Session 5 SMV: CTL and MCCTL

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1 Extended syntax: CTL Formula

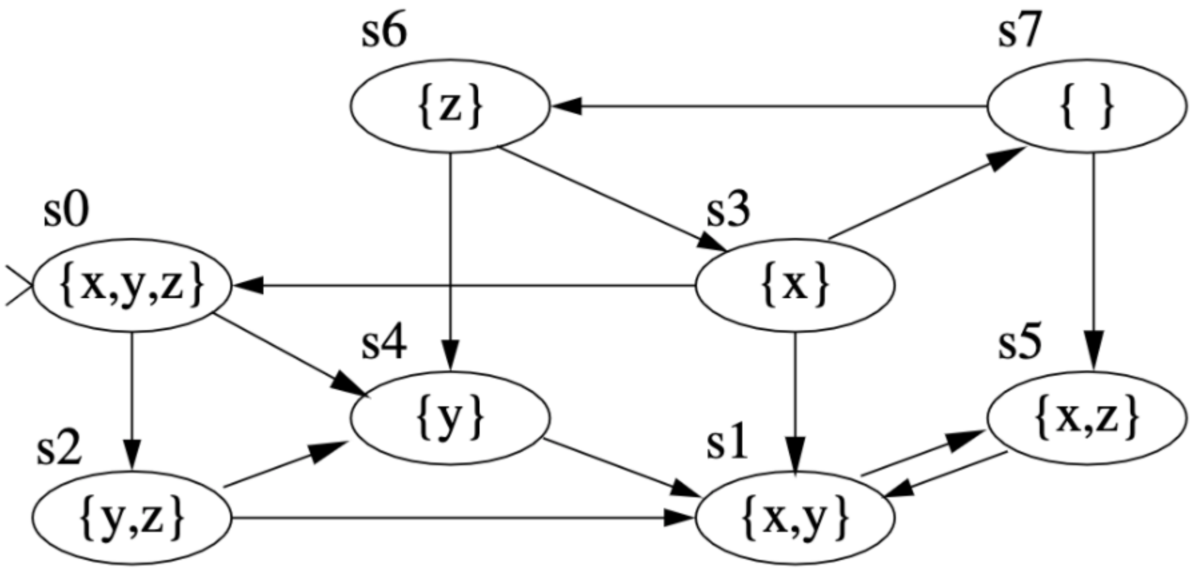
$$
\begin{aligned}
\phi_1 \wedge \phi_2 &\equiv \neg(\neg\phi_1 \vee \neg\phi_2) \\
\mathbf{AX}\,\phi &\equiv \neg\,\mathbf{EX}\,\neg\phi \\
\mathbf{true} &\equiv a \vee \neg a \\
\mathbf{AG}\,\phi &\equiv \neg\,\mathbf{EF}\,\neg\phi \\
\mathbf{false} &\equiv \neg\mathbf{true} \\
\mathbf{AF}\,\phi &\equiv \neg\,\mathbf{EG}\,\neg\phi \\
\phi_1\ \mathbf{EW}\ \phi_2 &\equiv \mathbf{EG}\,\phi_1 \vee (\phi_1\ \mathbf{EU}\ \phi_2) \\
\phi_1\ \mathbf{AW}\ \phi_2 &\equiv \neg(\neg\phi_2\ \mathbf{EU}\ \neg(\phi_1 \vee \phi_2)) \\
\mathbf{EF}\,\phi &\equiv \mathbf{true}\ \mathbf{EU}\ \phi \\
\phi_1\ \mathbf{AU}\ \phi_2 &\equiv \mathbf{AF}\,\phi_2 \wedge (\phi_1\ \mathbf{AW}\ \phi_2))
\end{aligned}
$$

In CTL formula, the extended syntax allows new operators to be introduced using existing operators. The idea is quite similar as the Boolean operators, which helps you express formulas in an easier way. In addition, implementing these extended operators can be translated into the basic operators. Care, these new operators are **not more expressive**.
For instance, $A \vee B$ (where A and B are predicates), which is equivalent to $\neg(\neg A \wedge \neg B)$.

1. Normalize formulas below (You do not need to rewrite true and false): Normalize means to get a form where there is no more extended syntax anymore.

    1. $true$ EW ϕ
    2. EF $true$
    3. AX $true \wedge$ AG $false$

2 Compute CTL formulas: Intuition

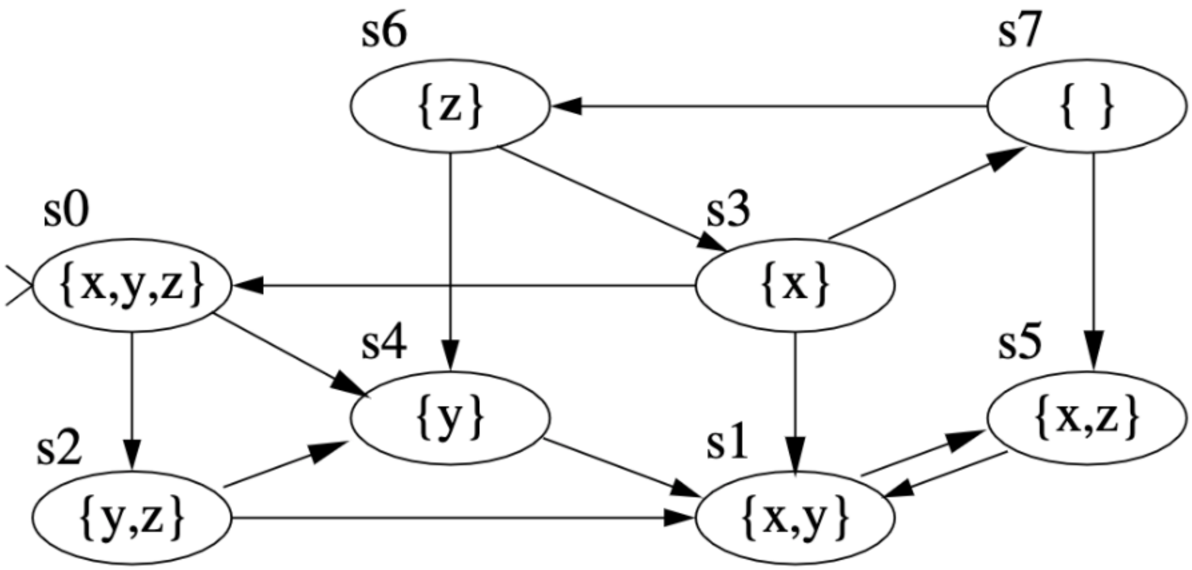


Let suppose that we have the following Kripke structure $\mathcal{K} = (S, \rightarrow, s^0, AP, \nu)$ where:

- $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$
- $\rightarrow = \{(s_0, s_2), (s_0, s_4), (s_1, s_5), (s_2, s_4), (s_2, s_1), (s_3, s_0), (s_3, s_1), (s_3, s_7), (s_4, s_1), (s_5, s_1), (s_6, s_3), (s_6, s_4), (s_7, s_5), (s_7, s_6)\}$
- $s^0 = s_0$
- $AP = \{x, y, z\}$
- $\nu(x) = \{s_0, s_1, s_3, s_5\}, \nu(y) = \{s_0, s_1, s_2, s_4\}, \nu(z) = \{s_0, s_2, s_5, s_6\}$

1. Compute the following CTL formulas:
    1. $[\![z]\!]_{\mathcal{K}}$
    2. $[\![x \vee y]\!]_{\mathcal{K}}$
    3. $[\![EX\ x]\!]_{\mathcal{K}}$
    4. $[\![AX\ x]\!]_{\mathcal{K}}$
    5. $[\![AG\ AF\ x]\!]_{\mathcal{K}}$
    6. $[\![\mathrm{E}[z\ U\ x]]\!]_{\mathcal{K}}$
    7. $[\![\mathrm{A}[z\ U\ x]]\!]_{\mathcal{K}}$
    8. $[\![EG\ \neg y]\!]$
    9. $[\![\mathrm{E}[x\ U\ (EG\ \neg y)]]\!]$

3 Compute CTL formulas: Fix point



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$$
\begin{aligned}
[\![\mathbf{EF}\,\phi]\!] &= \mu Y.[\![\phi]\!] \cup pre_{\exists}(Y) \\
[\![\mathbf{EG}\,\phi]\!] &= \nu Y.[\![\phi]\!] \cap pre_{\exists}(Y) \\
[\![\mathbf{AF}\,\phi]\!] &= \mu Y.[\![\phi]\!] \cup pre_{\forall}(Y) \\
[\![\mathbf{AG}\,\phi]\!] &= \nu Y.[\![\phi]\!] \cap pre_{\forall}(Y) \\
[\![\mathbf{E}[\phi U \psi]]\!] &= \mu Y.[\![\psi]\!] \cup ([\![\phi]\!] \cap pre_{\exists}(Y)) \\
[\![\mathbf{A}[\phi U \psi]]\!] &= \mu Y.[\![\psi]\!] \cup ([\![\phi]\!] \cap pre_{\forall}(Y))
\end{aligned}
$$

μ is for the least fix point, in this case the initial value of Y is an empty set $\{\}$.

ν is for the greatest fix point, in this case the initial value of Y is the set containing all states. For the above example: $\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

Example to compute: $[\![AG\, x]\!]_{\mathcal{K}}$:

- Step 1: $[\![x]\!] = \{s_0, s_1, s_3, s_5\}$
- Step 2: $[\![AG\, x]\!]_{\mathcal{K}} = \nu Y.[\![x]\!] \cap pre_{\forall}(Y)$
- Step 3: $\nu Y.\{s_0, s_1, s_3, s_5\} \cap pre_{\forall}(\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\})$ where $Y = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$

- Step 3.1: $pre_{\forall}(\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}) = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$
- Step 3.2: $\{s_0, s_1, s_3, s_5\} \cap \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = \{s_0, s_1, s_3, s_5\}$
- Step 4: $\nu Y.\{s_0, s_1, s_3, s_5\} \cap pre_{\forall}(\{s_0, s_1, s_3, s_5\})$ where $Y = \{s_0, s_1, s_3, s_5\}$
- Step 4.1: $pre_{\forall}(\{s_0, s_1, s_3, s_5\}) = \{s_1, s_4, s_5\}$
- Step 4.2: $\{s_0, s_1, s_3, s_5\} \cap \{s_1, s_4, s_5\} = \{s_1, s_5\}$
- Step 5: $\nu Y.\{s_0, s_1, s_3, s_5\} \cap pre_{\forall}(\{s_1, s_5\})$ where $Y = \{s_1, s_5\}$
- Step 5.1: $pre_{\forall}(\{s_1, s_5\}) = \{s_1, s_4, s_5\}$
- Step 5.2: $\{s_0, s_1, s_3, s_5\} \cap \{s_1, s_4, s_5\} = \{s_1, s_5\}$
- Result of 4.2 et 5.2 is the same, we can finish the algorithm.

1. Compute the following CTL formulas using the fix point operation.

   1. $[\![AF\ AG\ x]\!]_{\mathcal{K}}$
   2. $[\![EF\ x]\!]_{\mathcal{K}}$
   3. $[\![\mathrm{E}[z\ U\ x]]\!]_{\mathcal{K}}$
   4. $[\![\mathrm{A}[z\ U\ x]]\!]_{\mathcal{K}}$
   5. $[\![EG\ \neg y]\!]$
   6. $[\![\mathrm{E}[x\ U\ (EG\ \neg y)]]\!]$