# Session 3 SMV: Equational Theories

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## 1 Equational theories: proofs

In this section, we focus on the equational theories and tools that are given to prove new theorems. To prove these new theorems, we will use following definitions:

- Axioms: An axiom can be used as a theorem.
- Reflexivity: A term is equal to itself and can be used to create a theorem.
- Symmetry: Look at the course definition
- Transitivity: Look at the course definition
- Substitutivity: Look at the course definition
- Substitution: Look at the course definition

We give the following ADT:

```
ADT Natural;
Interface Booleans
  Sorts natural;
  Operations
  Generator:
    0: -> natural;
    s _ : natural -> natural
  Modifier:
    _ + _: natural, natural -> natural
  Axioms:
    (1) x + 0 = x
    (2) x + s(y) = s(x + y)
  Where x,y: natural
```

*Proof.* Using the latter rules and ADT, let's prove that: $s(0+0) = s(0)$:

1. Using *Axioms* rule, we use axiom (1) as a theorem. $x + 0 = x \in Ax \Rightarrow x + 0 = x \in Th(Spec)$ **(3)**
2. Using the substitution on the precedent theorem (3) with the binding $[x := 0]$, we get: $0 + 0 = 0$ **(4)**
3. Using the substitutivity on (4) with the successor operation, we get: $s(0 + 0) = s(0)$

$\square$

1. Prove that:

```
a) s(0) = s(0)
b) s(x + y) = x + s(y)
c) 0 + s(s(0)) = s(0 + s(0))
d) s(0) + 0 = s(0 + 0)
e) s(0 + s(0)) + 0 = 0 + s(s(0))
```

## 2 Equational theories: inductive proofs

An induction proof is decomposed by the following steps:

- Write the hypothesis that you want to prove.
- Prove the basic case(s).
- Suppose that the hypothesis is true.
- Prove the successor case.
- Conclude

Example of an induction proof for: $0 + x = x$

*Proof.* `We want to prove that: 0 + x = x (Hypothesis) (3)`

```
First of all, let's prove the basic case which is: 0 + 0 = 0.
Apply substitution of x by 0 in theorem (1), the result is: 0 + 0 = 0.
So, the basic case is proved.

Let's suppose that the hypothesis 0 + x = x is true.
Let's prove that the successor case: 0 + s(x) = s(x) is true.

WARNING: Do not make any substitutions within the hypothesis,
until you have completely proven it.

Apply substitution of x by 0 and y by x in theorem (2):
x + s(y) = s(x + y) -> 0 + s(x) = s(0 + x) (4)

Apply the substitutivity with the operation successor on the hypothesis (3):
s(0 + x) = s(x) (5)

Apply the transitivity between (4) and (5):
(4) 0 + s(x) = s(0 + x)
(5) s(0 + x) = s(x)
-> 0 + s(x) = s(x)

Hence: 0 + x = x
```

□

1. Proof by induction (or not if you think it is possible):

```
0 + x = x + 0
s(x) + y = s(x + y)
x + s(0) = s(0) + x
```

2. Using all the theorems that you proved, prove:

```
x + y = y + x
```