Computational Graphics: Lecture 7

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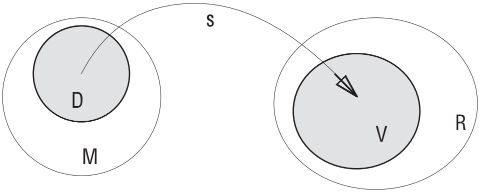
Outline: LAR1

Solid Modeling

Representation scheme: definition

mapping $s : M \rightarrow R$ from a space M of mathematical models to a space R of computer representations



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A.~Requicha, Representations for Rigid Solids: Theory, Methods, and Systems, ACM Comput. Surv., 1980.

V. Shapiro, Solid Modeling, In Handbook of Computer Aided Geometric Design, 2001

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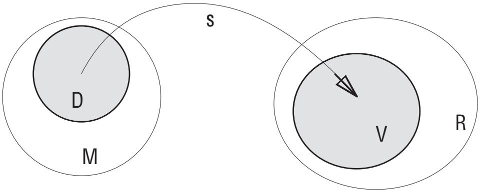

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2. The R set contains the symbolic representations, i.e. the proper data structures, built according to a suitable grammar

A.~Requicha, Representations for Rigid Solids: Theory, Methods, and Systems, ACM Comput. Surv., 1980.

V. Shapiro, Solid Modeling, In Handbook of Computer Aided Geometric Design, 2001

Representation schemes

Most of such papers introduce or discuss one or more representation schemes . . .

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and much more . . .

Space decompositions

Join of pointsets

The join of two sets $P, Q \subset \mathbb{E}^n$ is the convex hull of their points:

$$PQ = join(P, Q) := \{\gamma p + \lambda q,\ p \in P,\ q \in Q\}$$
$$\gamma, \lambda \in \mathbb{R},\ \gamma, \lambda \geq 0,\ \gamma + \lambda = 1$$

.

The join operation is associative and commutative.

Join of pointsets: examples

```
pts = [[0,0],[.5,0],[0,.5],[.5,.5],
       [1,.5],[1.5,.5],[1.5,1],[.25,1]]        # coords
P = AA(MK)(pts)                                # 0-polyhedra
S = AA(JOIN)([P[0:4],P[4:7],P[7]])             # array of d-polyhedra
H = JOIN(S)                                    # 2-polyhedron

VIEW(STRUCT(AA(SKELETON(1))(S)))
VIEW(H)
```

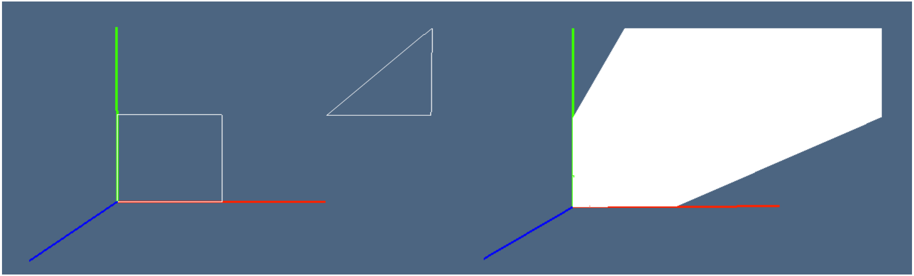

Figure : (a) 1-skeleton of pointsets in S; (b) convex hull H of pointset P

Simplex

A simplex $\sigma \subset \mathbb{E}^n$ of order d, or d-simplex, is the join of $d + 1$ affinely independent points, called vertices.

The $n + 1$ points $p_0, \ldots, p_n$ are affinely independent when the n vectors $p_1 - p_0, \ldots, p_n - p_0$ are linearly independent.

A d-simplex can be seen as a d-dimensional triangle: 0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, 3-simplex is a tetrahedron, and so on.

Simplex: examples

```
s0,s1,s2,s3 = [SIMPLEX(d) for d in range(4)]    # array of standard d-simplices
VIEW(s1); VIEW(s2); VIEW(s3);

points = [[1,1,1],[0,1,1],[1,0,0],[1,1,0]]      # coords of 4 points
tetra = JOIN(AA(MK)(points))                     # 3-simplex
VIEW(tetra)
```

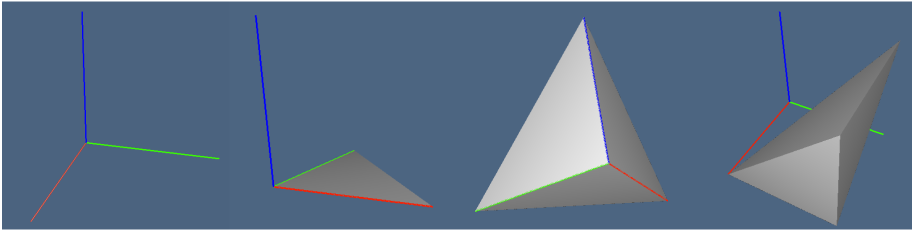

Figure : (a,b,c) 1-, 2-, and 3-standard simplex; (d) 3-simplex defined by 4 points

Simplicial complex

Any subset of $s+1$ vertices $(0 \geq s \geq d)$ of a d-simplex σ defines an s-simplex, which is called s-face of σ.

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Geometric carrier $|\Sigma|$ is the pointset union of simplices in Σ.

Simplicial complex: examples

```
from larcc import *                                  # import LAR library
V,CV = larSimplexGrid([5,5,5])                       # structured simplicial grid
FV = larSimplexFacets(CV)                            # 2-simplicial grid
EV = larSimplexFacets(FV)                            # 1-simplicial grid
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,CV))))
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,FV))))
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,EV))))

BV = [FV[t] for t in boundaryCells(CV,FV)]           # boundary 2-simplices
VIEW(EXPLODE(1.5,1.5,1.5)(MKPOLS((V,BV))))
```

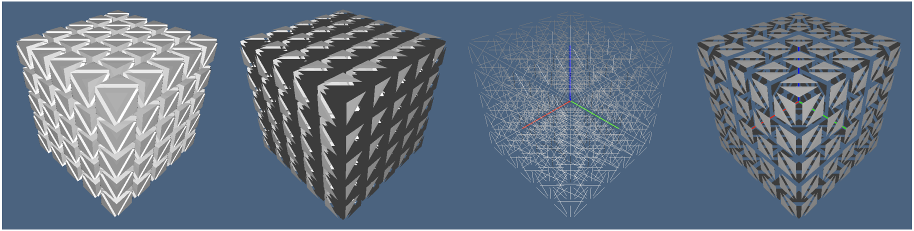

Figure : (a) 3-complex; (b) 2-subcomplex; (c) 1-subcomplex; (d) 2-boundary.

Simplicial complex: examples

(see Disk Point Picking)

```
from larcc import *; from random import random as rand
points = [[2*PI*rand(),rand()] for k in range(1000)]
V = [[SQRT(r)*COS(alpha),SQRT(r)*SIN(alpha)] for alpha,r in points]
cells = [[k+1] for k,v in enumerate(V)]
VIEW(MKPOL([V,cells,None]))

from scipy.spatial import Delaunay
FV = Delaunay(array(V)).vertices
VIEW(EXPLODE(1.2,1.2,1)(MKPOLS((V,FV))))
VIEW(SKELETON(1)(STRUCT(MKPOLS((V,FV)))))
```

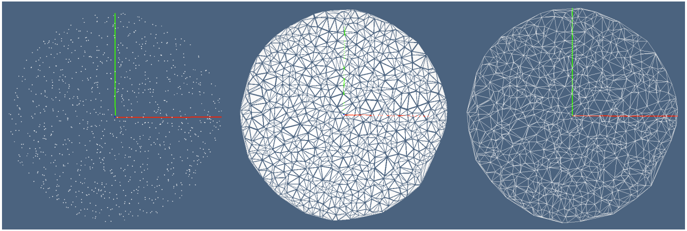

Figure : (a) Points; (b) Delaunay triangulation; (c) 1-skeleton.

Simplicial complex: examples (see Disk Point Picking)

```
from larcc import *; from random import random as rand
points = [[2*rand()-1,2*rand()-1,2*rand()-1] for k in range(30000)]
V = [p for p in points if VECTNORM(p) <= 1]
VIEW(STRUCT(MKPOLS((V,AA(LIST)(range(len(V)))))))
```

```
from scipy.spatial import Delaunay
CV = Delaunay(array(V)).vertices
def test(tetra): return AND([v[-1] < 0 for v in tetra])
CV = [cell for cell in CV if test([V[v] for v in cell]) ]
VIEW(STRUCT(MKPOLS((V,CV))))
```

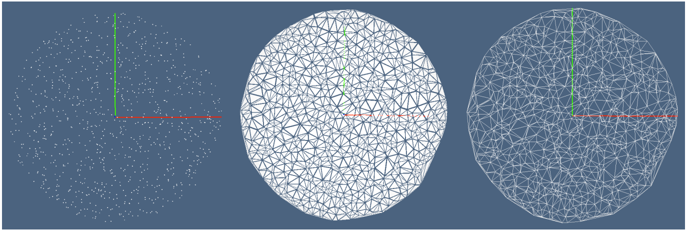

Figure : (a) Points; (b) Delaunay triangulation; (c) 1-skeleton

Cellular complex

CW-complex

- Roughly speaking, a CW-complex is made of basic building blocks called cells.

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n-Dimensional disk:

$$D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Interior of $D^n \subset \mathbb{R}^n$:

$$int(D^n) = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Boundary of $D^n \subset \mathbb{R}^n$:

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$$

CW-complex: inductive definition

Let X be a topological space.

Let $\Lambda(X) = \cup_p X_{p \in \mathbb{N}}$ be a partition of X, with X_p the set of p-cells.

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Definition (CW-structure)

A CW-structure on the space X is a filtration

$$\emptyset = X^{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = \Lambda(X),$$

such that, for each n, the space X_n is homeomorphic to a space obtained from X_{n-1} by attachment of n-cells.

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A cellular complex is a space endowed with a CW-structure.

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Definition (Cellular complex)

A cellular complex is a space endowed with a CW-structure.

A cellular complex is finite when it contains a finite number of cells.

Politopal, simplicial, cuboidal complexes

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2. In simplicial complexes cells are simplices, i.e.˜d-polyedra with $d+1$ facets ($(d-1)$-faces) and $d+1$ vertices.
3. In cuboidal complexes cells are cuboids, (in general, sets homeomorphic to) Cartesian products of intervals, i.e.˜d-polyedra with $2d$ facets and $2d$ vertices.

Numbers of vertices and facets

A d-simplex, or d-dimensional simplex, has $d+1$ extremal points called vertices and $d+1$ facets.

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A d-cuboid has conversely 2^d vertices and $2d$ facets:

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- a quadrilateral (2-cuboid) has $2^2=4$ vertices and 4 facets;
- a hexahedron (3-cuboids) has $2^3=8$ vertices and 8 facets, etc.

Cellular complex: other definitions

Support $|K|$ of a cellular complex K is the union of points of its cells

A triangulation of a polytope P is a simplicial complex K whose support is $|K| = P$

For example, a triangulation of a polygon is a subdivision in triangles

Simplices and cuboids are polytopes.

A polytope is always triangulable;

For example, a quadrilateral by be divided in two triangles, and a cube in either 5 or 6 tetrahedra without adding new vertices

Simplicial mapping

Simplicial mapping: definition

Definition

A simplicial map is a map between simplicial complexes with the property that the images of the vertices of a simplex always span a simplex.

Remarks

Simplicial maps are determined by their effects on vertices
for a precise definition of Simplicial Map look at Wolfram MathWorld

MAP operator in plasm

Map operator

`MAP(fun)(domain)`

Semantics

1. domain (HPC value) is decomposed into a simplicial complex

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Semantics

1. domain (HPC value) is decomposed into a simplicial complex
2. fun (a simplicial function) is applied to the domain vertices
3. the mapped domain is returned

MAP examples: 1-sphere (S^1) and 2-disk (D^2)

```
def sphere1(p): return [COS(p[0]), SIN(p[0])] # point function
def domain(n): return INTERVALS(2*PI)(n)      # generator of domain decomp
VIEW( MAP(sphere1)(domain(32)) )              # geometric value (HPC type)

def disk2D(p):                                           # point function
    u,v = p
    return [v*COS(u), v*SIN(u)]                          # coordinate functions
domain2D = PROD([INTERVALS(2*PI)(32), INTERVALS(1)(3)]) # 2D domain decompos
VIEW( MAP(disk2D)(domain2D) )
VIEW( SKELETON(1)(MAP(disk2D)(domain2D)) )
```

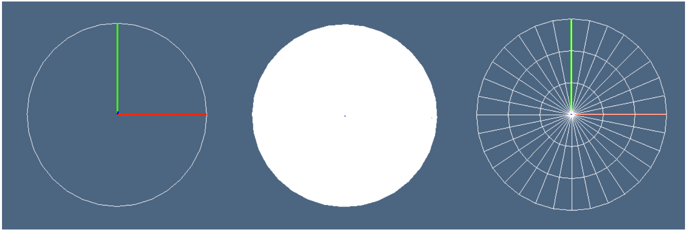

Figure : (a) sphere S^1 (b) disk D^2; (c) 1-skeleton.

References

References

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