Chapter 11 Angular Momentum

14. To find the center of mass speed v on the plateau, we use the projectile motion equations of Chapter 4. With $v_{oy} = 0$ (and using "h" for h_2) Eq. 4-22 gives the time-of-flight as $t = \sqrt{2h/g}$. Then Eq. 4-21 (squared, and using d for the horizontal displacement) gives $v^2 = gd^2/2h$. Now, to find the speed v_p at point P, we apply energy conservation, that is, mechanical energy on the plateau is equal to the mechanical energy at P. With Eq. 11-5, we obtain

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}I_{\text{com}}\,\omega^2 + mgh_1 = \tfrac{1}{2}mv_{\text{p}}^{\,2} + \tfrac{1}{2}I_{\text{com}}\,\omega_{\text{p}}^{\,2}\,.$$

Using item (f) of Table 10-2, Eq. 11-2, and our expression (above) $v^2 = gd^2/2h$, we obtain

$$gd^2/2h + 10gh_1/7 = v_{\text{p}}^{\,2}$$

which yields (using the values stated in the problem) $v_{\text{p}} = 1.34$ m/s.

15. (a) We choose clockwise as the negative rotational sense and rightward as the positive translational direction. Thus, since this is the moment when it begins to roll smoothly, Eq. 11-2 becomes

$$v_{\text{com}} = -R\omega = (-0.11\ \text{m})\omega.$$

This velocity is positive-valued (rightward) since ω is negative-valued (clockwise) as shown in the figure.

(b) The force of friction exerted on the ball of mass m is $-\mu_k mg$ (negative since it points left), and setting this equal to ma_{com} leads to

$$a_{\text{com}} = -\mu g = -(0.21)\,(9.8\ \text{m/s}^2) = -2.1\ \text{m/s}^2$$

where the minus sign indicates that the center of mass acceleration points left, opposite to its velocity, so that the ball is decelerating.

(c) Measured about the center of mass, the torque exerted on the ball due to the frictional force is given by $\tau = -\mu mgR$. Using Table 10-2(f) for the rotational inertia, the angular acceleration becomes (using Eq. 10-45)

$$\alpha = \frac{\tau}{I} = \frac{-\mu mgR}{2mR^2/5} = \frac{-5\mu g}{2R} = \frac{-5(0.21)(9.8\ \text{m/s}^2)}{2(0.11\ \text{m})} = -47\ \text{rad/s}^2$$

where the minus sign indicates that the angular acceleration is clockwise, the same direction as ω (so its angular motion is "speeding up'').

(d) The center of mass of the sliding ball decelerates from $v_{\text{com},0}$ to v_{com} during time t according to Eq. 2-11: $v_{\text{com}} = v_{\text{com},0} - \mu gt$. During this time, the angular speed of the ball increases (in magnitude) from zero to $|\omega|$ according to Eq. 10-12:

$$|\omega| = |\alpha|t = \frac{5\mu gt}{2R} = \frac{v_{\text{com}}}{R}$$

where we have made use of our part (a) result in the last equality. We have two equations involving v_{com}, so we eliminate that variable and find

$$t = \frac{2v_{\text{com},0}}{7\mu g} = \frac{2(8.5\ \text{m/s})}{7(0.21)(9.8\ \text{m/s}^2)} = 1.2\ \text{s}.$$

(e) The skid length of the ball is (using Eq. 2-15)

$$\Delta x = v_{\text{com},0}t - \frac{1}{2}(\mu g)t^2 = (8.5\ \text{m/s})(1.2\ \text{s}) - \frac{1}{2}(0.21)(9.8\ \text{m/s}^2)(1.2\ \text{s})^2 = 8.6\ \text{m}.$$

(f) The center of mass velocity at the time found in part (d) is

$$v_{\text{com}} = v_{\text{com},0} - \mu gt = 8.5\ \text{m/s} - (0.21)(9.8\ \text{m/s}^2)(1.2\ \text{s}) = 6.1\ \text{m/s}.$$

16. Using energy conservation with Eq. 11-5 and solving for the rotational inertia (about the center of mass), we find

$$I_{\text{com}} = MR^2[2g(H-h)/v^2 - 1]\ .$$

Thus, using the β notation suggested in the problem, we find

$$\beta = 2g(H-h)/v^2 - 1.$$

To proceed further, we need to find the center of mass speed v, which we do using the projectile motion equations of Chapter 4. With $v_{oy} = 0$, Eq. 4-22 gives the time-of-flight as $t = \sqrt{2h/g}$. Then Eq. 4-21 (squared, and using d for the horizontal displacement) gives $v^2 = gd^2/2h$. Plugging this into our expression for β gives

$$2g(H-h)/v^2 - 1 = 4h(H-h)/d^2 - 1.$$

Therefore, with the values given in the problem, we find β = 0.25.

17. (a) The derivation of the acceleration is found in §11-4; Eq. 11-13 gives

$$a_{\text{com}} = -\frac{g}{1 + I_{\text{com}} / MR_0^2}$$

where the positive direction is upward. We use $I_{\text{com}} = 950 \ \text{g} \cdot \text{cm}^2$, M =120g, R_0 = 0.320 cm, and g = 980 cm/s^2 and obtain

$$|a_{\text{com}}| = \frac{980 \text{ cm/s}^2}{1 + \left(950 \text{ g} \cdot \text{cm}^2\right) / (120 \text{ g})(0.32 \text{ cm})^2} = 12.5 \text{ cm/s}^2 \approx 13 \text{ cm/s}^2.$$

(b) Taking the coordinate origin at the initial position, Eq. 2-15 leads to $y_{\text{com}} = \frac{1}{2} a_{\text{com}} t^2$. Thus, we set y_{com} = – 120 cm, and find

$$t = \sqrt{\frac{2y_{\text{com}}}{a_{\text{com}}}} = \sqrt{\frac{2(-120 \text{ cm})}{-12.5 \text{ cm/s}^2}} = 4.38 \text{ s} \approx 4.4 \text{ s}.$$

(c) As it reaches the end of the string, its center of mass velocity is given by Eq. 2-11:

$$v_{\text{com}} = a_{\text{com}} t = \left(-12.5 \text{ cm/s}^2\right)(4.38 \text{ s}) = -54.8 \text{ cm/s},$$

so its linear speed then is approximately $|v_{\text{com}}| = 55$ cm/s.

(d) The translational kinetic energy is

$$K_{\text{trans}} = \frac{1}{2} m v_{\text{com}}^2 = \frac{1}{2}(0.120 \text{ kg})(0.548 \text{ m/s})^2 = 1.8 \times 10^{-2} \text{ J}.$$

(e) The angular velocity is given by $\omega = -v_{\text{com}}/R_0$ and the rotational kinetic energy is

$$K_{\text{rot}} = \frac{1}{2} I_{\text{com}} \omega^2 = \frac{1}{2} I_{\text{com}} \left(\frac{v_{\text{com}}}{R_0}\right)^2 = \frac{1}{2}(9.50 \times 10^{-5} \text{ kg} \cdot \text{m}^2)\left(\frac{0.548 \text{ m/s}}{3.2 \times 10^{-3} \text{ m}}\right)^2 \approx 1.4 \text{ J}.$$

(b) The angle turned is $\theta = \omega_0 t + \alpha t^2/2$. If the angular acceleration α is uniform, then so is the torque and $\alpha = \tau/I$. Furthermore, $\omega_0 = L_i/I$, and we obtain

$$\theta = \frac{L_i t + \tau t^2/2}{I} = \frac{\left(3.00\,\text{kg}\cdot\text{m}^2/\text{s}\right)\left(1.50\,\text{s}\right) + \left(-1.467\,\text{N}\cdot\text{m}\right)\left(1.50\,\text{s}\right)^2/2}{0.140\,\text{kg}\cdot\text{m}^2}$$
$$= 20.4\,\text{rad}.$$

(c) The work done on the wheel is

$$W = \tau\theta = \left(-1.47\,\text{N}\cdot\text{m}\right)\left(20.4\,\text{rad}\right) = -29.9\ \text{J}$$

where more precise values are used in the calculation than what is shown here. An equally good method for finding W is Eq. 10-52, which, if desired, can be rewritten as $W = \left(L_f^2 - L_i^2\right)/2I$.

(d) The average power is the work done by the flywheel (the negative of the work done on the flywheel) divided by the time interval:

$$P_{\text{avg}} = -\frac{W}{\Delta t} = -\frac{-29.8\ \text{J}}{1.50\,\text{s}} = 19.9\,\text{W}.$$

40. Torque is the time derivative of the angular momentum. Thus, the change in the angular momentum is equal to the time integral of the torque. With $\tau = (5.00 + 2.00t)\ \text{N}\cdot\text{m}$, the angular momentum (in units $\text{kg}\cdot\text{m}^2/\text{s}$) as a function of time is

$$L(t) = \int \tau\, dt = \int (5.00 + 2.00t)\, dt = L_0 + 5.00t + 1.00t^2 .$$

Since $L = 5.00\ \text{kg}\cdot\text{m}^2/\text{s}$ when $t = 1.00\ \text{s}$, the integration constant is $L_0 = -1$. Thus, the complete expression of the angular momentum is

$$L(t) = -1 + 5.00t + 1.00t^2 .$$

At $t = 3.00\ \text{s}$, we have $L(t = 3.00) = -1 + 5.00(3.00) + 1.00(3.00)^2 = 23.0\ \text{kg}\cdot\text{m}^2/\text{s}.$

41. (a) For the hoop, we use Table 10-2(h) and the parallel-axis theorem to obtain

$$I_1 = I_{\text{com}} + mh^2 = \frac{1}{2}mR^2 + mR^2 = \frac{3}{2}mR^2 .$$

Of the thin bars (in the form of a square), the member along the rotation axis has (approximately) no rotational inertia about that axis (since it is thin), and the member farthest from it is very much like it (by being parallel to it) except that it is displaced by a distance h; it has rotational inertia given by the parallel axis theorem:

$$I_2 = I_{\text{com}} + mh^2 = 0 + mR^2 = mR^2 .$$

Now the two members of the square perpendicular to the axis have the same rotational inertia (that is $I_3 = I_4$). We find I_3 using Table 10-2(e) and the parallel-axis theorem:

$$I_3 = I_{\text{com}} + mh^2 = \frac{1}{12}mR^2 + m\left(\frac{R}{2}\right)^2 = \frac{1}{3}mR^2 .$$

Therefore, the total rotational inertia is

$$I_1 + I_2 + I_3 + I_4 = \frac{19}{6}mR^2 = 1.6\,\text{kg}\cdot\text{m}^2 .$$

(b) The angular speed is constant:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{2.5} = 2.5\,\text{rad/s}.$$

Thus, $L = I_{\text{total}}\omega = 4.0\,\text{kg}\cdot\text{m}^2/\text{s}.$

42. The results may be found by integrating Eq. 11-29 with respect to time, keeping in mind that $\vec{L}_i = 0$ and that the integration may be thought of as "adding the areas" under the line-segments (in the plot of the torque versus time, with "areas" under the time axis contributing negatively). It is helpful to keep in mind, also, that the area of a triangle is $\frac{1}{2}$ (base)(height).

(a) We find that $\vec{L} = 24\ \text{kg}\cdot\text{m}^2/\text{s}$ at $t = 7.0$ s.

(b) Similarly, $\vec{L} = 1.5\ \text{kg}\cdot\text{m}^2/\text{s}$ at $t = 20$ s.

43. We assume that from the moment of grabbing the stick onward, they maintain rigid postures so that the system can be analyzed as a symmetrical rigid body with center of mass midway between the skaters.

(a) The total linear momentum is zero (the skaters have the same mass and equal and opposite velocities). Thus, their center of mass (the middle of the 3.0 m long stick) remains fixed and they execute circular motion (of radius $r = 1.5$ m) about it.

(b) Using Eq. 10-18, their angular velocity (counterclockwise as seen in Fig. 11-47) is

$$\omega = \frac{v}{r} = \frac{1.4\text{ m/s}}{1.5\text{ m}} = 0.93\text{ rad/s}.$$

where I_{rod} = $ML^2/12$ by Table 10-2(e), with M = 4.0 kg and L = 0.5 m. Angular momentum conservation leads to

$$rmv\sin\theta = \left(\frac{1}{12}ML^2 + mr^2\right)\omega.$$

Thus, with ω = 10 rad/s, we obtain

$$v = \frac{\left(\frac{1}{12}(4.0\text{ kg})(0.5\text{ m})^2 + (0.003\text{ kg})(0.25\text{ m})^2\right)(10\text{ rad/s})}{(0.25\text{ m})(0.003\text{ kg})\sin 60°} = 1.3\times 10^3\text{ m/s}.$$

54. We denote the cat with subscript 1 and the ring with subscript 2. The cat has a mass $m_1 = M/4$, while the mass of the ring is $m_2 = M$ = 8.00 kg. The moment of inertia of the ring is $I_2 = m_2(R_1^2 + R_2^2)/2$ (Table 10-2), and $I_1 = m_1r^2$ for the cat, where r is the perpendicular distance from the axis of rotation.

Initially the angular momentum of the system consisting of the cat (at $r = R_2$) and the ring is

$$L_i = m_1v_{1i}r_{1i} + I_2\omega_{2i} = m_1\omega_0R_2^2 + \frac{1}{2}m_2(R_1^2+R_2^2)\omega_0 = m_1R_2^2\,\omega_0\left[1+\frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)\right].$$

After the cat has crawled to the inner edge at $r = R_1$ the final angular momentum of the system is

$$L_f = m_1\omega_fR_1^2 + \frac{1}{2}m_2(R_1^2+R_2^2)\omega_f = m_1R_1^2\,\omega_f\left[1+\frac{1}{2}\frac{m_2}{m_1}\left(1+\frac{R_2^2}{R_1^2}\right)\right].$$

Then from $L_f = L_i$ we obtain

$$\frac{\omega_f}{\omega_0} = \left(\frac{R_2}{R_1}\right)^2\frac{1+\frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)}{1+\frac{1}{2}\frac{m_2}{m_1}\left(1+\frac{R_2^2}{R_1^2}\right)} = (2.0)^2\frac{1+2(0.25+1)}{1+2(1+4)} = 1.273.$$

Thus, $\omega_f = 1.273\omega_0$. Using ω_0 =8.00 rad/s, we have ω_f =10.2 rad/s. By substituting $I = L/\omega$ into $K = I\omega^2/2$, we obtain $K = L\omega/2$. Since $L_i = L_f$, the kinetic energy ratio becomes

$$\frac{K_f}{K_i} = \frac{L_f\omega_f/2}{L_i\omega_i/2} = \frac{\omega_f}{\omega_0} = 1.273.$$

which implies $\Delta K = K_f - K_i = 0.273K_i$. The cat does positive work while walking toward the center of the ring, increasing the total kinetic energy of the system.

Since the initial kinetic energy is given by

$$K_i = \frac{1}{2}\left[m_1R_2^2 + \frac{1}{2}m_2(R_1^2+R_2^2)\right]\omega_0^2 = \frac{1}{2}m_1R_2^2\omega_0^2\left[1+\frac{1}{2}\frac{m_2}{m_1}\left(\frac{R_1^2}{R_2^2}+1\right)\right]$$
$$= \frac{1}{2}(2.00\text{ kg})(0.800\text{ m})^2(8.00\text{ rad/s})^2[1+(1/2)(4)(0.5^2+1)]$$
$$=143.36\text{ J},$$

the increase in kinetic energy is

$$\Delta K = (0.273)(143.36\text{ J}) = 39.1\text{ J}.$$

55. For simplicity, we assume the record is turning freely, without any work being done by its motor (and without any friction at the bearings or at the stylus trying to slow it down). Before the collision, the angular momentum of the system (presumed positive) is $I_i\omega_i$ where $I_i = 5.0\times10^{-4}\text{ kg}\cdot\text{m}^2$ and $\omega_i = 4.7\text{ rad/s}$. The rotational inertia afterward is

$$I_f = I_i + mR^2$$

where m = 0.020 kg and R = 0.10 m. The mass of the record (0.10 kg), although given in the problem, is not used in the solution. Angular momentum conservation leads to

$$I_i\omega_i = I_f\omega_f \Rightarrow \omega_f = \frac{I_i\omega_i}{I_i+mR^2} = 3.4\text{ rad / s}.$$

56. Table 10-2 gives the rotational inertia of a thin rod rotating about a perpendicular axis through its center. The angular speeds of the two arms are, respectively,

$$\omega_1 = \frac{(0.500\text{ rev})(2\pi\text{ rad/rev})}{0.700\text{ s}} = 4.49\text{ rad/s}$$
$$\omega_2 = \frac{(1.00\text{ rev})(2\pi\text{ rad/rev})}{0.700\text{ s}} = 8.98\text{ rad/s}.$$

Treating each arm as a thin rod of mass 4.0 kg and length 0.60 m, the angular momenta of the two arms are

$$L_1 = I\omega_1 = mr^2\omega_1 = (4.0\text{ kg})(0.60\text{ m})^2(4.49\text{ rad/s}) = 6.46\text{ kg}\cdot\text{m}^2\text{/s}$$
$$L_2 = I\omega_2 = mr^2\omega_2 = (4.0\text{ kg})(0.60\text{ m})^2(8.98\text{ rad/s}) = 12.92\text{ kg}\cdot\text{m}^2\text{/s}.$$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f} = \frac{2.10\ \text{kg}\cdot\text{m}^2}{1.38\ \text{kg}\cdot\text{m}^2} = 1.52.$$

65. If we consider a short time interval from just before the wad hits to just after it hits and sticks, we may use the principle of conservation of angular momentum. The initial angular momentum is the angular momentum of the falling putty wad.

The wad initially moves along a line that is $d/2$ distant from the axis of rotation, where d is the length of the rod. The angular momentum of the wad is $mvd/2$ where m and v are the mass and initial speed of the wad. After the wad sticks, the rod has angular velocity ω and angular momentum $I\omega$, where I is the rotational inertia of the system consisting of the rod with the two balls (each having a mass M) and the wad at its end. Conservation of angular momentum yields $mvd/2 = I\omega$ where

$$I = (2M + m)(d/2)^2 .$$

The equation allows us to solve for ω.

(a) With M = 2.00 kg, d = 0.500 m, m = 0.0500 kg, and v = 3.00 m/s, we find the angular speed to be

$$\omega = \frac{mvd}{2I} = \frac{2mv}{(2M+m)d} = \frac{2(0.0500\ \text{kg})(3.00\ \text{m/s})}{(2(2.00\ \text{kg}) + 0.0500\ \text{kg})(0.500\ \text{m})}$$
$$= 0.148\ \text{rad/s}.$$

(b) The initial kinetic energy is $K_i = \frac{1}{2}mv^2$, the final kinetic energy is $K_f = \frac{1}{2}I\omega^2$, and their ratio is

$$K_f/K_i = I\omega^2/mv^2 .$$

When $I = (2M + m)d^2/4$ and $\omega = 2mv/(2M + m)d$ are substituted, the ratio becomes

$$\frac{K_f}{K_i} = \frac{m}{2M+m} = \frac{0.0500\ \text{kg}}{2(2.00\ \text{kg}) + 0.0500\ \text{kg}} = 0.0123.$$

(c) As the rod rotates, the sum of its kinetic and potential energies is conserved. If one of the balls is lowered a distance h, the other is raised the same distance and the sum of the potential energies of the balls does not change. We need consider only the potential energy of the putty wad. It moves through a 90° arc to reach the lowest point on its path, gaining kinetic energy and losing gravitational potential energy as it goes. It then swings up through an angle θ, losing kinetic energy and gaining potential energy, until it momentarily comes to rest. Take the lowest point on the path to be the zero of potential energy. It starts a distance $d/2$ above this point, so its initial potential energy is

$U_i = mg(d/2)$. If it swings up to the angular position θ, as measured from its lowest point, then its final height is $(d/2)(1 - \cos\theta)$ above the lowest point and its final potential energy is

$$U_f = mg(d/2)(1 - \cos\theta).$$

The initial kinetic energy is the sum of that of the balls and wad:

$$K_i = \frac{1}{2} I\omega^2 = \frac{1}{2}(2M+m)(d/2)^2\,\omega^2.$$

At its final position, we have $K_f = 0$. Conservation of energy provides the relation:

$$U_i + K_i = U_f + K_f \quad \Rightarrow \quad mg\frac{d}{2} + \frac{1}{2}(2M+m)\left(\frac{d}{2}\right)^2 \omega^2 = mg\frac{d}{2}(1-\cos\theta).$$

When this equation is solved for $\cos\theta$, the result is

$$\begin{aligned}\cos\theta &= -\frac{1}{2}\left(\frac{2M+m}{mg}\right)\left(\frac{d}{2}\right)\omega^2 \\ &= -\frac{1}{2}\left(\frac{2(2.00\text{ kg}) + 0.0500\text{ kg}}{(0.0500\text{ kg})(9.8\text{ m/s}^2)}\right)\left(\frac{0.500\text{ m}}{2}\right)(0.148\text{ rad/s})^2 \\ &= -0.0226.\end{aligned}$$

Consequently, the result for θ is 91.3°. The total angle through which it has swung is 90° + 91.3° = 181°.

During the rotation, the potential energy depends only on the position of the wad.
Let θ be the angle of the rod above the horizontal when the system momentarily stops.
Conservation of energy gives
$K_f = (mgL\sin\theta)/2$, with $K_f = (1/2)\,(2M+m)(L\omega/2)^2$.
Substituting the values of M, m, g, L and ω , we get $\sin\theta = 0.0226$ and $\theta = 1.30°$.
So the system rotates through an angle 180 + 1.30 = 181.3°.