期中试卷参考答案

A.

（1）let $f(x)=\dfrac{1}{(x+1)(x+2)}$， find $f^{(n)}(0)$

Solution:

$$f(x)=\frac{1}{(x+1)(x+2)}=\frac{1}{x+1}-\frac{1}{x+2}$$
$$f^{(n)}(x)=(-1)^n\frac{n!}{(x+1)^{n+1}}-(-1)^n\frac{n!}{(x+2)^{n+1}}$$
$$f^{(n)}(0)=(-1)^n n!-(-1)^n\frac{n!}{2^{n+1}}=(-1)^n n!(1-\frac{1}{2^{n+1}})$$

（2）find the limit of $\lim\limits_{n\to\infty}(2^n+3^n+4^n)^{\frac{1}{n}}=$

Solution:

$$4^n<2^n+3^n+4^n<3\cdot4^n$$
$$\lim_{n\to\infty}(4^n)\frac{1}{n}=4,\lim_{n\to\infty}(3\cdot4^n)\frac{1}{n}=4$$

Hence, $\lim\limits_{n\to\infty}(2^n+3^n+4^n)^{\frac{1}{n}}=4$

（3）Calculate $\int\limits_{-2007}^{2007}(|x|\sin^{15}x+xe^{x^4}+1)dx$

Solution:

$$\int\limits_{-2007}^{2007}(|x|\sin^{15}x+xe^{x^4}+1)dx$$
$$=\int\limits_{-2007}^{-2007}|x|\sin^{15}xdx+\int\limits_{-2007}^{-2007}xe^{x^4}dx+\int\limits_{-2007}^{-2007}1dx$$
$$=0+0+4014$$
$$=4014$$

（4）find the dy/dx,where $x(t)=\int\limits_0^t e^{t^2}dt, y(t)=t^t$

Solution:

$$\ln y=t\ln t\Rightarrow\frac{1}{y}\frac{dy}{dt}=\ln t+1\Rightarrow\frac{dy}{dt}=t^t(\ln t+1),\ and\quad \frac{dx}{dt}=e^{t^2}$$

Hence, $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{t^t(\ln t+1)}{e^{t^2}}$

（**5**）find the limit $\lim\limits_{x\to\infty}(\frac{x-2}{x+2})^x$

Solution: $\lim\limits_{x\to\infty}(\frac{x-2}{x+2})^x=\lim\limits_{x\to\infty}(1+\frac{-4}{x+2})^x=\lim\limits_{x\to\infty}[(1+\frac{-4}{x+2})^{\frac{x+2}{-4}}]^{\frac{-4x}{x+2}}=\lim\limits_{x\to\infty}e^{\frac{-4x}{x+2}}=e^{-4}$

（**6**）find the dy/dx,where $x^y+y^x=\int_3^x e^{-t^2}dt+\int_2^y \sin t^2 dt+1$

Solution:

Let $Let\ \ y=f(x),then\ \ \mathrm{y}'=f'(x)=\frac{dy}{dx}$

$$(x^y+y^x)'=(e^{y\ln x}+e^{x\ln y})'=e^{y\ln x}(y'+\frac{y}{x})+e^{x\ln y}(\ln y+\frac{x}{y}y')$$

$$e^{y\ln x}(y'+\frac{y}{x})+e^{x\ln y}(\ln y+\frac{x}{y}y')=e^{-x^2}+\sin y^2\cdot y'$$

$$\Rightarrow(x^y+\frac{x}{y}y^x-\sin y^2)y'=e^{-x^2}-\frac{y}{x}x^y-\ln y\cdot y^x$$

$$\Rightarrow\frac{dy}{dx}=\frac{e^{-x^2}-yx^{y-1}-\ln y\cdot y^x}{x^y+xy^{x-1}-\sin y^2}$$

（**7**）find the length of the curve $y=\int_1^x\sqrt{u^3-1}du,\ \ 1\le \mathrm{x}\le 2$

Solution:

$$y'=\sqrt{x^3-1}$$

$$length=\int_1^2\sqrt{1+(y')^2}dx=\int_1^2\sqrt{x^3}dx=\frac{2}{5}(4\sqrt{2}-1)$$

（**8**）Find the dy/dx where $y=\frac{(x^2+3)^{\frac{2}{3}}(3x+1)^x}{(\arcsin x)^4}$

Solution:

$$\ln y=\frac{2}{3}\ln(x^2+3)+x\ln(3x+1)-4\ln(\arcsin x)$$

$$\Rightarrow\frac{y'}{y}=\frac{4x}{3(x^2+3)}+\ln(3x+1)+\frac{3x}{3x+1}-\frac{4}{\arcsin x\sqrt{1-x^2}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{(x^2+3)^{\frac{2}{3}}(3x+1)^x}{(\arcsin x)^4}(\frac{4x}{3(x^2+3)}+\ln(3x+1)+\frac{3x}{3x+1}-\frac{4}{\arcsin x\sqrt{1-x^2}})$$

（**9**）Prove the limit $\lim_{x\to 4}(2x+3)=11$

Solution:

$$\forall \varepsilon > 0, choose\ \ \delta=\frac{\varepsilon}{2}, when\ \ 0<\left|x-4\right|<\delta,\text{we have:}$$
$$\left|(2x+3)-11\right|=\left|2x-8\right|=2\left|x-4\right|<2\delta=\varepsilon$$

Hence， $\lim_{x\to 4}(2x+3)=11$

（**10**） $\int \frac{1}{x(1+x^{2007})}dx=$

Solution:

$$\begin{aligned}
&\int \frac{1}{x(1+x^{2007})}dx \\
&=\int \frac{x^{2006}}{x^{2007}(1+x^{2007})}dx=\frac{1}{2007}\int \frac{1}{x^{2007}(1+x^{2007})}dx^{2007} \\
&\underline{\underline{let\ u=x^{2007}}}\frac{1}{2007}\int \frac{1}{u(1+u)}du=\frac{1}{2007}\int (\frac{1}{u}-\frac{1}{u+1})du \\
&=\frac{1}{2007}(\ln u-\ln(u+1))+C=\frac{1}{2007}\ln(\frac{u}{u+1})+C \\
&=\frac{1}{2007}\ln(\frac{x^{2007}}{x^{2007}+1})+C
\end{aligned}$$

B

（**1**）Find the limit $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$

Solution:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})^2}\frac{1}{n}=\int_0^1\frac{1}{1+x^2}dx=\arctan\Big|_0^1=\frac{\pi}{4}$$

（**2**）Find the volume of the solid generated by revolving the region bounded by the curves $x=\sqrt{y}$ and x=$\frac{y^3}{32}$ about the x-axis .

Solution:

$$\begin{cases} x=\sqrt{y} \\ x=\dfrac{y^3}{32} \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \text{ or } \begin{cases} x=2 \\ y=4 \end{cases}$$

$$V = 2\pi\int_0^4 y(\sqrt{y} - \frac{y^3}{32})dy$$
$$= 2\pi(\frac{2}{5}y^{\frac{5}{2}} - \frac{y^5}{160})\Big|_0^4$$
$$= \frac{64\pi}{5}$$

（3）Solve integration $\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$

Solution:

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$
$$= \int \frac{1}{(\frac{1}{x^2}+1)^{\frac{3}{2}}} \cdot \frac{1}{x^3}dx = -\frac{1}{2}\int \frac{1}{(\frac{1}{x^2}+1)^{\frac{3}{2}}} d(\frac{1}{x^2}+1) \ \ , \ \ let\ \text{u=}\frac{1}{x^2}+1$$
$$= -\frac{1}{2}\int u^{-\frac{3}{2}}du = u^{-\frac{1}{2}} + C$$
$$= (\frac{1}{x^2}+1)^{-\frac{1}{2}} + C$$

（4）Evaluating the integration $\int_c^{2c} \frac{xdx}{\sqrt{x^2+cx-2c^2}}, c > 0$

Solution:

$$\int_c^{2c} \frac{xdx}{\sqrt{x^2+cx-2c^2}}$$
$$= \int_c^{2c} \frac{xdx}{\sqrt{(x+\frac{c}{2})^2 - \frac{9}{4}c^2}} \overset{let\ \text{t=x+c/2}}{=\!=\!=} \int_{\frac{3}{2}c}^{\frac{5}{2}c} \frac{t-\frac{c}{2}}{\sqrt{t^2-\frac{9}{4}c^2}}dt$$
$$= \int_{\frac{3}{2}c}^{\frac{5}{2}c} \frac{t}{\sqrt{t^2-\frac{9}{4}c^2}}dt - \frac{c}{2}\int_{\frac{3}{2}c}^{\frac{5}{2}c} \frac{1}{\sqrt{t^2-\frac{9}{4}c^2}}dt = \sqrt{t^2-\frac{9}{4}c^2}\Big|_{\frac{3}{2}c}^{\frac{5}{2}c} - \frac{c}{2}\int_{\frac{3}{2}c}^{\frac{5}{2}c} \frac{1}{\sqrt{t^2-(\frac{3}{2}c)^2}}dt$$
$$= 2c - \frac{c}{2}\int_{\frac{3}{2}c}^{\frac{5}{2}c} \frac{1}{\sqrt{t^2-(\frac{3}{2}c)^2}}dt = 2c - \frac{c}{2}(\ln\left|t+\sqrt{t^2-\frac{9}{4}c^2}\right|)\Big|_{\frac{3}{2}c}^{\frac{5}{2}c}$$
$$= (2 - \frac{\ln 3}{2})c$$

$(Note: \int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C,$ You can prove it by supposing x=a$*$sec(t))

（5）Determine the monotonicity and concavity of function $f(x)=\dfrac{x}{1+x^2}$

Solution:

$$f(x)=\frac{x}{1+x^2},\ f'(x)=\frac{(1+x^2)-2x^2}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2},\ let\ f(x)=0\Rightarrow x=1\ or\ x=-1$$

$when\ \ x\in(-\infty,-1),\ f'(x)<0; when\ \ x\in(-1,1),\ f'(x)>0; when\ \ x\in(1,+\infty),\ f'(x)<0;$

hence, $f(x)$ is monotone increasing on $(-1,1)$, monotone decreasing on $(-\infty,-1)$, $(1,+\infty)$

$$f''(x)=\frac{(1+x^2)^2(-2x)-4x(1+x^2)(1-x^2)}{(1+x^2)^4}=\frac{2x(1+x^2)(x^2-3)}{(1+x^2)^4},\ f''(x)=0\Rightarrow x=0\ \ or\ \ x=\pm\sqrt{3}$$

x	$(-\infty,-\sqrt{3})$	$\sqrt{3}$	$(-\sqrt{3},0)$	**0**	$(0,\sqrt{3})$	$\sqrt{3}$	$(\sqrt{3},+\infty)$
$f'(x)$	↘		↗		↘		↗
$f''(x)$	−	**0**	+	**0**	−	**0**	+

hence, $f(x)$ is concave up on $(-\sqrt{3},0),(\sqrt{3},+\infty)$; concave down on $(-\infty,-\sqrt{3}),(0,\sqrt{3})$

（6）Find $G'(x), if\ G(x)=\displaystyle\int_{\cos x}^{\sin x}\frac{xdu}{\sqrt{u^2+c^2}}$

$$G(x)=\int_{\cos x}^{\sin x}\frac{xdu}{\sqrt{u^2+c^2}}=x\int_{\cos x}^{\sin x}\frac{1}{\sqrt{u^2+c^2}}du$$

$$G'(x)=\int_{\cos x}^{\sin x}\frac{1}{\sqrt{u^2+c^2}}du+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$$=(\ln\left|\sqrt{u^2+c^2}+u\right|)\Big|_{\cos x}^{\sin x}+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$$=\ln\left|\frac{\sqrt{\sin^2 x+c^2}+\sin x}{\sqrt{\cos^2 x+c^2}+\cos x}\right|+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$(Note:\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln\left|\sqrt{x^2+a^2}+x\right|+C,$ You can prove it by supposing x=a * tan (t))

（7）Determine constants a,b,c,so that $\lim\limits_{x\to1}\dfrac{ax^4+bx^3+1}{(x-1)\sin\pi x}=c$

Solution:

$$\because\ \lim_{x\to1}\frac{ax^4+bx^3+1}{(x-1)\sin\pi x}=c, \text{and} \lim_{x\to1}(x-1)\sin\pi x=0,$$

$$\therefore\ \lim_{x\to1}(ax^4+bx^3+1)=0\Rightarrow a+b+1=0$$

$$\because\lim_{x\to1}\frac{ax^4+bx^3+1}{(x-1)\sin\pi x}=\lim_{x\to1}\frac{4ax^3+3bx^2}{\sin\pi x+\pi(x-1)\cos\pi x}=0,$$

$$\therefore \quad \lim_{x\to 1}(4a+3b)=0 \Rightarrow 4a+3b=0$$

$$\begin{cases} a+b+1=0 \\ 4a+3b=0 \end{cases} \Rightarrow \begin{cases} a=3 \\ b=-4 \end{cases}$$

$$c=\lim_{x\to 1}\frac{12ax^2+6bx}{\pi\cos\pi x+\pi\cos\pi x-\pi^2(x-1)\sin\pi x}=-\frac{6}{\pi}$$

(**8**) assume that $u_1=\sqrt{3}, u_{n+1}=\sqrt{3+u_n}$, detemine a convergent sequence and find $\lim_{x\to\infty}u_n$

Solution:

$$u_1=\sqrt{3}<\sqrt{3+\sqrt{3}}=u_2$$

suppose $u_k=u_{k+1}$, then $u_{k+1}=\sqrt{3+u_k}<\sqrt{3+u_{k+1}}=u_{k+2}$

By induction, $\{u_n\}$ is an increasing sequence.

$$u_1=\sqrt{3}<3$$

supppse $u_k<3$, then $u_{k+1}=\sqrt{3+u_k}<\sqrt{3+3}<3$

By induction, $\{u_n\}$ is bounded

$\therefore \{u_n\}$ is a convergent sequence

Assume $\lim_{n\to\infty}u_n=a$

$$\because u_{n+1}=\sqrt{3+u_n}$$

$$\therefore \lim_{n\to\infty}u_{n+1}=\lim_{n\to\infty}\sqrt{3+u_n}\Rightarrow a=\sqrt{3+a}$$

$$\because u_{n+1}>u_n>\cdots>u_1>0 \quad \therefore a=\frac{1+\sqrt{13}}{2}, \lim_{n\to\infty}u_n=\frac{1+\sqrt{13}}{2}$$

(**9**) Proof the limit $\lim_{n\to\infty}\int_0^1\frac{x^n}{1+x}dx=0$

Solution:

when $x\in[0,1], 0<\frac{x^n}{1+x}<x^n$

$$\therefore 0<\int_0^1\frac{x^n}{1+x}dx<\int_0^1 x^n dx=\frac{1}{n+1}$$

$$\because \lim_{n\to\infty}\frac{1}{n+1}=0 \qquad \therefore \lim_{n\to\infty}\int_0^1\frac{x^n}{1+x}dx=0$$

(10) Find the limit $\lim_{x\to\infty}(\frac{x-2022}{x+2022})^{\sin x}$

Solution:

$$\lim_{x\to\infty}(\frac{x-2022}{x+2022})^{\sin x}$$

$$=\lim_{x\to\infty}(1+\frac{-4044}{x+2022})^{(\frac{x+2022}{-4044})\cdot(\frac{-4044\sin x}{x+2022})}$$

$$=\lim_{x\to\infty}e^{\frac{-4044\sin x}{x+2022}}$$

$$=1$$