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Answer 1

We know that for a cube graph Q_n (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 655 example 8), there are 2^n nodes in the graph where each node is labeled with a binary string of length n. And we know that there exists an edge between two vertices if their node labels differ in exactly one digit.

To construct a recursion relation for Q_n's number of edges, let's first look at how cube graphs behave for smaller n's such that:

For $n = 0$, the cube graph Q_0 has $2^0 = 1$ node and 0 edges.
For $n = 1$, the cube graph Q_1 has $2^1 = 2$ nodes and 1 edge.
For $n = 2$, the cube graph Q_2 has $2^2 = 4$ nodes and 4 edges.
For $n = 3$, the cube graph Q_3 has $2^3 = 8$ nodes and 12 edges.

Now let's think about the n^{th} case where each node's label has n binary digits in it. We know that each node will have an edge to a node with a label differing from it's label by exactly 1 digit. This means each node will have an edge to n different nodes, as there are n digits in each binary string label.

Also, we know that there are 2^n nodes in the graph in the n^{th} case and each node will have n edges connected to others. We can multiply these numbers and divide them by 2 (we divide by 2 since if we individually try to sum all edges, we will count all of the edges twice, as an edge connects exactly 2 nodes) to find the total number of edges. Then we have a formula, the number of edges in a Q_n graph is (for $n > 0$):

$$a_n = \frac{n * 2^n}{2} = n * 2^{n-1}$$

Then to construct a recurrence relation, we can look at what happens in the $n - 1$ case for $n > 1$:

$$a_{n-1} = (n-1) * 2^{n-2}, \text{ then we can write by substituting:}$$

$$a_n = \frac{2 * n * a_{n-1}}{n-1} = \frac{2 * (n - 1 + 1) * a_{n-1}}{n-1} = 2 * a_{n-1} + \frac{2 * a_{n-1}}{n-1} \text{ then substitute } a_{n-1} \text{ back:}$$

$$a_n = 2 * a_{n-1} + \frac{(n-1) * 2 * 2^{n-2}}{n-1} = 2 * a_{n-1} + 2^{n-1}$$

$$a_n = 2 * a_{n-1} + 2^{n-1} \text{ for } n > 1, \text{ and for } 0 \leq n \leq 1 \text{ we have } a_0 = 0, \quad a_1 = 1$$

Thus we have constructed a recurrence relation for a_n, the number of edges in a cube graph Q_n.

Answer 2

First of all we know that (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 542, table 1, 6th row):

$$< 1, 1, 1, 1, \cdots, 1, \cdots > \quad \longleftrightarrow \quad \frac{1}{1-x}$$

Then we can take the derivative of $\frac{1}{1-x}$ which corresponds to (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 542, table 1, 9th row):

$$< 1, 2, 3, 4, \cdots, (n+1), \cdots > \quad \longleftrightarrow \quad \frac{1}{(1-x)^2}$$

Multiplying by 3 gives us:

$$< 3, 6, 9, 12, \cdots, 3(n+1), \cdots > \quad \longleftrightarrow \quad \frac{3}{(1-x)^2}$$

Then if we multiply this statement with x, it corresponds to shifting the sequence by 1 digit to the right (as $a_0 + a_1x + \cdots$ becomes $0 + a_0x + a_1x^2 + \cdots$):

$$< 0, 3, 6, 9, 12, \cdots, 3n, \cdots > \quad \longleftrightarrow \quad \frac{3x}{(1-x)^2}$$

Finally, we can add $\frac{1}{1-x}$ to our current statement, which gives (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 538, Theorem 1):

$$< 1, 4, 7, 10, 13, \cdots, 3n+1, \cdots > \quad \longleftrightarrow \quad \frac{3x}{(1-x)^2} + \frac{1}{1-x}$$

Then our final result is:

$$< 1, 4, 7, 10, 13, \cdots, 3n+1, \cdots > \quad \longleftrightarrow \quad \frac{2x+1}{(1-x)^2}$$

The given sequence corresponds to the generating function $F(x) = \dfrac{2x+1}{(1-x)^2}$ in closed form.

Answer 3

$$a_n = a_{n-1} + 2^n, \quad n > 1 \text{ and the initial condition is } a_0 = 1$$

To solve the given recurrence relation, let's first define a function $F(x)$ for $|x| < 1$ such that:

$$< a_0, a_1, \cdots, a_n, \cdots > \quad \longleftrightarrow F(x) = \sum_{i=0}^{\infty} a_n x^n = a_0 + \sum_{i=1}^{\infty} a_n x^n = 1 + \sum_{i=1}^{\infty} a_n x^n$$

Then for the given recurrence relation, let's multiply both sides with x^n and take them into summations such that:

$$\sum_{i=1}^{\infty} a_n x^n = \sum_{i=1}^{\infty}(a_{n-1} + 2^n)x^n = \sum_{i=1}^{\infty} a_{n-1}x^n + \sum_{i=1}^{\infty} 2^n x^n \tag{1}$$

Notice that we can rewrite $\sum_{i=1}^{\infty} a_n x^n$ as $F(x) - 1$,and $\sum_{i=1}^{\infty} a_{n-1}x^n$ as $\sum_{i=0}^{\infty} a_n x^{n+1} = x * \sum_{i=0}^{\infty} a_n x^n$. Also, we know that for $|x| < 1$ (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 542, table 1, 7th row):

$$\sum_{i=0}^{\infty}(2x)^n = 1 + (2x) + (2x)^2 \cdots = \frac{1}{1-2x} \text{ which means } \sum_{i=1}^{\infty} 2^n x^n = \sum_{i=1}^{\infty}(2x)^n = \frac{1}{1-2x} - 1$$

So if we substitute these statements with their counterparts, (1) gives us:

$$F(x) - 1 = x * \sum_{i=0}^{\infty} a_n x^n + \frac{1}{1-2x} - 1 = xF(x) + \frac{1}{1-2x} - 1$$

Substract $xF(x)$ and -1 from both sides:

$$F(x) - xF(x) = (1-x)F(x) = \frac{1}{1-2x}$$

Divide both sides by $(1-x)$ (doable since $|x| < 1$):

$$F(x) = \frac{1}{1-2x} * \frac{1}{1-x} = \frac{2}{1-2x} + \frac{-1}{1-x} \text{ by Partial Fraction Decomposition}$$

Then since we know $< 1, 1, 1, 1, \cdots, n, \cdots > \quad \longleftrightarrow \quad \frac{1}{1-x}$ (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 542, table 1, 6th row):

$$\text{Multiplying by } -1 \text{ gives: } < -1, -1, -1, -1, \cdots, -1, \cdots > \quad \longleftrightarrow \quad \frac{-1}{1-x} \tag{2}$$

$$< 1, 2, 4, 8, \cdots, 2^n, \cdots > \quad \longleftrightarrow \quad \frac{1}{1-2x}$$

$$\text{Multiplying the above statement by 2 gives: } < 2, 4, 8, 16, \cdots, 2^{n+1}, \cdots > \quad \longleftrightarrow \quad \frac{2}{1-2x} \tag{3}$$

If we sum up (2) and (3) we get (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 538, Theorem 1):

$$< 1, 3, 7, 15, \cdots, 2^{n+1} - 1, \cdots > \quad \longleftrightarrow \quad \frac{2}{1-2x} + \frac{-1}{1-x} = F(x)$$

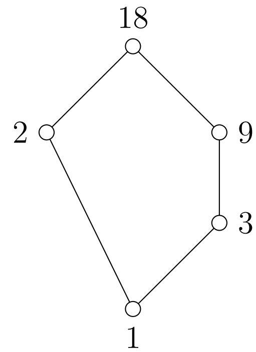
Then we can write a_n as:

$$a_n = 2^{n+1} - 1 \text{ for } n \geq 0$$

Thus we have solved the recurrence relation using generating functions.

Answer 4

a) The Hasse Diagram of the given relation on A is:



b) Let M_R be the matrix representation of the relation on the set A. Then we can build M_R such that where v_i and v_j are elements from the set A, $M_{ij} = 1$ means $v_i R v_j$ and $M_{ij} = 0$ otherwise:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

c) Note that we know R on the set A is a partially ordered set (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 623, example 12) since R has the properties of being reflexive, antisymmetric and transitive.

Then yes, the partially ordered set (A, R) is a Lattice, since every pair of distinct elements has a unique least upper bound and a unique greatest lower bound. We can show the unique LUB's and GLB's with a table:

	Least Upper Bound	Greatest Lower Bound
$\{1, 2\}$	2	1
$\{1, 3\}$	3	1
$\{1, 9\}$	9	1
$\{1, 18\}$	18	1
$\{2, 3\}$	18	1
$\{2, 9\}$	18	1
$\{2, 18\}$	18	2
$\{3, 9\}$	9	3
$\{3, 18\}$	18	3
$\{9, 18\}$	18	9

d) The symmetric closure of R, denoted as R_S, can be found such that $R_S = R \cup R^{-1}$ (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 638, in the Results section).

We also know that for a binary relation's matrix representation on a set, $R^{-1} = R^T$ (since we want to get $[M_{ji}]$, where v_i and v_j are elements from the set A, $M_{ji} = 1$ means $v_i R v_j$ and $M_{ji} = 0$ otherwise). From part (a) we have M_R, so:

$$M_R^{-1} = M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

We know that the matrix representation of R_S, denoted as M_S, can be obtained by $M_R \vee M_R^{-1}$ since $R_S = R \cup R^{-1}$. Then if we use "logical or" on the matrices, we get:

$$M_R \vee M_R^{-1} = M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 9 \\ 18 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 9 & 18 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Note that this is indeed the symmetric closure of R, since the matrix representation itself is symmetric.

e) For a pair of elements a, b to be comparable in a poset such as (A, R), the elements must be in the relation such that either aRb or bRa is true.The corollary of this statement is that in the Hasse Diagram associated with this poset, two elements are comparable if and only if there exists a strictly upward-directed path between them with one or the other being the upper element.

The integers 2 and 9 are incomparable, since $2 \nmid 9$ and $9 \nmid 2$. Also, no strictly upward-directed path in the Hasse Diagram goes from 2 to 9 or from 9 to 2.

The integers 3 and 18 are comparable, since $3 \mid 18$. Also, there exists a strictly upward-directed path in the Hasse Diagram which goes from 3 to 18.

Answer 5

Note that we can represent the reflexive binary relations on a set A with n elements as a matrix which has a diagonal full of 1's, such that (where * denotes undetermined 1's and 0's and $a_i \in A$):

$$M_{ref} = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & \cdots & a_n \end{array} \\ \begin{bmatrix} 1 & * & \cdots & * \\ * & 1 & \cdots & * \\ & & \ddots & \\ * & * & \cdots & 1 \end{bmatrix} \end{array}$$

a) The number of binary relations which are both reflexive and symmetric on A can be determined by organizing the matrix above, such that:

$$M_{ref,sym} = \begin{array}{c c} & \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} & \begin{bmatrix} 1 & b_{21} & b_{31} & \cdots & b_{n1} \\ b_{21} & 1 & & \cdots & b_{n2} \\ b_{31} & b_{32} & 1 & \cdots & b_{n3} \\ & & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 1 \end{bmatrix} \end{array}$$

Note that b_{ij} denotes the cell being 1 or 0 and since the matrix has to be symmetric, $b_{ij} = b_{ji}$ so I only denoted b_{ij}'s in the matrix representation.

To determine the number of possibilities for such a binary relation, we can go column by column to count the cells to be determined:

$$(n-1) + (n-2) + (n-3) + \cdots + 1 + 0 = \frac{n*(n-1)}{2} = \frac{n^2-n}{2}$$

Then we have $\frac{n^2-n}{2}$ cells to determine, each can have 2 values, which makes the total number of possibilities:

$2*2*\cdots*2 = 2^{\frac{n^2-n}{2}}$ is the number of different reflexive and symmetric binary relations on A.

b) Similar to the part (a), we can determine the number of binary relations which are both reflexive and antisymmetric on A by organizing the matrix above, such that:

$$M_{ref,asym} = \begin{array}{c c} & \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{array} & \begin{bmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & 1 & & \cdots & b_{2n} \\ b_{31} & b_{32} & 1 & \cdots & b_{3n} \\ & & & \ddots & \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 1 \end{bmatrix} \end{array}$$

Note that b_{ij} denotes the cell being 1 or 0 and since the matrix has to be antisymmetric, we can only have 3 possibilities for b_{ij} and b_{ji} such that they can be $\{0,0\}$, $\{1,0\}$ or $\{0,1\}$. Hence for every $\{b_{ij}, b_{ji}\}$ pair, we have 3 possibilities.

The total number of such distinct $\{b_{ij}, b_{ji}\}$ pairs can be found by going column by column and summing up the yet unpaired cells:

$$(n-1) + (n-2) + (n-3) + \cdots + 1 + 0 = \frac{n*(n-1)}{2} = \frac{n^2-n}{2}$$

Since we have $\frac{n^2-n}{2}$ many distinct pairs to determine, and each pair can have 3 different states, the total number of possibilities can be found by:

$3*3*\cdots*3 = 3^{\frac{n^2-n}{2}}$ is the number of different reflexive and antisymmetric binary relations on A.

Answer 6

No, the transitive closure of an antisymmetric relation is not always antisymmetric. To disprove this given claim, I will use a counter-example.

Let R be an antisymmetric relation on a set $A = \{a, b, c\}$ such that its matrix representation corresponds to:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then we know the transitive closure of R corresponds to $R^* = \bigcup_{i=1}^{|A|} R^i$ (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 601). Then as $|A| = 3$, we can write the matrix represantation of the transitive closure, M_T, as (as mentioned in the Rosen Discrete Mathematics Edition 8th, page 602, Theorem 3):

$$M_T = M_R \vee M_R^2 \vee M_R^3$$

$$M_R^2 = M_R * M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$M_R^3 = M_R^2 * M_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$M_T = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Note that this means the transitive closure of R is not antisymmetric, since $aRb \wedge bRa \rightarrow a = b$ is not true, as a and b are two different objects from the set A. Hence the given claim is wrong, proven by a counter-example.