Linear algebra and analytic geometry 2021-2022 Final Exam

I. 1. $A=\begin{bmatrix}2&0&2&0\\2&2&0&0\\2&0&0&2\end{bmatrix}\sim\begin{bmatrix}2&0&2&0\\0&2&-2&0\\0&0&-2&2\end{bmatrix}\sim\begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&-1\end{bmatrix}$ 2 分

Thus Col (A) has a basis $\left\{\begin{bmatrix}2\\2\\2\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$. 1 分

The solution of $A\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-x_4\\x_4\\x_4\\x_4\end{bmatrix}=x_4\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$, x_4 is any scalar.

Thus Nul (A) has a basis $\left\{\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$. 2 分

2. $AA^T=\begin{bmatrix}2&0&2&0\\2&2&0&0\\2&0&0&2\end{bmatrix}\begin{bmatrix}2&2&2\\0&2&0\\2&0&0\\0&0&2\end{bmatrix}=\begin{bmatrix}8&4&4\\4&8&4\\4&4&8\end{bmatrix}$ 5 分

3. $\det(AA^T)=\begin{vmatrix}8&4&4\\4&8&4\\4&4&8\end{vmatrix}=4^3\begin{vmatrix}2&1&1\\1&2&1\\1&1&2\end{vmatrix}=64\begin{vmatrix}0&-3&-1\\1&2&1\\0&-1&1\end{vmatrix}=-64\begin{vmatrix}-3&-1\\-1&1\end{vmatrix}=256$ (5 分)

II.

1. If $a_1(4+5x+2x_2)+a_2(3x+x^2)+a_3x^2=0$, then

$4a_1+(5a_1+3a_2)x+(2a_1+a_2+a_3)x^2=0$, which directly yields $a_1=a_2=a_3=0$.

And thus the set $\mathfrak{B}$ is linearly independent.

Observing that the dimension of $\mathbb{P}_2(\mathbb{R})$ is 3 and $\mathfrak{B}$ contains 3 vectors, we can conclude that $\mathfrak{B}$ is a basis of $\mathbb{P}_2(\mathbb{R})$. 6 分

2. The change-of coordinates matrix is $P_{\mathfrak{B}\leftarrow\mathcal{E}}=[[1]_{\mathcal{B}}\quad[x]_{\mathcal{B}}\quad[x^2]_{\mathcal{B}}]$. We can find it in the following way

$$\begin{bmatrix}4&0&0&1&0&0\\5&3&0&0&1&0\\2&1&1&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/4&0&0\\0&3&0&-5/4&1&0\\0&1&1&-1/2&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&1/4&0&0\\0&1&0&-5/12&1/3&0\\0&0&1&-1/12&-1/3&1\end{bmatrix}$$

$P_{\mathfrak{B}\leftarrow\mathcal{E}}=\begin{bmatrix}1/4&0&0\\-5/12&1/3&0\\-1/12&-1/3&1\end{bmatrix}$. Or

$P_{\mathfrak{B}\leftarrow\mathcal{E}}=P^{-1}_{\mathcal{E}\leftarrow\mathfrak{B}}=\begin{bmatrix}4&0&0\\5&3&0\\2&1&1\end{bmatrix}^{-1}=\begin{bmatrix}1/4&0&0\\-5/12&1/3&0\\-1/12&-1/3&1\end{bmatrix}$. 6分

3. The matrix $A=[[T(1)]_{\mathcal{E}}\quad[T(x)]_{\mathcal{E}}\quad[T(x^2)]_{\mathcal{E}}]=\begin{bmatrix}2&3&0\\2&-3&0\\2&3&3\end{bmatrix}$ 6分

4. $\det(A-\lambda I)=\begin{vmatrix}2-\lambda&3&0\\2&-3-\lambda&0\\2&3&3-\lambda\end{vmatrix}=-(\lambda+4)(\lambda-3)^2$.

So A has two distinct eigenvalues 3 and -4. 3分

A basis of the eigenspace of $\lambda = 3$ is $\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. 3分

A basis of the Eigenspace of $\lambda = -4$ is $\boldsymbol{v}_2 = \begin{bmatrix} 7 \\ -14 \\ 4 \end{bmatrix}$. 3分

5. The matrix is not diagonalizable.

（题目有问题，只要知道如何利用特征向量和特征值构造P矩阵和对角矩阵即可得3分）。

III.

1. $\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 4 \\ 0 & -5 & h+4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & h-6 \end{bmatrix}$. 3分

These three vectors are linearly dependent if and only if $h = 6$.

2. $\begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$. 3分

IV.

1. Proof.

Suppose $U = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \cdots \quad \boldsymbol{u}_n]$. Then $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$ is an orthonormal set.

If $c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n = \boldsymbol{0}$, then

$0 = \boldsymbol{0} \cdot \boldsymbol{u}_1 = (c_1\boldsymbol{u}_1 + \cdots + c_n\boldsymbol{u}_n) \cdot \boldsymbol{u}_1 = c_1(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) + \cdots + c_n(\boldsymbol{u}_n \cdot \boldsymbol{u}_1) = c_1$.

Similarly, $c_2, \dots, c_n$ must be zero. Thus the columns of U are linearly independent.

And thus U is invertible.

2. Proof.

Let $U = [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3]$.

$$U^T U = \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \boldsymbol{u}_3^T \end{bmatrix} [\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3] = \begin{bmatrix} \boldsymbol{u}_1^T\boldsymbol{u}_1 & \boldsymbol{u}_1^T\boldsymbol{u}_2 & \boldsymbol{u}_1^T\boldsymbol{u}_3 \\ \boldsymbol{u}_2^T\boldsymbol{u}_1 & \boldsymbol{u}_2^T\boldsymbol{u}_2 & \boldsymbol{u}_2^T\boldsymbol{u}_3 \\ \boldsymbol{u}_3^T\boldsymbol{u}_1 & \boldsymbol{u}_3^T\boldsymbol{u}_2 & \boldsymbol{u}_3^T\boldsymbol{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Proof

$(U\boldsymbol{x}) \cdot (U\boldsymbol{y}) = (U\boldsymbol{x})^T(U\boldsymbol{y}) = \boldsymbol{x}^T U^T U \boldsymbol{y} = \boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}$

VI.

1. The orthogonal projection of $\boldsymbol{u}_1$ onto $\boldsymbol{u}_2$ is

$\hat{\boldsymbol{u}}_1 = \frac{\boldsymbol{u}_1 \cdot \boldsymbol{u}_2}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\boldsymbol{u}_2 = -\frac{100}{254}\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} = \frac{50}{127}\begin{bmatrix} 3 \\ -14 \\ 7 \end{bmatrix}$ 6分

2. Let $\boldsymbol{u}_3 = \boldsymbol{u}_1 - \hat{\boldsymbol{u}}_1 = \frac{1}{127}\begin{bmatrix} 231 \\ 192 \\ 285 \end{bmatrix} = \begin{bmatrix} 1.8189 \\ 1.5118 \\ 2.2441 \end{bmatrix}$. 3分(构造出正交基)

The orthogonal projection of $\boldsymbol{y}$ onto the plane spanned by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is

$\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_2}{\boldsymbol{u}_2 \cdot \boldsymbol{u}_2}\boldsymbol{u}_2 + \frac{\boldsymbol{y} \cdot \boldsymbol{u}_3}{\boldsymbol{u}_3 \cdot \boldsymbol{u}_3}\boldsymbol{u}_3 = -\frac{176}{254}\begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix} + \frac{852}{171450}\begin{bmatrix} 231 \\ 192 \\ 285 \end{bmatrix} = \begin{bmatrix} 3.2267 \\ -8.7467 \\ 6.2667 \end{bmatrix}$. 3 分

$$\mathbf{y}-\hat{\boldsymbol{y}}=\begin{bmatrix}5\\-9\\5\end{bmatrix}-\begin{bmatrix}3.2267\\-8.7467\\6.2667\end{bmatrix}=\begin{bmatrix}1.7733\\-0.2533\\-1.2667\end{bmatrix}.$$

By the Best Approximate Theorem, the distance from $\boldsymbol{y}$ to the plane spanned by $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is $\|\mathbf{y}-\hat{\boldsymbol{y}}\|$=2.1939. 3 分

VI.

1. Proof.

A has three eigenvalues counting multiplicity. Suppose they are λ_1,λ_2 and λ_3. Then $\det A=\lambda_1\lambda_2\lambda_3$. Since λ_1,λ_2 and λ_3 are all positive, $\det A>0$. 6分

2. Proof. Since λ is an eigenvalue of A, there exists a nonzero vector $\boldsymbol{v}$ such that $A\boldsymbol{v}=\lambda\boldsymbol{v}$. Then we use A^{-1} to left multiply both size of $A\boldsymbol{v}=\lambda\boldsymbol{v}$ to find that $\boldsymbol{v}=\lambda A^{-1}\boldsymbol{v}$ which implies that $A^{-1}\boldsymbol{v}=\lambda^{-1}\boldsymbol{v}$. And thus λ^{-1} is an eigenvalue of A^{-1}. 6分

3. Since A is symmetric, there exists an othornonal matrix P such that

$$A=P\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}P^{-1}.$$ 2分

Let $\Lambda=\begin{bmatrix}\sqrt{\lambda_1}&0&0\\0&\sqrt{\lambda_2}&0\\0&0&\sqrt{\lambda_3}\end{bmatrix}$, $M=P\Lambda P^{-1}$.

Then $\Lambda^2=\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}$, and

$$M^2=(P\Lambda P^{-1})(P\Lambda P^{-1})=P\begin{bmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{bmatrix}P^{-1}=A.$$ 1分