Indefinite integral II

1. Find the following indefinite integral

(1) $\int \frac{dx}{4x-3}$

Solution: $\int \frac{dx}{4x-3} = \int \frac{\frac{1}{4}d(4x-3)}{4x-3} \quad (u = 4x-3)$

$$= \frac{1}{4}\int \frac{du}{u}$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|4x-3| + C$$

(2) $\int \frac{dx}{\sqrt{1-2x^2}}$

Solution: $\int \frac{dx}{\sqrt{1-2x^2}} = \int \frac{\frac{1}{\sqrt{2}}d(\sqrt{2}x)}{\sqrt{1-(\sqrt{2}x)^2}} \quad (u = \sqrt{2}x)$

$$= \frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{\sqrt{2}}\arcsin u + C$$

$$= \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}x) + C$$

(3) $\int \frac{dx}{e^x - e^{-x}}$

Solution: $\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x dx}{e^{2x}-1}$

$$= \int \frac{d(e^x)}{(e^x)^2-1} \quad (u = e^x)$$

$$= \int \frac{du}{u^2-1}$$

$$= \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

(4) $\int e^{3x+2}\,dx$

Solution: $\int e^{3x+2}\,dx = \int \frac{1}{3}e^{3x+2}\,d(3x+2) \quad (u = 3x+2)$

$$= \frac{1}{3}\int e^u \, du$$

$$= \frac{1}{3} e^u + C$$

$$= \frac{1}{3} e^{3x+2} + C$$

(5) $\int (2^x + 3^x)^2 \, dx$

Solution: $\int (2^x + 3^x)^2 \, dx = \int (4^x + 2 \cdot 6^x + 9^x) \, dx$

$$= \frac{4^x}{2\ln 2} + \frac{2 \cdot 6^x}{\ln 6} + \frac{9^x}{2\ln 3} + C$$

(6) $\int \frac{1}{2+5x^2} dx$

Solution: $\int \frac{1}{2+5x^2} dx = \int \frac{\sqrt{\frac{2}{5}} d\left(\sqrt{\frac{5}{2}}x\right)}{2\left[1+\left(\sqrt{\frac{5}{2}}x\right)^2\right]} \quad (u = \sqrt{\frac{5}{2}}x)$

$$= \frac{1}{\sqrt{10}} \int \frac{du}{1+u^2}$$

$$= \frac{1}{\sqrt{10}} \arctan u + C$$

$$= \frac{1}{\sqrt{10}} \arctan(\sqrt{\frac{5}{2}}x) + C$$

(7) $\int \sin^5 x \, dx$

Solution: $\int \sin^5 x \, dx = \int \sin^4 x \cdot \sin x \, dx$

$$= \int -(1-\cos^2 x)^2 d(\cos x) \quad (u = \cos x)$$

$$= -\int (1-u^2)^2 du$$

$$= -\int (1-2u^2+u^4) du$$

$$= -\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C$$

$$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$$

(8) $\int \tan^{10} x \sec^2 x \, dx$

Solution: $\int \tan^{10} x \sec^2 x \, dx = \int \tan^{10} x \, d(\tan x) \quad (u = \tan x)$

$$= \int u^{10} du$$

$$= \frac{1}{11}u^{11} + C$$

$$= \frac{1}{11}\tan^{11} x + C$$

(9) $\int \sin 5x \cos 3x \, dx$

Solution: $\int \sin 5x \cos 3x \, dx = \frac{1}{2} \int (\sin 8x + \sin 2x) dx$

$$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$$

(10) $\int \cos^2 5x \, dx$

Solution: $\int \cos^2 5x \, dx = \frac{1}{2}\int (1 + \cos 10x) dx$

$$= \frac{1}{2}x + \frac{1}{20}\sin 10x + C$$

(11) $\int \frac{(2x+4)dx}{(x^2+4x+5)^2}$

Solution: $\int \frac{(2x+4)dx}{(x^2+4x+5)^2} = \int \frac{d(x^2+4x+5)}{(x^2+4x+5)^2} \quad (u = x^2 + 4x + 5)$

$$= \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{x^2+4x+5} + C$$

(12) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Solution: $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx = \int 2\sin\sqrt{x}\, d\left(\sqrt{x}\right) \quad \left(u = \sqrt{x}\right)$

$$= \int 2\sin u \, du$$

$$= -2\cos u + C$$

$$= -2\cos\sqrt{x} + C$$

(13) $\int \frac{x^2 dx}{\sqrt[4]{1-2x^3}}$

Solution: $\int \frac{x^2 dx}{\sqrt[4]{1-2x^3}} = \int \frac{-\frac{1}{6}d(1-2x^3)}{\sqrt[4]{1-2x^3}} \quad (u = 1 - 2x^3)$

$$= -\frac{1}{6}\int \frac{du}{u^{\frac{1}{4}}}$$

$$= -\frac{2}{9}u^{\frac{3}{4}} + C$$

$$= -\frac{2}{9}(1-2x^3)^{\frac{3}{4}} + C$$

(14) $\int \frac{1}{1-\sin x}\,dx$

Solution: $\int \frac{1}{1-\sin x}\,dx = \int \frac{1}{\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}}\,dx$

$$= \int \frac{1}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}\,dx$$

$$= \int \frac{1}{\left[\sqrt{2}\sin\left(\frac{x}{2} - \frac{\pi}{4}\right)\right]^2}\,dx$$

$$= \int \csc^2\left(\frac{x}{2} - \frac{\pi}{4}\right) d\left(\frac{x}{2} - \frac{\pi}{4}\right) \quad \left(u = \frac{x}{2} - \frac{\pi}{4}\right)$$

$$= \int \csc^2 u\,du$$

$$= -\cot u + C$$

$$= -\cot\left(\frac{x}{2} - \frac{\pi}{4}\right) + C$$

(15) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\,dx$

Solution: $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\,dx = \int \frac{d(\sin x - \cos x)}{\sqrt[3]{\sin x - \cos x}} \quad (u = \sin x - \cos x)$

$$= \int \frac{du}{\sqrt[3]{u}}$$

$$= \frac{3}{2}u^{\frac{2}{3}} + C$$

$$= \frac{3}{2}(\sin x - \cos x)^{\frac{2}{3}} + C$$

(16) $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}}$

Solution: $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}} = \int \frac{d(\arcsin x)}{(\arcsin x)^2} \quad (u = \arcsin x)$

$$= \int \frac{du}{u^2}$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\arcsin x} + C$$

(17) $\int \frac{dx}{x^2-2x+2}$

Solution: $\int \frac{dx}{x^2-2x+2} = \int \frac{d(x-1)}{1+(x-1)^2} \quad (u = x-1)$

$$= \int \frac{du}{1+u^2}$$

$$= \arctan u + C$$

$$= \arctan(x-1) + C$$

(18) $\int \frac{1-x}{\sqrt{9-4x^2}} dx$

Solution: $\int \frac{1-x}{\sqrt{9-4x^2}} dx = \int \frac{1}{\sqrt{9-4x^2}} dx - \int \frac{x}{\sqrt{9-4x^2}} dx$

$$= \int \frac{\frac{3}{2}}{3\sqrt{1-\left(\frac{2}{3}x\right)^2}} d(\frac{2}{3}x) - \int \frac{-\frac{1}{8}}{\sqrt{9-4x^2}} d(9-4x^2)$$

$$(u = \frac{2}{3}x, v = 9-4x^2)$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} + \frac{1}{8}\int \frac{dv}{\sqrt{v}}$$

$$= \frac{1}{2}\arcsin u + \frac{1}{4}\sqrt{v} + C$$

$$= \frac{1}{2}\arcsin \frac{2}{3}x + \frac{1}{4}\sqrt{9-4x^2} + C$$

(19) $\int \tan\sqrt{1+x^2} \frac{x}{\sqrt{1+x^2}} dx$

Solution: $\int \tan\sqrt{1+x^2} \frac{x}{\sqrt{1+x^2}} dx$

$$= \int \tan\sqrt{1+x^2}\, d\left(\sqrt{1+x^2}\right) \quad (u = \sqrt{1+x^2})$$

$$= \int \tan u \, du$$

$$= -\ln|\cos u| + C$$

$$= -\ln\left|\cos\sqrt{1+x^2}\right| + C$$

(20) $\int \frac{\sin x \cos x}{1+\sin^4 x} dx$

Solution: $\int \frac{\sin x \cos x}{1+\sin^4 x} dx = \int \frac{\sin x}{1+\sin^4 x} d(\sin x) \quad (u = \sin x)$

$$= \int \frac{u}{1+u^4} du$$

$$= \int \frac{\frac{1}{2}d(u^2)}{1+(u^2)^2} \quad (v = u^2)$$

$$= \frac{1}{2}\int \frac{dv}{1+v^2}$$

$$= \frac{1}{2}\arctan v + C$$

$$= \frac{1}{2}\arctan(\sin^2 x) + C$$

2. Find the following indefinite integral

(1) $\int \frac{dx}{\sqrt{1+e^{2x}}}$

Solution: $\int \frac{dx}{\sqrt{1+e^{2x}}} = \int \frac{dx}{e^x\sqrt{1+e^{-2x}}}$

$$= \int \frac{-d(e^{-x})}{\sqrt{1+(e^{-x})^2}} \quad (u = e^{-x})$$

$$= -\int \frac{du}{\sqrt{1+u^2}} \quad (u = \tan t\,, \sec t = \sqrt{1+u^2})$$

$$= -\int \frac{\sec^2 t dt}{\sec t}$$

$$= -\int \sec t\, dt$$

$$= -\ln|\sec t + \tan t| + C$$

$$= -\ln\left|\sqrt{1+u^2} + u\right| + C$$

$$= -\ln\left|\sqrt{1+e^{-2x}} + e^{-x}\right| + C$$

$$= \ln|\sqrt{e^{2x}+1} - 1| - x + C$$

(2) $\int \frac{dx}{x\sqrt{1+x^2}}$

Solution: $\int \frac{dx}{x\sqrt{1+x^2}} \quad (x > 0)$

$$= \int \frac{dx}{x^2\sqrt{x^{-2}+1}}$$

$$= \int \frac{-d(x^{-1})}{\sqrt{(x^{-1})^2+1}} \quad (u = x^{-1})$$

$$= \int \frac{-du}{\sqrt{u^2+1}} \quad \left(u = \tan t\,, \sec t = \sqrt{1+u^2}\right)$$

$$= -\int \frac{\sec^2 t dt}{\sec t}$$

$$= -\int \sec t\, dt$$

$$= -\ln|\sec t + \tan t| + C$$

$= -\ln\left|\sqrt{1+u^2}+u\right| + C$

$= -\ln\left|\sqrt{1+x^{-2}}+x^{-1}\right| + C$

$= \ln\left|\sqrt{1+x^2}-1\right| - \ln|x| + C$

$Check \quad \left(\ln\left|\sqrt{1+x^2}-1\right| - \ln|x|\right)' = \frac{1}{x\sqrt{1+x^2}}$

$\Rightarrow \int \frac{dx}{x\sqrt{1+x^2}} = \ln\left|\sqrt{1+x^2}-1\right| - \ln|x| + C$

(3) $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx$

Solution: $\int \frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}dx \quad \left(t=\sqrt{x}, x=t^2, dx=2tdt\right)$

$= \int \frac{\arctan t}{t(1+t^2)}2tdt$

$= 2\int \frac{\arctan t}{(1+t^2)}dt$

$= 2\int \arctan t\, d(\arctan t) \quad (u=\arctan t)$

$= 2\int udu$

$= u^2 + C$

$= \arctan^2\sqrt{x} + C$

(4) $\int \frac{1+\ln x}{(x\ln x)^2}dx$

Solution: $\int \frac{1+\ln x}{(x\ln x)^2}dx = \int \frac{d(x\ln x)}{(x\ln x)^2} \quad (u = x\ln x)$

$= \int \frac{du}{u^2}$

$= -\frac{1}{u} + C$

$= -\frac{1}{x\ln x} + C$

(5) $\int (x-1)(x+2)^{20}\,dx$

Solution: $\int (x-1)(x+2)^{20}\,dx \quad (t = x+2, x = t-2, dx = dt)$

$= \int (t-3)t^{20}\,dt$

$= \int (t^{21} - 3t^{20})\,dt$

$= \frac{1}{22} t^{22} - \frac{1}{7} t^{21} + C$

$= \frac{1}{22} (x+2)^{22} - \frac{1}{7} (x+2)^{21} + C$

(6) $\int x^2 (x+1)^n dx$

Solution: $\int x^2 (x+1)^n dx \quad (t = x+1, x = t-1, dx = dt)$

$= \int (t-1)^2 t^n dt$

$= \int (t^{n+2} - 2t^{n+1} + t^n)\, dt$

$= \frac{1}{n+3} t^{n+3} - \frac{2}{n+2} t^{n+2} + \frac{1}{n+1} t^{n+1} + C$

$= \frac{1}{n+3} (x+1)^{n+3} - \frac{2}{n+2} (x+1)^{n+2} + \frac{1}{n+1} (x+1)^{n+1} + C$

(7) $\int \frac{dx}{x^4\sqrt{1+x^2}}$

Solution: $\int \frac{dx}{x^4\sqrt{1+x^2}} \quad (x = \tan t\,, dx = \sec^2 t\, dt)$

$= \int \frac{\sec^2 t dt}{\tan^4 t \cdot \sec t}$

$= \int \frac{\cos^3 t dt}{\sin^4 t}$

$= \int \frac{1-\sin^2 t}{\sin^4 t} d(\sin t) \quad \left(u = \sin t\,, \sin t = \frac{x}{\sqrt{1+x^2}}\right)$

$= \int \frac{1-u^2}{u^4} du$

$= \int \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du$

$= -\frac{1}{3u^3} + \frac{1}{u} + C$

$= -\frac{\sqrt{(1+x^2)^3}}{3x^3} + \frac{\sqrt{1+x^2}}{x} + C$

(8) $\int \frac{\sqrt{x^2-9}}{x} dx$

Solution: $\int \frac{\sqrt{x^2-9}}{x} dx \quad (x = 3\sec t\,, dx = 3\sec t \tan t\, dt)$

$= \int \frac{3\tan t}{3\sec t} 3\sec t \tan t\, dt$

$= 3\int \tan^2 t\, dt$

$= 3\int (\sec^2 t - 1) dt$

$$= 3(\tan t - t) + C \quad \left(t = \arccos\frac{3}{x}, \tan t = \frac{1}{3}\sqrt{x^2 - 9}\right)$$

$$= \sqrt{x^2 - 9} - 3\arccos\frac{3}{x} + C$$

(9) $\int \frac{dx}{\sqrt{(1-x^2)^3}}$

Solution: $\int \frac{dx}{\sqrt{(1-x^2)^3}} \quad (x = \sin t, dx = \cos t\, dt)$

$$= \int \frac{\cos t dt}{\cos^3 t}$$

$$= \int \sec^2 t\, dt$$

$$= \tan t + C \quad \left(\tan t = \frac{x}{\sqrt{1-x^2}}\right)$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

(10) $\int \frac{dx}{\sqrt{(x^2+a^2)^3}}$

Solution: $\int \frac{dx}{\sqrt{(x^2+a^2)^3}} \quad (x = a\tan t, dx = a\sec^2 t\, dt)$

$$= \int \frac{a\sec^2 t dt}{a^3 \sec^3 t}$$

$$= \frac{1}{a^2}\int \cos t\, dt$$

$$= \frac{1}{a^2}\sin t + C \quad \left(\sin t = \frac{x}{\sqrt{x^2+a^2}}\right)$$

$$= \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + C$$

(11) $\int \sqrt{\frac{x-a}{x+a}}\, dx$

Solution: $\int \sqrt{\frac{x-a}{x+a}}\, dx = \int \frac{x-a}{\sqrt{x^2-a^2}}\, dx$

$$= \int \frac{xdx}{\sqrt{x^2-a^2}} - a\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= \int \frac{\frac{1}{2}d(x^2-a^2)}{\sqrt{x^2-a^2}} - a\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$(u = x^2 - a^2, x = a\sec t)$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{u}} - a\int \frac{a\sec t\tan t}{a\tan t}\, dt$$

$$= \sqrt{u} - a\int \sec t\, dt$$

$$= \sqrt{x^2 - a^2} - a\ln|\sec t + \tan t| + C$$

$$= \sqrt{x^2 - a^2} - a\ln\left|\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right| + C$$

$$= \sqrt{x^2 - a^2} - a\ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

(12) $\int x\sqrt{\frac{x}{2a-x}}dx$

Solution: $\int x\sqrt{\frac{x}{2a-x}}dx \quad (x > 0)$

$$= \int \frac{x^2}{\sqrt{2ax-x^2}}dx$$

$$= \int \frac{x^2-2ax}{\sqrt{2ax-x^2}}dx + \int \frac{2ax-2a^2}{\sqrt{2ax-x^2}}dx + \int \frac{2a^2}{\sqrt{2ax-x^2}}dx$$

$$= -\int \sqrt{a^2 - (x-a)^2}\,dx + \int \frac{-ad(2ax-x^2)}{\sqrt{2ax-x^2}} + \int \frac{2a^2}{\sqrt{a^2-(x-a)^2}}dx$$

$$= -a^2\int \cos^2 t\,dt - 2a\sqrt{2ax-x^2} + \int \frac{2a^3 d\left(\frac{x-a}{a}\right)}{a\sqrt{1-\left(\frac{x-a}{a}\right)^2}}$$

$$= -\frac{a^2}{2}\int(1+\cos 2t)dt - 2a\sqrt{2ax-x^2} + 2a^2\arcsin\frac{x-a}{a}$$

$$= \frac{3}{2}a^2\arcsin\frac{x-a}{a} - \frac{1}{4}(x+3a)\sqrt{2ax-x^2} + C$$

(13) $\int \frac{dx}{1+\sqrt{2x}}$

Solution: $\int \frac{dx}{1+\sqrt{2x}} \quad \left(t = \sqrt{2x}, x = \frac{1}{2}t^2, dx = tdt\right)$

$$= \int \frac{tdt}{1+t}$$

$$= \int \left(1 - \frac{1}{1+t}\right)dt$$

$$= t - \ln(1+t) + C$$

$$= \sqrt{2x} - \ln\left(1 + \sqrt{2x}\right) + C$$

(14) $\int x^2 \cdot \sqrt[3]{1-x}dx$

Solution: $\int x^2 \cdot \sqrt[3]{1-x}dx \quad (t = 1-x, x = 1-t, dx = -dt)$

$$= -\int (1-t)^2 \cdot t^{\frac{1}{3}}dt$$

$$= -\int \left(t^{\frac{7}{3}} - 2t^{\frac{4}{3}} + t^{\frac{1}{3}}\right) dt$$
$$= -\frac{3}{10}t^{\frac{10}{3}} + \frac{6}{7}t^{\frac{7}{3}} - \frac{3}{4}t^{\frac{4}{3}} + C$$
$$= -\frac{3}{10}(1-x)^{\frac{10}{3}} + \frac{6}{7}(1-x)^{\frac{7}{3}} - \frac{3}{4}(1-x)^{\frac{4}{3}} + C$$

(15) $\int \frac{dx}{x\sqrt{x^2-1}}$

Solution: $\int \frac{dx}{x\sqrt{x^2-1}} \quad (x = \sec t\,, dx = \sec t \tan t\, dt)$

$$= \int \frac{\sec t \tan t dt}{\sec t \tan t}$$
$$= \int dt$$
$$= t + C$$
$$= \arccos\frac{1}{x} + C$$

(16) $\int \frac{x^2}{\sqrt{a^2-x^2}} dx$

Solution: $\int \frac{x^2}{\sqrt{a^2-x^2}} dx \quad (x = a\sin t\,, dx = a\cos t\, dt)$

$$= \int \frac{a^2\sin^2 t}{a\cos t} a\cos t\, dt$$
$$= a^2 \int \sin^2 t\, dt$$
$$= \frac{a^2}{2}\int (1 - \cos 2t)\, dt$$
$$= \frac{a^2}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$
$$= \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + C$$

(17) $\int \frac{\sqrt{a^2-x^2}}{x^4} dx$

Solution: $\int \frac{\sqrt{a^2-x^2}}{x^4} dx \quad (x = a\sin t\,, dx = a\cos t\, dt)$

$$= \int \frac{a\cos t}{a^4\sin^4 t} a\cos t\, dt$$
$$= \frac{1}{a^2}\int \frac{\cos^2 t}{\sin^4 t} dt$$
$$= \frac{1}{a^2}\int \cot^2 t \csc^2 t\, dt$$
$$= -\frac{1}{a^2}\int \cot^2 t\, d(\cot t)$$

$$= -\frac{1}{3a^2}\cot^3 t + C$$

$$= -\frac{1}{3a^2}\frac{\sqrt{(a^2-x^2)^3}}{x^3} + C$$

(18) $\int \frac{dx}{1+\sqrt{1-x^2}}$

Solution: $\int \frac{dx}{1+\sqrt{1-x^2}} \quad (x = \sin t\,, dx = \cos t\, dt)$

$$= \int \frac{\cos t dt}{1+\cos t}$$

$$= \int \left(1 - \frac{1}{1+\cos t}\right) dt$$

$$= t - \int \frac{1}{2}\sec^2\frac{t}{2}\, dt$$

$$= t - \tan\frac{t}{2} + C \quad (t = \arcsin x)$$

$$= \arcsin x - \tan\left(\frac{\arcsin x}{2}\right) + C$$

(19) $\int \frac{x^{15}}{(x^4-1)^3}\, dx$

Solution: $\int \frac{x^{15}}{(x^4-1)^3}\, dx = \int \frac{x^{12}}{(x^4-1)^3}\frac{1}{4}\ d(x^4-1) \quad (u = x^4-1)$

$$= \frac{1}{4}\int \frac{(u+1)^3}{u^3}\, du$$

$$= \frac{1}{4}\int \left(1 + \frac{3}{u} + \frac{3}{u^2} + \frac{1}{u^3}\right) du$$

$$= \frac{1}{4}\left(u + 3\ln|u| - \frac{3}{u} - \frac{1}{2u^2}\right) + C$$

$$= \frac{1}{4}\left(x^4 - 1 + 3\ln(x^4-1) - \frac{3}{x^4-1} - \frac{1}{2(x^4-1)}\right) + C$$

(20) $\int \frac{1}{x(x^n+1)}\, dx$

Solution: $\int \frac{1}{x(x^n+1)}\, dx = \int \frac{x^{n-1}dx}{x^n(x^n+1)}$

$$= \int \frac{\frac{1}{n}d(x^n)}{x^n(x^n+1)} \quad (u = x^n)$$

$$= \frac{1}{n}\int \frac{du}{u(u+1)}$$

$$= \frac{1}{n}\int \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \frac{1}{n}\ln\left|\frac{u}{u+1}\right| + C$$

$$= \frac{1}{n}\ln\left|\frac{x^n}{x^n+1}\right| + C$$

3. Find the following indefinite integral

(1) $\int x\, e^{2x} dx$

Solution: $\int x\, e^{2x} dx = \int \frac{1}{2} x\, d(e^{2x})$

$$= \frac{1}{2} x e^{2x} - \frac{1}{2}\int e^{2x}\, dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

(2) $\int x \ln(x-1)\, dx$

Solution: $\int x \ln(x-1)\, dx = \int \frac{1}{2} \ln(x-1)\, d(x^2)$

$$= \frac{1}{2} x^2 \ln(x-1) - \frac{1}{2}\int \frac{x^2}{x-1} dx$$

$$= \frac{1}{2} x^2 \ln(x-1) - \frac{1}{2}\int \left(x + 1 + \frac{1}{x-1}\right) dx$$

$$= \frac{1}{2}(x^2-1)\ln(x-1) - \frac{1}{4}x^2 - \frac{1}{2}x + C$$

(3) $\int x^2 \sin 3x\, dx$

Solution: $\int x^2 \sin 3x\, dx = \int -\frac{1}{3} x^2 d(\cos 3x)$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\int x \cos 3x\, dx$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{9}\int x\, d(\sin 3x)$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{9} x \sin 3x - \frac{2}{9}\int \sin 3x\, dx$$

$$= -\frac{1}{3}x^2 \cos 3x + \frac{2}{9} x \sin 3x + \frac{2}{27}\cos 3x + C$$

(4) $\int \frac{x}{\sin^2 x} dx$

Solution: $\int \frac{x}{\sin^2 x} dx = \int -x\, d(\cot x)$

$$= -x \cot x + \int \cot x\, dx$$

$$= -x \cot x + \ln|\sin x| + C$$

(5) $\int x \cos^2 x\, dx$

Solution: $\int x\cos^2 x\,dx = \frac{1}{2}\int x\,(1+\cos 2x)dx$

$$= \frac{1}{4}x^2 + \frac{1}{2}\int x\cos 2x\,dx$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x - \frac{1}{4}\int \sin 2x\,dx$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$$

(6) $\int \arcsin x\,dx$

Solution: $\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$

$$= x\arcsin x - \int \frac{-\frac{1}{2}d(1-x^2)}{\sqrt{1-x^2}} \quad (u = 1-x^2)$$

$$= x\arcsin x + \frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= x\arcsin x + \sqrt{u} + C$$

$$= x\arcsin x + \sqrt{1-x^2} + C$$

(7) $\int \arctan x\,dx$

Solution: $\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}dx$

$$= x\arctan x - \int \frac{1}{2}\frac{d(1+x^2)}{1+x^2} \quad (u = 1+x^2)$$

$$= x\arctan x - \frac{1}{2}\int \frac{du}{u}$$

$$= x\arctan x - \frac{1}{2}\ln|u| + C$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

(8) $\int x^2\arctan x\,dx$

Solution: $\int x^2\arctan x\,dx = \int \frac{1}{3}\arctan x\,d(x^3)$

$$= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$$

$$= \frac{1}{3}x^3\arctan x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$$

(9) $\int x\tan^2 x\,dx$

Solution: $\int x \tan^2 x \, dx = \int x\,(\sec^2 x - 1) dx$

$$= \int x \, d(\tan x) - \frac{1}{2}x^2$$

$$= x \tan x - \int \tan x \, dx - \frac{1}{2}x^2$$

$$= x \tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

(10) $\int \frac{\arcsin x}{\sqrt{1-x}} dx$

Solution: $\int \frac{\arcsin x}{\sqrt{1-x}} dx = \int -2 \arcsin x \, d\left(\sqrt{1-x}\right)$

$$= -2\sqrt{1-x} \arcsin x + 2\int \frac{\sqrt{1-x}}{\sqrt{1-x^2}} dx$$

$$= -2\sqrt{1-x} \arcsin x + 2\int \frac{1}{\sqrt{1+x}} dx$$

$$= -2\sqrt{1-x} \arcsin x + 4\sqrt{1+x} + C$$

(11) $\int \ln^2 x \, dx$

Solution: $\int \ln^2 x \, dx = x \ln^2 x - \int x \cdot 2\ln x \cdot \frac{1}{x} dx$

$$= x \ln^2 x - \int 2 \ln x \, dx$$

$$= x \ln^2 x - 2x \ln x + 2\int x \cdot \frac{1}{x} dx$$

$$= x \ln^2 x - 2x \ln x + 2x + C$$

(12) $\int x^2 \ln x \, dx$

Solution: $\int x^2 \ln x \, dx = \int \frac{1}{3} \ln x \, d(x^3)$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \cdot \frac{1}{x} dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

(13) $\int e^{-x} \sin 5x \, dx$

Solution: $\int e^{-x} \sin 5x \, dx = -\int \sin 5x \, d(e^{-x})$

$$= -e^{-x} \sin 5x + 5\int e^{-x} \cos 5x \, dx$$

$$= -e^{-x}\sin 5x - 5\int \cos 5x\, d(e^{-x})$$

$$= -e^{-x}\sin 5x - 5e^{-x}\cos 5x -$$

$$25\int e^{-x}\sin 5x\, dx$$

$$\Rightarrow \int e^{-x}\sin 5x\, dx = -\frac{1}{26}e^{-x}\sin 5x - \frac{5}{26}e^{-x}\cos 5x + C$$

(14) $\int e^{x}\sin^2 x\, dx$

Solution: $\int e^{x}\sin^2 x\, dx = \frac{1}{2}\int e^{x}(1-\cos 2x)dx$

$$= \frac{1}{2}e^{x} - \frac{1}{2}\int e^{x}\cos 2x\, dx$$

$$\int e^{x}\cos 2x\, dx = \int \cos 2x\, d(e^{x})$$

$$= e^{x}\cos 2x + 2\int e^{x}\sin 2x\, dx$$

$$= e^{x}\cos 2x + 2\int \sin 2x\, d(e^{x})$$

$$= e^{x}\cos 2x + 2e^{x}\sin 2x - 4\int e^{x}\cos 2x\, dx$$

$$\Rightarrow \int e^{x}\cos 2x\, dx = \frac{1}{5}e^{x}\cos 2x + \frac{2}{5}e^{x}\sin 2x + C$$

$$\Rightarrow \int e^{x}\sin^2 x\, dx = \frac{1}{2}e^{x} - \frac{1}{10}e^{x}\cos 2x - \frac{1}{5}e^{x}\sin 2x + C$$

(15) $\int \frac{\ln^3 x}{x^2}dx$

Solution: $\int \frac{\ln^3 x}{x^2}dx = \int -\ln^3 x\, d\left(\frac{1}{x}\right)$

$$= -\frac{1}{x}\ln^3 x + \int \frac{3\ln^2 x}{x^2}dx$$

$$= -\frac{1}{x}\ln^3 x - 3\int \ln^2 x\, d\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x}\ln^3 x - \frac{3\ln^2 x}{x} + 6\int \frac{\ln x}{x^2}dx$$

$$= -\frac{1}{x}\ln^3 x - \frac{3\ln^2 x}{x} - 6\int \ln x\, d\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x}\ln^3 x - \frac{3\ln^2 x}{x} - \frac{6\ln x}{x} + 6\int \frac{1}{x^2}dx$$

$$= -\frac{1}{x}\ln^3 x - \frac{3\ln^2 x}{x} - \frac{6\ln x}{x} - \frac{6}{x} + C$$

(16) $\int \cos(\ln x)\, dx$

Solution: $\int \cos(\ln x)\,dx \quad (t = \ln x\,, x = e^t, dx = e^t dt)$

$= \int e^t \cos t\,dt$

$= \frac{1}{2}(\sin t + \cos t)e^t + C$

$= \frac{1}{2}x(\sin \ln x + \cos \ln x) + C$

(17) $\int (\arcsin x)^2\,dx$

Solution: $\int (\arcsin x)^2\,dx = x(\arcsin x)^2 - 2\int \frac{x \arcsin x}{\sqrt{1-x^2}}\,dx$

$= x(\arcsin x)^2 + 2\int \arcsin x\, d\left(\sqrt{1-x^2}\right)$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(18) $\int \sqrt{x}e^{\sqrt{x}}dx$

Solution: $\int \sqrt{x}e^{\sqrt{x}}dx \quad \left(t = \sqrt{x}, x = t^2, dx = 2tdt\right)$

$= \int 2t^2 e^t\,dt$

$= 2t^2e^t - 4\int te^t\,dt$

$= 2t^2e^t - 4te^t + 4\int e^t\,dt$

$= 2t^2e^t - 4te^t + 4e^t + C$

$= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$

(19) $\int e^{\sqrt{x+1}}dx$

Solution: $\int e^{\sqrt{x+1}}dx \quad \left(t = \sqrt{x+1}, x = t^2 - 1, dx = 2tdt\right)$

$= \int 2te^t\,dt$

$= 2te^t - 2\int e^t\,dt$

$= 2te^t - 2e^t + C$

$= 2\sqrt{x+1}e^{\sqrt{x+1}} - 2e^{\sqrt{x+1}} + C$

(20) $\int \ln(x + \sqrt{1+x^2})\,dx$

Solution: $\int \ln(x+\sqrt{1+x^2})\,dx = x\ \ln(x+\sqrt{1+x^2}) -$

$$\int \frac{x(1+\frac{2x}{2\sqrt{1+x^2}})}{x+\sqrt{1+x^2}}\,dx$$

$$= x\ \ln\left(x+\sqrt{1+x^2}\right) - \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$= x\ \ln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + C$$