Chinese Name________ Student ID □□□□□□□□□□□□ School ________ Major ________ Sit NO. ________

(DONOY WRITE YOUR ANSWER IN THIS AREA)

Sale Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Middle Exam

2023-2024 《Calculus II》

Notice: 1. Make sure that you have filled the form on the left side of seal line.
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 95 minutes.

Finish the following questions. (1-8: $5'\times 8=40'$; 9-18: $6'\times 10=60'$)

1. Suppose the plane *M* passes through the origin and (1,-3,2) and is perpendicular to the plane $2x-4y+z=7$, find the equation of plane *M*.

Solution: normal vector: $<1,-3,2>\times<2,-4,1>=<5,3,2>$

M: $5x+3y+2z=0$

2. Suppose two lines $l_1: x-1=\frac{y-5}{-2}=z+8$ and l_2 is the intersection of planes $x-y=5$ and $2y+z=1$. Find the acute angle between l_1 and l_2.

Solution:the direction vector of l_1:$<1,-2,1>$, the direction vector of l_2:$<-1,-1,2>$,

the acute angle between l_1 and l_2 is θ:

$$\cos\theta=\frac{<-1,-1,2>\cdot<1,-2,1>}{\|<-1,-1,2>\|\|<1,-2,1>\|}=\frac{1}{2},\ \theta=\frac{\pi}{3}.$$

3. Suppose $l: z=ky(k>0)$ is a line on the *yz*-plane and *S* is the surface by revolving *l* about *z*-axis. Find the equation of *S*.

Solution: the equation of *S*: $z=\pm k\sqrt{x^2+y^2}\,(k>0)$

4. Determine if the series $\sum_{n=1}^{\infty}(1-\cos\frac{1}{\sqrt{n}})^q \quad (q>0)$ converges or not.

Solution:

$$\lim_{n\to\infty}\frac{(1-\cos\frac{1}{\sqrt{n}})^q}{(\frac{1}{2n})^q}=1, \text{ then } \sum_{n=1}^{\infty}(1-\cos\frac{1}{\sqrt{n}})^q \text{ and } \sum_{n=1}^{\infty}(\frac{1}{2n})^q \text{ converge or diverge together.}$$

$$q>1, \sum_{n=1}^{\infty}(1-\cos\frac{1}{\sqrt{n}})^q \text{ converges;}$$

$$0<q\le 1, \sum_{n=1}^{\infty}(1-\cos\frac{1}{\sqrt{n}})^q \text{ diverges.}$$

5. Classify the following series as absolutely convergent, conditionally convergent or divergent:

$$\sum_{n=2}^{\infty}(-1)^n\frac{\sqrt[n]{n}}{\ln n}.$$

Solution:

$$\lim_{n\to\infty}\frac{\frac{\sqrt[n]{n}}{\ln n}}{\frac{1}{\sqrt{n}}}=+\infty, \sum_{n=2}^{\infty}\frac{1}{\sqrt{n}} \text{ diverges, so } \sum_{n=2}^{\infty}\frac{\sqrt[n]{n}}{\ln n} \text{ diverges.}$$

$$\sum_{n=2}^{\infty}(-1)^n\frac{\sqrt[n]{n}}{\ln n} \text{ is convergent in terms of Alternating Series Test,}$$

$$\text{therefore } \sum_{n=2}^{\infty}(-1)^n\frac{\sqrt[n]{n}}{\ln n} \text{ is conditionally convergent.}$$

6. Find the convergence set and identify the sum as a function of the following power series

$$\sum_{n=1}^{\infty}\frac{(2n-1)x^{2n-2}}{2^n}.$$

Solution:

$$R=\lim_{n\to\infty}\sqrt{\frac{(2n-1)/2^n}{(2n+1)/2^{n+1}}}=\sqrt{2},$$

$$x=\sqrt{2},\sum_{n=1}^{\infty}\frac{(2n-1)2^{n-1}}{2^n}\text{ diverges.}$$

$$x=-\sqrt{2},\sum_{n=1}^{\infty}\frac{(2n-1)(-\sqrt{2})^{2n-2}}{2^n}\text{ diverges.}$$

Convergent set is $(-\sqrt{2},\sqrt{2})$.

$$\text{Let } t=x^2<2,\ S(t)=\sum_{n=1}^{\infty}\frac{(2n-1)t^{n-1}}{2^n}=\sum_{n=1}^{\infty}n(\frac{t}{2})^{n-1}-\frac{1}{t}\sum_{n=1}^{\infty}(\frac{t}{2})^n,\ t\neq 0$$

$$=\sum_{n=1}^{\infty}2[(\frac{t}{2})^n]'-\frac{1}{t}\sum_{n=1}^{\infty}(\frac{t}{2})^n=2[\sum_{n=1}^{\infty}(\frac{t}{2})^n]'-\frac{1}{t}\cdot\frac{t}{2-t}$$

$$=2(\frac{t}{2-t})'-\frac{1}{2-t}=\frac{2+t}{(2-t)^2}$$

$$\sum_{n=1}^{\infty}\frac{(2n-1)x^{2n-2}}{2^n}=S(x^2)=\begin{cases}\dfrac{2+x^2}{(2-x^2)^2}, & x\neq 0 \text{ and } |x|<\sqrt{2}\\ \dfrac{1}{2}, & x=0\end{cases}$$

NOTE: $x=0$, sum function still holds. $S(x^2)=\dfrac{2+x^2}{(2-x^2)^2},|x|<\sqrt{2}.$

7.Let $f(x)=\dfrac{x}{1-x^2}$, find $f^{(2024)}(0)$.

Solution:

$$f(x)=\frac{x}{1-x^2}=\sum_{n=1}^{\infty}x^{2n-1},\quad f^{(2024)}(0)=0.$$

8.Write the Taylor series at $x=4$ for the function $f(x)=\dfrac{1}{x^2-x-6}$.

Solution:

$$f(x)=\frac{1}{5}\left(\frac{1}{x-3}-\frac{1}{x+2}\right)=\frac{1}{5}\left[\frac{1}{1+(x-4)}-\frac{1}{6}\cdot\frac{1}{1+\frac{x-4}{6}}\right]$$

$$=\sum_{n=1}^{\infty}\frac{1}{5}(-1)^{n-1}(x-4)^{n-1}-\sum_{n=1}^{\infty}\frac{1}{30}(-1)^{n-1}\left(\frac{x-4}{6}\right)^{n-1}$$

$$=\sum_{n=1}^{\infty}\frac{1}{5}(-1)^{n-1}\left(1-\frac{1}{6^n}\right)(x-4)^{n-1}$$

Where $x\in(3,5)$.

9.Calculate $\iint\limits_D e^{\frac{x}{x+y}}\mathrm{d}x\mathrm{d}y$, where $D=\{(x,y)\,|\,x+y\le 1, x\ge 0, y\ge 0\}$.

Solution :

$$let\begin{cases}u=x\\ v=x+y\end{cases}\Rightarrow\begin{cases}x=u\\ y=v-u\end{cases}$$

$$D\to D'$$

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}=1$$

$\iint\limits_D e^{\frac{x}{x+y}}\mathrm{d}x\mathrm{d}y=\iint\limits_{D'} e^{\frac{u}{v}}\mathrm{d}u\mathrm{d}v$, D' is bounded by $v=u, u=0$ and $v=1$

$$=\int_0^1 dv\int_0^v e^{\frac{u}{v}}\mathrm{d}u=\frac{1}{2}(e-1).$$

10.Use symmetry to compute the integral $\iint_D [\cos^2(x^2+\mathrm{y})+\sin^2(x+y^2)]\mathrm{d}A,$ where D is

$\{(x,y)|(x-1)^2+(y-1)^2\le 3\}$.

Solution: D is symmetric about line y=x, so

$$\iint_D [\cos^2(x^2+\mathrm{y})+\sin^2(x+y^2)]\mathrm{d}A=\iint_D [\cos^2(y^2+x)+\sin^2(y+x^2)]\mathrm{d}A$$

$$=\frac{1}{2}\iint_D [\cos^2(x^2+\mathrm{y})+\sin^2(x+y^2)+\cos^2(y^2+x)+\sin^2(y+x^2)]\mathrm{d}A$$

$$=\iint_D \mathrm{d}A=3\pi$$

11. The function $f(u,v)$ has second-order continuous partial derivatives, and $df\,\big|_{(1,1)}=3du+4dv$, let

$y=f(\cos x,1+x^2)$, try to find $\frac{d^2y}{dx^2}\big|_{x=0}$.

Solu:

$$\text{By } df\,\big|_{(1,1)}=3du+4dv\Rightarrow f_u'(1,1)=3,\ f_v'(1,1)=4.$$

$$\frac{dy}{dx}=f_u'\cdot(-\sin x)+f_v'\cdot 2x$$

$$\Rightarrow\frac{d^2y}{dx^2}=[f_{uu}''\cdot(-\sin x)+f_{uv}''\cdot 2x](-\sin x)+f_u'\cdot(-\cos x)$$

$$+[f_{vu}''\cdot(-\sin x)+f_{vv}''\cdot 2x](2x)+2f_v'$$

$$\Rightarrow\frac{d^2y}{dx^2}\big|_{x=0}=f_u'(1,1)\cdot(-1)+2f_v'(1,1)=-3+8=5.$$

12. Find the extreme values of $f(x,y)=2x^3-9x^2-6y^4+12x+24y$.

Solu:

$$f(x,y)=2x^3-9x^2-6y^4+12x+24y$$

$$\Rightarrow\begin{cases}f_x'=6x^2-18x+12=0\\ f_y'=-24y^3+24=0\end{cases}\Rightarrow(1,1)\text{and}(2,1)$$

$$H(x,y)=\begin{pmatrix}f_{xx}'' & f_{xy}''\\ f_{yx}'' & f_{yy}''\end{pmatrix}=\begin{pmatrix}12x-18 & f_{xy}''\\ f_{yx}'' & -72y^2\end{pmatrix}$$

$$\Rightarrow H(1,1)=\begin{pmatrix}-6 & 0\\ 0 & -72\end{pmatrix}>0,\ f_{xx}''=-6<0;$$

The extreme maximum value is $f(1,1)=23$ at point(1,1),

$$\Rightarrow H(2,1)=\begin{pmatrix}6 & 0\\ 0 & -72\end{pmatrix}<0;\text{No extreme value at point}(2,1).$$

13. Find $\iint\limits_D \frac{x}{\sqrt{x^2+y^2}}dxdy, D=\left\{(x,y)\middle|\sqrt{1-y^2}\le x\le 1,-1\le y\le 1\right\}$.

Solu:

D is symmetrical about x-axis,

$f(x,y)$ is an even function about y.

$$\iint\limits_D \frac{x}{\sqrt{x^2+y^2}}dxdy = 2\iint\limits_{D_1} \frac{x}{\sqrt{x^2+y^2}}dxdy = 2\int_0^1 dy\int_{\sqrt{1-y^2}}^1 \frac{x}{\sqrt{x^2+y^2}}dx$$

$$=\int_0^1 dy\int_{\sqrt{1-y^2}}^1 \frac{1}{\sqrt{x^2+y^2}}dx^2$$

$$=2\int_0^1\sqrt{1+y^2}dy-2=\left[y\sqrt{1+y^2}+\ln(y+\sqrt{1+y^2})\right]\Big|_0^1-2$$

$$=\sqrt{2}-2+\ln(1+\sqrt{2}).$$

14. Find the maximum and minimum values of $u=x-2y+2z$ subjected to the condition

$x^2+y^2+z^2=1$.

Solu：Let $L=x-2y+2z+\lambda\left(x^2+y^2+z^2-1\right)$

then $\begin{cases} L_x=1+2\lambda x=0 \\ L_y=-2+2\lambda y=0 \\ L_z=2+2\lambda z=0 \\ x^2+y^2+z^2=1 \end{cases} \Rightarrow \begin{cases} x=\frac{-1}{2\lambda} \\ y=\frac{1}{\lambda} \\ z=\frac{-1}{\lambda} \end{cases} \Rightarrow \lambda=\pm\frac{3}{2}, \begin{cases} x=\frac{-1}{3} \\ y=\frac{2}{3} \\ z=\frac{-2}{3} \end{cases}$ or $\begin{cases} x=\frac{1}{3} \\ y=\frac{-2}{3} \\ z=\frac{2}{3} \end{cases}$

$u\left(-\frac{1}{3},\frac{2}{3},-\frac{2}{3}\right)=-\frac{1}{3}-\frac{4}{3}-\frac{4}{3}=-3, u\left(\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)=\frac{1}{3}+\frac{4}{3}+\frac{4}{3}=3$, Max 3，min -3

15. Let $f(x\,,\,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2\neq 0\\ 0 & x^2+y^2=0\end{cases}$

Prove：1）the partial derivatives of $f(x,y)$ exist at (0,0); 2） $f(x,y)$ is not differentiable at (0,0).

Proof: 1）by $f_x(0\,,\,0)=\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{0-0}{\Delta x}=0$

$$f_y(0\,,\,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

$f(x,y)$ has partial derivative at $(0\,,\,0)$

2）by $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\Delta z-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}$

take $\Delta y=k\Delta x$ 时

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}=\lim_{\Delta x\to 0}\frac{k^2}{1+k^2}=\frac{k^2}{1+k^2}$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta z-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\neq 0$$

So, it is not differentiable.

16. Let $z=xf\left(xy\,,\,\dfrac{y}{x}\right)$, f has second-order continuous partial derivatives, find $\dfrac{\partial z}{\partial y}$, $\dfrac{\partial^2 z}{\partial y\partial x}$.

Solu: $\dfrac{\partial z}{\partial x}=f+x\left(yf_1'-\dfrac{y}{x^2}f_2'\right)$ $\dfrac{\partial z}{\partial y}=x\left(xf_1'+\dfrac{1}{x}f_2'\right)$, ,

By the given conditon, it has second-order continuous partial derivatives, we have

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$$

$$\frac{\partial^2 z}{\partial y\partial x}=2xf_1'+x^2\left(yf_{11}''-\frac{y}{x^2}f_{12}''\right)+yf_{21}''-\frac{y}{x^2}f_{22}''=2xf_1'+x^2yf_{11}''-\frac{y}{x^2}f_{22}''$$

17. Let $G(u\ ,\ v)$ be differentiable, and the equation $G\left(\frac{x}{z}\ ,\ \frac{y}{z}\right)=0$ implies that z is a function of x, y (namely $z=z(x,y)$), please compute $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$.

Solu：**Method A**：Chain rule,

$$G\left(\frac{x}{z},\frac{y}{z}\right)=0 \Rightarrow G_1'\left(\frac{1}{z}+x(-\frac{1}{z^2})\frac{\partial z}{\partial x}\right)+G_2'\left(y(-\frac{1}{z^2})\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{zG_1'}{xG_1'+yG_2'},$$

Similarly, $$\Rightarrow \frac{\partial z}{\partial y}=\frac{zG_2'}{xG_1'+yG_2'},$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\cdot\frac{zG_1'}{xG_1'+yG_2'}+y\cdot\frac{zG_2'}{xG_1'+yG_2'}=z.$$

Method B：Implicit function,

Let $F(x,y,z)=G\left(\frac{x}{z},\frac{y}{z}\right)=0$

$$\Rightarrow\begin{cases} F_x=G_1'\cdot\frac{1}{z} \\ F_y=G_2'\cdot\frac{1}{z} \\ F_z=G_1'x(-\frac{1}{z^2})+G_2'y(-\frac{1}{z^2}) \end{cases} \Rightarrow\begin{cases} \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{zG_1'}{xG_1'+yG_2'} \\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{zG_2'}{xG_1'+yG_2'} \end{cases}$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\cdot\frac{zG_1'}{xG_1'+yG_2'}+y\cdot\frac{zG_2'}{xG_1'+yG_2'}=z.$$

18. Let $z=xyf\left(\frac{y}{x}\right)$, and f is differentiable, if $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=y^2(\ln y-\ln x)$ find $f(1), f'(1)$.

Solu:

$$\frac{\partial z}{\partial x}=yf(\frac{y}{x})+xyf'(\frac{y}{x})\cdot(-\frac{y}{x^2})=yf(\frac{y}{x})-\frac{y^2}{x}f'(\frac{y}{x})$$

$$\frac{\partial z}{\partial y}=xf(\frac{y}{x})+xyf'(\frac{y}{x})\cdot\frac{1}{x}=xf(\frac{y}{x})+yf'(\frac{y}{x})$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xyf(\frac{y}{x})=y^2(\ln\frac{y}{x})$$

$$f(\frac{y}{x})=\frac{1}{2}\frac{y}{x}\ln(\frac{y}{x})\Rightarrow f(u)=\frac{1}{2}u\ln u\Rightarrow f(x)=\frac{1}{2}x\ln x,\ f'(x)=\frac{1}{2}(\ln x+1)$$

$$x=1\Rightarrow f(1)=0,\ f'(1)=\frac{1}{2}.$$