# Indefinite integral I

一、 Find the following indefinite integral

(1) $\int\left(x^3+2x^2-5\sqrt{x}\right)dx$

Solution: $\int\left(x^3+2x^2-5\sqrt{x}\right)dx=\frac{1}{4}x^4+\frac{2}{3}x^3-\frac{10}{3}x^{\frac{3}{2}}+C$

(2) $\int(\sin x+3e^x)dx$

Solution: $\int(\sin x+3e^x)dx=-\cos x+3e^x+C$

(3) $\int(x^a+a^x)dx$

Solution: $\int(x^a+a^x)dx=\frac{1}{a+1}x^{a+1}+\frac{a^x}{\ln a}+C$

(4) $\int(2+\cot^2 x)dx$

Solution: $\int(2+\cot^2 x)dx=\int(1+\csc^2 x)dx=x-\cot x+C$

(5) $\int(2\csc^2 x-\sec x\tan x)dx$

Solution: $\int(2\csc^2 x-\sec x\tan x)dx=-2\cot x-\sec x+C$

(6) $\int(x^2-2)^3dx$

Solution: $\int(x^2-2)^3dx=\int(x^6-6x^4+12x^2-8)dx$

$$=\frac{1}{7}x^7-\frac{6}{5}x^5+4x^3-8x+C$$

(7) $\int\left(x+\frac{1}{x}\right)^2dx$

Solution: $\int\left(x+\frac{1}{x}\right)^2dx=\int(x^2+2+\frac{1}{x^2})dx=\frac{1}{3}x^3+2x-\frac{1}{x}+C$

(8) $\int\left(\sqrt{x}+\frac{1}{\sqrt[3]{x^2}}+1\right)\left(\frac{1}{\sqrt{x}}+1\right)dx$

Solution: $\int\left(\sqrt{x}+\frac{1}{\sqrt[3]{x^2}}+1\right)\left(\frac{1}{\sqrt{x}}+1\right)dx$

$=\int(1+x^{-\frac{7}{6}}+x^{-\frac{1}{2}}+x^{\frac{1}{2}}+x^{-\frac{2}{3}}+1)dx$

$=2x-6x^{-\frac{1}{6}}+2x^{\frac{1}{2}}+\frac{2}{3}x^{\frac{3}{2}}+3x^{\frac{1}{3}}+C$

(9) $\int\left(2^x+\frac{1}{3^x}\right)^2 dx$

Solution: $\int\left(2^x+\frac{1}{3^x}\right)^2 dx=\int\left[4^x+2\cdot\left(\frac{2}{3}\right)^x+\left(\frac{1}{9}\right)^x\right]dx$

$$=\frac{4^x}{2\ln 2}+\frac{2}{\ln 2-\ln 3}\left(\frac{2}{3}\right)^x-\frac{1}{2\ln 3}\left(\frac{1}{9}\right)^x+C$$

(10) $\int\frac{2\cdot 3^x-5\cdot 2^x}{3^x}dx$

Solution: $\int\frac{2\cdot 3^x-5\cdot 2^x}{3^x}dx=\int[2-5\cdot\left(\frac{2}{3}\right)^x]dx=2x-\frac{5}{\ln 2-\ln 3}\left(\frac{2}{3}\right)^x+C$

(11) $\int\frac{\cos 2x}{\cos x-\sin x}dx$

Solution: $\int\frac{\cos 2x}{\cos x-\sin x}dx=\int\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}dx$

$$=\int(\cos x+\sin x)dx$$

$$=\sin x-\cos x+C$$

(12) $\int\left(\frac{2}{1+x^2}-\frac{3}{\sqrt{1-x^2}}\right)dx$

Solution: $\int\left(\frac{2}{1+x^2}-\frac{3}{\sqrt{1-x^2}}\right)dx=2\arctan x-3\arcsin\ x+C$

(13) $\int(1-x^2)\sqrt{x\sqrt{x}}dx$

Solution: $\int(1-x^2)\sqrt{x\sqrt{x}}dx=\int(1-x^2)x^{\frac{3}{4}}dx$

$$=\int\left(x^{\frac{3}{4}}-x^{\frac{11}{4}}\right)dx$$

$$=\frac{4}{7}x^{\frac{7}{4}}-\frac{4}{15}x^{\frac{15}{4}}+C$$

(14) $\int\frac{\cos 2x}{\cos^2 x\sin^2 x}dx$

Solution: $\int\frac{\cos 2x}{\cos^2 x\sin^2 x}dx=\int\frac{\cos^2 x-\sin^2 x}{\cos^2 x\sin^2 x}dx$

$$=\int(\csc^2 x-\sec^2 x)dx$$

$$=-\cot x-\tan x+C$$

(15) $\int\frac{dx}{(x-a)^n}(n>1)$

Solution: $\int\frac{dx}{(x-a)^n}=\int\frac{d(x-a)}{(x-a)^n}\quad(u=x-a)$

$$=\int\frac{du}{u^n}$$

$$= -\frac{1}{n-1} \cdot \frac{1}{u^{n-1}} + C$$
$$= -\frac{1}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + C$$

(16) $\int \frac{dx}{x^2+a^2}$

Solution: $\int \frac{dx}{x^2+a^2} = \int \frac{a \cdot d\left(\frac{x}{a}\right)}{a^2\left[1+\left(\frac{x}{a}\right)^2\right]} \quad (u = \frac{x}{a})$

$$= \frac{1}{a}\int \frac{du}{1+u^2}$$
$$= \frac{1}{a}\arctan u + C$$
$$= \frac{1}{a}\arctan\frac{x}{a} + C$$

(17) $\int \tan x \, dx$

Solution: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$

$$= \int -\frac{d(\cos x)}{\cos x} \quad (u = \cos x)$$
$$= -\int \frac{du}{u}$$
$$= -\ln|u| + C$$
$$= -\ln|\cos x| + C$$

(18) $\int \sec x \, dx$

Solution: $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} \quad (u = \sec x + \tan x)$$
$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sec x + \tan x| + C$$

(19) $\int \frac{dx}{\sqrt{x}(1+x)}$

Solution: $\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{2 \cdot d(\sqrt{x})}{\left[1+\left(\sqrt{x}\right)^2\right]} \quad (u = \sqrt{x})$

$$= 2\int \frac{du}{(1+u^2)}$$

$$= 2\arctan u + C$$

$$= 2\arctan\sqrt{x} + C$$

(20) $\int \sin mx \cos nx \, dx (m \neq n)$

Solution: $\int \sin mx \cos nx \, dx = \int \frac{1}{2}[\sin(m+n)x + \sin(m-n)x] dx$

$$= -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right] + C$$

(21) $\int \sqrt{a^2 - x^2} dx$

Solution: $\int \sqrt{a^2 - x^2} dx \quad \left(x = a\sin t, \sin t = \frac{x}{a}, t = \arcsin\frac{x}{a}\right)$

$$= \int \sqrt{a^2 - a^2\sin^2 t} \cdot a\cos t \, dt$$

$$= \int a^2 \cos^2 t \, dt$$

$$= \frac{a^2}{2}\int (1 + \cos 2t) dt$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C \quad (\cos t = \sqrt{1 - \frac{x^2}{a^2}}, \sin 2t = \frac{2x\sqrt{a^2-x^2}}{a^2})$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

(22) $\int \frac{dx}{\sqrt{x^2-a^2}}$

Solution: $\int \frac{dx}{\sqrt{x^2-a^2}} \quad (x = a\sec t, \sec t = \frac{x}{a})$

$$= \int \frac{a\sec t \tan t}{\sqrt{a^2\sec^2 t - a^2}} dt$$

$$= \int \frac{a\sec t \tan t}{a\tan t} dt$$

$$= \int \sec t \, dt$$

$$= \ln|\sec t + \tan t| + C \quad \left(\tan t = \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right)$$

$$= \ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right| + C$$

$$= \ln|x + \sqrt{x^2 - a^2}| + C$$

(23) $\int \frac{dx}{\sqrt{x^2+a^2}}$

Solution: $\int \frac{dx}{\sqrt{x^2+a^2}}$ $(x = a\tan t\,, \tan t = \frac{x}{a})$

$= \int \frac{a\sec^2 t dt}{\sqrt{a^2\tan^2 t + a^2}}$

$= \int \frac{a\sec^2 t dt}{a\sec t}$

$= \int \sec t\, dt$

$= \ln|\sec t + \tan t| + C$ $(\sec t = \sqrt{\left(1+\left(\frac{x}{a}\right)^2\right)})$

$= \ln\left|\sqrt{\left(1+\left(\frac{x}{a}\right)^2\right)} + \frac{x}{a}\right| + C$

$= \ln\left|\sqrt{x^2+a^2} + x\right| + C$

(24) $\int x(2x-1)^{100}dx$

Solution: $\int x(2x-1)^{100}dx$ $(t = 2x-1, x = \frac{t+1}{2}, dx = \frac{1}{2}dt)$

$= \int \frac{t+1}{2} \cdot t^{100} \cdot \frac{1}{2}dt$

$= \frac{1}{4}\int (t^{101} + t^{100})dt$

$= \frac{1}{4}\left(\frac{1}{102}t^{102} + \frac{1}{101}t^{101}\right) + C$

$= \frac{(2x-1)^{101}}{4}\left(\frac{2x-1}{102} + \frac{1}{101}\right) + C$

(25) $\int \frac{dx}{x^2\sqrt{1+x^2}}$

Solution: $\int \frac{dx}{x^2\sqrt{1+x^2}}$ $(x = \tan t)$

$= \int \frac{\sec^2 t\, dt}{\tan^2 t\sqrt{1+\tan^2 t}}$

$= \int \frac{\sec t\, dt}{\tan^2 t}$

$= \int \frac{\cos t\, dt}{\sin^2 t}$

$= \int \frac{d(\sin t)}{\sin^2 t}$ $(u = \sin t\,, \sin t = \frac{x}{\sqrt{1+x^2}})$

$= \int \frac{du}{u^2}$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\sin t} + C$$

$$= -\frac{\sqrt{1+x^2}}{x} + C$$

(26) $\int x \cos x \, dx$

Solution: $\int x \cos x \, dx = \int x \, d(\sin x)$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

(27) $\int x^2 \, e^x dx$

Solution: $\int x^2 \, e^x dx = \int x^2 \, d(e^x)$

$$= x^2 e^x - 2\int x \, e^x dx$$

$$= x^2 e^x - 2xe^x + 2\int e^x \, dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

(28) $\int \ln x \, dx$

Solution: $\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx$

$$= x \ln x - x + C$$

(29) $\int x \arctan x \, dx$

Solution: $\int x \arctan x \, dx = \frac{1}{2}\int \arctan x \, d(x^2)$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{x^2+1} dx$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}x + C$$

(30) $\int \frac{x}{1+\cos x} dx$

Solution: $\int \frac{x}{1+\cos x} dx = \int \frac{x}{2\cos^2\frac{x}{2}} dx$

$$= \int \frac{1}{2} x \sec^2 \frac{x}{2} dx$$

$$= \int x d(\tan \frac{x}{2})$$

$$= x \tan \frac{x}{2} - \int \tan \frac{x}{2} dx$$

$$= x \tan \frac{x}{2} + 2 \ln |\cos \frac{x}{2}| + C$$

(31) $\int e^x \sin x \, dx$

Solution: $\int e^x \sin x \, dx = \int \sin x \, d(e^x)$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{1}{2}(\sin x - \cos x) e^x + C$$

(32) $\int \sqrt{x^2 + a^2} \, dx$

Solution: $\int \sqrt{x^2 + a^2} \, dx = x\sqrt{x^2 + a^2} - \int x \cdot \frac{2x}{2\sqrt{x^2+a^2}} dx$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}} dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2} \, dx +$$

$$a^2 \int \frac{1}{\sqrt{x^2+a^2}} dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2} \, dx +$$

$$a^2 \ln |x + \sqrt{x^2 + a^2}| + C$$

$$\Rightarrow \int \sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln |x + \sqrt{x^2 + a^2}| + C$$

(33) Find $I_n = \int \frac{dx}{(x^2+a^2)^n}$

Solution: $I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2} \int \frac{x^2+a^2-x^2}{(x^2+a^2)^n} dx$$

$$= \frac{1}{a^2} I_{n-1} - \frac{1}{a^2} \int \frac{x^2}{(x^2+a^2)^n} dx$$

$$= \frac{1}{a^2} I_{n-1} + \frac{1}{a^2} \int \frac{1}{2(n-1)} x d\left(\frac{1}{(x^2+a^2)^{n-1}}\right)$$

$$= \frac{1}{a^2} I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2+a^2)^{n-1}} -$$

$$\frac{1}{2a^2(n-1)} \int \frac{1}{(x^2+a^2)^{n-1}} dx$$

$$= \frac{2n-3}{2a^2(n-1)} I_{n-1} + \frac{1}{2a^2(n-1)} \cdot \frac{x}{(x^2+a^2)^{n-1}}$$

(34) $\int (x+1)\sqrt{x^2-2x+5}\, dx$

Solution: $\int (x+1)\sqrt{x^2-2x+5}\, dx$

$$= \int (x-1)\sqrt{x^2-2x+5}\, dx + 2\int \sqrt{x^2-2x+5}\ dx$$

$$= \frac{1}{2}\int \sqrt{x^2-2x+5}\ d(x^2-2x+5) + 2\int \sqrt{(x-1)^2+4}\ dx$$

$$= \frac{1}{3}(x^2-2x+5)^{\frac{3}{2}} + 2\int 2\sec t \cdot 2\sec^2 t\, dt \quad (x-1 = 2\tan t)$$

$$= \frac{1}{3}(x^2-2x+5)^{\frac{3}{2}} + 8\int \sec^3 t\, dt$$

$$(\tan t = \frac{x-1}{2}, \sec t = \sqrt{1+\left(\frac{x-1}{2}\right)^2} = \frac{1}{2}\sqrt{x^2-2x+5})$$

$$\int \sec^3 t\, dt = \int \sec t\, d(\tan t)$$

$$= \sec t \tan t - \int \sec t \tan^2 t\, dt$$

$$= \sec t \tan t - \int \sec t\,(\sec^2 t - 1) dt$$

$$= \sec t \tan t - \int \sec^3 t\, dt + \int \sec t\, dt$$

$$= \sec t \tan t - \int \sec^3 t\, dt + \ln|\sec t + \tan t|$$

$$\Rightarrow \int \sec^3 t\, dt = \frac{1}{2}\sec t \tan t + \frac{1}{2}\ln|\sec t + \tan t| + C$$

$$\Rightarrow \int (x+1)\sqrt{x^2-2x+5}\, dx$$

$$= \frac{1}{3}(x^2-2x+5)^{\frac{3}{2}} + 4\sec t \tan t + 4\ln|\sec t + \tan t| + C$$

$$= \frac{1}{3}(x^2-2x+5)^{\frac{3}{2}} + (x-1)\sqrt{x^2-2x+5} +$$

$$4\ln\left|\frac{1}{2}\sqrt{x^2-2x+5} + \frac{x-1}{2}\right| + C$$

(35) $\int \frac{dx}{\sqrt{x^2+2ax+b^2}}$

Solution: $\int \frac{dx}{\sqrt{x^2+2ax+b^2}}$ $(x^2+2ax+b^2=(x+a)^2+b^2-a^2)$

$$= \int \frac{dx}{\sqrt{(x+a)^2+b^2-a^2}}$$

$$= \int \frac{d(x+a)}{\sqrt{(x+a)^2 \pm |b^2-a^2|}}$$

$$= \ln\left|(x+a)+\sqrt{x^2+2ax+b^2}\right| + C$$

(36) $\int \frac{4x^3-13x^2+3x+8}{(x+1)(x-2)(x-1)^2} dx$

Solution: $\frac{4x^3-13x^2+3x+8}{(x+1)(x-2)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$

$$\Rightarrow 4x^3-13x^2+3x+8 = A(x-2)(x-1)^2 +$$

$$B(x+1)(x-1)^2 + C(x+1)(x-2)(x-1) + D(x+1)(x-2)$$

$$Let\ x=-1, \Rightarrow -1 = 12A \Rightarrow A = 1$$

$$x = 2 \ , \Rightarrow -6 = 3B \ \ \Rightarrow B = -2$$

$$x = 1 \ , \Rightarrow 2 = -2D \ \ \Rightarrow D = -1$$

$$Consider\ x^3, \Rightarrow 4 = A+B+C \Rightarrow C = 5$$

$$\Rightarrow \int \frac{4x^3-13x^2+3x+8}{(x+1)(x-2)(x-1)^2} dx = \int \left(\frac{1}{x+1} - \frac{2}{x-2} + \frac{5}{x-1} - \frac{1}{(x-1)^2}\right) dx$$

$$= \ln\left|\frac{(x+1)(x-1)^5}{(x-2)^2}\right| + \frac{1}{x-1} + C$$

(37) $\int \frac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)} dx$

Solution: $\frac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$$\Rightarrow x^4+x^3+3x^2-1 = A(x^2+1)^2 + (Bx+C)(x-1)$$

$$(x^2+1) + (Dx+E)(x-1)$$

$$Let\ x = 1, \Rightarrow 4 = 4A \Rightarrow A = 1$$

$$Consider\ x^4, \Rightarrow 1 = A + B \Rightarrow B = 0$$

$$Consider\ x^3, \Rightarrow 1 = C \Rightarrow C = 1$$

$$Let\ x^2 = -1 (x = i) \Rightarrow -3 - i = -(D+E) + (E-D)i$$

$$\Rightarrow \begin{cases} D+E=3 \\ D-E=1 \end{cases} \Rightarrow \begin{cases} D=2 \\ E=1 \end{cases}$$

$$\Rightarrow \int \frac{x^4+x^3+3x^2-1}{(x^2+1)^2(x-1)} dx$$

$$= \int \left(\frac{1}{x-1} + \frac{1}{x^2+1} + \frac{2x+1}{(x^2+1)^2}\right) dx$$

$$= \ln|x-1| + \arctan x + \int \frac{2x}{(x^2+1)^2} dx + \int \frac{1}{(x^2+1)^2} dx$$

$$= \ln|x-1| + \arctan x - \frac{1}{x^2+1} + \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)} + C$$

$$= \ln|x-1| + \frac{3}{2}\arctan x + \frac{x-2}{2(x^2+1)} + C$$

(38) $\int \frac{xdx}{\sqrt{4x-3}}$

Solution: $\int \frac{xdx}{\sqrt{4x-3}} \quad \left(t=\sqrt{4x-3}, x=\frac{1}{4}(t^2+3), dx=\frac{1}{2}tdt\right)$

$$= \int \frac{\frac{1}{4}(t^2+3)\cdot\frac{1}{2}tdt}{t}$$

$$= \int \frac{1}{8}(t^2+3)dt$$

$$= \frac{1}{24}t^3 + \frac{3}{8}t + C$$

$$= \frac{1}{24}(4x-3)^{\frac{3}{2}} + \frac{3}{8}\sqrt{4x-3} + C$$

(39) $\int \frac{dx}{x(\sqrt[3]{x}-\sqrt{x})}$

Solution: $\int \frac{dx}{x(\sqrt[3]{x}-\sqrt{x})} \quad (x=t^6, dx=6t^5dt)$

$$= \int \frac{6t^5dt}{t^6(t^2-t^3)}$$

$$= -\int \frac{6}{t^3(t-1)} dt$$

$$= 6\int \left(-\frac{1}{t-1} + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}\right) dt$$

$$= 6\left(\ln\left|\frac{t}{t-1}\right| - \frac{1}{t} - \frac{1}{2t^2}\right) + C$$

$$= 6\left(\ln\left|\frac{\sqrt[6]{x}}{\sqrt[6]{x}-1}\right| - \frac{1}{\sqrt[6]{x}} - \frac{1}{2\sqrt[3]{x}}\right) + C$$

(40) $\int \frac{\sqrt{1+x}}{x\sqrt{1-x}} dx$

Solution: $\int \frac{\sqrt{1+x}}{x\sqrt{1-x}} dx \quad \left(t=\sqrt{\frac{1+x}{1-x}}, x=\frac{t^2-1}{t^2+1}, dx=\frac{4t}{(t^2+1)^2}dt\right)$

$$= \int \frac{t}{\frac{t^2-1}{t^2+1}} \cdot \frac{4t}{(t^2+1)^2} dt$$

$$= \int \frac{4t^2}{(t^2+1)(t^2-1)} dt$$

$$= 2\int \left(\frac{1}{t^2-1} + \frac{1}{t^2+1}\right) dt$$

$$= \int \left(\frac{1}{t-1} - \frac{1}{t+1} + \frac{2}{t^2+1}\right) dt$$

$$= \ln\left|\frac{t-1}{t+1}\right| + 2\arctan t + C$$

$$= \ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + 2\arctan\sqrt{\frac{1+x}{1-x}} + C$$

(41) $\int \frac{dx}{\sqrt[3]{(x-1)^2(x+1)^4}}$

Solution: $\int \frac{dx}{\sqrt[3]{(x-1)^2(x+1)^4}} = \int \frac{1}{(x+1)^2} \sqrt[3]{\left(\frac{x+1}{x-1}\right)^2} dx$

$$\left(t = \sqrt[3]{\frac{x+1}{x-1}}, x = \frac{t^3+1}{t^3-1}, dx = -\frac{6t^2}{(t^3-1)^2} dt\right)$$

$$\int \frac{dx}{\sqrt[3]{(x-1)^2(x+1)^4}}$$

$$= \int \frac{1}{\left(\frac{t^3+1}{t^3-1}+1\right)^2} t^2 \cdot -\frac{6t^2}{(t^3-1)^2} dt$$

$$= \int -\frac{3}{2t^2} dt$$

$$= \frac{3}{2t} + C$$

$$= \frac{3}{2} \cdot \sqrt[3]{\frac{x-1}{x+1}} + C$$

(42) $\int \frac{dx}{4+4\sin x+\cos x}$

Solution: $\int \frac{dx}{4+4\sin x+\cos x}$

$$\left(t = \tan\frac{x}{2}, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt\right)$$

$$\int \frac{dx}{4+4\sin x+\cos x}$$

$$= \int \frac{\frac{2}{1+t^2}dt}{4+4\cdot\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$= \int \frac{2dt}{(3t+5)(t+1)}$$

$$= \int \left(\frac{1}{t+1} - \frac{3}{3t+5}\right) dt$$

$$= \ln|t+1| - 3\ln|3t+5| + C$$

$$= \ln\left|\tan\frac{x}{2} + 1\right| - 3\ln\left|3\tan\frac{x}{2} + 5\right| + C$$

(43) $\int \frac{\cot x dx}{1+\sin x}$

Solution: $\int \frac{\cot x dx}{1+\sin x} = \int \frac{\cos x dx}{\sin x(1+\sin x)}$

$$= \int \frac{d(\sin x)}{\sin x(1+\sin x)} \quad (u = \sin x)$$

$$= \int \frac{du}{u(1+u)}$$

$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$

$$= \ln\left|\frac{u}{u+1}\right| + C$$

$$= \ln\left|\frac{\sin x}{\sin x+1}\right| + C$$