Computational Graphics: Lecture 4

The Cvdlab Team

Mon, Mar 10, 2014

Outline: Algebra1

1. Linear spaces
2. Linear combinations
3. Subspaces
4. Spans
5. Bases

Linear spaces

Definition

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8. $1 \cdot \mathbf{v} = \mathbf{v}$. (neutral element of product)

Example: vector space of real matrices

Let $\mathcal{M}_n^m(\mathbb{R})$ be the set of $m \times n$ matrices with elements in the field $\mathbb{R}$. An element A in such a set is denoted as

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Addition and multiplication by a scalar are defined component-wise:

$$A + B = (\alpha_{ij}) + (\beta_{ij}) = (\alpha_{ij} + \beta_{ij})$$

$$\gamma A = \gamma(\alpha_{ij}) = (\gamma \alpha_{ij})$$

Linear combinations

Linear combination

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathcal{V}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$,

The vector

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \in \mathcal{V}$$

is called a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$

Subspaces

Subspace

Let $(\mathcal{V}, +, \cdot)$ be a vector space on the field $\mathcal{F}$.

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$\mathcal{U} \neq \emptyset$;

for each $\alpha \in \mathcal{F}$ and $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$, $\alpha \mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U}$

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Question

Examples of codimension? in 1D, 2D, 3D

Spans

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- If a subspace $\mathcal{U}$ of $\mathcal{V}$ can be generated as the span of a set S of vectors in $\mathcal{V}$, then S is called a generating set or a spanning set for $\mathcal{U}$.

Linear independence

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if

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  implies that $\alpha_i = 0$ for each i
- As a consequence, a set of vectors is linearly independent when none of them belongs to the span of the others.

Bases

Bases and coordinates

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- each element of the space can be represented uniquely as linear combination of basis elements
- this leads to a parametrization of the space, i.e. to represent each element by a sequence of scalars, called its coordinates with respect to the chosen basis.

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2. $\mathcal{V} = \mathrm{lin}\,\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

Bases

- Every two bases of $\mathcal{V}$ have the same number of elements, that is called the dimension of $\mathcal{V}$ and is denoted

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  3. each linearly independent set of vectors is contained in a basis;
  4. each maximal set of linearly independent vectors is a basis;

Example: vector space of polynomials of degree $\leq n$

A linear space we make often use of in Computer Graphics and Geometric modeling is the space of dimension $n + 1$:

$$\mathbb{P}^n(\mathbb{R}) = \{p : \mathbb{R} \to \mathbb{R} : u \mapsto \sum_{i=1}^{n} a_i p^i, a_i \in \mathbb{R}\}$$

of univariate polynomials of degree $\leq n$ on the real field (with real coefficients), with $p^i \in P_n$, where

$$P_n = (p^n, p^{n-1}, ..., p^1, p^0) \quad \text{and} \quad p^i : u \mapsto u^i$$

is the power basis.

Components

If $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is an ordered basis for $\mathcal{V}$, then for each $\mathbf{v} \in \mathcal{V}$ there exists a unique n-tuple of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$ such that

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i.$$

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The n-tuple of scalars (α_i) is called the components of $\mathbf{v}$ with respect to the ordered basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.

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- But this one would imply $\sum(\alpha_i - \beta_i)\mathbf{e}_i = \mathbf{0}$, hence $(\alpha_i - \beta_i) = 0$,
- i.e. $\alpha_i = \beta_i$, for every i.

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- and hence:

$$[T] = [V]^{-1}$$

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- the $[B_3]$ matrix in the P_3 basis is

$$[B_3]_{P_3} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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- WHY ?

Example

```
# Open a file
fo = open("foo.txt", "r+")
str = fo.read(10);
print "Read String is : ", str

# Check current position
position = fo.tell();
print "Current file position : ", position

# Reposition pointer at the beginning once again
position = fo.seek(0, 0);
str = fo.read(10);
print "Again read String is : ", str
# Close opend file
fo.close()
```

References

Linear Algebra Done Right book

NumPy tutorial