Polyhedral geometry 2

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Affine spaces

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Affine space

The idea of affine space corresponds to that of a set of points where the displacement from a point $\mathbf{x}$ to another point $\mathbf{y}$ is obtained by summing a vector $\mathbf{v}$ to the $\mathbf{x}$ point.

Definition

A set $\mathcal{A}$ of points is called an affine space modeled on the vector space $\mathcal{V}$ if there is a function

$$\mathcal{A} \times \mathcal{V} \to \mathcal{A} : (\mathbf{x}, \mathbf{v}) \mapsto \mathbf{x} + \mathbf{v}$$

called affine action, with the properties:

1. $(\mathbf{x} + \mathbf{v}) + \mathbf{w} = \mathbf{x} + (\mathbf{v} + \mathbf{w})$ for each $\mathbf{x} \in \mathcal{A}$ and each $\mathbf{v}, \mathbf{w} \in \mathcal{V}$;
2. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in \mathcal{A}$, where $\mathbf{0} \in \mathcal{V}$ is the null vector;
3. for each pair $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ there is a unique $(\mathbf{y} - \mathbf{x}) \in \mathcal{V}$ such that

$$\mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}.$$

Dimension

The affine space $\mathcal{A}$ is said of dimension n if modeled on a vector space $\mathcal{V}$ of dimension n.

Vector sum vs affine action

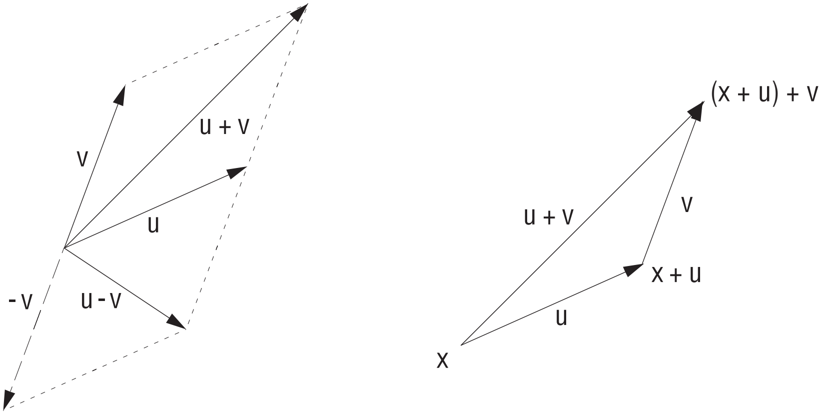

Figure : (a) Vector sum and difference are given by the parallelogram rule (b) associativity of displacement (point and vector sum) in an affine space

Operations on vectors and points

- The addition of vectors is a primitive operation in a vector space.
- The difference of vectors is defined through the two primitive operations:
$$\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1 + (-1)\mathbf{v}_2.$$
- Addition and difference of vectors are geometrically produced by the parallelogram rule
- notice also the associative property of the affine action on a point space.

Operations on vectors and points

The sum of a set $\{\mathbf{v}_i\}$ of vectors $(i = 1, \ldots, n)$ can be geometrically obtained, in an affine space:

- by setting $\mathbf{p}_0 = \mathbf{0}$
- $\mathbf{p}_i = \mathbf{p}_{i-1} + \mathbf{v}_i$,
- so that

$$\sum_i \mathbf{v}_i = \mathbf{p}_n - \mathbf{p}_0$$

Remark

1. the addition of points is not defined;
2. the difference of two points is a vector;
3. the sum of a point and a vector is a point.

Positive, affine and convex combinations

Three types of combinations of vectors or points can be defined. They lead to the concepts of cones, hyperplanes and convex sets, respectively.

Positive combination

Let $\mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}^+ \cup \{0\}$.

The vector

$$\alpha_0 \mathbf{v}_0 + \cdots + \alpha_d \mathbf{v}_d = \sum_{i=0}^{d} \alpha_i \mathbf{v}_i$$

is called a positive combination of such vectors.

The set of all the positive combinations of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ is called the positive hull of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ and denoted $\operatorname{pos}\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$.

This set is also called the cone generated by the given vectors

Affine combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$, such that $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\sum_{i=0}^{d} \alpha_i \mathbf{p}_i := \mathbf{p}_0 + \sum_{i=1}^{d} \alpha_i (\mathbf{p}_i - \mathbf{p}_0)$$

is called an affine combination of the points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

Affine combination

The set of all affine combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is an affine subspace, denoted by $\operatorname{aff}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$

It is easy to verify that:

$$\operatorname{aff}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\} = \mathbf{p}_0 + \operatorname{lin}\{\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_d - \mathbf{p}_0\}.$$

Affine combination

1. The dimension of an affine subspace is the dimension of the corresponding linear vector space.
2. Affine subspaces of $\mathbb{E}^d$ with dimensions 0, 1, 2 and $d-1$ are called points, lines, planes and hyperplanes, respectively.
3. Affine subspaces are also called flats.

Double description

Every affine subspace can be described either as

- the intersection of affine hyperplanes, or as
- the affine hull of a finite set of points.

Convex combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \geq 0$, with $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\alpha_0 \mathbf{p}_0 + \cdots + \alpha_d \mathbf{p}_d = \sum_{i=0}^{d} \alpha_i \mathbf{p}_i$$

is called a convex combination of points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

A convex combinations is both positive and affine.

Convex hull

The set of all convex combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is a convex set, called convex hull of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$, and is denoted by $\operatorname{conv}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$.

Properties

- the convex hull of a set of points is the intersection of all convex sets that contain them
- the convex hull of a set of points is the smallest set that contains them