

Mid-Term Test

2022-2023-1

Name: _____ Class: _____ Student ID: _____

1. (5') The equation $|x|^{\frac{1}{4}} + |x|^{\frac{1}{2}} - \frac{1}{2}\cos x = 0$ has _____ root(s) on $(-\infty, +\infty)$.

2. (5') Find the limit $\lim_{n \rightarrow +\infty} \left(\frac{1}{n^k} + \frac{2}{n^k} + \cdots + \frac{n}{n^k} \right), k \in \mathbb{R}$.

3. (5') Let $x_n = \frac{1}{3} + \frac{1}{15} + \cdots + \frac{1}{4n^2-1}$, please find $\lim_{n \rightarrow +\infty} x_n$.

4. (5') Try to prove that $\lim_{n \rightarrow +\infty} (1 + 2^n + 3^n)^{\frac{1}{n}} = 3$.

5. (5') Find the limit $\lim_{x \rightarrow +\infty} \frac{\ln(x \ln x)}{x^a}, a > 0$.

6. (7') If $f(x) = \lim_{n \rightarrow +\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ is continuous for $x \in \mathbb{R}$, try to find a and b .

7. (7') If $f(x)$ satisfies $f(x_1 + x_2) = f(x_1) + f(x_2)$, $\forall x_1, x_2 \in \mathbb{R}$, and $f(x)$ is continuous at $x = 0$. Try to prove that $f(x)$ is continuous at any point $x_0 \in \mathbb{R}$.

8. (5') Let $\begin{cases} x = f'(t), \\ y = tf'(t) - f(t), \end{cases}$ please find $\frac{d^2y}{dx^2}$.

9. (5') If $y = \left(\frac{a}{b}\right)^x \left(\frac{b}{x}\right)^a \left(\frac{x}{a}\right)^b$, $a > 0, b > 0$, please find $\frac{dy}{dx}$.

10. (5') If $F(x) = \min\left\{x, \frac{1}{x}\right\}$, and $x \in (0, 2)$, try to find $F'(x)$.

11. (5') If $y = (x^2 + 2x + 3) \cdot e^x$, please find $y^{(n)}(x)$.

12. (7') Let $f(x)$ be differentiable on $(a, +\infty)$, and $\lim_{x \rightarrow +\infty} f(x)$ exists, prove that $\lim_{x \rightarrow +\infty} f'(x) = 0$.

13. (7') Try to prove that $\left(1 + \frac{1}{x}\right)^{x+1} > e$, if $x > 0$.

14. (7') If the equation $ax + \frac{1}{x^2} = 1$ has only one root when $x > 0$. Try to find a .

15. (5') Find the integral $\int \frac{dx}{x^4 + x^6}$.

16. (5') Find the integral $\int \frac{\tan^3 x + \tan^2 x - \tan x - 1}{\tan x + 1} dx$.

17. (5') Find the general solution of the following differential equation

$$y' = xy + x + y + 1.$$

18. (5') Try to prove that $\frac{2}{3} < \int_0^1 \frac{dx}{\sqrt{2+x-x^2}} < \frac{1}{\sqrt{2}}$.