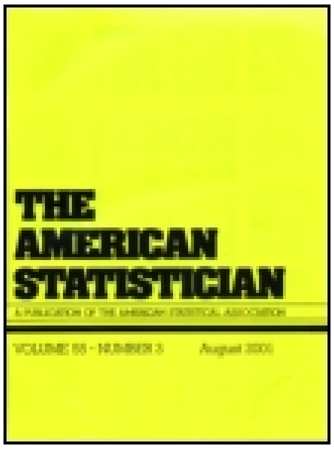

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The Existence of the First Negative Moment

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The question of the existence of negative moments of a continuous probability density function is explored. A sufficient condition for the existence of the first negative moment is given. The condition is easy to verify, as it involves limits rather than integrals. An example is given, however, that shows that this simple condition is not necessary for the existence of the first negative moment. The delicacy of the characterization of existence is explored further with some results concerning the existence of moments surrounding the first negative moment.

KEY WORDS: Inverse moment; Sufficient conditions; Limit conditions.

1. INTRODUCTION

In an introductory (postcalculus) probability and statistics course, students spend much time in evaluating expectations of random variables. For the most part, such evaluations are confined to positive moments of well-behaved distributions, and hence the question of the existence of these moments is rarely an issue. With the possible exception of the Cauchy distribution, the existence of at least two positive moments is usually a foregone conclusion. This is not the case, however, if one is attempting the evaluation of negative moments, for here nonexistence is a much more common occurrence.

In many practical applications, one is led quite naturally to the evaluation of negative moments of a random variable. Mendenhall and Lehman (1960) gave examples of such an application in life testing problems, and Savage (1980, sec. 12) gave a short history of how the problem developed from his days at the National Bureau of Standards. We cite three simple examples:

Example 1.1. The exponential distribution with parameter λ, given by

$$f(x) = \lambda\exp(-\lambda x), \qquad 0 \le x < \infty, \tag{1.1}$$

is often used as a model in reliability analysis. If one observation is taken from this distribution, the standard method of maximum likelihood estimation would lead to the use of $1/x$ as an estimate of λ. Does the expected value $E[X^{-1}]$ exist?

Example 1.2. In simple linear regression, one may assume the model $E[y] = \alpha + \beta(x - \bar{x})$. Usually, the interest is in predicting y from x, using a fitted equation $\hat{y} = \hat{\alpha} + \hat{\beta}(x - \bar{x})$, where under standard assumptions, $\hat{\alpha}$ and $\hat{\beta}$ are independent and normally distributed. Sometimes, however, the interest is in inverse regression; predicting x from y. The natural estimate of x, for a given y, is then $\bar{x} + (y - \hat{\alpha})/\hat{\beta}$ with expectation $\bar{x} + E[y - \hat{\alpha}]E[\hat{\beta}^{-1}]$. Does $E[\hat{\beta}^{-1}]$ exist?

Example 1.3. If a sample is drawn from a normal population with mean μ and standard deviation σ, a measure of the percentage of variation in the population per unit mean (which is popular in the engineering sciences) is the *coefficient of variation* $\sigma/|\mu|$. A natural estimate of this quantity is $s/|\bar{x}|$, where $\bar{x}$ and s are the sample mean and sample standard deviation, respectively. It is well known that in this setting, $\bar{X}$ and S are independently distributed, so $E[S/|\bar{X}|] = E[S] \cdot E[|\bar{X}|^{-1}]$. Does $E[|\bar{X}|^{-1}]$ (and hence this estimator's expectation) exist?

In general, the theory behind the existence of negative moments is difficult and not nearly as complete as that involving positive moments. There, one is led into the (quite elegant) theory of characteristic functions (see Zygmund 1947 and Pitman 1956). Alternatively, one could operate on the moment-generating function of the random variable. Cressie et al. (1981) discussed this approach for negative moments and gave a number of interesting examples.

In Section 2 we present a theorem that gives a sufficient condition for the existence of the first negative moment of a positive random variable. The condition is easy to check; it does not involve evaluation of an integral. In Section 3 we explore the delicacy of the characterization of existence by presenting a family of densities for which the first negative moment may or may not exist, depending on the parameter configuration of the density. We will also present two simple results pertaining to the moments surrounding $E[X^{-1}]$.

2. CONDITIONS FOR THE EXISTENCE OF NEGATIVE MOMENTS

The difficulties experienced in calculating expressions such as $E[X^{-1}]$ will occur near $X = 0$. For example, if X has a discrete probability density function (pdf) with positive mass at $X = 0$, $E[X^{-1}]$ will be infinite. Chao and Strawderman (1972) sidestepped this problem by adding a value to the random variable that makes it positive with probability one. We will only be concerned here with the continuous case, which in some respects is more complex.

If $f(0) > 0$ then $E[|X|^{-1}]$ will be infinite. Thus continuous random variables that take on positive and negative values will not have a first negative moment. [One could, of course, construct a distribution continuous on $(-\infty, \infty)$, with $f(0)$

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= 0, such that the first negative moment does exist, but this would be a rather uncommon distribution.] The far more interesting case, to which we will restrict our attention, is that of positive random variables.

If $f(x)$ is a continuous pdf on $(0, \infty)$, it is natural to first inquire whether

$$\lim_{x \downarrow 0} f(x) = 0$$

is a sufficient condition for $E[X^{-1}] < \infty$. This is not the case, for it is the rate at which $f(x)$ approaches zero that is important.

Theorem 2.1. Let $f(x)$ be a continuous pdf on $(0, \infty)$. If

$$\lim_{x \to 0} [f(x)/x^{\alpha}] < \infty \qquad \text{for some } \alpha > 0, \tag{2.1}$$

then $E[X^{-1}] < \infty$.

Proof. If $\alpha > 0$ satisfies (2.1), then there exist finite constants M and $\delta > 0$ such that $|f(x)/x^{\alpha}| \leq M$ when $0 \leq x \leq \delta$. Hence

$$\int_0^{\delta} \frac{f(x)dx}{x} = \int_0^{\delta} x^{\alpha-1} \cdot \frac{f(x)}{x^{\alpha}}\, dx$$

$$\leq M \int_0^{\delta} x^{\alpha-1} dx = \frac{M}{\alpha} \cdot \delta^{\alpha}. \tag{2.2}$$

So

$$E[X^{-1}] < \frac{M}{\alpha} \cdot \delta^{\alpha} + \int_{\delta}^{\infty} \frac{1}{x} \cdot f(x)dx$$

$$< \frac{M}{\alpha} \cdot \delta^{\alpha} + \frac{1}{\delta} < \infty. \tag{2.3}$$

Theorem 2.1 has an immediate corollary for differentiable density functions:

Corollary 2.1. Let $f(x)$ be a continuous pdf on $(0, \infty)$ with $f(0) = 0$. If $f'(0)$ exists and is finite, then $E[X^{-1}] < \infty$.

Proof. From the definition of a derivative,

$$f'(0) = \lim_{x \to 0} ([f(x) - f(0)]/x)$$

$$= \lim_{x \to 0} (1/x)f(x), \tag{2.4}$$

since $f(0) = 0$. The existence of $E[X^{-1}]$ now follows from Theorem 2.1 with $\alpha = 1$.

As mentioned before, the results of this section are strong enough to determine the existence of first moments for many common distributions. We illustrate this with some examples.

Example 2.1. The gamma distribution with parameters $r > 0$ and $\lambda > 0$ has pdf

$$f(x) = [\lambda^r/\Gamma(r)]x^{r-1} \exp(-\lambda x), \qquad 0 \leq x < \infty, \tag{2.5}$$

where $\Gamma(\cdot)$ denotes the gamma function. (The exponential distribution of Ex. 1.1 is a special case, having $r = 1$.) It is easy to check that $f(0) > 0$ if $r \leq 1$, suggesting the nonexistence of $E[X^{-1}]$ if $r \leq 1$. If $r > 1$, then (e.g.)

$$\lim_{x \downarrow 0} f(x)x^{-(r-1)/2} = 0,$$

showing that $E[X^{-1}] < \infty$ if and only if $r > 1$.

Example 2.2. If $f(x)$ is the normal density with mean μ and standard deviation σ, such that

$$f(x) = [1/(2\pi\sigma^2)^{1/2}] \exp[-\tfrac{1}{2}(x - \mu)^2/\sigma^2],$$
$$-\infty < x < \infty, \tag{2.6}$$

then $f(0) > 0$; and as alluded to earlier, $E[X^{-1}]$ does not exist. This example covers the situations outlined in Examples 1.2 (if the population is assumed to be normal) and 1.3 (since the density of $|\overline{X}|$ is positive at zero). Consequently, neither of the "natural" estimates presented in those examples has finite expectation.

Although Theorem 2.1 gives a simple sufficient condition for the existence of the first negative moment, this condition is unfortunately not a necessary one. Indeed, it seems unlikely that a necessary condition can be expressed in terms of limits.

3. THE NONEXISTENCE OF NEGATIVE MOMENTS

The establishment of necessary and sufficient conditions for the existence of moments is, in general, a difficult endeavor. Indeed, when dealing with positive moments, one is led quickly into the theory of characteristic functions (see Pitman 1956). The theory of the existence of negative moments cannot be tied in succinctly with that of characteristic functions and so in a sense is less elegant. Cressie et al. (1981) gave an interesting approach to the calculation of negative moments using moment-generating functions but, in general, did not address the question of existence.

We now give an example of a family of density functions that all violate condition (2.1), with some members having a finite first negative moment and others not having one.

Example 3.1. For any constant $a \in (0, 1)$, define the family of densities $f_n(x)$, $n = 1, 2, \ldots$, by

$$f_n(x) = |\log^n x|^{-1} \Big/ \int_0^a |\log^n t|^{-1} dt, \quad 0 < x < a. \tag{3.1}$$

It is easy to check that for every a, $0 < a < 1$, and $n = 1, 2, \ldots$,

$$\int_0^a |\log^n t|^{-1} dt < \infty,$$

$$\lim_{x \to 0} f_n(x) = 0, \quad \text{and}$$

$$\lim_{x \downarrow 0} f_n(x)/x^{\alpha} = \infty \quad \text{for every} \quad \alpha > 0; \tag{3.2}$$

hence Theorem 2.1 does not apply to $f_n(x)$. Consider now the existence of $E[X^{-1}]$. Clearly, $E[X^{-1}]$ is finite if and only if

$$\int_0^a \frac{dx}{x|\log^n(x)|} < \infty. \tag{3.3}$$

For $n = 1$, the integral in (3.3) can be evaluated as $[|\log(\log[x])|]_0^a$, which diverges for all $a \in (0, 1)$. For $n = 2, 3, \ldots$, however, the integral becomes

$$[|\log^{1-n} x|/(n - 1)]_0^a = |\log^{1-n} a|/(n - 1),$$

which is finite for any $0 < a < 1$. Thus $E[X^{-1}] < \infty$ if and only if $n > 1$, showing that condition (2.1) is not a necessary condition.

In general, a condition involving only limits is not likely to yield a necessary and sufficient condition for the existence of moments; the condition must itself involve an integral. Feller (1971, sec. V.6) presented a theorem that gives a necessary and sufficient condition for the existence of the first positive moment. In our context, this theorem translates as saying that if X is a positive, continuous random variable, then $E[X^{-1}]$ exists if and only if

$$\int_0^\infty x^{-2}F(x)dx < \infty, \tag{3.4}$$

where $F(x)$ is the cdf of X. Although this result is somewhat elegant, in practice it provides little computational simplification.

To further illustrate the delicacy of the question of the existence of the first negative moment, we present two simple results relating to the moments surrounding $E[X^{-1}]$.

Theorem 3.1. Let $f(x)$ be a continuous density on $(0, \infty)$, with $f(0)$ bounded near zero. If $0 < \alpha < 1$, then $E[X^{-\alpha}]$ exists.

Proof. We need only establish the existence of $\int_0^t x^{-\alpha}f(x)dx$ for some $t > 0$. Since $f(x)$ is bounded near zero, there exists some finite constant, $m > 0$, such that $f(x) \leq m$ for all $x < t$. Then

$$\int_0^t \frac{f(x)}{x^\alpha}\,dx \leq m\int_0^t x^{-\alpha}dx < \infty.$$

Thus negative moments of orders strictly less than one exist for all continuous, positive densities bounded near zero. In the other direction, we now show that the negation of condition (2.1) implies the nonexistence of all negative moments of orders greater than one.

Theorem 3.2. Let $f(x)$ be a continuous density on $(0, \infty)$. If $\lim_{x\to 0} f(x)x^{-\alpha}$ diverges for all $\alpha > 0$, then $E[X^{-(1+\delta)}]$ fails to exist for all $\delta > 0$.

Proof. Using integration by parts, we can write

$$\int_0^t \frac{f(x)}{x^{1+\delta}}\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^t \frac{f(x)}{x^{1+\delta}}\,dx$$

$$= \lim_{\varepsilon\to 0}\left\{\left[\frac{F(x)}{x^{1+\delta}}\right]_\varepsilon^t + \int_\varepsilon^t \frac{(1+\delta)F(x)}{x^{2+\delta}}\,dx\right\}.$$

Since $F(x)$ is nondecreasing, we have $F(x) \geq F(\varepsilon)$ on (ε, t); hence

$$\lim_{\varepsilon\to 0}\int_\varepsilon^t \frac{f(x)}{x^{1+\delta}}\,dx$$

$$\geq \lim_{\varepsilon\to 0}\left\{\left[\frac{F(x)}{x^{1+\delta}}\right]_\varepsilon^t + (1+\delta)F(\varepsilon)\int_\varepsilon^t \frac{dx}{x^{2+\delta}}\right\}$$

$$= \lim_{\varepsilon\to 0}\left\{\frac{F(t)}{t^{1+\delta}} - \frac{F(\varepsilon)}{t^{1+\delta}}\right\} = \frac{F(t)}{t^{1+\delta}}.$$

We thus have

$$\int_0^t \frac{f(x)}{x^{1+\delta}}\,dx > \frac{F(t)}{x^{1+\delta}}.$$

But using the hypothesis of the theorem together with l'Hospital's rule, we have $F(t)/t^{(1+\delta)} \to \infty$ as $t \to 0$, showing the nonexistence of $E[X^{-(1+\delta)}]$.

In summary: For a continuous, positive density function f, f bounded near zero implies that $E[X^{-\alpha}]$ exists for every $0 < \alpha < 1$;

$$\lim_{x\to 0} f(x)/x^\alpha < \infty$$

for some $\alpha > 0$ implies that $E[X^{-1}] < \infty$; and $\lim_{x\to 0} f(x)/x^\alpha = \infty$ for every $\alpha > 0$ implies that $E[X^{-(1+\delta)}] = \infty$ for every $\delta > 0$.

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