This article was downloaded by: [New York University]

On: 16 July 2015, At: 18:15 Publisher: Taylor & Francis

Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: 5 Howick Place, London,

SW1P 1WG



### Journal of the American Statistical Association

Publication details, including instructions for authors and subscription information: <a href="http://www.tandfonline.com/loi/uasa20">http://www.tandfonline.com/loi/uasa20</a>

## Negative Moments of Positive Random Variables

M. T. Chao <sup>a</sup> & W. E. Strawderman <sup>b</sup>

<sup>a</sup> Bell Telephone Laboratories , Holmdel , N. J. , 07733 , USA

b Department of Statistics, Rutgers—the State University of New Jersey, New Brunswick, N. J.,

Published online: 05 Apr 2012.

To cite this article: M. T. Chao & W. E. Strawderman (1972) Negative Moments of Positive Random Variables, Journal of the American Statistical Association, 67:338, 429-431

To link to this article: <a href="http://dx.doi.org/10.1080/01621459.1972.10482404">http://dx.doi.org/10.1080/01621459.1972.10482404</a>

#### PLEASE SCROLL DOWN FOR ARTICLE

Taylor & Francis makes every effort to ensure the accuracy of all the information (the "Content") contained in the publications on our platform. However, Taylor & Francis, our agents, and our licensors make no representations or warranties whatsoever as to the accuracy, completeness, or suitability for any purpose of the Content. Any opinions and views expressed in this publication are the opinions and views of the authors, and are not the views of or endorsed by Taylor & Francis. The accuracy of the Content should not be relied upon and should be independently verified with primary sources of information. Taylor and Francis shall not be liable for any losses, actions, claims, proceedings, demands, costs, expenses, damages, and other liabilities whatsoever or howsoever caused arising directly or indirectly in connection with, in relation to or arising out of the use of the Content.

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden. Terms & Conditions of access and use can be found at <a href="http://www.tandfonline.com/page/terms-and-conditions">http://www.tandfonline.com/page/terms-and-conditions</a>

# Negative Moments of Positive Random Variables

#### M. T. CHAO and W. E. STRAWDERMAN\*

We investigate the problem of finding the expected value of functions of a random variable X of the form  $f(X)=(X+A)^{-n}$  where X+A>0 a.s. and n is a non-negative integer. The technique is to successively integrate the probability generating function and is suggested by the well-known result that successive differentiation leads to the positive moments. The technique is applied to the problem of finding E[1/(X+A)] for the binomial and Poisson distributions.

#### 1. INTRODUCTION

We investigate the problem of finding the expected value of functions of a random variable X, of the form

$$f(X) = (X + A)^{-n}, (1.1)$$

where X+A>0 a.s., and n is a non-negative integer. The technique is to successively integrate the probability generating function, and is suggested by the well-known result that successive differentiation leads to the positive moments. We develop the technique in Section 2 and apply it to finding E[1/(X+A)] for the binomial and Poisson distributions in Section 3. Section 4 contains some further observations.

Negative moments are useful in applications in several contexts, notably in life testing problems, and survey sampling problems where ratio estimates are used. See [1, 2, 3, 4, 6] for some applications.

The results in the literature seem to have been confined primarily to the case of the truncated Poisson and binomial distributions [5, 6, 8, 9].

#### 2. THE BASIC RESULTS

Let X be a random variable defined on a probability space  $(\mathfrak{X}, \mathfrak{A}, P)$  and suppose  $X+A>\delta>0$  a.s. [P]. Define the probability generating function of X+A-1 as

$$g_1(t) = E(t^{X+A-1}) \qquad 0 \le t \le 1.$$
 (2.1)

Now inductively define  $g_{k+1}(t)$ , for  $k = 1, 2, \cdots$  as follows:

$$g_{k+1}(t) = t^{-1} \int_0^t g_k(u) du. \qquad 0 \le t \le 1.$$
 (2.2)

Clearly (2.1) and (2.2) exist under the assumption on X. We have the following result.

Theorem 1. For 
$$0 \le t \le 1$$
,

$$E\left[\left(\frac{1}{X+A}\right)^t t^{X+A}\right] = \int_0^t g_k(u) du.$$

Proof:

Since

$$\frac{1}{t} \int_0^t u^{\Delta - 1} du = \frac{t^{\Delta - 1}}{\Delta} \quad \text{for } \Delta > 0$$

we have

$$rac{t^{\Delta-1}}{\Delta^k} = rac{1}{t} \int_0^t \left[ rac{1}{t_k} \int_0^{t_k} \left[ \cdots \left[ rac{1}{t_2} \right. \right. \left. \left. \int_0^{t_2} t_1^{\Delta-1} dt_1 \left] dt_2 \left. \right] \cdots \right] dt_n \right] dt.$$

Letting  $\Delta = X - A$  and taking expectations gives Theorem 1.

We have immediately, the

Corollary:

$$E\left(\frac{1}{X+A}\right)^k = \int_0^1 g_k(u)du.$$

The potential applicability of the corollary is immediately evident. In Section 3 we apply it in the binomial and Poisson cases.

#### 3. APPLICATIONS

#### 3.1 Binomial Distribution

Let X be a binomially distributed random variable with parameters n and p. It is easy to show

$$g_1(t) = t^{A-1}(q+pt)^n,$$
 (3.1)

and using successive integrations by parts we are led to

$$\int_{0}^{t} g_{1}(u)du = \int_{0}^{t} u^{A-1} (q+pu)^{n} du$$

$$= q^{n} \left(\frac{q}{p}\right)^{A}$$

$$\cdot \left[\sum_{k=1}^{r} (-1)^{k+1} \frac{(A-1)(A-2) \cdot \cdot \cdot (A-k+1)}{(n+1)(n+2) \cdot \cdot \cdot \cdot (n+k)} \right]$$

$$b^{A-k} (1+b)^{n+k}$$
(3.2)

<sup>\*</sup> M. T. Chao is member of technical staff, Bell Telephone Laboratories, Holmdel, N. J. 07733. W. E. Strawderman is assistant professor, Department of Statistics, Rutgers—the State University of New Jersey, New Brunswick, N. J.

$$+ (-1)^{r} \frac{(A-1)(A-2) \cdot \cdot \cdot (A-r)}{(n+1)(n+2) \cdot \cdot \cdot (n+r)} \cdot \int_{0}^{b} v^{A-r-1} (1+v)^{n+r} dv \bigg],$$

where, by convention,  $(A-1)(A-2) \cdot \cdot \cdot (A-k) = 1$  if k=0; and where A-r>0, and b=pt/q.

For integral A, the formula leads to an exact result. Using relation (3.2) and the Corollary we have for integral A,

$$E\left(\frac{1}{X+A}\right) = q^{n} \left(\frac{q}{p}\right)^{A}$$

$$\cdot \left[\sum_{k=1}^{A-1} (-1)^{k+1} \frac{(A-1)(A-2) \cdot \cdot \cdot \cdot (A-k+1)}{(n+1)(n+2) \cdot \cdot \cdot \cdot (n+k)} \right]$$

$$\cdot b^{A-k} (1+b)^{n+k}$$

$$+ (-1)^{A-1} \frac{(A-1)!}{(n+1)(n+2) \cdot \cdot \cdot \cdot (n+A-1)} \frac{1}{n+A}$$

$$\cdot ((1+b)^{n+A}-1) \right],$$
(3.3)

where b = p/q.

In particular,

$$E\left(\frac{1}{X+1}\right) = \frac{1-q^{n+1}}{(n+1)p} . \tag{3.4}$$

and

$$E\left(\frac{1}{X+2}\right) = q^{n} \left(\frac{q}{p}\right)^{2} \left[\frac{1}{n+1} \left(\frac{p}{q}\right)^{1} \left(\frac{1}{q}\right)^{n+1} - \frac{1}{(n+1)(n+2)} \left(\frac{1}{q^{n+2}} - 1\right)\right]$$

$$= \frac{1}{(n+1)p} - \frac{1}{(n+1)(n+2)p^{2}}$$

$$\cdot (1 - q^{n+2})$$

$$= \frac{1}{(n+1)p} \left[1 - \frac{(1 - q^{n+2})}{(n+2)p}\right].$$
(3.5)

Unfortunately (3.2) does not seem particularly useful for computing purposes when A is non-integral. Also the higher negative moments do not seem to be easily computed using our procedure, although the "negative factorial moments" are more tractable, a fact we will not prove here.

#### 3.2 The Poisson Distribution

Let X be a random variable with the Poisson distribution with parameter  $\lambda$ .

Here

$$g_1(t) = t^{A-1}e^{-\lambda+\lambda t}.$$

$$\int_0^t g_1(u)du = \int_0^t u^{A-1}e^{-\lambda+\lambda u}du$$
(3.6)

Journal of the American Statistical Association, June 1972

$$= e^{-\lambda} \int_{0}^{t} u^{A-1} e^{\lambda u} du, \qquad \text{for } A > 0$$

$$= e^{-\lambda} \left[ \frac{t^{A-1} e^{\lambda t}}{\lambda} - \frac{A-1}{\lambda} \right]$$

$$\cdot \int_{0}^{t} u^{A-2} e^{\lambda u} du \qquad \text{for } A > 1.$$
(3.7)

Letting t=1, we have

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda} \left[1 - (A-1)E\left(\frac{1}{X+A-1}\right)\right]$$
for  $A > 1$ . (3.8)

When A = 1 we have directly

$$E\left(\frac{1}{X+1}\right) = \frac{e^{-\lambda}}{\lambda} \left[e^{\lambda} - 1\right] = \frac{1 - e^{-\lambda}}{\lambda}$$
 (3.9)

(3.8) and (3.9) together allow an inductive calculation of E[1/(X+A)] for any integer  $A \ge 1$ .

The formula is

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda} \left[ 1 + \sum_{j=1}^{r} \frac{\prod_{i=1}^{j} (A-i)}{\lambda^{j}} (-1)^{j} + (-1)^{r+1} \frac{\prod_{i=1}^{r+1} (A-i)}{\lambda^{r}} E\left(\frac{1}{X+A-r-1}\right) \right]$$

for A > r+1. When A = r+2,

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda} \left[ 1 + \sum_{j=1}^{A-2} \frac{\prod_{i=1}^{j} (A-i)}{\lambda^{j}} (-1)^{j} + (-1)^{A-1} \frac{\prod_{i=1}^{A-1} (A-i)}{\lambda^{A-2}} \left(\frac{1}{\lambda} - \frac{e^{-\lambda}}{\lambda}\right) \right]$$

$$= \frac{1}{\lambda} \left[ 1 + \sum_{j=1}^{A-1} \frac{\prod_{i=1}^{j} (A-i)}{\lambda^{j}} (-1)^{j} + (-1)^{A} \frac{(A-1)!e^{-\lambda}}{\lambda^{A-1}} \right].$$
(3.10)

For non-integral A the first equation in (3.10) will serve to reduce the problem to one of finding E[1/(X+A)] 0 < A < 1. We include a graph of E[1/(X+A)] for such values of A. We also indicate an asymptotic expression for E[1/(X+A)] for small values of A.

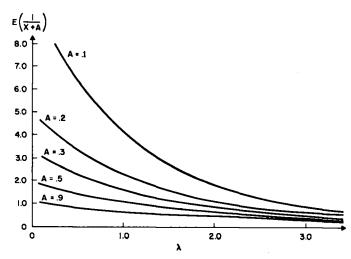
$$E\left(\frac{1}{X+A}\right) = \sum_{k=0}^{\infty} \frac{1}{k+A} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{A} e^{-\lambda} + \left[\sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!(k+A)}\right].$$
(3.11)

If we let

$$F(\lambda, A) = \sum_{k=1}^{\infty} \frac{1}{k!k} \lambda^k e^{-\lambda} + \frac{1}{A} e^{-\lambda},$$
 (3.12)





we have

$$\left| E\left(\frac{1}{X+A}\right) - F(\lambda, A) \right| \\
= A \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k! k(k+A)} \\
\leq A \left[ \frac{\lambda e^{-\lambda}}{A+1} + \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k+1)!} \right] \\
\leq A \left[ \frac{1}{2} \lambda e^{-\lambda} + \frac{1 - e^{-\lambda}}{\lambda} - e^{-\lambda} \right] \\
= A f(\lambda) \quad \text{for } A \leq 1.$$
(3.13)

Note that  $f(\lambda) \to 0$  as  $\lambda \to 0$  or  $\lambda \to \infty$ . Also max  $f(\lambda) \doteq .464$  (at  $\lambda \doteq 1.35$ ), and hence with small error we may approximate E[1/(X+A)] by  $F(\lambda, A)$  for small A.

But  $F(\lambda, A)$ , although it doesn't have a simple closed form, can be put in the form of well tabulated functions. In particular, we have

$$F(\lambda, A) = \frac{1}{A} e^{-\lambda} + (1 - e^{-\lambda}) E\left(\frac{1}{X} \mid X > 0\right). \quad (3.14)$$

Now for a random variable Y with the truncated Poisson distribution (with A = 0)

$$g_1(t) = \frac{e^{-\lambda}(e^{\lambda t} - 1)}{(1 - e^{-\lambda})t} = E(t^{Y-1}).$$

Hence,

$$E\left(\frac{1}{X} \mid X > 0\right) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \int_{0}^{1} \frac{e^{\lambda t} - 1}{t} dt. \quad (3.15)$$

This integral may be expressed in terms of *Ei*, the exponential integral [7, NBS Tables, pp. 228-51]. Specifically,

$$\int_{0}^{1} \frac{e^{\lambda t} - 1}{t} dt = Ei(\lambda) - \log \lambda - \gamma, \qquad (3.16)$$

where  $\gamma$  is Euler's Constant (.5772 · · · ). Hence  $F(\lambda, A)$  may be evaluated with reasonable simplicity. Also we have a simple derivation, as a bonus, of the expression for the expectation of the inverse of the truncated Poisson used by Grab and Savage [5] in calculating their tables.

[Received June 1970. Revised February 1971.]

#### REFERENCES

- [1] Bartholomew, A. J., "A Problem in Life Testing," Journal of the American Statistical Association, 52 (September 1957), 350-5.
- [2] Epstein, B. and Sobel, M., "Some Theorems Relevant to Life Testing," The Annals of Mathematical Statistics, 25 (June 1954), 373-81.
- [3] Epstein, B. and Sobel, M., "Life Testing," Journal of the American Statistical Association, 48 (September 1953), 486– 502.
- [4] Deming, W. E., Some Theory of Sampling, New York: John Wiley and Sons, Inc., 1950, 449-54.
- [5] Grab, E. L. and Savage, I. R., "Tables of the Expected Value of 1/X for Positive Bernoulli and Poisson Variables," Journal of the American Statistical Association, 49, (March 1954), 169-77.
- [6] Mendenhall, W. and Lehmann, E. H., Jr., "An Approximation to the Negative Moments of the Positive Binomial Useful in Life Testing," Technometrics, 2 (May 1960), 227-41.
- [7] Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics Series, 55, 1964.
- [8] Stephan, F. F., "The Expected Value and Variance of the Reciprocal and other Negative Powers of a Positive Bernoullian Variate," The Annals of Mathematical Statistics, 16 (March 1945), 50-61.
- [9] Tiku, M. L., "A Note on the Negative Moments of a Truncated Poisson Variate," Journal of the American Statistical Association, 59 (December 1964), 1220-4.