座位号________ 专业班级________ 学院________ 学号 □□□□□□□□□□□□ 姓名________

（密 封 线 内 不 答 题）

诚信应考，考试作弊将带来严重后果！

华南理工大学本科生考试

2023-2024-1 学期期中考试《(双语）微积分 II（一)》

注意事项：1. 开考前请将密封线内各项信息填写清楚；
2. 所有答案请直接答在答题纸上；
3．考试形式：闭卷；
4. 本试卷共 二 大题，满分 100 分， 考试时间 95 分钟。

I Please fill the correct answers in the following blanks.（$4'\times 6=24'$）

1. $f(x)=\begin{cases}\frac{\ln(1+x)}{ax}, & x\neq 0\\ -1, & x=0\end{cases}$ is continuous at $x=0$, then $a=\underline{-1}$

2. $\lim\limits_{n\to\infty}\left(\frac{n}{n^2+e}+\frac{n}{n^2+2e}+\cdots+\frac{n}{n^2+ne}\right)=\underline{\quad 1\quad}$;

3. $e^{x\cos x^2}-e^x\sim kx^n(x\to 0)$，then $k=\underline{\ -1/2\ }$, $n=\underline{\ 5\ }$;

4. $f'(x_0)=3$, $\lim\limits_{h\to 0}\frac{f(x_0+h^2)-f(x_0-h^2)}{h^2}=\underline{\quad 6\quad}$;

5. Let $y=f(e^{\sqrt{x^2+1}})$, where f is differentiable，then $dy=\underline{\frac{xe^{\sqrt{x^2+1}}f'(e^{\sqrt{x^2+1}})}{\sqrt{x^2+1}}dx}$;

6. The equation of tangent line of $r=1+\cos\theta$ at $\theta=\frac{\pi}{2}$ is $\underline{\ y=x+1\ }$.

II Finish the following questions($76'$).

7.（$7'$） Please describe the intuitive definition and $\varepsilon-\delta$ definition of $\lim\limits_{x\to x_0}f(x)$, then prove $\lim\limits_{x\to 4}\frac{x+2}{x-1}=2$ by $\varepsilon-\delta$ definition.

Solu. Intuitive definition: As x approaches x_0, $f(x)$ tends to some constant L.

$\varepsilon-\delta$ definition: $\lim\limits_{x\to x_0}f(x)=L\Leftrightarrow\forall\varepsilon>0,\exists\delta>0,\text{s.t. }0<|x-x_0|<\delta\Rightarrow|f(x)-L|<\varepsilon$.

Proof. $\because$ $x\to 4$, assume $|x-4|<\delta_1=1\Rightarrow 3<x<5\Rightarrow 0<\frac{1}{|x-1|}<\frac{1}{2}$,

$\forall\varepsilon>0$,

$$|f(x)-L|=\left|\frac{x+2}{x-1}-2\right|=\frac{|x-4|}{|x-1|}<\frac{1}{2}|x-4|,$$

Let $\delta=\min\{1,2\varepsilon\}$, then when $0<|x-4|<\delta$, we have

$$\left|\frac{x+2}{x-1}-2\right|<\frac{1}{2}|x-4|<\varepsilon,$$

so

$$\lim_{x\to 4}\frac{x+2}{x-1}=2.$$

8. (7′) Find $\lim\limits_{x\to 0}\frac{\sin x - x}{x\ln(1+x^2)}$.

Solu. $\mathrm{Lim}_{x\to 0}\frac{\sin x - x}{x\ln(1+x^2)} = \lim\limits_{x\to 0}\frac{\sin x - x}{x\cdot x^2} = \lim\limits_{x\to 0}\frac{\sin x - x}{x^3} = \lim\limits_{x\to 0}\frac{\cos x - 1}{3x^2} = \lim\limits_{x\to 0}\frac{-\frac{1}{2}x^2}{3x^2} = -\frac{1}{6}$.

9. (7′) Suppose the sequence $\{x_n\}$ satisfies the conditions: $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2 + x_n}$, find $\lim\limits_{n\to\infty} x_n$.

Solu. $x_n \le x_{n+1}$, $\{x_n\}$ is monotonic;

$x_1 = \sqrt{2} < 2$, assume that $x_k < 2$, then $x_{k+1} < \sqrt{2+2} = 2$, i.e. $\{x_n\}$ is bounded above,

so $\lim\limits_{n\to\infty} x_n$ exsits. Suppose $\lim\limits_{n\to\infty} x_n = L$,

$$x_{n+1}^2 = 2 + x_n \Rightarrow L^2 = 2 + L \Rightarrow L = 2 \text{ or } -1(\text{deleted}),$$

so, $\lim\limits_{n\to\infty} x_n = 2$.

10. (7′) Find $\lim\limits_{x\to+\infty}[x - x^2\ln(1+\frac{1}{x})]$.

Solu. Let $\frac{1}{x} = t$, then

$$\lim_{x\to+\infty}[x - x^2\ln(1+\frac{1}{x})] = \lim_{t\to 0^+}[\frac{1}{t} - \frac{1}{t^2}\ln(1+t)] = \lim_{t\to 0^+}\frac{t - \ln(1+t)}{t^2} = \lim_{t\to 0^+}\frac{1 - \frac{1}{1+t}}{2t} = \frac{1}{2}.$$

11. (7′) Let $\boldsymbol{y} = \frac{(\boldsymbol{x}+\mathbf{1})\cdot\sqrt[3]{x-1}}{(\boldsymbol{x}+\mathbf{4})^2}(x > 1)$, find y'.

Solu. $\ln y = \ln(x+1) + \frac{1}{3}\ln(x-1) - 2\ln(x+4)$

$$\frac{1}{y}\cdot y' = \frac{1}{x+1} + \frac{1}{3(x-1)} - \frac{2}{x+4}$$

$$\therefore\quad y' = \frac{(x+1)\sqrt[3]{x-1}}{(x+4)^2}\left[\frac{1}{x+1} + \frac{1}{3(x-1)} - \frac{2}{x+4}\right].$$

12. (**7′**) Let $f(x)=\begin{cases} x\mathrm{e}^{-\frac{1}{x^2}}, x\neq 0, \\ 0, x=0 \end{cases}$

(1) find $f'(x)$; (2) discuss the continuity of $f(x)$ and $f'(x)$ at the point $x=0$.

Solu. (1) when $x\neq 0, f'(x)=(1+\frac{2}{x^2})\mathrm{e}^{-\frac{1}{x^2}}$,

$$\text{when } x=0, f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x\mathrm{e}^{-\frac{1}{x^2}}-0}{x}=0,$$

$$\therefore\ f'(x)=\begin{cases}(1+\dfrac{2}{x^2})\mathrm{e}^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0\end{cases}.$$

(2) $\because \lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}x\mathrm{e}^{-\frac{1}{x^2}}=0=f(0)$,

$\therefore f(x)$ is continuous at $x=0$.

$$\because \lim_{x\to 0}f'(x)=\lim_{x\to 0}\left(1+\frac{2}{x^2}\right)\mathrm{e}^{-\frac{1}{x^2}}=\lim_{t\to+\infty}(1+2t)\mathrm{e}^{-t}=\lim_{t\to+\infty}\frac{1+2t}{\mathrm{e}^t}=0=f'(0),$$

$\therefore\ f'(x)$ is also continuous at $x=0$.

13. (**7′**) Let $y=x\mathrm{e}^{-x}$, find $y^{(n)}$.

Solu. $y'=1\cdot\mathrm{e}^{-x}+x\cdot\mathrm{e}^{-x}\cdot(-1)=(-1)^1(x-1)e^{-x}$,

$y''=(-1)^2(x-2)\mathrm{e}^{-x}$,

$y'''=(-1)^3(x-2)\mathrm{e}^{-x}$,

……

$y^{(n)}=(-1)^n(x-n)\mathrm{e}^{-x}$

14. (**7′**) Suppose that $y=y(x)$ is determined by $\begin{cases}x=t-\ln(1+t)\\ y=t^2+t^3\end{cases}$, find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Solu. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{2t+3t^2}{1-\frac{1}{1+t}}=(t+1)(2+3t)=3t^2+5t+2$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}(3t^2+5t+2)\cdot\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}=(6t+5)\cdot\frac{1}{1-\frac{1}{1+t}}=6t+\frac{5}{t}+11.$$

15. （7′） Determine the monotonicity and concavity of $f(x) = \frac{x}{1+x^2}$.

Solu.

$$f(x)=\frac{x}{1+x^2}, f'(x)=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}, let\ f(x)=0 \Rightarrow x=1\ or\ x=-1$$

$when\ \ x\in(-\infty,-1), f'(x)<0; when\ \ x\in(-1,1), f'(x)>0; when\ \ x\in(1,+\infty), f'(x)<0;$

hence, $f(x)$ is monotone increasing on $(-1,1)$, monotone decreasing on $(-\infty,-1)$, $(1,+\infty)$

$$f''(x)=\frac{(1+x^2)^2(-2x)-4x(1+x^2)(1-x^2)}{(1+x^2)^4}=\frac{2x(1+x^2)(x^2-3)}{(1+x^2)^4}, f''(x)=0 \Rightarrow x=0\ \ or\ \ x=\pm\sqrt{3}$$

x	$(-\infty,-\sqrt{3})$	$\sqrt{3}$	$(-\sqrt{3},0)$	**0**	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},+\infty)$
$f'(x)$	↘		↗		↘		↗
$f''(x)$	−	**0**	+	**0**	−	**0**	+

hence, $f(x)$ is concave up on $(-\sqrt{3},0),(\sqrt{3},+\infty)$; concave down on $(-\infty,-\sqrt{3}),(0,\sqrt{3})$

16. （7′） Find a point M of the curve $y=x^2+1$ such that the distance of M and $P(5,0)$ is the smallest.

Solu. Let $M(x, x^2+1)$, $d(x)$ be the distance of M and P,

$$f(x) \triangleq d^2(x) = (x-5)^2 + (x^2+1-0)^2 = x^4 + 3x^2 - 10x + 26,$$

Let $f'(x) = 4x^3 + 6x - 10 = 2(x-1)(2x^2+2x+5) = 0 \Rightarrow x = 1.$

Since $x = 1$ is the unique stationary point, when M is (1, 2), the distance is the smallest.

17. （6′） Let $x > 0$， prove $x\ln x \ge (x+1)\ln\frac{x+1}{2}$.

Proof. Let $f(x) = x\ln x - (x+1)\ln\frac{x+1}{2} = x\ln x - (x+1)\ln(x+1) + (x+1)\ln 2$，

then $f(x)$ is continuous, differentiable as $x>0$ and $f'(x) = \ln x - \ln(x+1) + \ln 2$.

Let $f'(x) = \ln x - \ln(x+1) + \ln 2 = 0$， we have $x = 1$.

Since $f''(x) = \frac{1}{x} - \frac{1}{x+1}$, $f''(1) = 1 - \frac{1}{2} > 0$， so $f(1) = 0$ is both the local minimum value and the global minimum value. Therefore, $x>0 \Rightarrow f(x) \ge f(1) = 0$， i.e.

$$x\ln x \ge (x+1)\ln\frac{x+1}{2} .$$