Q) 10 - Explain the difference between the caracteristic function and moment generating function.

- Moments : mean, variance, skewness, kurtosis
  - 1^{st} mean, 2^{nd} variance
  - For continuous random variable (pdf) &
  - For discrete RV (pmf) → (distributions)

- For some distributions, only two first moments are sufficient (eg Gaussian dist) to completely define the distribution. pdf $\rightleftarrows$ moments

- For some other distribution, we can comput it moments but from the moment we can not find the pdf pdf → moment, ←X– from only few moments. (not enough to do an approximation).

- CF of a pdf: minus in Fourier T. $j = \sqrt{-1}$ complex fct

$$\Phi(t) = \int_{-}^{+} f(x) e^{jtx} dx = E_{f(x)}[e^{jtx}]$$

- Pdf of a CF (to recover the pdf)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+} \Phi(t) e^{-jtx} dt$$

- Moment generating function $M(t) = E_{f(x)}[e^{tx}]$ (reel function) no j

- Those exponents can be extended w the taylor serie expansion $\sum_{0}^{\infty} \frac{x^k}{k!}$

$$e(tx) = \sum_{n=0}^{\infty} \frac{t^n . x^n}{n!} \; ; \; e(jtx) = \sum_{n=0}^{\infty} \frac{(jt)^n . x^n}{n!}$$

- Then we can rewritte $\Phi(t)$ and $M(t)$:

$$\Phi(t) = 1 + \frac{jt E_{f(x)}[X]}{1} - \frac{t^2 E_{f(x)}[X^2]}{2!} + \dots + \frac{(jt)^n E_{f(x)}[X^n]}{n!}$$

(+/-)

$$M(t) = 1 + \underbrace{\frac{t E_{f_x}[X]}{1}}_{mean} + \underbrace{\frac{t^2 E_{f_x}[X^2]}{2!}}_{variance} + \dots + \frac{t^n E_{f_x}[X^n]}{n!}$$

(+ only)

(if we substract the mean ?)

⇒ We can get the caracteristic fct from the distribution f(x).

⇒ Sometimes we can cut the previous formula and use only few moment to approximate the distribution (other moments might be small or ≈ 0)

! Sometimes we don't have the distribution actually most of the time we have a lot of data but not the distribution. (Δ i.i.d data!)

⇒ We derive from the data an estimation of our moments :

- Nth moment estimation from data:

$$\hat{m}_n = \frac{1}{N} \sum_{i=1}^{N} x_i^n \; ; \; n \geq 1$$

When n is very large (lot of data) $\xrightarrow{cvg}$ true mean.

⇒ From data we can compute moments directly w/o knowing the true dist

⇒ From moment, i can approximate the CF (or moment gen fct) (idea)

⇒ From CF i can compute the pdf with the inverse Fourier T.

Apply: Steganalysis Fake detection

Summary

$$\begin{array}{ccc} pdf & \underset{Fourier^{-1}}{\overset{Fourier}{\longleftrightarrow}} & C.F \; \Phi(t) \\ \downarrow & & \downarrow \\ \text{Moment of pdf} & & \text{Moments of CF} \\ m_n = E[X^n] = \int_{-}^{+} f(x) x^n dx & & M_N = \int_{-}^{+} \Phi(t) t^N dt \\ \downarrow & & \downarrow \\ \text{Empirical estimation from iid data} & & \text{Empirical CF} \\ \hat{m}_n = \frac{1}{N} \sum_{i=1}^{N} x_i^n & & \end{array}$$

- Need DWT of PCA to have I.I.d data.

- Use moments for discrimination of data to distinguish btw two img for example (pdf will be the same)