NATIONAL BUREAU OF STANDARDS REPORT

1993

TABLES OF E(1/X) FOR POSITIVE BERNOULLI AND POISSON VARIABLES

by

Edwin Grab

U. S. DEPARTMENT OF COMMERCE
NATIONAL BUREAU OF STANDARDS

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NBS

FOREWORD

These tables were prepared as part of a continuing program of research on mathematical statistics and its applications carried out at the National Bureau of Standards under the general supervision of Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory. The Statistical Engineering Laboratory is Section 11.3 of the National Applied Mathematics Laboratories (Division 11, National Bureau of Standards), and is concerned with the development and application of modern statistical methods in the physical sciences and engineering.

J. H. Curtiss
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National Bureau of Standards

TABLES OF E(1/X) FOR POSITIVE BERNOULLI AND POISSON VARIABLES

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INTRODUCTION

The random variable X is said to have a positive Bernoulli distribution [1] if the probability that X=x is equal to $\binom{n}{x}p^x(1-p)^{n-x}/[1-(1-p)^n]$ for $x=1,2,\ldots,n$ and $0<p<1$. Similarly the variable X is said to have a positive Poisson distribution if the probability that X=x is equal to $e^{-m}m^x/x!(1-e^{-m})$ for $x=1,2,\ldots,$ and $m>0$. This report tabulates the functions:

$$(1)\quad E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x}p^x(1-p)^{n-x}/x[1-(1-p)^n]$$

$$(2)\quad E(1/X|m) = \sum_{x=1}^{\infty} e^{-m}m^x/x!x(1-e^{-m}) \quad .$$

E(1/X|n,p) was tabulated for the following values of the parameters:

$n = 2(1)20(5)30$, $p = .01, .05(.05).95, .99$;

$n = 21(1)24$, $p = .01, .05(.05).50$;

$n = 26(1)29$, $p = .01, .05(.05).45$;

$n = 35$ and 40, $p = .01, .05(.05).35$.

$E(1/X|m)$ was tabulated for these values:

$$m = .01(.01).20(.10)1(1)10(5)20.$$

All tables are given to five decimals.

The need for tables of the above functions arises in many problems of sampling when zero is an inadmissible value of the variable [1], [2].

COMPUTATION METHODS AND USE OF TABLES

The computation of $E(1/X)^*$ for positive Bernoulli variables is a laborious task on a hand calculator. For the ranges of the parameters covered by TABLE I (n is small), there is no simple approximation of $E(1/X)$. Stephan [1] presents a factorial series as an approximation of $E(1/X)$. Finkner [2], from Monte Carlo experimentation, suggests $1/(np-1)$ as an overestimate of the function with $1/np$ the lower bound. We used as an estimate of $E(1/X)$

(3) $$1/(np-q)$$

(where $q = 1-p$).

Included with the tables are graphs of D_i $(i=1,2,3)$ for p equal to .50 and .90.

$$D_1 = E(1/X) - 1/np$$
$$D_2 = E(1/X) - 1/(np-q)$$
$$D_3 = E(1/X) - 1/(np-1)$$

Table II gives the relative error $[R_i = D_i/E(1/X)]$ when n=15 and 30 for the various values of p.

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* $E(1/X)$ will for convenience be used to denote $E(1/X|n,p)$ or $E(1/X|m)$ when there is no chance for confusion.

Linear interpolation within TABLE I will in most cases produce two significant figures, while equation (3), for the probabilities indicated by a footnote to the tables, is a better approximation than $1/np$ or $1/(np-1)$ and produces accuracies of two or three decimals if not two significant figures. As the magnitude of n increases, $1/(np-q)$ rapidly approaches $E(1/X)$.

$E(1/X|n,p)$ was computed by summing the probabilities of the xth term of the binomial series [3] divided by x, with the resulting summation divided by $1-(1-p)^n$.

Two methods were used in calculating $E(1/X|m)$. Poisson tables in Fry [4] were used for m = .1(.1)1(1)10(5)20. The calculation of $E(1/X|m)$ using Poisson tables is done in like manner to the calculation of the Bernoulli reciprocal using the binomial tables. An alternate method, used in the parameter range of .01(.01).20, is

(4) $$E(1/X|m) = [Ei(m)-\gamma-\log_e m]e^{-m}/(1-e^{-m})$$

[5], [6], [7]. Values of $E(1/X)$ for m=.1 and .2 provided checks as to similarity of the methods. The more inclusive tables of the Poisson distribution by Molina [8] and Kitagawa [9] could have been used [and are easier to work with then formula (4)] for very small m values. The results are given in TABLE III.

Linear interpolation within the table will generally produce two decimal place accuracy. It is suggested for the range $10<m<40$ that $1/(m-1)$ be used for the approximation and $1/m$ be used for values of $m\geq 40$.

REFERENCES

[1] F. F. Stephan, "The expected value and variance of the reciprocal and other negative powers of a positive Bernoullian variate", Ann. Math. Stat., 16, 50-61, (1945).

[2] A. L. Finkner, "Further investigation on the theory and application of sampling for scarcity items", Institute of Statistics, University of North Carolina, Mimeo. Series 30.

[3] National Bureau of Standards, Table of the Binomial Probability Distribution, Applied Mathematics Series 6.

[4] T. C. Fry, Probability and It's Engineering Uses, D. van Nostrand Company, Inc., 458-462, (1928).

[5] National Bureau of Standards, Tables of the Exponential Function e^x, Applied Mathematics Series 14.

[6] National Bureau of Standards, Tables of Sine, Cosine, and Exponential Integrals, 1, MT 5, U. S. Government Printing Office, Washington, D. C.

[7] National Bureau of Standards, Tables of Natural Logarithms, MT 10, U. S. Government Printing Office, Washington, D. C.

[8] E. C. Molina, Poisson's Exponential Binomial Limit, D. van Nostrand Company, Inc., (1947).

[9] T. Kitagawa, Tables of Poisson Distribution, Baifukan, Tokyo, Japan, (1952).

TABLE I

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x} / x[1-(1-p)^n]$$

p \ n	2	3	4	5	6
.01	.99749	.99498	.99247	.98997	.98747
.05	.98718	.97444	.96178	.94920	.93671
.10	.97368	.94772	.92214	.89696	.87220
.15	.95946	.91983	.88117	.84357	.80708
.20	.94444	.89071	.83898	.78940	.74210
.25	.92857	.84479	.79571	.73489	.67806
.30	.91176	.82877	.75158	.68055	.61583
.35	.89394	.79594	.70683	.62697	.55629
.40	.87500	.75744	.66176	.57474	.50026
.45	.85484	.72672	.61676	.52451	.44843
.50	.83333	.69048	.57222	.47688	.40132
.55	.81034	.65330	.52862	.43241	.35890
.60	.78571	.61538	.48645	.39156	.32231
.65	.75926	.57697	.44622	.35465	.29037
.70	.73077	.53869	.40843	.32183	.26305
.75	.70375	.49543	.37353	.29311	.23989
.80	.66666	.46237	.34188	.26829	.22031
.85	.63043	.42608	.31373	.24704	.20372*
.90	.59091	.39189	.28915	.22891*	.18956
.95	.54762	.36065	.26803*	.21340	.17734
.99	.50990	.33843	.25338	.20253	.16869

* 1/(np-q) produces accuracies of two or three decimals, or two significant figures, in predicting E(1/x) at this point and improves as p increases.

TABLE I (Continued)

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^{x}(1-p)^{n-x}/x[1-(1-p)^{n}]$$

p \ n	7	8	9	10	11
.01	.98497	.98247	.97998	.97749	.97501
.05	.92431	.91200	.89979	.88767	.87565
.10	.84786	.82400	.80060	.77768	.75526
.15	.77176	.73763	.70475	.67312	.64277
.20	.69715	.65461	.61450	.57682	.54152
.25	.62529	.57657	.53184	.49095	.45371
.30	.55736	.50492	.45819	.41674	.38010
.35	.49441	.44067	.39429	.35440	.32018
.40	.43725	.38436	.34016	.30327	.27243
.45	.38637	.33604	.29523	.26204	.23487
.50	.34194	.29530	.25847	.22911	.20541
.55	.30378	.26048	.22864	.20285	.18217
.60	.27147	.23348	.20447	.18177	.16361
.65	.24436	.21049	.18477	.16467	.14854
.70	.22173	.19151	.16856	.15057	.13608*
.75	.20282	.17570	.15504	.13876*	.12560
.80	.18692	.16238*	.14359*	.12872	.11665
.85	.17341*	.15101	.13376	.12006	.10892
.90	.16181	.14118	.12523	.11252	.10216
.95	.15172	.13259	.11774	.10589	.09620
.99	.14454	.12645	.11238	.10112	.09192

TABLE I (Continued)

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x} / x[1-(1-p)^n]$$

p \ n	12	13	14	15	16
.01	.97253	.97004	.96757	.96509	.96262
.05	.86373	.85191	.84020	.82859	.81709
.10	.73334	.71193	.69105	.67069	.65087
.15	.61370	.58591	.55938	.53412	.51007
.20	.50856	.47785	.44932	.42284	.39832
.25	.41990	.38930	.36163	.33667	.31415
.30	.34780	.31937	.29433	.27228	.25282
.35	.29081	.26557	.24382	.22502	.20868
.40	.24655	.22471	.20619	.19035	.17668
.45	.21243	.19372	.17795	.16453	.15299
.50	.18601	.16992	.15638	.14486	.13493
.55	.16530	.15131	.13952	.12945*	.12075*
.60	.14878	.13643	.12600*	.11707	.10933
.65	.13532*	.12429*	.11493	.10689	.09991
.70	.12416	.11417	.10568	.09837	.09201
.75	.11473	.10561	.09783	.09112	.08528
.80	.10666	.09826	.09108	.08488	.07948
.85	.09967	.09187	.08521	.07945	.07442
.90	.09355	.08628	.08006	.07467	.06997
.95	.08814	.08133	.07550	.07044	.06602
.99	.08425	.07777	.07221	.06739	.06317

TABLE I (Continued)

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x} / x[1-(1-p)^n]$$

p \ n	17	18	19	20
.01	.96015	.95769	.95523	.95277
.05	.80570	.79443	.78326	.77222
.10	.63158	.61282	.59460	.57968
.15	.48723	.46556	.44502	.42557
.20	.37565	.35480	.33535	.31750
.25	.29384	.27552	.25898	.24403
.30	.23562	.22038	.20682	.19472
.35	.19442	.18193	.17085	.16104
.40	.16482	.15444	.14529	.13717
.45	.14296	.13419	.12643	.11954*
.50	.12629	.11870*	.11198*	.10599
.55	.11316*	.10648	.10055	.09525
.60	.10255	.09658	.09126	.08650
.65	.09379	.08841	.08357	.07925
.70	.08643	.08148	.07708	.07312
.75	.08014	.07559	.07153	.06788
.80	.07472	.07050	.06673	.06335
.85	.06999	.06605	.06254	.05938
.90	.06582	.06214	.05885	.05588
.95	.06212	.05866	.05556	.05278
.99	.05946	.05615	.05319	.05053

TABLE I (Continued)

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x} / x[1-(1-p)^n]$$

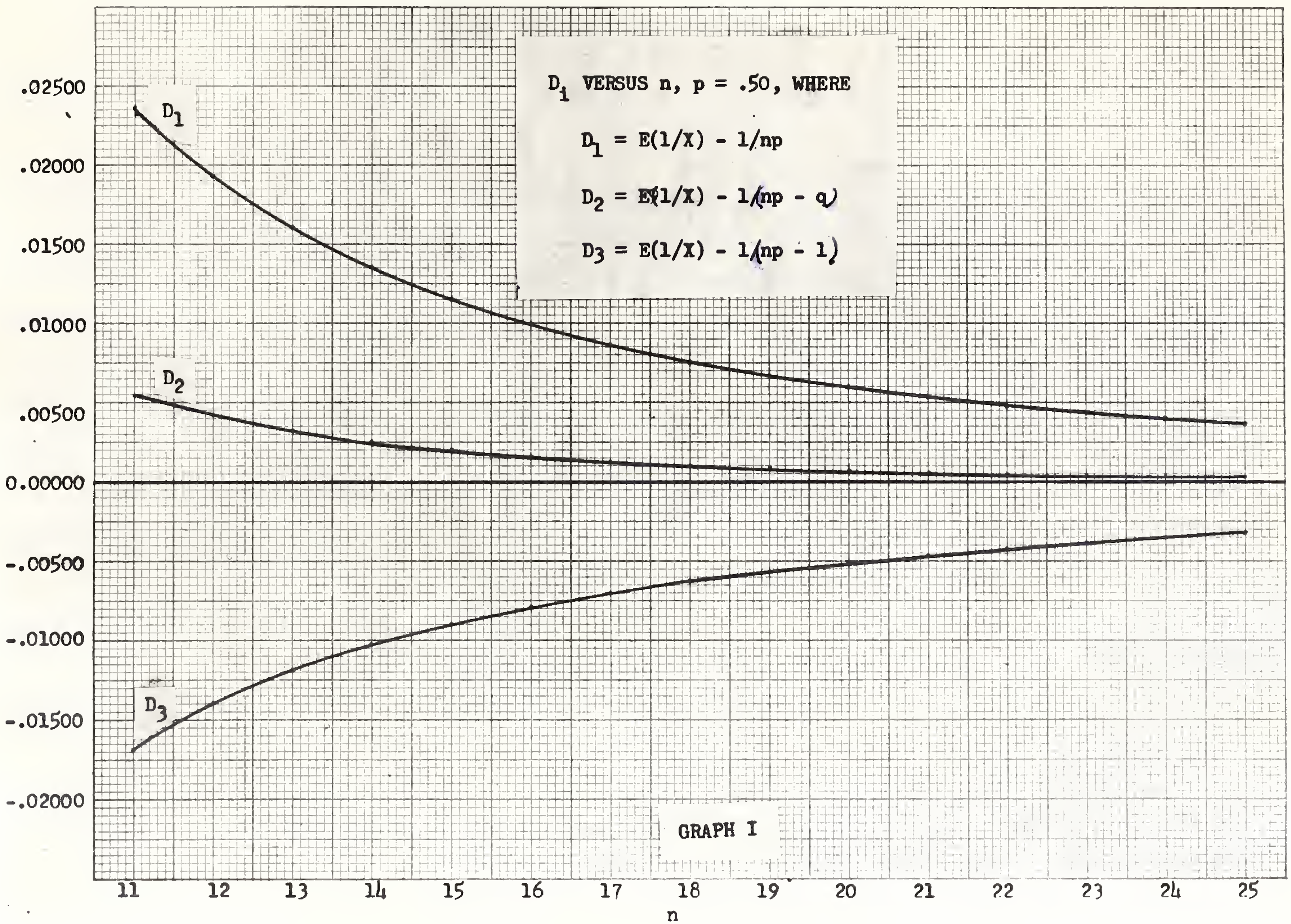
p \ n	21	22	23	24
.01	.95031	.94786	.94541	.94297
.05	.76128	.75047	.73977	.72920
.10	.55975	.54312	.52700	.51140
.15	.40718	.38980	.37338	.35788
.20	.30103	.28287	.27180	.25885
.25	.23049	.21822	.20079	.19689
.30	.18389	.17416	.16537	.15742
.35	.15229	.14444	.13736	.13095
.40	.12992	.12341	.11752	.11218*
.45	.11336*	.10780*	.10277*	.09819
.50	.10061	.09576	.09136	.08735

p \ n	26	27	28	29
.01	.93809	.93565	.93322	.93079
.05	.70841	.69820	.68811	.67814
.10	.48171	.46760	.45397	.44081
.15	.32947	.31647	.30421	.29264
.20	.23582	.22558	.21609	.20729
.25	.17917	.17140	.16425	.15766
.30	.14359	.13755	.13199	.12688
.35	.11979*	.11490*	.11040*	.10624*
.40	.10285	.09874	.09496	.09148
.45	.09016	.08663	.08336	.08033

TABLE I (Continued)

$$E(1/X|n,p) = \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x} / x[1(1-p)^n]$$

p \ n	25	30	35	40
.01	.94053	.92837	.91629	.90431
.05	.71874	.66830	.62098	.57680
.10	.49630	.42811	.37110	.32381
.15	.34326	.28174	.23584	.20130
.20	.24688	.19911	.16588	.14190
.25	.18765	.15157	.12703	.10937*
.30	.15019	.12217	.10300*	.08909
.35	.12512	.10239*	.08671	.07523
.40	.10731 *	.08821		
.45	.09400	.07752		
.50	.08367	.06915		
.55	.07541	.06243		
.60	.06864	.05690		
.65	.06299	.05227		
.70	.05820	.04835		
.75	.05410	.04497		
.80	.05053	.04203		
.85	.04741	.03946		
.90	.04465	.03718		
.95	.04220	.03515		
.99	.04042	.03368		



D_i VERSUS n, p = .50, WHERE
$D_1 = E(1/X) - 1/np$
$D_2 = E(1/X) - 1/(np - q)$
$D_3 = E(1/X) - 1/(np - 1)$
D_1
D_2
D_3
.02500
.02000
.01500
.01000
.00500
0.00000
-.00500
-.01000
-.01500
-.02000
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
n

GRAPH I

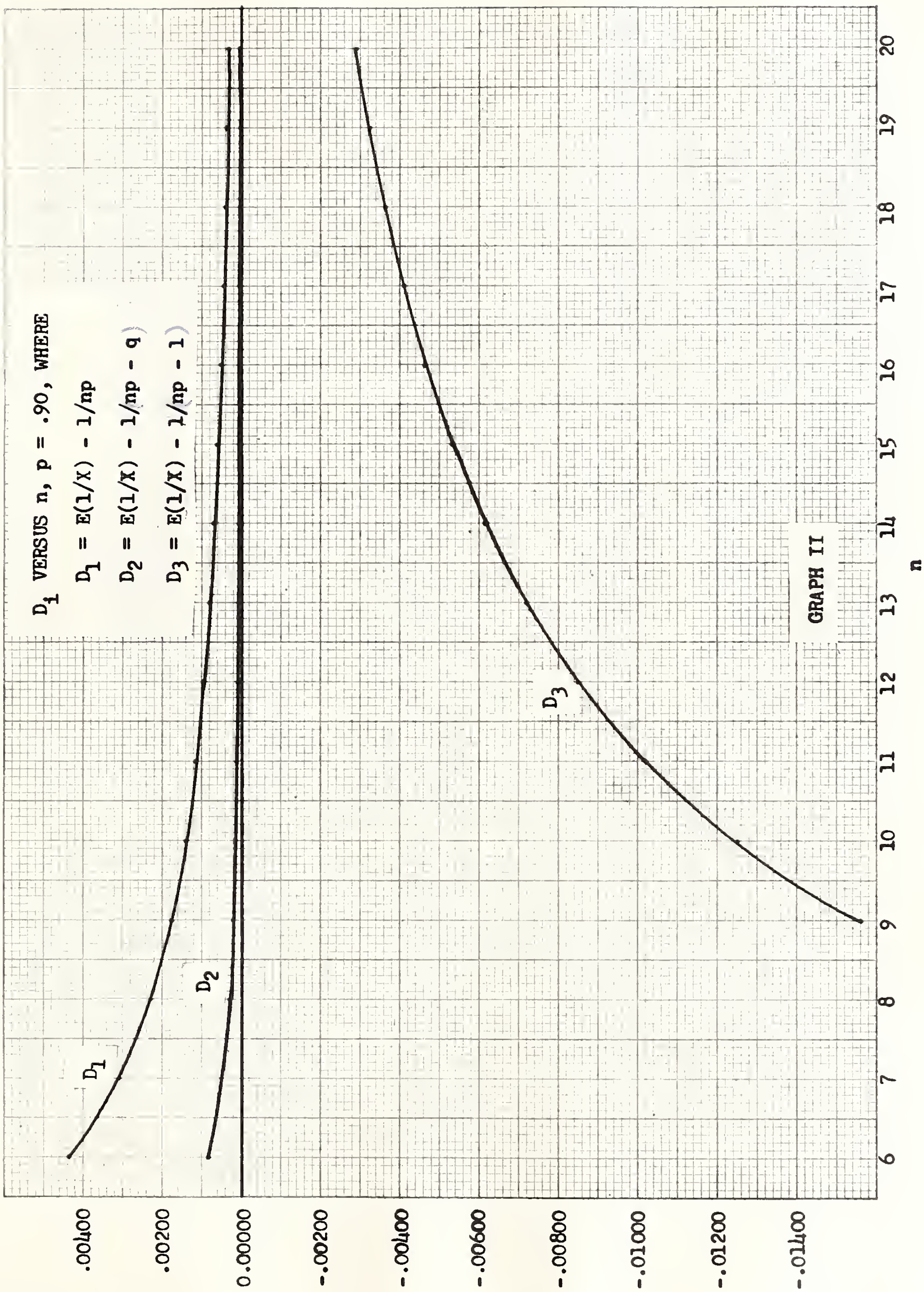

D_1 VERSUS n, p = .90, WHERE
D_1 = E(1/X) - 1/np
D_2 = E(1/X) - 1/np - q
D_3 = E(1/X) - 1/np - 1
D_1
D_2
D_3
GRAPH II
n
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
.00400
.00200
0.00000
-.00200
-.00400
-.00600
-.00800
-.01000
-.01200
-.01400

TABLE II

R_i VERSUS p, n = 15, 30

p	n = 15			n = 30		
	R_1	R_2	R_3	R_1	R_2	R_3
.01	-5.90782	2.23354	2.21903	-2.59052	2.56110	2.53879
.05	0.60916	7.03435	5.82748	.00244	-1.72060	-1.99267
.10	.00599	-1.48501	-1.98200	.22139	-0.13567	-0.16792
.15	.18662	-0.33732	-0.49779	.21126	.02758	-0.01409
.20	.21169	-0.07499	-0.18248	.16293	.03415	-0.00447
.25	.20792	.00992	-0.08011	.12034	.02256	-0.01504
.30	.18385	.03349	-0.04932	.09053	.01383	-0.02316
.35	.15350	.03391	-0.04564	.06983	.00850	-0.02803
.40	.12440	.02711	-0.05070	.05532	.00555	-0.03061
.45	.09348	.01969	-0.05701	.04450	.00387	-0.03199
.50	.07959	.01381	-0.06206	.03586	.00260	-0.03297
.55	.06365	.00958	-0.06551	.02915	.00192	-0.03348
.60	.05091	.00675	-0.06774	.02355	.00141	-0.03374
.65	.04051	.00477	-0.06923	.01894	.00096	-0.03405
.70	.03121	.00335	-0.07004	.01510	.00083	-0.03413
.75	.02458	.00230	-0.07068	.01179	.00067	-0.03425
.80	.01826	.00153	-0.07104	.00857	.00036	-0.03450
.85	.01284	.00101	-0.07124	.00634	.00023	-0.03447
.90	.00804	.00054	-0.07138	.00377	.00014	-0.03443
.95	.00369	.00028	-0.07141	.00171	.00006	-0.03442
.99	.00074	.00005	-0.07138	.00035	.00001	-0.03444

TABLE III

$$E(1/x|m) = \sum_{x=1}^{\infty} e^{-m}\, m^x/x!\,x(1-e^{-m})$$

m	E(1/X|m)	m	E(1/X|m)
.01	.99750	.30	.92636
.02	.99501	.40	.90244
.03	.99251	.50	.87889
.04	.99002	.60	.85571
.05	.98754	.70	.83292
.06	.98505	.80	.81052
.07	.98257	.90	.78854
.08	.98009	1.0	.76699
.09	.97759	2.0	.57659
.10	.97514	3.0	.43268
.11	.97267	4.0	.32963
.12	.97021	5.0	.25777
.13	.96774	6.0	.20779
.14	.96528	7.0	.17249
.15	.96282	8.0	.14689
.16	.96037	9.0	.12776
.17	.95792	10.0	.11302
.18	.95547	15.0	.07181
.19	.95302	20.0	.05280
.20	.95058		

THE NATIONAL BUREAU OF STANDARDS

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Information on the Bureau's publications can be found in NBS Circular 460, Publications of the National Bureau of Standards ($1.00). Information on calibration services and fees can be found in NBS Circular 483, Testing by the National Bureau of Standards (25 cents). Both are available from the Government Printing Office. Inquiries regarding the Bureau's reports and publications should be addressed to the Office of Scientific Publications, National Bureau of Standards, Washington 25, D. C.

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