Computational Graphics: Lecture 10

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1. 2D Affine Transformations – (1)
   - Translation
   - Scaling
   - Reflection

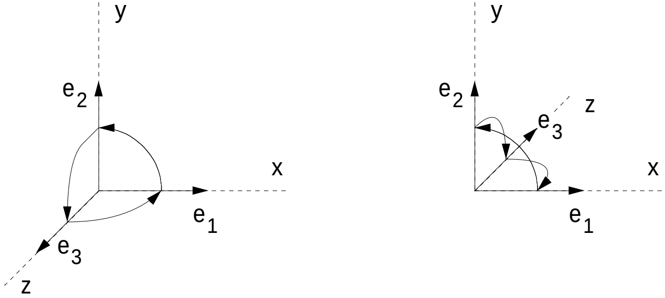
2. 2D Affine transformations – (2)
   - Rotation
   - Shearing
   - General transformations
   - Representation of tensors

Introduction

- Affine transformations are used to map a figure or model into another of different size, position or orientation;
- they reduce to an invertible linear transformationby using homogeneous coordinates
- fixed a reference system, they are represented by squared invertible matrices, said transformation matrices
- we study the structure and properties of "elementary" transformations of 2D plane and 3D space.

Assumptions

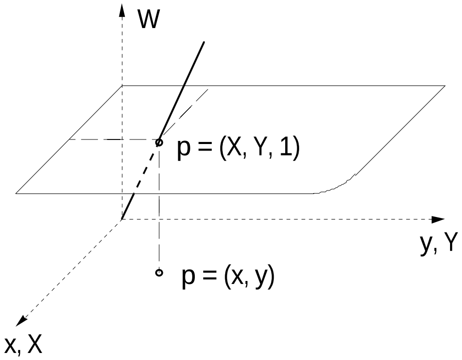
- vectors and points are represented as column vectors
- transformations are given by left products by a matrix
- the reference frame is assumed left-handed



positive rotations: (a) right-handed frame (b) left-handed frame

Homogeneous coordinates

define a bijective mapping between the set of points of Cartesian plane and the set of lines through the origin **o** of 3D space



Homogeneous coordinates of 2D plane

Homogeneous coordinates

in such $\mathbb{E}^2 \to \mathbb{E}^3$ mapping, every point $(x, y)^T \in \mathbb{E}^2$ is represented as the set of points

$$\{(X, Y, W)^T \in \mathbb{E}^3 \mid x = X/W,\ y = Y/W,\ W \neq 0\}$$

to transform the homogeneous point $\mathbf{p}' = (X, Y, W)$ into the Cartesian point $\mathbf{p} = (x, y)$ two divisions by the homogeneous coordinate W are needed.

to avoid this computation we use the homogeneous normalized representation $(X, Y, 1)^T$, such that

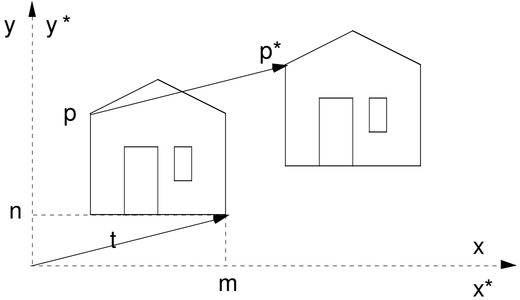
$$x = X, \qquad y = Y$$

the point $(x, y)^T$ of plane is represented by a vector $\lambda(x, y, 1)^T$, with $\lambda \in \mathbb{R}$ e $\lambda \neq 0$.

Translation

A translation of 2D plane is a function $\mathbf{T} : \mathbb{E}^2 \to \mathbb{E}^2$, where a fixed vector $\mathbf{t} = (m, n)^T$ is summed to each point $\mathbf{p} = (x, y)^T$, so that

$$\mathbf{p}^* = \mathbf{T}(\mathbf{p}) = \mathbf{p} + \mathbf{t} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} x + m \\ y + n \end{pmatrix}.$$



Traslation

A movement of origin implies that the translation is not a linear transformation. Therefore, it cannot be represented in coordinates by a matrix

the translation is linear when using homogeneous coordinates. In fact, the translation that maps the **p** point to

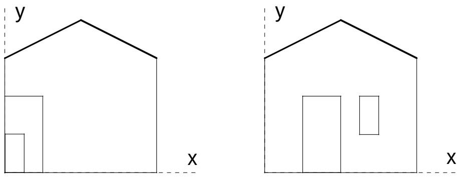
$$\mathbf{p}^* = \mathbf{p} + \mathbf{t},$$

with $\mathbf{t} = (m, n)^T$, becomes, in homogeneous coordinates:

$$\mathbf{p}^* = \mathbf{T}\ \mathbf{p} = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + m \\ y + n \\ 1 \end{pmatrix}$$

Translation

Higher-order functions need a double application over (a) integer specificators and (b) real parameters, in order to generate the transformation tensor



write the code to do the above example

Scaling

definition

A scaling **S** is a transformation tensor represented by a diagonal matrix with positive coefficients, so that:

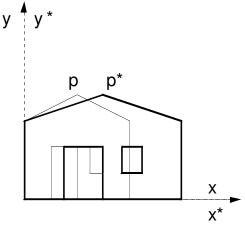
$$\mathbf{p}^* = \mathbf{S}\ \mathbf{p} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ by \end{pmatrix}, \qquad a, b > 0$$

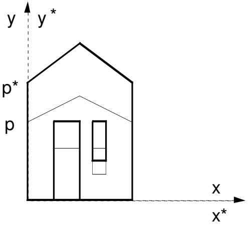
- if $a, b > 1$, then **S** is a dilatation tensor
- if $a = b = 1$, then **S** is the identity tensor
- if $a, b < 1$, then **S** is a compression tensor

Scaling

elementary scalings

$$\mathbf{p}^* = \mathbf{S}_x\,\mathbf{p} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ y \end{pmatrix}$$





$$\mathbf{p}^* = \mathbf{S}_y\,\mathbf{p} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ by \end{pmatrix}$$

Scaling

homogeneous coordinates

the homogeneous normalized coordinate matrix $\mathbf{S}' \in \mathbb{R}^3_3$ of a 2D scaling tensor may be easily derived from the non-homogeneous matrix $\mathbf{S} \in \mathbb{R}^2_2$, by adding a unit row and column:

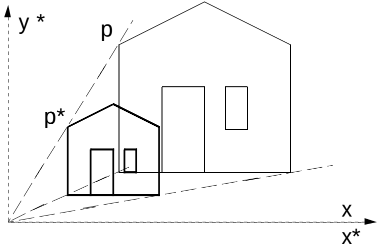
$$\mathbf{p}^* = \mathbf{S}'\mathbf{p} = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} ax \\ by \\ 1 \end{pmatrix}.$$

Scaling

uniform scaling

When $a = b$ the scaling is said uniform or homothetic transformation

1. with $a = b = 0.5$ the length of all segments is halved
2. the image $\mathbf{p}^*$ of each $\mathbf{p}$ goes on the line through $\mathbf{p}$ and the origin
3. the transformed figure is also closer to the origin



action of a tensor of uniform scaling

Reflection

definition

Linear transformation defined by a matrix that differs from the identity since one of diagonal coefficients is -1

Two elementary reflections $\mathbf{M}_x$ e $\mathbf{M}_y$ may be defined in the plane $\mathbb{E}^2$

$$\mathbf{M}_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{M}_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

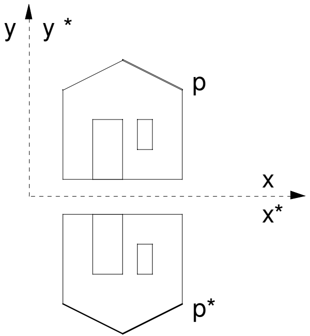
The action of a reflection tensor inverts the sign of one of coordinates of points

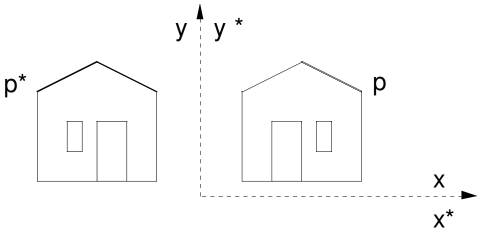
Reflection

homogeneous representation

As usual, the normalized homogeneous representation of such transformations is obtained by adding a unit row and column to their matrices

$$\mathbf{M}'_x = \begin{pmatrix} \mathbf{M}_x & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \qquad \mathbf{M}'_y = \begin{pmatrix} \mathbf{M}_y & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$$

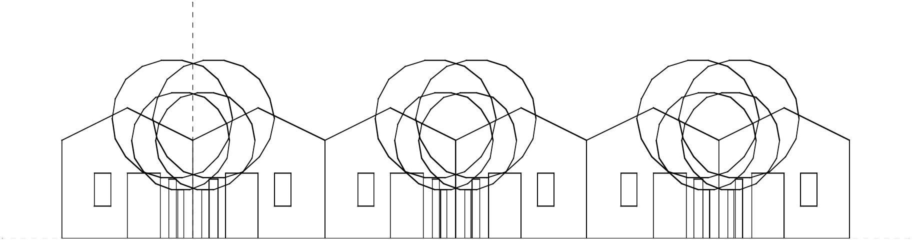




Reflection

example

Let us continue the house example by adding simmetry to the scene



Elementary rotation of plane

An elementary rotation of 2D plane is a linear function that maps every point $\mathbf{p} \in \mathbb{E}^2$ to the second extreme $\mathbf{p}^* = \mathbf{R}(\mathbf{p})$ of a circle arc with first extreme in $\mathbf{p}$, center in the origin and constant angle α

The matrix of a rotation tensor is easily computed by considering the images of basis vectors $(\mathbf{e}_i)$

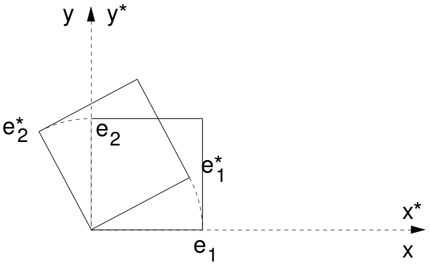
$$\begin{pmatrix} \mathbf{e}_1^* & \mathbf{e}_2^* \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix}.$$

where $\mathbf{R}$ is the unknown rotation matrix

Elementary rotation of plane

more explicitly:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \mathbf{R} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$



Elementary rotation of plane

in homogeneous coordinates

The normalized homogeneous matrix $\mathbf{R}' \in \operatorname{lin} \mathbb{R}^3$ of a plane rotation is obtained from the non-homogeneous matrix $\mathbf{R} \in \operatorname{lin} \mathbb{R}^2$

$$\mathbf{p}^* = \mathbf{R}'\mathbf{p} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \ \cos\alpha + y \ \sin\alpha \\ -x \ \sin\alpha + y \ \cos\alpha \\ 1 \end{pmatrix}$$

in the usual way, by adding a unit row and column ...

Shearing

elementary

The plane is seen as a bundle of lines parallel to a coordinate axis

A 2D elementary shearing is a tensor which maps the points of a line in other points of the same line, in a way such that:

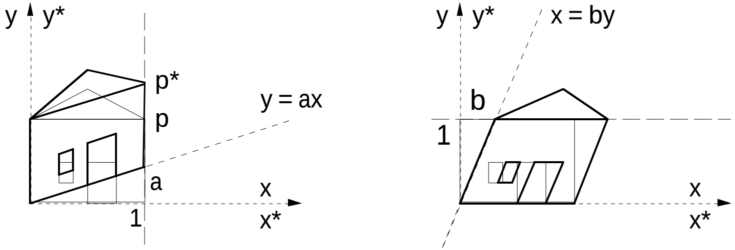
1. all points of a line translate by the same vector
2. only the coordinate axis parallel to the line bundle remains fixed
3. the translation of each line is proportional to its distance to the fixed line

Shearing

An elementary shearing tensor does not change one coordinate, whereas the other changes linearly with the value of the fixed coordinate

$$\mathbf{p}^* = \mathbf{H}_x\ \mathbf{p} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + ax \end{pmatrix},$$

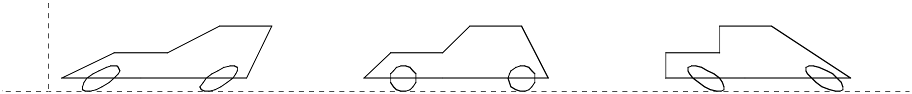
$$\mathbf{p}^* = \mathbf{H}_y\ \mathbf{p} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + by \\ y \end{pmatrix}.$$



Action of $\mathbf{H}_x$, normal to the x axis, and $\mathbf{H}_y$, normal to the y axis

Shearing

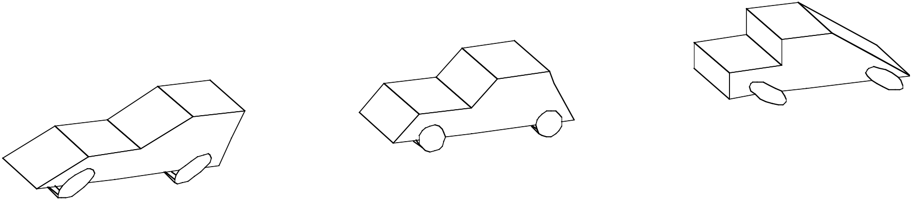
example



Shearing

example

Three keyframes of the storyboard of 3D animation entitled: “My wife’s car”



Arbitrary linear transformation

Let consider the action of a general $\mathbf{Q}$ tensor on the unit square built on the basis of the Cartesian frame $(\mathbf{o}, \mathbf{e}_i)$, with

$$\mathbf{Q} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

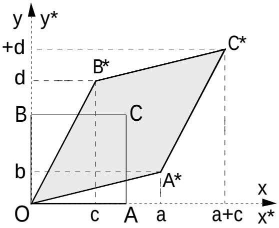
arbitrary, but invertible matrix

such arbitrary linear transformation:

1. does not move the origin;
2. maps parallel lines to parallel lines;
3. does'nt conserve, in general, the size of areas.

General transformation

action of a general tensor on the unit standard square



$$\begin{pmatrix} \mathbf{o}^* & \mathbf{a}^* & \mathbf{b}^* & \mathbf{c}^* \end{pmatrix} = \mathbf{Q} \begin{pmatrix} \mathbf{o} & \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix},$$

or, by using the corresponding coordinates:

$$\begin{pmatrix} 0 & a & c & a+c \\ 0 & b & d & b+d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Transformation with fixed point

different from the origin

Every invertible linear transformation $\mathbf{Q}$ has the origin $\mathbf{o}$ of the Cartesian frame as its unique fixed point, i.e. $\mathbf{Q}(\mathbf{o}) = \mathbf{o}$

To have a fixed point $\mathbf{q}$ different from origin we must compose three transformations, such that:

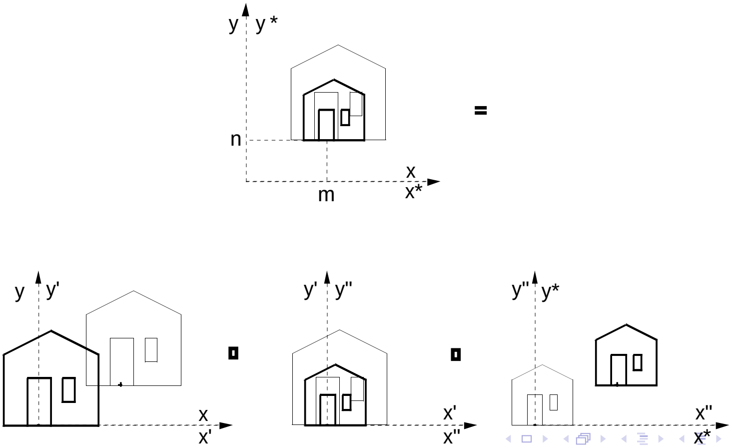
1. map $\mathbf{q}$ to the origin $\mathbf{o}$;
2. apply the required transformation;
3. map back $\mathbf{o}$ to $\mathbf{q}$.

Transformation with fixed point

scaling

Let consider a scaling tensor with fixed point $\mathbf{q} = (m, n)^T \neq \mathbf{o}$:

$$\mathbf{S}_{\mathbf{q}}(m, n, a, b) = \mathbf{T}_{xy}(m, n) \circ \mathbf{S}_{xy}(a, b) \circ \mathbf{T}_{xy}(-m, -n).$$

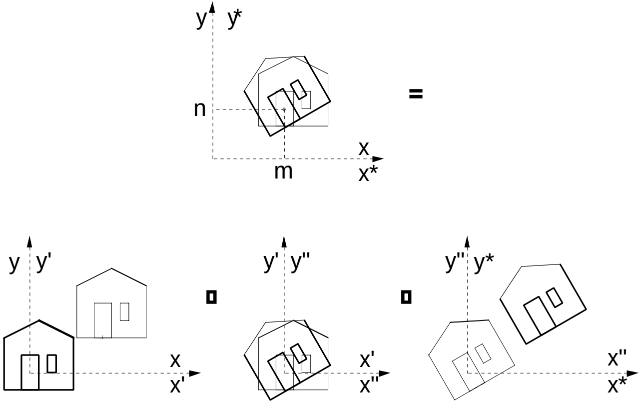


Transformation with fixed point

rotation

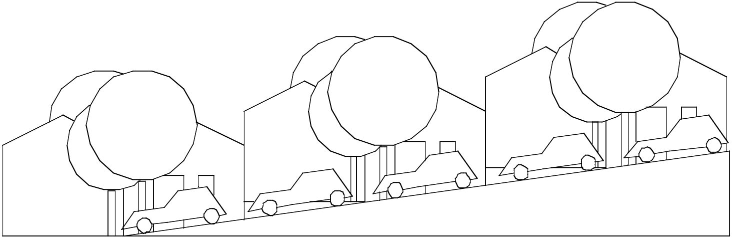
Let consider a rotation tensor with fixed point $\mathbf{q} = (m, n)^T \neq \mathbf{o}$:

$$\mathbf{R_q}(m, n, \alpha) = \mathbf{T}_{xy}(m, n) \circ \mathbf{R}_{xy}(\alpha) \circ \mathbf{T}_{xy}(-m, -n).$$



2D Affine transformations

example



Remark (Assignment)

Convert the example on pages 230-231 (chapter 6) of book GP4CAD from classic PLaSM (FL style) to pyplasm

Representation of tensors

Tensors are represented in PLaSM by applying the predefined function `MAT` to the tensor matrix (list of lists of coordinates)

$$\texttt{MAT} : \mathbb{R}^3_3 \rightarrow \mathrm{lin}\, \mathbb{R}^3$$

Tensors, defined as linear endomorphisms of a vector space, have first-grade citizenship in PLaSM, and can be composed to generate new tensors. For example:

Tensors can be applied to polyhedral complexes of arbitrary dimensions (d, n)

Representation of tensors

example

```
from pyplasm import *

wall = MKPOL([  [[0,0],[4,0],[4,4],[2,6],[0,4]],
[[1,2,3,4,5]],None  ])
Q = MAT([[1,0,0],[0,1,0.5],[0,0,1]])

VIEW(Q(wall))
```

Remember that:

- we use homogeneous coordinates (2D matrices are 3×3);
- in PlaSM the homogeneous coordinate is the first