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Answer 1

a. Our claim is that $2^{3n} - 3^n$ is divisible by 5 for all integers n such that $n \geq 1$. To prove this claim, I will use mathematical induction:

Base Case: For $n = 1$ we have $2^3 - 3 = 5$, and $5|5$.
Inductive Step:
Inductive Hypothesis: Assume that $2^{3k} - 3^k$ is divisible by 5 for some $k \geq 1$. Then for the $k+1$ case;

$$2^{3(k+1)} - 3^{k+1} = 2^{3k}8 - 3^k 3 = 2^{3k}5 + 3(2^{3k} - 3^k)$$

Note that since $5|(2^{3k}5)$; and by the Inductive Hypothesis, $5|(2^{3k} - 3^k)$ we can say that $5|(2^{3k}5 + 3(2^{3k} - 3^k))$ by the properties of dividers. Hence:

$$5|(2^{3(k+1)} - 3^{k+1}) \text{ which concludes my proof by induction.}$$

b. Our claim is that $4^n - 7n - 1 > 0$ for all integers n such that $n \geq 2$. To prove this claim, I will use mathematical induction:

Base Case: For $n = 2$ we have $4^2 - 14 - 1 = 1 > 0$
Inductive Step:
Inductive Hypothesis: Assume that $4^k - 7k - 1 > 0$ for some $k \geq 2$. Then for the $k+1$ case:

$$4^{k+1} - 7(k+1) - 1 = 4^k 4 - 7k - 8$$

By our initial assumption $k \geq 2$, $21k \geq 42$ which means $21k > 4$. By adding $4^k 4 - 28k - 8$ to each side of the inequality, we get:

$4^k 4 - 7k - 8 > 4^k 4 - 28k - 4 = 4(4^k - 7k - 1)$ and by the inductive hypothesis, we know that:

$$4^k 4 - 7k - 8 = 4^{k+1} - 7(k+1) - 1 > 4(4^k - 7k - 1) > 0$$

$$4^{k+1} - 7(k+1) - 1 > 0 \text{ which concludes my proof by induction.}$$

Answer 2

a. We know that permutations of n objects, from which a many are identical to each other in their group and $n - a$ many are identical to each other in their group, is:

$$\frac{n!}{(n-a)!a!}$$

Then to construct bit strings of length 10 which have at least 7 1's in them, we need to look at each case separately (note that we know 1's and 0's are identical to each other in their corresponding groups):

Case 1: Bit strings of length 10 which have 7 1's in them: $\dfrac{10!}{7!3!} = 120$
Case 2: Bit strings of length 10 which have 8 1's in them: $\dfrac{10!}{8!2!} = 45$
Case 3: Bit strings of length 10 which have 9 1's in them: $\dfrac{10!}{9!1!} = 10$
Case 4: Bit strings of length 10 which have 10 1's in them: $\dfrac{10!}{10!0!} = 1$

Note that since each case is disjoint from the others (there are no common elements between every two cases), by the Inclusion-Exclusion Principle, the total number of bitstrings of length 10 which have at least 7 1's in them can be found by summing each case up:

$120 + 45 + 10 + 1 = 176$ thus, there are **176** bitstrings of length 10 which have at least 7 1's in them.

b. Since all 4 Discrete Mathematics textbooks are identical to each other and all 5 Statistical Methods textbooks are identical to each other; there is no difference between choosing a Discrete Mathematics textbook over another, the same is true for Statistical Methods textbooks too. So, to make a collection (a set) of 4 books, which contains at least 1 Discrete Mathematics textbook, and at least 1 Statistical Methods textbook; we have three cases:

Case 1: 1 Discrete Mathematics textbook, 3 Statistical Methods textbooks.
Case 2: 2 Discrete Mathematics textbook. 2 Statistical Methods textbooks.
Case 3: 3 Discrete Mathematics textbooks, 1 Statistical Methods textbook.

Since all cases are disjoint from each other, we can conclude that there are **3** ways to make a collection of 4 books (since the books are identical inside their respective classes), such that there is at least 1 Discrete Mathematics and 1 Statistical Methods textbook inside the collection.

c. To find the number of onto functions from a set with 5 elements to a set with 3 elements, we can first find the number of all possible functions from the first set to the second and then we can subtract the non-surjective functions from that number.

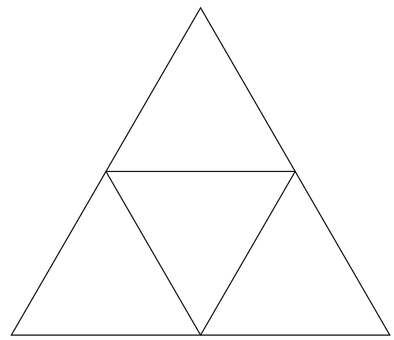
1) The number of all functions from a set with 5 elements to a set with 3 elements is: $3^5 = 243$

2) To find the case where only two elements are matched from the second set, we can first find the number of all onto functions from a set with 5 elements to a set with 2 elements, which is: $2^5 - 2 = 30$. Since we need to choose the 2 elements that are matched from a total of 3 elements, there are 3 possibilities, so total number of such functions become: $30 * 3 = 90$

3) To find the case where only one element is matched from the second set, we can first find the number of all functions from a set with 5 elements to a set with 1 element, which is: 1. Since this 1 element is chosen from 3 elements, the total number of such functions is: 3.

Then finally we can subtract the numbers we got from 2nd and 3rd parts, from the 1st part, which gives: $243 - 90 - 3 = \mathbf{150}$ which is the total number of onto functions from a set with 5 elements to a set with 3 elements.

Answer 3



The smaller equilateral triangles have a sidelength of 250 meters.

I have put borders on the open air circus area, and thus partitioned the area to 4 identical equilateral triangles of sidelength 250 meters. Note that if we think of the 5 children as pigeons and the 4 regions of smaller equilateral triangles as pigeonholes, by the Pigeonhole Principle, at least one of the regions must hold two children in it.

Since we know there exists two children inside an equilateral triangle of sidelength 250 meters, this means these two children are always within 250 meters of each other (since the longest distance between two points in an equilateral triangle is at most equal to its sidelength), which proves the claim of the question.

Answer 4

Considering the given recurrence relation $a_n = 3a_{n-1} + 5^{n-1}$ with the initial condition $a_1 = 4$:

a. The homogeneous recurrence relation can be written as:

$$a_n = 3a_{n-1} \text{ or in other words } a_n - 3a_{n-1} = 0$$

Then if we take $a_n^{(h)} = A\alpha^n$, the characteristic equation of this homogeneous recurrence relation can be written in the form of $\alpha - 3 = 0$. Which means:

$$\alpha = 3, \quad a_n^{(h)} = A3^n$$

Note that we cannot use the given initial condition to find A here as it is given for the general solution, not the homogeneous solution.

b. We can guess the partial solution to the given recurrence relation as:

$$a_n^{(p)} = B5^n \text{ then plugging this guess into the relation:}$$

$$a_n^{(p)} - 3a_{n-1}^{(p)} = B5^n - 3B5^{n-1} = 5^{n-1} \text{ dividing both sides by } 5^{n-1} \text{ gives:}$$

$$5B - 3B = 2B = 1, \quad B = 1/2$$

Then the partial solution can be written as:

$$a_n^{(p)} = \frac{1}{2}5^n$$

c. Combining the answers from part (a) and (b), we can write the general solution as:

$$a_n = a_n^{(h)} + a_n^{(p)} = A3^n + \frac{1}{2}5^n$$

Then using the initial condition, we can find A:

$$a_1 = 4 = 3A + \frac{5}{2}, \quad 3A = \frac{3}{2}, \quad A = \frac{1}{2}$$

So we claim that $a_n = \frac{1}{2}3^n + \frac{1}{2}5^n$ is the general solution to this recurrence relation. To prove this claim, I will use mathematical induction:

Base Case: For $n = 0$ we have $a_0 = \frac{1}{2} + \frac{1}{2} = 1$, which is consistent with the given recurrence relation, as $a_1 = 3a_0 + 5^{1-1} = 4$, and by plugging in $a_0 = 1$, we also get $a_1 = 4$ from using the general solution we derived.

Inductive Step:

Inductive Hypothesis: Assume that $a_k = \frac{1}{2}3^k + \frac{1}{2}5^k$ is true for some $k \geq 0$.

Then for the $k+1$ case:

$$a_{k+1} = \frac{1}{2}3^{k+1} + \frac{1}{2}5^{k+1} = 3\left(\frac{1}{2}3^k + \frac{1}{2}5^k\right) + 5^k$$

Then by the Inductive Hypothesis:

$$a_{k+1} = 3a_k + 5^k \text{ note that we can swap } k \text{ with } (n-1) \text{ such that:}$$

$$a_n = 3a_{n-1} + 5^{n-1} \quad \text{for } n \geq 1$$

Thus we have proven by induction that the general solution we derived is true for this system of recurrence relations.