Computational Graphics: Lecture 5

The CVDlab Team

Tue, Mar 11, 2014

Outline: Algebra2

1. Affine spaces
2. Affine combinations
3. Convex combinations

Affine spaces

Affine space

The idea of affine space corresponds to that of a set of points where the displacement from a point $\mathbf{x}$ to another point $\mathbf{y}$ is obtained by summing a vector $\mathbf{v}$ to the $\mathbf{x}$ point.

Definition

A set $\mathcal{A}$ of points is called an affine space modeled on the vector space $\mathcal{V}$ if there is a function

$$\mathcal{A} \times \mathcal{V} \to \mathcal{A} : (\mathbf{x}, \mathbf{v}) \mapsto \mathbf{x} + \mathbf{v}$$

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1. $(\mathbf{x} + \mathbf{v}) + \mathbf{w} = \mathbf{x} + (\mathbf{v} + \mathbf{w})$ for each $\mathbf{x} \in \mathcal{A}$ and each $\mathbf{v}, \mathbf{w} \in \mathcal{V}$;

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2. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in \mathcal{A}$, where $\mathbf{0} \in \mathcal{V}$ is the null vector;
3. for each pair $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ there is a unique $(\mathbf{y} - \mathbf{x}) \in \mathcal{V}$ such that

$$\mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}.$$

Dimension

The affine space $\mathcal{A}$ is said of dimension n if modeled on a vector space $\mathcal{V}$ of dimension n.

Vector sum vs affine action

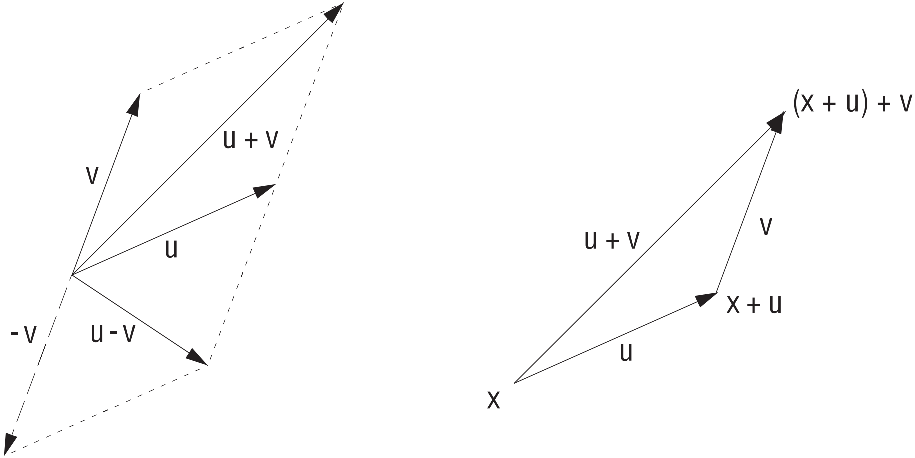

Figure : (a) Vector sum and difference are given by the parallelogram rule (b) associativity of displacement (point and vector sum) in an affine space

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- Addition and difference of vectors are geometrically produced by the parallelogram rule
- notice also the associative property of the affine action on a point space.

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Remark: operations on points

1. the addition of points is not defined;
2. the difference of two points is a vector;
3. the sum of a point and a vector is a point.

Affine combinations

Positive, affine and convex combinations

Three types of combinations of vectors or points can be defined. They lead to the concepts of cones, hyperplanes and convex sets, respectively.

Positive combination

Let $\mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}^+ \cup \{0\}$.

The vector

$$\alpha_0 \mathbf{v}_0 + \cdots + \alpha_d \mathbf{v}_d = \sum_{i=0}^{d} \alpha_i \mathbf{v}_i$$

is called a positive combination of such vectors.

The set of all the positive combinations of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ is called the positive hull of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ and denoted $\mathrm{pos}\,\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$.

This set is also called the cone generated by the given vectors

Affine combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$, such that $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\sum_{i=0}^{d} \alpha_i \mathbf{p}_i := \mathbf{p}_0 + \sum_{i=1}^{d} \alpha_i (\mathbf{p}_i - \mathbf{p}_0)$$

is called an affine combination of the points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

Affine combination

The set of all affine combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is an affine subspace, denoted by $\mathrm{aff}\,\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$

It is easy to verify that:

$$\mathrm{aff}\,\{\mathbf{p}_0, \ldots, \mathbf{p}_d\} = \mathbf{p}_0 + \mathrm{lin}\,\{\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_d - \mathbf{p}_0\}.$$

Affine combination

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Double description

Every affine subspace can be described either as

- the intersection of affine hyperplanes, or as
- the affine hull of a finite set of points.

Convex combinations

Convex combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \geq 0$, with $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\alpha_0 \mathbf{p}_0 + \cdots + \alpha_d \mathbf{p}_d = \sum_{i=0}^{d} \alpha_i \mathbf{p}_i$$

is called a convex combination of points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

A convex combinations is both positive and affine.

Convex hull

The set of all convex combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is a convex set, called convex hull of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$, and is denoted by $\mathrm{conv}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$.

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Properties

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Properties

- the convex hull of a set of points is the intersection of all convex sets that contain them
- the convex hull of a set of points is the smallest set that contains them

EXERCISE

- Produce (and draw) 100 random points within the unit square $[0, 1]^2$;

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- Produce (and draw) 100 random points within the unit square $[0,1]^2$;
- Produce (and draw) 1000 random points within S_1, the 1D sphere (circle) of unit radius centered at the origin $(0,0)$;

References

GP4CAD book