Name ________ Student ID [ ] School ________ Major/Class ________ Seat No. ________

(DON NOT WRITE YOUR ANSWER IN THIS AREA)

Seal line .... Seal line .... Seal line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

[redacted]

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Notice: **1. Make sure that you have filled the form on the left side of the seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts [redacted]

Question No.	**I**	**II**	**III**	**IV**	**V**	**VI**	**Sum**
Score							

I. (20 points) Consider the matrix

Score

$$A = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}.$$

1. Find a basis for $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$.
2. Calculate AA^T.
3. Calculate the determinant of the matrix AA^T.

II. (30 points) Let $\mathbb{P}_2(\mathbb{R})$ be the space of polynomials of degree at most two with real numbers as the coefficients.

Score

1. Prove that $\mathscr{B} = \{4+5x+2x^2, 3x+x^2, x^2\}$ is a basis of $\mathbb{P}_2(\mathbb{R})$.
2. Let $\mathscr{E} = \{1, x, x^2\}$ be the standard basis of $\mathbb{P}_2(\mathbb{R})$. Find the change-of-coordinates matrix from $\mathscr{E}$ to $\mathscr{B}$.
3. Let $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be a linear transformation such that
$$T(a+bx+cx^2) = (2a+3b) + (2a-3b)x + (2a+3b+3c)x^2.$$
Write down the matrix $A = [T]_{\mathscr{E}}$ of the linear transformation T corresponding to the basis $\mathscr{E}$.
4. Find the eigenvalues and eigenvectors of the matrix A.
5. Diagonalize the matrix A.

IV. (10 points) Consider the following vectors

Score

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ h \end{pmatrix}.$$

1. Find the value h such that these three vectors are linearly dependent.
2. Based on the value h you find in the first question, write the third vector as a linear combination of the first two vectors.

Score

IV. (15 points) Let U be an $n \times n$ matrix whose columns form an orthonormal set.

1. Prove that U is invertible.
2. Let $n = 3$. Prove that $U^T U = I$.
3. Let $n = 3$, and take two vectors x and y in $\mathbb{R}^3$. Show that $(Ux) \cdot (Uy) = x \cdot y$.

Score

V. (15 points) Consider three vectors $y = \begin{pmatrix} 5 \\ -9 \\ 5 \end{pmatrix}, u_1 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}, u_2 = \begin{pmatrix} -3 \\ 14 \\ -7 \end{pmatrix}$.

1. Find the orthogonal projection of u_1 onto u_2;
2. Find the distance from y to the plane in $\mathbb{R}^3$ spanned by u_1 and u_2.

Score

VI. (15 points) If a matrix M is symmetric (i.e. $M = M^T$) and all of its eigenvalues are positive, then we say that the matrix M is *positive definite*. Let A be a 3×3 positive definite matrix.

1. Prove that the determinant of A is positive.
2. Let λ be an eigenvalue of A. Prove that λ^{-1} is an eigenvalue of A^{-1}.
3. Prove that there exists a matrix M such that $M^2 = A$.