Answers to 2020-2021- middle TEST

$1-18(5'*18=90')$; Select one question form 19 and 20. $(10'*1=10')$

1. If $y=\dfrac{1}{1+2x}$, then $y^{(6)}(x)=$ $\underline{(-2)^6\dfrac{6!}{(1+2x)^7}}$

2. If $y=\ln\dfrac{\sqrt{x^2+1}}{\sqrt[3]{x-2}}(x>2)$, then $\mathrm{d}y=$ $\underline{(\dfrac{x}{x^2+1}-\dfrac{1}{3(x-2)})dx}$

3. Determine $f(x)=\begin{cases}\dfrac{x}{1+\mathrm{e}^{\frac{1}{x}}}, & x\neq 0\\ 0, & x=0\end{cases}$ is continuous or not at $x=0$.

Ans：

(1) $f(0)=0;$

(2) $f(0+0)=\lim\limits_{x\to 0^+}\dfrac{x}{1+\mathrm{e}^{\frac{1}{x}}}=0 \quad (\dfrac{0}{1+\infty}\to 0)$

$f(0-0)=\lim\limits_{x\to 0^-}\dfrac{x}{1+\mathrm{e}^{\frac{1}{x}}}=0 \quad (\dfrac{0}{1+0}\to 0)$

$\lim\limits_{x\to 0}f(x)=0$

(3) $\lim\limits_{x\to 0}f(x)=f(0)$

$\therefore$ continuous at $x=0$.

4. Find the greatest volume that a right circular cylinder can have, if it is inscribed in a sphere of radius r.

Ans：

Let the height of cylinder be $2h$, radius be r , volume be V.

Then, the objective function is $V=\pi r^2\cdot 2h$.

By $r^2+h^2=R^2,$ we have $V=2\pi(R^2-h^2)\cdot h,\ 0<h<R$.

To find maximum point, since $V_h'=2\pi(R^2-3h^2)$,

let $V_h'=0,$ we can obtain $h=\dfrac{R}{\sqrt{3}}$. (delete negative value) (Unique stationary point: the maximum volume of the cylinder must be obtained.)

So the unique stationary point $h=\dfrac{R}{\sqrt{3}}$ is the maximum value point.

The maximum volume is: $V=2\pi(R^2-\dfrac{R^2}{3})\cdot\dfrac{R}{\sqrt{3}}=\dfrac{4\pi}{3\sqrt{3}}R^3.$

5. Prove that $\dfrac{x}{1+x} < \ln\left(1+x\right) < x$, when $x > 0$.

Ans：

Let $f\left(x\right) = \ln\left(1+x\right)$,by Mean Value Theorem for Derivative , we have

$$f(x) - f(0) = f'(\xi)(x-0),\ 0<\xi<x \because f(0)=0,\ f'(x)=\frac{1}{1+x},\ \text{ we have}$$

$$\ln(1+x) = \frac{x}{1+\xi},\ \text{by}\ \ 0<\xi<x \Rightarrow 1<1+\xi<1+x \Rightarrow \frac{1}{1+x}<\frac{1}{1+\xi}<1,$$

$$\therefore \frac{x}{1+x} < \frac{x}{1+\xi} < x,\ \ \text{namely}\ \ \frac{x}{1+x} < \ln(1+x) < x.$$

6. Find $\lim\limits_{x\to0}(\dfrac{3-e^x}{2+x})^{\frac{1}{\sin x}}$.

Ans :

$$\lim_{x\to0}(\frac{3-e^x}{2+x})^{\frac{1}{\sin x}} = e^A$$

$$A = \lim_{x\to0}\frac{\frac{3-e^x}{2+x}-1}{\sin x} = \lim_{x\to0}\frac{1-e^x-x}{(2+x)\sin x} = \frac{1}{2}\lim_{x\to0}\frac{1-e^x-x}{x}$$

$$= \frac{1}{2}(\lim_{x\to0}\frac{1-e^x}{x}-1) = \frac{1}{2}(-1-1) = -1$$

$$\Rightarrow \lim_{x\to0}(\frac{3-e^x}{2+x})^{\frac{1}{\sin x}} = e^{-1}$$

7. Find $\lim\limits_{x\to+\infty}\dfrac{e^x}{(1+\frac{1}{x})^{x^2}}$.

Ans :

$$\lim_{x\to+\infty}\frac{e^x}{(1+\frac{1}{x})^{x^2}} = \lim_{x\to+\infty} e^{x[1-x\ln(1+\frac{1}{x})]}$$

$$\lim_{x\to+\infty} x[1-x\ln(1+\frac{1}{x})] = \lim_{u\to0^+}\frac{1-\frac{\ln(1+u)}{u}}{u}\ (u=\frac{1}{x})$$

$$= \lim_{u\to0^+}\frac{u-\ln(1+u)}{u^2} = \frac{1}{2}$$

$$\Rightarrow \lim_{x\to+\infty}\frac{e^x}{(1+\frac{1}{x})^{x^2}} = e^{\frac{1}{2}}$$

8. Find $\lim\limits_{x\to0}\dfrac{e^{x^2}-\sqrt{\cos x}}{x^2}$.

Ans :

$$\lim_{x\to0}\frac{e^{x^2}-\sqrt{\cos x}}{x^2} = \lim_{x\to0}\frac{e^{x^2}-1+1-\sqrt{\cos x}}{x^2}$$

$$= \lim_{x\to0}\frac{e^{x^2}-1}{x^2} + \lim_{x\to0}\frac{1-\cos x}{x^2(1+\sqrt{\cos x})}$$

$$= \lim_{x\to0}\frac{x^2}{x^2} + \frac{1}{2}\lim_{x\to0}\frac{1-\cos x}{x^2}$$

$$= 1 + \frac{1}{4} = \frac{5}{4}$$

9. Find $\lim_{x\to 0}(\frac{4-\sin x-3\cos x}{1+x})^{\frac{1}{\tan x}}$.

Ans :

$$\lim_{x\to 0}(\frac{4-\sin x-3\cos x}{1+x})^{\frac{1}{\tan x}}=e^{A}$$

$$A=\lim_{x\to 0}\frac{1}{\tan x}\cdot\ln\left[\frac{4-\sin x-3\cos x}{1+x}\right]$$

$$=\lim_{x\to 0}\frac{1}{\tan x}\cdot\ln\left[\frac{1+x+3-x-\sin x-3\cos x}{1+x}\right]$$

$$=\lim_{x\to 0}\frac{1}{\tan x}\cdot\ln\left[1+\frac{3-x-\sin x-3\cos x}{1+x}\right]$$

$$=\lim_{x\to 0}\frac{1}{\tan x}\cdot\frac{3-x-\sin x-3\cos x}{(1+x)}$$

$$=\lim_{x\to 0}\frac{1}{x}\cdot\frac{3-x-\sin x-3\cos x}{(1+x)}$$

$$=\lim_{x\to 0}\frac{-1-\cos x+3\sin x}{1+2x}$$

$$=-2$$

$$\therefore \lim_{x\to 0}(\frac{4-\sin x-3\cos x}{1+x})^{\frac{1}{\tan x}}=e^{-2}.$$

10. Find $\lim_{n\to\infty}\frac{1}{n}(\sin\frac{1}{n}+\sin\frac{2}{n}+\ldots+\sin\frac{n}{n})$.

Ans :

$$\lim_{n\to\infty}\frac{1}{n}(\sin\frac{1}{n}+\sin\frac{2}{n}+\ldots+\sin\frac{n}{n})$$

$$=\lim_{n\to\infty}\frac{1}{n}(\sin\frac{1}{n}+\sin\frac{2}{n}+\ldots+\sin\frac{n}{n})$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}(\sin\frac{i}{n})$$

$$=\int_0^1\sin x\,dx$$

$$=1-\cos 1$$

11. If $y=\sqrt[3]{x^2}\sin x$, find y'.

Ans :

when $x\neq 0$, $y'=\frac{2}{3\sqrt[3]{x}}\sin x+\sqrt[3]{x^2}\cos x$

when $x=0$, $y'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\sqrt[3]{x^2}\cdot\frac{\sin x}{x}=0$

12. If $y=(1+x^2)^{\sin x}$, find y'.

Ans :

$$y=(1+x^2)^{\sin x}\Rightarrow \ln y=\sin x\ln(1+x^2)$$

$$\Rightarrow \frac{y'}{y}=\cos x\ln(1+x^2)+\frac{2x\sin x}{1+x^2}$$

$$\Rightarrow y'=(1+x^2)^{\sin x}[\cos x\ln(1+x^2)+\frac{2x\sin x}{1+x^2}]$$

13. If $f(x)=\begin{cases}(1+x)^{\frac{1}{x}}-e, & (x\neq 0)\\ 0, & (x=0)\end{cases}$, find $f'(0)$.

Ans :

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}$$

$$=e\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}=e\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}=e\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}=-\frac{e}{2}$$

14. If $f(x)=\begin{cases}x^3\sin\frac{1}{x}, & (x\neq 0)\\ 0, & (x=0)\end{cases}$, find $f'(x)$.

Ans :

when $x\neq 0$, $f'(x)=3x^2\sin\frac{1}{x}-x\cos\frac{1}{x}$

when $x=0$, $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^3\sin\frac{1}{x}}{x}=0$

15. If $y\sin x-\cos(x-y)=0$, find dy.

Ans :

$y\sin x-\cos(x-y)=0$

$\Rightarrow \sin x dy+y\cos x dx+\sin(x-y)(dx-dy)=0$

$\Rightarrow [\sin(x-y)-\sin x]dy=[y\cos x+\sin(x-y)]dx$

$\Rightarrow dy=\dfrac{y\cos x+\sin(x-y)}{\sin(x-y)-\sin x}dx$

16. $F(x)=\int_1^x[\frac{1}{t}\int_0^t f(u)du]dt$, find $F''(x)$ ($f(x)$ is continuous).

Ans :

let g(t)=$\frac{1}{t}\int_0^t f(u)du$

$\Rightarrow F(x)=\int_1^x g(t)dt \Rightarrow F'(x)=g(x)$

$\Rightarrow F''(x)=g'(x)=\dfrac{xf(x)-\int_0^x f(u)du}{x^2}$

17. If $F(x)=\int_{\frac{1}{x}}^{\ln x} f(t)dt$, $f(x)$ is continuous, find $F'(x)$.

Ans :

$F'(x)=f(\ln x)(\ln x)'-f(\frac{1}{x})(\frac{1}{x})'=\frac{1}{x}f(\ln x)+\frac{1}{x^2}f(\frac{1}{x})$

18. Find $I=\lim\limits_{x\to 0}\dfrac{\int_0^x(\int_0^{\tan^2 y}\frac{\sin t}{t}dt)dy}{x^3}$.

Ans :

$$I=\lim_{x\to 0}\frac{\int_0^x(\int_0^{\tan^2 y}\frac{\sin t}{t}dt)dy}{x^3}=\lim_{x\to 0}\frac{\int_0^{\tan^2 x}\frac{\sin t}{t}dt}{3x^2}$$

$$=\lim_{x\to 0}\frac{\frac{\sin(\tan^2 x)}{\tan^2 x}2\tan x\cdot\frac{1}{\cos^2 x}}{6x}$$

$$=\frac{1}{3}$$

19. If $f(x)=\begin{cases}\dfrac{\ln(1+2x^2)}{x}, & (x>0),\\ (1+x^2)^{\frac{4}{3}}+\sin 2x-1, & (x\le 0).\end{cases}$ (1) Find $f'(x)$, (2) Is $f'(x)$ differentiable at $x=0$?

Ans :

(1)when $x\le 0$, $f'(x)=\frac{8}{3}x(1+x^2)^{\frac{1}{3}}+2\cos 2x \Rightarrow f_-'(0)=2$

when $x>0$, $f'(x)=\dfrac{\frac{4x}{1+2x^2}\cdot x-\ln(1+2x^2)}{x^2}=\dfrac{4}{1+2x^2}-\dfrac{\ln(1+2x^2)}{x^2}$

$\Rightarrow \lim\limits_{x\to0^+} f'(x)=\lim\limits_{x\to0^+}\dfrac{4}{1+2x^2}-\lim\limits_{x\to0^+}\dfrac{\ln(1+2x^2)}{x^2}=4-2=2$

$\lim\limits_{x\to0^+} f(x)=\lim\limits_{x\to0^+}\dfrac{\ln(1+2x^2)}{x}=0=f(0)\Rightarrow f(x)$ is right continuous at $x=0$

$\Rightarrow f_+'(0)=2\Rightarrow f'(0)=2$

$f'(x)=\dfrac{4}{1+2x^2}-\dfrac{\ln(1+2x^2)}{x^2},(x>0);\ f'(x)=\dfrac{8}{3}x(1+x^2)^{\frac{1}{3}}+2\cos 2x,(x\le 0)$

(2) $f_-''(0)=[\frac{8}{3}x(1+x^2)^{\frac{1}{3}}+2\cos 2x]'\Big|_{x=0}=\dfrac{8}{3}$

$$f_+''(0)=\lim_{x\to0+}\frac{f'(x)-f'(0)}{x}=\lim_{x\to0+}\frac{\frac{4}{1+2x^2}-\frac{\ln(1+2x^2)}{x^2}-2}{x}$$

$$=\lim_{x\to0+}\frac{4x^2-(1+2x^2)\ln(1+2x^2)-2x^2(1+2x^2)}{x^3}$$

$$=\lim_{x\to0+}\frac{2x^2-\ln(1+2x^2)}{x^3}-\lim_{x\to0+}\frac{2x^2\ln(1+2x^2)+4x^4}{x^3}$$

$$=\lim_{x\to0+}\frac{4x-\frac{4x}{1+2x^2}}{3x^2}=\frac{4}{3}\lim_{x\to0+}\frac{2x^2}{x}=0$$

$\Rightarrow f_+''(0)\ne f_-''(0)\Rightarrow f'(x)$ is not differentiable at $x=0$

20. Find the extremum point, inflection point, concave up and concave down interval and asymptote of $y(x)=xe^{\frac{1}{x-1}}$.

Ans :

$$y' = e^{\frac{1}{x-1}} \frac{x^2 - 3x + 1}{(x-1)^2} = e^{\frac{1}{x-1}} \frac{(x-x_1)(x-x_2)}{(x-1)^2} = 0$$

$$\Rightarrow x_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}, x_2 = \frac{3}{2} + \frac{\sqrt{5}}{2}$$

when $x < x_1$ or $x > x_2$, $y' > 0 \Rightarrow y$ is increasing function

when $x_1 < x < 1$ or $1 < x < x_2$, $y' < 0 \Rightarrow y$ is subtraction function

$\Rightarrow x = x_1$ is maximum point, $x = x_2$ is minimum point

$y''=e^{\frac{1}{x-1}} \frac{3x-2}{(x-1)^4} = 0 \Rightarrow x = \frac{2}{3}$, y''does not exist when $x = 1$

when $x < \frac{2}{3}$ $y'' < 0 \Rightarrow$ convex interval is $(-\infty, \frac{2}{3})$

when $\frac{2}{3} < x < 1$ or $1 < x$, $y'' > 0 \Rightarrow$ concave interval is $(\frac{2}{3}, 1) \cup (1, +\infty)$

Inflection point is $(\frac{2}{3}, \frac{2}{3} e^{-3})$

$\lim\limits_{x \to 1^-} x e^{\frac{1}{x-1}} = 0, \lim\limits_{x \to 1^+} x e^{\frac{1}{x-1}} = +\infty \Rightarrow x = 1$ is vertical asymptote.

$\lim\limits_{x \to \infty} x e^{\frac{1}{x-1}} = \infty \Rightarrow$ There is no horizontal asymptote

$$a = \lim_{x \to \infty} \frac{x e^{\frac{1}{x-1}}}{x} = 1, b = \lim_{x \to \infty} (x e^{\frac{1}{x-1}} - x) = \lim_{x \to \infty} \frac{e^{\frac{1}{x-1}} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{1}{x-1}}{\frac{1}{x}} = 1$$

$\Rightarrow y = x + 1$ is oblique asymptote