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5차시-벡터의 선형독립과 기저벡터

2.3 Linear Independence

$\Rightarrow$ linear independent

$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$

$\rightarrow$ only $c_1 = c_2 = \cdots = c_n = 0$

◎ If G.E. of A generates m non-zero rows $\rightarrow$ m independent column vectors in A

- Rank of A
  - = # of independent column vectors
  - = # of " row vectors
  - = # of pivots in G.E.
  - = Dim of $C(A)$

◎ Spanning

- all linear combinations of vectors $\{v_1, v_2, \cdots, v_n\}$ construct a vector space
- = $\{v_1, v_2, \cdots, v_n\}$ span vector space

◎ Basis (vectors)

- $\Rightarrow$ # of minimum linearly independent vectors to span the vector space
- $\Rightarrow$ linear combination is unique from basis
- Basis is not unique for a vector space

8차시-1차연립방정식 풀이와 직교벡터 구하기

orthogonal basis(vectors)

$v_1, v_2, \cdots, v_n \quad \|v_i\| = 1$

$v_i^Tv_j = 0$

$X = \sum_{i=1}^{n} c_iv_i \quad [v_1 v_2 \cdots v_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [X]$

if v_i orthogonal

$c_i = v_i^TX$

$= \dfrac{v_i^TX}{v_i^Tv_i}$

- If given independent vectors $a_1, a_2, a_3 \cdots$
  - $\rightarrow$ Find the orthonormal basis vectors
  - $\Rightarrow$ Gram-Schmidt orthogonalization

◎ Gram-Schmidt Orthogonalization

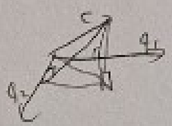
1) $a \rightarrow \dfrac{a}{\|a\|} = q_1$

2) project b onto q_1

$b - (q_1^Tb)q_1 \perp q_1$

$\dfrac{b - (q_1^Tb)q_1}{\|b - (q_1^Tb)q_1\|} = q_2$

$b = (q_1^Tb)q_1 + (q_2^Tb)q_2$



$c - \underbrace{\left((q_1^Tc)q_1 + (q_2^Tc)q_2\right)}_{}\perp \begin{matrix} q_1 \\ q_2 \end{matrix}$

$\downarrow$ normalization

q_3

$c = (q_1^Tc)q_1 + (q_2^Tc)q_2 + (q_3^Tc)q_3$

1) $q_1 = \dfrac{a}{\|a\|}$ 2) $a_j - \sum_{i=1}^{j-1}(q_i^Ta_j)q_i = A_j$ 3) $\dfrac{A_j}{\|A_j\|} = q_j$ $\therefore a_j = \sum_{i=1}^{j}(q_i^Ta_j)q_i$

8차시-일반 최소제곱법과 QR분할

- Let $q_1, q_2, \cdots, q_n$ be orthonormal

$q_i^Tq_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$Q = [q_1 q_2 \cdots q_n]$

$Q^TQ = \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix}\begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} = I$

$\Rightarrow Q^T = Q^{-1}$ (Left-inverse)

◎ Q examples

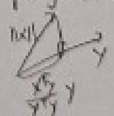
1) Rotation matrix
2) Permutation matrix

◎ Q transformation preserves the length and angle.

$\Rightarrow \|X\|^2 = X^TX \quad \|QX\|^2 = X^TQ^TQX = X^TX$

$\Rightarrow X^TY \quad (QX)^T(QY) = X^TQ^TQY = X^TY$

- Projection reduces the length



$\|X^Ty\| \leq \|X\|\|y\| \quad \|PX\| \leq \|X\|$

- for $q_1, q_2, \cdots, q_n \in \mathbb{R}^n$ (square sys)

$\Rightarrow X = \sum_{i=1}^{n} c_iq_i$

$X = [q_1 q_2 \cdots q_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = (Q)^{-1}(X) = Q^TX = Q^T\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$= \begin{bmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{bmatrix}X \quad c_i = \dfrac{q_i^TX}{q_i^Tq_i = 1}$

- for $q_1, q_2, \cdots, q_n \in \mathbb{R}^m$ (Rectangular sys)

$Q = [q_1 q_2 \cdots q_n] \quad m < n$

$Qx = b$

$x = \underbrace{(Q^TQ)^{-1}}_{I}Q^Tb$

$\boxed{Q}\boxed{Q^T} = \boxed{\begin{matrix} I & 0 \\ 0\,0\,0 & \end{matrix}}$

$n \times m \quad m \times n$

$\boxed{Q^T}\boxed{Q} = \boxed{I}$

Q: Left-inverse for rectangular system

- A: QR factorization

$\underset{A}{[a_1 a_2 \cdots a_n]} = \begin{bmatrix} (q_1^Ta_1)q_1 & (q_1^Ta_2)q_1 & \cdots & (q_1^Ta_n)q_1 \\ & +(q_2^Ta_2)q_2 & \cdots & +(q_2^Ta_n)q_2 \\ & & & \vdots \\ & & & +(q_n^Ta_n)q_n \end{bmatrix}$

$= [q_1 q_2 \cdots q_n]\begin{bmatrix} (q_1^Ta_1) & (q_1^Ta_2) & \cdots & (q_1^Ta_n) \\ 0 & (q_2^Ta_2) & \cdots & (q_2^Ta_n) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & (q_n^Ta_n) \end{bmatrix}$

$Ax = b$

$X = (A^TA)^{-1}A^Tb$

$= (R^T\underbrace{Q^TQ}_{I}R)^{-1}R^TQ^Tb$

$= (R^TR)R^TQ^Tb$

11차시 고유값과 고유벡터 및 대각화

$\underset{n\times n}{A}x = \lambda X \rightarrow$ eigenvector

$\downarrow$

scalar multiplication

$\rightarrow$ eigenvalue

$\Rightarrow (A-\lambda I)X = 0$

for non-zero $X \rightarrow N(A-\lambda I)$

$\det(A-\lambda I) = 0$

$\rightarrow$ singular

$\rightarrow$ how to find null space (eigenvectors)

$\rightarrow$ Row Reduced Echelon form!

- for triangular (or diagonal) matrix

$\rightarrow$ eigenvalues = diagonal elements of A

$\det A = \prod_{i=1}^{n} \text{pivot}_i = \prod_{i=1}^{n} d_i = \prod_{i=1}^{n} \lambda_i$

- Trace of $A = \sum_{i=1}^{n} a_{ii} = (a_{11} + a_{22} + \cdots + a_{nn})$

5.2 Diagonalization of matrix

1) $A = LU$ 2) $A = QR$

3) $A = S\Lambda S^{-1}$ $S = [e_1\ e_2 \cdots e_n]$ $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$Ae_i = \lambda_i e_i$

$[Ae_1\ Ae_2 \cdots Ae_n] = [\lambda_1 e_1\ \lambda_2 e_2 \cdots \lambda_n e_n]$

$\rightarrow A[e_1\ e_2 \cdots e_n] = [e_1\ e_2 \cdots e_n]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

$AS = S\Lambda$

$\therefore A = S\Lambda S^{-1}$

Remark 1>

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are different

then, $e_1, e_2, \cdots, e_n$ are linearly independent!

Remark 2>

S is not unique

since $ke \rightarrow$ eigenvector

Remark 3>

The order of eigenvalues is same with that of eigenvectors

Remark 4>

Not all matrices have n linearly independent eigenvectors

$\rightarrow A = S\Lambda S^{-1}$ is not always established.

- Power

$A \rightarrow \lambda, e$

$A^k \rightarrow \lambda^k, e$

$Ae = \lambda e$

$A^2 e = \lambda A e = \lambda^2 e$

$A^k = (S\Lambda S^{-1})^k$

$= S\Lambda^k S^{-1}$

$\downarrow$

$\begin{bmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \ \lambda_n^k \end{bmatrix}$