Sit NO.______ Major______ Student ID______ Chinese Name______

（DONOY WRITE YOUR ANSWER IN THIS AREA）

Sale Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

2021-2022-2《Calculus II》 Exam Paper B

Notice: **1. Make sure that you have filled the form on the left side of seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	1-10	11-18	Sum
Score			

一. Finish the following questions. (1-10: $6'\times 10=60'$ **)**

1. Let $\vec{a}=<2,1,6>$, $\vec{b}=<9,8,5>$, find (1) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$;(2) $(2\vec{a}+\vec{b})\times(3\vec{a}-\vec{b})$.

Ans：

(1) $\because\ \vec{a}-\vec{b}=<-7,-7,1>,\ \vec{a}+\vec{b}=<11,9,11>,$

$\therefore (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=-129;$ $3'$

(2) $\because\ 2\vec{a}+\vec{b}=<13,10,7>,\ 3\vec{a}-\vec{b}=<-3,-5,13>,$

$\therefore (2\vec{a}+\vec{b})\times(3\vec{a}-\vec{b})=<215,-220,-35>.$ $3'$

2. Determine the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{4n^2-1}$ is absolutely convergent, conditionally convergent, or divergent and give the proof.

Ans:

$\sum_{n=1}^{\infty}|(-1)^{n-1}\frac{1}{4n^2-1}|=\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$, note that

$\because \frac{\frac{1}{4n^2-1}}{\frac{1}{n^2}}\to\frac{1}{4}<1,(n\to\infty)$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $4'$

$\therefore \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{4n^2-1}$ is absolutely convergent. $2'$

3. Expand the function $f(x)=\dfrac{1}{x^2-5x+6}$ into Maclaurin series (the power series of x).

Ans:

$$f(x)=\frac{1}{x^2-5x+6}=\frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}}-\frac{1}{3}\cdot\frac{1}{1-\frac{x}{3}}$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n \qquad 4'$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}}-\frac{1}{3^{n+1}}\right)x^n,(-2<x<2). \qquad 2'$$

4. Find the convergent radius, convergent set and sum function $S(x)$ for the series $\sum_{n=1}^{\infty}\dfrac{x^n}{n(n+1)}$.

Ans:

$$a_n=\frac{1}{n(n+1)},\ R=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=1,$$

As $x=\pm1$, $\sum_{n=1}^{\infty}\dfrac{x^n}{n(n+1)}=\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$ converges,

So, the covergence set is $[-1,1]$.

$$S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}=\sum_{n=1}^{\infty}\frac{x^n}{n}-\sum_{n=1}^{\infty}\frac{x^n}{n+1}=\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{x}\sum_{n=2}^{\infty}\frac{x^n}{n}, \qquad 3'$$

$$\because g(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=-\ln(1-x),$$

$$\therefore S(x)=-\ln(1-x)-\frac{1}{x}[-\ln(1-x)-x]=1+(\frac{1}{x}-1)\ln(1-x),$$

when $x=0$, $\sum_{n=1}^{\infty}\dfrac{x^n}{n(n+1)}=0$; when $x=1$, $\sum_{n=1}^{\infty}\dfrac{x^n}{n(n+1)}=\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}=1$;

$$\therefore S(x)=\begin{cases}1+(\frac{1}{x}-1)\ln(1-x), -1\le x<1 \text{ and } x\ne 0,\\ 0, \qquad x=0,\\ 1,, \qquad x=1.\end{cases} \qquad 3'$$

5. Find the parametric equations of the line $\begin{cases} x-y-z=1 \\ 3x+y-z=3 \end{cases}$.

Ans:

It is to test that the line goes through $A(0,1,-2)$ and $B(1,0,0)$

$\overrightarrow{AB} = <1,-1,2>$, $3'$

the parametric equation is : $x=1+t, y=-t, z=2t$. $3'$

(NOTE: The answer is NOT unique.

NOTE: You still can obtain the correct answers by cross product of two normal vectors.)

6. Let $f(x,y)=\begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}$ (1) Prove $f(x,y)$ is continuous at $(0,0)$;

(2) Find the partials $f_x(0,0)$ and $f_y(0,0)$; (3) Prove $f(x,y)$ is not differentiable at $(0,0)$.

Ans:

(1) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$, f is continous at $(0,0)$. $2'$

(2) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-0}{h} = 0$,

Similarly, $f_y(0,0)=0$. $2'$

(3) At $(0,0)$,

$$\lim_{\rho\to 0}\frac{\Delta z-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]}{\rho} = \lim_{\rho\to 0}\frac{2\Delta x\Delta y}{\Delta x^2+\Delta y^2}$$

does not exist, so $f(x,y)$ is not differentiable at $(0,0)$. $2'$

7. Suppose $z=z(x,y)$ is determined by the equation $y=f(x,y,z)$, where f is differentiable, and $f_z(x,y,z)\neq 0$, find $\mathrm{d}z$.

Ans:

$$\mathrm{d}y = \mathrm{d}f(x,y,z) = f_x\mathrm{d}x + f_y\mathrm{d}y + f_z\mathrm{d}z, \quad 4'$$

$$\mathrm{d}z = \frac{1}{f_z}\left[-f_x\mathrm{d}x + (1-f_y)\mathrm{d}y\right]. \quad 2'$$

8. Let $z = y^4 f(xy, \frac{x}{y})$, where f has the second-order continuous partial derivatives, find $\frac{\partial^2 z}{\partial x \partial y}$.

Ans:

$\frac{\partial z}{\partial x} = y^5 f_1' + y^3 f_2'$; $3'$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 5y^4 f_1' + 3y^2 f_2' + xy^5 f_{11}'' - xy f_{22}''$. $3'$

(NOTE $f_{12}'' = f_{21}''$)

9. Find the maximum of $u = \ln x + \ln y + 3\ln z$ subjected to the constraint $x^2 + y^2 + z^2 = 5r^2$

$(x > 0, y > 0, z > 0, r > 0)$.

Ans:

Let $L(x, y, z, \lambda) = \ln x + \ln y + 3\ln z + \lambda(x^2 + y^2 + z^2 - 5r^2)$,

$$\begin{cases} L_x = \frac{1}{x} + 2\lambda x = 0, \\ L_y = \frac{1}{y} + 2\lambda y = 0, \\ L_z = \frac{3}{z} + 2\lambda z = 0, \\ L_\lambda = x^2 + y^2 + z^2 - 5r^2 = 0, \end{cases} \Rightarrow \begin{cases} x = r \\ y = r \\ z = \sqrt{3}r \end{cases} .3'$$

$$u_{\max} = \ln r + \ln r + 3\ln\sqrt{3}r = \ln\left(3\sqrt{3}r^5\right). \quad 3'$$

10. Find the directional derivative of $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ at $(1,1,1)$ in the direction toward

$(1, -2, 5)$.

Ans:

Let $P = (1,1,1), Q = (1,-2,5)$. $\overrightarrow{PQ} = < 0, -3, 4 >, \boldsymbol{u}_{\overrightarrow{PQ}} = \frac{1}{5} < 0, -3, 4 >$, $3'$

$\nabla f(1,1,1) = < \frac{2}{3}, \frac{2}{3}, \frac{2}{3} >, \frac{\partial u}{\partial l} = \nabla f(1,1,1) \cdot u_{\overrightarrow{PQ}} = 0 \times \frac{2}{3} + \left(-\frac{3}{5}\right) \times \frac{2}{3} + \frac{4}{5} \times \frac{2}{3} = \frac{2}{15}$. $3'$

二．**Finish the following questions. (11-18:** $5'\times 8=40'$)

11. Find the double integral $I=\int_0^1 dx\int_x^{\sqrt{x}}\frac{\sin y}{y}dy$.

Ans:

$$I=\int_0^1 dx\int_x^{\sqrt{x}}\frac{\sin y}{y}dy$$

$$=\int_0^1\frac{\sin y}{y}dy\int_{y^2}^{y}dx$$

$$=\int_0^1(1-y)\sin ydy \qquad 3'$$

$$=\int_0^1(y-1)d\cos y$$

$$=(y-1)\cos y\Big|_0^1-\int_0^1\cos ydy$$

$$=1-\sin 1 \qquad 2'$$

12. Find $\iint\limits_D e^{\max\{x^2,y^2\}}dxdy$, $D=\{(x,y)\,|\,0\le x\le 1,0\le y\le 1\}$.

Ans:

$D_1=\{(x,y)\,|\,0\le x\le 1,0\le y\le x\},D_2=\{(x,y)\,|\,0\le x\le 1,x\le y\le 1\}$

$$\iint\limits_D e^{\max\{x^2,y^2\}}dxdy=\iint\limits_{D_1}e^{\max\{x^2,y^2\}}dxdy+\iint\limits_{D_2}e^{\max\{x^2,y^2\}}dxdy$$

$$=\iint\limits_{D_1}e^{x^2}dxdy+\iint\limits_{D_2}e^{y^2}dxdy$$

$$=\int_0^1 dx\int_0^x e^{x^2}dy+\int_0^1 dy\int_0^y e^{y^2}dx \qquad 3'$$

$$=\int_0^1 xe^{x^2}dx+\int_0^1 ye^{y^2}dy$$

$$=e-1 \qquad 2'$$

13. Find $\iiint\limits_\Omega z^2dV,\Omega=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$.

Ans :

According to the symmetrical properties, we have

$$\iiint\limits_\Omega z^2dv=\frac{1}{3}\iiint\limits_\Omega(x^2+y^2+z^2)dv$$

$$=\frac{1}{3}\int_0^\pi d\varphi\int_0^{2\pi}d\theta\int_0^1 r^2\cdot r^2\sin\varphi dr \qquad 3'$$

$$=\frac{1}{3}\times 2\times 2\pi\times\frac{1}{5}$$

$$=\frac{4}{15}\pi \qquad 2'$$

14. L is the curve of $y=x^2(0\le x\le\sqrt{2})$, find $\int_L xds$.

Ans :

$$\int_L xds=\int_0^{\sqrt{2}} x\sqrt{1+(2x)^2}dx$$

$$=\frac{1}{8}\int_0^{\sqrt{2}}\sqrt{1+4x^2}d(1+4x^2) \qquad 3'$$

$$=\frac{1}{12}(1+4x^2)^{\frac{3}{2}}\Big|_0^{\sqrt{2}}=\frac{13}{6} \qquad 2'$$

15. $G=\{(x,y,z)\,|\,x+y+z=1,x\ge 0,y\ge 0,z\ge 0\}$, find $\iint\limits_G y^2dS$.

Ans :

$D=\{(x,y)\,|\,x+y\le 1,x\ge 0,y\ge 0\}$

$$\iint\limits_\Sigma y^2dS=\iint\limits_D y^2\bullet\sqrt{3}dxdy \qquad 3'$$

$$=\sqrt{3}\int_0^1 dy\int_0^{1-y} y^2dx=\frac{\sqrt{3}}{12} \qquad 2'$$

16. Σ is the outside of surface $x^2+y^2+z^2=a^2(a>0)$, calculate $\oiint\limits_\Sigma x^3dydz+y^3dzdx+z^3dxdy$.

Ans :

From Gauss formula, and use spherical coordinates to calculate triple integral.

$$\Rightarrow I=3\iiint\limits_\Omega (x^2+y^2+z^2)dv(\Omega\text{is the sapce region enclosed by }\Omega) \qquad 2'$$

$$=3\int_0^{2\pi} d\theta\int_0^\pi \sin\varphi d\varphi\int_0^a r^2\cdot r^2dr=\frac{12}{5}\pi a^5. \qquad 3'$$

17. If the curve integral $\int_L \frac{xdx-aydy}{x^2+y^2-1}$ is independent of path in $D=\{(x,y)\,|\,x^2+y^2<1\}$, find a.

Ans :

The curve integral $\int_L \frac{xdx-aydy}{x^2+y^2-1}$ is independent

of the path in the region $D=\{(x,y)\,|\,x^2+y^2<1\}$.

$$P=\frac{x}{x^2+y^2-1},\quad Q=\frac{-ay}{x^2+y^2-1}$$

$$\frac{\partial P}{\partial y}=\frac{-2xy}{(x^2+y^2-1)^2},\ \frac{\partial Q}{\partial x}=\frac{2axy}{(x^2+y^2-1)^2} \qquad 3'$$

it is independent of the path $\Rightarrow \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}\Rightarrow a=-1$ $\quad 2'$

18. Find the general solution to $2y''+y'-y=2e^{x}$.

Ans:

(1) The characteristic equation is

$$2r^{2}+r-1=0,(2r-1)(r+1)=0,r_{1}=-1,r_{2}=\frac{1}{2},Y=c_{1}e^{\frac{1}{2}x}+c_{2}e^{-x} \qquad 2'$$

(2) $f(x)=P_{m}(x)e^{\lambda x},m=0,\lambda=1$ is not the characteristic root, so we have $k=0$.

The special solution is $y^{*}=x^{k}Q_{m}(x)e^{\lambda x}=ae^{x}$, we have $a=1$, $y^{*}=e^{x}$ $2'$.

(3) The general solution is $y=Y+y^{*}=c_{1}e^{\frac{1}{2}x}+c_{2}e^{-x}+e^{x}$. $1'$