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# THE MOMENT GENERATING FUNCTION HAS ITS MOMENTS

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*Abstract:* Traditionally, the moment generating function of a random variable $X$, is used to generate (positive integer) moments of $X$. However, moments of quite general transformations of $X$ can be obtained by judicious differentiations and weighted integration of the moment generating function. The particular case of $X^{\gamma}$, $-\infty<\gamma<\infty$, is treated in detail, and applications are given.

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## 1. Introduction

That the moment generating function (provided it is defined) easily yields all positive integer moments, is well known. However it is *not* well known that the moment generating function also contains a wealth of knowledge about arbitrary real moments. The case of negative integer moments was treated by Cressie, Davis, Folks and Policello (1981). The present article derives new results for arbitrary moments and beyond (see Sections 2 and 3), giving a simple general approach to gleaning more from the moment generating function.

Let $X$ be a random variable with distribution function $F$. Then the expectation of any real measurable function $h$, is defined by

$$E(h(X))=\int h(x)\,\mathrm{d}F(x). \tag{1}$$

In particular, the moment generating function (m.g.f.) is

$$M_X(t)=E(\mathrm{e}^{tX}), \tag{2}$$

provided the expectation is finite for $0<|t|<\delta$, $\delta>0$.

In Section 2 we introduce the fractional calculus which will allow us to write $D^\gamma M_X(0)=E(X^\gamma)$, $-\infty<\gamma<\infty$, provided certain integrability conditions hold. Section 3 shows this to be a special case of a more general approach involving Laplace transforms, that says that moments of *functions* of $X$ are also obtainable from the m.g.f. Applications are discussed in Section 4.

## 2. All real moments

### 2.1. Integer moments

It is well known (see e.g. Hogg and Craig, 1970) that provided the m.g.f. is defined in an open neighbourhood of the origin, all nonnegative integer moments exist, are finite, and are given by

$$E(X^n) = D^n M_X(0), \quad n=0,1,2,\ldots, \tag{3}$$

where $D^n M_X(t)$ denotes the $n$-th derivative of $M_X$ evaluated at $t$.

Cressie et al. (1981) give formulas involving *negative* integer moments. Let $X$ be a positive random variable; then

$$E(X^{-m}) = \Gamma(m)^{-1}\int_0^\infty t^{m-1}M_X(-t)\,dt, \quad m=1,2,\ldots, \tag{4}$$

and

$$E(Y/X) = \int_0^\infty \lim_{t_2\to 0-} (\partial/\partial t_2) M_{X,Y}(-t_1,t_2)\,dt_1, \tag{5}$$

where $M_{X,Y}(t_1,t_2)=E(e^{t_1X+t_2Y})$ is the joint m.g.f. of $X>0$ and $Y$. They indicate that all moments might be obtained by implementing a fractional differential calculus for m.g.f.'s (where for example, integration becomes negative integer differentiation).

### 2.2. Fractional calculus

For the history of the fractional calculus, we refer the reader to Oldham and Spanier (1974), and for more recent developments, see Ross (1975). Its potential uses in probability theory have been set out by Gomez and Pestana (1977). Several definitions of differentiation to all real orders have been proposed; we shall use Riemann's approach here, since it is the natural generalization of the $n$-th order integral of a function, seen in (4).

**Definition 1.** Let $\mu>0$. Define the $\mu$-th order integral of the function $f$ as

$$I^\mu f(t) \equiv \Gamma(\mu)^{-1}\int_{-\infty}^t (t-z)^{\mu-1}f(z)\,dz, \tag{6}$$

provided the integral exists.

**Definition 2.** Let $\alpha>0$ be nonintegral, $n$ a positive integer, and $0<\lambda<1$ such that $\alpha=n-\lambda$. Then the $\alpha$-th order derivative of $f$ is defined by

$$\mathrm{D}^{\alpha}f(t) \equiv \Gamma(\lambda)^{-1} \int_{-\infty}^{t} (t-z)^{\lambda-1} (\mathrm{d}^n f(z)/\mathrm{d}z^n)\, \mathrm{d}z, \tag{7}$$

provided the derivative to order $n$ exists and the integral exists. When $\alpha=n$ is a positive integer, $\mathrm{D}^n f(t) \equiv \mathrm{d}^n f(t)/\mathrm{d}t^n$ as usual.

**Definition 3.** For $\mu>0$, define

$$\mathrm{D}^{-\mu} \equiv I^{\mu}. \tag{8}$$

Notice from Definition 1 and (4) that $I^m M_X(0)=E(X^{-m})$, where $X$ is a positive random variable. We already know that $\mathrm{D}^n M_X(0)=E(X^n)$. Definition 2 is essentially saying $\mathrm{D}^{\alpha} \equiv I^{\lambda}\mathrm{D}^n$, or in the language of Definition 3, $\mathrm{D}^{\alpha} \equiv \mathrm{D}^{-\lambda}\mathrm{D}^n$.

Thus the following propositions are easily proved:

**Proposition 1.** *Suppose* $\alpha=n-\lambda>0$, $0<\lambda<1$, *and* $\mathrm{D}^{\alpha}f(t)$ *exists; then*

$$\mathrm{D}^{\alpha}f(t)=\mathrm{D}^{-\lambda}\mathrm{D}^n f(t)=\mathrm{D}^n\mathrm{D}^{-\lambda}f(t).$$

**Proposition 2.** *Suppose* $\mu$, $\upsilon>0$, *and* $\mathrm{D}^{-(\mu+\upsilon)}f(t)$ *exists; then*

$$\mathrm{D}^{-(\mu+\upsilon)}f(t)=\mathrm{D}^{-\mu}\mathrm{D}^{-\upsilon}f(t)=\mathrm{D}^{-\upsilon}\mathrm{D}^{-\mu}f(t).$$

**Proposition 3.** *Suppose* $\alpha$, $\beta>0$, *and* $\mathrm{D}^{\alpha+\beta}f(t)$ *exists; then*

$$\mathrm{D}^{\alpha+\beta}f(t)=\mathrm{D}^{\alpha}\mathrm{D}^{\beta}f(t)=\mathrm{D}^{\beta}\mathrm{D}^{\alpha}f(t).$$

**Proposition 4.** $\mathrm{D}^{\gamma}(\mathrm{e}^{ct})=c^{\gamma}\,\mathrm{e}^{ct}$, $-\infty<t<\infty$, *for* $c>0$ *and all real* $\gamma$.

The last proposition demonstrates a property of this fractional differentiation, not shared by other definitions.

### *2.3. Real moments of positive X*

Suppose for now that $X$ is a positive random variable (we will discuss generalizations later) with distribution function $F$, and assume that $E(X^n)<\infty$ for some positive integer $n$. Let $\alpha=n-\lambda$, $0<\lambda<1$; then clearly $E(X^{\alpha})<\infty$. Thus if $M_X$ is defined, then $E(X^{\alpha})$ exists and is finite for all $\alpha>0$.

**Proposition 5.** *Suppose* $M_X$ *is defined. Then for* $\alpha>0$,

$$\mathrm{D}^{\alpha}M_X(0)=E(X^{\alpha})<\infty. \tag{9}$$

**Proof.** It is easy to see that $M_X$ is analytic in $(-\infty,0]$, and hence the following interchange holds: $D^\alpha M_X(t)=\int_0^\infty D^\alpha(e^{tx})\,dF(x)$, $t\leq 0$. Thus from Proposition 4 the result follows easily. □

With negative moments, in contrast to positive moments, it is not enough for $M_X$ to be defined to ensure their existence. An additional integrability condition is needed.

**Proposition 6.** *Suppose $t^{\mu-1}M_X(-t)$, $\mu>0$, is integrable over $(0,\infty)$. Then*

$$D^{-\mu}M_X(0)=E(X^{-\mu})<\infty. \tag{10}$$

**Proof.**

$$\begin{aligned} D^{-\mu}M_X(0)&=\Gamma(\mu)^{-1}\int_0^\infty z^{\mu-1}M_X(-z)\,dz\\ &=\Gamma(\mu)^{-1}\int_0^\infty\int_0^\infty z^{\mu-1}\,e^{-xz}\,dz\,dF(x), \end{aligned}$$

the interchange being justified by assumption. Recognizing the inner integral as a gamma type integral yields the result. □

We bring together (9) and (10) in the form of:

**Theorem 1.** *For a positive random variable $X$, whose m.g.f. $M_X$ satisfies the regularity conditions of either Proposition 5 or Proposition 6,*

$$D^\gamma M_X(0)=E(X^\gamma),\quad -\infty<\gamma<\infty, \tag{11}$$

*where $D^\gamma$ is defined in Definition 2 and Definition 3.*

Note that when the random variable $X$ has *nonpositive* realizations (i.e. $F(0)>0$), there may be trouble in defining $E(X^\alpha)$ for *all* real $\alpha$. For example suppose $X$ is normal: $E(X^{1/2})$ is not defined, however $E(X^{1/3})$ could be defined using the convention $x^{1/3}=\operatorname{sgn}(x)\,|x|^{1/3}$. Unfortunately Proposition 5 is not true even in this case, since $M_X(t)=e^{t^2/2}$, and $\int_0^\infty t^{2/3}\,e^{t^2/2}\,dt=\infty$. The correct way to handle this is to work with the joint m.g.f. $M_{|X|,Y}(t_1,t_2)$, of $|X|$ and $Y=\operatorname{sgn}(X)$ (Cressie et al., 1981). We will not pursue the matter here. A *nonnegative* random variable in Proposition 5 causes no problems at all, but Proposition 6 generally would not be true if $F(0)>0$ (remember $F(0-)=0$, for nonnegative random variables).

### 2.4. The characteristic function approach

A brief mention should be made of the approach of Laue (1980), who worked with a fractional calculus applied to characteristic functions. He used a slightly different definition for $D^\alpha f(t)$. Let $\alpha=l+\kappa$, $l$ a nonnegative integer and $0<\kappa<1$; he defined

$$\mathrm{D}^{\alpha} f(t) \equiv \Gamma(1-\kappa)^{-1} \int_{-\infty}^{t} \frac{\mathrm{D}^{l} f(t)-\mathrm{D}^{l} f(z)}{(t-z)^{1+\kappa}} \, \mathrm{d}z .$$

However in our definition we work with the decomposition $\alpha=(l+1)-\lambda$, $0<\lambda<1$. Although it appears as if one more derivative is needed in our definition of $\mathrm{D}^{\alpha}$ (see (7)), Laue actually needs differentiality of $\mathrm{D}^{l} f$ at $t$, for his defining integral to exist.

Suppose $\psi(t) \equiv E(\mathrm{e}^{\mathrm{i}tx})$ is the characteristic function of the nonnegative random variable $X$. Laue proves for example that

$$E(X^{\alpha})=\operatorname{Re}[\mathrm{i}^{l} \mathrm{D}^{\alpha} \psi(0)] / \cos(\kappa \pi / 2),$$

where $\alpha>0$, $\alpha=l+\kappa$, and $0<\kappa<1$. He also looks at absolute moments of arbitrary random variables. Furthermore he promises results for all real moments, but unfortunately does not give them.

Our Propositions 5 and 6 address the problem of *all* real moments, showing how to obtain them from the *moment* generating function.

## 3. A generalization

Consider the following formal manipulations: Let $w(\cdot)$ be a function defined on $[0, \infty)$. Then,

$$\begin{aligned} \int_{0}^{\infty} M_{X}(-z) w(z) \, \mathrm{d}z &= \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-zx} w(z) \, \mathrm{d}z \, \mathrm{d}F(x) \\ &= \int_{0}^{\infty} L_{w}(x) \, \mathrm{d}F(x)=E(L_{w}(X)), \end{aligned} \tag{12}$$

where $L_{w}$ is the Laplace transform of the function equal to $w(x)$ on $[0, \infty)$ (and zero elsewhere). Now if we could find a $w$ whose Laplace transform is $L_{w}(x)=x^{-\mu}$, then (12) would yield $E(X^{-\mu})$.

In fact, it is a simple matter to look through a table of functions and their Laplace transforms (e.g. Abramowitz and Stegun, 1965, p. 1021), focussing on the Laplace transform column to find interesting functions $L_{w}(x)$. Then read across the table to find the function $w(z)$ to integrate against $M_{X}(-z)$. A variety of examples are set out in Table 1. For example (formally),

$$E(\log(1+a X^{-1}))=\int_{0}^{\infty} M_{X}(-z)\{z^{-1}-z^{-1} \mathrm{e}^{-az}\} \, \mathrm{d}z,$$

although there is likely to be little need for such a moment. The most important example is of course (formally),

$$E(\Gamma(\mu) X^{-\mu})=\int_{0}^{\infty} M_{X}(-t) t^{\beta-1} \, \mathrm{d}t .$$

Table 1
Examples of Laplace transforms and their inverses (mostly taken from Abramowitz and Stegun, 1965, p. 1021 ff)

| $L_w(x)$ | $w(z)$ |
|---|---|
| $x^{-n}$ | $z^{n-1}/(n-1)!,\ n=1,2,\ldots$ |
| $x^{-1/2}$ | $(\pi z)^{-1/2}$ |
| $x^{-3/2}$ | $2(z/\pi)^{1/2}$ |
| $x^{-(n+1/2)}$ | $\dfrac{2^n z^{n-1/2}}{1\cdot 3\cdots(2n-1)\pi^{1/2}},\ n=1,2,\ldots$ |
| $\Gamma(\mu)/x^{\mu}$ | $z^{\mu-1},\ \mu>0$ |
| $\Gamma(\mu)/(x+a)^{\mu}$ | $z^{\mu-1}\,\mathrm{e}^{-az},\ \mu>0$ |
| $(x^2-a^2)^{-1}$ | $a^{-1}\sinh az$ |
| $\log(1+a/x)$ | $z^{-1}-z^{-1}\,\mathrm{e}^{-az}$ |

## 4. Concluding remarks and applications

Moments of random variables are not always easy to evaluate. This paper gives an alternative technique to try (see Theorem 1) when the m.g.f. is known. It may be that *only* the m.g.f. is known (e.g. when independent random variables with known m.g.f.'s are added to form a new random variable).

Fractional moments have proved extremely useful in 'method of moments' estimation procedures. Instead of matching $E(X), E(X^2), \ldots$ to sample moments $\bar{X}, \sum_{i=1}^{n} X_i^2/n, \ldots$, in certain circumstances more stability is attainable through matching $E(X^{\alpha_1}), E(X^{\alpha_2}), \ldots$, to $\sum_{i=1}^{n} X_i^{\alpha_1}/n, \sum_{i=1}^{n} X_i^{\alpha_2}/n, \ldots$, (Tallis and Light, 1968; Kumar, Nicklin and Paulson, 1979; Brockwell and Brown, 1981; Srivastava and Bhatnagar, 1981). One needs therefore, methods to evaluate the chosen fractional moments.

Another application arises when variables are transformed (e.g. to obtain additivity, or homoscedasticity, or normality). Then it is often the moments of the transformed variables which are of interest. For example Cressie and Hawkins (1980) consider $(X_1, X_2)$ jointly normal with mean $\mu$, variance $\sigma^2$ and correlation $\varrho$. Therefore $(X_1-X_2)^2$ is distributed as $c\times\chi_1^2$, where $\chi_n^2$ denotes a chi-squared random variable on $n$ ($n$ a positive integer) degrees of freedom; $c=2\sigma^2(1-\varrho)$. They then look for a power transformation for which $\{(X_1-X_2)^2\}^{\lambda}$ is approximately normal. The Wilson–Hilferty transformation (Wilson and Hilferty, 1931) has the property that for $n>3$ (and possibly even $n=2$), $(\chi_n^2)^{1/3}$ is approximately normal. One would expect for $(\chi_1^2)^{\lambda}$, something more severe than $\lambda=\frac{1}{3}$, but not as severe as $\lambda=0$ ($\log\chi_1^2$ is skewed in the *opposite* direction to $\chi_1^2$). From third and fourth moment considerations, Cressie and Hawkins choose $\lambda=\frac{1}{4}$, and have to evaluate $E((\chi_1^2)^{1/4})$, $\operatorname{var}((\chi_1^2)^{1/4})$.

Now $M_{\chi_1^2}(t)=(1-2t)^{-1/2}$, $t<\frac{1}{2}$. We want $\mathrm{D}^{1/4}M_{\chi_1^2}(0)$, i.e. $\alpha=1-\frac{3}{4}$. Hence from Theorem 1,

$$\begin{aligned}
E((\chi_1^2)^{1/4}) &= \Gamma(\tfrac{3}{4})^{-1}\int_0^\infty z^{-1/4}(1+2z)^{-3/2}\,\mathrm{d}z \\
&= \Gamma(\tfrac{3}{4})^{-1}\int_0^1 2^{3/4}w^{-1/4}(1-w)^{-1/4}\,\mathrm{d}w, \quad z=(1-w)/2w, \\
&= 2^{1/4}\Gamma(\tfrac{3}{4})/(\pi)^{1/2} = 0.82216,
\end{aligned}$$

and

$$\operatorname{var}((\chi_1^2)^{1/4}) = 2^{1/2}\{\pi^{-1/2}-\Gamma^2(\tfrac{3}{4})/\pi\} = 0.12192.$$

In applications that involve a countable number of possible realizations, a probability generating function $P_X(s)=\sum_{j=0}^\infty \Pr\{X=j\}s^j$, is often known. Through the relationship $M_X(t)=P_X(\mathrm{e}^t)$, the results of this paper can be equally applied to obtain $E(X^\gamma)$ from $P_X$.

Finally the m.g.f. *does* have *its* moments. Suppose

$$\Gamma(m)^{-1}\int_0^\infty z^{m-1}M_X(-z)\,\mathrm{d}z, \quad m=1,2,\ldots,$$

is finite. Then $\int_0^\infty M_X(-z)\,\mathrm{d}z$ is finite, and hence

$$\int_0^\infty z^{m-1}\left\{M_X(-z)\Big/\int_0^\infty M_X(-w)\,\mathrm{d}w\right\}\mathrm{d}z = \Gamma(m)E(X^{-m})/E(X^{-1}).$$

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