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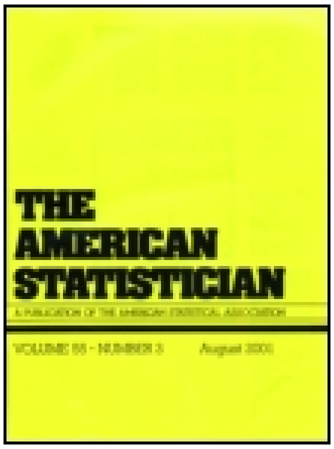

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The Moment-Generating Function and Negative Integer Moments

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The Moment-Generating Function and Negative Integer Moments

NOEL CRESSIE, ANNE S. DAVIS, J. LEROY FOLKS, AND GEORGE E. POLICELLO II*

It does not appear to be widely known that the moment-generating function $M_X(t)$ contains information about negative as well as positive integer moments. Proofs of this are provided, extensions are indicated, and earlier literature deriving and using this type of result is briefly noted. Some examples are given.

KEY WORDS: Bias of ratio estimators; Laplace transform; Mellin transform.

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1. INTRODUCTION

Typical undergraduate courses in probability theory introduce the moment-generating function (mgf) of a random variable X as an extremely portable way of carrying around all the (positive integer) moments of X. Mere repeated differentiation of the mgf, and evaluation of the derivative at the origin, *generates the moments* about the origin; hence the name. As numerous authors have pointed out, one really only needs to know the mgf in some small neighborhood of the origin. Indeed, its definition (e.g., Hogg and Craig 1970, Sec. 1.10; Feller 1968, p. 285) is stated as follows.

The moment-generating function of a random variable X is defined as $M_X(t) = E(e^{tX})$, provided the expectation $E(\ )$ exists in an interval $|t| < h$. Furthermore, $\lim_{t\to 0-} (d^nM_X(t)/dt^n) = E(X^n)$. We will now show that the mgf also generates negative moments, provided certain regularity conditions are met. Williams (1941) is the earliest appearance of this type of result we could find. Analogously, Chao and Strawderman (1972) have used the probability-generating function to find the negative integer moments of $X + A > 0$, where X is a random variable and A a constant; see also Kabe (1976) and Schuh (1981) for an application in a branching process problem.

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2. THE RESULTS

Suppose for the moment that X is a positive random variable. Since $x \equiv \int_{-\infty}^{0} e^{t/x}dt$ $(x > 0)$,

$$\begin{aligned} E(X) &= \int_0^\infty x dF(x) = \int_0^\infty \int_{-\infty}^0 e^{t/x} dt dF(x) \\ &= \int_{-\infty}^0 \int_0^\infty e^{t/x} dF(x) dt = \int_{-\infty}^0 M_{X^{-1}}(t) dt \\ &= \int_0^\infty M_{X^{-1}}(-t) dt. \end{aligned}$$

The interchange of this order of integration is subject to $E(e^{-t/X})$ being integrable from $t = 0$ to $t = \infty$.

Finally, by substituting X^{-1} for X, we find

$$E(X^{-1}) = \int_0^\infty M_X(-t)dt, \tag{1}$$

if either integral exists. Performing the integration analytically may not be easy; however (1) does give an alternative way of evaluating an inverse moment (perhaps even numerically).

There are two natural ways to generalize (1) to $E(X^{-n})$; one way gives

$$E(X^{-n}) = \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{n-1}}^\infty M_X(-t_n)dt_n \cdots dt_2 dt_1, \tag{2}$$

while the second way gives

$$E(X^{-n}) = \Gamma(n)^{-1} \int_0^\infty t^{n-1} M_X(-t)dt. \tag{3}$$

Probably the most important extension from (1) is

$$E(Y/X) = \int_0^\infty \lim_{t_2 \to 0-} (\partial/\partial t_2) M_{X,Y}(-t_1, t_2)dt_1, \tag{4}$$

if either integral exists, where $M_{X,Y}(t_1, t_2) = E(e^{t_1X+t_2Y})$ is the joint mgf of $X > 0$ and Y. Equation (4) can be very useful, since ratio statistics and questions concerning their bias arise frequently in statistical analyses. For example, Williams (1941) looked at moments of the ratio of the mean squared successive difference to the mean squared difference from a normal population, using a variant of (4). Similarly, Halperin and Gurian (1971) calculated bias and mean squared error for the usual least squares slope estimator when both variables are subject to error.

Now relax the assumption that X be a positive random variable, although the restriction that $F(0+) = F(0)$ is necessary to avoid degeneracy: define $\text{sgn}(x) = 1$ if $x \geq 0$, $= -1$ if $x < 0$. Then $X^{-1} = Y/$

$|X|$ almost surely, where $Y = \text{sgn}(X)$. Thus $E(X^{-1})$ is given by (4), after calculating $M_{|X|,Y}(t_1, t_2)$.

The interpretation of (1) to (4) deserves some comment. These equations are merely expressing the well-known duality between function space and transform space. Negative moments clearly pertain to the behavior of the distribution at the origin, which in turn suggests something about the behavior of the transform at infinity. Also, there is the pleasing symmetry that whereas positive moments are generated by successive differentiations of the mgf, negative moments are a consequence of successive integrations. The next section expands a little on this.

3. INVERSE MOMENTS AND LAPLACE AND MELLIN TRANSFORMS

Suppose (in the terminology of Abramowitz and Stegun 1965) that the original function is $G(t)$ and the image function (Laplace transform of G) is $g(s) = \int_0^\infty e^{-st}G(t)dt$. Then in Abramowitz and Stegun (1965, p. 1021), for example, we see that if the original function is $t^{-1}G(t)$, its Laplace transform is $\int_s^\infty g(x)dx$; that is,

$$\int_s^\infty g(x)dx = \int_s^\infty e^{-st}t^{-1}G(t)dt.$$

If we put $s = 0$, and interpret $G(t)$ as the density function of a positive random variable X with mgf $M_X(s) = g(-s)$, we then have exactly (1). Thus the result is by no means new, although most statisticians have probably not been aware of it.

The Mellin transform (e.g., Oberhettinger 1974) $h(z) = \int_0^\infty H(x)x^{z-1}dx$ of the function $H(x)$ is a function of the *complex* parameter z. If we interpret H as a density function of a positive random variable X, then knowledge of the Mellin transform tells us all moments of X, positive integer, negative integer, fractional, and so forth. This is a hint then that *all* moments are probably obtainable from the mgf by generalizing differentiation to fractional differentiation, including integration as a special case of negative integer differentiation. Thus the αth moment can be obtained from the αth fractional derivative of the mgf, $\alpha \in \mathbb{R}$ (see, e.g., Oldham and Spanier 1974). Indeed, Laue (1980) has considered this idea for characteristic functions; fractional derivatives are used for the formulation of new conditions on the existence of positive real moments of nonnegative random variables. We will not pursue this matter here, since we believe that it detracts from the simplicity and thrust of (1), (2), (3), and (4).

4. EXAMPLES

Example 1. The inverse moment of $aX + b$ is easily found by using (1):

$$E((aX + b)^{-1}) = \int_0^\infty e^{-bt}M_X(-at)dt.$$

Example 2. Suppose X is gamma distributed with scale parameter $\alpha > 0$ and shape parameter $\lambda > 0$. Then

$$M_X(t) = (1 - \alpha t)^{-\lambda}.$$

Therefore, from (1),

$$E(X^{-1}) = \alpha^{-1}(\lambda - 1)^{-1},$$

provided $\lambda > 1$, and from formulas (2) or (3),

$$\text{var}(X^{-1}) = \alpha^{-2}(\lambda - 1)^{-2}(\lambda - 2)^{-1}, \quad \lambda > 2.$$

Example 3. Consider the important problem of estimating the success probability for a negative binomial distribution. In this familiar situation, we let N denote the random number of trials required to obtain a fixed number r, of successes. Let p be the probability of a success on any trial. Then

$$M_N(t) = [pe^t/(1 - qe^t)]^r, \quad \infty < t < -\log q,$$

where $q = 1 - p$. The maximum likelihood estimator $\hat{p}$ of p is r/N, whose expectation is

$$E(\hat{p}) = rE(N^{-1}) = r\int_0^\infty p^r/(e^t - q)^r dt,$$

using (1). Putting $u = 1 - qe^{-t}$ and then expanding $(1 - u)^{r-1}$ by using the binomial theorem gives

$$E(\hat{p}) = r(-1)^r(p/q)[\log p - \sum_{s=1}^{r-1} (-1)^s\binom{r-1}{s}(1 - p^s)/sp^s].$$

Example 4. A beta random variable B, with parameters $\lambda > 0$, $\mu > 0$, can be written as $B = U/(U + V)$, where U, V are independent gamma random variables with parameters α, λ, and α, μ, respectively ($\alpha > 0$). Then $M_{U+V,U}(t_1, t_2) = (1 - \alpha t_1)^{-\mu}\cdot(1 - \alpha(t_1 + t_2))^{-\lambda}$. Partial differentiation with respect to t_2 and integration with respect to t_1 yields, using (4), $E(B) = \lambda/(\lambda + \mu)$, which is a well-known result.

5. CONCLUSION

Sometimes a solution distribution can only be written in terms of its mgf, the inversion being too difficult. In particular, the use of mgf's arises when independent random variables are being added. Although the distribution might be inaccessible (positive), moments are easily derived from differentiation. This article shows that negative moments are also hidden in the mgf. Indeed, the most general result we can present is for $X > 0$, Y random variables with joint mgf $M_{X,Y}(t_1, t_2)$. Then it can be easily seen that (1), (3), and (4) are special cases of

$$E(Y^j/X^k) = \Gamma(k)^{-1}\int_0^\infty t_1^{k-1} \lim_{t_2 \to 0-} \partial^j M_{X,Y}(-t_1, t_2)/\partial t_2^j dt_1,$$

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where $j = 0, 1, 2, \ldots, k = 1, 2, 3, \ldots$, and when either integral exists. The result has been in the literature in various forms for some time, but it is certainly not well known.

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Eye Fitting Straight Lines

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Because little is known about properties of lines fitted by eye, we designed and carried out an empirical investigation. Inexperienced graduate and postdoctoral students instructed to locate a line for estimating y from x for four sets of points tended to choose slopes near that of the first principal component (major axis) of the data, and their lines passed close to the centroids. Students had a slight tendency to choose consistently either steeper or shallower slopes for all sets of data.

KEY WORDS: Least squares; Regression; Subjective fitting; Principal components.

1. INTRODUCTION

The properties of least squares and other computed lines are well understood, but surprisingly little is known about the commonly used method of fitting by eye. This method involves maneuvering a string, black thread, or ruler until the fit seems satisfactory, and then drawing the line. We report one systematic investigation of eye fitting lines.

Students fitted lines by eye to four sets of points given in an experimental design to help us discover the properties of their fitted lines and whether order of fitting or practice made a difference. Other populations of subjects may produce different results. These sets of data were not unusual in curvature or in having outlying points or patterns. Thus additional populations of data sets could profitably be investigated.

The principal quantitative reference on fitting straight lines by eye is Finney (1951). He found that a mathematical iteration starting with slopes provided by scientists, inexperienced with probit analysis, gave satisfactory approximations to the relative potency in a bioassay.

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