Polyhedral geometry 2

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Computational Graphics 2012

Section 1

Affine spaces

Affine space

The idea of affine space corresponds to that of a set of points where the displacement from a point $\mathbf{x}$ to another point $\mathbf{y}$ is obtained by summing a vector $\mathbf{v}$ to the $\mathbf{x}$ point.

Definition

A set $\mathcal{A}$ of points is called an affine space modeled on the vector space $\mathcal{V}$ if there is a function

$$\mathcal{A} \times \mathcal{V} \to \mathcal{A} : (\mathbf{x}, \mathbf{v}) \mapsto \mathbf{x} + \mathbf{v}$$

called affine action, with the properties:

1. $(\mathbf{x} + \mathbf{v}) + \mathbf{w} = \mathbf{x} + (\mathbf{v} + \mathbf{w})$ for each $\mathbf{x} \in \mathcal{A}$ and each $\mathbf{v}, \mathbf{w} \in \mathcal{V}$;

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2. $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in \mathcal{A}$, where $\mathbf{0} \in \mathcal{V}$ is the null vector;
3. for each pair $\mathbf{x}, \mathbf{y} \in \mathcal{A}$ there is a unique $(\mathbf{y} - \mathbf{x}) \in \mathcal{V}$ such that

$$\mathbf{x} + (\mathbf{y} - \mathbf{x}) = \mathbf{y}.$$

Dimension

The affine space $\mathcal{A}$ is said of dimension n if modeled on a vector space $\mathcal{V}$ of dimension n.

Vector sum vs affine action

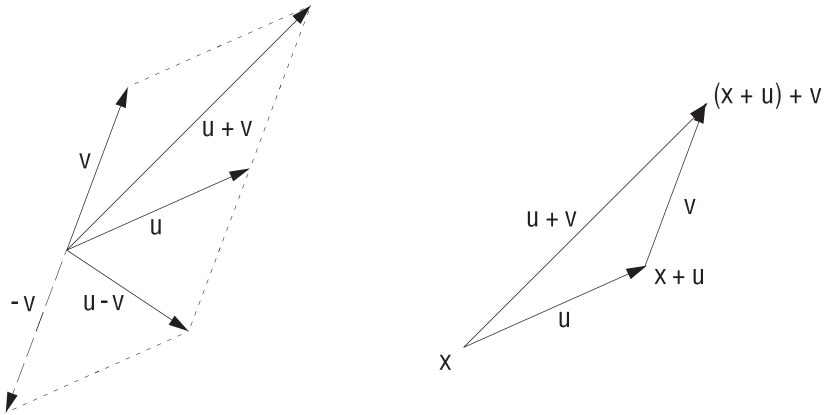

Figure: (a) Vector sum and difference are given by the parallelogram rule (b) associativity of displacement (point and vector sum) in an affine space

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- Addition and difference of vectors are geometrically produced by the parallelogram rule
- notice also the associative property of the affine action on a point space.

Operations on vectors and points

The sum of a set $\{\mathbf{v}_i\}$ of vectors $(i = 1, \ldots, n)$ can be geometrically obtained, in an affine space:

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Remark

1. the addition of points is not defined;
2. the difference of two points is a vector;
3. the sum of a point and a vector is a point.

Positive, affine and convex combinations

Three types of combinations of vectors or points can be defined. They lead to the concepts of cones, hyperplanes and convex sets, respectively.

Positive combination

Let $\mathbf{v}_0, \ldots, \mathbf{v}_d \in \mathbb{R}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}^+ \cup \{0\}$.

The vector

$$\alpha_0 \mathbf{v}_0 + \cdots + \alpha_d \mathbf{v}_d = \sum_{i=0}^{d} \alpha_i \mathbf{v}_i$$

is called a positive combination of such vectors.

The set of all the positive combinations of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ is called the positive hull of $\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$ and denoted $\mathrm{pos}\,\{\mathbf{v}_0, \ldots, \mathbf{v}_d\}$.

This set is also called the cone generated by the given vectors

Affine combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$, such that $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\sum_{i=0}^{d} \alpha_i \mathbf{p}_i := \mathbf{p}_0 + \sum_{i=1}^{d} \alpha_i (\mathbf{p}_i - \mathbf{p}_0)$$

is called an affine combination of the points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

Affine combination

The set of all affine combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is an affine subspace, denoted by $\operatorname{aff}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$

It is easy to verify that:

$$\operatorname{aff}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\} = \mathbf{p}_0 + \operatorname{lin}\{\mathbf{p}_1 - \mathbf{p}_0, \ldots, \mathbf{p}_d - \mathbf{p}_0\}.$$

Affine combination

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Double description

Every affine subspace can be described either as

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Double description

Every affine subspace can be described either as

- the intersection of affine hyperplanes, or as
- the affine hull of a finite set of points.

Convex combination

Let $\mathbf{p}_0, \ldots, \mathbf{p}_d \in \mathbb{E}^n$ and $\alpha_0, \ldots, \alpha_d \geq 0$, with $\alpha_0 + \cdots + \alpha_d = 1$.

The point

$$\alpha_0 \mathbf{p}_0 + \cdots + \alpha_d \mathbf{p}_d = \sum_{i=0}^{d} \alpha_i \mathbf{p}_i$$

is called a convex combination of points $\mathbf{p}_0, \ldots, \mathbf{p}_d$.

A convex combinations is both positive and affine.

Convex hull

The set of all convex combinations of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$ is a convex set, called convex hull of $\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$, and is denoted by $\operatorname{conv}\{\mathbf{p}_0, \ldots, \mathbf{p}_d\}$.

Properties

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Properties

- the convex hull of a set of points is the intersection of all convex sets that contain them
- the convex hull of a set of points is the smallest set that contains them

Computational Geometry

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- since very large datasets contain tens or hundreds of millions of points, the focus is on computational complexity
- for large data sets, the difference between $O(n^2)$ and $O(n \log n)$ may be between days and seconds of computation.

Convex hull algorithms: Gift wrapping (or Jarvis march)

$O(nh)$: where n is the number of points and h is the number of points on the convex hull

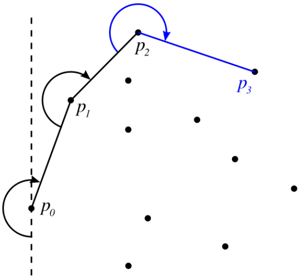

Figure: begin with $i = 0$, at point p_0 known to be on the convex hull, and select the point p_{i+1} such that all points are to the right of the line $[p_i, p_{i+1}]$

Linear, affine and convex independence

A set of vectors is said to be linearly independent if none of them can be obtained as a linear combination of the other ones.

Analogously, a set of points is said affinely/convexely independent if none of them can be obtained as an affine/convex combination of the other ones, respectively.

Affine combination

The set of affine combinations of two affinely independent (i.e. distinct) points $\mathbf{p}_0, \mathbf{p}_1 \in \mathbb{E}^n$, $\mathbf{p}_0 \neq \mathbf{p}_1$, is their line:

$$\begin{aligned} \alpha_0 \mathbf{p}_0 + \alpha_1 \mathbf{p}_1 &:= (1 - \alpha_1)\mathbf{p}_0 + \alpha_1 \mathbf{p}_1 \\ &= \mathbf{p}_0 + \alpha_1(\mathbf{p}_1 - \mathbf{p}_0) \end{aligned}$$

Remember that a difference of points is a vector, and that the sum of a point and a vector is a point.

Convex combination (examples)

1. The set of convex combinations of two affinely independent (i.e. distinct) points $\mathbf{p}_0, \mathbf{p}_1 \in \mathbb{E}^n$, $\mathbf{p}_0 \neq \mathbf{p}_1$, is the line segment joining $\mathbf{p}_0$ with $\mathbf{p}_1$:

$$\begin{aligned} \mathbf{p}(\beta) &= (1-\beta)\mathbf{p}_0 + \beta\mathbf{p}_1 \\ &= \mathbf{p}_0 + \beta(\mathbf{p}_1 - \mathbf{p}_0), \qquad 0 \leq \beta \leq 1 \end{aligned}$$

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2. The set of convex combinations of three affinely independent points is their triangle, i.e. the triangle whose vertices are those points.

Plasm.js: Exercise 10 (Plane for three points)

Let $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{E}^3$ be three non collinear points, i.e. such that:

$$\det \begin{pmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} \neq 0.$$

The vector $\mathbf{n}$, normal to the bundle of planes parallel to the 3 points, may be computed as

$$\mathbf{n} = (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)$$

The Cartesian equation $ax + by + cz + d = 0$ of the plane for $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$, with

$$a = n_x, \quad b = n_y, \quad c = n_z$$

is finally fixed by imposing the passage for one of such points—say $\mathbf{p}_0 = (x_0, y_0, z_0)^T$:

$$ax_0 + by_0 + cz_0 + d = 0 \quad \text{and hence} \quad d = -(n_x\, x_0 + n_y\, y_0 + n_z\, z_0)$$

Affine coordinates

Given a point $\mathbf{p} \in \operatorname{aff}\{\mathbf{p_0}, \ldots, \mathbf{p_d}\}$, the scalars $\alpha_0, \ldots, \alpha_d$ such that:

$$\mathbf{p} = \alpha_0 \mathbf{p_0} + \cdots + \alpha_d \mathbf{p_d},$$

with $\alpha_0 + \cdots + \alpha_d = 1$, are called affine coordinates of $\mathbf{p}$.

Affine coordinates are unique if the points $\mathbf{p_0}, \ldots, \mathbf{p_d}$ are affinely independent.

Remarks

1. A d-flat of $\mathbb{E}^n$, $(1 \leq d \leq n)$ is parameterized by $d + 1$ affine coordinates

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2. Points of the d-flat are associated bijectively with such $(d + 1)$-tuples of coordinates

Convex coordinates

Given a point $\mathbf{p} \in \operatorname{conv}\{\mathbf{p_0}, \ldots, \mathbf{p_d}\}$, the scalars $\alpha_0, \ldots, \alpha_d$ such that:

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with $\alpha_0, \ldots, \alpha_d \geq 0$ and $\alpha_0 + \cdots + \alpha_d = 1$, are called convex coordinates of $\mathbf{p}$.

Convex coordinates are unique if the points $\mathbf{p_0}, \ldots, \mathbf{p_d}$ are affinely independent.

Affine coordinates

The affine coordinates of $\mathbb{E}^d$ are $(d+1)$-tuples of numbers summing to one.

A linear mapping between $\mathbb{E}^d$ points and the $\mathbb{E}^{d+1}$ hyperplane $x_1 + x_2 + \cdots + x_{d+1} = 0$ is used.

The transformation of coordinates depends on $d+1$ affinely independent points $(\mathbf{p}_0, \ldots, \mathbf{p}_d) \in \mathbb{E}^d$ that, embedded in $x_d = 1$, are mapped to the standard basis:

$$[\mathbf{I}] = [\mathbf{T}] \begin{pmatrix} \mathbf{p}_0 & \ldots & \mathbf{p}_d \\ 1 & \ldots & 1 \end{pmatrix}$$

so that

$$[\mathbf{T}] = \begin{pmatrix} \mathbf{p}_0 & \ldots & \mathbf{p}_d \\ 1 & \ldots & 1 \end{pmatrix}^{-1}$$