Section 3.1 Sequences and Their Limits

A sequence in a set S is a function whose domain is the set $\mathbb{N}$ of natural numbers, and whose range is contained in the set S. In this chapter, we will be concerned with sequences in $\mathbb{R}$ and will discuss what we mean by the convergence of these sequences.

3.1.1 Definition A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

In other words, a sequence in $\mathbb{R}$ assigns to each natural number $n = 1, 2, \ldots$ a uniquely determined real number. If $X : \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n rather than using the function notation $X(n)$. The values x_n are also called the **terms** or the **elements** of the sequence. We will denote this sequence by the notations

$$X, \quad (x_n), \quad (x_n : n \in \mathbb{N}).$$

Of course, we will often use other letters, such as $Y = (y_k)$, $Z = (z_i)$, and so on, to denote sequences.

We purposely use parentheses to emphasize that the ordering induced by the natural order of $\mathbb{N}$ is a matter of importance. Thus, we distinguish notationally between the sequence $(x_n : n \in \mathbb{N})$, whose infinitely many terms have an ordering, and the set of values $\{x_n : n \in \mathbb{N}\}$ in the range of the sequence that are not ordered. For example, the sequence $X := ((-1)^n : n \in \mathbb{N})$ has infinitely many terms that alternate between -1 and 1, whereas the set of values $\{(-1)^n : n \in \mathbb{N}\}$ is equal to the set $\{-1, 1\}$, which has only two elements.

Sequences are often defined by giving a formula for the nth term x_n. Frequently, it is convenient to list the terms of a sequence in order, stopping when the rule of formation seems evident. For example, we may define the sequence of reciprocals of the even numbers by writing

$$X := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right),$$

though a more satisfactory method is to specify the formula for the general term and write

$$X := \left(\frac{1}{2n} : n \in \mathbb{N}\right)$$

or more simply $X = (1/2n)$.

Another way of defining a sequence is to specify the value of x_1 and give a formula for $x_{n+1} (n \geq 1)$ in terms of x_n. More generally, we may specify x_1 and give a formula for obtaining x_{n+1} from $x_1, x_2, \ldots, x_n$. Sequences defined in this manner are said to be **inductively** (or **recursively**) defined.

3.1.2 Examples **(a)** If $b \in \mathbb{R}$, the sequence $B := (b, b, b, \ldots)$, all of whose terms equal b, is called the **constant sequence** b. Thus the constant sequence 1 is the sequence $(1, 1, 1, \ldots)$, and the constant sequence 0 is the sequence $(0, 0, 0, \ldots)$.

(b) If $b \in \mathbb{R}$, then $B := (b^n)$ is the sequence $B = (b, b^2, b^3, \ldots, b^n, \ldots)$. In particular, if $b = \frac{1}{2}$, then we obtain the sequence

$$\left(\frac{1}{2^n} : n \in \mathbb{N}\right) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right).$$

(c) The sequence of $(2n : n \in \mathbb{N})$ of even natural numbers can be defined inductively by

$$x_1 := 2, \qquad x_{n+1} := x_n + 2,$$

or by the definition

$$y_1 := 2, \qquad y_{n+1} := y_1 + y_n.$$

(d) The celebrated **Fibonacci sequence** $F := (f_n)$ is given by the inductive definition

$$f_1 := 1, \qquad f_2 := 1, \qquad f_{n+1} := f_{n-1} + f_n \quad (n \geq 2).$$

Thus each term past the second is the sum of its two immediate predecessors. The first ten terms of F are seen to be $(1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots)$. □

The Limit of a Sequence

There are a number of different limit concepts in real analysis. The notion of limit of a sequence is the most basic, and it will be the focus of this chapter.

3.1.3 Definition A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

Note The notation $K(\varepsilon)$ is used to emphasize that the choice of K depends on the value of ε. However, it is often convenient to write K instead of $K(\varepsilon)$. In most cases, a "small" value of ε will usually require a "large" value of K to guarantee that the distance $|x_n - x|$ between x_n and x is less than ε for all $n \geq K = K(\varepsilon)$.

When a sequence has limit x, we will use the notation

$$\lim X = x \quad \text{or} \quad \lim(x_n) = x.$$

We will sometimes use the symbolism $x_n \to x$, which indicates the intuitive idea that the values x_n "approach" the number x as $n \to \infty$.

3.1.4 Uniqueness of Limits *A sequence in $\mathbb{R}$ can have at most one limit.*

Proof. Suppose that x' and x'' are both limits of (x_n). For each $\varepsilon > 0$ there exist K' such that $|x_n - x'| < \varepsilon/2$ for all $n \geq K'$, and there exists K'' such that $|x_n - x''| < \varepsilon/2$ for all $n \geq K''$. We let K be the larger of K' and K''. Then for $n \geq K$ we apply the Triangle Inequality to get

$$\begin{aligned} |x' - x''| &= |x' - x_n + x_n - x''| \\ &\leq |x' - x_n| + |x_n - x''| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $x' - x'' = 0$. Q.E.D.

For $x \in \mathbb{R}$ and $\varepsilon > 0$, recall that the ε-neighborhood of x is the set

$$V_\varepsilon(x) := \{u \in \mathbb{R} : |u - x| < \varepsilon\}.$$

(See Section 2.2.) Since $u \in V_\varepsilon(x)$ is equivalent to $|u - x| < \varepsilon$, the definition of convergence of a sequence can be formulated in terms of neighborhoods. We give several different ways of saying that a sequence x_n converges to x in the following theorem.

3.1.5 Theorem *Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent.*

(a) *X converges to x.*

(b) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $|x_n - x| < \varepsilon$.*

(c) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $x - \varepsilon < x_n < x + \varepsilon$.*

(d) *For every ε-neighborhood $V_\varepsilon(x)$ of x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\varepsilon(x)$.*

Proof. The equivalence of (a) and (b) is just the definition. The equivalence of (b), (c), and (d) follows from the following implications:

$$|u - x| < \varepsilon \iff -\varepsilon < u - x < \varepsilon \iff x - \varepsilon < u < x + \varepsilon \iff u \in V_\varepsilon(x).$$

Q.E.D.

With the language of neighborhoods, one can describe the convergence of the sequence $X = (x_n)$ to the number x by saying: *for each ε-neighborhood $V_\varepsilon(x)$ of x, all but a finite number of terms of X belong to $V_\varepsilon(x)$.* The finite number of terms that may not belong to the ε-neighborhood are the terms $x_1, x_2, \ldots, x_{K-1}$.

Remark The definition of the limit of a sequence of real numbers is used to verify that a proposed value x is indeed the limit. It does *not* provide a means for initially determining what that value of x might be. Later results will contribute to this end, but quite often it is necessary in practice to arrive at a conjectured value of the limit by direct calculation of a number of terms of the sequence. Computers can be helpful in this respect, but since they can calculate only a finite number of terms of a sequence, such computations do not in any way constitute a proof of the value of the limit.

The following examples illustrate how the definition is applied to prove that a sequence has a particular limit. In each case, a positive ε is given and we are required to find a K, depending on ε, as required by the definition.

3.1.6 Examples **(a)** $\lim(1/n) = 0$.

If $\varepsilon > 0$ is given, then $1/\varepsilon > 0$. By the Archimedean Property 2.4.3, there is a natural number $K = K(\varepsilon)$ such that $1/K < \varepsilon$. Then, if $n \geq K$, we have $1/n \leq 1/K < \varepsilon$. Consequently, if $n \geq K$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

Therefore, we can assert that the sequence $(1/n)$ converges to 0.

(b) $\lim(1/(n^2+1)) = 0$.

Let $\varepsilon > 0$ be given. To find K, we first note that if $n \in \mathbb{N}$, then

$$\frac{1}{n^2+1} < \frac{1}{n^2} \le \frac{1}{n}.$$

Now choose K such that $1/K < \varepsilon$, as in (a) above. Then $n \ge K$ implies that $1/n < \varepsilon$, and therefore

$$\left|\frac{1}{n^2+1} - 0\right| = \frac{1}{n^2+1} < \frac{1}{n} < \varepsilon.$$

Hence, we have shown that the limit of the sequence is zero.

(c) $\lim\left(\dfrac{3n+2}{n+1}\right) = 3$.

Given $\varepsilon > 0$, we want to obtain the inequality

$$(1) \qquad \left|\frac{3n+2}{n+1} - 3\right| < \varepsilon$$

when n is sufficiently large. We first simplify the expression on the left:

$$\left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}.$$

Now if the inequality $1/n < \varepsilon$ is satisfied, then the inequality (1) holds. Thus if $1/K < \varepsilon$, then for any $n \ge K$, we also have $1/n < \varepsilon$ and hence (1) holds. Therefore the limit of the sequence is 3.

(d) $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.

We multiply and divide by $\sqrt{n+1} + \sqrt{n}$ to get

$$\begin{aligned}\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} &= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \\ &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \le \frac{1}{\sqrt{n}}\end{aligned}$$

For a given $\varepsilon > 0$, we obtain $1/\sqrt{n} < \varepsilon$ if and only if $1/n < \varepsilon^2$ or $n > 1/\varepsilon^2$. Thus if we take $K > 1/\varepsilon^2$, then $\sqrt{n+1} - \sqrt{n} < \varepsilon$ for all $n > K$. (For example, if we are given $\varepsilon = 1/10$, then $K > 100$ is required.)

(e) If $0 < b < 1$, then $\lim(b^n) = 0$.

We will use elementary properties of the natural logarithm function. If $\varepsilon > 0$ is given, we see that

$$b^n < \varepsilon \iff n \ln b < \ln \varepsilon \iff n > \ln \varepsilon / \ln b.$$

(The last inequality is reversed because $\ln b < 0$.) Thus if we choose K to be a number such that $K > \ln \varepsilon/\ln b$, then we will have $0 < b^n < \varepsilon$ for all $n \ge K$. Thus we have $\lim(b^n) = 0$.

For example, if $b = .8$, and if $\varepsilon = .01$ is given, then we would need $K > \ln .01/\ln .8 \approx 20.6377$. Thus $K = 21$ would be an appropriate choice for $\varepsilon = .01$. □

Remark **The $K(\varepsilon)$ Game** In the notion of convergence of a sequence, one way to keep in mind the connection between the ε and the K is to think of it as a game called the $K(\varepsilon)$ Game. In this game, Player A asserts that a certain number x is the limit of a sequence (x_n). Player B challenges this assertion by giving Player A a specific value for $\varepsilon > 0$. Player A must respond to the challenge by coming up with a value of K such that $|x_n - x| < \varepsilon$ for all $n > K$. If Player A can always find a value of K that works, then he wins, and the sequence is convergent. However, if Player B can give a specific value of $\varepsilon > 0$ for which Player A cannot respond adequately, then Player B wins, and we conclude that the sequence does not converge to x.

In order to show that a sequence $X = (x_n)$ does *not* converge to the number x, it is enough to produce one number $\varepsilon_0 > 0$ such that no matter what natural number K is chosen, one can find a particular n_K satisfying $n_K \geq K$ such that $|x_{n_K} - x| \geq \varepsilon_0$. (This will be discussed in more detail in Section 3.4.)

3.1.7 Example The sequence $(0, 2, 0, 2, \ldots, 0, 2, \ldots)$ does *not* converge to the number 0.

If Player A asserts that 0 is the limit of the sequence, he will lose the $K(\varepsilon)$ Game when Player B gives him a value of $\varepsilon < 2$. To be definite, let Player B give Player A the value $\varepsilon_0 = 1$. Then no matter what value Player A chooses for K, his response will not be adequate, for Player B will respond by selecting an even number $n > K$. Then the corresponding value is $x_n = 2$ so that $|x_n - 0| = 2 > 1 = \varepsilon_0$. Thus the number 0 is not the limit of the sequence. □

Tails of Sequences

It is important to realize that the convergence (or divergence) of a sequence $X = (x_n)$ depends only on the "ultimate behavior" of the terms. By this we mean that if, for any natural number m, we drop the first m terms of the sequence, then the resulting sequence X_m converges if and only if the original sequence converges, and in this case, the limits are the same. We will state this formally after we introduce the idea of a "tail" of a sequence.

3.1.8 Definition If $X = (x_1, x_2, \ldots, x_n, \ldots)$ is a sequence of real numbers and if m is a given natural number, then the ***m*-tail** of X is the sequence

$$X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \ldots)$$

For example, the 3-tail of the sequence $X = (2, 4, 6, 8, 10, \ldots, 2n, \ldots)$, is the sequence $X_3 = (8, 10, 12, \ldots, 2n + 6, \ldots)$.

3.1.9 Theorem *Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case,* $\lim X_m = \lim X$.

Proof. We note that for any $p \in \mathbb{N}$, the pth term of X_m is the $(p + m)$th term of X. Similarly, if $q > m$, then the qth term of X is the $(q - m)$th term of X_m.

Assume X converges to x. Then given any $\varepsilon > 0$, if the terms of X for $n \geq K(\varepsilon)$ satisfy $|x_n - x| < \varepsilon$, then the terms of X_m for $k \geq K(\varepsilon) - m$ satisfy $|x_k - x| < \varepsilon$. Thus we can take $K_m(\varepsilon) = K(\varepsilon) - m$, so that X_m also converges to x.

Conversely, if the terms of X_m for $k \geq K_m(\varepsilon)$ satisfy $|x_k - x| < \varepsilon$, then the terms of X for $n \geq K(\varepsilon) + m$ satisfy $|x_n - x| < \varepsilon$. Thus we can take $K(\varepsilon) = K_m(\varepsilon) + m$.

Therefore, X converges to x if and only if X_m converges to x. Q.E.D.

We shall sometimes say that a sequence X *ultimately* has a certain property if some tail of X has this property. For example, we say that the sequence $(3, 4, 5, 5, 5, \ldots, 5, \ldots)$ is "ultimately constant." On the other hand, the sequence $(3, 5, 3, 5, \ldots, 3, 5, \ldots)$ is not ultimately constant. The notion of convergence can be stated using this terminology: A sequence X converges to x if and only if the terms of X are ultimately in every ε-neighborhood of x. Other instances of this "ultimate terminology" will be noted later.

Further Examples

In establishing that a number x is the limit of a sequence (x_n), we often try to simplify the difference $|x_n - x|$ before considering an $\varepsilon > 0$ and finding a $K(\varepsilon)$ as required by the definition of limit. This was done in some of the earlier examples. The next result is a more formal statement of this idea, and the examples that follow make use of this approach.

3.1.10 Theorem *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have*

$$|x_n - x| \le C a_n \quad \text{for all} \quad n \ge m,$$

then it follows that $\lim(x_n) = x$.

Proof. If $\varepsilon > 0$ is given, then since $\lim(a_n) = 0$, we know there exists $K = K(\varepsilon/C)$ such that $n \ge K$ implies

$$a_n = |a_n - 0| < \varepsilon/C.$$

Therefore it follows that if both $n \ge K$ and $n \ge m$, then

$$|x_n - x| \le C a_n < C(\varepsilon/C) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $x = \lim(x_n)$. Q.E.D.

3.1.11 Examples **(a)** If $a > 0$, then $\lim\left(\dfrac{1}{1+na}\right) = 0$.

Since $a > 0$, then $0 < na < 1 + na$, and therefore $0 < 1/(1 + na) < 1/(na)$. Thus we have

$$\left|\frac{1}{1+na} - 0\right| \le \left(\frac{1}{a}\right)\frac{1}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\lim(1/n) = 0$, we may invoke Theorem 3.1.10 with $C = 1/a$ and $m = 1$ to infer that $\lim\ (1/(1 + na)) = 0$.

(b) If $0 < b < 1$, then $\lim(b^n) = 0$.

This limit was obtained earlier in Example 3.1.6(e). We will give a second proof that illustrates the use of Bernoulli's Inequality (see Example 2.1.13(c)).

Since $0 < b < 1$, we can write $b = 1/(1 + a)$, where $a := (1/b) - 1$ so that $a > 0$. By Bernoulli's Inequality, we have $(1 + a)^n \ge 1 + na$. Hence

$$0 < b^n = \frac{1}{(1+a)^n} \le \frac{1}{1+na} < \frac{1}{na}.$$

Thus from Theorem 3.1.10 we conclude that $\lim(b^n) = 0$.

In particular, if $b = .8$, so that $a = .25$, and if we are given $\varepsilon = .01$, then the preceding inequality gives us $K(\varepsilon) = 4/(.01) = 400$. Comparing with Example 3.1.6(e), where we obtained $K = 21$, we see this method of estimation does not give us the "best" value of K. However, for the purpose of establishing the limit, the size of K is immaterial.

(c) If $c > 0$, then $\lim(c^{1/n}) = 1$.

The case $c = 1$ is trivial, since then $(c^{1/n})$ is the constant sequence $(1, 1, \ldots)$, which evidently converges to 1.

If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n \leq (c-1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \leq (c-1)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1 + nh_n} < \frac{1}{nh_n},$$

from which it follows that $0 < h_n < 1/nc$ for $n \in \mathbb{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \frac{h_n}{1 + h_n} < h_n < \frac{1}{nc}$$

so that

$$|c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + \tfrac{1}{2}n(n-1)k_n^2 + \cdots \geq 1 + \tfrac{1}{2}n(n-1)k_n^2,$$

whence it follows that

$$n - 1 \geq \tfrac{1}{2}n(n-1)k_n^2.$$

Hence $k_n^2 \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$. □

Exercises for Section 3.1

1. The sequence (x_n) is defined by the following formulas for the nth term. Write the first five terms in each case:

   (a) $x_n := 1 + (-1)^n$, (b) $x_n := (-1)^n/n$,

   (c) $x_n := \dfrac{1}{n(n+1)}$, (d) $x := \dfrac{1}{n^2+2}$.

2. The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term x_n.
   (a) $5, 7, 9, 11, \ldots,$ (b) $1/2, -1/4, 1/8, -1/16, \ldots,$
   (c) $1/2, 2/3, 3/4, 4/5, \ldots,$ (d) $1, 4, 9, 16, \ldots.$
3. List the first five terms of the following inductively defined sequences.
   (a) $x_1 := 1, \quad x_{n+1} := 3x_n + 1,$
   (b) $y_1 := 2, \quad y_{n+1} := \frac{1}{2}(y_n + 2/y_n),$
   (c) $z_1 := 1, \quad z_2 := 2,\ z_{n+2} := (z_{n+1} + z_n)/(z_{n+1} - z_n),$
   (d) $s_1 := 3, \quad s_2 := 5,\ s_{n+2} := s_n + s_{n+1}.$
4. For any $b \in \mathbb{R}$, prove that $\lim(b/n) = 0$.
5. Use the definition of the limit of a sequence to establish the following limits.

   (a) $\lim\left(\dfrac{n}{n^2+1}\right) = 0,$ (b) $\lim\left(\dfrac{2n}{n+1}\right) = 2,$

   (c) $\lim\left(\dfrac{3n+1}{2n+5}\right) = \dfrac{3}{2},$ (d) $\lim\left(\dfrac{n^2-1}{2n^2+3}\right) = \dfrac{1}{2}.$
6. Show that

   (a) $\lim\left(\dfrac{1}{\sqrt{n+7}}\right) = 0,$ (b) $\lim\left(\dfrac{2n}{n+2}\right) = 2,$

   (c) $\lim\left(\dfrac{\sqrt{n}}{n+1}\right) = 0,$ (d) $\lim\left(\dfrac{(-1)^n n}{n^2+1}\right) = 0.$
7. Let $x_n := 1/\ln(n+1)$ for $n \in \mathbb{N}$.
   (a) Use the definition of limit to show that $\lim(x_n) = 0$.
   (b) Find a specific value of $K(\varepsilon)$ as required in the definition of limit for each of (i) $\varepsilon = 1/2$, and (ii) $\varepsilon = 1/10$.
8. Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).
9. Show that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim(x_n) = 0$, then $\lim(\sqrt{x_n}) = 0$.
10. Prove that if $\lim(x_n) = x$ and if $x > 0$, then there exists a natural number M such that $x_n > 0$ for all $n \geq M$.
11. Show that $\lim\left(\dfrac{1}{n} - \dfrac{1}{n+1}\right) = 0.$
12. Show that $\lim(\sqrt{n^2+1} - n) = 0$.
13. Show that $\lim(1/3^n) = 0$.
14. Let $b \in \mathbb{R}$ satisfy $0 < b < 1$. Show that $\lim(nb^n) = 0$. [*Hint*: Use the Binomial Theorem as in Example 3.1.11(d).]
15. Show that $\lim\left((2n)^{1/n}\right) = 1$.
16. Show that $\lim(n^2/n!) = 0$.
17. Show that $\lim(2^n/n!) = 0$. [*Hint*: If $n \geq 3$, then $0 < 2^n/n! \leq 2\left(\frac{2}{3}\right)^{n-2}$.]
18. If $\lim(x_n) = x > 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}x < x_n < 2x$.

Section 3.2 Limit Theorems

In this section we will obtain some results that enable us to evaluate the limits of certain sequences of real numbers. These results will expand our collection of convergent sequences rather extensively. We begin by establishing an important property of convergent sequences that will be needed in this and later sections.

3.2.1 Definition A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Thus, the sequence (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}\}$ of its values is a bounded subset of $\mathbb{R}$.

3.2.2 Theorem *A convergent sequence of real numbers is bounded.*

Proof. Suppose that $\lim(x_n) = x$ and let $\varepsilon := 1$. Then there exists a natural number $K = K(1)$ such that $|x_n - x| < 1$ for all $n \geq K$. If we apply the Triangle Inequality with $n \geq K$ we obtain

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

If we set

$$M := \sup\{|x_1|, |x_2|, \ldots, |x_{K-1}|, 1 + |x|\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Q.E.D.

Remark We can also prove a convergent sequence (x_n) is bounded using the language of neighborhoods. If $V_\varepsilon(x)$ is a given neighborhood of the limit x, then all but a finite number of terms of the sequence belong to $V_\varepsilon(x)$. Therefore, since $V_\varepsilon(x)$ is clearly bounded and finite sets are bounded, it follows that the sequence is bounded.

We will now examine how the limit process interacts with the operations of addition, subtraction, multiplication, and division of sequences. If $X = (x_n)$ and $Y = (y_n)$ are sequences of real numbers, then we define their **sum** to be the sequence $X + Y := (x_n + y_n)$, their **difference** to be the sequence $X - Y := (x_n - y_n)$, and their **product** to be the sequence $X \cdot Y := (x_n y_n)$. If $c \in \mathbb{R}$, we define the **multiple** of X by c to be the sequence $cX := (cx_n)$. Finally, if $Z = (z_n)$ is a sequence of real numbers with $z_n \neq 0$ for all $n \in \mathbb{N}$, then we define the **quotient** of X and Z to be the sequence $X/Z := (x_n/z_n)$.

For example, if X and Y are the sequences

$$X := (2,\ 4,\ 6, \ldots, 2n, \ldots), \quad Y := \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right),$$

then we have

$$\begin{aligned}
X + Y &= \left(\frac{3}{1}, \frac{9}{2}, \frac{19}{3}, \ldots, \frac{2n^2+1}{n}, \cdots\right), \\
X - Y &= \left(\frac{1}{1}, \frac{7}{2}, \frac{17}{3}, \ldots, \frac{2n^2-1}{n}, \cdots\right), \\
X \cdot Y &= \quad (2, 2, 2, \ldots, 2, \ldots), \\
3X &= (6, 12, 18, \ldots, 6n, \ldots), \\
X/Y &= (2, 8, 18, \ldots, 2n^2, \ldots).
\end{aligned}$$

We note that if Z is the sequence

$$Z := \Big(0, 2, 0, \ldots, 1 + (-1)^n, \ldots\Big),$$

then we can define $X + Z$, $X - Z$ and $X \cdot Z$, but X/Z is not defined since some of the terms of Z are zero.

We now show that sequences obtained by applying these operations to convergent sequences give rise to new sequences whose limits can be predicted.

3.2.3 Theorem **(a)** *Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then the sequences $X + Y$, $X - Y$, $X \cdot Y$, and cX converge to $x + y$, $x - y$, xy, and cx, respectively.*

(b) *If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to z and if $z \neq 0$, then the quotient sequence X/Z converges to x/z.*

Proof. (a) To show that $\lim(x_n + y_n) = x + y$, we need to estimate the magnitude of $|(x_n + y_n) - (x + y)|$. To do this we use the Triangle Inequality 2.2.3 to obtain

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |(x_n - x) + (y_n - y)| \\ &\leq |x_n - x| + |y_n - y|. \end{aligned}$$

By hypothesis, if $\varepsilon > 0$ there exists a natural number K_1 such that if $n \geq K_1$, then $|x_n - x| < \varepsilon/2$; also there exists a natural number K_2 such that if $n \geq K_2$, then $|y_n - y| < \varepsilon/2$. Hence if $K(\varepsilon) := \sup\{K_1, K_2\}$, it follows that if $n \geq K(\varepsilon)$ then

$$\begin{aligned} |(x_n + y_n) - (x + y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we infer that $X + Y = (x_n + y_n)$ converges to $x + y$.

Precisely the same argument can be used to show that $X - Y = (x_n - y_n)$ converges to $x - y$.

To show that $X \cdot Y = (x_n y_n)$ converges to xy, we make the estimate

$$\begin{aligned} |x_n y_n - xy| &= |(x_n y_n - x_n y) + (x_n y - xy)| \\ &\leq |x_n(y_n - y)| + |(x_n - x)y| \\ &= |x_n||y_n - y| + |x_n - x||y|. \end{aligned}$$

According to Theorem 3.2.2 there exists a real number $M_1 > 0$ such that $|x_n| \leq M_1$ for all $n \in \mathbb{N}$ and we set $M := \sup\{M_1, |y|\}$. Hence we have

$$|x_n y_n - xy| \leq M|y_n - y| + M|x_n - x|.$$

From the convergence of X and Y we conclude that if $\varepsilon > 0$ is given, then there exist natural numbers K_1 and K_2 such that if $n \geq K_1$ then $|x_n - x| < \varepsilon/2M$, and if $n \geq K_2$ then $|y_n - y| < \varepsilon/2M$. Now let $K(\varepsilon) = \sup\{K_1, K_2\}$; then, if $n \geq K(\varepsilon)$ we infer that

$$\begin{aligned} |x_n y_n - xy| &\leq M|y_n - y| + M|x_n - x| \\ &< M(\varepsilon/2M) + M(\varepsilon/2M) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves that the sequence $X \cdot Y = (x_n y_n)$ converges to xy.

The fact that $cX = (cx_n)$ converges to cx can be proved in the same way; it can also be deduced by taking Y to be the constant sequence $(c, c, c, \ldots)$. We leave the details to the reader.

(b) We next show that if $Z = (z_n)$ is a sequence of nonzero numbers that converges to a nonzero limit z, then the sequence $(1/z_n)$ of reciprocals converges to $1/z$. First let $\alpha := \frac{1}{2}|z|$ so that $\alpha > 0$. Since $\lim(z_n) = z$, there exists a natural number K_1 such that if $n \geq K_1$ then $|z_n - z| < \alpha$. It follows from Corollary 2.2.4(a) of the Triangle Inequality that $-\alpha \leq -|z_n - z| \leq |z_n| - |z|$ for $n \geq K_1$, whence it follows that $\frac{1}{2}|z| = |z| - \alpha \leq |z_n|$ for $n \geq K_1$. Therefore $1/|z_n| \leq 2/|z|$ for $n \geq K_1$ so we have the estimate

$$\begin{aligned} \left|\frac{1}{z_n} - \frac{1}{z}\right| &= \left|\frac{z - z_n}{z_n z}\right| = \frac{1}{|z_n z|}|z - z_n| \\ &\leq \frac{2}{|z|^2}|z - z_n| \quad \text{for all} \quad n \geq K_1. \end{aligned}$$

Now, if $\varepsilon > 0$ is given, there exists a natural number K_2 such that if $n \geq K_2$ then $|z_n - z| < \frac{1}{2}\varepsilon|z|^2$. Therefore, it follows that if $K(\varepsilon) = \sup\{K_1, K_2\}$, then

$$\left|\frac{1}{z_n} - \frac{1}{z}\right| < \varepsilon \quad \text{for all} \quad n > K(\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim\left(\frac{1}{z_n}\right) = \frac{1}{z}.$$

The proof of (b) is now completed by taking Y to be the sequence $(1/z_n)$ and using the fact that $X \cdot Y = (x_n/z_n)$ converges to $x(1/z) = x/z$. Q.E.D.

Some of the results of Theorem 3.2.3 can be extended, by Mathematical Induction, to a finite number of convergent sequences. For example, if $A = (a_n)$, $B = (b_n), \ldots, Z = (z_n)$ are convergent sequences of real numbers, then their sum $A + B + \cdots + Z = (a_n + b_n + \cdots + z_n)$ is a convergent sequence and

(1) $$\lim(a_n + b_n + \cdots + z_n) = \lim(a_n) + \lim(b_n) + \cdots + \lim(z_n).$$

Also their product $A \cdot B \cdots Z := (a_n b_n \cdots z_n)$ is a convergent sequence and

(2) $$\lim(a_n b_n \cdots z_n) = (\lim(a_n))\,(\lim(b_n)) \cdots (\lim(z_n)).$$

Hence, if $k \in \mathbb{N}$ and if $A = (a_n)$ is a convergent sequence, then

(3) $$\lim(a_n^k) = (\lim(a_n))^k.$$

We leave the proofs of these assertions to the reader.

3.2.4 Theorem *If $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.*

Proof. Suppose the conclusion is not true and that $x < 0$; then $\varepsilon := -x$ is positive. Since X converges to x, there is a natural number K such that

$$x - \varepsilon < x_n < x + \varepsilon \quad \text{for all} \quad n \geq K.$$

In particular, we have $x_K < x + \varepsilon = x + (-x) = 0$. But this contradicts the hypothesis that $x_n \geq 0$ for all $n \in \mathbb{N}$. Therefore, this contradiction implies that $x \geq 0$. Q.E.D.

We now give a useful result that is formally stronger than Theorem 3.2.4.

3.2.5 Theorem *If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$.*

Proof. Let $z_n := y_n - x_n$ so that $Z := (z_n) = Y - X$ and $z_n \geq 0$ for all $n \in \mathbb{N}$. It follows from Theorems 3.2.3 and 3.2.4 that

$$0 \leq \lim Z = \lim(y_n) - \lim(x_n),$$

so that $\lim(x_n) \leq \lim(y_n)$. Q.E.D.

The next result asserts that if all the terms of a convergent sequence satisfy an inequality of the form $a \leq x_n \leq b$, then the limit of the sequence satisfies the same inequality. Thus if the sequence is convergent, one may "pass to the limit" in an inequality of this type.

3.2.6 Theorem *If $X = (x_n)$ is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim(x_n) \leq b$.*

Proof. Let Y be the constant sequence $(b, b, b, \ldots)$. Theorem 3.2.5 implies that $\lim X \leq \lim Y = b$. Similarly one shows that $a \leq \lim X$. Q.E.D.

The next result asserts that if a sequence Y is squeezed between two sequences that converge to the *same limit*, then it must also converge to this limit.

3.2.7 Squeeze Theorem *Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \quad \textit{for all} \quad n \in \mathbb{N},$$

and that $\lim(x_n) = \lim(z_n)$. *Then $Y = (y_n)$ is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

Proof. Let $w := \lim(x_n) = \lim(z_n)$. If $\varepsilon > 0$ is given, then it follows from the convergence of X and Z to w that there exists a natural number K such that if $n \geq K$ then

$$|x_n - w| < \varepsilon \qquad \text{and} \qquad |z_n - w| < \varepsilon.$$

Since the hypothesis implies that

$$x_n - w \leq y_n - w \leq z_n - w \quad \text{for all} \quad n \in \mathbb{N},$$

it follows (why?) that

$$-\varepsilon < y_n - w < \varepsilon$$

for all $n \geq K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\lim(y_n) = w$. Q.E.D.

Remark Since any tail of a convergent sequence has the same limit, the hypotheses of Theorems 3.2.4, 3.2.5, 3.2.6, and 3.2.7 can be weakened to apply to the tail of a sequence. For example, in Theorem 3.2.4, if $X = (x_n)$ is "ultimately positive" in the sense that there exists $m \in \mathbb{N}$ such that $x_n \geq 0$ for all $n \geq m$, then the same conclusion that $x \geq 0$ will hold. Similar modifications are valid for the other theorems, as the reader should verify.

3.2.8 Examples **(a)** The sequence (n) is divergent.

It follows from Theorem 3.2.2 that if the sequence $X := (n)$ is convergent, then there exists a real number $M > 0$ such that $n = |n| < M$ for all $n \in \mathbb{N}$. But this violates the Archimedean Property 2.4.3.

(b) The sequence $((-1)^n)$ is divergent.

This sequence $X = ((-1)^n)$ is bounded (take $M := 1$), so we cannot invoke Theorem 3.2.2. However, assume that $a := \lim X$ exists. Let $\varepsilon := 1$ so that there exists a natural number K_1 such that

$$|(-1)^n - a| < 1 \quad \text{for all} \quad n \geq K_1.$$

If n is an odd natural number with $n \geq K_1$ this gives $|-1 - a| < 1$, so that $-2 < a < 0$. On the other hand, if n is an even natural number with $n \geq K_1$, this inequality gives $|1 - a| < 1$ so that $0 < a < 2$. Since a cannot satisfy both of these inequalities, the hypothesis that X is convergent leads to a contradiction. Therefore the sequence X is divergent.

(c) $\lim\left(\dfrac{2n+1}{n}\right) = 2.$

If we let $X := (2)$ and $Y := (1/n)$, then $((2n+1)/n) = X + Y$. Hence it follows from Theorem 3.2.3(a) that $\lim(X + Y) = \lim X + \lim Y = 2 + 0 = 2$.

(d) $\lim\left(\dfrac{2n+1}{n+5}\right) = 2.$

Since the sequences $(2n + 1)$ and $(n + 5)$ are not convergent (why?), it is not possible to use Theorem 3.2.3(b) directly. However, if we write

$$\frac{2n+1}{n+5} = \frac{2 + 1/n}{1 + 5/n},$$

we can obtain the given sequence as one to which Theorem 3.2.3(b) applies when we take $X := (2 + 1/n)$ and $Z := (1 + 5/n)$. (Check that all hypotheses are satisfied.) Since $\lim X = 2$ and $\lim Z = 1 \neq 0$, we deduce that $\lim((2n + 1)/(n + 5)) = 2/1 = 2$.

(e) $\lim\left(\dfrac{2n}{n^2+1}\right) = 0.$

Theorem 3.2.3(b) does not apply directly. (Why?) We note that

$$\frac{2n}{n^2+1} = \frac{2}{n + 1/n},$$

but Theorem 3.2.3(b) does not apply here either, because $(n + 1/n)$ is not a convergent sequence. (Why not?) However, if we write

$$\frac{2n}{n^2+1} = \frac{2/n}{1 + 1/n^2},$$

then we can apply Theorem 3.2.3(b), since $\lim(2/n) = 0$ and $\lim(1 + 1/n^2) = 1 \neq 0$. Therefore $\lim(2n/(n^2 + 1)) = 0/1 = 0$.

(f) $\lim\left(\dfrac{\sin n}{n}\right) = 0.$

We cannot apply Theorem 3.2.3(b) directly, since the sequence (n) is not convergent [neither is the sequence $(\sin n)$]. It does not appear that a simple algebraic manipulation will enable us to reduce the sequence into one to which Theorem 3.2.3 will apply. However, if we note that $-1 \leq \sin n \leq 1$, then it follows that

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \qquad \text{for all} \quad n \in \mathbb{N}.$$

Hence we can apply the Squeeze Theorem 3.2.7 to infer that $\lim(n^{-1}\sin n) = 0$. (We note that Theorem 3.1.10 could also be applied to this sequence.)

(g) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Let p be a polynomial; for example, let

$$p(t) := a_k t^k + a_{k-1} t^{k-1} + \cdots + a_1 t + a_0,$$

where $k \in \mathbb{N}$ and $a_j \in \mathbb{R}$ for $j = 0, 1, \ldots, k$. It follows from Theorem 3.2.3 that the sequence $(p(x_n))$ converges to $p(x)$. We leave the details to the reader as an exercise.

(h) Let $X = (x_n)$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Let r be a rational function (that is, $r(t) := p(t)/q(t)$, where p and q are polynomials). Suppose that $q(x_n) \neq 0$ for all $n \in \mathbb{N}$ and that $q(x) \neq 0$. Then the sequence $(r(x_n))$ converges to $r(x) = p(x)/q(x)$. We leave the details to the reader as an exercise. □

We conclude this section with several results that will be useful in the work that follows.

3.2.9 Theorem *Let the sequence $X = (x_n)$ converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$. That is, if $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.*

Proof. It follows from the Triangle Inequality (see Corollary 2.2.4(a)) that

$$\big||x_n| - |x|\big| \leq |x_n - x| \qquad \text{for all} \quad n \in \mathbb{N}.$$

The convergence of $(|x_n|)$ to $|x|$ is then an immediate consequence of the convergence of (x_n) to x. Q.E.D.

3.2.10 Theorem *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \geq 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.*

Proof. It follows from Theorem 3.2.4 that $x = \lim(x_n) \geq 0$ so the assertion makes sense. We now consider the two cases: (i) $x = 0$ and (ii) $x > 0$.

Case (i) If $x = 0$, let $\varepsilon > 0$ be given. Since $x_n \to 0$ there exists a natural number K such that if $n \geq K$ then

$$0 \leq x_n = x_n - 0 < \varepsilon^2.$$

Therefore [see Example 2.1.13(a)], $0 \leq \sqrt{x_n} < \varepsilon$ for $n \geq K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

Case (ii) If $x > 0$, then $\sqrt{x} > 0$ and we note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \geq \sqrt{x} > 0$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \left(\frac{1}{\sqrt{x}}\right)|x_n - x|.$$

The convergence of $\sqrt{x_n} \to \sqrt{x}$ follows from the fact that $x_n \to x$. Q.E.D.

For certain types of sequences, the following result provides a quick and easy "ratio test" for convergence. Related results can be found in the exercises.

3.2.11 Theorem *Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.*

Proof. By 3.2.4 it follows that $L \geq 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \geq K$ then

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon.$$

It follows from this (why?) that if $n \geq K$, then

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r.$$

Therefore, if $n \geq K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set $C := x_K / r^K$, we see that $0 < x_{n+1} < C r^{n+1}$ for all $n \geq K$. Since $0 < r < 1$, it follows from 3.1.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D.

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

so that $\lim(x_{n+1}/x_n) = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

Exercises for Section 3.2

1. For x_n given by the following formulas, establish either the convergence or the divergence of the sequence $X = (x_n)$.
   (a) $x_n := \dfrac{n}{n+1}$, (b) $x_n := \dfrac{(-1)^n n}{n+1}$,
   (c) $x_n := \dfrac{n^2}{n+1}$, (d) $x_n := \dfrac{2n^2+3}{n^2+1}$.
2. Give an example of two divergent sequences X and Y such that:
   (a) their sum $X + Y$ converges, (b) their product XY converges.
3. Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.
4. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.
5. Show that the following sequences are not convergent.
   (a) (2^n), (b) $\left((-1)^n n^2\right)$.
6. Find the limits of the following sequences:
   (a) $\lim\left((2 + 1/n)^2\right)$, (b) $\lim\left(\dfrac{(-1)^n}{n+2}\right)$,
   (c) $\lim\left(\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right)$, (d) $\lim\left(\dfrac{n+1}{n\sqrt{n}}\right)$.

7. If (b_n) is a bounded sequence and $\lim(a_n) = 0$, show that $\lim(a_n b_n) = 0$. Explain why Theorem 3.2.3 *cannot* be used.
8. Explain why the result in equation (3) before Theorem 3.2.4 *cannot* be used to evaluate the limit of the sequence $((1 + 1/n)^n)$.
9. Let $y_n := \sqrt{n+1} - \sqrt{n}$ for $n \in \mathbb{N}$. Show that $(\sqrt{n}y_n)$ converges. Find the limit.
10. Determine the limits of the following sequences.
    (a) $(\sqrt{4n^2 + n} - 2n)$,
    (b) $(\sqrt{n^2 + 5n} - n)$.
11. Determine the following limits.
    (a) $\lim\left((3\sqrt{n})^{1/2n}\right)$,
    (b) $\lim\left((n+1)^{1/\ln(n+1)}\right)$.
12. If $0 < a < b$, determine $\lim\left(\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$.
13. If $a > 0, b > 0$, show that $\lim\left(\sqrt{(n+a)(n+b)} - n\right) = (a+b)/2$.
14. Use the Squeeze Theorem 3.2.7 to determine the limits of the following,
    (a) (n^{1/n^2}),
    (b) $\left((n!)^{1/n^2}\right)$.
15. Show that if $z_n := (a^n + b^n)^{1/n}$ where $0 < a < b$, then $\lim(z_n) = b$.
16. Apply Theorem 3.2.11 to the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) (a^n),
    (b) $(b^n/2^n)$,
    (c) (n/b^n),
    (d) $(2^{3n}/3^{2n})$.
17. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_{n+1}/x_n) = 1$.
    (b) Give an example of a divergent sequence with this property. (Thus, this property cannot be used as a test for convergence.)
18. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) = L > 1$. Show that X is not a bounded sequence and hence is not convergent.
19. Discuss the convergence of the following sequences, where a, b satisfy $0 < a < 1, b > 1$.
    (a) $(n^2 a^n)$,
    (b) (b^n/n^2),
    (c) $(b^n/n!)$,
    (d) $(n!/n^n)$.
20. Let (x_n) be a sequence of positive real numbers such that $\lim(x_n^{1/n}) = L < 1$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in \mathbb{N}$. Use this to show that $\lim(x_n) = 0$.
21. (a) Give an example of a convergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$.
    (b) Give an example of a divergent sequence (x_n) of positive numbers with $\lim(x_n^{1/n}) = 1$. (Thus, this property cannot be used as a test for convergence.)
22. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists M such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?
23. Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent. (See Exercise 2.2.18.)
24. Show that if (x_n), (y_n), (z_n) are convergent sequences, then the sequence (w_n) defined by $w_n := \operatorname{mid}\{x_n, y_n, z_n\}$ is also convergent. (See Exercise 2.2.19.)

Section 3.3 Monotone Sequences

Until now, we have obtained several methods of showing that a sequence $X = (x_n)$ of real numbers is convergent:

(i) We can use Definition 3.1.3 or Theorem 3.1.5 directly. This is often (but not always) difficult to do.

(ii) We can dominate $|x_n - x|$ by a multiple of the terms in a sequence (a_n) known to converge to 0, and employ Theorem 3.1.10.

(iii) We can identify X as a sequence obtained from other sequences that are known to be convergent by taking tails, algebraic combinations, absolute values, or square roots, and employ Theorems 3.1.9, 3.2.3, 3.2.9, or 3.2.10.

(iv) We can "squeeze" X between two sequences that converge to the same limit and use Theorem 3.2.7.

(v) We can use the "ratio test" of Theorem 3.2.11.

Except for (iii), all of these methods require that we already know (or at least suspect) the value of the limit, and we then verify that our suspicion is correct.

There are many instances, however, in which there is no obvious candidate for the limit of a sequence, even though a preliminary analysis may suggest that convergence is likely. In this and the next two sections, we shall establish results that can be used to show a sequence is convergent even though the value of the limit is not known. The method we introduce in this section is more restricted in scope than the methods we give in the next two, but it is much easier to employ. It applies to sequences that are monotone in the following sense.

3.3.1 Definition Let $X = (x_n)$ be a sequence of real numbers. We say that X is **increasing** if it satisfies the inequalities

$$x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots.$$

We say that X is **decreasing** if it satisfies the inequalities

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots.$$

We say that X is **monotone** if it is either increasing or decreasing.

The following sequences are increasing:

$$\begin{aligned}&(1, 2, 3, 4, \ldots, n, \ldots), \quad (1, 2, 2, 3, 3, 3, \ldots),\\ &(a, a^2, a^3, \ldots, a^n, \ldots) \quad \text{if} \quad a > 1.\end{aligned}$$

The following sequences are decreasing:

$$\begin{aligned}&(1, 1/2, 1/3, \ldots, 1/n, \ldots), \quad (1, 1/2, 1/2^2, \ldots, 1/2^{n-1}, \ldots),\\ &(b, b^2, b^3, \ldots, b^n, \ldots) \qquad\quad \text{if} \quad 0 < b < 1.\end{aligned}$$

The following sequences are not monotone:

$$\big(+1, -1, +1, \ldots, (-1)^{n+1}, \ldots\big), \quad \big(-1, +2, -3, \ldots, (-1)^n n \ldots\big)$$

The following sequences are not monotone, but they are "ultimately" monotone:

$$(7, 6, 2, 1, 2, 3, 4, \ldots), \quad (-2, 0, 1, 1/2, 1/3, 1/4, \ldots).$$

3.3.2 Monotone Convergence Theorem *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*

(a) *If* $X = (x_n)$ *is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}.$$

(b) *If* $Y = (y_n)$ *is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}.$$

Proof. It was seen in Theorem 3.2.2 that a convergent sequence must be bounded.

Conversely, let X be a bounded monotone sequence. Then X is either increasing or decreasing.

(a) We first treat the case where $X = (x_n)$ is a bounded, increasing sequence. Since X is bounded, there exists a real number M such that $x_n \leq M$ for all $n \in \mathbb{N}$. According to the Completeness Property 2.3.6, the supremum $x^* = \sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$; we will show that $x^* = \lim(x_n)$.

If $\varepsilon > 0$ is given, then $x^* - \varepsilon$ is not an upper bound of the set $\{x_n : n \in \mathbb{N}\}$, and hence there exists x_K such that $x^* - \varepsilon < x_K$. The fact that X is an increasing sequence implies that $x_K \leq x_n$ whenever $n \geq K$, so that

$$x^* - \varepsilon < x_K \leq x_n \leq x^* < x^* + \varepsilon \quad \text{for all} \quad n \geq K.$$

Therefore we have

$$|x_n - x^*| < \varepsilon \quad \text{for all} \quad n \geq K.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (x_n) converges to x^*.

(b) If $Y = (y_n)$ is a bounded decreasing sequence, then it is clear that $X := -Y = (-y_n)$ is a bounded increasing sequence. It was shown in part (a) that $\lim X = \sup\{-y_n : n \in \mathbb{N}\}$, Now $\lim X = -\lim Y$ and also, by Exercise 2.4.4(b), we have

$$\sup\{-y_n : n \in \mathbb{N}\} = -\inf\{y_n : n \in \mathbb{N}\}.$$

Therefore $\lim Y = -\lim X = \inf\{y_n : n \in \mathbb{N}\}$. Q.E.D.

The Monotone Convergence Theorem establishes the existence of the limit of a bounded monotone sequence. It also gives us a way of calculating the limit of the sequence *provided* we can evaluate the supremum in case (a), or the infimum in case (b). Sometimes it is difficult to evaluate this supremum (or infimum), but once we know that it exists, it is often possible to evaluate the limit by other methods.

3.3.3 Examples **(a)** $\lim(1/\sqrt{n}) = 0$.

It is possible to handle this sequence by using Theorem 3.2.10; however, we shall use the Monotone Convergence Theorem. Clearly 0 is a lower bound for the set $\{1/\sqrt{n} : n \in \mathbb{N}\}$, and it is not difficult to show that 0 is the infimum of the set $\{1/\sqrt{n} : n \in \mathbb{N}\}$; hence $0 = \lim(1/\sqrt{n})$.

On the other hand, once we know that $X := (1/\sqrt{n})$ is bounded and decreasing, we know that it converges to some real number x. Since $X = (1/\sqrt{n})$ converges to x, it follows from Theorem 3.2.3 that $X \cdot X = (1/n)$ converges to x^2. Therefore $x^2 = 0$, whence $x = 0$.

(b) Let $h_n := 1 + 1/2 + 1/3 + \cdots + 1/n$ for $n \in \mathbb{N}$.

Since $h_{n+1} = h_n + 1/(n+1) > h_n$, we see that (h_n) is an increasing sequence. By the Monotone Convergence Theorem 3.3.2, the question of whether the sequence is convergent or not is reduced to the question of whether the sequence is bounded or not. Attempts to use direct numerical calculations to arrive at a conjecture concerning the possible boundedness of the sequence (h_n) lead to inconclusive frustration. A computer run will reveal the approximate values $h_n \approx 11.4$ for $n = 50,000$, and $h_n \approx 12.1$ for $n = 100{,}000$. Such numerical facts may lead the casual observer to conclude that the sequence is bounded. However, the sequence is in fact divergent, which is established by noting that

$$\begin{aligned}
h_{2^n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right)\\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right)\\
&= 1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}\\
&= 1+\frac{n}{2}.
\end{aligned}$$

Since (h_n) is unbounded, Theorem 3.2.2 implies that it is divergent. (This proves that the infinite series known as the *harmonic series* diverges. See Example 3.7.6(b) in Section 3.7.)

The terms h_n increase extremely slowly. For example, it can be shown that to achieve $h_n > 50$ would entail approximately 5.2×10^{21} additions, and a normal computer performing 400 million additions a second would require more than 400,000 years to perform the calculation (there are 31,536,000 seconds in a year). A supercomputer that can perform more than a trillion additions a second would take more than 164 years to reach that modest goal. And the IBM Roadrunner supercomputer at a speed of a quadrillion operations per second would take over a year and a half. □

Sequences that are defined inductively must be treated differently. If such a sequence is known to converge, then the value of the limit can sometimes be determined by using the inductive relation.

For example, suppose that convergence has been established for the sequence (x_n) defined by

$$x_1 = 2, \qquad x_{n+1} = 2 + \frac{1}{x_n}, \quad n \in \mathbb{N}.$$

If we let $x = \lim(x_n)$, then we also have $x = \lim(x_{n+1})$ since the 1-tail (x_{n+1}) converges to the same limit. Further, we see that $x_n \geq 2$, so that $x \neq 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Therefore, we may apply the limit theorems for sequences to obtain

$$x = \lim(x_{n+1}) = 2 + \frac{1}{\lim(x_n)} = 2 + \frac{1}{x}.$$

Thus, the limit x is a solution of the quadratic equation $x^2 - 2x - 1 = 0$, and since x must be positive, we find that the limit of the sequence is $x = 1 + \sqrt{2}$.

Of course, the issue of convergence must not be ignored or casually assumed. For example, if we assumed the sequence (y_n) defined by $y_1 := 1,\ y_{n+1} := 2y_n + 1$ is convergent with limit y, then we would obtain $y = 2y + 1$, so that $y = -1$. Of course, this is absurd.

In the following examples, we employ this method of evaluating limits, but only after carefully establishing convergence using the Monotone Convergence Theorem. Additional examples of this type will be given in Section 3.5.

3.3.4 Examples **(a)** Let $Y = (y_n)$ be defined inductively by $y_1 := 1,\ y_{n+1} := \frac{1}{4}(2y_n + 3)$ for $n \geq 1$. We shall show that $\lim Y = 3/2$.

Direct calculation shows that $y_2 = 5/4$. Hence we have $y_1 < y_2 < 2$. We show, by Induction, that $y_n < 2$ for all $n \in \mathbb{N}$. Indeed, this is true for $n = 1, 2$. If $y_k < 2$ holds for some $k \in \mathbb{N}$, then

$$y_{k+1} = \tfrac{1}{4}(2y_k + 3) < \tfrac{1}{4}(4 + 3) = \tfrac{7}{4} < 2,$$

so that $y_{k+1} < 2$. Therefore $y_n < 2$ for all $n \in \mathbb{N}$.

We now show, by Induction, that $y_n < y_{n+1}$ for all $n \in \mathbb{N}$. The truth of this assertion has been verified for $n = 1$. Now suppose that $y_k < y_{k+1}$ for some k; then $2y_k + 3 < 2y_{k+1} + 3$, whence it follows that

$$y_{k+1} = \tfrac{1}{4}(2y_k + 3) < \tfrac{1}{4}(2y_{k+1} + 3) = y_{k+2}.$$

Thus $y_k < y_{k+1}$ implies that $y_{k+1} < y_{k+2}$. Therefore $y_n < y_{n+1}$ for all $n \in \mathbb{N}$.

We have shown that the sequence $Y = (y_n)$ is increasing and bounded above by 2. It follows from the Monotone Convergence Theorem that Y converges to a limit that is at most 2. In this case it is not so easy to evaluate $\lim(y_n)$ by calculating $\sup\{y_n : n \in \mathbb{N}\}$. However, there is another way to evaluate its limit. Since $y_{n+1} = \frac{1}{4}(2y_n + 3)$ for all $n \in \mathbb{N}$, the nth term in the 1-tail Y_1 of Y has a simple algebraic relation to the nth term of Y. Since, by Theorem 3.1.9, we have $y := \lim Y_1 = \lim Y$, it therefore follows from Theorem 3.2.3 (why?) that

$$y = \tfrac{1}{4}(2y + 3),$$

from which it follows that $y = 3/2$.

(b) Let $Z = (z_n)$ be the sequence of real numbers defined by $z_1 := 1$, $z_{n+1} := \sqrt{2z_n}$ for $n \in \mathbb{N}$. We will show that $\lim(z_n) = 2$.

Note that $z_1 = 1$ and $z_2 = \sqrt{2}$; hence $1 \le z_1 < z_2 < 2$. We claim that the sequence Z is increasing and bounded above by 2. To show this we will show, by Induction, that $1 \le z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$. This fact has been verified for $n = 1$. Suppose that it is true for $n = k$; then $2 \le 2z_k < 2z_{k+1} < 4$, whence it follows (why?) that

$$1 < \sqrt{2} \le z_{k+1} = \sqrt{2z_k} < z_{k+2} = \sqrt{2z_{k+1}} < \sqrt{4} = 2.$$

[In this last step we have used Example 2.1.13(a).] Hence the validity of the inequality $1 \le z_k < z_{k+1} < 2$ implies the validity of $1 \le z_{k+1} < z_{k+2} < 2$. Therefore $1 \le z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$.

Since $Z = (z_n)$ is a bounded increasing sequence, it follows from the Monotone Convergence Theorem that it converges to a number $z := \sup\{z_n\}$. It may be shown directly that $\sup\{z_n\} = 2$, so that $z = 2$. Alternatively we may use the method employed in part (a). The relation $z_{n+1} = \sqrt{2z_n}$ gives a relation between the nth term of the 1-tail Z_1 of Z and the nth term of Z. By Theorem 3.1.9, we have $\lim Z_1 = z = \lim Z$. Moreover, by Theorems 3.2.3 and 3.2.10, it follows that the limit z must satisfy the relation

$$z = \sqrt{2z}.$$

Hence z must satisfy the equation $z^2 = 2z$, which has the roots $z = 0, 2$. Since the terms of $z = (z_n)$ all satisfy $1 \le z_n \le 2$, it follows from Theorem 3.2.6 that we must have $1 \le z \le 2$. Therefore $z = 2$. □

The Calculation of Square Roots

We now give an application of the Monotone Convergence Theorem to the calculation of square roots of positive numbers.

3.3.5 Example Let $a > 0$; we will construct a sequence (s_n) of real numbers that converges to $\sqrt{a}$.

Let $s_1 > 0$ be arbitrary and define $s_{n+1} := \frac{1}{2}(s_n + a/s_n)$ for $n \in \mathbb{N}$. We now show that the sequence (s_n) converges to $\sqrt{a}$. (This process for calculating square roots was known in Mesopotamia before 1500 B.C.)

We first show that $s_n^2 \geq a$ for $n \geq 2$. Since s_n satisfies the quadratic equation $s_n^2 - 2s_{n+1}s_n + a = 0$, this equation has a real root. Hence the discriminant $4s_{n+1}^2 - 4a$ must be nonnegative; that is, $s_{n+1}^2 \geq a$ for $n \geq 1$.

To see that (s_n) is ultimately decreasing, we note that for $n \geq 2$ we have

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) = \frac{1}{2} \cdot \frac{(s_n^2 - a)}{s_n} \geq 0.$$

Hence, $s_{n+1} \leq s_n$ for all $n \geq 2$. The Monotone Convergence Theorem implies that $s := \lim(s_n)$ exists. Moreover, from Theorem 3.2.3, the limit s must satisfy the relation

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right),$$

whence it follows (why?) that $s = a/s$ or $s^2 = a$. Thus $s = \sqrt{a}$.

For the purposes of calculation, it is often important to have an estimate of *how rapidly* the sequence (s_n) converges to $\sqrt{a}$. As above, we have $\sqrt{a} \leq s_n$ for all $n \geq 2$, whence it follows that $a/s_n \leq \sqrt{a} \leq s_n$. Thus we have

$$0 \leq s_n - \sqrt{a} \leq s_n - a/s_n = (s_n^2 - a)/s_n \quad \text{for} \quad n \geq 2.$$

Using this inequality we can calculate $\sqrt{a}$ to any desired degree of accuracy. □

Euler's Number

We conclude this section by introducing a sequence that converges to one of the most important ''transcendental'' numbers in mathematics, second in importance only to π.

3.3.6 Example Let $e_n := (1 + 1/n)^n$ for $n \in \mathbb{N}$. We will now show that the sequence $E = (e_n)$ is bounded and increasing; hence it is convergent. The limit of this sequence is the famous *Euler number e*, whose approximate value is 2.718 281 828 459 045 . . . , which is taken as the base of the ''natural'' logarithm.

If we apply the Binomial Theorem, we have

$$e_n = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3}$$
$$+ \cdots + \frac{n(n-1) \cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n}.$$

If we divide the powers of n into the terms in the numerators of the binomial coefficients, we get

$$
\begin{aligned}
e_n = 1 + 1 &+ \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \\
&+ \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
e_{n+1} = 1 + 1 &+ \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \\
&+ \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) \\
&+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right).
\end{aligned}
$$

Note that the expression for e_n contains $n + 1$ terms, while that for e_{n+1} contains $n + 2$ terms. Moreover, each term appearing in e_n is less than or equal to the corresponding term in e_{n+1}, and e_{n+1} has one more positive term. Therefore we have $2 \le e_1 < e_2 < \cdots < e_n < e_{n+1} < \cdots$, so that the terms of E are increasing.

To show that the terms of E are bounded above, we note that if $p = 1, 2, \ldots, n$, then $(1 - p/n) < 1$. Moreover $2^{p-1} \le p!$ [see 1.2.4(e)] so that $1/p! \le 1/2^{p-1}$. Therefore, if $n > 1$, then we have

$$2 < e_n < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Since it can be verified that [see 1.2.4(f)]

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 - \frac{1}{2^{n-1}} < 1,$$

we deduce that $2 < e_n < 3$ for all $n \in \mathbb{N}$. The Monotone Convergence Theorem implies that the sequence E converges to a real number that is between 2 and 3. We define the number e to be the limit of this sequence.

By refining our estimates we can find closer rational approximations to e, but we cannot evaluate it *exactly*, since e is an irrational number. However, it is possible to calculate e to as many decimal places as desired. The reader should use a calculator (or a computer) to evaluate e_n for "large" values of n. □

Leonhard Euler

Leonhard Euler (1707–1783) was born near Basel, Switzerland. His clergyman father hoped his son would follow him into the ministry, but when Euler entered the University of Basel at age 14, he studied medicine, physics, astronomy, and mathematics as well as theology. His mathematical talent was noticed by Johann Bernoulli, who became his mentor. In 1727, Euler traveled to Russia to join Bernoulli's son, Daniel, at the new St. Petersburg Academy. There he met and married Katharina Gsell, the daughter of a Swiss artist. During their 40-year marriage, they had 13 children, but only five survived childhood.

In 1741, Euler accepted an offer from Frederick the Great to join the Berlin Academy, where he stayed for 25 years. During this period he wrote landmark books on a relatively new subject called calculus and a steady stream of papers on mathematics and science. In response to a request for instruction in science from the Princess of Anhalt-Dessau, he wrote her nearly 200 letters on science that later became famous in a book titled *Letters to a German Princess*. When Euler lost vision in one eye, Frederick thereafter referred to him as his mathematical "cyclops."

In 1766, he happily returned to Russia at the invitation of Catherine the Great. His eyesight continued to deteriorate and in 1771 he became totally blind following an eye operation. Incredibly, his blindness made little impact on his mathematics output, for he wrote several books and over 400 papers while blind. He remained active until the day of his death.

Euler's productivity was remarkable. He wrote textbooks on physics, algebra, calculus, real and complex analysis, and differential geometry. He also wrote hundreds of papers, many winning prizes. A current edition of his collected works consists of 74 volumes.

Exercises for Section 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.
2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. show that (x_n) is bounded and monotone. Find the limit.
3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.
4. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.
5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. [*Hint:* One upper bound is $1 + 2\sqrt{p}$.]
6. Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that (z_n) converges and find the limit.
7. Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine whether (x_n) converges or diverges.
8. Let (a_n) be an increasing sequence, (b_n) be a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.
9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.
10. Establish the convergence or the divergence of the sequence (y_n), where
$$y_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad \text{for} \quad n \in \mathbb{N}.$$
11. Let $x_n := 1/1^2 + 1/2^2 + \cdots + 1/n^2$ for each $n \in \mathbb{N}$. Prove that (x_n) is increasing and bounded, and hence converges. [*Hint:* Note that if $k \geq 2$, then $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$.]
12. Establish the convergence and find the limits of the following sequences.
    (a) $\left((1 + 1/n)^{n+1}\right)$, (b) $\left((1 + 1/n)^{2n}\right)$,
    (c) $\left(\left(1 + \dfrac{1}{n+1}\right)^n\right)$, (d) $\left((1 - 1/n)^n\right)$.
13. Use the method in Example 3.3.5 to calculate $\sqrt{2}$, correct to within 4 decimals.
14. Use the method in Example 3.3.5 to calculate $\sqrt{5}$, correct to within 5 decimals.
15. Calculate the number e_n in Example 3.3.6 for $n = 2, 4, 8, 16$.
16. Use a calculator to compute e_n for $n = 50$, $n = 100$, and $n = 1000$.

Section 3.4 Subsequences and the Bolzano-Weierstrass Theorem

In this section we will introduce the notion of a subsequence of a sequence of real numbers. Informally, a subsequence of a sequence is a selection of terms from the given sequence such that the selected terms form a new sequence. Usually the selection is made for a definite purpose. For example, subsequences are often useful in establishing the convergence or the divergence of the sequence. We will also prove the important existence theorem known as the Bolzano-Weierstrass Theorem, which will be used to establish a number of significant results.

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

For example, if $X := \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right)$, then the selection of even indexed terms produces the subsequence

$$X' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2k}, \ldots\right),$$

where $n_1 = 2,\ n_2 = 4, \ldots, n_k = 2k, \ldots$. Other subsequences of $X = (1/n)$ are the following:

$$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k-1}, \ldots\right), \quad \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \ldots, \frac{1}{(2k)!}, \ldots\right).$$

The following sequences are *not* subsequences of $X = (1/n)$:

$$\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots\right), \quad \left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right).$$

A tail of a sequence (see 3.1.8) is a special type of subsequence. In fact, the m-tail corresponds to the sequence of indices

$$n_1 = m+1, n_2 = m+2, \ldots, n_k = m+k, \ldots.$$

But, clearly, not every subsequence of a given sequence need be a tail of the sequence.

Subsequences of convergent sequences also converge to the same limit, as we now show.

3.4.2 Theorem *If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Proof. Let $\varepsilon > 0$ be given and let $K(\varepsilon)$ be such that if $n \geq K(\varepsilon)$, then $|x_n - x| < \varepsilon$. Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easily proved (by Induction) that $n_k \geq k$. Hence, if $k \geq K(\varepsilon)$, we also have $n_k \geq k \geq K(\varepsilon)$ so that $|x_{n_k} - x| < \varepsilon$. Therefore the subsequence (x_{n_k}) also converges to x. Q.E.D.

3.4.3 Examples **(a)** $\lim(b^n) = 0$ if $0 < b < 1$.

We have already seen, in Example 3.1.11(b), that if $0 < b < 1$ and if $x_n := b^n$, then it follows from Bernoulli's Inequality that $\lim(x_n) = 0$. Alternatively, we see that since

$0 < b < 1$, then $x_{n+1} = b^{n+1} < b^n = x_n$ so that the sequence (x_n) is decreasing. It is also clear that $0 \le x_n \le 1$, so it follows from the Monotone Convergence Theorem 3.3.2 that the sequence is convergent. Let $x := \lim x_n$. Since (x_{2n}) is a subsequence of (x_n) it follows from Theorem 3.4.2 that $x = \lim(x_{2n})$. Moreover, it follows from the relation $x_{2n} = b^{2n} = (b^n)^2 = x_n^2$ and Theorem 3.2.3 that

$$x = \lim(x_{2n}) = \big(\lim(x_n)\big)^2 = x^2.$$

Therefore we must have either $x = 0$ or $x = 1$. Since the sequence (x_n) is decreasing and bounded above by $b < 1$, we deduce that $x = 0$.

(b) $\lim(c^{1/n}) = 1$ for $c > 1$.

This limit has been obtained in Example 3.1.11(c) for $c > 0$, using a rather ingenious argument. We give here an alternative approach for the case $c > 1$. Note that if $z_n := c^{1/n}$, then $z_n > 1$ and $z_{n+1} < z_n$ for all $n \in \mathbb{N}$. (Why?) Thus by the Monotone Convergence Theorem, the limit $z := \lim(z_n)$ exists. By Theorem 3.4.2, it follows that $z = \lim(z_{2n})$. In addition, it follows from the relation

$$z_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = z_n^{1/2}$$

and Theorem 3.2.10 that

$$z = \lim(z_{2n}) = \big(\lim(z_n)\big)^{1/2} = z^{1/2}.$$

Therefore we have $z^2 = z$ whence it follows that either $z = 0$ or $z = 1$. Since $z_n > 1$ for all $n \in \mathbb{N}$, we deduce that $z = 1$.

We leave it as an exercise to the reader to consider the case $0 < c < 1$. □

The following result is based on a careful negation of the definition of $\lim(x_n) = x$. It leads to a convenient way to establish the divergence of a sequence.

3.4.4 Theorem *Let $X = (x_n)$ be a sequence of real numbers, Then the following are equivalent:*

(i) *The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.*

(ii) *There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \ge k$ and $|x_{n_k} - x| \ge \varepsilon_0$.*

(iii) *There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \ge \varepsilon_0$ for all $k \in \mathbb{N}$.*

Proof. (i) $\Rightarrow$ (ii) If (x_n) does not converge to x, then for some $\varepsilon_0 > 0$ it is impossible to find a natural number k such that for all $n \ge k$ the terms x_n satisfy $|x_n - x| < \varepsilon_0$. That is, for each $k \in \mathbb{N}$ it is *not true* that for *all* $n \ge k$ the inequality $|x_n - x| < \varepsilon_0$ holds. In other words, for each $k \in \mathbb{N}$ there exists a natural number $n_k \ge k$ such that $|x_{n_k} - x| \ge \varepsilon_0$.

(ii) $\Rightarrow$ (iii) Let ε_0 be as in (ii) and let $n_1 \in \mathbb{N}$ be such that $n_1 \ge 1$ and $|x_{n_1} - x| \ge \varepsilon_0$. Now let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ and $|x_{n_2} - x| \ge \varepsilon_0$; let $n_3 \in \mathbb{N}$ be such that $n_3 > n_2$ and $|x_{n_3} - x| \ge \varepsilon_0$. Continue in this way to obtain a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \ge \varepsilon_0$ for all $k \in \mathbb{N}$.

(iii) $\Rightarrow$ (i) Suppose $X = (x_n)$ has a subsequence $X' = (x_{n_k})$ satisfying the condition in (iii). Then X cannot converge to x; for if it did, then, by Theorem 3.4.2, the subsequence X' would also converge to x. But this is impossible, since none of the terms of X' belongs to the ε_0-neighborhood of x. Q.E.D.

Since all subsequences of a convergent sequence must converge to the same limit, we have part (i) in the following result. Part (ii) follows from the fact that a convergent sequence is bounded.

3.4.5 Divergence Criteria *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) *X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.*
(ii) *X is unbounded.*

3.4.6 Examples **(a)** The sequence $X := ((-1)^n)$ is divergent.

The subsequence $X' := ((-1)^{2n}) = (1, 1, \ldots)$ converges to 1, and the subsequence $X'' := ((-1)^{2n-1}) = (-1, -1, \ldots)$ converges to -1. Therefore, we conclude from Theorem 3.4.5(i) that X is divergent.

(b) The sequence $\left(1, \frac{1}{2}, 3, \frac{1}{4}, \ldots\right)$ is divergent.

This is the sequence $Y = (y_n)$, where $y_n = n$ if n is odd, and $y_n = 1/n$ if n is even. It can easily be seen that Y is not bounded. Hence, by Theorem 3.4.5(ii), the sequence is divergent.

(c) The sequence $S := (\sin n)$ is divergent.

This sequence is not so easy to handle. In discussing it we must, of course, make use of elementary properties of the sine function. We recall that $\sin(\pi/6) = \frac{1}{2} = \sin(5\pi/6)$ and that $\sin x > \frac{1}{2}$ for x in the interval $I_1 := (\pi/6, 5\pi/6)$. Since the length of I_1 is $5\pi/6 - \pi/6 = 2\pi/3 > 2$, there are at least two natural numbers lying inside I_1; we let n_1 be the first such number. Similarly, for each $k \in \mathbb{N}$, $\sin x > \frac{1}{2}$ for x in the interval

$$I_k := \big(\pi/6 + 2\pi(k-1), 5\pi/6 + 2\pi(k-1)\big).$$

Since the length of I_k is greater than 2, there are at least two natural numbers lying inside I_k; we let n_k be the first one. The subsequence $S' := (\sin n_k)$ of S obtained in this way has the property that all of its values lie in the interval $\left[\frac{1}{2}, 1\right]$.

Similarly, if $k \in \mathbb{N}$ and J_k is the interval

$$J_k := \big(7\pi/6 + 2\pi(k-1),\ 11\pi/6 + 2\pi(k-1)\big),$$

then it is seen that $\sin x < -\frac{1}{2}$ for all $x \in J_k$ and the length of J_k is greater than 2. Let m_k be the first natural number lying in J_k. Then the subsequence $S'' := (\sin m_k)$ of S has the property that all of its values lie in the interval $\left[-1, -\frac{1}{2}\right]$.

Given any real number c, it is readily seen that at least one of the subsequences S' and S'' lies entirely outside of the $\frac{1}{2}$-neighborhood of c. Therefore c cannot be a limit of S. Since $c \in \mathbb{R}$ is arbitrary, we deduce that S is divergent. □

The Existence of Monotone Subsequences

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.*

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_k}, \ldots$. Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots.$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$. Let $s_1 := m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (x_{s_k}) of X. Q.E.D.

It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing.

The Bolzano-Weierstrass Theorem

We will now use the Monotone Subsequence Theorem to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence. Because of the importance of this theorem we will also give a second proof of it based on the Nested Interval Property.

3.4.8 The Bolzano-Weierstrass Theorem *A bounded sequence of real numbers has a convergent subsequence.*

First Proof. It follows from the Monotone Subsequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{n_k})$ that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 3.3.2 that the subsequence is convergent. Q.E.D.

Second Proof. Since the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.

We now bisect I_1 into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:

$$A_1 := \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}, \quad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}.$$

If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_1. If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I_1''$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I_2' and I_2'', and divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n, \in I_2'\}, \quad B_2 := \{n \in \mathbb{N} : n > n_2, x_n \in I_2''\}$$

If A_2 is infinite, we take $I_3 := I_2'$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 := I_2''$ and let n_3 be the smallest natural number in B_2.

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \leq (b-a)/2^{k-1},$$

whence it follows that the subsequence (x_{n_k}) of X converges to ξ. Q.E.D.

Theorem 3.4.8 is sometimes called the Bolzano-Weierstrass Theorem *for sequences*, because there is another version of it that deals with bounded sets in $\mathbb{R}$ (see Exercise 11.2.6).

It is readily seen that a bounded sequence can have various subsequences that converge to different limits or even diverge. For example, the sequence $((-1)^n)$ has subsequences that converge to -1, other subsequences that converge to $+1$, and it has subsequences that diverge.

Let X be a sequence of real numbers and let X' be a subsequence of X. Then X' is a sequence in its own right, and so it has subsequences. We note that if X'' is a subsequence of X', then it is also a subsequence of X.

3.4.9 Theorem *Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.*

Proof. Suppose $M > 0$ is a bound for the sequence X so that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If X does not converge to x, then Theorem 3.4.4 implies that there exist $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

$$(1) \qquad |x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Since X' is a subsequence of X, the number M is also a bound for X'. Hence the Bolzano-Weierstrass Theorem implies that X' has a convergent subsequence X''. Since X'' is also a subsequence of X, it converges to x by hypothesis. Thus, its terms ultimately belong to the ε_0-neighborhood of x, contradicting (1). Q.E.D.

Limit Superior and Limit Inferior

A bounded sequence of real numbers (x_n) may or may not converge, but we know from the Bolzano-Weierstrass Theorem 3.4.8 that there will be a convergent subsequence and possibly many convergent subsequences. A real number that is the limit of a subsequence of (x_n) is called a *subsequential limit of* (x_n). We let S denote the set of all subsequential limits of the bounded sequence (x_n). The set S is bounded, because the sequence is bounded.

For example, if (x_n) is defined by $x_n := (-1)^n + 2/n$, then the subsequence (x_{2n}) converges to 1, and the subsequence (x_{2n-1}) converges to -1. It is easily seen that the set of subsequential limits is $S = \{-1,\ 1\}$. Observe that the largest member of the sequence itself is $x_2 = 2$, which provides no information concerning the limiting behavior of the sequence.

An extreme example is given by the set of all rational numbers in the interval [0, 1]. The set is denumerable (see Section 1.3) and therefore it can be written as a sequence (r_n). Then it follows from the Density Theorem 2.4.8 that every number in [0, 1] is a subsequential limit of (r_n). Thus we have $S = [0, 1]$.

A bounded sequence (x_n) that diverges will display some form of oscillation. The activity is contained in decreasing intervals as follows. The interval $[t_1,\ u_1]$, where $t_1 :=$

$\inf\{x_n : n \in \mathbb{N}\}$ and $u_1 := \sup\{x_n : n \in \mathbb{N}\}$, contains the entire sequence. If for each $m = 1, 2, \ldots$, we define $t_m := \inf\{x_n : n \geq m\}$ and $u_m := \sup\{x_n : n \geq m\}$, the sequences (t_m) and (u_m) are monotone and we obtain a nested sequence of intervals $[t_m, u_m]$ where the mth interval contains the m-tail of the sequence.

The preceding discussion suggests different ways of describing limiting behavior of a bounded sequence. Another is to observe that if a real number v has the property that $x_n > v$ for *at most* a finite number of values of n, then no subsequence of (x_n) can converge to a limit larger than v because that would require infinitely many terms of the sequence be larger than v. In other words, if v has the property that there exists N_v such that $x_n \leq v$ for all $n \geq N_v$, then no number larger than v can be a subsequential limit of (x_n).

This observation leads to the following definition of limit superior. The accompanying definition of limit inferior is similar.

3.4.10 Definition Let $X = (x_n)$ be a bounded sequence of real numbers.

(a) The **limit superior** of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. It is denoted by

$$\limsup(x_n) \quad \text{or} \quad \limsup X \quad \text{or} \quad \overline{\lim}(x_n).$$

(b) The **limit inferior** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ such that $x_m < w$ for at most a finite number of $m \in \mathbb{N}$. It is denoted by

$$\liminf(x_n) \quad \text{or} \quad \liminf X \quad \text{or} \quad \underline{\lim}(x_n).$$

For the concept of limit superior, we now show that the different approaches are equivalent.

3.4.11 Theorem *If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.*

(a) $x^* = \limsup(x_n)$.

(b) *If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number* of $n \in \mathbb{N}$ *such that $x^* - \varepsilon < x_n$.*

(c) *If $u_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.*

(d) *If S is the set of subsequential limits of (x_n), then $x^* = \sup S$.*

Proof. (a) implies (b). If $\varepsilon > 0$, then the fact that x^* is an infimum implies that there exists a v in V such that $x^* \leq v < x^* + \varepsilon$. Therefore x^* also belongs to V, so there can be at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$. On the other hand, $x^* - \varepsilon$ is not in V so there are an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

(b) implies (c). If (b) holds, given $\varepsilon > 0$, then for all sufficiently large m we have $u_m < x + \varepsilon$. Therefore, $\inf\{u_m : m \in \mathbb{N}\} \leq x^* + \varepsilon$. Also, since there are an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$, then $x^* - \varepsilon < u_m$ for all $m \in \mathbb{N}$ and hence $x^* - \varepsilon \leq \inf\{u_m : m \in \mathbb{N}\}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $x^* = \inf\{u_m : m \in \mathbb{N}\}$. Moreover, since the sequence (u_m) is monotone decreasing, we have $\inf(u_m) = \lim(u_m)$.

(c) implies (d). Suppose that $X' = (x_{n_k})$ is a convergent subsequence of $X = (x_n)$. Since $n_k \geq k$, we have $x_{n_k} \leq u_k$ and hence $\lim X' \leq \lim(u_k) = x^*$. Conversely, there exists n_1 such that $u_1 - 1 \leq x_{n_1} \leq u_1$. Inductively choose $n_{k+1} > n_k$ such that

$$u_k - \frac{1}{k+1} < x_{n_{k+1}} \leq u_k.$$

Since $\lim\,(u_k) = x^*$, it follows that $x^* = \lim(x_{n_k})$, and hence $x^* \in S$.

(d) implies (a). Let $w = \sup S$. If $\varepsilon > 0$ is given, then there are at most finitely many n with $w + \varepsilon < x_n$. Therefore $w + \varepsilon$ belongs to V and $\lim\sup(x_n) \le w + \varepsilon$. On the other hand, there exists a subsequence of (x_n) converging to some number larger than $w - \varepsilon$, so that $w - \varepsilon$ is not in V, and hence $w - \varepsilon \le \lim\sup(x_n)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $w = \lim\sup(x_n)$. Q.E.D.

As an instructive exercise, the reader should formulate the corresponding theorem for the limit inferior of a bounded sequence of real numbers.

3.4.12 Theorem A bounded sequence (x_n) is convergent if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.

We leave the proof as an exercise. Other basic properties can also be found in the exercises.

Exercises for Section 3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.
2. Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.
3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.
4. Show that the following sequences are divergent.
   (a) $\left(1 - (-1)^n + 1/n\right)$, (b) $(\sin n\pi/4)$.
5. Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1, z_2 := y_1, \ldots, z_{2n-1} := x_n, z_{2n} := y_n, \ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.
6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.
   (a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \ge 3$. (See Example 3.3.6.) Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.
   (b) Use the fact that the subsequence (x_{2n}) also converges to x to conclude that $x = 1$.
7. Establish the convergence and find the limits of the following sequences:
   (a) $\left((1 + 1/n^2)^{n^2}\right)$, (b) $\left((1 + 1/2n)^n\right)$,
   (c) $\left((1 + 1/n^2)^{2n^2}\right)$, (d) $\left((1 + 2/n)^n\right)$.
8. Determine the limits of the following.
   (a) $\left((3n)^{1/2n}\right)$, (b) $\left((1 + 1/2n)^{3n}\right)$.
9. Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.
10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \ge n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.
11. Suppose that $x_n \ge 0$ for all $n \in \mathbb{N}$ and that $\lim\left((-1)^n x_n\right)$ exists. Show that (x_n) converges.
12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.
13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 := [-1, 1]$.

14. Let (x_n) be a bounded sequence and let $s := \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.
15. Let (I_n) be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to give a proof of the Nested Intervals Property 2.5.2.
16. Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.
17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by

$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \ldots).$$

Determine the values of $\lim\sup(x_n)$ and $\lim\inf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.
18. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.
19. Show that if (x_n) and (y_n) are bounded sequences, then

$$\lim\sup(x_n + y_n) \leq \lim\sup(x_n) + \lim\sup(y_n).$$

Give an example in which the two sides are not equal.

Section 3.5 The Cauchy Criterion

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

However, we will first highlight the definition of Cauchy sequence in the following examples.

3.5.2 Examples **(a)** The sequence $(1/n)$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \geq H$, we have $1/n \leq 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if $m, n \geq H$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.

The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \geq \varepsilon_0$. For the terms $x_n := 1 + (-1)^n$, we observe that if n is even, then $x_n = 2$ and $x_{n+1} = 0$. If we take $\varepsilon_0 = 2$, then for any H we can choose an even number $n > H$ and let $m := n + 1$ to get

$$|x_n - x_{n+1}| = 2 = \varepsilon_0.$$

We conclude that (x_n) is not a Cauchy sequence. □

Remark We emphasize that to prove a sequence (x_n) is a Cauchy sequence, we may not assume a relationship between m and n, since the required inequality $|x_n - x_m| < \varepsilon$ must hold for *all* $n, m \geq H(\varepsilon)$. But to prove a sequence is *not* a Cauchy sequence, we may specify a relation between n and m as long as arbitrarily large values of n and m can be chosen so that $|x_n - x_m| \geq \varepsilon_0$.

Our goal is to show that the Cauchy sequences are precisely the convergent sequences. We first prove that a convergent sequence is a Cauchy sequence.

3.5.3 Lemma *If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.*

Proof. If $x := \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$ then $|x_n - x| < \varepsilon/2$. Thus, if $H(\varepsilon) := K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a Cauchy sequence. Q.E.D.

In order to establish that a Cauchy sequence is convergent, we will need the following result. (See Theorem 3.2.2.)

3.5.4 Lemma *A Cauchy sequence of real numbers is bounded.*

Proof. Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup\{|x_1|, |x_2|, \ldots, |x_{H-1}|, |x_H| + 1\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Q.E.D.

We now present the important Cauchy Convergence Criterion.

3.5.5 Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_k})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\varepsilon/2)$ such that if n, $m \geq H(\varepsilon/2)$ then

(1) $$|x_n - x_m| < \varepsilon/2.$$

Since the subsequence $X' = (x_{n_k})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_K| < \varepsilon/2 \qquad \text{for} \quad n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Sincee $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent. Q.E.D.

We will now give some examples of applications of the Cauchy Criterion.

3.5.6 Examples **(a)** Let $X = (x_n)$ be defined by

$$x_1 := 1, \quad x_2 := 2, \quad \text{and} \quad x_n := \frac{1}{2}(x_{n-2} + x_{n-1}) \quad \text{for} \quad n > 2.$$

It can be shown by Induction that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. (Do so.) Some calculation shows that the sequence X is not monotone. However, since the terms are formed by averaging, it is readily seen that

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}} \quad \text{for} \quad n \in \mathbb{N}.$$

(Prove this by Induction.) Thus, if $m > n$, we may employ the Triangle Inequality to obtain

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \\ &= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^{n-2}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $1/2^n < \varepsilon/4$ and if $m \geq n$, then it follows that $|x_n - x_m| < \varepsilon$. Therefore, X is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion 3.5.5 we infer that the sequence X converges to a number x.

To evaluate the limit x, we might first "pass to the limit" in the rule of definition $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ to conclude that x must satisfy the relation $x = \frac{1}{2}(x + x)$, which is true, but not informative. Hence we must try something else.

Since X converges to x, so does the subsequence X' with odd indices. By Induction, the reader can establish that [see 1.2.4(f)]

$$\begin{aligned} x_{2n+1} &= 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} \\ &= 1 + \frac{2}{3}\left(1 - \frac{1}{4^n}\right). \end{aligned}$$

It follows from this (how?) that $x = \lim X = \lim X' = 1 + \frac{2}{3} = \frac{5}{3}$.

(b) Let $Y = (y_n)$ be the sequence of real numbers given by

$$y_1 := \frac{1}{1!}, \quad y_2 := \frac{1}{1!} - \frac{1}{2!}, \ldots, \quad y_n := \frac{1}{1!} - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}, \cdots$$

Clearly, Y is not a monotone sequence. However, if $m > n$, then

$$y_m - y_n = \frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}.$$

Since $2^{r-1} \le r!$ [see 1.2.4(e)], it follows that if $m > n$, then (why?)

$$\begin{aligned} |y_m - y_n| &\le \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\ &\le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}. \end{aligned}$$

Therefore, it follows that (y_n) is a Cauchy sequence. Hence it converges to a limit y. At the present moment we cannot evaluate y directly; however, passing to the limit (with respect to m) in the above inequality, we obtain

$$|y_n - y| \le 1/2^{n-1}.$$

Hence we can calculate y to any desired accuracy by calculating the terms y_n for sufficiently large n. The reader should do this and show that y is approximately equal to 0.632 120 559. (The exact value of y is $1 - 1/e$.)

(c) The sequence $\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}\right)$ diverges.

Let $H := (h_n)$ be the sequence defined by

$$h_n := \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \qquad \text{for} \quad n \in \mathbb{N},$$

which was considered in 3.3.3(b). If $m > n$, then

$$h_m - h_n = \frac{1}{n+1} + \cdots + \frac{1}{m}.$$

Since each of these $m - n$ terms exceeds $1/m$, then $h_m - h_n > (m-n)/m = 1 - n/m$. In particular, if $m = 2n$ we have $h_{2n} - h_n > \frac{1}{2}$. This shows that H is not a Cauchy sequence (why?); therefore H is *not* a convergent sequence. (In terms that will be introduced in Section 3.7, we have just proved that the "harmonic series" $\sum_{n=1}^{\infty} 1/n$ is divergent.) □

3.5.7 Definition We say that a sequence $X = (x_n)$ of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. The number C is called the **constant** of the contractive sequence.

3.5.8 Theorem *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Proof. If we successively apply the defining condition for a contractive sequence, we can work our way back to the beginning of the sequence as follows:

$$\begin{aligned}|x_{n+2} - x_{n+1}| &\le C|x_{n+1} - x_n| \le C^2|x_n - x_{n-1}| \\ &\le C^3|x_{n-1} - x_{n-2}| \le \cdots \le C^n|x_2 - x_1|.\end{aligned}$$

For $m > n$, we estimate $|x_m - x_n|$ by first applying the Triangle Inequality and then using the formula for the sum of a geometric progression (see 1.2.4(f)). This gives

$$\begin{aligned}|x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\le \left(C^{m-2} + C^{m-3} + \cdots + C^{n-1}\right)|x_2 - x_1| \\ &= C^{n-1}\left(\frac{1 - C^{m-n}}{1 - C}\right)|x_2 - x_1| \\ &\le C^{n-1}\left(\frac{1}{1 - C}\right)|x_2 - x_1|.\end{aligned}$$

Since $0 < C < 1$, we know $\lim(C^n) = 0$ [see 3.1.11(b)]. Therefore, we infer that (x_n) is a Cauchy sequence. It now follows from the Cauchy Convergence Criterion 3.5.5 that (x_n) is a convergent sequence. Q.E.D.

3.5.9 Example We consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$ where $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$. (See Example 3.1.2(d).) The first few terms are $x_1 = 1$, $x_2 = 1/2$, $x_3 = 2/3$, $x_4 = 3/5$, $x_5 = 5/8$, and so on. It is shown that the sequence (x_n) is given inductively by the equation $x_{n+1} = 1/(1 + x_n)$ as follows:

$$x_{n+1} = \frac{f_{n+1}}{f_{n+2}} = \frac{f_{n+1}}{f_{n+1} + f_n} = \frac{1}{1 + \dfrac{f_n}{f_{n+1}}} = \frac{1}{1 + x_n}.$$

An induction argument establishes $1/2 \le x_n \le 1$ for all n, so that adding 1 and taking reciprocals gives us the inequality $1/2 \le 1/(1 + x_n) \le 2/3$ for all n. It then follows that

$$|x_{n+1} - x_n| = \frac{|x_n - x_{n-1}|}{(1 + x_n)(1 + x_{n-1})} \le \frac{2}{3} \cdot \frac{2}{3}|x_n - x_{n-1}| = \frac{4}{9}|x_n - x_{n-1}|.$$

Hence, the sequence (x_n) is contractive and therefore converges by Theorem 3.5.8. Passing to the limit $x = \lim(x_n)$, we obtain the equation $x = 1/(1 + x)$, so that x satisfies the equation $x^2 + x - 1 = 0$. The quadratic formula gives us the positive solution $x = (-1 + \sqrt{5})/2 = 0.618034\ldots$.

The reciprocal $1/x = (1 + \sqrt{5})/2 = 1.618034\ldots$ is often denoted by the Greek letter φ and referred to as the Golden Ratio in the history of geometry. In the artistic theory of the ancient Greek philosophers, a rectangle having φ as the ratio of the longer side to the shorter side is the rectangle most pleasing to the eye. The number also has many interesting mathematical properties. (A historical discussion of the Golden Ratio can be found on Wikipedia.) □

In the process of calculating the limit of a contractive sequence, it is often very important to have an estimate of the error at the nth stage. In the next result we give two