Modular forms and Dirichlet series

M2 – Agrégation

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Abstract

In this article, we introduce the general results of modular forms. The aim is to study some properties of L–series associated to modular forms. In particular, we will estimate the Fourier coefficients of entire modular forms.

1 Reminders

Definition 1.1 (Möbius transformation): *Let $a, b, c, d \in \mathbb{Z}$. We define the general Möbius transformation by*

$$\forall z \in \mathbb{C} \setminus \{\frac{-d}{c}\},\ f(z) = \frac{az+b}{cz+d}.$$

We extend the definition of f to all of $\widehat{\mathbb{C}}$ by posing

$$f\left(-\frac{d}{c}\right) = \infty \text{ and } f(\infty) = \frac{a}{c}$$

with the convention : $z/0 = \infty$ if $z \neq 0$.

✍ **Notation 1.** For any Möbius transformation, with $ad - bc = 1$, we associate the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $\det(A) = 1$.

Definition 1.2 (Modular group): *We denote by Γ the set $\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1\}$. Γ is called the modular group or the Möbius group.*

Theorem 1.1 (Generators of Γ): *Let*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is noted that $S^2 = I_2$. One has, for each $\tau \in \widehat{\mathbb{C}}$,

- $T\tau = \tau + 1$;
- $S\tau = -\frac{1}{\tau}$.

The modular group Γ is generated by S and T, i.e. for each $A \in \Gamma$, there exists $k \in \mathbb{N}$ et $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$A = T^{n_1} S T^{n_2} S \cdots S T^{n_k}.$$

This decomposition is unique.

✍ **Notation 2.** In this article, H denotes the upper half-plane $\{z \in \mathbb{C},\ \mathrm{Im} z > 0\}$. Let G be a subgroup of Γ.

Definition 1.3 (Equivalence between two points): *Let $\tau, \tau' \in H$. τ and τ' are said to be equivalent under G if for some $A \in G$,*

$$\tau' = A\tau.$$

This relation is an equivalence relation.

Definition 1.4 (Fundamental region of G): *An open subset R_G is called a fundamental region of G if the two following conditions are satisfied :*

1. *It contains only one representative per equivalence class.*
2. *If $\tau \in H$, there is a point τ' in the closure of R_G such that τ and τ' are equivalent under G.*

In particular, the fundamental region of Γ, R_Γ is the set $\{\tau \in H,\ |\tau| > 1,\ |\tau + \overline{\tau}| < 1\}$ which is illustrated in the following figure :

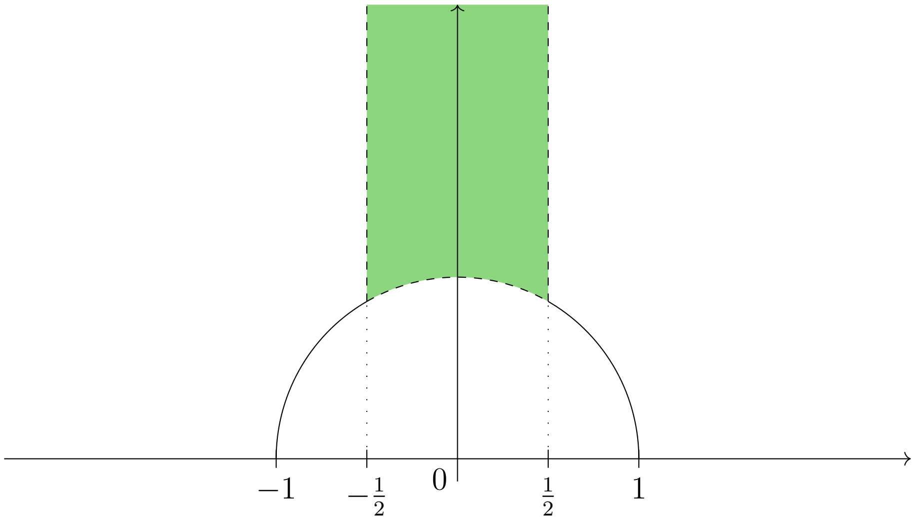

Figure 1. Fundamental region of Γ

2 Modular forms with multiplicative coefficients

This section focuses on a type of functions.

Definition 2.1 (Eisenstein series): *Let $k \in \mathbb{Z}$ such that $k \geqslant 2$. We define the Eisenstein series of weight $2k$ by*

$$\forall \tau \in H,\ G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}.$$

This function satisfies the relation, for all $\tau \in H$,

$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2k} G_{2k}(\tau)$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$.

Definition 2.2 (Modular forms of weight k): *Let $k \in \mathbb{N}$.*
A function $f : H \longrightarrow \mathbb{C}$ is an entire modular form of weight k if it satisfies the following conditions :

1. *f is analytic in the upper half-plane H.*
2. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$ *whenever* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
3. *Its Fourier expansion has the form* $f(\tau) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n\tau}$.

Remarks.

- We have
$$f(T\tau) = f(\tau+1) = f(\tau).$$
Hence, f is 1–periodic.
- The Fourier expansion of a 1–periodic function is defined as its Laurent expansion near the origin 0 where $q = e^{2\pi i\tau}$.
- The condition 3 states that the Laurent expansion of an entire modular form does not contain negative power of q. It means that an entire modular form is analytic everywhere in H and at $i\infty$.
- Let f be an entire modular form of weight k. For $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$,
$$\forall \tau \in H,\ f(A\tau) = f(\tau) = (-1)^k f(\tau).$$
So, if k is odd, then f is zero.
- The only modular forms of weight 0 are the constant functions.

The constant term $c(0)$ is called the value of f at $i\infty$, denoted by $f(i\infty)$. If $c(0) = 0$, the function f is called a cusp form and the smallest integers that satisfies $c(r) \neq 0$ is called the order of the zero of f at $i\infty$.

3 Estimates for the Fourier coefficients of entire modular forms

Let f be an entire form with Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} c(n) q^n$$

where $q = e^{2\pi i\tau}$. By writing $\tau = u + iv$, where $(u, v) \in \mathbb{R} \times \mathbb{R}_+^*$, then $q = e^{-2\pi v}e^{2\pi iu}$. For a fixed $v > 0$, as u varies from 0 to 1, x traces out the circle $C(v)$ of radius $e^{-2\pi v}$ with center at 0. By Cauchy's theorem, one has

$$c(n) = \frac{1}{2\pi i}\int_{C(v)} \frac{f(\tau)}{q^{n+1}}\,\mathrm{d}q = \int_0^1 \frac{f(u+iv)}{q^n}\,\mathrm{d}u. \tag{1}$$

We shall use this formula to estimate the order or magnitude of $|c(n)|$.

✍ **Notation 3.** We note M_k the space of entire modular forms of weight k and $M_{k,0}$ the subspace of the cusp forms of weight k in M_k.

Theorem 3.1: *If $f \in M_{2k,0}$, then $c(n) = \underset{n\to\infty}{O}(n^k)$.*

Proof. The series $\sum_{n\geqslant 0} c(n)q^n$ converges absolutely if $|q| < 1$, *i.e.* $v > 0$. Since $c(0) = 0$,

$$|f(\tau)| = |q|\left|\sum_{n=1}^{\infty} c(n)q^{n-1}\right| \leqslant |q|\sum_{n=1}^{\infty}|c(n)||q|^{n-1}.$$

Let us remind that the fundamental region of Γ is $R_\Gamma = \{\tau \in H,\ |\tau| > 1,\ |\tau + \overline{\tau}| < 1\}$. The conditions ($\tau \in H,\ |\tau| > 1$ and $|\tau + \overline{\tau}| < 1$) are equivalent to ($\mathrm{Im}\tau > 0$ and $\mathrm{Re}\tau < 1/2$) and $\mathrm{Re}(\tau)^2 + \mathrm{Im}(\tau)^2 > 1$.

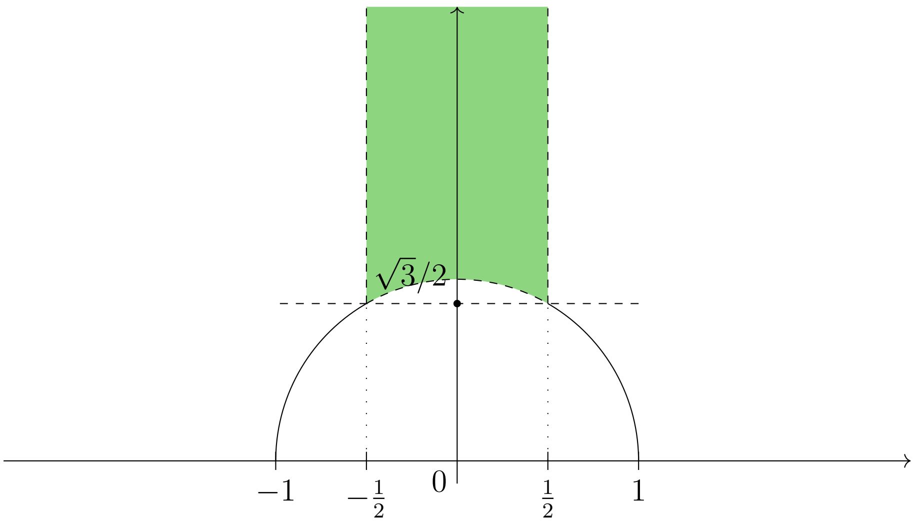

Figure 2. Fundamental region of Γ

Hence, if $\tau \in R_\Gamma$, by writing $\tau = u + iv$, $v > \frac{\sqrt{3}}{2} > \frac{1}{2}$, so

$$|q| = e^{-2\pi v} < e^{-\pi}$$

.
So,

$$|f(\tau)| \leqslant \sum_{n=1}^{\infty} |c(n)| e^{-(n-1)\pi} e^{-2\pi v}.$$

Let $C := \sum_{n=1}^{\infty} |c(n)| e^{-(n-1)\pi}$.
This implies

$$|f(\tau)| v^k \leqslant C v^k e^{-2\pi v}.$$

Let $g : \tau \in H \longmapsto \frac{1}{2}|\tau - \overline{\tau}| = v$.
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
For $\tau \in H$,

$$\begin{aligned} g(A\tau) &= g\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= \frac{1}{2}\left|\frac{a\tau + b}{c\tau + d} - \frac{a\overline{\tau} + b}{c\overline{\tau} + d}\right| \\ &= \frac{1}{2}\frac{|(a\tau + b)(c\overline{\tau} + d) - (a\overline{\tau} + b)(c\tau + d)|}{|c\tau + d|^2} \\ &= \frac{1}{2}|c\tau + d|^{-2}|(ad - bc)(\tau - \overline{\tau})| \\ &= |c\tau + d|^{-2} g(\tau). \end{aligned}$$

Hence

$$g(A\tau)^k = |c\tau + d|^{-2k} g(\tau)^k.$$

If we consider $\varphi : \tau \longmapsto |f(\tau)| g(\tau)^k = |f(\tau)| v^k$,

$$\varphi(\tau) = |f(\tau)| g(\tau)^k = |f(\tau)| v^k$$

is invariant under the transformations of Γ. Moreover, φ is continuous in $\overline{R_\Gamma}$ and $\lim_{v \to +\infty} \varphi(\tau) = 0$. So, φ is bounded in $\overline{R_\Gamma}$ and as φ is invariant under Γ, φ is bounded in H, so, there is $M \geqslant 0$ such that

$$|\varphi(\tau)| \leqslant M$$

for all $\tau \in H$.
Therefore,

$$|f(\tau)| \leqslant M v^{-k}$$

for all $\tau \in H$.
Using the equality 1,

$$|c(n)| \leqslant \int_0^1 |f(u+iv)q^{-n}|\,\mathrm{d}u \leqslant Mv^{-k}|q|^{-n} = Mv^{-k}e^{2\pi nv}.$$

This inequality is true for all $v > 0$, so it applies to $v = 1/n$ that gives

$$|c(n)| \leqslant Mn^k e^{2\pi} = \underset{n\to\infty}{O}(n^k).$$

□

Let's make some recall before the following result.
The Fourier expansion of G_{2k} is

$$\forall \tau \in H,\ G_{2k}(\tau) = 2\zeta(2k) + \sum_{n=1}^{+\infty} \frac{2(2i\pi)^{2k}}{(2k-1)!}\sigma_{2k-1}(n)e^{2i\pi n\tau}$$

where ζ is the Zeta function and

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha = \sum_{d|n} \left(\frac{n}{\alpha}\right)^\alpha,\ \alpha \in \mathbb{N}$$

σ is multiplicative : for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$,

$$f(mn) = f(m)f(n).$$

Theorem 3.2: *If $f \in M_{2k}$ and f is not a cusp form, then*

$$c(n) = \underset{n\to\infty}{O}(n^{2k-1}).$$

Proof. If $f = G_{2k}$, then each coefficient $c(n)$ is of the form $\alpha\sigma_{2k-1}(n)$.
Now,

$$\sigma_{2k-1}(n) = \sum_{d|n} \left(\frac{n}{d}\right)^{2k-1} = n^{2k-1}\sum_{d|n}\frac{1}{d^{2k-1}} \leqslant n^{2k-1}\sum_{d=1}^{\infty}\frac{1}{d^{2k-1}} = O(n^{2k-1}).$$

For a general non cuspform in M_{2k}, one proves that f can be expressed by a linear combination of G_{2k} and a cusp form. Let $\lambda = \frac{f(i\infty)}{G_{2k}(i\infty)}$. Then, $f - \lambda G_{2k}$ is a cusp form so

$$f = \lambda G_{2k} + g$$

where $g \in M_{2k,0}$.

Therefore, by applying the particular case of G_{2k} and the Théorème 3.1,

$$|c(n)| = \underset{n\to\infty}{O}(n^{2k-1}).$$

$\square$

4 Modular forms and Dirichlet series

The aim of this section is to study the properties of Dirichlet series associated to the coefficients of a modular form.

Definition 4.1 (Dirichlet series): *If f is an entire modular form with the Fourier expansion, for all $\tau \in H$,*

$$f(\tau) = c(0) + \sum_{n=1}^{+\infty} c(n) e^{2i\pi n\tau}.$$

We consider the following sum

$$\varphi(s) = \sum_{n=1}^{+\infty} \frac{c(n)}{n^s} \tag{2}$$

where the series converges.
The function φ is called a Dirichlet series.

If $f \in M_{2k,0}$, then $c(n) = \underset{n\to\infty}{O}(n^k)$. So, the series 2 converges absolutely for $\mathrm{Re}(s) > k+1$. Indeed,

$$\left|\frac{c(n)}{n^s}\right| = \frac{|c(n)|}{n^{\mathrm{Re}(s)}} = \underset{n\to\infty}{O}\left(\frac{1}{n^{\mathrm{Re}(s)-k}}\right).$$

In the same way, if $f \in M_{2k}$, and f is not a cusp form, as $c(n) = \underset{n\to\infty}{O}(n^{2k-1})$, then $\mathrm{Re}(s) > 2k$,

$$\left|\frac{c(n)}{n^s}\right| = \underset{n\to\infty}{O}\left(\frac{1}{n^{\mathrm{Re}(s)-2k+1}}\right)$$

and the series of 2 converges absolutely.
First, let us recall that if $(u_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers, the infinite product $\prod_{n\geqslant 0}(1+u_n)$ converges absolutely if, and only if the series $\sum_{n\geqslant 0} u_n$ converges absolutely.

An arithmetical function is a function $f : \mathbb{N} \longrightarrow \mathbb{C}$. This function is multiplicative if for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$, $f(mn) = f(m)f(n)$.

Remarks. For a multiplicative arithmetical function, one has

1. $\forall n \geqslant 1,\ f(n) = f(n)f(1)$.
2. If $p_1, \dots, p_k \in \mathcal{P}$ are distinct and $\alpha_1, \dots, \alpha_k \in \mathbb{N}$, $f(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = f(p_1^{\alpha_1}) \cdots f(p_k^{\alpha_k})$.

Theorem 4.1: *$\mathcal{P}$ denotes the set of prime numbers. Let f be a non zero multiplicative function such that $\sum_{n\geqslant 0} f(n)$ converges absolutely. Then, the product*

$$\prod_{p\in\mathcal{P}} (1 + f(p) + f(p^2) + \cdots) \tag{3}$$

converges absolutely, and

$$\sum_{n=1}^{+\infty} f(n) = \prod_{p\in\mathcal{P}} \left(\sum_{k=0}^{+\infty} f(p^k) \right).$$

Proof. For $x \in \mathbb{N}$, let $P(x) = \prod_{\substack{p\leqslant x \\ p\in\mathcal{P}}} \sum_{k=0}^{+\infty} f(p^k)$, the partial product.
Since $\{p^k,\ p \in \mathcal{P},\ k \in \mathbb{N}\} \subset \mathbb{N}$ and $\sum_{n\geqslant 0} f(n)$ converges absolutely, the series $\sum_{k\geqslant 0} f(p^k)$ converges absolutely.
A term of this product has the form of $f(p_1^{\alpha_1}) \cdots f(p_r^{\alpha_r}) = f(p_1^{\alpha_1} \cdots p_r^{\alpha_r})$.
By the fundamental theorem of arithmetic,

$$P(x) = \sum_{n\in A} f(n)$$

where $A = \{n \in \mathbb{N}^*,\ \text{ the prime factors of } n \text{ are } \leqslant x\}$.
Hence,

$$\sum_{n=1}^{+\infty} f(n) - P(x) = \sum_{n\in B} f(n)$$

where $B = \{n \in \mathbb{N}^*,\ \text{ the prime factors of } n \text{ are strictly greater than } x\}$.
For all $x \in \mathbb{N}^*$,

$$\left| \sum_{n=1}^{+\infty} f(n) - P(x) \right| \leqslant \sum_{n\in B} |f(n)| \leqslant \sum_{n>x} |f(n)|$$

with $x \to \infty$, since the series $\sum_{n\geqslant 0} |f(n)|$ converges,

$$P(x) \underset{x\to\infty}{\longrightarrow} \sum_{n=1}^{+\infty} f(n) < +\infty.$$

Now, since

$$\sum_{p\leqslant x}|f(p)+f(p^2)+\cdots|\leqslant\sum_{p\leqslant x}(|f(p)|+|f(p^2)|+\cdots)\leqslant\sum_{n=2}^{+\infty}|f(n)|<+\infty.$$

That inequality shows that the series $\sum_{p\in\mathcal{P}}|f(p)+f(p^2)+\cdots|$ converges because it is bounded, and with positive terms.
Then the product 3 converges absolutely. □

Corollary 4.1: *Let $f\in M_{2k}$ which is not a cusp form such that*

$$\forall\tau\in H,\ f(\tau)=c(0)+\sum_{n=1}^{+\infty}c(n)e^{2i\pi n\tau}$$

(with $c(1)=1$) and $\varphi(s)=\sum_{n=1}^{+\infty}\frac{c(n)}{n^s}$.
Suppose that $(c(n))_{n\geqslant 1}$ satisfies

$$c(m)c(n)=\sum_{d|(m,n)}d^{2k-1}c\left(\frac{mn}{d^2}\right) \tag{4}$$

The Dirichlet series $\varphi(s)$ has an Euler product representation of the form

$$\varphi(s)=\prod_{p\in\mathcal{P}}\frac{1}{1-c(p)p^{-s}+p^{2k-1}p^{-2s}} \tag{5}$$

Remarks. 5 implies that for all $m,n\in\mathbb{N}^*$ such that $(m,n)=1$,

$$c(mn)=c(m)c(n).$$

Proof. By the previous theorem,

$$\varphi(s)=\prod_{p\in\mathcal{P}}(1+\sum_{n=1}^{\infty}c(p^n)p^{-ns})$$

where the Dirichlet series congerges absolutely.
By 4,

$$c(p)c(p^n)=c(p^{n+1})+p^{2k-1}c(p^{n-1}).$$

Then, for $|x|<1$,

$$(1-c(p)x+p^{2k-1}x^2)(1+\sum_{n=1}^{+\infty}c(p^n)x^n)\ =1+\sum_{n=1}^{\infty}c(p^n)x^n-c(p)x-\sum_{n=1}^{+\infty}c(p)c(p^n)x^{n+1}$$

$$
\begin{aligned}
&+p^{2k-1}x^2+\sum_{n=1}^{+\infty}p^{2k-1}c(p^n)x^{n+2}\\
&=1+\sum_{n=1}^{\infty}c(p^n)x^n-c(p)x-\sum_{n=2}^{+\infty}(c(p^n)+p^{2k-1}c(p^{n-2}))x^n\\
&+p^{2k-1}x^2+\sum_{n=3}^{\infty}p^{2k-1}c(p^{n-2})x^n\\
&=1+c(p)x-c(p)x+c(p^2)x^2-c(p^2)x^2\\
&-p^{2k-1}c(1)x^2+p^{2k-1}x^2\\
&=1
\end{aligned}
$$

since $c(1)=1$. For $x=p^{-s}$, $|x|<1$, we have 5. □

The following result is proved by Hecke.

Theorem 4.2: *Let k be an even number greater than 4. Let $f\in M_k$ with the Fourier expansion defined previously, $\varphi(s)$ is defined for $\sigma=\mathrm{Re}(s)>k$. Then, $\varphi(s)$ can be continued analytically beyond the line $\{\sigma=k\}$ with the properties :*

(a) If $c(0)=0$, then φ is an entire function.

(b) If $c(0)\neq 0$, φ is analytic for all s except a simple pole at $s=k$, and

$$\mathrm{Res}(f,k)=\frac{(-1)^{k/2}c(0)(2\pi)^k}{\Gamma(k)}.$$

Proof. From the integral representation of $\Gamma(s)$, one has

$$
\begin{aligned}
\Gamma(s)(2\pi n)^{-s}&=\int_0^{+\infty}t^{s-1}(2\pi n)^{-s}e^{-t}\,\mathrm{d}t\\
&=\int_0^{+\infty}\left(\frac{t}{2\pi n}\right)^s\frac{1}{t}e^{-t}\,\mathrm{d}t\\
&\underset{u=\frac{t}{2\pi n}}{=}\int_0^{+\infty}u^s\frac{1}{2\pi n u}e^{-2\pi n u}(2\pi n)\,\mathrm{d}u\\
&=\int_0^{+\infty}e^{-2\pi n u}u^{s-1}\,\mathrm{d}u
\end{aligned}
$$

$$= \int_0^{+\infty} e^{-2\pi n u} u^{s-1} \, \mathrm{d}u$$

if $\sigma = \mathrm{Re}(s) > 0$.
Therefore, if $\sigma > k$, then

$$\forall n \geqslant 1,\ (2\pi)^{-s} \frac{c(n)}{n^s} \Gamma(s) = \int_0^{+\infty} e^{-2\pi n y} y^{s-1} c(n) \, \mathrm{d}y.$$

Hence,

$$\begin{aligned} (2\pi)^{-s} \varphi(s) \Gamma(s) &= \int_0^{+\infty} \sum_{n=1}^{+\infty} c(n) e^{-2\pi n y} y^{s-1} \, \mathrm{d}y \\ &= \int_0^{+\infty} (f(iy) - c(0)) y^{s-1} \, \mathrm{d}y. \end{aligned}$$

Moreover, since $f \in M_k$,

$$f\left(\frac{i}{y}\right) = (iy)^k f(iy).$$

So,

$$(2\pi)^{-s} \varphi(s) \Gamma(s) = \int_1^{+\infty} (f(iy) - c(0)) y^{s-1} \, \mathrm{d}y + \int_0^1 (f(iy) - c(0) y^{s-1} \, \mathrm{d}y$$

and

$$\begin{aligned} \int_0^1 (f(iy) - c(0)) y^{s-1} \, \mathrm{d}y &= \int_0^1 f(iy) y^{s-1} \, \mathrm{d}y - \frac{c(0)}{s} \\ &= \int_0^1 (iy)^{-k} f\left(\frac{i}{y}\right) y^{s-1} \, \mathrm{d}y - \frac{c(0)}{s} \\ &\underset{w=\frac{1}{y}}{=} \int_1^{+\infty} \left(\frac{i}{w}\right)^{-k} f(iw) \frac{1}{w^{s-1}} \frac{\mathrm{d}w}{w^2} - \frac{c(0)}{s} \\ &= \int_1^{+\infty} i^{-k} f(iw) w^{k-s-1} \, \mathrm{d}w - \frac{c(0)}{s}. \end{aligned}$$

Hence,

$$\begin{aligned} (2\pi)^{-s} \Gamma(s) \varphi(s) = & \int_1^{+\infty} (f(iy) - c(0)) y^{s-1} \, \mathrm{d}y + (-1)^{k/2} \int_1^{+\infty} (f(iw) - c(0)) w^{k-s-1} \, \mathrm{d}w \\ = & + (-1)^{k/2} c(0) \int_1^{+\infty} w^{k-s-1} \, \mathrm{d}w - \frac{c(0)}{s} \\ & = \int_1^{+\infty} (f(iy) - c(0)) (y^s + (-1)^{k/2} y^{k-s}) \frac{\mathrm{d}y}{y} - c(0) \left(\frac{1}{s} + \frac{(-1)^{k/2}}{k-s}\right). \end{aligned}$$

That relation is proved under the assumption $\sigma > k$, the member on the right $c(0)\left(\frac{1}{s} + \frac{(-1)^{k/2}}{k-s}\right)$ is meaningful for all complex number s. That gives the analytic continuation of φ beyond the line $\sigma = k$ and (a) and (b). $\square$

References

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