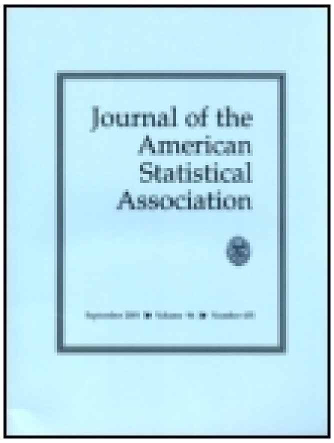

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Negative Moments of Positive Random Variables

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Negative Moments of Positive Random Variables

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We investigate the problem of finding the expected value of functions of a random variable X of the form $f(X)=(X+A)^{-n}$ where $X+A>0$ a.s. and n is a non-negative integer. The technique is to successively integrate the probability generating function and is suggested by the well-known result that successive differentiation leads to the positive moments. The technique is applied to the problem of finding $E[1/(X+A)]$ for the binomial and Poisson distributions.

1. INTRODUCTION

We investigate the problem of finding the expected value of functions of a random variable X, of the form

$$f(X) = (X + A)^{-n}, \qquad (1.1)$$

where $X+A>0$ a.s., and n is a non-negative integer. The technique is to successively integrate the probability generating function, and is suggested by the well-known result that successive differentiation leads to the positive moments. We develop the technique in Section 2 and apply it to finding $E[1/(X+A)]$ for the binomial and Poisson distributions in Section 3. Section 4 contains some further observations.

Negative moments are useful in applications in several contexts, notably in life testing problems, and survey sampling problems where ratio estimates are used. See [1, 2, 3, 4, 6] for some applications.

The results in the literature seem to have been confined primarily to the case of the truncated Poisson and binomial distributions [5, 6, 8, 9].

2. THE BASIC RESULTS

Let X be a random variable defined on a probability space $(\mathfrak{X}, \mathfrak{A}, P)$ and suppose $X+A>\delta>0$ a.s. $[P]$. Define the probability generating function of $X+A-1$ as

$$g_1(t) = E(t^{X+A-1}) \qquad 0 \le t \le 1. \qquad (2.1)$$

Now inductively define $g_{k+1}(t)$, for $k=1, 2, \cdots$ as follows:

$$g_{k+1}(t) = t^{-1}\int_0^t g_k(u)du. \qquad 0 \le t \le 1. \qquad (2.2)$$

Clearly (2.1) and (2.2) exist under the assumption on X. We have the following result.

Theorem 1. For $0\le t\le 1$,

$$E\left[\left(\frac{1}{X+A}\right)^k t^{X+A}\right] = \int_0^t g_k(u)du.$$

Proof:

Since

$$\frac{1}{t}\int_0^t u^{\Delta-1}du = \frac{t^{\Delta-1}}{\Delta} \quad \text{for } \Delta > 0$$

we have

$$\frac{t^{\Delta-1}}{\Delta^k} = \frac{1}{t}\int_0^t\left[\frac{1}{t_k}\int_0^{t_k}\left[\cdots\left[\frac{1}{t_2}\cdot\int_0^{t_2} t_1^{\Delta-1}dt_1\right]dt_2\right]\cdots\right]dt_n\right]dt.$$

Letting $\Delta=X-A$ and taking expectations gives Theorem 1.

We have immediately, the

Corollary:

$$E\left(\frac{1}{X+A}\right)^k = \int_0^1 g_k(u)du.$$

The potential applicability of the corollary is immediately evident. In Section 3 we apply it in the binomial and Poisson cases.

3. APPLICATIONS

3.1 Binomial Distribution

Let X be a binomially distributed random variable with parameters n and p. It is easy to show

$$g_1(t) = t^{A-1}(q + pt)^n, \qquad (3.1)$$

and using successive integrations by parts we are led to

$$\int_0^t g_1(u)du = \int_0^t u^{A-1}(q+pu)^n du$$

$$= q^n\left(\frac{q}{p}\right)^A \cdot\left[\sum_{k=1}^{r}(-1)^{k+1}\frac{(A-1)(A-2)\cdots(A-k+1)}{(n+1)(n+2)\cdots(n+k)} \qquad (3.2)\right.$$

$$b^{A-k}(1+b)^{n+k}$$

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$$+ (-1)^r \frac{(A-1)(A-2)\cdots(A-r)}{(n+1)(n+2)\cdots(n+r)} \cdot \int_0^b v^{A-r-1}(1+v)^{n+r}dv \Bigg],$$

where, by convention, $(A-1)(A-2)\cdots(A-k)=1$ if $k=0$; and where $A-r>0$, and $b=pt/q$.

For integral A, the formula leads to an exact result. Using relation (3.2) and the Corollary we have for integral A,

$$\begin{aligned} E\left(\frac{1}{X+A}\right) &= q^n \left(\frac{q}{p}\right)^A \\ &\cdot \Bigg[\sum_{k=1}^{A-1} (-1)^{k+1} \frac{(A-1)(A-2)\cdots(A-k+1)}{(n+1)(n+2)\cdots(n+k)} \\ &\cdot b^{A-k}(1+b)^{n+k} \\ &+ (-1)^{A-1} \frac{(A-1)!}{(n+1)(n+2)\cdots(n+A-1)} \frac{1}{n+A} \\ &\cdot ((1+b)^{n+A}-1) \Bigg], \end{aligned} \tag{3.3}$$

where $b=p/q$.

In particular,

$$E\left(\frac{1}{X+1}\right) = \frac{1-q^{n+1}}{(n+1)p} \cdot \tag{3.4}$$

and

$$\begin{aligned} E\left(\frac{1}{X+2}\right) &= q^n \left(\frac{q}{p}\right)^2 \left[\frac{1}{n+1}\left(\frac{p}{q}\right)^1 \left(\frac{1}{q}\right)^{n+1} \right. \\ &\quad \left. - \frac{1}{(n+1)(n+2)}\left(\frac{1}{q^{n+2}} - 1\right)\right] \\ &= \frac{1}{(n+1)p} - \frac{1}{(n+1)(n+2)p^2} \\ &\quad \cdot (1-q^{n+2}) \\ &= \frac{1}{(n+1)p}\left[1 - \frac{(1-q^{n+2})}{(n+2)p}\right]. \end{aligned} \tag{3.5}$$

Unfortunately (3.2) does not seem particularly useful for computing purposes when A is non-integral. Also the higher negative moments do not seem to be easily computed using our procedure, although the "negative factorial moments" are more tractable, a fact we will not prove here.

3.2 The Poisson Distribution

Let X be a random variable with the Poisson distribution with parameter λ.

Here

$$g_1(t) = t^{A-1}e^{-\lambda+\lambda t}. \tag{3.6}$$

$$\begin{aligned} \int_0^t g_1(u)du &= \int_0^t u^{A-1}e^{-\lambda+\lambda u}du \\ &= e^{-\lambda}\int_0^t u^{A-1}e^{\lambda u}du, \qquad \text{for } A>0 \\ &= e^{-\lambda}\left[\frac{t^{A-1}e^{\lambda t}}{\lambda} - \frac{A-1}{\lambda} \cdot \int_0^t u^{A-2}e^{\lambda u}du\right] \qquad \text{for } A>1. \end{aligned} \tag{3.7}$$

Letting $t=1$, we have

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda}\left[1-(A-1)E\left(\frac{1}{X+A-1}\right)\right] \qquad \text{for } A>1. \tag{3.8}$$

When $A=1$ we have directly

$$E\left(\frac{1}{X+1}\right) = \frac{e^{-\lambda}}{\lambda}[e^{\lambda}-1] = \frac{1-e^{-\lambda}}{\lambda} \cdot \tag{3.9}$$

(3.8) and (3.9) together allow an inductive calculation of $E[1/(X+A)]$ for any integer $A \geq 1$.

The formula is

$$E\left(\frac{1}{X+A}\right) = \frac{1}{\lambda}\left[1 + \sum_{j=1}^{r} \frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^{r+1}\frac{\prod_{i=1}^{r+1}(A-i)}{\lambda^r} E\left(\frac{1}{X+A-r-1}\right)\right]$$

for $A>r+1$. When $A=r+2$,

$$\begin{aligned} E\left(\frac{1}{X+A}\right) &= \frac{1}{\lambda}\left[1 + \sum_{j=1}^{A-2}\frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^{A-1}\frac{\prod_{i=1}^{A-1}(A-i)}{\lambda^{A-2}}\left(\frac{1}{\lambda}-\frac{e^{-\lambda}}{\lambda}\right)\right] \\ &= \frac{1}{\lambda}\left[1 + \sum_{j=1}^{A-1}\frac{\prod_{i=1}^{j}(A-i)}{\lambda^j}(-1)^j + (-1)^A\frac{(A-1)!e^{-\lambda}}{\lambda^{A-1}}\right]. \end{aligned} \tag{3.10}$$

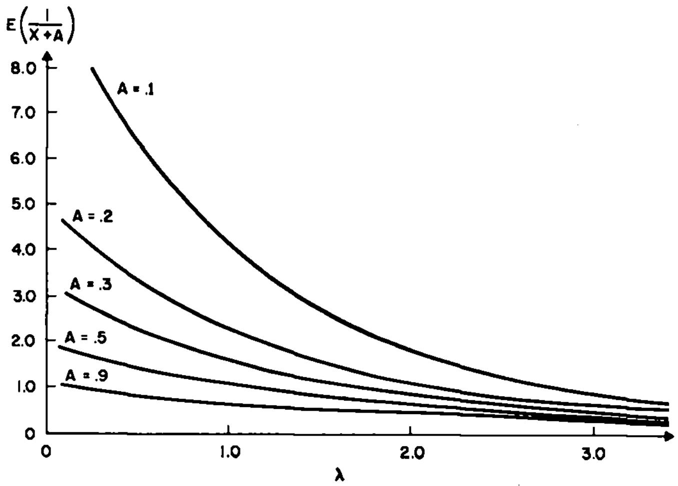
For non-integral A the first equation in (3.10) will serve to reduce the problem to one of finding $E[1/(X+A)]$ $0<A<1$. We include a graph of $E[1/(X+A)]$ for such values of A. We also indicate an asymptotic expression for $E[1/(X+A)]$ for small values of A.

$$\begin{aligned} E\left(\frac{1}{X+A}\right) &= \sum_{k=0}^{\infty}\frac{1}{k+A}\frac{e^{-\lambda}\lambda^k}{k!} \\ &= \frac{1}{A}e^{-\lambda} + \left[\sum_{k=1}^{\infty}\frac{\lambda^k e^{-\lambda}}{k!(k+A)}\right]. \end{aligned} \tag{3.11}$$

If we let

$$F(\lambda, A) = \sum_{k=1}^{\infty}\frac{1}{k!k}\lambda^k e^{-\lambda} + \frac{1}{A}e^{-\lambda}, \tag{3.12}$$

$X \sim POISSON\ (\lambda)$



we have

$$\left|E\left(\frac{1}{X+A}\right) - F(\lambda, A)\right|$$

$$\begin{aligned} &= A \textstyle\sum_{k=1}^{\infty} \dfrac{\lambda^k e^{-\lambda}}{k!k(k+A)} \\ &\leq A\left[\frac{\lambda e^{-\lambda}}{A+1} + \textstyle\sum_{k=2}^{\infty} \dfrac{\lambda^k e^{-\lambda}}{(k+1)!}\right] \\ &\leq A\left[\frac{1}{2}\lambda e^{-\lambda} + \frac{1-e^{-\lambda}}{\lambda} - e^{-\lambda}\right] \\ &= Af(\lambda) \qquad \text{for } A \leq 1. \end{aligned} \tag{3.13}$$

Note that $f(\lambda) \to 0$ as $\lambda \to 0$ or $\lambda \to \infty$. Also max $f(\lambda) \doteq .464$ (at $\lambda \doteq 1.35$), and hence with small error we may approximate $E[1/(X+A)]$ by $F(\lambda, A)$ for small A.

But $F(\lambda, A)$, although it doesn't have a simple closed form, can be put in the form of well tabulated functions. In particular, we have

$$F(\lambda, A) = \frac{1}{A} e^{-\lambda} + (1 - e^{-\lambda}) E\left(\frac{1}{X} \mid X > 0\right). \tag{3.14}$$

Now for a random variable Y with the truncated Poisson distribution (with $A = 0$)

$$g_1(t) = \frac{e^{-\lambda}(e^{\lambda t} - 1)}{(1 - e^{-\lambda})t} = E(t^{Y-1}).$$

Hence,

$$E\left(\frac{1}{X} \mid X > 0\right) = \frac{e^{-\lambda}}{1 - e^{-\lambda}} \int_0^1 \frac{e^{\lambda t} - 1}{t} dt. \tag{3.15}$$

This integral may be expressed in terms of Ei, the exponential integral [7, NBS Tables, pp. 228–51]. Specifically,

$$\int_0^1 \frac{e^{\lambda t} - 1}{t} dt = Ei(\lambda) - \log \lambda - \gamma, \tag{3.16}$$

where γ is Euler's Constant (.5772 $\cdots$). Hence $F(\lambda, A)$ may be evaluated with reasonable simplicity. Also we have a simple derivation, as a bonus, of the expression for the expectation of the inverse of the truncated Poisson used by Grab and Savage [5] in calculating their tables.

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