Quantitative methods and simulation in Finance

Chapter 2 - Linear Time Series

Introduction

- we discuss basic theories of linear time series analysis, introduce some simple econometric models useful for analyzing financial data, and apply the models to financial time series such as asset returns.

- Discussions of the concepts are brief with emphasis on those relevant to financial applications.

- Understanding the simple time series models introduced here will go a long way to better appreciate the more sophisticated financial econometric models of the later chapters.

Introduction

- Treating an asset return (e.g., log return r_t of a stock) as a collection of random variables over time, we have a time series $\{r_t\}$.

- Linear time series analysis provides a natural framework to study the dynamic structure of such a series.

- The theories of linear time series discussed include stationarity, dynamic dependence, autocorrelation function, modeling, and forecasting.

- For an asset return r_t , simple models attempt to capture the linear relationship between r_t and information available prior to time t.

- As such, correlation plays an important role in understanding these models. In particular, correlations between the variable of interest and its past values become the focus of linear time series analysis.

- These correlations are referred to as serial correlations or autocorrelations. They are the basic tool for studying a stationary time series.

STATIONARITY

- *strictly stationary*:

The joint distribution of $(r_{t1}, \dots, r_{tk})$ is identical to that of $(r_{t_1+t}, \dots, r_{t_k+t})$ for all t, where k is an arbitrary positive integer and $(t_1, \dots, t_k)$ is a collection of k positive integers.

In other words, strict stationarity requires that the joint distribution of $(r_{t1}, \dots, r_{tk})$ is invariant under time shift.

- *weakly stationary* :

(a) $E(r_t) = \mu$, which is a constant.

(b) $Cov(r_t, r_{t-l}) = \gamma_l$, which only depends on l.

s.t. $\longrightarrow$ w.s. w.s. $\longrightarrow$ s.t. (normal distribution)

STATIONARITY

- The weak stationarity implies that the time plot of the data would show that the T values fluctuate with constant variation around a fixed level.

- In applications, weak stationarity enables one to make inference concerning future observations (e.g., prediction).

- We are mainly concerned with weakly stationary series

STATIONARITY

- autocovariance of r_t:

$$\gamma_l = Cov(r_t, r_{t-l})$$

- It has two important properties:

(a) $\gamma_0 = Var(r_t)$

(b) $\gamma_{-l} = \gamma_l$.

- In the finance literature, it is common to assume that an asset return series is weakly stationary. This assumption can be checked empirically provided that a sufficient number of historical returns are available.

Autocorrelation Function (ACF)

- Correlation and autocorrelation

- Correlation measures the strength of linear dependence between variables

$$\rho_\ell = \frac{\mathrm{Cov}(r_t, r_{t-\ell})}{\sqrt{\mathrm{Var}(r_t)\mathrm{Var}(r_{t-\ell})}} = \frac{\mathrm{Cov}(r_t, r_{t-\ell})}{\mathrm{Var}(r_t)} = \frac{\gamma_\ell}{\gamma_0},$$

$$\hat{\rho}_1 = \frac{\sum_{t=2}^{T}(r_t - \bar{r})(r_{t-1} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2}.$$

$$\hat{\rho}_\ell = \frac{\sum_{t=\ell+1}^{T}(r_t - \bar{r})(r_{t-\ell} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2}, \qquad 0 \le \ell < T - 1.$$

Testing Individual ACF

- $$H_0: \rho_l = 0 \text{ vs. } H_a: \rho_l \neq 0 .$$

$$t \text{ ratio} = \frac{\hat{\rho}_\ell}{\sqrt{(1 + 2\sum_{i=1}^{\ell-1} \hat{\rho}_i^2)/T}}.$$

- If $\{r_t\}$ is a stationary Gaussian series satisfying $\rho_j = 0$ for $j > l$, the t ratio is asymptotically distributed as a standard normal random variable.

- For simplicity, many software packages use $1/T$ as the asymptotic variance of $\hat{\rho}_l$.

sample autocorrelation function (ACF)

- A linear time series model can be characterized by its ACF, and linear time series modeling makes use of the sample ACF to capture the linear dynamic of the data. (IBM)

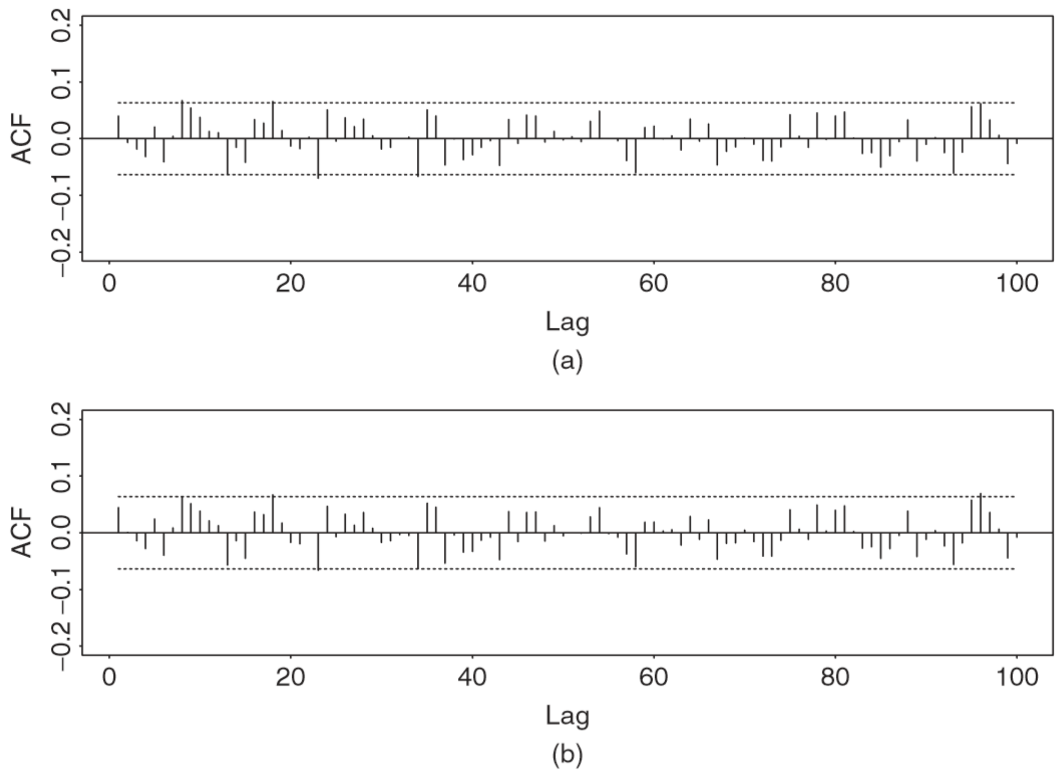

Figure 2.1 Sample autocorrelation functions of monthly (a) simple returns and (b) log returns of IBM stock from January 1926 to December 2008. In each plot, two horizontal dashed lines denote two standard error limits of sample ACF.

sample autocorrelation function (ACF)

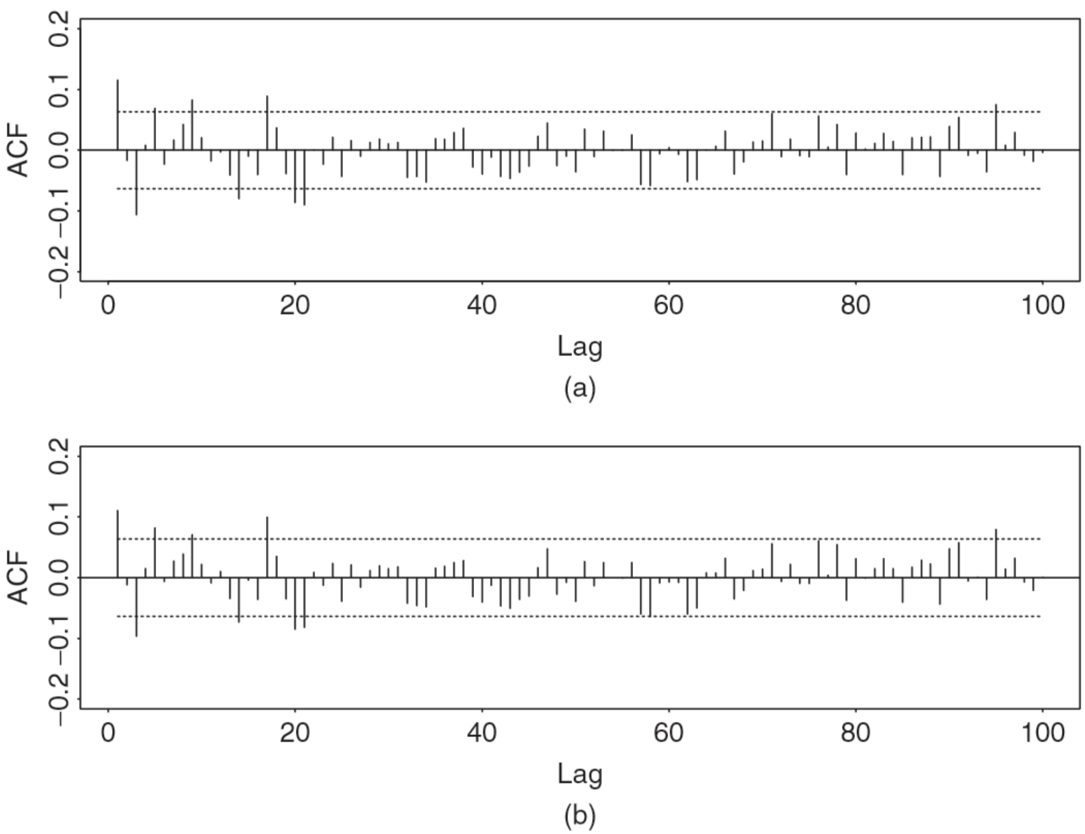

Figure 2.2 Sample autocorrelation functions of monthly (a) simple returns and (b) log returns of value-weighted index of U.S. markets from January 1926 to December 2008. In each plot, two horizontal dashed lines denote two standard error limits of sample ACF.

Portmanteau Test

- Box and Pierce (1970) propose the Portmanteau statistic

$$Q^*(m) = T \sum_{\ell=1}^{m} \hat{\rho}_\ell^2$$

as a test statistic for the null hypothesis $H_0 : \rho_1 = \cdots = \rho_m = 0$ against the alternative hypothesis H_a: $\rho_i \neq 0$ for some $i \in \{1, \ldots, m\}$.

Under the assumption that $\{r_t\}$ is an iid sequence with certain moment conditions, $Q^*(m)$ is asymptotically a chi-squared random variable with m degrees of freedom.

Portmanteau Test

- Ljung and Box (1978) modify the $Q^*(m)$ statistic as below to increase the power of the test in finite samples.

$$Q(m) = T(T+2)\sum_{\ell=1}^{m}\frac{\hat{\rho}_\ell^2}{T-\ell}.$$

- Most software packages will provide the p value of Q(m).

- In the finance literature, a version of the capital asset pricing model (CAPM) theory is that the return $\{r_t\}$ of an asset is not predictable and should have no autocorrelations. Testing for zero autocorrelations has been used as a tool to check the efficient market assumption.

R Demonstration

```
> da=read.table("m-ibm3dx2608.txt",header=T) % Load data
> da[1,]   % Check the 1st row of the data
      date       rtn    vwrtn    ewrtn    sprtn
1 19260130 -0.010381 0.000724 0.023174 0.022472
> sibm=da[,2] % Get the IBM simple returns
> Box.test(sibm,lag=5,type='Ljung') % Ljung-Box statistic Q(5)
      Box-Ljung test

data: sibm
X-squared = 3.3682, df = 5, p-value = 0.6434

> libm=log(sibm+1)  % Log IBM returns
> Box.test(libm,lag=5,type='Ljung')

      Box-Ljung test

data: libm
X-squared = 3.5236, df = 5, p-value = 0.6198
```

time series model

- A time series r_t is called a white noise if $\{r_t\}$ is a sequence of independent and identically distributed random variables with finite mean and variance.

- In particular, if r_t is normally distributed with mean zero and variance σ 2, the series is called a Gaussian white noise. For a white noise series, all the ACFs are zero. In practice, if all sample ACFs are close to zero, then the series is a white noise series.

- Based on Figures 2.1 and 2.2, the monthly returns of IBM stock are close to white noise, whereas those of the value-weighted index are not.

Simple AR models

- The behavior of sample autocorrelations of the value-weighted index returns indicates that for some asset returns it is necessary to model the serial dependence

- The fact that the monthly return r_t of CRSP value-weighted index has a statistically significant lag-1 autocorrelation indicates that the lagged return $r_t - 1$ might be useful in predicting r_t.

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t,$$

where $\{a_t\}$ is assumed to be a white noise series with mean zero and variance σ^2.

- Linear regression form

Simple AR models

- The conditional mean

$$E(r_t | r_{t-1}) = \phi_0 + \phi_1 r_{t-1}, \qquad \mathrm{Var}(r_t | r_{t-1}) = \mathrm{Var}(a_t) = \sigma_a^2.$$

- This is a Markov property such that conditional on r_{t-1}, the return r_t is not correlated with $r_{t-i}$for $i > 1$.

- A straightforward generalization of the AR(1) model is the AR(p) model:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + a_t,$$

AR(1) Model

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t,$$

- Taking the expectation of the equation and because $E(a_t) = 0$, we obtain

$$E(r_t) = \phi_0 + \phi_1 E(r_{t-1}).$$

- Under the stationarity condition, $E(r_t) = E(r_{t-1}) = \mu$ and hence

$$\mu = \phi_0 + \phi_1 \mu \qquad \text{or} \qquad E(r_t) = \mu = \frac{\phi_0}{1 - \phi_1}.$$

- First, the mean of r_t exists if $\phi_1 \neq 1$.

  Second, the mean of r_t is zero if and only if $\phi_0 = 0$.

AR(1) model variance

$$\mathrm{Var}(r_t) = \phi_1^2 \mathrm{Var}(r_{t-1}) + \sigma_a^2,$$

$$\mathrm{Var}(r_t) = \frac{\sigma_a^2}{1 - \phi_1^2}$$

- The requirement of $-1 < \phi_1 < 1$ results from the fact that the variance of a random variable is bounded and nonnegative.
- In summary, the necessary and sufficient condition for the AR(1) model to be weakly stationary is $|\phi_1| < 1$.

ACF of an AR(1) Model

- Using $\phi_0 = (1-\phi_1)\mu$, one can rewrite a stationary AR(1) model as

$$r_t = (1-\phi_1)\mu + \phi_1 r_{t-1} + a_t.$$

$$\gamma_\ell = \begin{cases} \phi_1\gamma_1 + \sigma_a^2 & \text{if } \ell = 0 \\ \phi_1\gamma_{\ell-1} & \text{if } \ell > 0, \end{cases}$$

$$\rho_\ell = \phi_1\rho_{\ell-1}, \quad \text{for} \quad \ell > 0.$$

- Because $\rho_0 = 1$, we have $\rho_l = \phi_1^l$. This result says that the ACF of a weakly stationary AR(1) series decays exponentially with rate ϕ_1 and starting value $\rho_0 = 1$.

ACF of an AR(1) Model

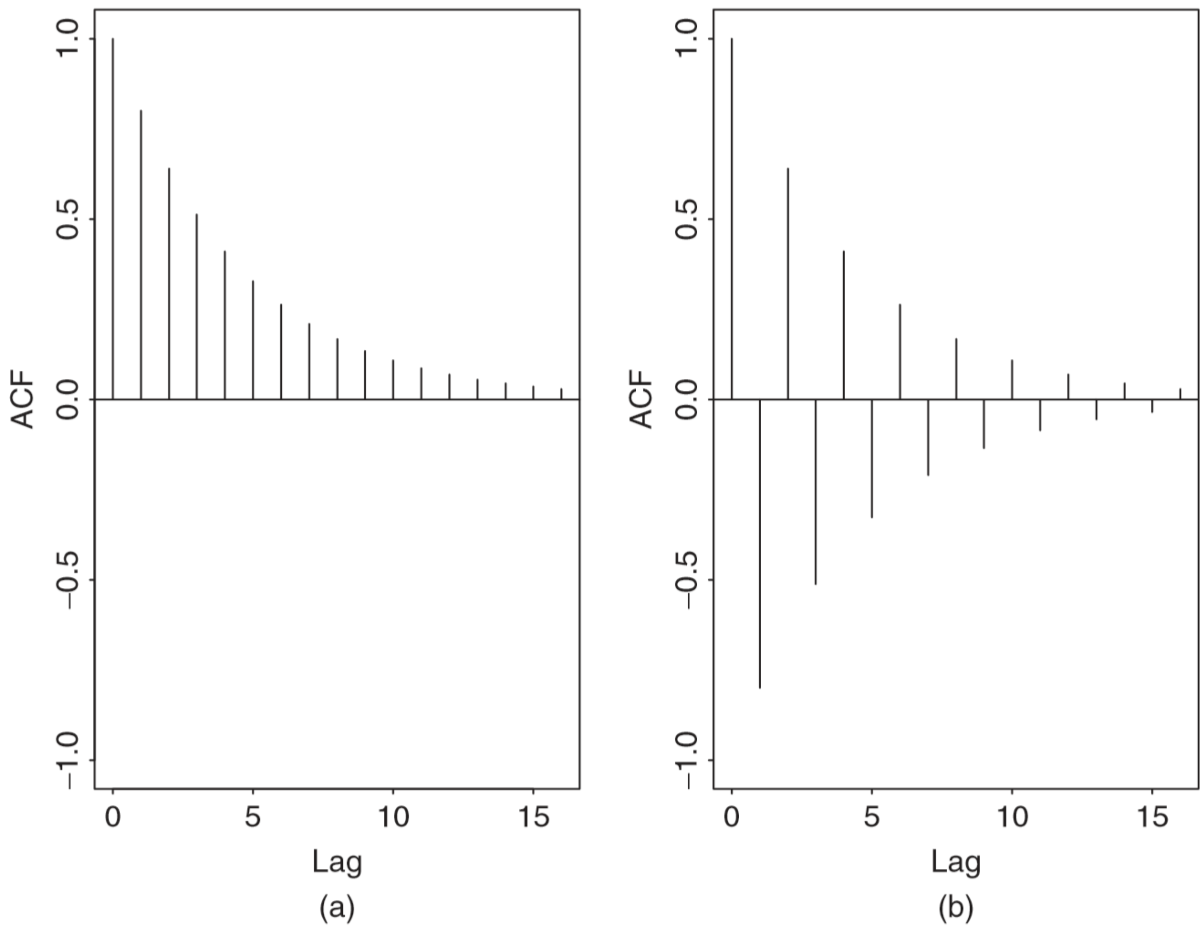

Figure 2.3 Autocorrelation function of an AR(1) model: (a) for $\phi_1 = 0.8$ and (b) for $\phi_1 = -0.8$.

ACF of an AR(2) Model

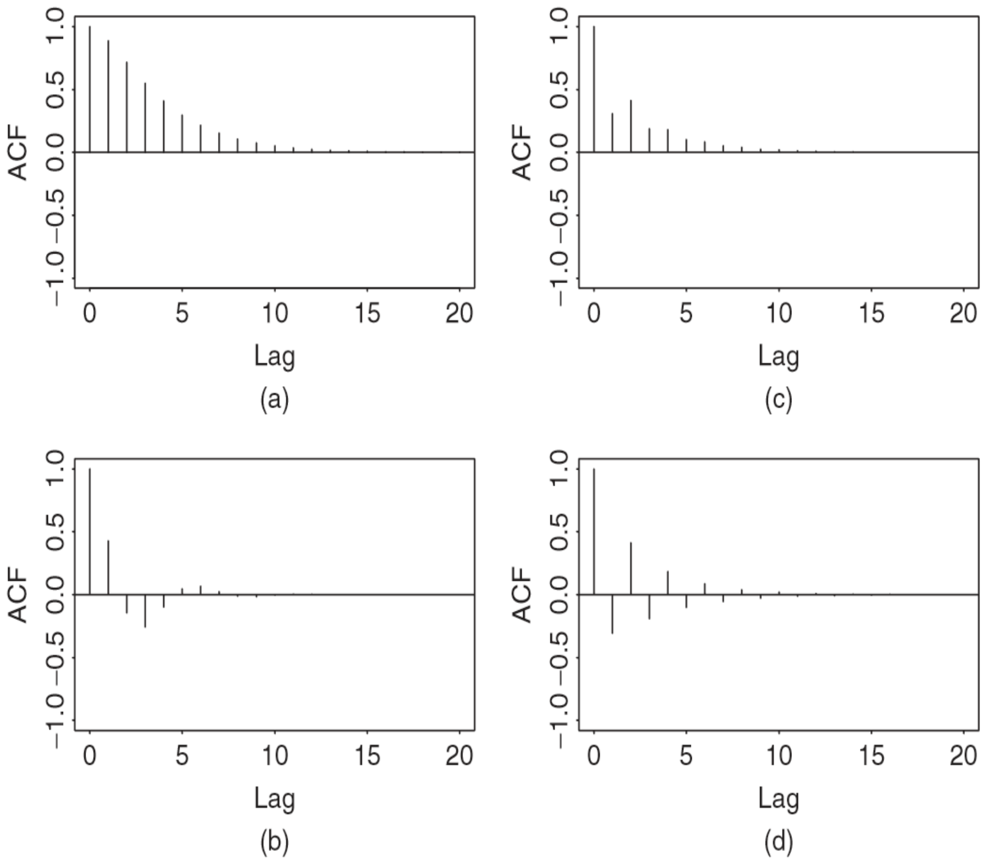

Figure 2.4 Autocorrelation function of an AR(2) model: (a) $\phi_1 = 1.2$ and $\phi_2 = -0.35$, (b) $\phi_1 = 0.6$ and $\phi_2 = -0.4$, (c) $\phi_1 = 0.2$ and $\phi_2 = 0.35$, and (d) $\phi_1 = -0.2$ and $\phi_2 = 0.35$.

How to choose the order

- Remove the effect of the previous lag.

$$
\begin{aligned}
r_t &= \phi_{0,1} + \phi_{1,1} r_{t-1} + e_{1t}, \\
r_t &= \phi_{0,2} + \phi_{1,2} r_{t-1} + \phi_{2,2} r_{t-2} + e_{2t}, \\
r_t &= \phi_{0,3} + \phi_{1,3} r_{t-1} + \phi_{2,3} r_{t-2} + \phi_{3,3} r_{t-3} + e_{3t}, \\
r_t &= \phi_{0,4} + \phi_{1,4} r_{t-1} + \phi_{2,4} r_{t-2} + \phi_{3,4} r_{t-3} + \phi_{4,4} r_{t-4} + e_{4t}, \\
&\vdots
\end{aligned}
$$

- Partial ACF

The estimate $\hat{\phi}_{1,1}$ of the first equation is called the lag-1 sample PACF of r_t.
The estimate $\hat{\phi}_{2,2}$ of the second equation is the lag-2 sample PACF of r_t.
The estimate $\hat{\phi}_{3,3}$ of the third equation is the lag-3 sample PACF of r_t, and so on.

How to choose the order

- For a stationary Gaussian AR(p) model, it can be shown that the sample PACF has the following properties:
  - $\hat{\phi}_{p,p}$ converges to ϕ_p as the sample size T goes to infinity.
  - $\hat{\phi}_{\ell,\ell}$ converges to zero for all $\ell > p$.
  - The asymptotic variance of $\hat{\phi}_{\ell,\ell}$ is $1/T$ for $\ell > p$.

- These results say that, for an AR(p) series, the sample PACF cuts off at lag p.

TABLE 2.1 Sample Partial Autocorrelation Function and Some Information Criteria for the Monthly Simple Returns of CRSP Value-Weighted Index from January 1926 to December 2008

p	1	2	3	4	5	6
PACF	0.115	−0.030	−0.102	0.033	0.062	−0.050
AIC	−5.838	−5.837	−5.846	−5.845	−5.847	−5.847
BIC	−5.833	−5.827	−5.831	−5.825	−5.822	−5.818
p	7	8	9	10	11	12
PACF	0.031	0.052	0.063	0.005	−0.005	0.011
AIC	−5.846	−5.847	−5.849	−5.847	−5.845	−5.843
BIC	−5.812	−5.807	−5.805	−5.798	−5.791	−5.784

- The sample size is 996, the asymptotic standard error of the sample PACF is 0.032 .

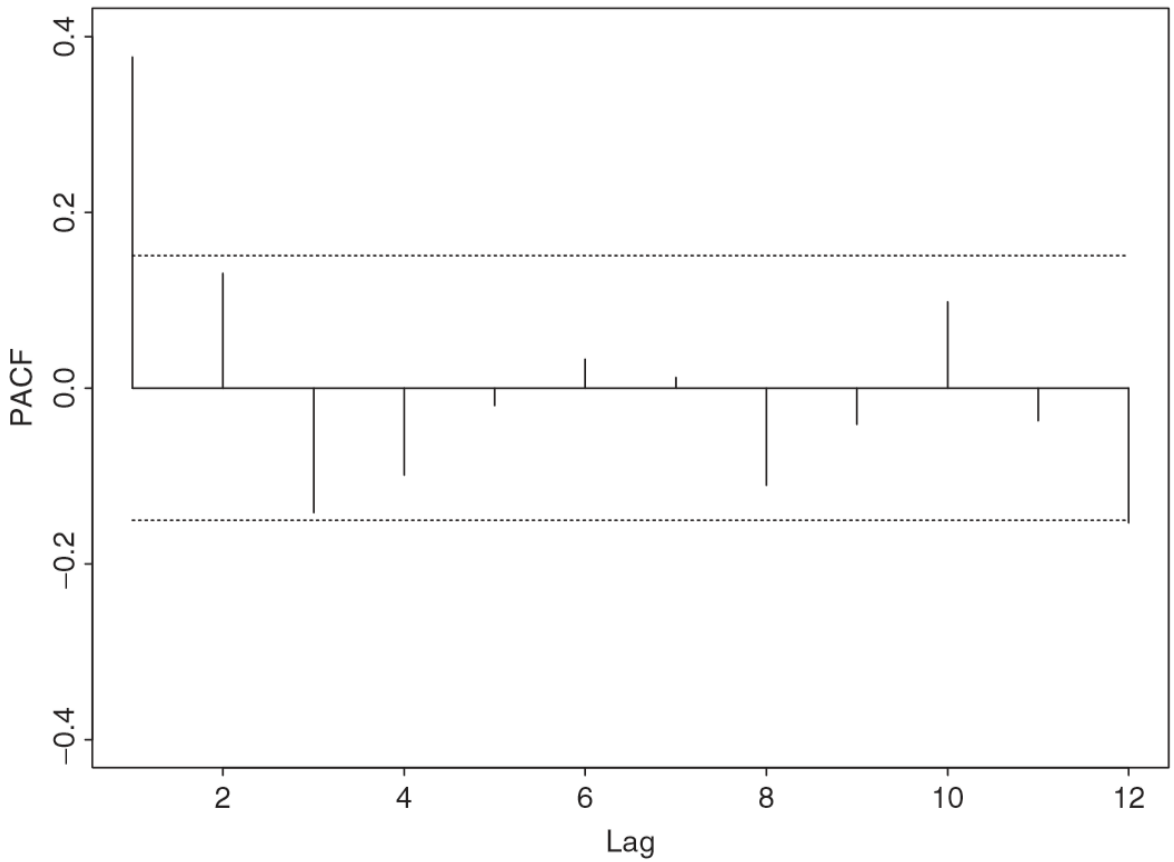

Figure 2.6 Sample partial autocorrelation function of U.S. quarterly real GNP growth rate from 1947.II to 1991.I. Dotted lines give approximate pointwise 95% confidence interval.

Parameter Estimation

- For a specified AR(p) model

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + a_t, \qquad t = p+1, \ldots, T,$$

which is in the form of a multiple linear regression and can be estimated by the least-squares method.

Denote the estimate of ϕ_i by $\hat{\phi}_i$. The fitted model is

$$\hat{r}_t = \hat{\phi}_0 + \hat{\phi}_1 r_{t-1} + \cdots + \hat{\phi}_p r_{t-p},$$

and the associated residual is

$$\hat{a}_t = r_t - \hat{r}_t.$$

Parameter Estimation

- For illustration, consider an AR(3) model for the monthly simple returns of the value-weighted index in Table 2.1. The fitted model is

$$r_t = 0.0091 + 0.116 r_{t-1} - 0.019 r_{t-2} - 0.104 r_{t-3} + \hat{a}_t, \qquad \hat{\sigma}_a = 0.054.$$

The standard errors of the coefficients are 0.002, 0.032, 0.032, and 0.032, respectively. Except for the lag-2 coefficient, all parameters are statistically significant at the 1% level.

Model Checking

- If the model is adequate, then the residual series should behave as a white noise.

The ACF and the Ljung–Box statistics of the residuals can be used to check the closeness of $\hat{a}_t$ to a white noise.

For an AR(p) model, the Ljung–Box statistic Q(m) follows asymptotically a chi-squared distribution with m−g degrees of freedom, where g denotes the number of AR coefficients used in the model.

Model Checking

- if some of the estimated AR coefficients are not significantly different from zero, then the model should be simplified by trying to remove those insignificant parameters.

- If residual ACF shows additional serial correlations, then the model should be extended to take care of those correlations.

- Most time series packages do not adjust the degrees of freedom when applying the Ljung–Box statistics Q(m) to a residual series.

Model Checking

- Consider the residual series of the fitted AR(3) model for the monthly value weighted simple returns. We have Q(12) =16.35 with a p value 0.060 based on its asymptotic chi-squared distribution with 9 degrees of freedom. Thus, the null hypothesis of no residual serial correlation in the first 12 lags is barely not rejected at the 5% level.

- However, since the lag-2 AR coefficient is not significant at the 5% level, one can refine the model as

$$r_t = 0.0088 + 0.114 r_{t-1} - 0.106 r_{t-3} + a_t, \qquad \hat{\sigma}_a = 0.0536,$$

- where all the estimates are now significant at the 1% level. The residual series gives Q(12) =16.83 with a p value 0.078 (based on $\chi 2$ 10). The model is adequate in modeling the dynamic linear dependence of the data.

R Demonstration

```
> vw=read.table('m-ibm3dx2608.txt',header=T)[,3]
> m3=arima(vw,order=c(3,0,0))
> m3
Call:
arima(x = vw, order = c(3, 0, 0))

Coefficients:
         ar1      ar2      ar3  intercept
      0.1158  -0.0187  -0.1042     0.0089
s.e.  0.0315   0.0317   0.0317     0.0017

sigma^2 estimated as 0.002875: log likelihood=1500.86,
   aic=-2991.73

> (1-.1158+.0187+.1042)*mean(vw) % Compute
   the intercept phi(0).
[1] 0.00896761
> sqrt(m3$sigma2)  % Compute standard error of residuals
[1] 0.0536189

> Box.test(m3$residuals,lag=12,type='Ljung')
```

R Demonstration

```
      Box-Ljung test

data: m3$residuals   % R uses 12 degrees of freedom

X-squared = 16.3525, df = 12, p-value = 0.1756

> pv=1-pchisq(16.35,9) % Compute p-value using 9 degrees
   of freedom
> pv
[1] 0.05992276
 % To fix the AR(2) coef to zero:
> m3=arima(vw,order=c(3,0,0),fixed=c(NA,0,NA,NA))
 % The subcommand 'fixed' is used to fix parameter values,
 % where NA denotes estimation and 0 means fixing the
    parameter to 0.
 % The ordering of the parameters can be found using m3$coef.
> m3
Call:
arima(x = vw, order = c(3, 0, 0), fixed = c(NA, 0, NA, NA))

Coefficients:
         ar1  ar2      ar3  intercept
      0.1136    0  -0.1063     0.0089
s.e.  0.0313    0   0.0315     0.0017

sigma^2 estimated as 0.002876: log likelihood=1500.69,
```

```
   aic=-2993.38
> (1-.1136+.1063)*.0089 % Compute phi(0)
[1] 0.00883503
> sqrt(m3$sigma2)  % Compute residual standard error
[1] 0.05362832

> Box.test(m3$residuals,lag=12,type='Ljung')

      Box-Ljung test

data: m3$residuals
X-squared = 16.8276, df = 12, p-value = 0.1562

> pv=1-pchisq(16.83,10)
> pv
[1] 0.0782113
```

Goodness of Fit

$$R^2 = 1 - \frac{\sum_{t=p+1}^{T} \hat{a}_t^2}{\sum_{t=p+1}^{T} (r_t - \bar{r})^2},$$

- For a given data set, it is well known that R^2 is a nondecreasing function of the number of parameters used.

1-Step-Ahead Forecast

$$r_{h+1} = \phi_0 + \phi_1 r_h + \cdots + \phi_p r_{h+1-p} + a_{h+1}.$$

$$\hat{r}_h(1) = E(r_{h+1}|F_h) = \phi_0 + \sum_{i=1}^{p} \phi_i r_{h+1-i},$$

$$e_h(1) = r_{h+1} - \hat{r}_h(1) = a_{h+1}.$$

2-Step-Ahead Forecast

$$r_{h+2} = \phi_0 + \phi_1 r_{h+1} + \cdots + \phi_p r_{h+2-p} + a_{h+2}.$$

- Taking conditional expectation, we have

$$\hat{r}_h(2) = E(r_{h+2}|F_h) = \phi_0 + \phi_1 \hat{r}_h(1) + \phi_2 r_h + \cdots + \phi_p r_{h+2-p}$$

$$e_h(2) = r_{h+2} - \hat{r}_h(2) = \phi_1[r_{h+1} - \hat{r}_h(1)] + a_{h+2} = a_{h+2} + \phi_1 a_{h+1}.$$

$$\mathrm{Var}[e_h(2)] = (1 + \phi_1^2)\sigma_a^2.$$

Multistep-Ahead Forecast

$$r_{h+\ell} = \phi_0 + \phi_1 r_{h+\ell-1} + \cdots + \phi_p r_{h+\ell-p} + a_{h+\ell}.$$

$$\hat{r}_h(\ell) = \phi_0 + \sum_{i=1}^{p} \phi_i \hat{r}_h(\ell - i),$$

- This forecast can be computed recursively using forecasts $\hat{r}_h(i)$ for $i = 1, \ldots, l-1$.
- The l-step-ahead forecast error is $e_h(l) = r_{h+l} - \hat{r}_h(l)$.
- It can be shown that for a stationary AR(p) model, $\hat{r}_h(l)$ converges to $E(r_t)$ as $l \to \infty$, meaning that for such a series long-term point forecast approaches its unconditional mean. This property is referred to as the mean reversion in the finance literature.

Real example

- the monthly simple return of the value-weighted index use an AR(3) model that was reestimated using the first 984 observations. The fitted model is

$$r_t = 0.0098 + 0.1024 r_{t-1} - 0.0201 r_{t-2} - 0.1090 r_{t-3} + a_t,$$

- The sample mean is 0.0095

Summary

- ACF and Box-Ljune test of the data
- Fit an AR model
- Use PACF of the data to identify the order
- check the residuals
- Select the lag and modify the model