagonalizing the Linear Swing Equations

May 28, 2020

We consider the linear equations,

$$\ddot{\theta}_i(t) = -\gamma\dot{\boldsymbol{\theta}}_i(t) + \sum_j A_{ij}(\theta_j - \theta_i(t)) + P_i, \tag{1}$$

$i = 1, ..., N$, $\gamma > 0$, $\sum_i P_i = 0$. We consider the vector $\boldsymbol{\theta} = [\theta_1, \theta_2, ...\theta_N]$ and the vector $\boldsymbol{P} = [P_1, P_2, ...P_N]$ and rewrite,

$$\ddot{\boldsymbol{\theta}}(t) = -\gamma\dot{\boldsymbol{\theta}}(t) - L\boldsymbol{\theta}(t) + \boldsymbol{P}, \tag{2}$$

where the Laplacian matrix $L = \{L_{ij}\}$, each entry $L_{ij} = -A_{ij} + \delta_{ij}\sum_j A_{ij}$ and δ_{ij} is the Kronecker delta. The Laplacian matrix L has the property that all of its rows sum to zero, which ensures it has one zero eigenvalue $\lambda_1 = 0$ with associated eigenvector $[1, 1, ..., 1]$ and the remaining eigenvalues $\lambda_2 \leq \lambda_3 \leq ....\lambda_N$ are all positive.Now we diagonalize L, $L = V\Lambda V^{-1}$, where $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$ is the matrix of the eigenvalues of L and $V = \{V_{ij}\}$ is the matrix of the eigenvectors of L. We multiply Eq. (2) on the left by V^{-1} and by calling $\boldsymbol{\eta}(t) = V^{-1}\boldsymbol{\theta}(t)$, and $\boldsymbol{Q} = V^{-1}\boldsymbol{P}$ we obtain,

$$\ddot{\boldsymbol{\eta}}(t) = -\gamma\dot{\boldsymbol{\eta}}(t) - \Lambda\boldsymbol{\eta}(t) + \boldsymbol{Q}, \tag{3}$$

which can be broken up in N independent equations,

$$\ddot{\eta}_i(t) = -\gamma\dot{\eta}_i(t) - \lambda_i\eta_i(t) + Q_i, \tag{4}$$

$i = 1, ..., N$. For $i = 1$, we obtain,

$$\ddot{\eta}_1(t) + \gamma\dot{\eta}_1(t) = 0, \tag{5}$$

indicating that $\eta_1(t)$ approaches a constant for large t. For $i > 1$, Eq. (4) is the equation of an underdamped ($0 < \zeta_i < 1$, $i = 2, ..., M$) or overdamped ($\zeta_i > 1$, $i = M + 1, ..., N$) second order system forced by a step function of amplitude Q_i. Overdamped systems don't give rise to overshoots, and so in what follows, we are going to neglect the indices $i > M$. For underdamped systems, we can write

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + {\omega_i}^2\eta_i(t) = Q_i, \tag{6}$$

where $\lambda_i = \omega_i^2$ and $\zeta_i = \gamma/(2\omega_i)$, $i = 2, ..., M$. The solution to (6) can be written as $\eta_i(t) = \bar{\eta}_i(t) + \tilde{\eta}_i(t)$, where $\bar{\eta}_i(t)$ is the free evolution and $\tilde{\eta}_i(t)$ is the forced evolution. The free evolution decays exponentially in time. Thus we can assume $\eta_i(t) \simeq \tilde{\eta}_i(t)$ and we can write:

$$\eta_i(t) = \frac{Q_i}{\omega_i^2}\Big[1 - \frac{e^{-\zeta_i \omega_i t}}{\sqrt{1-\zeta_i^2}} \sin\Big(\omega_i \sqrt{1-\zeta_i^2 t} - \cos^{-1}\zeta_i\Big)\Big], \tag{7}$$

which converges at steady state to

$$\eta_i^{ss} = \frac{Q_i}{\omega_i^2} \tag{8}$$

and for which the peak time can be computed

$$t_i = \frac{\pi}{\omega_i \sqrt{1-\zeta_i^2}} \tag{9}$$

and the peak

$$\eta_i^{\text{peak}} = \frac{Q_i}{\omega_i^2}\Big[1 + \exp(-\pi\zeta_i/\sqrt{1-\zeta_i^2})\Big]. \tag{10}$$

The larger is Q_i the higher is the peak. The smaller is $\omega_i^2 = \lambda_i$ the higher is the peak. Finally, the smaller is ζ_i, the higher is the peak. The eigenvalue λ_2 is associated with the smallest ω_2 and also the smallest ζ_2 and so is generically responsible for the largest peak.

Question: what if the λ_i are complex (case of asymmetric Laplacian)?

As we know each one of the $\eta_i(t)$, we can also compute each one of the $\theta_i(t)$, using the formula $\boldsymbol{\theta}(t) = V\boldsymbol{\eta}(t)$, or equivalently,

$$\theta_i(t) = \sum_j V_{ij}\eta_j(t), \tag{11}$$

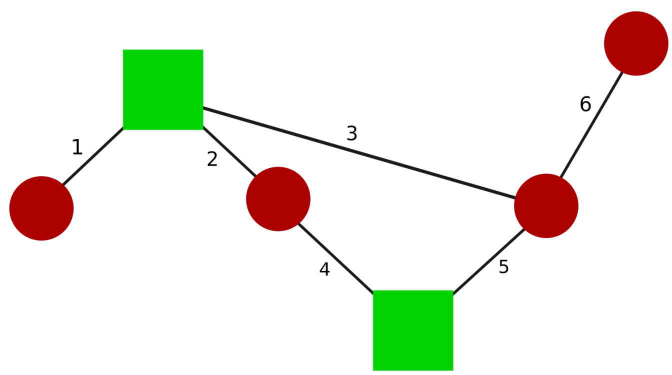
and it is important to note that the V_{ij} can be either positive or negative, and so are the Q_i.

Generically, the peak of $\theta_i(t)$ will be more strongly affected by $\eta_2(t)$, and then by $\eta_3(t)$, $\eta_4(t)$, etc., but it will also depend on the terms $V_{ij}Q_j$.

We can also compute the values of θ_i at steady state,

$$\theta_i^{ss} = \sum_j V_{ij}\eta_j^{ss} = \sum_j V_{ij}\frac{Q_j}{\omega_j^2} \tag{12}$$

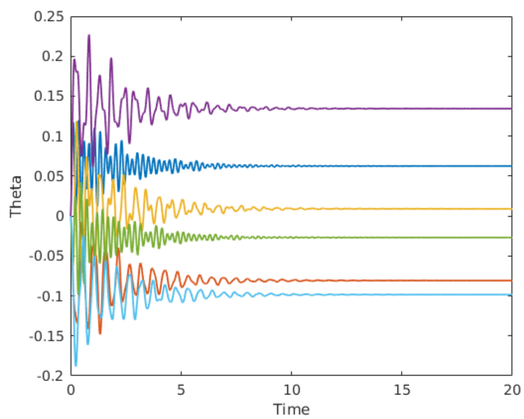
1 Example



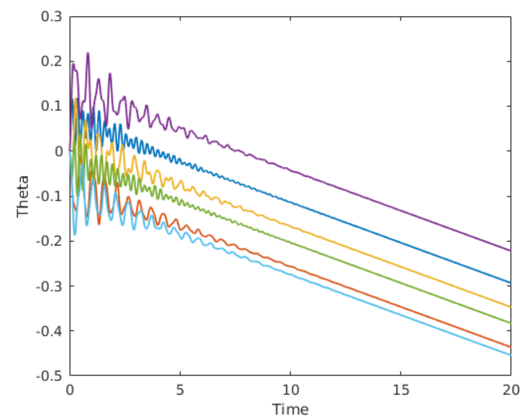
$$L=\begin{bmatrix} 420 & -140 & -140 & 0 & -140 & -140 \\ -140 & 140 & 0 & 0 & 0 & 0 \\ -140 & 0 & 280 & -140 & 0 & 0 \\ 0 & 0 & -140 & 280 & -140 & 0 \\ -140 & 0 & 0 & -140 & 420 & -140 \\ 0 & 0 & 0 & 0 & -140 & 140 \end{bmatrix}$$

Laplacian Matrix.

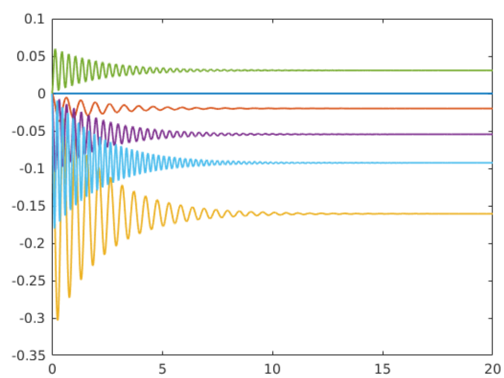
$$\text{V}=\underset{\text{Eigenvectors}}{\begin{bmatrix} -0.408 & -0.227 & 0.0 & 0.290 & 0.577 & -0.603 \\ -0.408 & -0.662 & 0.5 & -0.190 & -0.289 & 0.158 \\ -0.408 & -0.097 & -0.5 & 0.616 & -0.289 & 0.332 \\ -0.408 & 0.097 & -0.5 & -0.616 & -0.289 & -0.332 \\ -0.408 & 0.227 & 0.0 & -0.290 & 0.577 & 0.603 \\ -0.408 & 0.662 & 0.5 & 0.190 & -0.289 & -0.158 \end{bmatrix}} \quad \lambda=\underset{\text{EigenValues}}{\begin{bmatrix} 0 \\ 91.99 \\ 140.00 \\ 354.10 \\ 420.00 \\ 637.90 \end{bmatrix}}$$



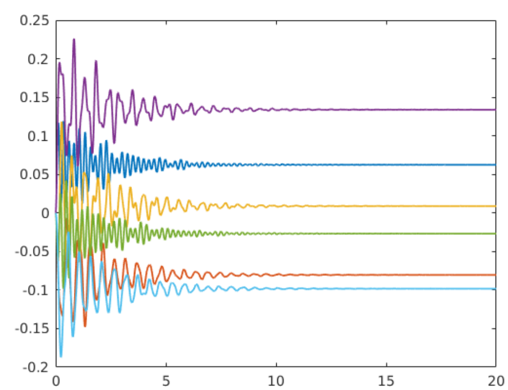
(a) Non-Linear Theta



(b) Linear Theta



(c) Eta



(d) Eta to Theta

Figure 1: Plots

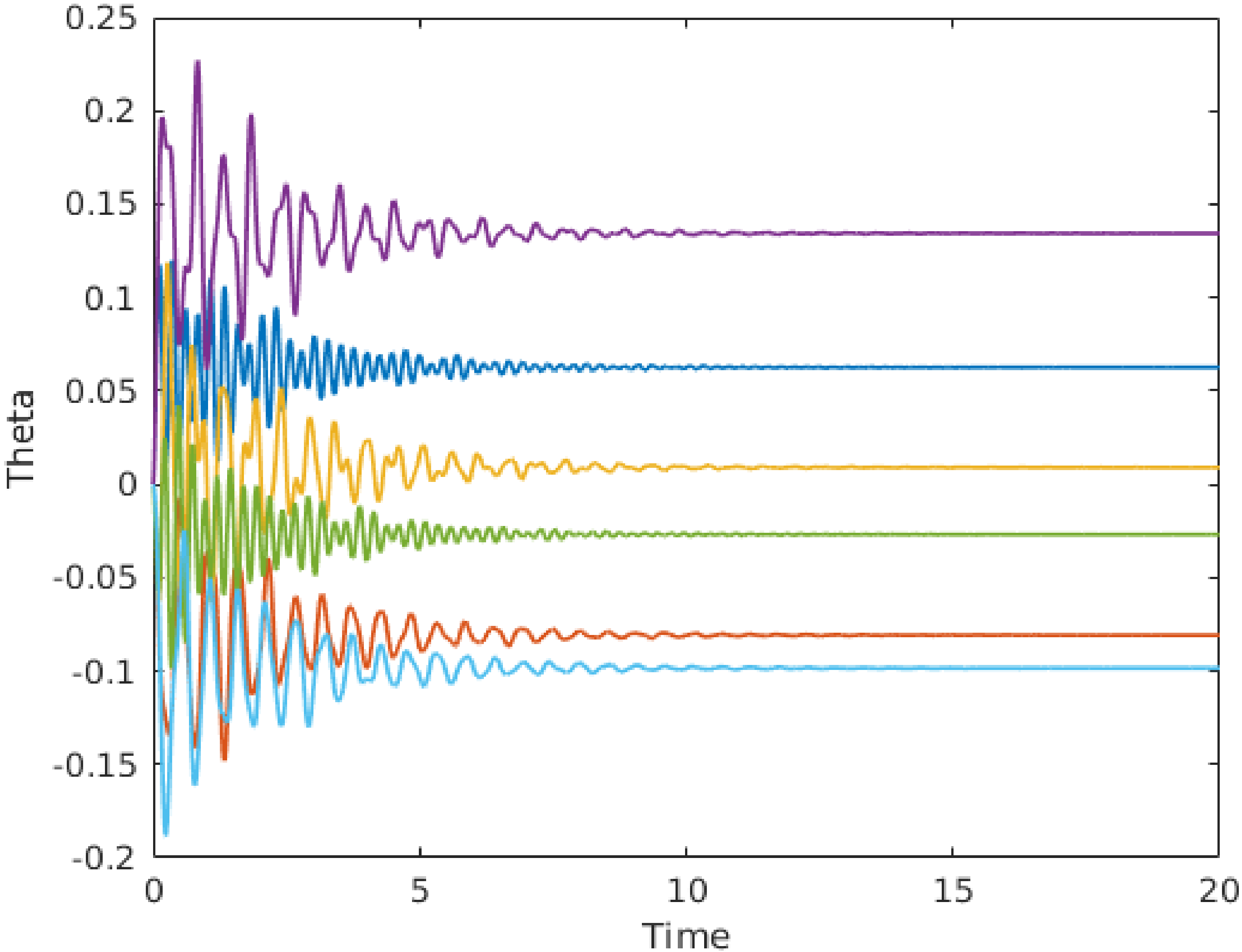


Figure 2: Non-linear Theta

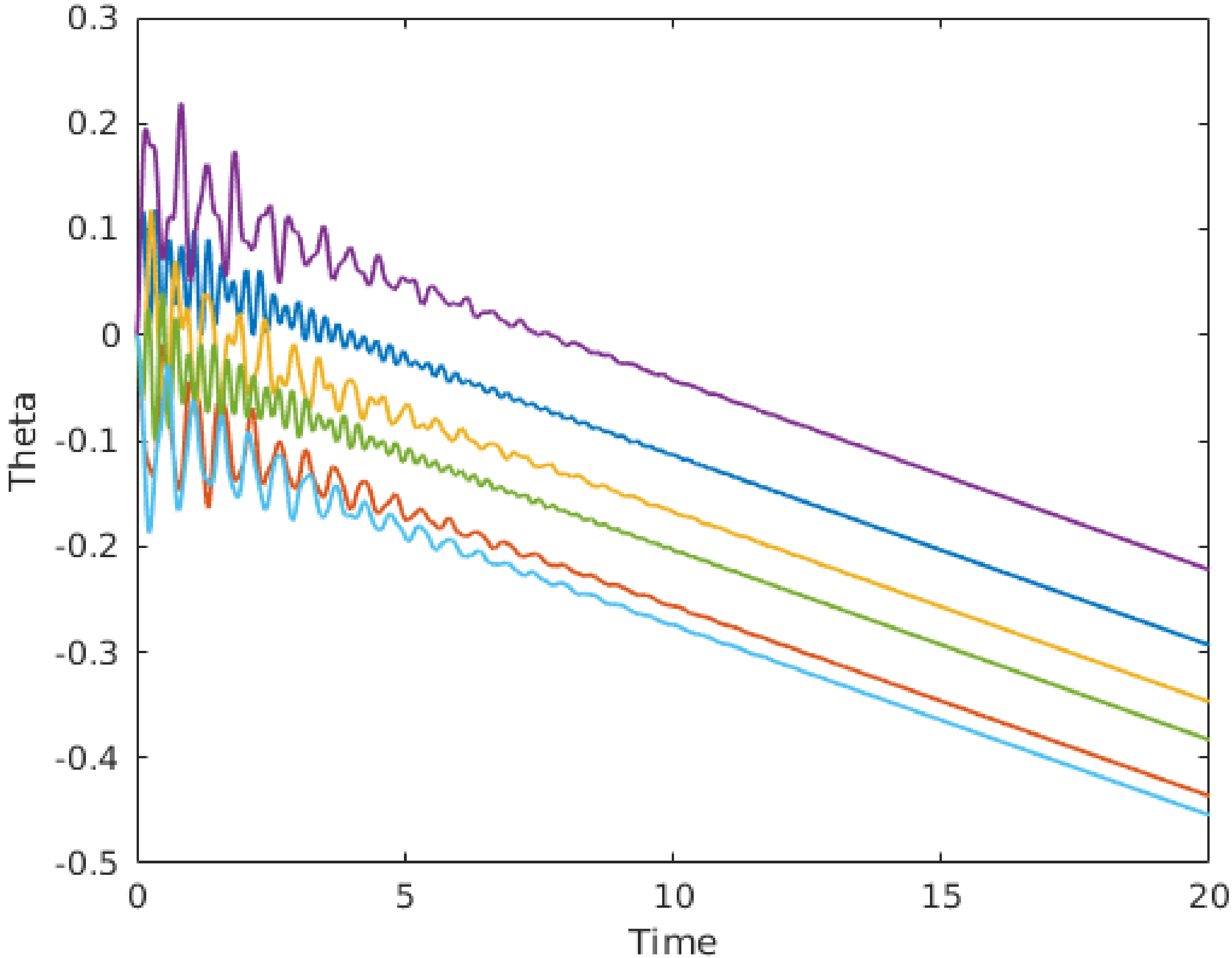


Figure 3: Linear Theta

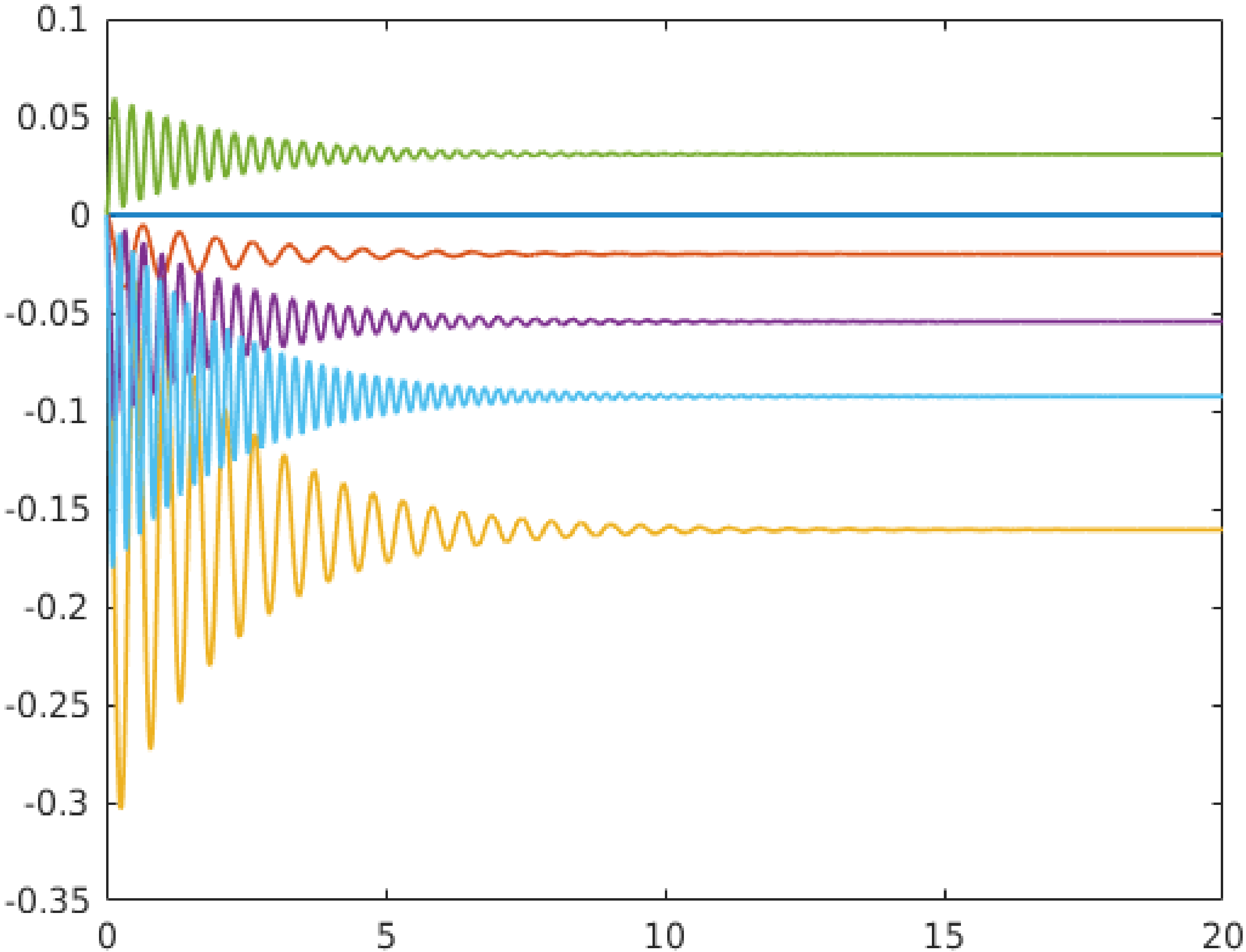


Figure 4: Eta

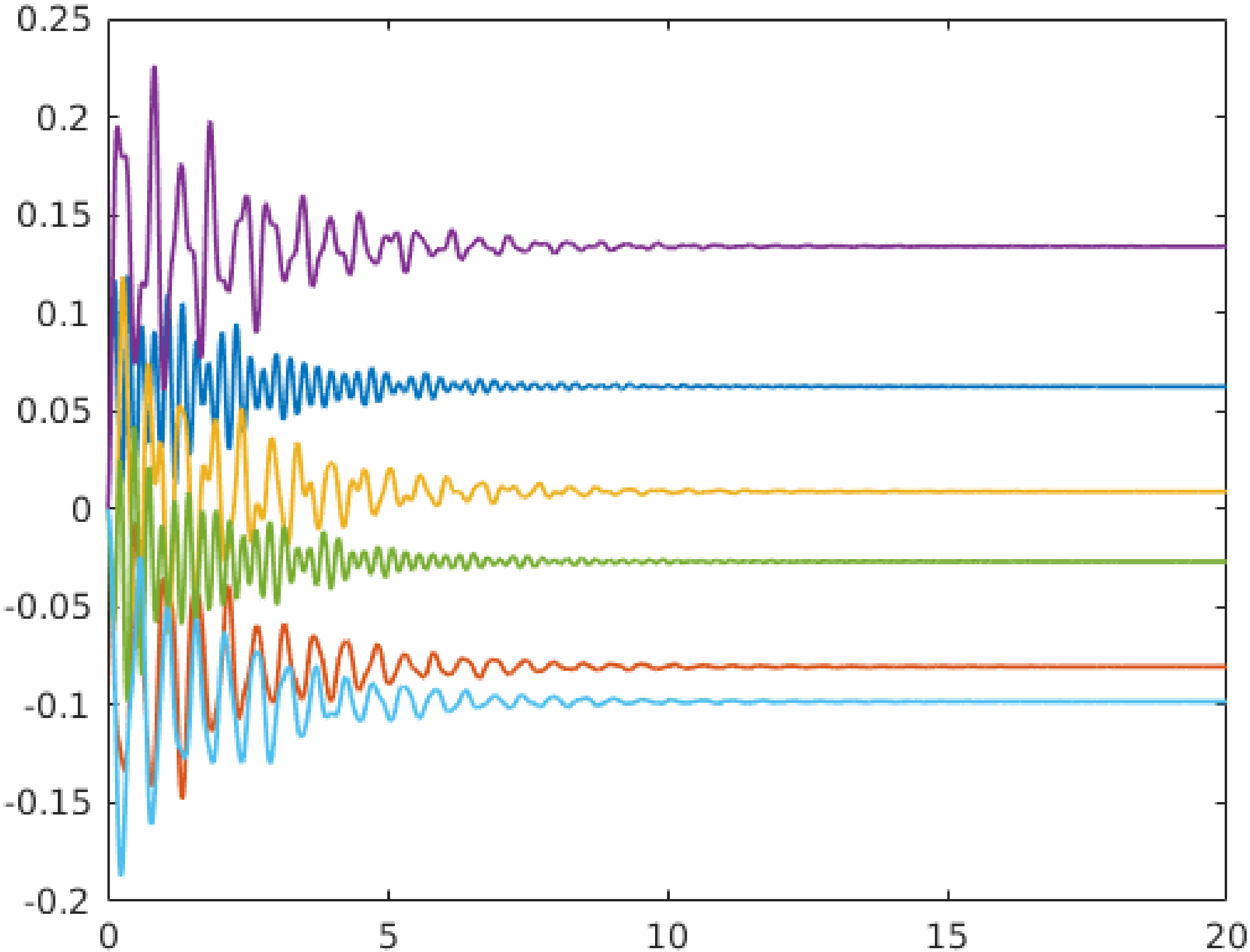


Figure 5: Eta To Theta

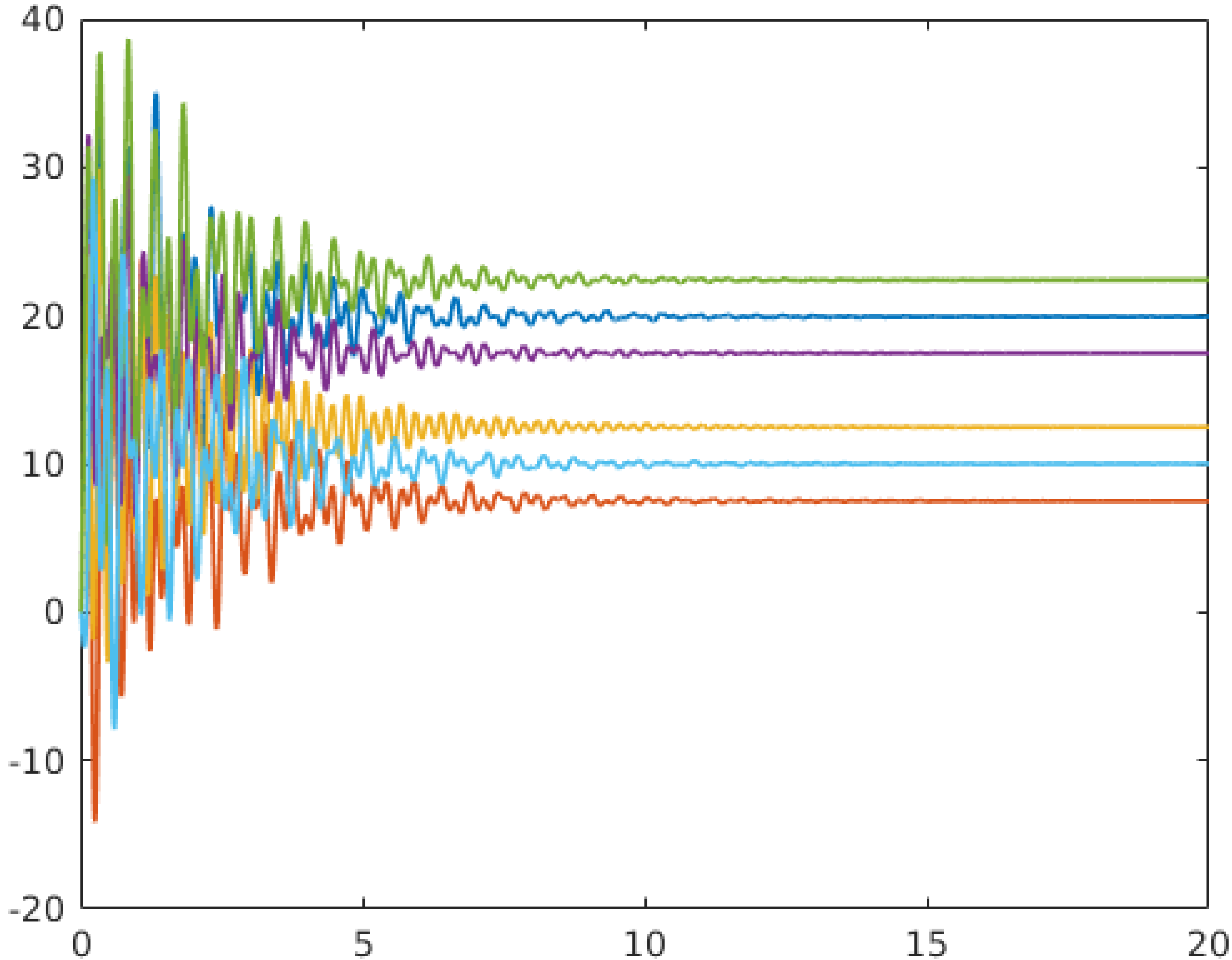


Figure 6: Eta Power

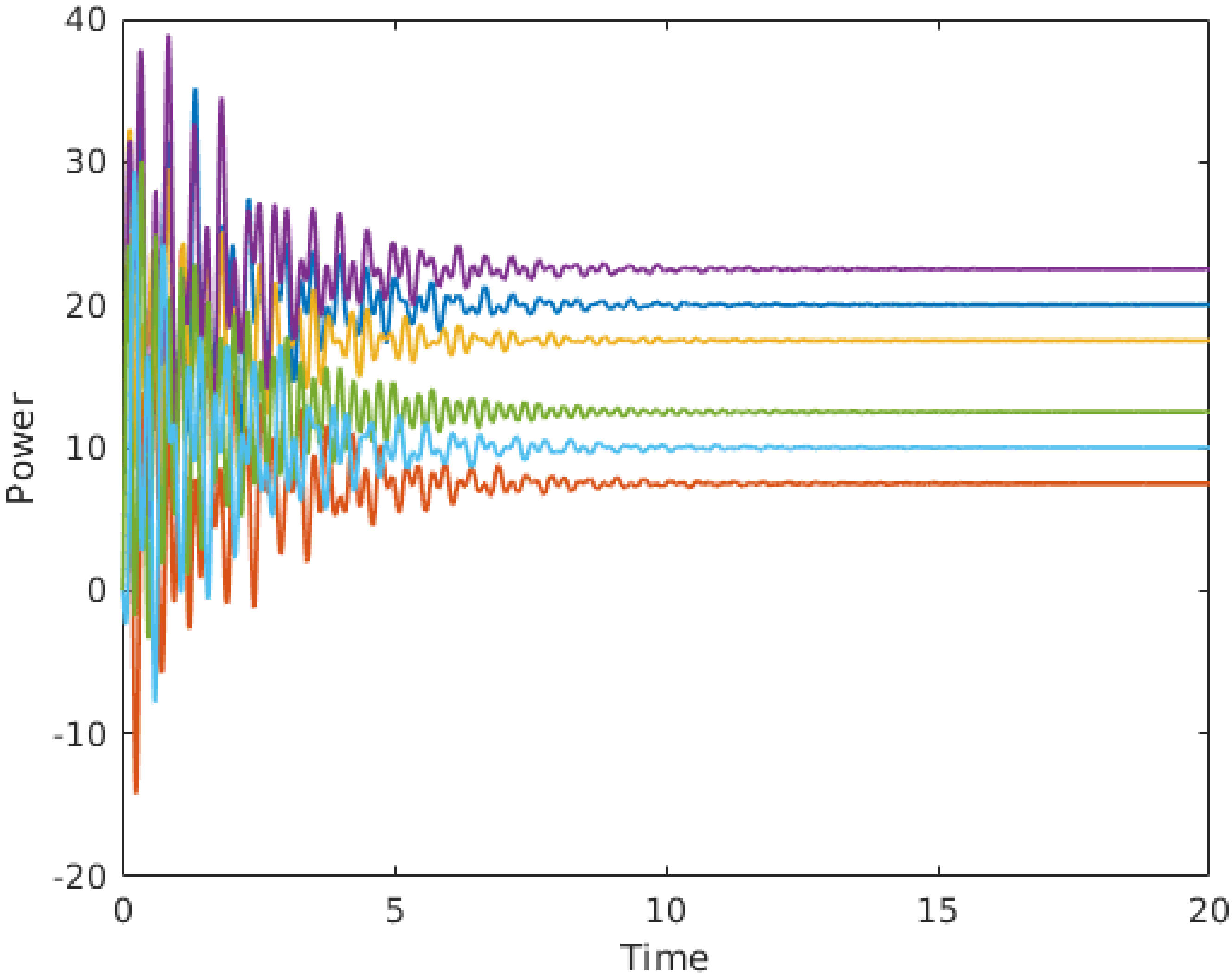


Figure 7: Theta power