2015005187 최철훈

2차시 - LU 분할

$Ax=b$ # of unknowns = # of equations

$E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$

ex) $2u+v+w=5$ ①
$4u-6v=-2$ ②
$-2u+7v+2w=9$ ③

$$\overset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}} \overset{x}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}} = \overset{b}{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}} \Rightarrow \begin{array}{l} 2u+v+w=5 \\ -8v-2w=-12 \cdots ②-2\times① \\ 8v+3w=14 \cdots ③+① \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ 0 & 8 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} 2u+v+w=5 \\ -8v-2w=-12 \\ w=2 \cdots ③+② \end{array} \quad \overset{E_{32}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = \overset{U}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}$$

$E_{32}E_{31}E_{21}A = U$ Upper triangular matrix

$A = \underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}_{L}U = LU$ Lower triangular matrix → 가우스 소거법으로 어떤 과정을 거쳤는지 알 수 있음

1) LU factorization is unique!

2) $Ax=b \to L^{-1}Ax = L^{-1}b = c$
$\to Ux=c \to Lc=b$

3) $\det(A) = \underset{1}{\det(L)} \times \underset{\text{대각성분곱}}{\det(U)}$

4) $U = \begin{bmatrix} d_1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & d_2 & & & \vdots \\ 0 & & & \ddots & d_n \end{bmatrix} = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots\ d_n \end{bmatrix}\begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdots & \frac{u_{1n}}{d_1} \\ & 1 & \frac{u_{23}}{d_2} & & \vdots \\ 0 & & & \ddots & 1 \end{bmatrix} = DU \quad A=LU=LDU$

$D^n = \begin{bmatrix} d_1^n & & 0 \\ & d_2^n & \\ 0 & & \ddots\ d_n^n \end{bmatrix}$

◎ Row Exchange (Pivoting)

- Permutation Matrix
  - → has the same rows with I
  - → There is a single "1" in every row and column

$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad P^{-1} = P^T$

$A = LU$
$PA = LU$
$A = P^TLU$ → 피벗이 0이 아닌 non-singular만 가능

4차시 - 벡터공간과 열벡터 공간

of unknowns > # of equations ⇒ infinitely many solution or no solution

◎ Vector Space & Subspace

- Space ⇒ set closed under addition & scalar multiplication
  - for any vectors $x, y \in \mathbb{R}^n$
  - for any scalar $c \in \mathbb{R}$

$x, y \in V \ \begin{pmatrix} x+y \in V \\ cx \in V \end{pmatrix} c_1x + c_2y \in V$ (vector space)

1) $x+y = y+x$ 2) $x+(y+z) = (x+y)+z$ 3) There is a zero-vector such that $x+0=0+x=x$ 4) For each vector x, $x+(-x) = (-x)+x = 0$ ⇒ $-x$ unique! 5) $1 \cdot x = x$ 6) $c(x+y) = cx+cy$
7) $(c_1+c_2)x = c_1x + c_2x$

- Subspace
  - subset of the whole V.S. that satisfies the condition of V.S.

V, S, 0

$s_1, s_2 \in S \subset V$
$c_1s_1 + c_2s_2 \in S$

ex) Lower triangular Matrix in $\mathbb{R}^{n\times n}$ (Upper)

ex) $y = mx \ (m \neq 0) \ (x,y) \in \mathbb{R}^2$

◎ Column Space of A ($C(A)$)

⇒ set of all linear combinations from column vectors in A

$A = [a_1\ a_2 \cdots a_n] \Rightarrow \{\sum_{i=1}^{n} c_i a_i\}$

$Ax=b$
$\Rightarrow [a_1\ a_2 \cdots a_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$
$= x_1a_1 + x_2a_2 + \cdots + x_na_n = b$

- if $b \in C(A)$
  then, there is at least one solution $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

else if $b \notin C(A)$ then, no solution

if $b_1, b_2 \in C(A)$
$Ax_1 = b_1$
$Ax_2 = b_2$ $\quad Ax_1 + Ax_2 = A\underbrace{(x_1+x_2)}_{X} = b$
$b_1 + b_2 = b$

$Ax_1 = b_1$ $\quad A(cx_1) = cb_1 = b$
$cb_1 = b$

4차시 - 영벡터 공간과 해집합

$Ax=b$ if A^{-1} exist, always $b \in C(A) \Rightarrow C(A)$: whole space

whole space is constructed by linear combinations → span

◎ Null Space of A ($N(A)$)

⇒ set of vectors such that $Ax=0$

$N(A)=\{x \mid Ax=0\}$

(1) closed under addition

for $Ax_1=0$, $Ax_2=0$

$x_1+x_2 \in N(A)$

$Ax_1+Ax_2=0$

$A(x_1+x_2)=0 \quad x_1+x_2 \in N(A)$

(2) closed under Scalar multiplication

for $Ax=0$ for any c

$cx \in N(A)$

$\rightarrow A(cx)=0$

$cAx=0$

$cx \in N(A)$

◎ Solving $Ax=0$ & $Ax=b$

◦ Echelon form U

ex) $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $C(A) \subset \mathbb{R}^m$, $N(A) \subset \mathbb{R}^n$

Row Reduced form R

$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \underset{\text{all pivots}=1}{\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $Ax=0 \rightarrow Ux=0 \rightarrow Rx=0$

$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

◦ pivot variables: u, w

◦ free variables: v, z

$u+3v-z=0$, $w+z=0$ $\Rightarrow$ $u=-3v+z$, $w=-z$

$\begin{bmatrix} u \\ v \\ w \\ z \end{bmatrix}=\begin{bmatrix} -3v+z \\ v \\ -z \\ z \end{bmatrix}=v\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}+z\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \in N(A)$

e.g. 식을 만족하는 특정 벡터로 나타낸 식인데, special solution

$A^Ty=0$

column space - left null space

null space - row space

$\mathrm{Dim}(N(A))=2$

나머지 2차원은 Rowspace

7차시 - 벡터의 직교성과 직선투영

◎ Orthogonal vectors & Subspace

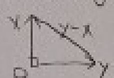
◦ why orthogonal vectors?

⇒ independent basis

⇒ easy calculation in linear combination

◦ length of vectors $\|x\| \Rightarrow \|x\|^2 = x^Tx=\sum_{i=1}^{n} x_i^2$

◦ By Pythagoras Theorem



$\|y-x\|^2=\|x\|^2+\|y\|^2$

$(y-x)^T(y-x)=x^Tx+y^Ty$

$\rightarrow y^Ty-y^Tx-x^Ty+x^Tx=x^Tx+y^Ty$

$x^Ty=y^Tx$

$x^Ty=0 \quad \therefore \vec{x}\cdot\vec{y}=0$

◦ for vector innerproduct x^Ty

1) $x^Ty=0 \rightarrow$ angle $=90°$

2) $x^Ty<0 \rightarrow$ angle $>90°$

3) $x^Ty>0 \rightarrow$ angle $<90°$

◦ If non-zero vectors $v_1, v_2, \cdots, v_n$ are orthogonal

$v_i^Tv_j=0$ $(i \neq j)$, then the vectors are linearly independent!

→ 가우스소거에서 나오는 피벗의 수가 벡터의 수와 같으면 independent

$c_1v_1+c_2v_2+\cdots+c_nv_n=0 \rightarrow v_i^T(c_1v_1+c_2v_2+\cdots+c_nv_n)=0$

$\Rightarrow c_1=c_2=\cdots=c_n=0$

$c_i\|v_i\|^2=0$ for $v_i \rightarrow c_i=0$

⇒ orthogonal vectors → basis vectors

$x=\sum_{i=1}^{n} c_iv_i \quad v_i^Tx=c_i\|v_i\|^2 \quad \|v_i\|=1 \rightarrow$ orthonormal

◎ Orthogonal Subspaces

⇒ Every vector in one subspace is orthogonal to every vector in the other subspace

ex) $\mathbb{R}^3$

◎ Row space ⊥ Null Space ◎ Column space ⊥ Left Null Space ◎ Dim of V.S = # of independent vectors to span the V.S

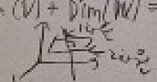
= rank of A (r)

$\mathrm{Dim}(C(A))+\mathrm{Dim}(N(A^T))=m$, with $\mathrm{Dim}(C(A))=r$, $\mathrm{Dim}(N(A^T))=m-r$

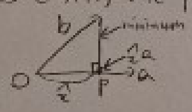
$\mathrm{Dim}(C(A^T))+\mathrm{Dim}(N(A))=n$, with $\mathrm{Dim}(C(A^T))=r$, $\mathrm{Dim}(N(A))=n-r$

◎ Orthogonal complement subspace → $V \perp W$ and $\mathrm{Dim}(V)+\mathrm{Dim}(W)=n$

$v \in V, w \in W$ in $\mathbb{R}^n$ → 해당 벡터공간을 스팬하지는 않음



◎ Cosines and Projection onto line



$(b-\hat{x}a) \perp a$

$a^T(b-\hat{x}a)=0$

$a^Tb=a^T\hat{x}a$

$\hat{x}=\frac{a^Tb}{a^Ta}$

◦ Project b onto a

$P=\hat{x}a=\frac{a^Tb}{a^Ta}a=\underbrace{\frac{aa^T}{a^Ta}}_{\mathbb{P}\text{(Projection matrix)}}b \rightarrow \mathbb{P}b=P$ (linear transform)

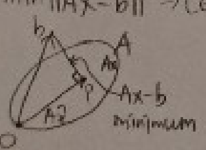
$\mathbb{P}^T=\mathbb{P}$

$\rightarrow \mathbb{P}^2=\mathbb{P}$

9차시 - 벡터투영과 최소제곱법

# of unknowns < # of equations

$\min\|Ax-b\|^2 \rightarrow$ Least square



$b-A\hat{x} \perp a_i \rightarrow$ column vectors in A

$a_1^T(b-A\hat{x})=0$

$a_2^T(b-A\hat{x})=0$

$\vdots$

$a_n^T(b-A\hat{x})=0$

$A^T(b-A\hat{x})=0$

$A^Tb=A^TA\hat{x}$

$\hat{x}=(A^TA)^{-1}A^Tb$

best estimate

pseudo inverse

$P=A\hat{x}$

$=A(A^TA)^{-1}A^Tb$

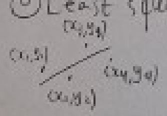
$=\mathbb{P}b$

1) If $b \in C(A)$

$\mathbb{P}b=b$

2) $\mathbb{P}^2=\mathbb{P}$

◎ Least square for line fitting



$y_1=ax_1+b$

$y_2=ax_2+b$

$\vdots$

$y_n=ax_n+b$

$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

$Ax-b=\begin{bmatrix} ax_1+b-y_1 \\ ax_2+b-y_2 \\ \vdots \\ ax_n+b-y_n \end{bmatrix}$

$\min\|Ax-b\|^2=\sum_{i=1}^{n}(ax_i+b-y_i)^2$

$f(x)=(x-a_1)^2+(x-a_2)^2+\cdots+(x-a_n)^2$

$\min f(x) \Rightarrow x=\frac{\sum_{i=1}^{n} a_i}{n}$ 평균