Seat NO. ______ Major ______ Student ID ______ Chinese Name ______

（DONOT WRITE YOUR ANSWER IN THIS AREA）

Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

2021-2022-1《Calculus I》 Exam Paper A

Notice:
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	1	2	3	4	5	6	7	8	9	Sum
Score										

1. Answer the questions (25 marks):

(1) Let $f(x)=(x^2+1)5^x$, find $f^{(n)}(0)$.

Solution:

$$f^{(n)}(x) = C_n^0(x^2+1)(5^x)^{(n)} + C_n^1(2x)(5^x)^{(n-1)} + C_n^2 2(5^x)^{(n-2)}$$

$$f^{(n)}(0) = (\ln 5)^n + n(n-1)(\ln 5)^{n-2}$$

(2) Find the limit $\lim\limits_{n\to+\infty} n(\sqrt[n]{2021}-1)$.

Solution:

$$\text{The limit} = \lim_{n\to+\infty}\frac{e^{\frac{1}{n}\ln 2021}-1}{\frac{1}{n}} = \lim_{n\to+\infty}\frac{\frac{1}{n}\ln 2021}{\frac{1}{n}} = \ln 2021$$

(3) Calculate $\int_0^{\frac{\pi}{2}}\sin^4 x\,dx$.

Solution:

$$\text{The integral} = \int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2x}{2}\right)^2 dx = \int_0^{\frac{\pi}{2}}\left(\frac{1}{4}-\frac{\cos 2x}{2}+\frac{1}{4}\cdot\frac{1+\cos 4x}{2}\right)dx = \frac{3\pi}{16}$$

or

$$\text{The integral} = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi}{16}$$

(4) Find the $\frac{d^2y}{dx^2}$, where $x^3+y^3=3xy$.

Solution:

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

$$\frac{d^2y}{dx^2} = \frac{(y' - 2x)(y^2 - x) - (y - x^2)(2yy' - 1)}{(y^2 - x)^2}$$

$$= \frac{-2xy + 6x^2y^2 - 2xy^4 - 2x^4y}{(y^2 - x)^3}$$

(5) Evaluate $\int \frac{dx}{(16+x^2)^{3/2}}$.

Solution: Let $x = 4\tan t,\ t \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ dx = 4\sec^2 t dt$

$$\int \frac{dx}{(16 + x^2)^{3/2}} = \int \frac{4\sec^2 t dt}{(16 + 16\tan^2 t)^{3/2}} = \frac{1}{16}\int \cos t dt = \frac{1}{16}\sin t + C$$

$$= \frac{1}{16}\frac{x}{\sqrt{16 + x^2}} + C$$

2. Evaluate the problems (20 marks):

(1) If $Q(x) = \int_1^{x-1}[\int_1^t f(z)dz]dt - \int_1^x f(x + t)dt$, Find $\frac{dQ}{dx}$.

Solution: $\int_1^x f(x + t)dt \overset{u=x+t}{=} \int_{x+1}^{2x} f(u)du$

$$\frac{dQ}{dx} = \int_1^{x-1} f(z)dz - 2f(2x) + f(x + 1)$$

(2) Find the volume of the solid generated by revolving the region bounded by the curves $y = x^2$ and $y^2 = 8x$ about the x-axis.

Solution:

$$\begin{cases} y = x^2 \\ y^2 = 8x \end{cases} \Rightarrow \text{intersection points}: (0, 0), (2, 4)$$

$$V = \pi\int_0^2 (8x - x^4)dx = \frac{48}{5}\pi$$

(3) Determine the concavity of the following function

$$f(x) = \frac{1}{x^2 + 1}, \quad x \in (-\infty, +\infty).$$

And find, if possible, the inflection point(s) of the given function on the indicated interval.

Solution:

$$f'(x) = -\frac{2x}{(x^2+1)^2},\ f''(x) = \frac{6x^2-2}{(x^2+1)^3} \triangleq 0 \Rightarrow x = \pm\frac{\sqrt{3}}{3}$$

The split points are $\frac{\sqrt{3}}{3}$ and $-\frac{\sqrt{3}}{3}$.

$x \in (-\infty, -\frac{\sqrt{3}}{3}) \cup (\frac{\sqrt{3}}{3}, +\infty)$, $f'' > 0$;

$x \in (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, $f'' < 0$.

So, $f(x)$ is concave up on $(-\infty, -\frac{\sqrt{3}}{3}], [\frac{\sqrt{3}}{3}, +\infty)$, and concave down on $[-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$.

Inflection points are $(\frac{\sqrt{3}}{3}, \frac{3}{4})$, $(-\frac{\sqrt{3}}{3}, \frac{3}{4})$.

(4) Evaluate $\lim\limits_{x\to 0^+}\left(\frac{2^x+3^x+5^x+9^x}{4}\right)^{\frac{1}{x}}$.

Solution: The limit

$$= \lim_{x\to 0^+} e^{\frac{1}{x}\ln(\frac{2^x+3^x+5^x+9^x}{4})}$$

$$= e^{\lim\limits_{x\to 0^+}\frac{\ln(\frac{2^x+3^x+5^x+9^x}{4})}{x}} = e^{\lim\limits_{x\to 0^+}\frac{4}{2^x+3^x+5^x+9^x}\cdot\frac{2^x\ln 2+3^x\ln 3+5^x\ln 5+9^x\ln 9}{4}}$$

$$= e^{\frac{\ln 2+\ln 3+\ln 5+\ln 9}{4}} = \sqrt[4]{270}$$

3. Evaluate the following integrals(25 marks):

(1) $\int \frac{1}{x^4-1}\,dx$,

Solution: The integral

$$= \int\left(\frac{1}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$$

(2) $\int_{-\pi}^{\pi}(\sin x + \cos x)^3\,dx$,

Solution: The integral

$$= \int_{-\pi}^{\pi} (\sin^3 x + 3\sin^2 x\cos x + 3\sin x\cos^2 x + \cos^3 x)dx$$

$$= 2\int_{0}^{\pi} (3\sin^2 x\cos x + \cos^3 x)dx$$

$$= 6 \cdot \frac{1}{3}\sin^3 x \Big|_0^{\pi} + 2[\sin x - \frac{1}{3}\sin^3 x]_0^{\pi} = 0$$

(3) $\int \frac{\ln\tan x}{\sin x\cos x}dx$,

Solution: The integral

$$= \int \frac{\ln\tan x}{\tan x\cos^2 x}dx = \int \frac{\ln\tan x}{\tan x}d\tan x$$

$$= \int \ln\tan x\, d(\ln\tan x) = \frac{1}{2}(\ln\tan x)^2 + C$$

(4) $\int_0^1 \frac{\ln(1+x)}{(1+x)^2}dx$,

Solution: The integral

$$= [\ln(1+x) \cdot \frac{-1}{1+x}]_0^1 - \int_0^1 \frac{-1}{1+x} \cdot \frac{1}{1+x}dx$$

$$= -\frac{1}{2}\ln 2 + \int_0^1 \frac{1}{(1+x)^2}dx$$

$$= -\frac{1}{2}\ln 2 + (\frac{-1}{1+x})\Big|_0^1 = \frac{1}{2} - \frac{1}{2}\ln 2$$

(5) $\int_0^{+\infty} e^{-x}\sin x dx$.

Solution:

$$\int_0^{+\infty} e^{-x}\sin x dx = \lim_{b\to+\infty}[-\frac{1}{2}e^{-x}(\sin x + \cos x)]_0^b = \frac{1}{2}$$

4. (5 marks) If $f(x)$ is continuous and $f(x) = \begin{cases} ae^{2x}, & x \geq 0 \\ \dfrac{e^{\sin x} - 1}{\arcsin\frac{x}{2}}, & x < 0 \end{cases}$, please find the number a.

Solution:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} ae^{2x} = a,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\sin x} - 1}{\arcsin\frac{x}{2}} = \lim_{x\to 0^-} \frac{\sin x}{\frac{x}{2}} = 2$$

$$\Rightarrow a = 2$$

5. (5 marks) If $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, find its derivative $f'(x)$.

Solution:

$$x \neq 0,\ f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}};$$

$$x = 0,\ f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{x\to 0} \frac{x}{2e^{\frac{1}{x^2}}} = 0$$

6. (5 marks) Find the area of the region between the cardioid $r = a(1+\cos\theta)$ and the circle $r = a$, here $a > 0$.

Solution:

$$A = 2\,\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} a^2(1 + \cos\theta)^2 d\theta + \frac{\pi a^2}{2}$$

$$= a^2\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{2}}^{\pi} + \frac{\pi a^2}{2}$$

$$= \frac{5\pi a^2}{4} - 2a^2$$

7. (5 marks) Let $c > 0$, evaluate, if possible, the integral $\int_{2c}^{4c} \frac{dx}{\sqrt{x^2 - 4c^2}}$.

Solution: The integral

$$= \lim_{a\to(2c)^+} \int_a^{4c} \frac{1}{\sqrt{(\frac{x}{2c})^2 - 1}} d\left(\frac{x}{2c}\right)$$

$$= \lim_{a\to(2c)^+} \left[\ln\left(\frac{x}{2c} + \sqrt{(\frac{x}{2c})^2 - 1}\right)\right]_a^{4c} = \ln(2 + \sqrt{3})$$

8. (5 marks) Prove that $\lim_{x\to 3}(x^2 + x - 5) = 7$ by using ε - δ definition.

Proof : $\forall\varepsilon > 0, \left|(x^2 + x - 5) - 7\right| = \left|x + 4\right| \cdot \left|x - 3\right|$

If $\left|x - 3\right| < \delta \le 1$, then $\left|x + 4\right| < \left|x - 3\right| + 7 < 8$

$\therefore \left|(x^2 + x - 5) - 7\right| < 8\left|x - 3\right|$

Choose $\delta = \min\{1, \frac{\varepsilon}{8}\}$, when $0 < \left|x - 3\right| < \delta$, we have

$\left|(x^2 + x - 5) - 7\right| < 8\left|x - 3\right| < 8\delta < \varepsilon,$

so $\lim\limits_{x\to 3}(x^2 + x - 5) = 7$

9. (5 marks) If $f(x) \in C[0,1]$, and $f(x)$ is differentiable on $(0,1)$. $f(0) = f(1), \left|f'(x)\right| < 1$.

Try to prove that for any $x_1, x_2 \in (0,1)$, we have $|f(x_1) - f(x_2)| < \frac{1}{2}$.

Proof : (1) $0 \le x_2 - x_1 < \frac{1}{2}$, by M.V.T.

$\left|f(x_1) - f(x_2)\right| = \left|f'(c)(x_1 - x_2)\right| < \left|x_1 - x_2\right| < \frac{1}{2}, \ c \in (x_1, x_2)$

(2) $\frac{1}{2} \le x_2 - x_1 < 1$, since $f(0) = f(1)$,

$\left|f(x_1) - f(x_2)\right| = \left|f(x_1) - f(0) + f(1) - f(x_2)\right|$

$\le \left|f'(c_1)\right| x_1 + \left|f'(c_2)\right|(1 - x_2) < 1 - (x_2 - x_1) < \frac{1}{2} \qquad c_1 \in (0, x_1), \ c_2 \in (x_2, 1)$