Answers to Middle Test

No.	1-5 $6'*5=30'$	6-7-8-9 $6'*4=24'$	10-11-12-13 $6'*4=24'$	14-15 $11'*2=22'$	Total
Marks					

1. In the parametric equation $\begin{cases} x=2e^t+t+1 \\ y=4(t-1)e^t+t^2 \end{cases}$, please find $\left.\dfrac{d^2y}{dx^2}\right|_{t=0}$.

Solu:

$$\frac{dy}{dx}=\frac{4te^t+2t}{2e^t+1}$$

$$\frac{d^2y}{dx^2}=\frac{(4e^t+4te^t+2)(2e^t+1)-(4te^t+2t)2e^t}{(2e^t+1)^3}$$

$$\frac{d^2y}{dx^2}\Big|_{t=0}=\frac{2}{3}.$$

2. Find the tangent plane of the surface $z=x+2y+\ln(1+x^2+y^2)$ at point $(0,0,0)$

Solu:

$$F=x+2y+\ln(1+x^2+y^2)-z$$

$$\vec{n}=(F'_x,F'_y,F'_z)=(1+\frac{2x}{1+x^2+y^2},2+\frac{2y}{1+x^2+y^2},-1)$$

$$\vec{n}_{(0,0,0)}=(1,2,-1)\Rightarrow$$

The tangent plane is: $x+2y-z=0$.

3. The maximum directional derivative of $f=x^2+2y^2$ at $(0,1)$

Solu:

The maximum directional derivative is the module of gradient. So we have

$$\left|(f_x,f_y)\right|=\left|(2x,4y)\right|_{(0,1)}=4.$$

4. Find the convergence field and summation function of the power series $\sum_{n=0}^{\infty}\frac{x^n}{n+1}$.

Solu:

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right|=1\Rightarrow R=1$$

$$x=-1,\sum_{n=0}^{\infty}\frac{1}{n+1}(-1)^n,\text{converge}$$

$$x=1,\quad \sum_{n=0}^{\infty}\frac{1}{n+1},\qquad \text{diverge}$$

$$\Rightarrow x\in[-1,1).$$

$$s(x)=\sum_{n=0}^{\infty}\frac{x^n}{n+1},\quad xs(x)=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}\frac{x^n}{n},$$

$$\left(xs(x)\right)'=\sum_{n=1}^{\infty}\left(\frac{x^n}{n}\right)'=\sum_{n=1}^{\infty}x^{n-1}=\frac{1}{1-x}$$

$$\int_0^x\left(xs(x)\right)'dx=\int_0^x\frac{1}{1-x}dx\Rightarrow xs(x)=-\ln(1-x)$$

$$\Rightarrow s(x)=\begin{cases}\dfrac{-\ln(1-x)}{x},0<|x|<1\text{ or }x=-1\\ 1,\qquad\qquad x=0.\end{cases}$$

5. Find the power series of x for the function $f\left(x\right)=\ln\frac{1+x}{1-x}$.

Solu: $f\left(x\right)=\ln\frac{1+x}{1-x}=2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1},\quad\left(-1<x<1\right)$

$$f(x)=\ln(1+x)-\ln(1-x)$$

$$f'(x)=\frac{1}{1+x}+\frac{1}{1-x}=\frac{2}{1-x^2},$$

$$f'(x)=\frac{2}{1-x^2}=2\sum_{n=0}^{\infty}(x^2)^n$$

$$\int_0^x f'(x)dx=2\int_0^x\sum_{n=0}^{\infty}(x^{2n})dx$$

$$f(x)-f(0)=2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1},\ f(0)=0\Rightarrow f(x)=2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}.$$

6. Find the maximum and minimum values of $u = x - 2y + 2z$ subject to the condition $x^2 + y^2 + z^2 = 1$.

Solu：令 $L = x - 2y + 2z + \lambda\left(x^2 + y^2 + z^2 - 1\right)$

then $\begin{cases} L_x = 1 + 2\lambda x = 0 \\ L_y = -2 + 2\lambda y = 0 \\ L_z = 2 + 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}, \Rightarrow \begin{cases} x = \dfrac{-1}{2\lambda} \\ y = \dfrac{1}{\lambda} \\ z = \dfrac{-1}{\lambda} \end{cases}, \Rightarrow \lambda = \pm\dfrac{3}{2}, \begin{cases} x = \dfrac{-1}{3} \\ y = \dfrac{2}{3} \\ z = \dfrac{-2}{3} \end{cases} or \begin{cases} x = \dfrac{1}{3} \\ y = \dfrac{-2}{3} \\ z = \dfrac{2}{3} \end{cases}$

$u\left(-\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3}\right) = -\dfrac{1}{3} - \dfrac{4}{3} - \dfrac{4}{3} = -3, u\left(\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3}\right) = \dfrac{1}{3} + \dfrac{4}{3} + \dfrac{4}{3} = 3$, Max 3，min -3

7. Let $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$

Prove：1）the partial derivatives of $f(x, y)$ exist at (0,0);

2）$f(x, y)$ is not differentiable at $(0, 0)$.

Proof：1）by $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0 - 0}{\Delta x} = 0$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0$$

$f(x, y)$ has partial derivative at $(0, 0)$

2）by $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta z - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2}$

take $\Delta y = k\Delta x$ 时

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x \Delta y}{(\Delta x)^2 + (\Delta y)^2} = \lim_{\Delta x \to 0} \frac{k^2}{1 + k^2} = \frac{k^2}{1 + k^2}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta z - f_x(0, 0)\Delta x - f_y(0, 0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \neq 0$$

not differentiable.

8. Let $z = xf\left(xy \ , \ \frac{y}{x} \right)$, f has second-order continuous partial derivatives, find $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial y \partial x}$

Solu: $\frac{\partial z}{\partial y} = x\left(xf_1' + \frac{1}{x} f_2' \right)$,

$$\frac{\partial z}{\partial x} = f + x\left(yf_1' - \frac{y}{x^2} f_2' \right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2xf_1' + x^2\left(yf_{11}'' - \frac{y}{x^2} f_{12}'' \right) + yf_{21}'' - \frac{y}{x^2} f_{22}'' = 2xf_1' + x^2 yf_{11}'' - \frac{y}{x^2} f_{22}''$$

9. Let $z = f(u, x, y), \quad u = xe^y$, and f has second-order continuous partial derivatives, find $\frac{\partial^2 z}{\partial x \partial y}$.

$$\frac{\partial z}{\partial x} = e^y f_1' + f_2', \frac{\partial^2 z}{\partial x \partial y} = xe^{2y} f_{11}'' + e^y f_{13}'' + xe^y f_{21}'' + f_{23}'' + e^y f_1'$$

10. Let $z = f(2x - y, y\sin x)$, and f has second-order continuous partial derivatives, find $\frac{\partial^2 z}{\partial x \partial y}$.

$$\frac{\partial z}{\partial x} = 2 f_1' + y\cos x f_2', \frac{\partial^2 z}{\partial x \partial y} = -2 f_{11}'' + \left(2\sin x - y\cos x\right) f_{12}'' + y\sin x\cos x f_{22}'' + \cos x f_2'$$

11. Let $G(u\ ,\ v)$ is differentiable, and the equation $G\left(\frac{x}{z}\ ,\ \frac{y}{z}\right)=0$ implying $z=z(x,y)$, compute $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$.

Solu：

(1)：chain rule，

$$G\left(\frac{x}{z}\ ,\ \frac{y}{z}\right)=0$$
$$G_1'\left(\frac{1}{z}+x(-\frac{1}{z^2})\frac{\partial z}{\partial x}\right)+G_2'\left(y(-\frac{1}{z^2})\frac{\partial z}{\partial x}\right)=0\Rightarrow\frac{\partial z}{\partial x}=?$$

In the similar way,

$$G\left(\frac{x}{z}\ ,\ \frac{y}{z}\right)=0$$
$$G_1'\left(x(-\frac{1}{z^2})\frac{\partial z}{\partial y}\right)+G_2'\left(\frac{1}{z}+y(-\frac{1}{z^2})\frac{\partial z}{\partial y}\right)=0\Rightarrow\frac{\partial z}{\partial y}=?$$

(2)

$$F(x,y,z)=G\left(\frac{x}{z}\ ,\ \frac{y}{z}\right)=0$$
$$F_x=G_1'\frac{1}{z}$$
$$F_y=G_2'\frac{1}{z}$$
$$F_z=G_1'x(-\frac{1}{z^2})+G_2'y(-\frac{1}{z^2})$$
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z};\frac{\partial z}{\partial y}=-\frac{F_y}{F_z};\text{take in OK}$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=?$$

12. Let $z = xyf\left(\frac{y}{x}\right)$, and f is derivative, if $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y^2(\ln y - \ln x)$ find $f(1), f'(1)$.

Solu:

$$\frac{\partial z}{\partial x} = yf(\frac{y}{x}) + xyf'(\frac{y}{x})\cdot(-\frac{y}{x^2}) = yf(\frac{y}{x}) - \frac{y^2}{x}f'(\frac{y}{x})$$

$$\frac{\partial z}{\partial y} = xf(\frac{y}{x}) + xyf'(\frac{y}{x})\cdot\frac{1}{x} = xf(\frac{y}{x}) + yf'(\frac{y}{x})$$

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2xyf(\frac{y}{x}) = y^2(\ln\frac{y}{x})$$

$$f(\frac{y}{x}) = \frac{1}{2}\frac{y}{x}\ln(\frac{y}{x}) \Rightarrow f(u) = \frac{1}{2}u\ln u$$

$$\Rightarrow f(x) = \frac{1}{2}x\ln x,\ f'(x) = \frac{1}{2}(\ln x + 1)$$

$$x = 1 \Rightarrow f(1) = 0,\ f'(1) = \frac{1}{2}.$$

13. Let *f* is differential, and $f(x+1, e^x) = x(x+1)^2$, $f(x, x^2) = 2x^2\ln x$, find $df(1,1)$.

Solu:

$$f_1'(x+1, e^x) + e^x f_2'(x+1, e^x) = (x+1)^2 + 2x(x+1), (1)$$

$$f_1'(x, x^2) + 2xf_2'(x, x^2) = 4x\ln x + 2x \qquad (2)$$

(0,0)and (1,1) in (1) and (2), have

$$f_1'(1,1) + f_2'(1,1) = 1, \text{and } f_1'(1,1) + 2f_2'(1,1) = 2$$

$$\Rightarrow f_1'(1,1) = 0,\ f_2'(1,1) = 1.$$

$$\Rightarrow \mathrm{d}\,f(1,1) = f_1'(1,1)dx + f_2'(1,1)dy = dy.$$

14.The convergence field is $(a,+\infty)$ of series $\sum_{n=1}^{\infty}\frac{n!}{n^n}e^{-nx}$, find a.

Solu:

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}e^{-nx}=\sum_{n=1}^{\infty}\frac{n!}{n^n}(e^{-x})^n$$

then

$$\lim_{n\to+\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to+\infty}\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}=\lim_{n\to+\infty}(1+\frac{1}{n})^n=e.$$

$$-e<e^{-x}<e\Rightarrow -x<1\Rightarrow x>-1\Rightarrow a=-1.$$

$NOTE:$

$x=-1$, doesn't converge

$\sum_{n=1}^{\infty}\frac{n!}{n^n}e^n$, we have

$$\frac{u_{n+1}}{u_n}=\frac{\frac{(n+1)!}{(n+1)^{n+1}}e^{n+1}}{\frac{n!}{n^n}e^n}=e\frac{1}{(1+\frac{1}{n})^n}>e\cdot\frac{1}{e}=1$$

doesn't tend to 0.diverge.

15. Let $u_n(\mathrm{x})=e^{-nx}+\frac{1}{n(n+1)}x^{n+1}(n=1,2,\cdots)$, find convergence field and sum function of $\sum_{n=1}^{\infty}u_n(x)$.

Solu:

$$\sum_{n=1}^{\infty}u_n(x)=S_1(x)+S_2(x)=\sum_{n=1}^{\infty}e^{-nx}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}x^{n+1}(n=1,2,\cdots),$$

(1)

$$S_1(x)=\sum_{n=1}^{\infty}e^{-nx},0<e^{-x}<1\Rightarrow x>0$$

$$S_2(x)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}x^{n+1}\Rightarrow -1\le x\le 1$$

the convergent field is: (0,1]

(2)

$$S_1(x)=\sum_{n=1}^{\infty}e^{-nx}=\frac{e^{-x}}{1-e^{-x}}$$

$$S_2(x)=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}x^{n+1}=\sum_{n=1}^{\infty}\frac{x^{n+1}}{n}-\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=-x\ln(1-x)-[-\ln(1-x)-x]$$

$$=(1-x)\ln(1-x)+x,\ x\in(0,1]$$

and $S_2(1)=\lim_{x\to 1}S_2(x)=1$

so sum function is

$$S(x)=\begin{cases}\dfrac{e^{-x}}{1-e^{-x}}+(1-x)\ln(1-x)+x, & x\in(0,1)\\ \dfrac{e}{1-e}, & x=1.\end{cases}$$