Some details about Rewriting rules

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Substitution

Let’s suppose to know the natural specification with axioms:

1. $x + 0 = x$
2. $x + s(y) = s(x+y)$

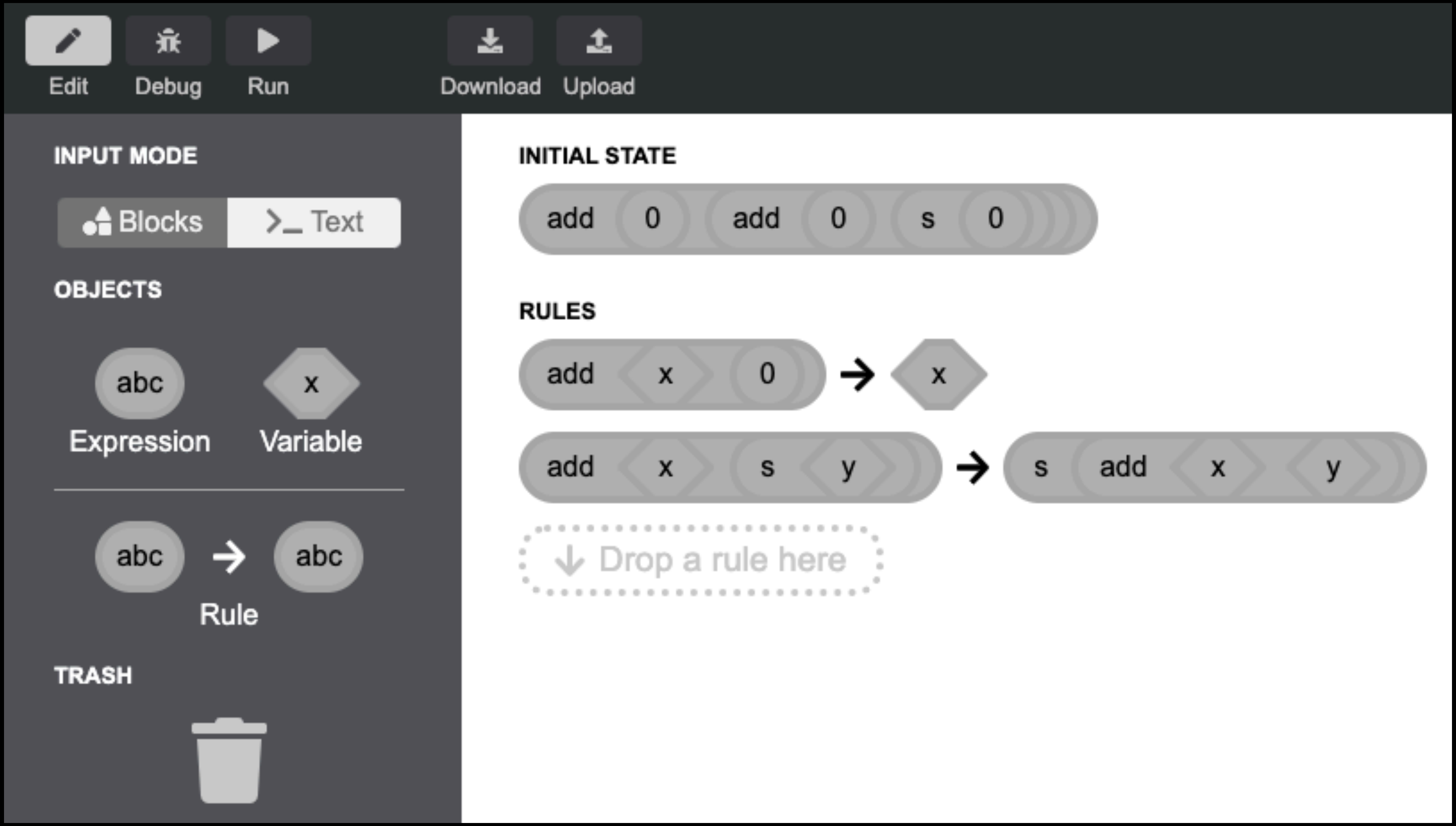
Substitution

We transform these axioms into rules:

1. $x + 0 \rightsquigarrow x$

2. $x + s(y) \rightsquigarrow s(x+y)$

FunBlock



https://blissful-bhaskara-4b1032.netlify.app/

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Substitution

How can we simplify the term

“s(s(0)+0)” using rules ?

Theory

Definition (Rewrite step)

Let $\Sigma = \langle S, F \rangle$ be a signature and X be a S-sorted set of variables and $l \rightsquigarrow r$,$l, r \in T_\Sigma(X)$ a rewrite rule.

- $filter(t, l) =< \sigma, c > \Leftrightarrow \exists \sigma \in S\ \exists c,$
  - $t = c[\sigma l]$ [a]
  - $t' = c[\sigma r]$
- $< t, t' > \in Rew_{l \rightsquigarrow r}$ a rewrite step

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a. $c[\_]$ denotes the context of a term, i.e. a term with a place holder

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Show that: s(s(0)+0) ↝ s(s(0))

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Rewriting rule: x + 0 ↝ x

σ = [s(0)/x] —> s(0) + 0 = s(0) (substitution of x by s(0))

c[s(0) + 0] = s(s(0) +0) where c = s(_) (where _ is the place holder)

c[s(0)] = s(s(0))

Show that: s(s(0)+0) ⤳ s(s(0))

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l = x + 0

t = s(s(0) + 0)

r = x

t’ = s(s(0))

Recap: σ = [s(0)/x], c[_] = s(_)

We obtain t ⤳ t’ ⇒ s(s(0)+0) ⤳ s(s(0))

When you **replace** a subterm in a rule,

think about these steps !

IT IS NOT MAGIC !

Training time !
Find the context and the substitution for examples below:

(1) $x + 0 \rightsquigarrow x$

(2) $x + s(y) \rightsquigarrow s(x + y)$

Ex 1: $s(0) + s(0) \rightsquigarrow s(s(0) + 0)$

Ex 2: $s(s(y) + 0) \rightsquigarrow s(s(y))$

Ex 3: $s(0 + s(x)) \rightsquigarrow s(s(0 + x))$

Termination and Confluence What is the problem ?

Random ADT

f: natural -> boolean;

Axiom:

(1) f(x) = b (where x:natural, b: boolean))

Termination and Confluence
What is the problem ?

(1) $x \rightsquigarrow x + 0$

(2) $x + s(y) \rightsquigarrow s(x + y)$