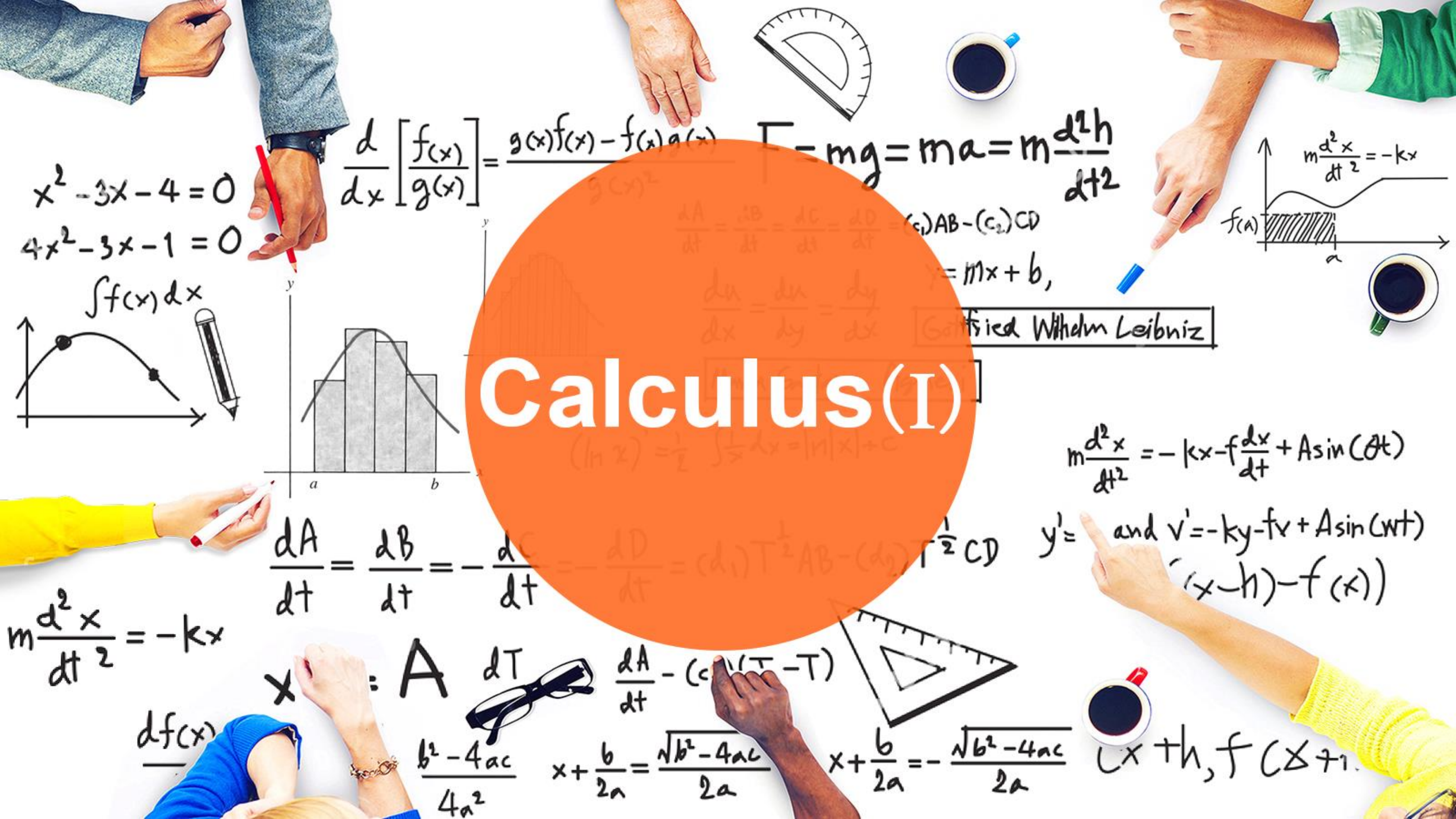
Calculus(I)



x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
F = mg = ma = m d²h/dt²
y = mx + b,
m d²x/dt² = -kx
m d²x/dt² = -kx - f dx/dt + A sin(θt)
y' = and v' = -ky - fv + A sin(wt)
dA/dt = dB/dt = - dC/dt
dA/dt - (c)(T - T)
x + b/2a = √b² - 4ac / 2a
x + b/2a = - √b² - 4ac / 2a
b² - 4ac / 4a²
df(x)

Answers to Middle Test

Lecturer: Xue Deng

1

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}.$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}=\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}$$

$$=e\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}$$

$$=e\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}$$

$$=e\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x^2}=\boxed{-\frac{e}{2}}.$$

2

$$\lim_{x\to0}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)$$

$$\boxed{\lim_{x\to0^-}}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\frac{2+0}{1+0}-1=1$$

$$\boxed{\lim_{x\to0^+}}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=0+1=1$$

$$\lim_{x\to0}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=1$$

3

$$\int_0^{2018\pi} \sqrt{1-\cos 2x}\,\mathrm{d}x$$

$$\int_0^{2018\pi} \sqrt{1-\cos 2x}\,\mathrm{d}x = \int_0^{2018\pi} \sqrt{2}\left|\sin x\right|\mathrm{d}x$$

$$= 2018\sqrt{2}\int_0^{\pi} \sin x\,\mathrm{d}x = 4036\sqrt{2}.$$

$$\int_{-2}^{2} \frac{x+|x|}{2+x^2}\,dx.$$

$$= 2\int_0^2 \frac{x}{2+x^2}\,dx$$

$$= 2\int_0^2 \frac{x}{2+x^2}\,dx = \ln(2+x^2)\Big|_0^2 = \ln 3$$

5

Find the average value of the function $y = x^2 + x\sqrt{a^2 - x^2}$ on $[-a, a]$.

$$\bar{y} = \frac{1}{a-(-a)}\int_{-a}^{a}\left(x^2 + x\sqrt{a^2 - x^2}\right)dx = \frac{1}{2a}\left[2\int_{0}^{a} x^2 dx + 0\right] = \frac{1}{a}\cdot\frac{1}{3}\cdot x^3\Big|_0^a = \frac{a^2}{3}$$

6

$$\int_{-1}^{1}\left(x+\sqrt[4]{1-x^2}\right)^2 dx$$

$$=\int_{-1}^{1}(x^2+\boxed{2x\sqrt[4]{1-x^2}}+\sqrt{1-x^2})dx$$

$$=\int_{-1}^{1}x^2dx+\boxed{\int_{-1}^{1}\sqrt{1-x^2}dx}$$

$$=\frac{2}{3}+\frac{\pi}{2}$$

7

$$\lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx$$

(1) $0 \le \dfrac{x^n}{\sqrt{1+x^2}} \le x^n$,

$0 \le \displaystyle\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx \le \int_0^1 x^n dx = \frac{1}{n+1}$, and $\displaystyle\lim_{n\to\infty} 0 = \lim_{n\to\infty}\frac{1}{n+1} = 0$,

So $\displaystyle\lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx = 0$ 。

7

$$\lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx$$

$$(2)\quad \lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx = \lim_{n\to\infty}\frac{1}{\sqrt{1+\xi^2}}\int_0^1 x^n dx = \lim_{n\to\infty}\frac{1}{\sqrt{1+\xi^2}}\frac{1}{n+1} = 0, \xi \in [0,1]$$

7

$$\lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx$$

(3) $\forall \varepsilon > 0$, $0 < \int_{1-\varepsilon/2}^{1} \frac{x^n}{\sqrt{1+x^2}}dx < \int_{1-\varepsilon/2}^{1} dx = \frac{\varepsilon}{2}$,

$\lim_{n\to\infty}\int_0^{1-\varepsilon/2} \frac{x^n}{\sqrt{1+x^2}}dx = \lim_{n\to\infty}\frac{\xi_n^n}{\sqrt{1+\xi_n^2}}\left(1-\frac{\varepsilon}{2}\right) = 0, 0 \le \xi_n \le 1-\frac{\varepsilon}{2} < 1$, take $\varepsilon_1 = \frac{\varepsilon}{2} > 0$, exist

$N > 0$, when $n > N$, we have $0 < \int_0^{1-\varepsilon/2} \frac{x^n}{\sqrt{1+x^2}}dx < \frac{\varepsilon}{2}$, so

For every $\varepsilon > 0$, there exists $N > 0$, when $n > N$, we have $\left|\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx - 0\right| < \varepsilon$,

So $\lim_{n\to\infty}\int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx = 0$.

8

Let $f(x+1)=\lim_{n\to\infty}\left(\frac{n+x}{n-2}\right)^n$,find the expression of $f(x)$.

Solu:

$$x\neq -2\ ,\quad f(x+1)=\lim_{n\to\infty}\left(\frac{n+x}{n-2}\right)^n=\lim_{n\to\infty}\left(1+\frac{x+2}{n-2}\right)^n=\lim_{n\to\infty}\left[\left(1+\frac{x+2}{n-2}\right)^{\frac{n-2}{x+2}}\right]^{\frac{n(x+2)}{n-2}}=e^{x+2}\ ,$$

$$x=-2,\quad f(x+1)=\lim_{n\to\infty}\left(\frac{n+x}{n-2}\right)^n=\lim_{n\to\infty}1^n=1=e^{x+2}\Big|_{x=-2}\ ,$$

so $\quad f(x+1)=e^{x+2}=e^{(x+1)+1},f(x)=e^{x+1}$.

9

Let $y = y(x)$ is decided by $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$, find $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$.

Solu. $\dfrac{dy}{dx} = \dfrac{-(-\sin t)}{1-\cos t} = \dfrac{\sin t}{1-\cos t}$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\sin t}{1-\cos t}\right) = \frac{1}{1-\cos t}\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right) = \frac{\cos t(1-\cos t)-\sin^2 t}{(1-\cos t)^3} = \frac{-1}{(1-\cos t)^2}$$

10

$f(x)=x^2\sin x\cos x$, find $f^{(2015)}(0)$.

$$f^{(n)}(x)=\frac{1}{2}\left(C_n^0x^2(\sin 2x)^{(n)}+C_n^1 2x(\sin 2x)^{(n-1)}+C_n^2 2(\sin 2x)^{(n-2)}\right)$$

$$=\frac{x^2}{2}\times 2^n\sin\left(2x+\frac{n\pi}{2}\right)+nx\times 2^{n-1}\sin\left(2x+\frac{(n-1)\pi}{2}\right)+\frac{n(n-1)}{2}\times 2^{n-2}\sin\left(2x+\frac{(n-2)\pi}{2}\right)$$

$$f^{(n)}(0)=\frac{n(n-1)}{2}\times 2^{n-2}\sin\left(\frac{(n-2)\pi}{2}\right)$$

$$f^{(2015)}(0)=2015\times 2014\times 2^{2012}\sin\left(1006\pi+\frac{\pi}{2}\right)=2015\times 2014\times 2^{2012}$$

11. Find the value of a, such that the plane region area bounded by $y=x^2$ and lines $x=a$, $x=a+1, y=0$ is minimum.

Solu: $S=\int_a^{a+1} x^2dx=\frac{1}{3}x^3\Big|_a^{a+1}=a^2+a+\frac{1}{3}=\left(a+\frac{1}{2}\right)^2+\frac{1}{12}\geq\frac{1}{12}, a=-\frac{1}{2}, s\left(-\frac{1}{2}\right)=\frac{1}{12}$

. Prove $\lim_{x\to 1}\frac{x+1}{x-2}=-2$ by $\varepsilon-\delta$.

Proof: $\forall\varepsilon>0$, and $\left|\frac{x+1}{x-2}-(-2)\right|=\left|\frac{3(x-1)}{x-2}\right|=\frac{3}{|x-2|}|x-1|$

let $|x-1|<\frac{1}{4}$, we have $\frac{3}{4}<x<\frac{5}{4}$, and $-1\frac{1}{4}<x-2<-\frac{3}{4}\Rightarrow|x-2|>\frac{3}{4}$

$\frac{3}{|x-2|}|x-1|<\frac{2}{\frac{3}{4}}|x-1|=\frac{8}{3}|x-1|(<\varepsilon)$ Take $\delta=\min\left\{\frac{1}{4},\frac{3\varepsilon}{8}\right\}$, when $0<|x-1|<\delta$, we

have $\left|\frac{x+1}{x-2}-(-2)\right|<\varepsilon$, so $\lim_{x\to 1}\frac{x+1}{x-2}=-2$.

Find volume by the region with curve $y = x^2$, lines $x = 2$ and x -axis that revolved by y-axis.

Solu: $V = \int_0^2 2\pi x \cdot x^2 \mathrm{d}x = 8\pi$

Find volume by the region with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>0,b>0)$ that revolved by x-axis.

Solu: bu symmetrical property, the volume is two times of volume by the first quadratant.

$$V=2\int_0^a \pi y^2 dx = 2\int_0^a \pi\left(b^2-\frac{b^2x^2}{a^2}\right)dx = 2\pi\left(b^2x-\frac{b^2x^3}{3a^2}\right)\Bigg|_0^a = 2\pi\left(b^2a-\frac{b^2a^3}{3a^2}\right)=\frac{4\pi}{3}ab^2$$

. Find the arc length of $y=\frac{1}{4}x^2-\frac{1}{2}\ln x\ (1\le x\le e)$.

$$L=\int_1^e\sqrt{1+y'^2}\,dx=\int_1^e\frac{1}{2}(x+\frac{1}{x})dx=\frac{1}{4}(e^2+1).$$

. Find the arc length of $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases} (0 \le \theta \le 2\pi)$.

Solu: $s = 4\int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\,\mathrm{d}t = 12\int_0^{\frac{\pi}{2}} \cos t \sin t \mathrm{d}t = 6$

The region A bounded by $x^2 + y^2 \le 2x$ and $y \ge x$, and A is revolved about the line $x = 2$.

Find the corresponding volume.

$$
\begin{aligned}
dV &= \pi r_1^{\,2} dy - \pi r_2^{\,2} dy \\
&= \pi(2 - x_1)^2 dy - \pi(2 - x_2)^2 dy \\
&= 2\pi[\sqrt{1 - y^2} - (1 - y)^2] dy
\end{aligned}
$$

$$
\begin{aligned}
V &= 2\pi \int_0^1 \sqrt{1 - y^2}\, dy - 2\pi \int_0^1 (1 - y)^2 dy \\
&= \frac{\pi^2}{2} - \frac{2}{3}\pi.
\end{aligned}
$$

. Let f is derivative at point $x=1$ and $f(1)=f'(1)=2$, find $\lim\limits_{x\to 0}\dfrac{f^3(1+x)-f^3(1)}{x}$.

Solu: $$\lim_{x\to 0}\frac{f^3(1+x)-f^3(1)}{x}=\lim_{x\to 0}\frac{f(1+x)-f(1)}{x}\left[f^2(1+x)+f(1+x)f(1)+f^2(1)\right]$$

$$=f'(1)\cdot 3f^2(1)=24$$

MIDDLE TEST II