

**BOOTSTRAPPING THE ILLIQUIDITY
MULTIPLE YIELD CURVES CONSTRUCTION
FOR MARKET COHERENT FORWARD RATES ESTIMATION**

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ABSTRACT. The large basis spreads observed on the interest rate market since the liquidity crisis of summer 2007 imply that different yield curves are required for market coherent estimation of forward rates with different tenors (e.g. Euribor 3 months, Euribor 6 months, etc.).

In this paper we review the methodology for bootstrapping multiple interest rate yield curves, each homogeneous in the underlying rate tenor, from non-homogeneous plain vanilla instruments quoted on the market, such as Deposits, Forward Rate Agreements, Futures, Swaps, and Basis Swaps. The approach includes turn of year effects and is robust to deliver smooth yield curves and to ensure non-negative rates also in highly stressed market situations, characterized by crazy roller coaster shapes of the market quotations.

The concrete EUR market case is analyzed in detail, using the open source QuantLib implementation of the proposed algorithms.

1. INTRODUCTION

Pricing complex interest rate derivatives requires modeling the future dynamics of the yield curve term structure. Most of the literature assumes the existence of the *current* yield curve as given, and its construction is often neglected, or even obscured, as it is considered more an art than a science. Actually any yield curve term structure modeling approach will fail to produce good/reasonable prices if the current term structure is not correct.

Financial institutions, software houses and practitioners have developed their own proprietary methodologies in order to extract the yield curve term structure from quoted prices of a finite number of liquid market instruments.

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“Best-fit” algorithms assume a smooth functional form for the term structure and calibrate its parameters such that to minimize the repricing error of the chosen set of calibration instruments. For instance, the European Central Bank publishes yield curves on the basis of the Soderlind and Svensson model [1], which is an extension of the Nelson-Siegel model (see e.g. refs. [2], [3] and [4]). Such approach is popular due to the smoothness of the curve, calibration easiness, intuitive financial interpretation of functional form parameters (level, slope, curvature) and correspondence with principal component analysis. On the other side, the fit quality is typically not good enough for trading purposes in liquid markets.

In practice “exact-fit” algorithms are often preferred: they fix the yield curve on a time grid of N points (pillars) in order to *exactly* reprice N pre-selected market instruments. The implementation of such algorithms is often incremental, extending the yield curve step-by-step with the increasing maturity of the ordered instruments, in a so called “bootstrap” approach. Intermediate yield curve values are obtained by interpolation on the bootstrapping grid. Here different interpolation algorithms are available but little attention has been devoted in the literature to the fact that interpolation is often already used during bootstrapping, not just after that, and that the interaction between bootstrapping and interpolation can be subtle if not nasty (see e.g. [5], [6]).

Whilst naive algorithms may fail to deal with market subtleties such as date conventions, the intra-day fixing of the first floating payment of a Swap, the turn-of-year effect, the Futures convexity adjustment, etc., even very sophisticated algorithms used in a naive way may fail to estimate correct forward Euribor rates in difficult market conditions, as those observed since the summer of 2007 in occasion of the so-called *subprime credit crunch crisis*. Namely using just one single curve is not enough to account for forward rates of different tenor, such as 1, 3, 6, 12 months, because of the large Basis Swap spreads presently quoted on the market.

The plan of the paper is as follows: In section 2 we fix the notation and nomenclature. In section 3 we describe in great detail the new post-credit crunch multi-curve approach; in particular in its 9 subsections we discuss the general features of the bootstrapping procedure, we review the (EUR) market instruments available for the bootstrap, and we deal with the crucial role played of interpolation. In section 3, that constitutes the central contribution of this work, we describe in great detail the new post-credit crunch multi-curve approach; in particular in its nine subsections we discuss the general features of the bootstrapping procedure, we review in detail the (EUR) market instruments available for yield curves construction, and we deal with some issues crucial for bootstrapping, in particular the fundamental role played by the interpolation scheme adopted (sec. 3.7) and the incorporation of the turn-of-year effect (3.8). Finally, in section 4 we show an example of numerical results for the Euribor1M, 3M, 6M and 12M

forward curves bootstrapping using the open source implementation released within the QuantLib framework. The conclusions are collected in section 5.

2. FIXING NOTATION AND NOMENCLATURE

In this section we fix notation and nomenclature for the multi-curve environment (see also refs. [7], [8]). We start by postulating the existence of N distinct yield curves \mathcal{C}_x in the form of a continuous term structure of discount factors,

$$\mathcal{C}_x^P = \{T \longrightarrow P_x(t_0, T), T \geq t_0\}, \quad (1)$$

where the superscript P stands for discount curve, t_0 is the reference date (e.g. today, or spot date), and $P_x(t, T)$ denotes the price at time $t \geq t_0$ of the \mathcal{C}_x^P -zero coupon bond for maturity T , such that $P_x(T, T) = 1$. The index x will take the values corresponding to the underlying rate tenors, e.g. $x = \{1M, 3M, 6M, 12M\}$.

Time intervals between couples of dates $[T_1, T_2]$ are measured as year fractions with a given day count convention $dc_x, \tau(T_1, T_2; dc_x)$.

We also define continuously compounded zero coupon rates $z_x(t_0, T)$ and simply compounded instantaneous forward rates¹ $f_x(t_0, T)$ such that

$$P_x(t_0, T) = \exp[-z_x(t_0, T) \tau_C(t_0, T)] = \exp\left[-\int_{t_0}^T f_x(t_0, u) du\right], \quad (2)$$

or, using the equivalent log notation,

$$\log P_x(t_0, T) = -z_x(t_0, T) \tau_C(t_0, T) = -\int_{t_0}^T f_x(t_0, u) du, \quad (3)$$

where

$$\tau_C(T_1, T_2) := \tau(T_1, T_2; dc_C) \quad (4)$$

and dc_C is the day count convention for the zero rate. The common choice is $dc_C = \text{actual}/365(\text{fixed})$ [9], such that

$$\tau_C(T_1, T_2) := \tau[T_1, T_2; \text{actual}/365(\text{fixed})] = \frac{T_2 - T_1}{365}. \quad (5)$$

This definition ensures monotonicity with increasing time intervals (otherwise one obtains spurious null forward rates), and additivity, such that

$$\tau_C(T_1, T_2) + \tau_C(T_2, T_3) = \tau_C(T_1, T_3). \quad (6)$$

From the relationships above it is immediate to observe that:

- $z_x(t_0, T)$ is the average of $f_x(t_0, u)$ over $[t_0, T]$;
- if rates are non-negative², $(\log) P(t_0, T)$ is a monotone non-increasing function of T such that $0 < P(t_0, T) \leq 1 \quad \forall T > t_0$.

¹par rates could be used too; we do not use them here as they are not frequently used and would not provide additional benefit anyway.

²this is generally true in all western markets and in the EUR market we consider in this paper

- the instantaneous forward curve \mathcal{C}_x^f is the most severe indicator of yield curve smoothness, since anything else is obtained through its integration, therefore being smoother by construction. We will discuss this point in section 3.7.

Eq. (2) or (3) allows to define other two rate curves associated to \mathcal{C}_x^P , precisely a zero curve and an instantaneous forward rate curve,

$$\mathcal{C}_x^z = \{T \longrightarrow z_x(t_0, T), T \geq t_0\}, \quad (7)$$

$$\mathcal{C}_x^f = \{T \longrightarrow f_x(t_0, T), T \geq t_0\}, \quad (8)$$

where

$$z_x(t_0, T) = -\frac{1}{\tau_C(t_0, T)} \log P_x(t_0, T), \quad (9)$$

$$\begin{aligned} f_x(t_0, T) &= -\frac{\partial}{\partial t} \log P_x(t_0, t)|_{t=T} \\ &= z_x(t_0, T) + \frac{\partial}{\partial t} z_x(t_0, t)|_{t=T} \tau_C(t_0, T), \end{aligned} \quad (10)$$

respectively. In the following we will denote with \mathcal{C}_x the generic curve and we will specify the particular typology (discount, zero or forward curve) if necessary.

The usual no arbitrage relation among discount factors holds,

$$P_x(t, T_2) = P_x(t, T_1) \times P_x(t, T_1, T_2), \quad \forall t_0 \leq t \leq T_1 < T_2, \quad (11)$$

where $P_x(t, T_1, T_2)$ denotes the forward discount factor from time T_2 to time T_1 , prevailing at any time $t \geq t_0$. The financial meaning of expression (11) is that, given a cashflow of one unit of currency at time T_2 , its corresponding value at time $t < T_2$ must be the same both if we discount in one single step from T_2 to t , using the discount factor $P_x(t, T_2)$, and if we discount in two steps, first from T_2 to T_1 , using the forward discount $P_x(t, T_1, T_2)$ and then from T_1 to t , using $P_x(t, T_1)$. Denoting with $F_x(t; T_1, T_2)$ the simple compounded annual forward rate associated to $P_x(t, T_1, T_2)$, resetting at time T_1 and covering the time interval $[T_1, T_2]$ with day count convention dc_F , we have

$$P_x(t, T_1, T_2) = \frac{P_x(t, T_2)}{P_x(t, T_1)} = \frac{1}{1 + F_x(t; T_1, T_2) \tau_F(T_1, T_2)}, \quad (12)$$

where we have defined

$$\tau_F(T_1, T_2) := \tau(T_1, T_2; dc_F). \quad (13)$$

The year fraction in eq. 13 is, for Euribor rates considered in this paper, $dc_F = \text{actual}/360$ [9] such that

$$\tau_F(T_1, T_2) := \tau[T_1, T_2; \text{actual}/360(\text{fixed})] = \frac{T_2 - T_1}{360}. \quad (14)$$

From eq. (11) we obtain the familiar no arbitrage expression

$$\begin{aligned} F_x(t; T_1, T_2) &= \frac{1}{\tau_F(T_1, T_2)} \left[\frac{1}{P_x(t, T_1, T_2)} - 1 \right] \\ &= \frac{P_x(t, T_1) - P_x(t, T_2)}{\tau_F(T_1, T_2) P_x(t, T_2)}. \end{aligned} \quad (15)$$

Regarding swap rates, given two increasing dates vectors $\mathbf{T} = \{T_0, \dots, T_n\}$, $\mathbf{S} = \{S_0, \dots, S_m\}$, $T_n = S_m > T_0 = S_0 \geq t_0$, and an interest rate Swap with a floating leg paying at times S_j , $j = 1, \dots, m$, the Euribor rate with tenor $[S_{j-1}, S_j]$ fixed at time S_{j-1} , plus a fixed leg paying a fixed rate at times T_i , $i = 1, \dots, n$, the corresponding simple compounded fair swap rate on curve \mathcal{C}_x with day count convention dc_S is given by

$$\begin{aligned} S_x(t, \mathbf{T}, \mathbf{S}) &= \frac{\sum_{j=1}^m P_x(t, S_j) \tau_F(S_{j-1}, S_j) F_x(t; S_{j-1}, S_j)}{A_x(t, \mathbf{T})} \\ &= \frac{P_x(t, T_0) - P_x(t, T_n)}{A_x(t, \mathbf{T})}, \quad t_0 \leq t \leq T_0 \end{aligned} \quad (16)$$

where

$$A_x(t, \mathbf{T}) = \sum_{i=1}^n P_x(t, T_i) \tau_S(T_{i-1}, T_i) \quad (17)$$

is the annuity on curve \mathcal{C}_x and we have defined

$$\tau_S(T_{i-1}, T_i) := \tau(T_{i-1}, T_i; dc_S). \quad (18)$$

Notice that on the r.h.s. of eq. (16) we have used the definition of forward rate from eq. (15) and the telescopic property of the summation.

3. BOOTSTRAPPING MULTIPLE YIELD CURVES

A summary of the standard bootstrapping methodology is given in common textbooks as, for instance, [10] and [11]. In the following subsections we discuss in detail the construction of multiple yield curves.

3.1. Market Instrument Selection. In the present market situation, distinct interest rate market areas, relative to different underlying rate tenors, are characterized by different internal dynamics, liquidity and credit risk premia, reflecting the different views and interests of the market players. Such more complex market mechanic generates the following features:

- similar market instruments insisting on different underlyings, for instance FRAs or Swaps on Euribor3M and Euribor6M, may display very different price levels;
- similar market instruments may display very different relative liquidities;
- even small idiosyncracies, asynchronism and inconsistencies in market quotations may result in erratic forward rates.

Instrument	Quote	Underlying	Start Date	Maturity	Settlement rule	Business Day Conv.	End of Month
OND	1.200	Euribor1D	Mon 16 Feb 2009	Tue 17 Feb 2009	Today	Following	False
TND	1.200	Euribor1D	Tue 17 Feb 2009	Wed 18 Feb 2009	Tomorrow	Following	False
SND	1.200	Euribor1D	Wed 18 Feb 2009	Thu 19 Feb 2009	Spot	Following	False
SWD	1.450	Euribor1W	Wed 18 Feb 2009	Wed 25 Feb 2009	Spot	Following	False
2WD	1.550	Euribor2W	Wed 18 Feb 2009	Wed 04 Mar 2009	Spot	Following	False
3WD	1.600	Euribor3W	Wed 18 Feb 2009	Wed 11 Mar 2009	Spot	Following	False
1MD	1.660	Euribor1M	Wed 18 Feb 2009	Wed 18 Mar 2009	Spot	Mod. Follow.	True
2MD	1.850	Euribor2M	Wed 18 Feb 2009	Mon 20 Apr 2009	Spot	Mod. Follow.	True
3MD	1.980	Euribor3M	Wed 18 Feb 2009	Mon 18 May 2009	Spot	Mod. Follow.	True
4MD	2.000	Euribor4M	Wed 18 Feb 2009	Thu 18 Jun 2009	Spot	Mod. Follow.	True
5MD	2.020	Euribor5M	Wed 18 Feb 2009	Mon 20 Jul 2009	Spot	Mod. Follow.	True
6MD	2.050	Euribor6M	Wed 18 Feb 2009	Tue 18 Aug 2009	Spot	Mod. Follow.	True
7MD	2.080	Euribor7M	Wed 18 Feb 2009	Fri 18 Sep 2009	Spot	Mod. Follow.	True
8MD	2.090	Euribor8M	Wed 18 Feb 2009	Mon 19 Oct 2009	Spot	Mod. Follow.	True
9MD	2.110	Euribor9M	Wed 18 Feb 2009	Wed 18 Nov 2009	Spot	Mod. Follow.	True
10MD	2.130	Euribor10M	Wed 18 Feb 2009	Fri 18 Dec 2009	Spot	Mod. Follow.	True
11MD	2.140	Euribor11M	Wed 18 Feb 2009	Mon 18 Jan 2010	Spot	Mod. Follow.	True
12MD	2.160	Euribor12M	Wed 18 Feb 2009	Thu 18 Feb 2010	Spot	Mod. Follow.	True

FIGURE 1. EUR Deposit strip. Source: Reuters page KLIEM, 16 Feb. 2009.

Hence, the first step for multiple yield curve construction is a very careful selection of the corresponding multiple sets of bootstrapping instruments. Different kinds of instruments can be selected for bootstrapping an yield curve term structure, and whilst they roughly cover different maturities, they overlap in significant areas. Therefore it is usually impossible to include all the available instruments, and the subset of the mostly non-overlapping contracts is selected, with preference given to more liquid ones with a tighter bid/ask spread. The mispricing level of the excluded instruments must thus be monitored as safety check (or cheap-rich analysis).

In the following subsections we examine these instruments in detail. In order to fix the data set once for all, we thoroughly refer to the EUR market quotes observed on the Reuters platform as of 16 Feb. 2009, close time (around 16.30 CET³). Obviously the discussion holds for other EUR market data sets and can be remapped to other major currencies with small changes.

3.2. Deposits. Interest rate Deposits (Depos) are Over-The-Counter (OTC) zero coupon contracts that start at reference date t_0 (today or spot), span the length corresponding to their maturity, and pay the interest accrued over the period with a given rate fixed at t_0 .

The EUR market quotes standard plain vanilla Deposits strip that start at spot date and span various periods up to 1 year. Exceptions are the first *over-night* (ON) and the second *tomorrow-next* (TN) one-day contracts, which start today and tomorrow, respectively, and span one day each, covering (without overlapping) the two business days interval between today and spot dates. The maturity date of Deposits shorter than one month obeys

³Central European Time, equal to Greenwich Mean Time (GMT) plus 1 hour

the *following* convention; for longer Deposits the convention is *modified following*. For the latter the *end-of-month* convention is also respected: if the start date is the last working day in a given month, the end date must be the last working date of the ending month too. In fig. 1 we report the EUR Depo strip quoted in Reuters page KLIEM.

Market Deposits can be selected as bootstrapping instruments for the construction of the short term structure section of the discount curves. Notice that, apart ON, TN and SN, each Depo admits its own underlying rate tenor, corresponding to its maturity. Hence each Depo should be selected, in principle, for the construction of a different curve.

If $R_x^{Depo}(t_0, T_i)$ is the quoted rate (annual, simply compounded) associated to the i -th Deposit with maturity T_i and underlying rate tenor $x = T_i - t_0$ months, the implied discount factor at time T_i is given by the following relation⁴

$$P_x(t_0, T_i) = \frac{1}{1 + R_x^{Depo}(t_0, T_i) \tau_F(t_0, T_i)}, \quad t_0 < T_i, \quad (19)$$

where τ_F is given by eq. (13). The expression (19) above can be used to bootstrap the yield curve \mathcal{C}_x at point T_i .

3.3. Forward Rate Agreements (FRAs). FRA contracts are forward starting Deposits. For instance the 3x9 FRA is a six months Deposit starting three months forward.

The EUR market quotes standard plain vanilla FRA strips with different forward start dates (i.e. the start date of the forward Depo), calculated with the same convention used for the end date of Deposits. So FRAs do concatenate exactly, e.g. the 6x9 FRA starts when the preceding 3x6 FRA ends. The underlying forward rate fixes two working days before the forward start date. In fig. 2 we report the four FRA strips on 3M, 6M, and 12M Euribor rate quoted in Reuters page ICAPSHORT2.

Market FRAs provide direct empirical evidence that a single curve cannot be used to estimate forward rates with different tenors. We can observe in fig. 2 that, for instance, the level of the market 1x4 FRA3M (spanning from 18th March to 18th June, $\tau_{F,1x4} = 0.25556$) was $F_{1x4}^{mkt} = 1.696\%$, the level of market 4x7 FRA3M (spanning from 18th June to 18th September, $\tau_{F,4x7} = 0.25556$) was $F_{4x7}^{mkt} = 1.580\%$. If one would compound these two rates to obtain the level of the implied 1x7 FRA6M (spanning from 18th March to 18th September, $\tau_{F,1x7} = 0.50556$) would obtain

$$F_{1x7}^{implied} = \frac{(1 + F_{1x4}^{mkt} \tau_{F,1x4}) \times (1 + F_{4x7}^{mkt} \tau_{F,4x7}) - 1.0}{\tau_{F,1x7}} = 1.641\%, \quad (20)$$

⁴here we keep the subscript x explicit also in order to be consistent with the following eq. (21).

Instrument	Quote	Underlying	Start Date	Maturity
Tod3MF	1.927	Euribor3M	Wed 18 Feb 2009	Mon 18 May 2009
Tom3MF	1.925	Euribor3M	Thu 19 Feb 2009	Tue 19 May 2009
1x4F	1.696	Euribor3M	Wed 18 Mar 2009	Thu 18 Jun 2009
2x5F	1.651	Euribor3M	Mon 20 Apr 2009	Mon 20 Jul 2009
3x6F	1.612	Euribor3M	Mon 18 May 2009	Tue 18 Aug 2009
4x7F	1.580	Euribor3M	Thu 18 Jun 2009	Fri 18 Sep 2009
5x8F	1.589	Euribor3M	Mon 20 Jul 2009	Tue 20 Oct 2009
6x9F	1.598	Euribor3M	Tue 18 Aug 2009	Wed 18 Nov 2009
Tod6MF	2.013	Euribor6M	Wed 18 Feb 2009	Tue 18 Aug 2009
Tom6MF	2.000	Euribor6M	Thu 19 Feb 2009	Wed 19 Aug 2009
1x7F	1.831	Euribor6M	Wed 18 Mar 2009	Fri 18 Sep 2009
2x8F	1.792	Euribor6M	Mon 20 Apr 2009	Tue 20 Oct 2009
3x9F	1.765	Euribor6M	Mon 18 May 2009	Wed 18 Nov 2009
4x10F	1.742	Euribor6M	Thu 18 Jun 2009	Fri 18 Dec 2009
5x11F	1.783	Euribor6M	Mon 20 Jul 2009	Wed 20 Jan 2010
6x12F	1.788	Euribor6M	Tue 18 Aug 2009	Thu 18 Feb 2010
12x18F	1.959	Euribor6M	Thu 18 Feb 2010	Wed 18 Aug 2010
18x24F	2.352	Euribor6M	Wed 18 Aug 2010	Fri 18 Feb 2011
12x24F	2.256	Euribor12M	Thu 18 Feb 2010	Fri 18 Feb 2011
IMM1x7F	98.169	Euribor6M	Wed 18 Feb 2009	Tue 18 Aug 2009
IMM2x8F	98.204	Euribor6M	Wed 18 Mar 2009	Fri 18 Sep 2009
IMM3x9F	98.236	Euribor6M	Wed 15 Apr 2009	Thu 15 Oct 2009
IMM4x10F	98.257	Euribor6M	Wed 20 May 2009	Fri 20 Nov 2009

FIGURE 2. EUR FRA strips on Euribor3M, Euribor6M, and Euribor12M. Source: Reuters page ICAPSHORT2, 16 Feb. 2009.

while the market quote for the 1x7 FRA6M was $F_{1x7}^{mkt} = 1.831\%$, 19 basis point larger. The difference is the liquidity/default risk premium seen by the market in post credit crunch times.

Market FRAs on x -tenor Euribor can be selected, together with the corresponding Depos, as bootstrapping instruments for the construction of the short term structure section of the yield curve \mathcal{C}_x . If $F_x(t; T_{i-1}, T_i)$ is the i -th Euribor forward rate resetting at time T_{i-1} with tenor $x = T_i - T_{i-1}$ months associated to the i -th FRA with maturity T_i , the implied discount factor at time T_i is obtained by eq. (15) as

$$P_x(t_0, T_i) = \frac{P_x(t_0, T_{i-1})}{1 + F_x(t_0; T_{i-1}, T_i) \tau_F(T_{i-1}, T_i)}, \quad t_0 < T_{i-1} < T_i, \quad (21)$$

where τ_F is given by eq. (13). The expression (21) above can be used to bootstrap the yield curve \mathcal{C}_x at point T_i once point T_{i-1} is known. Notice that FRAs collapse to Depos for shrinking $T_{i-1} - t_0$

$$\lim_{T_{i-1} \rightarrow t_0} F_x(t_0; T_{i-1}, T_i) = R_x^{Depo}(t_0, T_i), \quad (22)$$

and eq. (21) reduces to eq. (19).

3.4. Futures. Interest Rate Futures are the exchange-traded contracts equivalent to the over-the-counter FRAs. While FRAs have the advantage of being

Instrument	Quote	Convexity adjustment	Underlying	Underlying Start Date	Underlying End Date
FUT3MG9	98.0675	0.0000%	Euribor3M	Wed 18 Feb 2009	Mon 18 May 2009
FUT3MH9	98.3075	0.0001%	Euribor3M	Wed 18 Mar 2009	Thu 18 Jun 2009
FUT3MM9	98.4200	0.0007%	Euribor3M	Wed 17 Jun 2009	Thu 17 Sep 2009
FUT3MU9	98.3950	0.0016%	Euribor3M	Wed 16 Sep 2009	Wed 16 Dec 2009
FUT3MZ9	98.2550	0.0028%	Euribor3M	Wed 16 Dec 2009	Tue 16 Mar 2010
FUT3MH0	98.1625	0.0043%	Euribor3M	Wed 17 Mar 2010	Thu 17 Jun 2010
FUT3MM0	97.9725	0.0061%	Euribor3M	Wed 16 Jun 2010	Thu 16 Sep 2010
FUT3MU0	97.7675	0.0081%	Euribor3M	Wed 15 Sep 2010	Wed 15 Dec 2010
FUT3MZ0	97.5300	0.0104%	Euribor3M	Wed 15 Dec 2010	Tue 15 Mar 2011
FUT3MH1	97.3475	0.0131%	Euribor3M	Wed 16 Mar 2011	Thu 16 Jun 2011
FUT3MM1	97.1350	0.0159%	Euribor3M	Wed 15 Jun 2011	Thu 15 Sep 2011
FUT3MU1	96.9550	0.0193%	Euribor3M	Wed 21 Sep 2011	Wed 21 Dec 2011
FUT3MZ1	96.7650	0.0227%	Euribor3M	Wed 21 Dec 2011	Wed 21 Mar 2012

FIGURE 3. EUR Futures on Euribor 3M. The first serial contract (where “G9” stands for Feb. 09 expiry) and three IMM sets (where “H”, “M”, “U” and “Z” stand for March, June, September and December expiries, respectively) are displayed. Source: Reuters page 0#/FEI, 16 Feb. 2009. In column 3 are reported the corresponding convexity adjustments, calculated as discussed in the text.

more customizable, Futures are highly standardized contracts. In the EUR market the most common contracts (so called *IMM*⁵ *Futures*) insist on Euribor3M and expire every March, June, September and December (IMM dates). They fix the third Wednesday of the maturity month, the last trading day being the preceding Monday (because of the two days of settlement). Notice that such date grid is not regular: if S_i is the maturity date of the i -th Futures, then S_i and T_i , such that $\tau_F(S_i, T_i) = 3M$, are the underlying FRA3M start and end dates, respectively, and, in general, $T_i \neq S_{i+1}$. There are also so called *serial Futures*, expiring in the upcoming months not covered by the quarterly Futures. Any profit and loss is regulated through daily marking to market (so called *marking process*).

Such standard characteristics reduce the credit risk and the transaction costs, thus enhancing a very high liquidity. The first front contract is the most liquid interest rate instrument, with longer expiry contracts having very good liquidity up to the 8th-12th contract. Also the first serial contract is quite liquid, especially when it expires before the front contract.

In fig. 3 we report the quoted Futures strip on 3M Euribor rate up to 3 years maturity. As we can see, Futures are quoted in terms of prices instead of rates, the relation being

$$P_x^{Fut}(t_0, S_i, T_i) = 100 - R_x^{Fut}(t_0, S_i, T_i), \quad (23)$$

Because of their daily marking to market mechanism Futures do not have the same payoff of FRAs (including an unitary discount factor): an investor

⁵International Money Market of the Chicago Mercantile Exchange.

HW parameter	Value
Mean reversion	0.03
Volatility	0.709%

TABLE 1. Hull-White parameters values for Futures3M convexity adjustment at 16 Feb. 2009.

long a Futures contract will have a loss when the Futures price increases (and the Futures rate decreases) but he will finance such loss at lower rate; viceversa when the Futures price decreases the profit will be reinvested at higher rate. This means that the volatility of the forward rates and their correlation to the spot rates have to be accounted for, hence a *convexity adjustment* is needed to convert the rate R_x^{Fut} implied in the Futures price to its corresponding forward rate F_x ,

$$F_x(t_0, S_i, T_i) = R_x^{Fut}(t_0, S_i, T_i) - C_x(t_0, S_i, T_i) \quad (24)$$

(see e.g. ref. [12]). In other words, the trivial unit discount factor implied by daily margination introduces a pricing measure mismatch with respect to the corresponding FRA case that generates a volatility-correlation dependent convexity adjustment (see e.g. ch. 12 in ref. [13]).

The calculation of convexity adjustment thus requires a model for the evolution of the rates. While advanced approaches are available in literature (see e.g. refs. [12], [14], [13]), a standard practitioners' recipe is given in ref. [15], based on a simple short rate 1 factor Hull & White model [16]. This approach has been used in fig. 3 to calculate the adjustments, using the Hull-White parameters values given in table 1.

Market Futures on x -tenor Euribor can be selected as bootstrapping instruments for the construction of short-medium term structure section of the yield curve \mathcal{C}_x . Notice that Futures contracts have expiration dates gradually shrinking to zero and as such they generate rolling pillars that periodically jumps and overlap the fixed Depo and FRA pillars. Hence some *priority* rule must be used in order to decide which instruments must be excluded from the bootstrapping procedure.

Given the i -th Futures market quote $P_x^{Fut}(t_0, S_i, T_i)$ with underlying FRA maturity T_i , the implied discount factor at time T_i is obtained by eqs. (21), (23) and (24) as

$$P_x(t_0, T_i) = \frac{P_x(t_0, T_{i-1})}{1 + [R_x^{Fut}(t_0, S_i, T_i) - C_x(t_0, S_i, T_i)] \tau_F(S_i, T_i)}, \quad (25)$$

where τ_F is given by eq. (13). The expression above can be used to bootstrap the yield curve \mathcal{C}_x at point T_i once point S_i is known.

3.5. Swaps. Interest rate Swaps are Over-The-Counter (OTC) contracts in which two counterparties agree to exchange fixed against floating rate cash

Instrument	Quote	Underlying	Start Date	Maturity
AB6E1Y	1.933	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2010
AB6E15M	1.858	Euribor6M	Wed 18 Feb 2009	Tue 18 May 2010
AB6E18M	1.947	Euribor6M	Wed 18 Feb 2009	Wed 18 Aug 2010
AB6E21M	1.954	Euribor6M	Wed 18 Feb 2009	Thu 18 Nov 2010
AB6E2Y	2.059	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2011
AB6E3Y	2.350	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2012
AB6E4Y	2.604	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2013
AB6E5Y	2.808	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2014
AB6E6Y	2.983	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2015
AB6E7Y	3.136	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2016
AB6E8Y	3.268	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2017
AB6E9Y	3.383	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2018
AB6E10Y	3.488	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2019
AB6E11Y	3.583	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2020
AB6E12Y	3.668	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2021
AB6E13Y	3.738	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2022
AB6E14Y	3.793	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2023
AB6E15Y	3.833	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2024
AB6E16Y	3.861	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2025
AB6E17Y	3.877	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2026
AB6E18Y	3.880	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2027
AB6E19Y	3.872	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2028
AB6E20Y	3.854	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2029
AB6E21Y	3.827	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2030
AB6E22Y	3.792	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2031
AB6E23Y	3.753	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2032
AB6E24Y	3.713	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2033
AB6E25Y	3.672	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2034
AB6E26Y	3.635	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2035
AB6E27Y	3.601	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2036
AB6E28Y	3.569	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2037
AB6E29Y	3.539	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2038
AB6E30Y	3.510	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2039
AB6E35Y	3.377	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2044
AB6E40Y	3.266	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2049
AB6E50Y	3.145	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2059
AB6E60Y	3.076	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2069

FIGURE 4. EUR Swaps on Euribor6M. The codes “AB6En” in col. 1 label swaps receiving annually a fixed rate and paying semi-annually a floating rate on Euribor6M with maturity in n months/years. Source: Reuters page ICAPEURO, 16 Feb. 2009.

flows. These payment streams are called fixed and floating leg of the Swap, respectively.

The EUR market quotes standard plain vanilla Swaps starting at spot date with annual fixed leg versus floating leg indexed to x -months Euribor rate paid with x -months frequency. Such Swaps can be regarded as portfolio of FRA contracts (the first one being actually a Deposit). The day count convention for the quoted (fair) swap rates is $30/360$ (*bond basis*) [9].

Instrument	Quote	Underlying	Start Date	Maturity
1S1Y	1.668	Euribor3M	Wed 18 Mar 2009	Thu 18 Jun 2009
2S1Y	1.704	Euribor3M	Wed 17 Jun 2009	Thu 17 Sep 2009
3S1Y	1.817	Euribor3M	Wed 16 Sep 2009	Wed 16 Dec 2009
4S1Y	1.975	Euribor3M	Wed 16 Dec 2009	Tue 16 Mar 2010
1S2Y	1.910	Euribor3M	Wed 18 Mar 2009	Thu 18 Jun 2009
2S2Y	2.029	Euribor3M	Wed 17 Jun 2009	Thu 17 Sep 2009
1S3Y	2.256	Euribor3M	Wed 18 Mar 2009	Thu 18 Jun 2009

FIGURE 5. EUR IMM Swaps on Euribor3M. The codes “ $mSnY$ ” in col. 1 label $m = \text{Mar., Jun., Sep. and Dec.}$ IMM starting swaps receiving annually a fixed rate and paying quarterly a floating rate on Euribor3M with maturity in $n = 1, 2, 3$ years. The 1x1S instrument, not reported, is equivalent to the 1M Deposit in fig. 1. Source: Reuters page ICAP-SHORT2, 16 Feb. 2009.

Instrument	Quote	Underlying	Start Date	Maturity
2x1S	1.456	Euribor1M	Wed 18 Feb 2009	Mon 20 Apr 2009
3x1S	1.406	Euribor1M	Wed 18 Feb 2009	Mon 18 May 2009
4x1S	1.365	Euribor1M	Wed 18 Feb 2009	Thu 18 Jun 2009
5x1S	1.337	Euribor1M	Wed 18 Feb 2009	Mon 20 Jul 2009
6x1S	1.322	Euribor1M	Wed 18 Feb 2009	Tue 18 Aug 2009
7x1S	1.316	Euribor1M	Wed 18 Feb 2009	Fri 18 Sep 2009
8x1S	1.315	Euribor1M	Wed 18 Feb 2009	Mon 19 Oct 2009
9x1S	1.321	Euribor1M	Wed 18 Feb 2009	Wed 18 Nov 2009
10x1S	1.330	Euribor1M	Wed 18 Feb 2009	Fri 18 Dec 2009
11x1S	1.347	Euribor1M	Wed 18 Feb 2009	Mon 18 Jan 2010
12x1S	1.355	Euribor1M	Wed 18 Feb 2009	Thu 18 Feb 2010

FIGURE 6. EUR Swaps on Euribor1M. The codes “ $nx1S$ ” in col. 1 label n -months maturity swaps receiving a single fixed rate at maturity and paying monthly a floating rate on Euribor1M. Source: Reuters page ICAPSHORT2, 16 Feb. 2009.

In figures 4, 5 and 6 we report the quoted Swaps strips on 6M, 3M and 1M Euribor rates, respectively.

Market Swaps on x -tenor Euribor can be selected as bootstrapping instruments for the construction of the medium-long term structure section of the yield curve \mathcal{C}_x . Given the swap rate $S_x(t_0, T_i)$ quoted for maturity T_i , from equation (16) we obtain, setting $T_0 = S_0 = t = t_0$ and $T_n = S_m = T_i$,

$$\begin{aligned}
S_x(t_0, T_i) &= \frac{1 - P_x(t_0, T_i)}{A_x(t_0, T_i)} \\
&= \frac{1 - P_x(t_0, T_i)}{A_x(t_0, T_{i-1}) + P_x(t_0, T_i) \tau_S(T_{i-1}, T_i)}, \quad (26)
\end{aligned}$$

where the annuity $A_x(t_0, T_{i-1})$ is given by eq. (17). The expression above can be inverted to find the implied discount factor at time T_i as

$$P_x(t_0, T_i) = \frac{1 - S_x(t_0, T_i) A_x(t_0, T_{i-1})}{1 + S_x(t_0, T_i) \tau_S(T_{i-1}, T_i)}, \quad (27)$$

where τ_S is given by

$$\tau_S(T_1, T_2) := \tau[T_1, T_2; 30/360(\text{bondbasis})]. \quad (28)$$

Expression (27) can be used to bootstrap the yield curve \mathcal{C}_x at point T_i once the points $\mathbf{T} = \{T_1, \dots, T_{i-1}\}$ are known.

3.6. Basis Swaps. Interest rate (single currency) Basis Swaps are floating vs floating swaps admitting underlying rates with different tenors.

The EUR market quotes standard plain vanilla Basis Swaps as portfolios of two swaps with the same fixed legs and floating legs paying Euribor xM and yM, e.g. 3M vs 6M, 1M vs 6M, 6M vs 12M, etc. In fig. 7 we report three quoted Basis Swaps strips. The quotation convention is to provide the difference (in basis points) between the fixed rate of the higher frequency swap and the fixed rate of the lower frequency swap. At the moment such difference is positive and decreasing with maturity, reflecting the preference of market players for receiving payments with higher frequency (e.g. 3M instead of 6M, 6M instead of 12M, etc.) and shorter maturities.

Basis swaps are a fundamental element for long term multi-curve bootstrapping, because, starting from the quoted Swaps on Euribor 6M (fig. 4), they allow to imply levels for non-quoted Swaps on Euribor 1M, 3M, and 12M, to be selected as bootstrapping instruments for the corresponding yield curves construction. If $\Delta_{x,6M}(t_0, T_i)$ is the quoted basis spread for a basis swap receiving Euribor xM and paying Euribor 6M plus spread for maturity T_i , we simply have

$$S_x(t_0, T_i) = S_{6M}(t_0, T_i) + \Delta_{x,6M}(t_0, T_i), \quad (29)$$

with the obvious caveat that $\Delta_{6M,x}(t_0, T_i) = -\Delta_{x,6M}(t_0, T_i)$. In fig. 8 we report all the possible basis combinations obtained from fig. 7. Notice that basis swaps in fig. 7 are quoted up to 30 years, while swaps on Euribor 6M in fig. 4 are quoted up to 60 years. Thus the bootstrapping of yield curves different from \mathcal{C}_{6M} over 30 years maturity requires extrapolation of basis swap quotations. In the present market conditions, such extrapolation is not particularly critical, given the smooth and monotonic long term shape of the basis curves in fig. 7.

3.7. The Role of Interpolation. The interpolation scheme we choose for the given parametrization determines how reasonable the yield curve will be. For instance, linear interpolation of discount factors is an obvious but extremely poor choice. Linear interpolation of zero rates or log-discounts are popular choices leading to stable and fast bootstrapping procedures, but unfortunately they produce horrible forward curves, with a sawsaw or

Instrument	Quote (bps)	Underlying 1st leg	Underlying 2nd leg	Start Date	Maturity
1E6E1Y	55.1	Euribor1M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2010
1E6E2Y	38.7	Euribor1M	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2011
1E6E3Y	29.8	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2012
1E6E4Y	24.7	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2013
1E6E5Y	21.1	Euribor1M	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2014
1E6E6Y	18.5	Euribor1M	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2015
1E6E7Y	16.5	Euribor1M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2016
1E6E8Y	15.0	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2017
1E6E9Y	13.7	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2018
1E6E10Y	12.7	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2019
1E6E11Y	11.9	Euribor1M	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2020
1E6E12Y	11.2	Euribor1M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2021
1E6E15Y	9.6	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2024
1E6E20Y	7.9	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2029
1E6E25Y	6.9	Euribor1M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2034
1E6E30Y	6.2	Euribor1M	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2039
3E6E1Y	18.6	Euribor3M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2010
3E6E2Y	12.7	Euribor3M	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2011
3E6E3Y	9.7	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2012
3E6E4Y	8.0	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2013
3E6E5Y	6.7	Euribor3M	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2014
3E6E6Y	5.8	Euribor3M	Euribor6M	Wed 18 Feb 2009	Wed 18 Feb 2015
3E6E7Y	5.1	Euribor3M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2016
3E6E8Y	4.6	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2017
3E6E9Y	4.2	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2018
3E6E10Y	3.8	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 18 Feb 2019
3E6E11Y	3.5	Euribor3M	Euribor6M	Wed 18 Feb 2009	Tue 18 Feb 2020
3E6E12Y	3.3	Euribor3M	Euribor6M	Wed 18 Feb 2009	Thu 18 Feb 2021
3E6E15Y	2.8	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2024
3E6E20Y	2.2	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 19 Feb 2029
3E6E25Y	2.0	Euribor3M	Euribor6M	Wed 18 Feb 2009	Mon 20 Feb 2034
3E6E30Y	1.8	Euribor3M	Euribor6M	Wed 18 Feb 2009	Fri 18 Feb 2039
6E12E1Y	21.2	Euribor6M	Euribor12M	Wed 18 Feb 2009	Thu 18 Feb 2010
6E12E2Y	15.2	Euribor6M	Euribor12M	Wed 18 Feb 2009	Fri 18 Feb 2011
6E12E3Y	11.7	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 20 Feb 2012
6E12E4Y	9.7	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 18 Feb 2013
6E12E5Y	8.2	Euribor6M	Euribor12M	Wed 18 Feb 2009	Tue 18 Feb 2014
6E12E6Y	7.2	Euribor6M	Euribor12M	Wed 18 Feb 2009	Wed 18 Feb 2015
6E12E7Y	6.3	Euribor6M	Euribor12M	Wed 18 Feb 2009	Thu 18 Feb 2016
6E12E8Y	5.7	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 20 Feb 2017
6E12E9Y	5.1	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 19 Feb 2018
6E12E10Y	4.7	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 18 Feb 2019
6E12E11Y	4.4	Euribor6M	Euribor12M	Wed 18 Feb 2009	Tue 18 Feb 2020
6E12E12Y	4.1	Euribor6M	Euribor12M	Wed 18 Feb 2009	Thu 18 Feb 2021
6E12E15Y	3.5	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 19 Feb 2024
6E12E20Y	2.8	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 19 Feb 2029
6E12E25Y	2.5	Euribor6M	Euribor12M	Wed 18 Feb 2009	Mon 20 Feb 2034
6E12E30Y	2.2	Euribor6M	Euribor12M	Wed 18 Feb 2009	Fri 18 Feb 2039

FIGURE 7. EUR Basis Swaps. The codes “ $xEyEnY$ ” in col. 1 label basis swaps receiving Euribor xM and paying Euribor yM plus basis spread with n years maturity. Source: Reuters page ICAPEUROBASIS, 16 Feb. 2009.

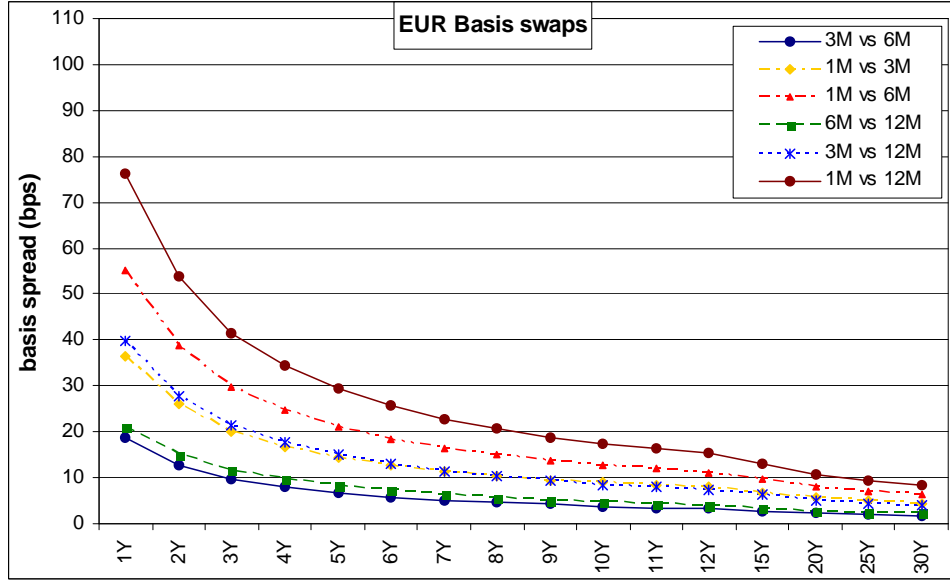


FIGURE 8. EUR Basis spreads from fig. 7. The spreads not explicitly quoted there have been deduced using eq. (29).

piecewise-constant shape (see e.g. [5], [6] for a review of available interpolation schemes). We show in fig. 9 one examples of such poor interpolation schemes. While zero curves (upper panel) display similar smooth behaviors, simple visual inspection of forward curves (lower panel) reveals different non-smooth behaviors, with oscillations larger than 100 basis points. Such discontinuities in the forward curves correspond to angle points in the zero curves (as pointed out in section 2), generated by linear interpolation that forces them to suddenly “turn” around a market point. Notice that only the most liquid Swaps from fig. 4, with maturities 3-10, 12, 15, 20, 25 and 30 years have been included in the bootstrapping of curve \mathcal{C}_{6M} . Often the remaining less liquid quotations for 11, 13, 14, 16-19, 21-24, 26-29 years maturity are included in the linear interpolations schemes to reduce the amplitude of the forward curve oscillations. The same can be done for longer maturities using interpolated quotes on the 30, 40, 50 and 60 years market pillars.

In fig. 9 the monotonic cubic spline interpolation on log-discounts is shown too, clearly ensuring a smooth and financially sound behavior of the forward curve. The choice of cubic interpolations is a very delicate issue. Simple splines (see e.g. [18]) suffer of well-documented problems such as spurious inflection points, excessive convexity, and lack of locality after input price perturbations (distributed sensitivities). Recently, Andersen [19] has addressed these issues through the use of shape-preserving splines from the class of generalized tension splines, while Hagan and West [5]-[6] have developed a new scheme based on positive preserving forward interpolation.

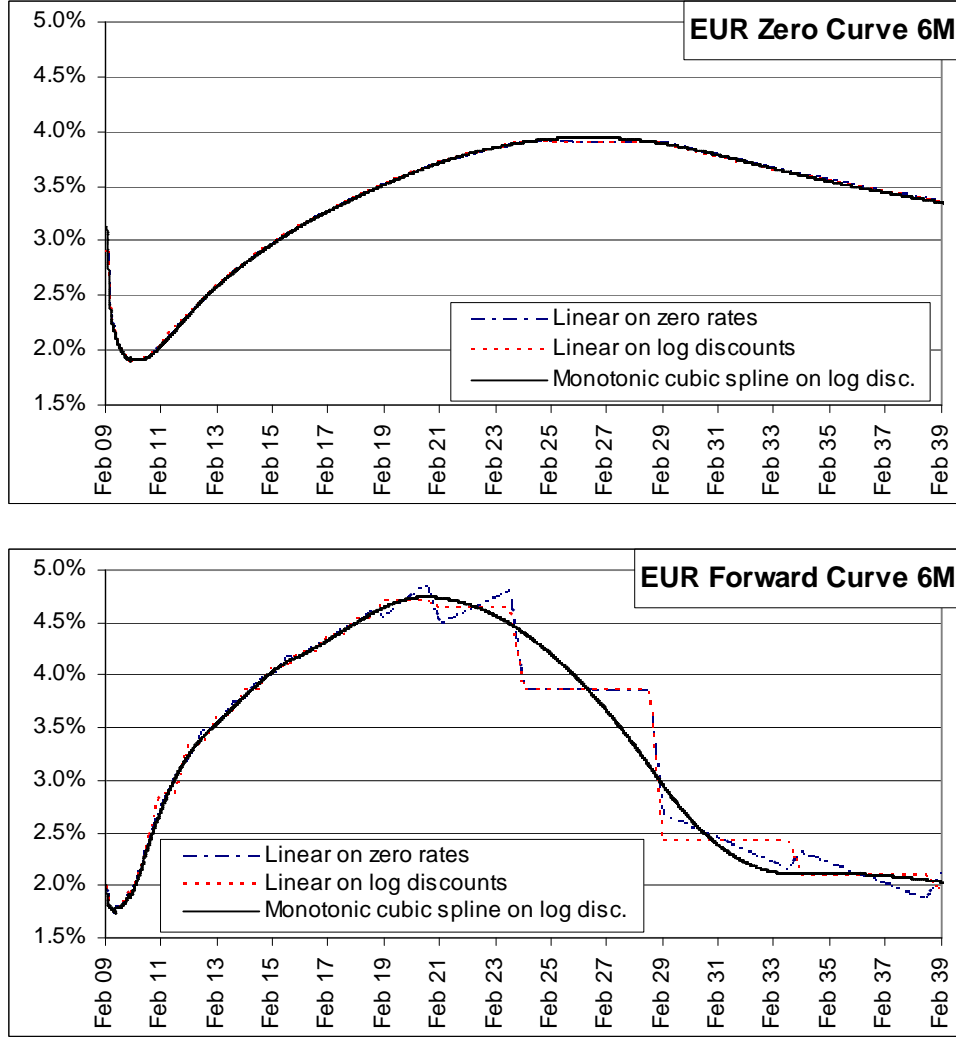


FIGURE 9. Examples of *bad* (but very popular!) interpolation schemes. Upper panel: different zero curves display similar smooth behaviors. Lower panel: forward curves reveals different non-smooth behaviors, with oscillations larger than 100 basis points. The smooth monotonic cubic spline interpolation on log-discounts (continuous black line) of fig. 15 is shown as a benchmark.

We found the classic Hyman monotonic cubic filter [17] applied to spline interpolation of log-discounts to be the easiest and best approach: its monotonicity ensures non-negative forward curves and actually remove most of the unpleasant waviness. Notice that the Hyman filter can be applied to any cubic interpolants: this helps to address the non-locality of spline using

Term	Zero rate	Capitalizat ion factor	Discount factor	Log Discount factor	Discrete forward	FRA
0.0	0.00%	1.000000	1.000000	0.000000		
0.1	8.10%	1.008133	0.991933	0.008100	8.1000%	8.1329%
1.0	7.00%	1.072508	0.932394	0.070000	6.8778%	7.0951%
4.0	4.40%	1.192438	0.838618	0.176000	3.5333%	3.7274%
9.0	7.00%	1.877611	0.532592	0.630000	9.0800%	11.4920%
20.0	4.00%	2.225541	0.449329	0.800000	1.5455%	1.6846%
30.0	3.00%	2.459603	0.406570	0.900000	1.0000%	1.0517%

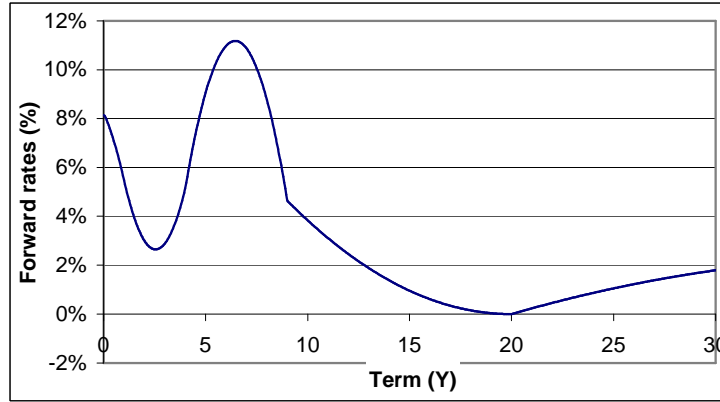


FIGURE 10. Example of nasty curve taken from ref. [5] (p. 98 and fig. 2, bottom right panel). Upper panel: the example curve. Lower panel: the forward curve obtained through Hyman monotonic cubic spline [17] applied on log-discounts. The forward rate at 20Y is null but no negative rates appear.

alternative more local cubic interpolations. In fig. 10 we show an example of particularly nasty curve taken from Hagan and West [5] (p. 98 and fig. 2, bottom right panel). The forward curve obtained through Hyman monotonic cubic spline [17] applied on log discounts (lower panel) is always non negative (there is a unique minimum at 20Y).

A peculiarity of using non-local interpolation inside the bootstrapping procedure is that the shape of the already bootstrapped part of the curve is altered by the addition of further pillars. This is usually remedied by cycling in iterative fashion: after a first bootstrap, which might even use a local interpolation scheme and build up the pillar grid one point at time, the resulting complete grid is altered one pillar at time using again the same bootstrapping algorithm, until convergence is reached. The first cycle can be even replaced by a good grid guess, the most natural one being just the grid previous state in a dynamically changing environment.

We stress that the focus on smooth discrete forward rate is the key point of state-of-the-art bootstrapping. For even the best interpolation schemes to be effective the forward rate curve must be smooth, i.e. any jump must be

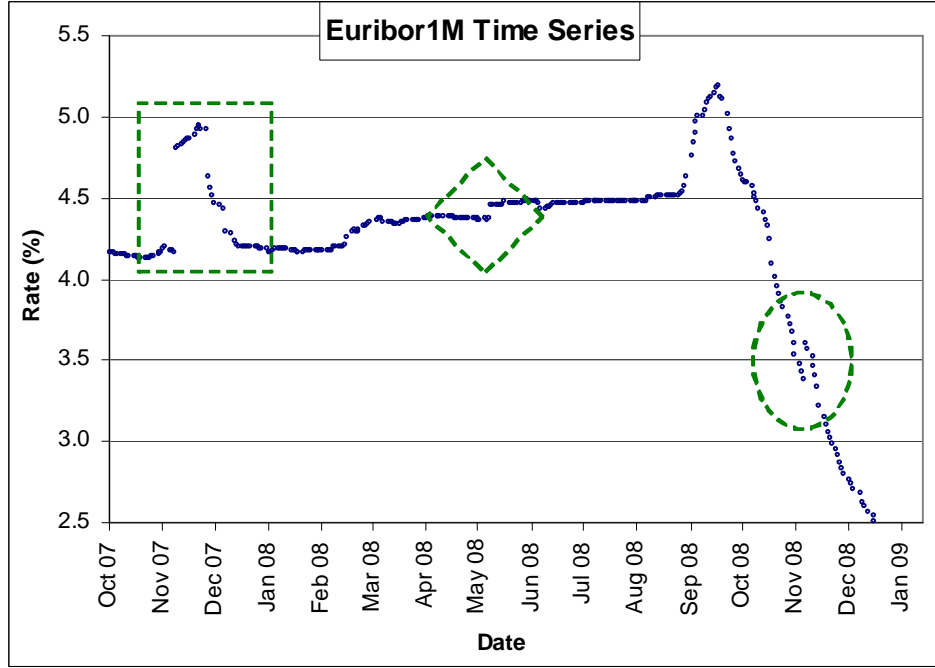


FIGURE 11. Turn of year effect on Euribor1M. The historical time series in the Oct. 2007 - Feb 2009 window is displayed. Three jumps can be identified: the 2007 turn of year (29th Nov. 2007, 64 bps, left rectangle); the 2008 turn of year (27th Nov. 2008, 22 bps, right oval); a smaller “end of semester effect” (29th May 2008, 9 bps, middle rhombus). Source: Reuters.

removed, and added back only at the end of the smooth curve construction. The most relevant jump in forward rates is the so-called turn of year effect, discussed in the next section.

3.8. The Turn of Year Effect. In the interest rate market the turn of year effect is a jump normally observed in market quotations of rates spanning across the end of a year. In fig. 11 we display the historical series of Euribor1M in the window October 2007 - February 2009. The 2007 turn of year jump (64 bps) is clearly visible on 29th Nov. 2007 (left rectangle), just when the spot starting 1M tenor rate spans the end of 2007, with rates reverting toward the previous levels one month later. The 2008 turn of year jump on 27th Nov. 2008 (22 bps, right oval) is partially hidden by the high market volatility realized in that period. Viceversa in lower volatility regimes even the much smaller “end of semester effect” may be observable, as seen on 29th of May 2008 (9 bps, middle rhombus).

In the EUR market the larger jump is observed the last working day of the year (e.g. 31th December) for the Overnight Deposit maturing the first

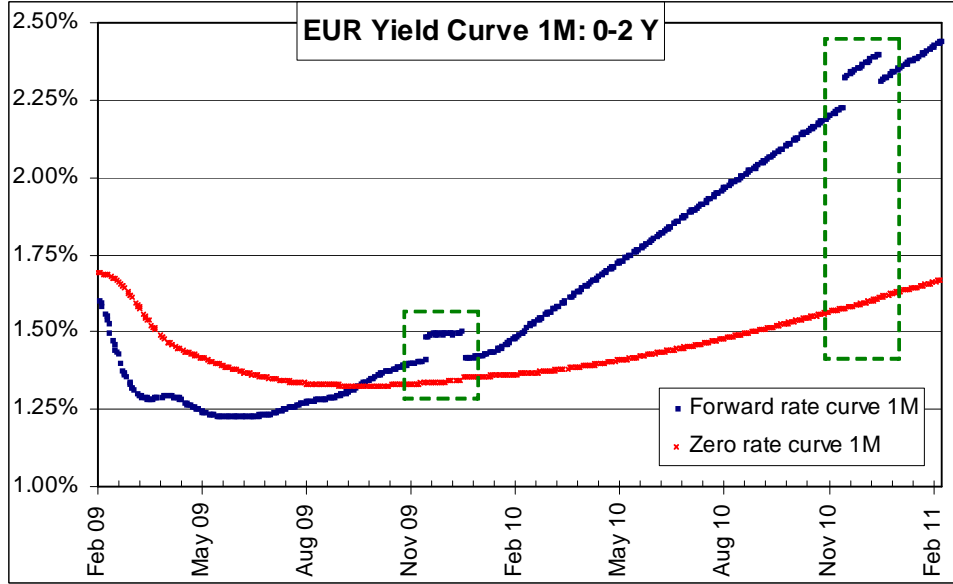


FIGURE 12. Detail from fig. 13, upper panel, showing the turn of year effect included in the short term bootstrapping of the forward rate curve \mathcal{C}_{1M}^f (blue line). The 2009 and 2010 turn of year jumps are clearly observable (left and right dashed rectangles, respectively). The same jumps are also present in the zero rate curve \mathcal{C}_{1M}^z (red line), but less visible because of the scale (see discussion in the text).

working day of the next year (e.g. 2nd January). The same happens for the Tomorrow Next and Spot Next Deposits one and two business days before, respectively (e.g. 30th and 29th December). Other instruments with longer underlying rate tenors display smaller jumps when their maturity crosses the same border: for instance, the 1M Deposit quotation jumps 2 business days before the 1st business day of December; the 12M Deposits always include a jump except 2 business days before the end of the year (due to the end of month rule); the December IMM Futures always include a jump, as well as the October and November serial Futures; 2Y Swaps always include two jumps; etc. The effect is generally observable at the first two ends of year and becomes negligible at the following crosses.

The decreasing jump with increasing underlying rate tenor can be easily understood once we distinguish between jumping rates and non-jumping rates. For instance, we may think to the 1M Deposit as a weighted average of 22 (business days in one month) overnight rates (plus a basis). If such Depo spans an end of year, there must be a single overnight rate, weighting $1/22$ th, that crosses that end of year and displays the jump, while the others do not. Considering rates with longer tenors, there are still single jumping overnight

rates, but with smaller weights. Hence longer deposits/FRAs display smaller jumps. The same holds for Swaps, as portfolios of Depos/FRAs.

From a financial point of view, the turn of year effect is due to the increased search for liquidity by financial institutions just after the periodic balance sheet strikes.

An yield curve term structure up to N years including the turn of year effect should contain, in principle, N discontinuities; in practice essentially the end of the current and the next year can be taken into account. The effect can be modeled simply through a multiplicative coefficient applied to discount factors, or, equivalently, an additive coefficient applied to zero rates, corresponding to all the dates following a given end of year. In this way we are allowed to estimate the coefficient using instruments with a given underlying rate tenor (e.g. those on Euribor3M used for \mathcal{C}_{3M}), and to apply it to any other curve \mathcal{C}_x taking into account the proper weights. Notice that, as stressed in the previous section, starting from a smooth and continuous yield curve is crucial for correctly take into account the discontinuity at the turn of year.

The jump coefficient can be estimated from market quotations using different approaches:

- jump in the 3M Futures strip: the (no-jump) end of year crossing forward is obtained through interpolation of non-crossing forwards; the jump coefficient is given by the difference between the latter and the quoted value. This approach always allows the estimation of the second turn of year. The first turn of year can be obtained only up to the third Wednesday of September, when the corresponding Futures expiries. In the period October-December there are no non-crossing Futures to interpolate and the first turn of year should be extrapolated from the second, making the method not robust;
- jump in the 6M FRA strip: this is equivalent to the approach above but it allows the estimation of the first turn of year up to June (included);
- jump in the 1M Swaps strip: this is equivalent to the approaches above and it allows the estimation of the first turn of year up to November (included);
- jump in the FRA strip quoted by brokers each Monday: this approach is valid all year long, but it allows only a discontinuous weekly update.

The empirical approaches above, when available at the same time, give estimates in excellent agreement with each other.

A numerical example of application of the methodology discussed above is given in fig. 12 (a detail from upper panel of fig. 13 reported in section 4), where we display the bootstrapping of the forward and zero rate curves \mathcal{C}_{1M}^f and \mathcal{C}_{1M}^z on Euribor1M. The 2009 turn of year jump is clearly observable in \mathcal{C}_{1M}^f from 27th Nov. 2009 (+7.5 bps) to 30th Dec. 2009 (-8.6 bps) for

Euribor1M rates spot starting on 1st Dec. 2009 and terminating on 4th Dec. 2010. Also the 2010 turn of year jump is observable in \mathcal{C}_{1M}^f from 29th Nov. 2010 (+9.3 bps) to 30th Dec. 2010 (-8.6 bps) for rates spot starting on 1st Dec. 2010 and terminating on 3rd Dec. 2011. The jumps are present also in the zero rate curve \mathcal{C}_{1M}^z , but they are less observable because of the scale in fig. 12. We stress that a single turn of the year induces *one* discontinuity in the zero rate and discount curves, and *two* discontinuities in the forward rate curve (remember that the forward rate is given by the ratio of two discounts).

The yield curve discontinuities induced by the turn of year effect may appear, to a non market-driven reader, a fuzzy effect broking the desired yield curve smoothness. On the contrary, we stress that they are neither a strangeness of the market quotations nor an accident of the bootstrapping, but correspond to true and detectable financial effects that should be included in any yield curve used to mark to market interest rate derivatives.

4. IMPLEMENTATION AND EXAMPLES OF BOOTSTRAPPING

Given the methodology discussed in the previous sections, we are able to bootstrap four yield curves \mathcal{C}_{1M} , \mathcal{C}_{3M} , \mathcal{C}_{6M} , \mathcal{C}_{12M} on Euribor 1M, 3M, 6M and 12M, respectively. In the four figures 13, 14, 15 and 16 below we show an example of bootstrapping using our personal selection of the market data discussed in sections 3.2-3.6. Only the forward curves are displayed, being the most significative bootstrapping test as discussed in section 3.7. The scales are the same across all figures, allowing a general comparison. The maximum maturity reported is 30 years, according to the basis swaps quotations (see fig. 7). As discussed in section 3.6, while for the \mathcal{C}_{6M} curve swaps market data are available up to 60 years (see fig. 4), the bootstrapping of other curves over 30 years maturity would require extrapolation of basis swap quotations (see figures 7 and 8).

The results discussed in this paper have been obtained using the QuantLib framework. The basic classes and methods (iterative bootstrapping, interpolations, market conventions, etc.) are implemented in the object oriented C++ QuantLib library [20]. The QuantLib objects and analytics are exposed to a variety of end-user platforms (including Excel and Calc) through the QuantLibAddin [21] and QuantLibXL [22] libraries. Market data are retrieved from the chosen provider and real time is ensured by the ObjectHandler in-memory repository [23]. The full framework described above is available open source. Anyone interested in the topic may download and test the implementation, posting to the QuantLib community forum any comment or suggestion to improve the job.

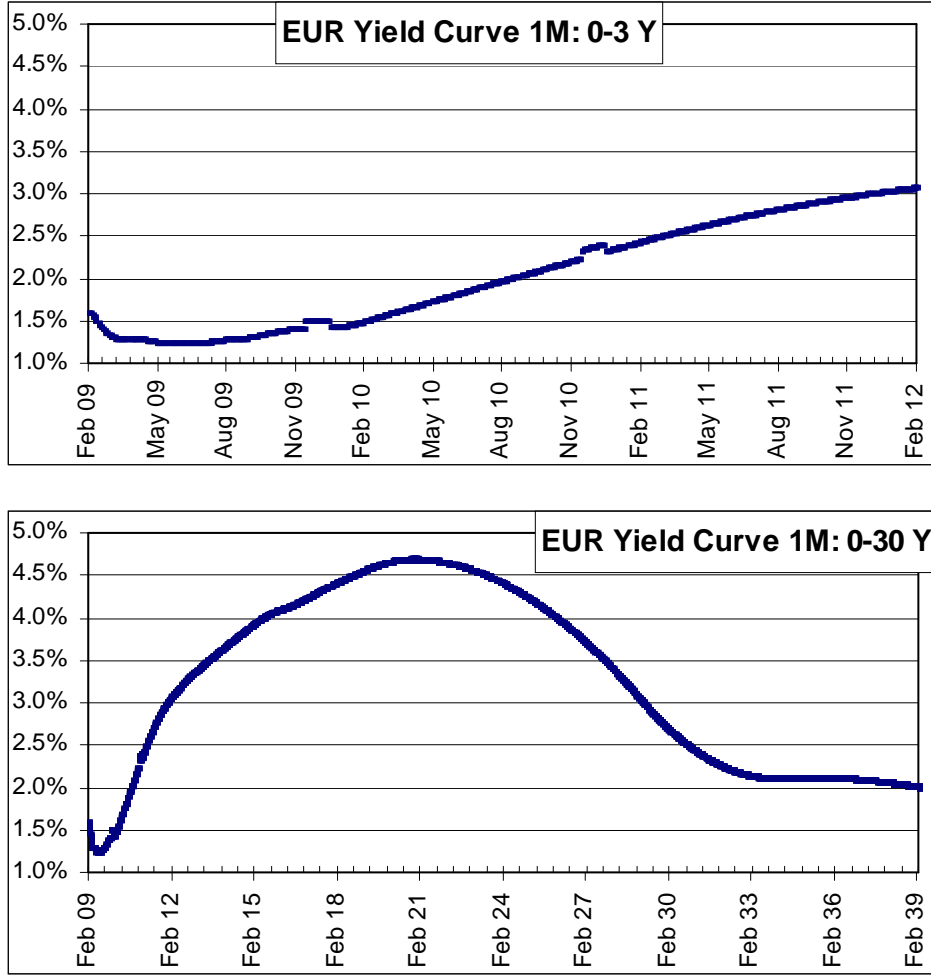


FIGURE 13. Forward curve C_{1M}^f on Euribor1M at 16 Feb. 2009, plotted with 1M-tenor forward rates $F(t_0; t, t+1M, act/360)$, t daily sampled and spot date $t_0 = \text{Feb. 18th, 2009}$. Upper panel: short term structure up to 3 years; lower panel: whole term structure up to 30 years. The two jumps observed in the curve correspond to the two turn of years for 1M tenor forward rates spot starting at 1st Dec. 2009 and 1st Dec. 2010.

5. CONCLUSIONS

We have illustrated a methodology for bootstrapping multiple interest rate yield curves, each homogeneous in the underlying rate tenor, from non-homogeneous plain vanilla instruments quoted on the market.

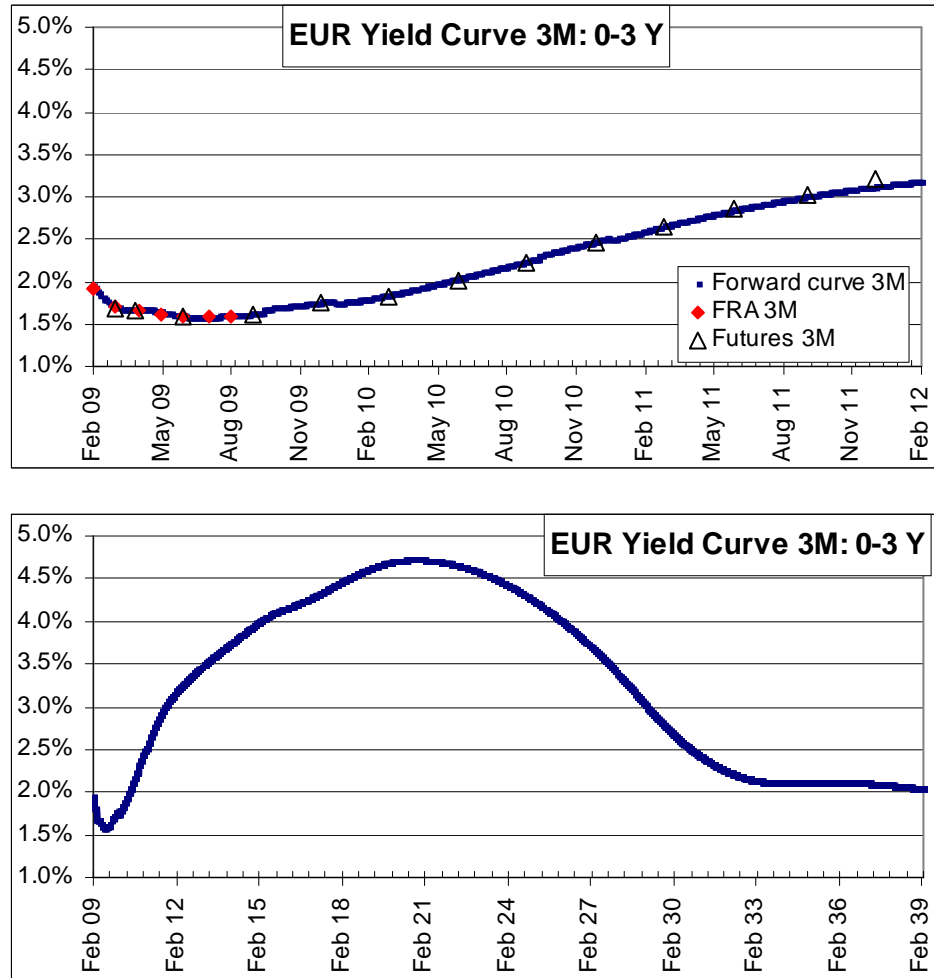


FIGURE 14. Forward curve C_{3M}^f on Euribor3M at 16 Feb. 2009. Plots as in fig. 13. Quoted 3M FRAs and 3M Futures are also reported.

Results for the concrete EUR market case have been analyzed in detail, showing how real quotations for interest rate instruments on Euribor1M, 3M, 6M and 12M tenor can be used in practice to construct stable, robust and smooth yield curves for pricing and hedging interest rate derivatives.

The full implementation of the work, comprehensive of C++ code and Excel workbooks, is available open source.

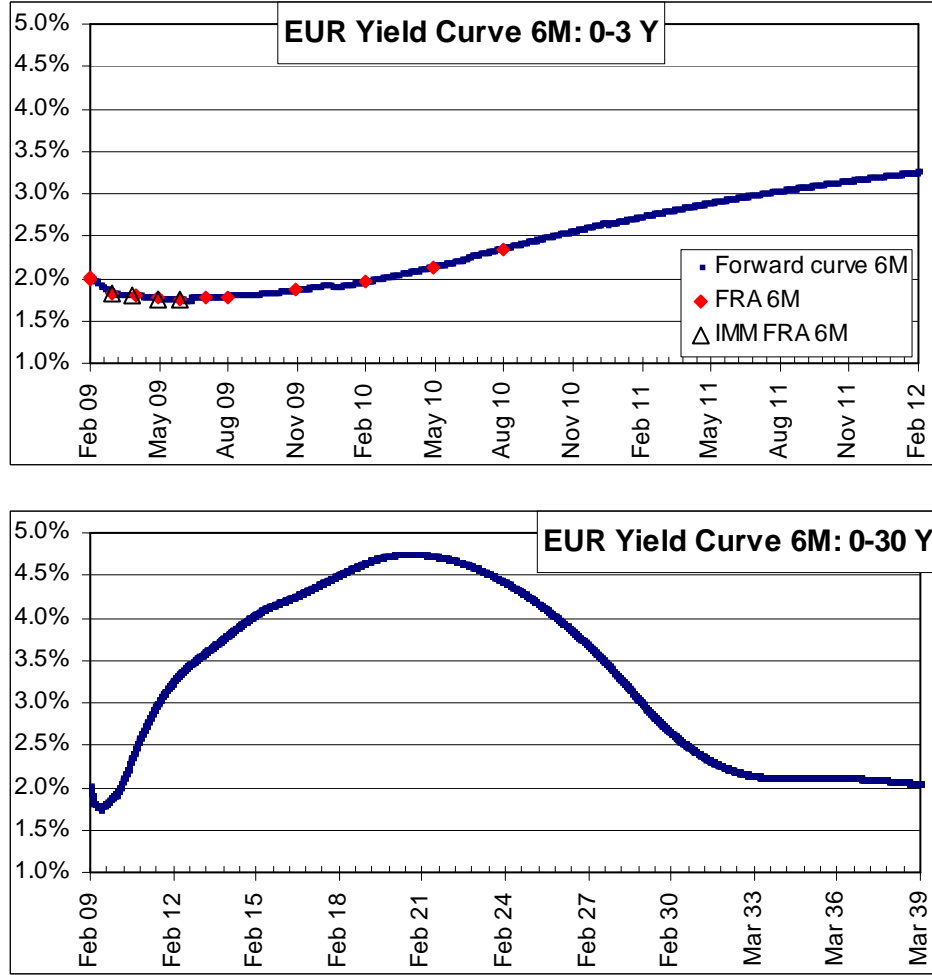


FIGURE 15. Forward curve \mathcal{C}_{6M}^f on Euribor6M at 16 Feb. 2009. Plots as in fig. 13. Quoted 6M FRAs and 6M IMM FRAs are also reported.

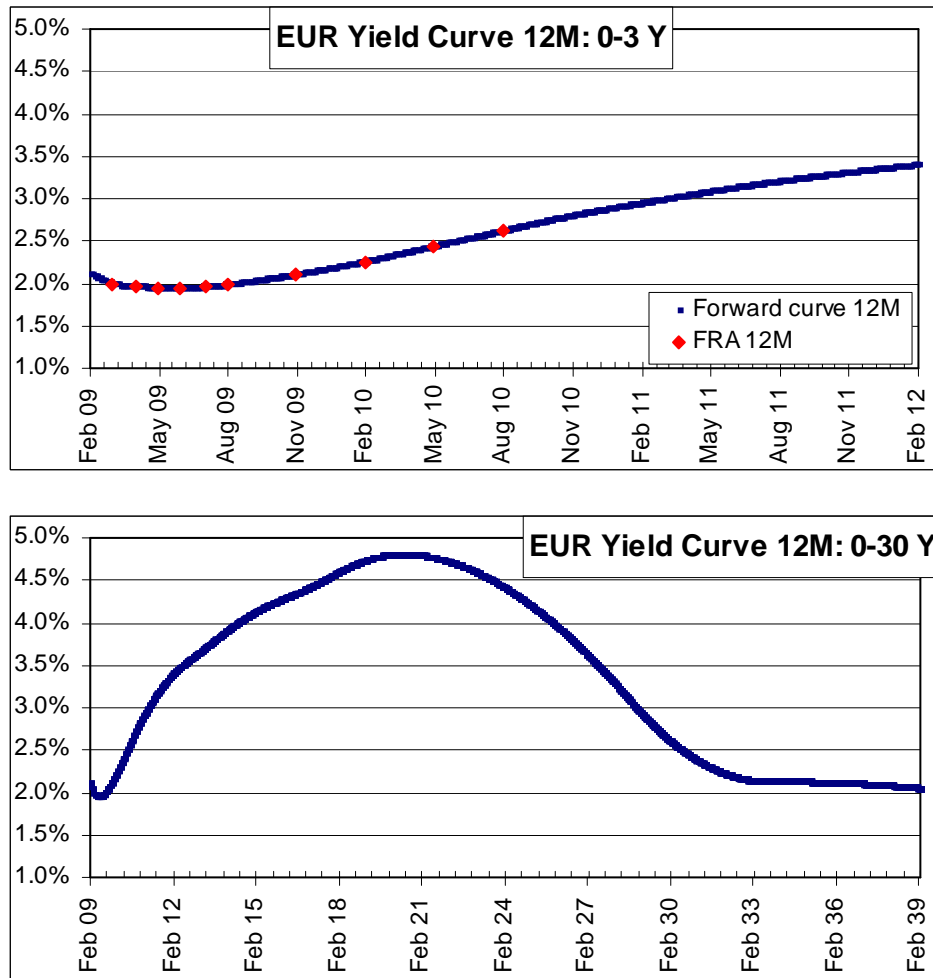


FIGURE 16. Forward curve C_{12M}^f on Euribor12M at 16 Feb. 2009. Plots as in fig. 13. Quoted 12M FRAs are also reported.

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