Class + Name + Homework Number

Question Number	1-12(6′)	13-16(7′)	Total score
Score			

1.Let $\lim\limits_{x\to x_0}\dfrac{f\left(\dfrac{x+x_0}{2}\right)-f\left(x_0\right)}{x-x_0}=a$, then find $f'\left(x_0\right)$.

Solu: $\lim\limits_{x\to x_0}\dfrac{f\left(\dfrac{x+x_0}{2}\right)-f\left(x_0\right)}{x-x_0}=a$

$t=x-x_0,\dfrac{x+x_0}{2}=x_0+\dfrac{t}{2},x\to x_0\Rightarrow t\to 0$

$\therefore \lim\limits_{t\to 0}\dfrac{\frac{1}{2}\cdot f\left(x_0+\frac{t}{2}\right)-f\left(x_0\right)}{\frac{t}{2}}=a$

$\Rightarrow \dfrac{1}{2}f'(x_0)=a\Rightarrow f'(x_0)=2a.$

2. If $f'\left(x_0\right)$ exists, then find $\lim\limits_{h\to 0}\dfrac{f\left(x_0+h^2\right)-f\left(x_0-h^2\right)}{\sin\left(h^2\right)}$

Solu: $\lim\limits_{h\to 0}\dfrac{f\left(x_0+h^2\right)-f\left(x_0-h^2\right)}{\arcsin\left(h^2\right)}$

$=\lim\limits_{h\to 0}\dfrac{f\left(x_0+h^2\right)-f\left(x_0-h^2\right)}{h^2}(\arcsin\left(h^2\right)\sim h^2)$

$=\lim\limits_{h\to 0}[\dfrac{f\left(x_0+h^2\right)-f(x_0)}{h^2}-\dfrac{f\left(x_0-h^2\right)-f(x_0)}{h^2}]$

$=2f'(x_0).$

3. Find $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{x^2}$.

Solu: $L=\lim\limits_{x\to 0}(\dfrac{\cos x-1}{x^2}-\dfrac{e^{-\frac{x^2}{2}}-1}{x^2})$

$=\lim\limits_{x\to 0}\dfrac{-\frac{x^2}{2}}{x^2}-\lim\limits_{x\to 0}\dfrac{-\frac{x^2}{2}}{x^2}$

$=0.$

4. Find $\lim\limits_{x\to+\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$.

$$\text{Solu:}L=\lim_{t=\frac{1}{x}\to0}\left[\frac{1}{t}-\frac{\ln(1+t)}{t^2}\right]=\lim_{t\to0}\frac{t-\ln(1+t)}{t^2}$$

$$=\lim_{t\to0}\frac{1-\frac{1}{1+t}}{2t}=\lim_{t\to0}\frac{1}{2(1+t)}=\frac{1}{2}.$$

5. Find $\lim\limits_{x\to0}\left(\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}\right)^{\frac{1}{x}}$.

$$\text{Solu:}L=e^{\lim\limits_{x\to0}\frac{\ln\frac{a_1^x+a_2^x+\cdots+a_n^x}{n}}{x}}\left(\frac{0}{0}\right)$$

$$=e^{\lim\limits_{x\to0}\frac{\frac{a_1^x\ln a_1+a_2^x\ln a_2+\cdots+a_n^x\ln a_n}{a_1^x+a_2^x+\cdots+a_n^x}}{1}}$$

$$=e^{\frac{\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}}{x}}=\sqrt[n]{a_1a_2\cdots a_n}.$$

The method is not unique.

6. Find $\lim\limits_{x\to0}\frac{x-\int_0^x\frac{\sin t}{t}dt}{x-\sin x}$.

$$\text{Solu:}L=\lim_{x\to0}\frac{x-\int_0^x\frac{\sin t}{t}dt}{x-\sin x}$$

$$=\lim_{x\to0}\frac{1-\frac{\sin x}{x}}{1-\cos x}=\lim_{x\to0}\frac{\frac{x-\sin x}{x}}{\frac{1}{2}x^2}=2\lim_{x\to0}\frac{x-\sin x}{x^3}$$

$$=2\lim_{x\to0}\frac{1-\cos x}{3x^2}=2\lim_{x\to0}\frac{\frac{1}{2}x^2}{3x^2}=\frac{1}{3}.$$

7. Find the extreme values of $f(x)=(x+1)^{\frac{2}{5}}(5-2x)$.

Solu: $f'(x)=\frac{2}{5}(x+1)^{\frac{-3}{5}}(5-2x)-2(x+1)^{\frac{2}{5}}=\frac{2}{5}(x+1)^{\frac{-3}{5}}(5-2x-5x-5)=-\frac{14x}{5}(x+1)^{\frac{-3}{5}}$

Get stationary point $x_1=0$， singular point $x_2=-1$，

$f'(x)<0, x\in(-\infty,-1)$, $f'(x)>0, x\in(-1,0)$, $f'(x)<0, x\in(0,+\infty)$

So the extreme minimum value $f_{\min}(-1)=0$， extreme maximum value $f_{\max}(0)=5$..

8. If $f(x)=\begin{cases} xe^{-\frac{1}{x^2}}, x\neq 0 \\ 0, x=0 \end{cases}$, (1) find $f'(x)$; (2) determines $f(x)$ and $f'(x)$ is continuous or not at $x=0$.

Solu:

(1) When $x=0$, $f'(0)=\lim\limits_{x\to 0}\dfrac{xe^{-\frac{1}{x^2}}-0}{x}=\lim\limits_{x\to 0}e^{-\frac{1}{x^2}}=0$

when $x\neq 0$, $f'(x)=e^{-\frac{1}{x^2}}+x\cdot e^{-\frac{1}{x^2}}\cdot\left(-\dfrac{-2}{x^3}\right)=e^{-\frac{1}{x^2}}\cdot\left(1+\dfrac{2}{x^2}\right)$, namely,

we have $f'(x)=\begin{cases} e^{-\frac{1}{x^2}}\cdot\left(1+\dfrac{2}{x^2}\right), & x\neq 0, \\ 0, & x=0. \end{cases}$

(2)Because $f'(0)$ exists, so $f(x)$ is continuous at $x=0$;

Because $\lim\limits_{x\to 0}f'(x)=\lim\limits_{x\to 0}e^{-\frac{1}{x^2}}\cdot\left(1+\dfrac{2}{x^2}\right)$

$$=\lim_{t\to\infty}e^{-t^2}\cdot\left(1+2t^2\right)=\lim_{t\to\infty}\frac{1+2t^2}{e^{t^2}}$$

$$=\lim_{t\to\infty}\frac{4t}{2te^{t^2}}=\lim_{t\to\infty}\frac{2}{e^{t^2}}=0=f'(0).$$

so $f'(x)$ is still continuous at $x=0$.

9. If $\begin{cases} x=\ln\sqrt{t^2+1} \\ y=t-\arctan t \end{cases}$ implies $y=y(x)$, find $\dfrac{d^2y}{dx^2}$.

Solu: $\dfrac{dy}{dx}=\dfrac{1-\dfrac{1}{1+t^2}}{\dfrac{1}{2}\dfrac{1}{1+t^2}\cdot 2t}=t,$

$$\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)'_t\cdot\frac{1}{x'(t)}=1\cdot\frac{1}{\dfrac{t}{1+t^2}}=\frac{1+t^2}{t}.$$

10. Let $y = x^2 \sin x$, find $y^{(100)}$.

Solu: Let $u = \sin x, v = x^2$, by Leibniz formula

$$y^{(100)} = (\sin x)^{(100)} x^2 + 100(\sin x)^{(99)} (x^2)'$$
$$+ \frac{100 \times 99}{2!} (\sin x)^{(98)} (x^2)'' + 0$$
$$= x^2 \sin(x + 100 \times \frac{\pi}{2}) + 200x \sin(x + 99 \times \frac{\pi}{2})$$
$$+ 100 \times 99 \sin(x + 98 \times \frac{\pi}{2})$$
$$= x^2 \sin x - 200x \cos x - 9900 \sin x.$$

10. Let $y = xe^{-x}$, find $y^{(n)}$.

Solu: $y' = e^{-x} - xe^{-x} = (1-x)e^{-x}$

$$y'' = -e^{-x} + (1-x)e^{-x}(-1) = (-1)(2-x)e^{-x}$$

$$y''' = (-1)\left[-e^{-x} + (2-x)e^{-x}(-1)\right] = (-1)^2 (3-x)e^{-x}$$

So $y^{(n)} = (-1)^{n-1}(n-x)e^{-x}$

The proof is as follows:

when $n = 1$, it is true

let $n = k$, we have $y^{(k)} = (-1)^{k-1}(k-x)e^{-x}$

then $y^{(k+1)} = (-1)^{k-1}\left[-e^{-x} + (k-x)e^{-x}(-1)\right] = (-1)^{(k+1)-1}\left[(k+1)-x\right]e^{-x}$, the result is proved.

11. If $\int e^{-x^2} dx = F(x) + C$ and $\int f(x)dx = F(\sqrt{\ln x}) + C$, find $f(x)$.

Solu: $\int e^{-x^2} dx = F(x) + C \Rightarrow F'(x) = e^{-x^2}$

$$\int f(x)dx = F(\sqrt{\ln x}) + C$$

$$\Rightarrow f(x) = F'(\sqrt{\ln x}) \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x}$$

$$\Rightarrow f(x) = e^{-\ln x} \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{x} \cdot \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x^2\sqrt{\ln x}}.$$

12. Let $y = y(x)$ is determined by $\begin{cases} x = 3t^2 + 2t + 3 \\ e^y \sin t - y + 1 = 0 \end{cases}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{t=0}$.

Solu: $t = 0, x = 3, y = 1, \begin{cases} x_t' = 6t + 2 \\ e^y y_t' \sin t + e^y \cos t - y_t' = 0 \end{cases} \Rightarrow \begin{cases} x_t' = 2 \\ e - y_t' = 0 \end{cases}, \quad \frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{t=0} = \frac{e}{2}.$

12. Let $y = y(x)$ is determined by $x^y + y^x = 17$, find $\frac{\mathrm{d}\,y}{\mathrm{d}\,x}$ at $x = 2, y = 3$.

Solu: $e^{y\ln x} + e^{x\ln y} = 17, x^y\left(y'\ln x + \frac{y}{x}\right) + y^x\left(\ln y + \frac{x}{y}y'\right) = 0,$

$$y'8\ln 2 + 12 + 9\ln 3 + 6y' = 0, \Rightarrow \frac{\mathrm{d}\,y}{\mathrm{d}\,x} \text{ at } x = 2, y = 3: \ -\frac{12 + 9\ln 3}{6 + 8\ln 2}.$$

13. Prove $e^x > \frac{e}{2}(x^2 + 1)$ when $x > 1$.

Proof: Let $f(x) = e^x - \frac{e}{2}(x^2 + 1)$, $f(x)$ is continuous on $[1, +\infty)$ and derivative in $(1, +\infty)$.

$f'(x) = e^x - ex$, $f''(x) = e^x - e > 0$, then $f'(x)$ in monotonic increasing on $[1, +\infty)$.

And $f'(x) > f'(1) = 0$, so $f(x)$ in monotonic increasing on $[1, +\infty)$, $f(x) > f(1) = 0$.

14. Prove $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}\,(x_0 > 0)$ by $\varepsilon - \delta$.

Proof: (The method in not unique)

$|x - x_0| \le x_0$ makes that $x \ge 0$

$$\because |f(x) - L| = \left|\sqrt{x} - \sqrt{x_0}\right| = \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| \le \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon,$$

$\forall \varepsilon > 0$, in order to $|f(x) - L| < \varepsilon$

Only need to $|x - x_0| < \sqrt{x_0}\,\varepsilon$ and $x > 0$

Take $\delta = \min\left\{x_0 \sqrt{x_0}\,\varepsilon\right\}$ when $0 < |x - x_0| < \delta$

we have $\left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon, \therefore \lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

15. Find a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant to minimize the triangle area bounded by the tangent line passing through the point and the two coordinate axes.

Solu:

Derivative for the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for x, then $\frac{2x}{a^2} + \frac{2y}{b^2} \cdot y' = 0, y' = -\frac{xb^2}{ya^2}$,

So, the tangent line passing through $(x_0, y_0): y - y_0 = -\frac{x_0 b^2}{y_0 a^2}(x - x_0)$, namely $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$

Let $x = 0$, we have $y = \frac{b^2}{y_0}$, let $y = 0$ we have $x = \frac{a^2}{x_0}$, the area of triangle is:

$$s = \frac{1}{2}\frac{a^2}{x_0}\frac{b^2}{y_0} = \frac{a^2 b^2}{2x_0\sqrt{b^2\left(1 - \frac{x_0^2}{a^2}\right)}} = \frac{a^3 b}{2x_0\sqrt{a^2 - x_0^2}}$$

Let $f(x)=x^2(a^2-x^2), x\in(0,a)$, and thenlet $f'(x)=(a^2x^2-x^4)'=2a^2x-4x^3=0$,we get the unique stationary point $x_0=\frac{\sqrt{2}}{2}a$, and $y_0=\frac{\sqrt{2}}{2}b$.

The unique stationary point is our minimum point, so the point $\left(\frac{\sqrt{2}}{2}a,\frac{\sqrt{2}}{2}b\right)$ is our desired poinit.

16. When $x\to 1$, the two infinitesimals $\alpha=\ln(x^3-3x^2+3x)$ and $\beta=A(\sqrt{x+3}-2)^k$ are equivalent, fined constants A,k.

Solu:

$$
\begin{aligned}
1&=\lim_{x\to1}\frac{\beta}{\alpha}\\
&=\lim_{x\to1}\frac{A(\sqrt{x+3}-2)^k}{\ln(1+x^3-3x^2+3x-1)}\\
&=\lim_{x\to1}\frac{A(\sqrt{x+3}-2)^k\cdot\left(\sqrt{x+3}+2\right)^k}{\ln(1+(x-1)^3)\cdot\left(\sqrt{x+3}+2\right)^k}\\
&=\lim_{x\to1}\frac{A(x-1)^k}{\left(\sqrt{x+3}+2\right)^k(x-1)^3}\Rightarrow\text{then we get}
\end{aligned}
$$

$k=3,$

$A=64.$