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Ashish Kumar^a & P.C. Consul^a

^a University of Calgary, Calgary, Alberta, Canada

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NEGATIVE MOMENTS OF A MODIFIED POWER SERIES DISTRIBUTION
AND BIAS OF THE MAXIMUM LIKELIHOOD ESTIMATOR

Ashish Kumar and P.C. Consul

University of Calgary
Calgary, Alberta, Canada

Key Words and Phrases: maximum likelihood estimate; modified power series distribution; bias; negative moments; Lagrangian Poisson distribution; Lagrangian binomial distribution.

ABSTRACT

The class of Modified Power Series distributions (MPSD) containing Lagrangian Poisson (LPD) (Consul and Jain, 1973) and Lagrangian binomial distributions (LBD) (Jain and Consul, 1971) was studied by Gupta (1974). We investigate the problem of finding the negative moments, $E[X^{-n}]$, of displaced and decapitated Modified Power Series Distributions. We derive the relationship between n^{th} and $(n-1)^{\text{th}}$ negative moments. The negative moments of the decapitated and displaced LPD are obtained. These results are, then, used to find the exact amount of bias in the ML estimators of the parameters in the LPD and the LBD. We have also given the variances of the ML estimator and the minimum variance unbiased estimator of the parameter in the LPD.

1. INTRODUCTION

Yule (1944) and Stephan (1945) demonstrated the importance of the study of negative or reciprocal moments of the discrete random variables taking positive values. Reciprocal moments were used in different contexts, notably in life testing (Bartholomew, 1957) and in the ratio estimators (Deming, 1950). Formulas and the tables for the mean and variance of the reciprocal random variables having decapitated binomial, Poisson, negative binomial and the hypergeometric distributions have been provided by Stephan (1945) and Rider (1962). Chao-Strawderman (1972) gave a new technique for finding the negative moments of the positive random variables and used it to find $E[(X+A)^{-1}]$ for the Poisson and the binomial distributions.

The first four moments of the ML estimator of the parameter θ in p.d.f. $f(x; \theta)$ were obtained by Haldane and Smith (1956). Gupta (1975) used their expression to find the bias of the ML estimator, $\hat{\theta}$, of θ in the Modified Power Series Distribution (MPSD) (Gupta, 1974). Since the terms of order higher than n^{-1} were neglected, his expression provides an approximate value of the bias. To calculate the term of the next higher order, one will use the results given by Shenton and Bowman (1962) which involve tedious computational work.

In section 2 of this paper, we give some known discrete distributions for ready reference. In section 3, we derive the main result concerning the relation between r^{th} and $(r-1)^{\text{th}}$ negative moments of the random variable $(X+k)$, where k is some constant such that $X+k \neq 0$, and X is an MPSD random variable. We have used the results to get the expressions for $E[X^{-r}]$, $r = 1, 2$; for the decapitated Lagrangian Poisson and the decapitated Lagrangian binomial distributions. Expressions for $E[(X+k)^{-1}]$ and $E[(X+k)^{-2}]$ are also obtained. In section 4, we obtain the exact amount of bias present in the ML estimators of the parameters of the Lagrangian Poisson and the Lagrangian binomial distributions. We also

obtain the variances of the ML estimator and the uniformly minimum variance unbiased estimator in the case of the LPD.

2. SOME KNOWN DISCRETE PROBABILITY DISTRIBUTIONS

Let X be a discrete random variable having a probability density function

$$P(X=x) = \frac{\alpha(x) \{g(\theta)\}^x}{f(\theta)} ; \quad x \in T, \quad (2.1)$$

$$= 0 \quad \text{elsewhere,}$$

where T is a subset of the set of non negative integers, $\alpha(x) > 0$; $g(\theta)$ and $f(\theta)$ are positive, finite and differentiable functions of θ .

Gupta (1974) studied some of its properties under the title 'Modified Power Series Distribution' (MPSD). Its mean is

$$\mu(\theta) = \frac{g(\theta) f'(\theta)}{g'(\theta) f(\theta)}, \quad (2.2)$$

where the primes denote the derivatives of the functions with respect to θ .

2.1 The Lagrangian Poisson Distribution (LPD)

The LPD (Consul and Jain, 1973), given by

$$P(X=x) = \frac{(1 + \alpha)^{x-1}}{x!} \frac{(\theta e^{-\alpha\theta})^x}{e^\theta}, \quad (2.3)$$

$x = 0, 1, 2, \dots$, and zero elsewhere for $\theta > 0$, $0 \leq \alpha < \theta^{-1}$; is an MPSD with

$$f(\theta) = e^\theta \text{ and } g(\theta) = \theta e^{-\alpha\theta}, \quad (2.4)$$

and the mean

$$\mu(\theta) = \theta/(1-\alpha\theta). \quad (2.5)$$

The Poisson and the Borel Tanner distributions are its particular cases given by putting $\alpha = 0$ and $\alpha = 1/\lambda_1$, $\theta = \lambda_1 \phi$ respectively in (2.3).

2.2 The Lagrangian Binomial Distribution (LBD)

The LBD (Jain and Consul, 1971), given by p.d.f.

$$P(X=x) = \frac{m \Gamma(m+\beta x)}{x! \Gamma(m+\beta x+1)} \frac{\{\theta(1-\theta)^{\beta-1}\}^x}{(1-\theta)^{-m}}, \quad (2.6)$$

$x = 0, 1, 2, \dots$ and zero elsewhere for $m > 0$, $0 < \theta < 1$, $0 \leq \beta < \theta^{-1}$ is also an MPSD with

$$f(\theta) = (1-\theta)^{-m} \text{ and } g(\theta) = \theta(1-\theta)^{\beta-1}, \quad (2.7)$$

and the mean

$$\mu(\theta) = \frac{m\theta}{1-\beta\theta}. \quad (2.8)$$

The binomial and the negative binomial distributions are particular cases of the LBD and are obtained by putting $\beta=0$ and $\beta=1$ respectively in (2.6).

3. RECURSIVE RELATION BETWEEN THE NEGATIVE MOMENTS

Let X be a random variable having an MPSD defined in (2.1) with $g(0) = 0$ and let k be a non-negative number such that $k+x \neq 0$ for $x \in T$.

For a positive integer r , let

$$\begin{aligned} M(r, k) &= E[(X+k)^{-r}] \\ &= \sum_{x \in T} \frac{\alpha(x) \{g(\theta)\}^x}{(x+k)^r f(\theta)}. \end{aligned} \quad (3.1)$$

On differentiating (3.1) with respect to θ , we get

$$M'(r, k) = \sum_{x \in T} \frac{\alpha(x)}{(x+k)^r} \frac{\{g(\theta)\}^x}{f(\theta)} \left[\frac{xg'(\theta)}{g(\theta)} - \frac{f'(\theta)}{f(\theta)} \right],$$

which can be easily put in the form

$$M'(r, k) = \frac{g'(\theta)}{g(\theta)} M(r-1, k) - \left[\frac{k g'(\theta)}{g(\theta)} + \frac{f'(\theta)}{f(\theta)} \right] M(r, k). \quad (3.2)$$

The above relation provides a simple linear differential equation

$$M'(r, k) + M(r, k) \left[\frac{k g'(\theta)}{g(\theta)} + \frac{f'(\theta)}{f(\theta)} \right] = \frac{g'(\theta)}{g(\theta)} M(r-1, k). \quad (3.3)$$

Multiplying it by the factor $f(\theta)\{g(\theta)\}^k$ and integrating from 0 to θ , we get the relation

$$M(r, k) f(\theta)\{g(\theta)\}^k = \int_0^\theta M(r-1, k) f(\theta) g'(\theta)\{g(\theta)\}^{k-1} d\theta. \quad (3.4)$$

It is clear from (3.1) that $M(0, k) = 1$. Assuming that

$$I(k) = f(\theta) g'(\theta)\{g(\theta)\}^{k-1}, \quad (3.5)$$

the relation (3.4) gives the following recursive relation between the negative moments about the point $-k$ for Modified Power Series Distribution,

$$M(r, k) = \left[f(\theta)\{g(\theta)\}^k \right]^{-1} \int_0^\theta M(r-1, k) I(k) d\theta. \quad (3.6)$$

3.1 The Negative Moments for the LPD Family

For the decapitated LPD (defined in (2.3) and truncated on the left at $x=0$), the recursive relation (3.6) gives the first negative moment as

$$\begin{aligned} E\left[\frac{1}{X}\right] &= M(1, 0) = (e^\theta - 1)^{-1} \int_0^\theta \left\{ \frac{e^\theta - 1}{\theta} \right\} (1 - \alpha\theta) d\theta \\ &= (e^\theta - 1)^{-1} \int_0^\theta \left\{ \frac{e^\theta - 1}{\theta} - \alpha(e^\theta - 1) \right\} d\theta \\ &= \frac{1}{e^\theta - 1} \left\{ E_i(\theta) - \log \theta - \gamma - \alpha(e^\theta - \theta - 1) \right\}, \quad (3.7) \end{aligned}$$

where E_i is the exponential integral ([11], NBS tables, pp 228-51) and γ is the Euler's constant ($.5772\cdots$).

On using the result (3.7) in (3.6) for $r = 2$, the second negative moment of the DLPD becomes

$$E\left[\frac{1}{X^2}\right] = \frac{1}{e^\theta - 1} \int_0^\theta \frac{1 - \alpha\theta}{\theta} \left\{ E_i(\theta) - \log \theta - \gamma - \alpha(e^\theta - \theta - 1) \right\} d\theta. \quad (3.8)$$

The first two negative moments of the decapitated Poisson and decapitated Borel-Tanner distributions are easily obtainable from the results (3.7) and (3.8) by putting $\alpha = 0$ and $\alpha = 1/\lambda_1$, $\theta = \lambda_1 \phi$ respectively. The first negative moment of the decapitated Poisson distribution,

$$E\left[\frac{1}{X}\right] = \frac{1}{e^{\frac{\theta}{\theta-1}} - 1} \left[E_i(\theta) - \log \theta - \gamma \right],$$

was used by Grab-Savage (1954) in calculating the tables.

The first negative moment about the point $-k$ ($k > 0$) for the LPD family (2.3) is given by the recursive formula (3.6) as

$$E\left[\frac{1}{X+k}\right] = \frac{1}{\theta^k e^{\theta(1-\alpha k)}} \left[\int_0^{\theta} e^{(1-\alpha k)} \theta^{k-1} d\theta - \alpha \int_0^{\theta} e^{\theta(1-\alpha k)} \theta^k d\theta \right]. \quad (3.9)$$

For $\alpha k = 1$, the above relation (3.9) simplifies to

$$E\left[\frac{1}{X+k}\right] = \frac{1}{k} - \frac{\alpha \theta}{k+1}. \quad (3.10)$$

If k is a positive integer and $\alpha k \neq 1$, the expression in (3.9) reduces to a finite sum given by

$$E\left[\frac{1}{X+k}\right] = -\frac{\alpha}{1-\alpha k} - \sum_{i=1}^k \frac{(-1)^i (k-1)!}{i!} \left\{ \frac{\theta^i (1-\alpha k)^{i+1}}{(k-i)!} \right\} + (-1)^k \frac{(k-1)!}{k!} \left\{ \frac{\theta^k (1-\alpha k)^{k+1}}{e^{\theta(1-\alpha k)}} \right\}. \quad (3.11)$$

However, if k is a positive real number, not necessarily an integer, and $\alpha k \neq 1$ then

$$E\left[\frac{1}{X+k}\right] = -\frac{\alpha}{1-\alpha k} + \frac{1}{e^{\theta(1-\alpha k)}} \sum_{i=0}^{\infty} \frac{(1-\alpha k)^{i-1} \theta^i}{i! (k+i)}.$$

It may be noted that the second term on the right hand side of the above equation is same as $(1-\alpha k)^{-1} E[(Y+k)^{-1}]$, where the random variable Y is distributed as Poisson with parameter $\theta(1-\alpha k)$.

If $\alpha k > 1$, the relation (3.9) may be written as

$$E\left[\frac{1}{X+k}\right] = \frac{e^{\theta(\alpha k-1)}}{\theta^k} \left[G_{\theta}(\alpha k-1, k) - \alpha G_{\theta}(\alpha k-1, k+1) \right],$$

where

$$G_{\theta}(\lambda, \alpha) = \int_0^{\theta} e^{-\lambda x} x^{\alpha-1} dx, \quad \lambda > 0, \alpha > 0$$

is the incomplete gamma function.

By (3.10) and (3.11)

$$E\left[\frac{1}{X+1}\right] = \begin{cases} 1 - \frac{\theta}{2}, & \text{for } \alpha = 1, \\ -\frac{\alpha}{1-\alpha} + \frac{1}{\theta(1-\alpha)^2} - \frac{e^{-\theta(1-\alpha)}}{\theta(1-\alpha)^2}, & \text{for } \alpha \neq 1. \end{cases}$$

Similar expressions can be written for other values of k .

Also, for $\alpha = 0$ the result (3.9) gives the first negative moment about $-k$ for the Poisson distribution as

$$E\left[\frac{1}{X+k}\right] = \frac{1}{e^{\theta} \theta^k} \int_0^{\theta} \theta^{k-1} e^{\theta} d\theta,$$

which easily gives the following recursive relation

$$E\left[\frac{1}{X+k}\right] = \frac{1}{\theta} \left\{ 1 - (k-1) E\left[\frac{1}{X+k-1}\right] \right\}, \quad k > 1,$$

proved by Chao-Strawderman (1972). Also, the results derived by Chao-Strawderman (1972), for the first negative moment of the Poisson distribution are particular cases of our results (3.11) and (3.12), given by putting $\alpha = 0$.

For $\alpha k = 1$, using equation (3.10) in (3.6) with $r = 2$, one can easily obtain the second negative moment of the LPD about the point $-k$ ($k > 0$), given as

$$E\left[\left(\frac{1}{X+k}\right)^2\right] = \frac{1}{k} E\left[\frac{1}{X+k}\right] + \frac{\alpha\theta}{k+1} \left[\frac{\alpha\theta}{k+2} - \frac{1}{k+1} \right]. \quad (3.12)$$

For k positive integer, substituting (3.11) in (3.6) with $r=2$, we get, after messy computation, the following expression for the second negative moment of the LPD about the point $-k$,

$$E\left[\left(\frac{1}{X+k}\right)^2\right] = \frac{-\alpha}{\beta} E\left[\frac{1}{X+k}\right] - \frac{1}{\theta^k e^{\beta\theta}} \sum_{i=1}^{k-1} \frac{\xi(k,i)}{\beta^{i+1}} \left[G_{\theta}^*(\beta, k-i) - \alpha G_{\theta}^*(\beta, k-i+1) \right] - \frac{\xi(k,k)}{\beta^{k+1}} \left[E_{\theta}(\beta\theta) - \log(\beta\theta) - \gamma - \frac{\alpha}{\beta} \left\{ e^{\beta\theta} - \beta\theta - 1 \right\} \right],$$

where $\beta = 1 - \alpha k$,

$$\xi(k, i) = (-1)^i (k-1)! / (k-i)! \quad , \quad i = 1, 2, \dots, k,$$

and

$$G_{\theta}^*(\beta, k) = \begin{cases} \int_0^{\theta} x^{k-1} e^{\beta x} dx, & \beta < 0, \\ 0 & \\ e^{\beta \theta} \sum_{j=1}^k \frac{(-1)^{j-1} \theta^{k-j}}{\beta^j} \frac{(k-1)!}{(k-j)!} + \frac{(-1)^k (k-1)!}{\beta^k}, & \beta > 0. \end{cases}$$

3.2 Negative Moments for the LBD Family

The first negative moment of the decapitated LBD (defined in (2.6) and truncated on the left at $x=0$), is given by the recursive formula (3.6) as

$$E\left[\frac{1}{X}\right] = \left\{ \frac{1}{(1-\theta)^{-m-1}} \right\} \int_0^{\theta} \left\{ \frac{1}{\theta(1-\theta)^{m+1}} - \frac{1}{\theta(1-\theta)} - \frac{\beta}{(1-\theta)^{m+1}} + \frac{\beta}{1-\theta} \right\} d\theta,$$

which reduces to

$$E\left[\frac{1}{X}\right] = \frac{(1-\theta)^m}{1-(1-\theta)^m} \left[\sum_{i=1}^m \binom{m}{i} \left\{ \frac{\theta}{1-\theta} \right\}^i / i - \frac{\beta}{m} \left\{ \frac{1-(1-\theta)^m}{(1-\theta)^m} \right\} - \beta \log(1-\theta) \right]. \quad (3.13)$$

The identity

$$\sum_{i=1}^m \binom{m}{i} \alpha^i / i = \sum_{i=1}^m \left\{ (1+\alpha)^i - 1 \right\} / i,$$

transforms the above result to the form given by Gupta (1974).

On substituting the result (3.13) in (3.6) for $k=0$ and $r=2$, we get the second negative moment of the decapitated LBD as

$$E\left[\frac{1}{X^2}\right] = \frac{(1-\theta)^m}{1-(1-\theta)^m} \int_0^{\theta} \left\{ \frac{1-\beta\theta}{\theta(1-\theta)} \right\} \left[\sum_{i=1}^m \binom{m}{i} \left\{ \frac{\theta}{1-\theta} \right\}^i / i - \frac{\beta}{m} \left\{ (1-\theta)^{-m} - 1 \right\} - \beta \log(1-\theta) \right] d\theta. \quad (3.14)$$

The results obtained by Rider (1962) and Govindarajulu (1962), concerning the first and second negative moments of the decapitated negative binomial distribution, are particular cases of the above two relations (3.13) and (3.14) with $\beta = 1$.

The first negative moment about the point $-k$ ($k > 0$) for the LBD is given by recursive formula (3.6) for $r = 1$ and the relations (2.7), as

$$E\left[\frac{1}{X+k}\right] = \frac{(1-\theta)^{m+k-\beta k}}{\theta^k} \int_0^\theta (1-\beta\theta) \theta^{k-1} (1-\theta)^{\beta k-k-m-1} d\theta. \quad (3.15)$$

For $\beta k - m - k = l > 0$, we have

$$E\left[\frac{1}{X+k}\right] = \frac{1}{\theta^k (1-\theta)^l} \left[B_\theta(k, l) - \beta B_\theta(k+1, l) \right], \quad (3.16)$$

where $B_\theta(p, q)$ is the incomplete beta function

$$B_\theta(p, q) = \int_0^\theta x^{p-1} (1-x)^{q-1} dx; \quad 0 < \theta < 1, p > 0, q > 0.$$

When $\beta k - k - m = l < 0$ and k is real positive number, the integral in (3.15) can be evaluated in the form of a convergent series and the first negative moment becomes,

$$E\left[\frac{1}{X+k}\right] = \frac{1}{(1-\theta)^l} \sum_{i=0}^{\infty} \binom{-l+i}{i} \left\{ \frac{1}{k+i} - \frac{\beta\theta}{k+i+1} \right\} \theta^i. \quad (3.17)$$

However, when k is a positive integer and $l \neq -1, -2, \dots, -k$, the integral in the relation (3.15) can be evaluated by parts and we have

$$E\left[\frac{1}{X+k}\right] = \frac{\beta}{\beta k - m} + \frac{m}{\beta k - m} \left[\sum_{i=1}^k \frac{(1-\theta)^{i-1}}{\theta^i (\beta k - k - m)_{(i)}} \frac{(k-1)!}{(k-i)!} - \frac{(k-1)!}{(\beta k - k - m)_{(k)}} \frac{(1-\theta)^{k+m-\beta k}}{\theta^k} \right], \quad (3.18)$$

where $l_{(i)} = l(l+1)(l+2) \dots (l+i-1)$.

For the binomial distribution ($\beta = 0$), (3.18) gives the recursive relation

$$E\left[\frac{1}{X+k}\right] = \frac{1}{\theta(m+k)} \left\{1 - (k-1)(1-\theta) E\left[\frac{1}{X+k-1}\right]\right\}.$$

Similarly, for the negative binomial distribution ($\beta=1$), we have

$$E\left[\frac{1}{X+k}\right] = \frac{1}{k-m} + \frac{1}{\theta(m-k)} \left\{1 - (k-1) E\left[\frac{1}{X+k-1}\right]\right\}.$$

4. AMOUNT OF BIAS IN THE ML ESTIMATORS

Let X_1, X_2, \dots, X_n denote a random sample from a population having a probability density function defined in (2.1).

Let $Y = \sum_{i=1}^n X_i$ denote the sum of sample observations. It can be easily shown that the random variable Y has the MPD given by

$$P(Y=y) = \frac{b(n,y) \{g(\theta)\}^y}{\{f(\theta)\}^n}; \quad y \in D \quad (4.1)$$

and zero elsewhere for $\theta > 0$, where $b(n,y)$ is the coefficient of $\{g(\theta)\}^y$ in expansion of $\{f(\theta)\}^n$ and the set D is defined as

$$D = \left\{y \mid y = \sum_{i=1}^n x_i, \quad x_i \in T\right\}.$$

4.1 The LPD Family

Let X_1, X_2, \dots, X_n be a random sample from the LPD defined in (2.3). Gupta (1975) showed that the ML estimator, $\hat{\theta}$, of θ in the LPD is given by

$$\hat{\theta} = \frac{\bar{X}}{1+\alpha\bar{X}} = \frac{1}{\alpha} - \frac{n}{\alpha^2} \left(\frac{1}{Y+n/\alpha} \right), \quad (4.2)$$

where $Y = \sum_{i=1}^n X_i$ is the sum of sample observations having LPD given by (4.1) as

$$P(Y=y) = \frac{(1+y\alpha/n)^{y-1}}{y!} \cdot \frac{(n\theta e^{-\alpha\theta})^y}{e^{n\theta}}, \quad (4.3)$$

$y = 0, 1, 2, \dots$, and zero elsewhere for $\theta > 0$, $0 \leq \alpha < \theta^{-1}$, and $\bar{X} = Y/n$ is the sample mean.

Comparing (4.3) with (2.3), we note that the distribution of the random variable Y defined above is an MPSD with parameters $f(\theta) = e^{n\theta}$ and $g(\theta) = n\theta e^{-(\alpha/n)(n\theta)}$ i.e. all the results derived in section 3.1 can be used for the random variable Y by replacing α by α/n and θ by $n\theta$.

Thus, by the relation (3.10)

$$\begin{aligned} E\left[\frac{1}{\bar{Y} + n/\alpha}\right] &= \frac{\alpha}{n} - \frac{(\alpha/n)(n\theta)}{(n/\alpha) + 1} \\ &= \frac{\alpha}{n} - \frac{\alpha^2\theta}{n+\alpha}. \end{aligned} \quad (4.4)$$

By using the above value in (4.2) we get the expected value of the ML estimator as

$$E\left[\hat{\theta}\right] = \frac{n\theta}{n+\alpha}. \quad (4.5)$$

Thus, the actual bias in the ML estimators, $\hat{\theta}$, of θ in the LPD becomes

$$b(\hat{\theta}) = E(\hat{\theta}) - \theta = -\frac{\alpha\theta}{n+\alpha}. \quad (4.6)$$

The bias of $\hat{\theta}$ obtained by Gupta (1975) is given as

$$b^*(\hat{\theta}) = -\frac{\alpha\theta}{n},$$

which is an approximation of order n^{-1} .

VARIANCE OF $\hat{\theta}$. We use the relations (3.12) and (3.10) with α replaced by α/n and θ replaced by $n\theta$ and get

$$E\left[\frac{1}{\bar{Y} + n/\alpha}\right]^2 = \frac{\alpha}{n} \left(\frac{\alpha}{n} - \frac{\alpha^2\theta}{n+\alpha}\right) + \frac{\alpha^2\theta}{n+\alpha} \left(\frac{\alpha^2\theta}{n+2\alpha} - \frac{\alpha}{n+\alpha}\right).$$

Using (4.4) and the above relation the variance of $\hat{\theta}$ becomes

$$\begin{aligned} V(\hat{\theta}) &= \frac{n^2}{\alpha^4} \left[E\left[\left\{\frac{1}{\bar{Y} + n/\alpha}\right\}^2\right] - \left\{E\left[\frac{1}{\bar{Y} + n/\alpha}\right]\right\}^2 \right] \\ &= \frac{n^2\theta}{(n+\alpha)^2} \left[\frac{1}{n} - \frac{\alpha\theta}{n+2\alpha} \right]. \end{aligned}$$

It can be easily seen that the approximation of order of n^{-1} of the above expression is given as $V(\theta) \doteq \frac{\theta(1-\alpha\theta)}{n}$, the expression given by Gupta (1975).

Note: Since \bar{X} is a complete and sufficient statistic for the parameter θ in the LPD (2.3), it is evident from the relations (4.2) and (4.5) that the statistic defined as

$$\tilde{\theta} = \frac{(n+\alpha) \bar{X}}{n(1+\alpha \bar{X})},$$

is the unbiased estimator and, therefore, is the uniformly minimum variance unbiased (UMVU) estimator for the parameter θ . Also, using the expression for the variance of the ML estimator, we can easily show that the variance of the UMVU estimator, $\tilde{\theta}$, of the parameter θ is given by

$$V(\tilde{\theta}) = \left[\frac{\theta}{n} - \frac{\alpha\theta^2}{n+2\alpha} \right].$$

4.2 The LBD Family

The ML estimator, $\hat{\theta}$, of θ based on the random sample X_1, X_2, \dots, X_n from the LBD defined in (2.6) is given by (Gupta, 1975)

$$\hat{\theta} = \frac{\bar{X}}{m+\beta\bar{X}} = \frac{1}{\beta} - \frac{mn}{\beta^2} \left[\frac{1}{Y+mn/\beta} \right], \quad (4.7)$$

where Y denotes the sum of sample values and \bar{X} is the sample mean.

The random variable Y has the LBD given by (4.1) as

$$P(Y=y) = \frac{mn \Gamma(mn+\beta y)}{y! \Gamma(mn+\beta y-y+1)} \frac{\{\theta(1-\theta)^{\beta-1}\}^y}{(1-\theta)^{-mn}}, \quad (4.8)$$

$y = 0, 1, 2, \dots$, and zero elsewhere for $m > 0$, $0 < \theta < 1$ and $0 < \beta < \theta^{-1}$.

It may be noted that the above distribution of the random variable Y is an MPSD similar to (2.6) with the only difference that m is replaced by mn .

In view of the above comment, we can use the relations (3.16) and (3.17) after changing m by mn for the random variable Y . Since

$k = mn/\beta$ gives $l = \beta k - mn - k$ to be negative, we use the relation (3.17) replacing m by mn to get

$$E\left[\frac{1}{Y+mn/\beta}\right] = \frac{\beta}{mn} - \beta \sum_{i=0}^{\infty} \binom{mn/\beta + i}{i} \frac{\theta^{i+1} (1-\theta)^{mn/\beta}}{(mn/\beta + i + 1)},$$

which, on substituting in (4.7), provides us the expected value of the ML estimator, $\hat{\theta}$, of θ in the LBD, given as

$$E\left[\hat{\theta}\right] = \frac{mn}{\beta} \sum_{i=0}^{\infty} \binom{mn/\beta + i}{i} \frac{\theta^{i+1} (1-\theta)^{mn/\beta}}{(mn/\beta + i + 1)}. \quad (4.9)$$

It may be noted that the above expression is same as $\frac{\theta mn}{\beta(1-\theta)} \times E\left[\left(Z + \frac{mn}{\beta} + 1\right)^{-1}\right]$, where the random variable Z has the negative binomial distribution given by (2.6) after putting $\beta = 1$ and replacing m by $\left(\frac{mn}{\beta} + 1\right)$.

Subtracting θ from both sides of above relation (4.9) we get the bias in ML estimator as

$$b(\hat{\theta}) = -\theta + \frac{mn}{\beta} \sum_{i=0}^{\infty} \binom{mn/\beta + i}{i} \frac{\theta^{i+1} (1-\theta)^{mn/\beta}}{(mn/\beta + i + 1)}. \quad (4.10)$$

However, if $k = mn/\beta$ is a positive integer, the relation (3.15) reduces to

$$\begin{aligned} E\left[\frac{1}{Y+mn/\beta}\right] &= E\left[\frac{1}{Y+k}\right] \\ &= \frac{1}{k} - \beta \sum_{i=0}^{k-1} \frac{(-1)^i}{(k-i)} \left(\frac{1-\theta}{\theta}\right)^i \\ &\quad + (-1)^{k-1} \log(1-\theta) \beta \left(\frac{1-\theta}{\theta}\right)^k. \end{aligned} \quad (4.11)$$

From equations (4.7) and (4.11), we get the expected value of the ML estimator, $\hat{\theta}$, of θ in the LBD, when $\frac{mn}{\beta} = k$ is a positive integer, given as

$$E[\hat{\theta}] = k \sum_{i=0}^{k-1} \frac{(-1)^i}{(k-i)} \left(\frac{1-\theta}{\theta}\right)^i + (-1)^k k \log(1-\theta) \left(\frac{1-\theta}{\theta}\right)^k. \quad (4.12)$$

Thus, the bias in $\hat{\theta}$ is given in a closed form as

$$b(\hat{\theta}) = (1-\theta) + k \sum_{i=1}^{k-1} \frac{(-1)^i}{(k-i)} \left(\frac{1-\theta}{\theta} \right)^i + (-1)^k k \log(1-\theta) \left(\frac{1-\theta}{\theta} \right)^k. \quad (4.13)$$

Now, we use the relation (4.10) to get the upper and lower bounds for the bias in $\hat{\theta}$. By decreasing the denominators by unity in every term of (4.10) except the first three terms, we have

$$\begin{aligned} b(\hat{\theta}) &\leq -\theta + \frac{N \theta (1-\theta)^N}{(N+1)} + \frac{N(N+1) \theta^2 (1-\theta)^N}{(N+2)} + N \theta (1-\theta)^N \sum_{i=2}^{\infty} \binom{N+i}{i} \frac{\theta^i}{N+i} \\ &= -\theta + \frac{N \theta (1-\theta)^N}{(N+1)} + \frac{N(N+1) \theta^2 (1-\theta)^N}{(N+2)} + \theta (1-\theta)^N \sum_{i=2}^{\infty} \binom{N+i-1}{N-1} \theta^i \\ &= -\theta + \frac{N \theta (1-\theta)^N}{N+1} + \frac{N(N+1) \theta^2 (1-\theta)^N}{N+2} + \theta (1-\theta)^N \left[(1-\theta)^{-N} - 1 - N\theta \right] \\ &= -\frac{\theta (1-\theta)^N}{N+2} - \frac{N \theta^2 (1-\theta)^N}{N+2} < 0. \end{aligned} \quad (4.14)$$

Also, by increasing the denominators by $\left(\frac{i-1}{N+1} \right)$ in each term of (4.10), except the first two terms, we have

$$\begin{aligned} b(\hat{\theta}) &> -\theta + \frac{N \theta (1-\theta)^N}{N+1} + N \theta (1-\theta)^N \sum_{i=1}^{\infty} \binom{N+i}{i} \frac{\theta^i}{\left(\frac{(N+2)(N+i)}{N+1} \right)} \\ &= -\theta + \frac{N \theta (1-\theta)^N}{N+1} + \left(\frac{N+1}{N+2} \right) \theta (1-\theta)^N \sum_{i=1}^{\infty} \binom{N+i-1}{N-1} \theta^i \\ &= -\theta + \frac{N \theta (1-\theta)^N}{N+1} + \left(\frac{N+1}{N+2} \right) \theta (1-\theta)^N \left[(1-\theta)^{-N} - 1 \right] \\ &= -\frac{\theta}{N+2} - \frac{\theta (1-\theta)^N}{(N+1)(N+2)} > -\frac{\theta}{N+1}. \end{aligned} \quad (4.15)$$

The inequalities (4.14) and (4.15) can be combined to give the bounds for the bias in $\hat{\theta}$ as

$$-\frac{\theta (1-\theta)^N}{(N+1)} - \frac{N \theta^2 (1-\theta)^N}{(N+2)} > b(\hat{\theta}) > -\frac{\theta}{N+2} - \frac{\theta (1-\theta)^N}{(N+1)(N+2)}, \quad (4.16)$$

where $N = \frac{m}{\beta}$.

The expression for the bias of the order n^{-1} was given by Gupta (1975) as

$$b^*(\hat{\theta}) = - \frac{\theta(1-\theta)}{N}.$$

By using a different method it can be shown by (4.7) that an approximation for the bias, of the order of n^{-2} , is given by

$$b(\hat{\theta}) \doteq \frac{\theta(1-\theta)}{N} + \frac{\theta(1-\theta) \{1-2\theta+\beta\theta(2-\theta)\}}{N^2(1-\beta\theta)}.$$

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Refereed by N. L. Johnson, University of South Carolina,
Chapel Hill, SC.