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An Approximation to the Negative Moments of the Positive Binomial Useful in Life Testing

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The purpose of this paper is to obtain the mean and variance of the maximum likelihood estimator of the scale parameter of a Weibull distribution where the sample is censored at a fixed time. It will be shown that these moments are functions of the negative moments of the positive binomial distribution. A simple approximation is obtained for the negative moments of the positive binomial, thus giving an approximate expression for the mean and variance of the estimator.

1. INTRODUCTION

A family of distributions useful in fitting failure populations is the function

$$f(t) = \left(\frac{m}{\alpha}\right)t^{m-1}e^{-t^m/\alpha} \quad \text{where} \quad t \geq 0, \alpha, m > 0. \tag{1.1}$$

It is known by engineers as the Weibull function as a result of its application by W. Weibull to several problems in engineering (10). The parameter m is called the shape parameter, the choice of which provides a wide range of distributional forms including the negative exponential ($m = 1$). The properties of the distribution as well as several methods of estimating the parameters, m and α, are discussed by Kao (7). In the case where the shape parameter is assumed known and we desire to estimate α, the maximum likelihood estimation of α reduces to the problem of estimating the parameters of the negative exponential distribution due to the invariance property of maximum likelihood estimators. Problems of estimation and tests concerning the parameter of a negative exponential distribution have been investigated by Epstein and Sobel (2, 3) and others. These results are applicable to the Weibull distribution when m is assumed known. An extensive list of references is given in (8).

Data obtained from a life test have the property that the observations are received in order of magnitude. The first item failing will be the smallest, the second will be the next to the smallest, and so on. We can take advantage of this property by concluding the test after either a fixed number of items fail or after a fixed length of time has elapsed. Such sampling procedures are called censored sampling. By increasing sample size, the length of time for a fixed number of units to fail will be reduced. One is therefore able to buy time at the expense of increased sample size. Censoring after a fixed number of items

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has failed has the advantage of providing a more or less uniform amount of information in repeated sampling with the disadvantage that the length of testing time varies from test to test. Censoring after a fixed length of testing time provides a constant length of test time in repeated tests with the amount of information varying from test to test. One advantage of the latter censoring procedure is that it simplifies the problem of test scheduling in a production process where information from periodic production lots must be obtained at regular time intervals.

In this paper we are interested in investigating the properties of the maximum likelihood estimator of the scale parameter, α, where it is assumed that m is known and that the sample is censored at a fixed time T. It can be shown that the maximum likelihood estimator of α is

$$\hat{\alpha} = \frac{\sum_{i=1}^{r} t_i^m + (n - r)T^m}{r} \tag{1.2}$$

where

n = number of items placed on test
T = length of test time
r = number failing before time T
$n - r$ = number surviving to time T
t_i = length of life of the ith ordered failure (where t_1 is the smallest, etc.)

In an experimental situation, the experimenter would like to know how large the sample size should be in order that the estimates fall within a given distance of the true value α. One method of expressing the "distance" numerically would be to use the mean square error, $E\{(\hat{\alpha} - \alpha)^2\}$, where E represents the operation of taking the average or expected value of the square of the difference $(\hat{\alpha} - \alpha)$. The problem may then be restated, "How large should the sample size be in order that $E\{(\hat{\alpha} - \alpha)^2\} = K$, where K is some desired value?" Or, we may ask the converse, "For a given sample size and censoring time, what will be the size of the mean square error?"

It will be noted that the mean square error is a function of the mean and variance of $\hat{\alpha}$:

$$E\{(\hat{\alpha} - \alpha)^2\} = V(\hat{\alpha}) + \{E(\hat{\alpha}) - \alpha\}^2 \tag{1.3}$$

where

$V(\hat{\alpha})$ = variance of $\hat{\alpha}$ and
$E(\hat{\alpha})$ = average or expected value of $\hat{\alpha}$.

We will derive an expression for the mean and variance of $\hat{\alpha}$, showing that both are functions of the negative moments of the positive binomial, i.e., functions of $E\{1/r^k\}$ where r is distributed as the truncated binomial,

$$P(r) = \frac{\binom{n}{r} p^r q^{n-r}}{\sum_{r=1}^{n} \binom{n}{r} p^r q^{n-r}}, \qquad r = 1, 2, \cdots, n. \tag{1.4}$$

A simple approximation to these negative moments will be derived and compared with some tabulated exact values for $k = 1, 2$.

2. The Mean and Variance of $\hat{\alpha}$

Let t equal the length of time to failure for an experimental unit and let the probability density function for t (failure distribution) be

$$f(t) = \left(\frac{m}{\alpha}\right)t^{m-1}e^{-t^m/\alpha}, \qquad t \geq 0 \tag{2.1}$$

A random sample of n units is tested until a predetermined time T at which time the test is concluded. Times to failure for r observations are observed where it will be noted that r is random. Thus we observe $t_1, t_2, \cdots t_r$ where t_j is the jth ordered observation. A total of r units failed and $(n - r)$ units survived.

For the sake of convenience and without loss of generality, we can measure time in units of size T. Then

$$x = \frac{t}{T},$$

and

$$f(x) = \frac{m}{\beta}x^{m-1}e^{-x^m/\beta}, \qquad \beta, m > 0, \tag{2.2}$$

where

$$x \geq 0 \quad \text{and} \quad \beta = \frac{\alpha}{T^m}$$

It can be shown that the maximum likelihood estimator of β is

$$\hat{\beta} = \left(\frac{1}{r}\right)\left[\sum_{i=1}^{r} x_i^m + (n - r)\right]. \tag{2.3}$$

It would follow that $\hat{\alpha} = T^m\hat{\beta}$.

In finding the mean and variance of $\hat{\beta}$, and hence $\hat{\alpha}$, we will restrict our results to the set of estimates for which $r > 0$. When $r = 0$, the maximum likelihood estimate is infinite. In order to obtain more information on α, either n or T (or both) should be increased.

The expected value of $\hat{\alpha}$ for the case $m = 1$ is given by Bartholomew (1). It can be shown that the expected value of $\hat{\beta}$ remains the same, regardless of the value of m and that

$$E(\hat{\beta}) = \beta - \frac{q}{p} + nE\left(\frac{1}{r}\right) - 1 \tag{2.4}$$

where

$$q = e^{-1/\beta}, \qquad p = 1 - q \quad \text{and} \quad r = 1, 2, \cdots, n.$$

The variance of $\hat{\beta}$ is equal to

$$V(\hat{\beta}) = V\left(\frac{1}{r}\sum_1^r x_i^m\right) + n^2 V\left(\frac{1}{r}\right) + 2n\,\text{Cov.}\left(\frac{1}{r}\sum_1^r x_i^m, \frac{1}{r}\right) \tag{2.5}$$

Denote the expectation or variance of a quantity, given r, as E_r or V_r, respectively, and let $P(j)$ be the probability that r equals j. Then considering the first term of (2.5),

$$V_r\left(\frac{1}{r}\sum_1^r x_i^m\right) = \frac{1}{r} V(x^m), \qquad x \leq 1, \tag{2.6}$$

and

$$\begin{aligned} V\left(\frac{1}{r}\sum_1^r x_i^m\right) &= P(1)V(x^m) + P(2)\frac{V(x^m)}{2} \\ &\quad + \cdots + P(n)\frac{V(x^m)}{n} \\ &= V(x^m)\sum_1^n \frac{1}{r}P(r) \\ &= E\left(\frac{1}{r}\right)V(x^m) \end{aligned} \tag{2.7}$$

It can be shown that the variance of x^m, given $x \leq 1$, is equal to

$$V(x^m) = \beta^2 - q/p^2 \tag{2.8}$$

from which it follows that

$$V\left(\frac{1}{r}\sum_1^r x_i^m\right) = (\beta^2 - q/p^2)E(1/r) \tag{2.9}$$

The third term of (2.5) involves the covariance of $1/r \sum_1^r x^m$ and $1/r$.

$$\text{Cov}\left(\frac{1}{r}\sum_1^r x_i^m, \frac{1}{r}\right) = E\left(\frac{1}{r^2}\sum_1^r x_i^m\right) - E\left(\frac{1}{r}\sum_1^r x_i^m\right)E\left(\frac{1}{r}\right) \tag{2.10}$$

Note that
$$E_r\left(\frac{1}{r}\sum_1^r x_i^m\right) = E(x^m) \tag{2.11}$$

and hence that
$$E\left(\frac{1}{r}\sum_1^r x_i^m\right) = \sum_{r=1}^n P(r)E_r\left(\frac{1}{r}\sum_1^r x_i^m\right) = E(x^m) \tag{2.12}$$

Also,
$$\begin{aligned} E\left(\frac{1}{r^2}\sum_1^r x_i^m\right) &= P(1)E_1(x^m) + \frac{P(2)}{2}E_2\left(\frac{1}{2}\sum_1^2 x_i^m\right) \\ &\quad + \cdots + \frac{P(n)}{n}E_n\left(\frac{1}{n}\sum_1^n x_i^m\right) \end{aligned} \tag{2.13}$$

Utilizing (2.11),
$$E\left(\frac{1}{r^2}\sum_1^r x_i^m\right) = E(x^m)E\left(\frac{1}{r}\right) \tag{2.13}$$

Substituting (2.12) and (2.14) into (2.10), we find that the

$$\operatorname{Cov}\left(\frac{1}{r}\sum_{1}^{r} x_i^m, \frac{1}{r}\right) = 0$$

and

$$\begin{aligned} V(\hat\beta) &= V\left(\frac{1}{r}\sum_{1}^{r} x^m\right) + n^2 V\left(\frac{1}{r}\right) \\ &= E\left(\frac{1}{r}\right)V(x^m) + n^2 V\left(\frac{1}{r}\right) \qquad (2.15) \\ &= n^2 V\left(\frac{1}{r}\right) + (\beta^2 - q/p^2)E\left(\frac{1}{r}\right) \end{aligned}$$

Summarizing; the mean, bias, and variance of $\hat\beta$ are

$$E(\hat\beta) = \beta - q/p + nE\left(\frac{1}{r}\right) - 1 \qquad (2.16)$$

Bias of $\hat\beta = E(\hat\beta) - \beta$

$$= nE\left(\frac{1}{r}\right) - q/p - 1 \qquad (2.17)$$

$$V(\hat\beta) = n^2 V\left(\frac{1}{r}\right) + (\beta^2 - q/p^2)E\left(\frac{1}{r}\right) \qquad (2.18)$$

It follows that

$$E(\hat\alpha) = T^m E(\hat\beta), \qquad (2.19$$

$$V(\hat\alpha) = T^{2m} V(\hat\beta), \qquad (2.20$$

and Bias $\hat\alpha = T^m$ (Bias of $\hat\beta$) (2.21)

The mean square error of $\hat\beta$ is given by the equation

$$E(\hat\beta - \beta)^2 = V(\hat\beta) + (\text{Bias } \hat\beta)^2 \qquad (2.22)$$

and it would follow from (2.20) and (2.21) that

$$E(\hat\alpha - \alpha)^2 = T^{2m} E(\hat\beta - \beta)^2. \qquad (2.23)$$

It is apparent that the mean square errors of $\hat\beta$ and $\hat\alpha$ are functions of the moments of $1/r$ since

$$V\left(\frac{1}{r}\right) = E\left(\frac{1}{r^2}\right) - \left[E\left(\frac{1}{r}\right)\right]^2 \qquad (2.24)$$

and can be computed given n, T, an approximate value of α, and the first two moments of $1/r$. Desk computation of $E(1/r)$ and $E(1/r^2)$ is exceedingly laborious. To overcome this difficulty, we present an abbreviated set of tables in Section 3 and a simple approximation formula in Section 4.

3. Negative Moments of the Positive Binomial Distribution

Examples of the use of the negative moments of the positive binomial distribution are given in papers by Stephan (9) and Grab and Savage (5). The latter paper discusses in particular the situation arising in sample survey theory where it is desired to find the variance of a sample mean,

$$\bar{y} = \frac{\sum_{1}^{r} y_i}{r} \quad \text{where the} \quad y_i\,, \qquad i = 1, 2, \cdots r,$$

are identically distributed with mean equal to μ, variance equal to σ^2 and r is a random variable distributed as a positive binomial variate. The variance of $\bar{y}$ is given by

$$V(\bar{y}) = \sigma^2 E\left(\frac{1}{r}\right), \quad \text{thus requiring}$$

the first negative moment of r. Reference is made to extensive discussions of this problem in (4) and (6). A table of $E(1/r)$ is given in (5) for

$$p = .01,\ .05(.05).95,\ .99 \quad \text{and} \quad n = 2(1)20.$$

Table 1.

Negative Moments of Positive Binomial Distribution $E(1/r^k)$

	$k = 1$					
n / $100\,p$	5	10	15	20	30	40
5	.94920	.88767	.82859	.77222	.66830	.57680
10	.89696	.77768	.67069	.57691	.42811	.32381
15	.84357	.67312	.53411	.42557	.28174	.20130
20	.78940	.57682	.42284	.31750	.19911	.14190
25	.73489	.49095	.33667	.24403	.15157	.10937
30	.68055	.41674	.27228	.19472	.12215	.08909
35	.62697	.35440	.22502	.16104	.10239	.07523
40	.57474	.30327	.19034	.13717	.08821	.06512
45	.52451	.26204	.16453	.11954	.07752	.05742
50	.47688	.22911	.14486	.10599	.06915	.05136
55	.43241	.20285	.12945	.09525	.06243	.04645
60	.39156	.18177	.11707	.08650	.05690	.04241
65	.35465	.16467	.10689	.07925	.05227	.03901
70	.32183	.15057	.09837	.07312	.04835	.03612
75	.29311	.13876	.09112	.06788	.04497	.03362
80	.26829	.12872	.08488	.06335	.04203	.03145
85	.24704	.12006	.07945	.05938	.03946	.02955
90	.22891	.11252	.07467	.05588	.03718	.02786
95	.21340	.10589	.07044	.05278	.03515	.02635

TABLE 1—*Continued*

n / 100 p	$k = 2$					
	5	10	15	20	30	40
5	.92438	.83489	.75122	.67352	.53616	.42214
10	.84785	.68013	.53810	.42095	.25209	.14993
15	.77099	.54003	.36857	.24789	.11260	.05498
20	.69446	.41791	.24342	.14155	.05290	.02453
25	.61905	.31567	.15720	.08136	.02811	.01353
30	.54559	.23352	.10116	.04881	.01708	.00863
35	.47499	.17014	.06622	.03130	.01152	.00601
40	.40815	.12303	.04488	.02155	.00834	.00444
45	.34594	.08915	.03184	.01577	.00634	.00342
50	.28915	.06540	.02365	.01209	.00498	.00272
55	.23840	.04902	.01831	.00959	.00403	.00221
60	.19410	.03775	.01464	.00782	.00332	.00183
65	.15642	.02991	.01201	.00650	.00279	.00154
70	.12526	.02434	.01005	.00549	.00238	.00132
75	.10023	.02026	.00854	.00470	.00205	.00114
80	.08069	.01717	.00736	.00407	.00178	.00099
85	.06582	.01477	.00640	.00356	.00157	.00088
90	.05470	.01285	.00563	.00314	.00139	.00078
95	.04637	.01129	.00498	.00279	.00124	.00070

and for

$$p = .01, .05(.05).50, .99 \quad \text{and} \quad n = 21(1)30.$$

In this paper we present tables of $E(1/r^k)$ for

$$p = .05\ (.05)\ .95, n = 5, 10, 15, 20, 30, 40, \quad \text{and}$$
$$k = 1, 2, 3, 4.$$

Table 1 presents the negative moments for $k = 1, 2$; Table 2 for $k = 3, 4$.

The tabulated results are obtained from the relation

$$E\left(\frac{1}{r^k}\right) = \frac{1}{1 - q^n} \sum_{r=1}^{n} \frac{1}{r^k} \binom{n}{r} p^r q^{n-r}$$

and are rounded to five decimal places.

4. AN APPROXIMATION TO THE NEGATIVE MOMENTS OF THE POSITIVE BINOMIAL DISTRIBUTION

Several approximations have been given for negative moments of the positive binomial distribution. Stephan (9) achieves an approximation by expanding $E(1/r)$ in a series of inverse factorials which converge to the true value of $E(1/r)$ as more and more terms are added. An extension of this result is utilized to obtain the higher negative moments. Stephan's procedure permits the calculation of $E(1/r^k)$ to any degree of accuracy and enables one to calculate upper and lower bounds for the exact value. Its disadvantage is that the formula for $E(1/r^k)$ lacks computational simplicity.

Numerous references to approximations for $E(1/r)$ are given in (5). Specifically, Grab and Savage mention the approximation

$$E\left(\frac{1}{r}\right) \approx \frac{1}{np - q} \tag{4.1}$$

and state that one can expect two place accuracy for values of $np > 10$. We will derive an approximation which is nearly as simple as (4.1), which can be used to approximate the negative moments for $k \geq 1$, and which appears to be more accurate than (4.1) for values of $np > 5$.

As previously mentioned, Stephan derives his approximation by expanding $1/r$ into a series and then finding the expected value of the first t terms. In other words, he obtains the expected value of an approximation to $1/r$. Rather than approximate $1/r$, we will approximate the probability distribution function of r and then find the exact value of $E(1/r)$ for the approximating distribution. In consideration of the moments desired, the Beta function is an obvious choice.

The Beta function is given by the equation

$$f(z) = \frac{z^{a-1}(1 - z)^{b-1}}{B(a, b)} \tag{4.2}$$

where

$$B(a, b) = \frac{(a - 1)!\,(b - 1)!}{(a + b - 1)!}\,; a, b > 0, \quad \text{and} \quad 0 \leq z \leq 1.$$

Let $y = r/n$ where r is the positive binomial variate. Then the probability distribution of y will be a histogram ranging from $1/2n$ to $(2n + 1)/2n$. We will approximate this function with the Beta function, $f(z)$, although obviously an error will be introduced because of the discrepancy between the ranges of y and z. This is negligible for large values of np.

The parameters of the approximating Beta function were obtained by equating the first two moments of the two distributions. The two equations are:

$$\frac{na}{(a + b)} = np/(1 - q^n) \tag{4.3}$$

and

$$n^2 \frac{a(a + 1)}{(a + b)(a + b + 1)} = (npq + n^2p^2)/(1 - q^n)$$

where the moments of $P(r)$ are obtained from (9). In order to simplify the solution for a and b, let us assume that np is large and hence that $(1 - q^n)$ is approximately equal to one. The values of a and b are then the simultaneous solution of and

$$\frac{na}{a+b} = np$$
$$\frac{n^2 a(a+1)}{(a+b)(a+b+1)} = npq + n^2p^2 \tag{4.3}$$

Solving,

$$a = (n-1)p$$
$$b = (n-1)q \tag{4.4}$$

The approximating distribution is then

$$f(y) = \frac{y^{a-1}}{B(a, b)}(1-y)^{b-1}$$
$$= \frac{y^{(n-1)p-1}(1-y)^{(n-1)q-1}}{B(a, b)} \quad \text{where} \quad 0 \leq y \leq 1. \tag{4.5}$$

The negative moments of y are easily obtained from (4.5) yielding

$$E\left(\frac{1}{y^k}\right) = \frac{B(a-k, b)}{B(a, b)} \tag{4.6}$$

$$= \frac{(a-k-1)!(b-1)!}{(a+b-k-1)!} \bigg/ \frac{(a-1)!(b-1)!}{(a+b-1)!}$$
$$= \frac{(a+b-1)(a+b-2)\cdots(a+b-k)}{(a-1)(a-2)\cdots(a-k)} \tag{4.7}$$

Specifically, from (4.7),

$$E\left(\frac{1}{y}\right) = \frac{(a+k-1)}{a-1} = \frac{n-2}{a-1} \tag{4.8}$$

and

$$E\left(\frac{1}{y^2}\right) = \frac{(n-2)(n-3)}{(a-1)(a-2)} \tag{4.9}$$

As noted in (4.1), in order for the moments (4.6) to exist, it is necessary that

$$a = (n-1)p > k. \tag{4.10}$$

The approximations to the moments of $1/r$ are easily obtained from the moments of $1/y$. Since $1/r = 1/(ny)$

$$E\left(\frac{1}{r^k}\right) \approx \frac{1}{n^k} E\left(\frac{1}{y^k}\right) \tag{4.11}$$

In particular,

$$E\left(\frac{1}{r}\right) \approx \frac{n-2}{n(a-1)} \tag{4.12}$$

and

$$E\left(\frac{1}{r^2}\right) \approx \frac{(n-2)(n-3)}{n^2(a-1)(a-2)} \tag{4.13}$$

Utilizing (4.12) and (4.13), it can be shown that

$$V\left(\frac{1}{r}\right) \approx \frac{(n-2)(n-a-1)}{n^2(a-1)^2(a-2)} \tag{4.14}$$

In order to distinguish these approximations from others, we will call them the Beta approximations.

The accuracy of these approximations will be discussed in Section 5. It might be well, however, to comment upon the errors introduced by our simplifying assumptions. The first error was caused by the error in approximating the range of the histogram of $P(r)$ as given by equation (1.4). The second was caused by the assumption that q^n is small and hence that the equations for the solution of a and b are approximately as given in (4.3). Both assumptions were made to obtain simplicity in the approximation and both errors diminish as np increases. We will see in Section 5 that the approximation is quite good for moderate sized values of np in spite of the assumptions.

5. A Comparison of the Approximation With Tabulated Values of $E(1/r)$ and $E(1/r^2)$

A comparison of the approximation to $E(1/r)$ with the tabulated values for various combinations of n and p is presented in Table 3. In addition, the approximation (4.1) is included. Proceeding from top to bottom, the three values in each cell are the tabulated value, the Beta approximation (4.12), and the approximation (4.1), respectively. All values are rounded in the sixth decimal place. The tabulated value for $n = 100$, $p = .90$ is omitted due to computing difficulties. Also, no approximation is given for the cells where $a \leq k$.

Examining Table 3, we note that the Beta approximation is correct to two decimal places for $np \geq 5$ and in several instances is correct to six decimal places. The approximation (4.1) also appears to be quite good although the Beta approximation would, in general, appear to be more accurate and hence more suitable for use in computing the mean square error of the estimator $\hat{\alpha}$ mentioned in Section 2. As n and p increase, $(a - 1) \rightarrow (n - 2)$, $q \rightarrow 0$, and as a result, both the Beta approximation and the approximation given by equation (4.1) approach $1/n$.

Stephan (8) gives two examples of the use of his approximation. In the first example, $n = 100$, $p = .1$ and $E(1/r) = .111527$, correct to six decimal places. Utilizing two terms of the series, he obtains $E(1/r) \approx .108675$; for three terms, $E(1/r) \approx .110548$. The Beta approximation is .110112. In the second example, $n = 1000$, $p = .3$ and $E(1/r)$, correct to nine places, is .00334116. Two terms of the series yield $E(1/r) \approx .00334108$ and for three terms, $E(1/r) \approx .00334116$. The Beta approximation is .00334114. Thus, in these two examples, it requires three terms of Stephan's series to equal the accuracy of the Beta approximation.

The tabulated value of $E(1/r^2)$ and the Beta approximation are given in

TABLE 2

Negative Moments of Positive Binomial Distribution $E(1/r^k)$

	$k = 3$					
n / $100\,p$	5	10	15	20	30	40
5	.91216	.80959	.71513	.62878	.47972	.36014
10	.82408	.63555	.48109	.35810	.19045	.09761
15	.73651	.48239	.30380	.18559	.06578	.02326
20	.65028	.35318	.18042	.08892	.02152	.00604
25	.56636	.24903	.10141	.04046	.00745	.00206
30	.48574	.16905	.05459	.01818	.00303	.00094
35	.40946	.11061	.02867	.00848	.00149	.00052
40	.33854	.07000	.01508	.00430	.00086	.00032
45	.27391	.04314	.00821	.00244	.00055	.00021
50	.21634	.02619	.00476	.00153	.00038	.00015
55	.16639	.01593	.00299	.00104	.00027	.00011
60	.12435	.00991	.00202	.00074	.00020	.00008
65	.09015	.00643	.00144	.00055	.00015	.00006
70	.06345	.00440	.00108	.00042	.00012	.00005
75	.04352	.00318	.00083	.00033	.00009	.00004
80	.02942	.00240	.00065	.00027	.00008	.00003
85	.01999	.00187	.00052	.00022	.00006	.00003
90	.01401	.00149	.00043	.00018	.00005	.00002
95	.01034	.00121	.00035	.00015	.00004	

	$k = 4$					
n / $100\,p$	5	10	15	20	30	40
5	.90611	.79729	.69792	.60784	.45436	.33346
10	.81245	.61457	.45536	.33099	.16640	.07942
15	.71985	.45627	.27647	.16130	.05059	.01494
20	.62925	.32509	.15588	.07080	.01341	.00255
25	.54167	.22155	.08167	.02833	.00332	.00048
30	.45820	.14399	.03987	.01053	.00085	.00013
35	.37993	.08902	.01825	.00374	.00026	.00005
40	.30790	.05225	.00793	.00134	.00011	.00002
45	.24305	.02909	.00334	.00052	.00005	.00001
50	.18612	.01539	.00142	.00024	.00003	.00001
55	.13758	.00779	.00065	.00012	.00002	.0000
60	.09760	.00384	.00033	.00008	.00001	.00000
65	.06599	.00190	.00019	.00005	.00001	.00000
70	.04221	.00099	.00012	.00003	.00001	.00000
75	.02535	.00057	.00008	.00002	.00000	.00000
80	.01426	.00036	.00006	.00002	.00000	.00000
85	.00762	.00025	.00004	.00001	.00000	.00000
90	.00408	.00018	.00003	.00001	.00000	.00000
95	.00240	.00013	.00002	.00001	.00000	

TABLE 3. *A Comparison of the Tabulated Value of $E(1/r)$, the Beta Approximation, and the Approximation Given by Equation (4.1)*
Entries: Tabulated Value
Beta Approximation
Approximation (4.1)

p \ n	5	10	20	30	40	100
.10			.576912	.428114	.323807	.111527
			1.00000	.491228	.327586	.110112
			.909091	.476190	.322581	.109890
.30	.680554	.416735	.194724	.122147	.089094	.034159
	3.00000	.470588	.191489	.121212	.088785	.034146
	1.25000	.434783	.188679	.120482	.088496	.034130
.50	.476882	.229109	.105990	.069152	.051356	.020206
	.600000	.228571	.105882	.069136	.051351	.020206
	.500000	.222222	.105263	.068966	.051282	.020202
.70	.321834	.150568	.073123	.048345	.036116	.014348
	.333333	.150943	.073171	.048359	.036122	.014348
	.312500	.149254	.072993	.048309	.036101	.014347
.90	.228913	.112523	.055885	.037180	.027857	
	.230769	.112676	.055901	.037185	.027859	.011124
	.227273	.112360	.055866	.037175	.027855	.011123

TABLE 4. *A Comparison of the Tabulated Value of $E(1/r^2)$ and the Beta Approximation*
Entries: Tabulated Value
Beta Approximation

p \ n	5	10	20	30	40	100
.1				.252093	.149937	.014559
				.049123	.159483	.013520
.3		.233527	.048813	.017077	.008627	.001198
		.470588	.043991	.016282	.008467	.001196
.5		.065403	.012090	.004985	.002716	.000413
		.064000	.012000	.004978	.002714	.000413
.7	.125262	.024343	.005489	.002375	.001320	.000207
	.166667	.024572	.005504	.002378	.001321	.000207
.9	.054698	.012847	.003143	.001388	.000778	
	.057692	.012930	.003147	.001389	.000779	.000124

Table 4, the tabulated value followed by the approximation. The agreement appears to be quite good for moderate sized values of np.

6. An Example of the Determination of the Mean Square Error of $\hat{\alpha}$

We plan to conduct a life test in which time, T, and sample size, n, are dictated by the amount and availability of material and equipment. We desire a measure of the goodness of estimation which the money represented by n and T will buy. As mentioned in Section 1, the mean square error of the estimator provides this measure.

For example, let us assume that we wish to estimate the parameter α of the Weibull distribution for the estimation problem discussed in Sections 1 and 2. Further, let us assume that from previous experience, we know that the failure distribution has a form represented by the Weibull distribution with shape parameter $m = 2$ and that the mean life lies somewhere in the range of 80 to 120 hours. We might choose $T = 200$ hours and sample size $n = 40$.

In calculating the mean square error of $\hat{\alpha}$, we will use equations (2.17, 2.18, and 2.22) to obtain the mean square error of $\hat{\beta}$. Utilizing (2.23), we can determine the mean square error of $\hat{\alpha}$.

Looking at equations (2.17 and 2.18), we see that we need to know the values $E(1/r)$, $V(1/r)$, β, p and q. Since β (or α) is unknown and is the parameter we wish to estimate, the best that we can do is to substitute a guessed value for β.

The mean value of t for the Weibull distribution is

$$E(t) = \alpha^{1/m}\Gamma\left(\frac{1}{m} + 1\right) \tag{6.1}$$

or

$$\alpha = \left[\frac{E(t)}{\Gamma\left(\frac{1}{m} + 1\right)}\right]^m \tag{6.2}$$

Since we expect the mean life to fall in the range 80 to 120 hours, we might choose $E(t) = 100$ as a guessed value. Then, for our example, the guessed value of α would be

$$\alpha = \left[\frac{100}{\Gamma\left(\frac{1}{2} + 1\right)}\right]^2$$

Tables of $\Gamma(x)$ can be found in most standard handbooks of mathematical tables. We note $\Gamma(1.5) = .88623$. Then

$$\alpha \approx \left(\frac{100}{.88623}\right)^2 \approx 12{,}732 \tag{6.3}$$

From (2.2) and (2.4)

$$\beta = \frac{\alpha}{T^m} = \frac{12{,}732}{(200)^2} = .31830 \tag{6.4}$$

$$q = e^{-1/\beta} = e^{-1/.31830} = .04320 \tag{6.5}$$

and

$$p = 1 - q = .95680 \tag{6.6}$$

The values of $E(1/r)$ and $V(1/r)$ can be determined from the Beta approximations (4.12) and (4.14).

$$E\left(\frac{1}{r}\right) = \frac{n-2}{n(a-1)} \quad \text{where} \quad a = (n-1)p = (39)(.9568) = 37.315$$

$$E\left(\frac{1}{r}\right) = \frac{38}{40(36.315)} = .0262$$

$$V\left(\frac{1}{r}\right) = \frac{(n-2)(n-a-1)}{n^2(a-1)^2(a-2)}$$

$$= \frac{(38)(40 - 37.315 - 1)}{(40)^2(36.315)^2(35.315)} = .00000085927$$

Substituting the values into (2.17) and (2.18), Bias $\hat{\beta} = nE(1/r) - q/p - 1$

$$= 40(.0262) - \frac{.0432}{.9568} - 1$$

$$= .0013$$

$$V(\hat{\beta}) = n^2 V\left(\frac{1}{r}\right) + (\beta^2 - q/p^2)E\left(\frac{1}{r}\right)$$

$$= (40)^2(.0000009) + (.318)^2 - \frac{.0432}{(.9568)^2}(.0262)$$

$$= .00284$$

The mean square error of $\hat{\beta}$ and $\hat{\alpha}$ can be calculated from (2.22) and (2.23) respectively.

$$E\{(\hat{\beta} - \beta)^2\} = V(\hat{\beta}) + (\text{Bias } \hat{\beta})^2$$

$$= .00284 + (.0013)^2 = .002849$$

In this case, the square of the bias is so small compared with the variance that it can be neglected.

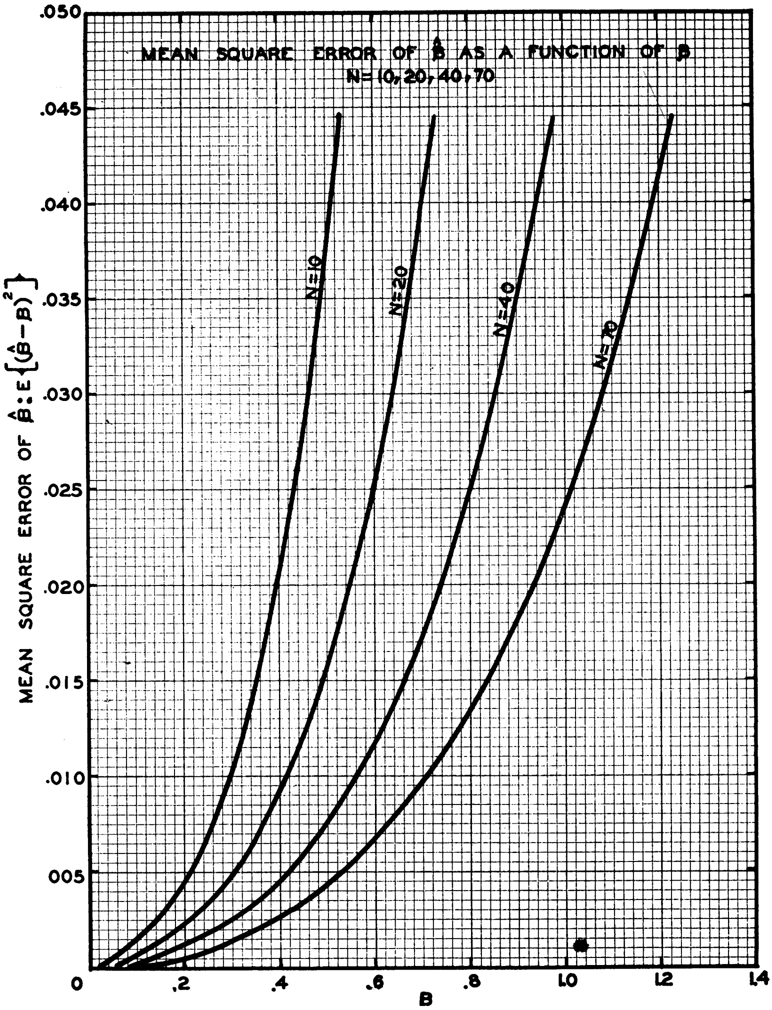
$$E\{(\hat{\alpha} - \alpha)^2\} = T^{2m}E\{(\hat{\beta} - \beta)^2\}$$

$$= (200)^4(.00285) = 4{,}574{,}000$$

When the bias equals zero, the mean square error equals the variance and consequently, for this case (bias approximately equal to zero), the standard deviation of the estimator, $\hat{\alpha}$, is 2139.

The mean square error as a function of β is presented in Figure 1 for $n = 10$,

FIGURE I



20, 40, and 70. For the above example, we read from Figure 1 ($n = 40$) the mean square error of β is approximately .0030 for $\hat{\beta} = .318$.

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