Linear algebra and analytic geometry 2022-2023 Final Exam

I. Solution:

1. $\det A = \begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 2 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$

2. $\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & -4 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right].$

And thus $\mathrm{A}^{-1} = \begin{bmatrix} 7 & 2 & -4 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}.$

3. $\mathrm{A}^{-1}\mathrm{A}^{-1} = \begin{bmatrix} 49 & 18 & -30 \\ -15 & -5 & 9 \\ -6 & -3 & 4 \end{bmatrix}.$

$$\mathrm{B}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ A^{-1}A^{-1} & -A^{-1} \end{bmatrix} = \begin{bmatrix} 7 & 2 & -4 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 49 & 18 & -30 & -7 & -2 & 4 \\ -15 & -5 & 9 & 2 & 0 & -1 \\ -6 & -3 & 4 & 1 & 1 & -1 \end{bmatrix}.$$

II. Solution:

1. Suppose $\vec{x}$ is a vector in H, then

$$\vec{x} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 3 \\ 2 \\ 4 \end{bmatrix} + d\begin{bmatrix} 5 \\ 8 \\ 5 \\ 10 \end{bmatrix}.$$

And thus H is a subspace of $\mathbb{R}^4$. $k = 4$.

2. Observe that H is the column space of $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 5 \\ 1 & 2 & 4 & 10 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 5 \\ 1 & 2 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 2 \\ 4 \end{bmatrix}$ form a set of basis for H and its dimension is 3.

3. The orthogonal compliment $H^{\perp}$ is the null space of $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 4 \end{bmatrix}$.

Since $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, each solution of

$$\begin{bmatrix}1&1&0&1\\1&2&1&2\\2&3&2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$ has the form $$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}-x_4\\0\\-x_4\\x_4\end{bmatrix}=x_4\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}.$$

Thus, $\left\{\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right\}$ is a set of basis for the orthogonal compliment $H^{\perp}$.

III. Solution:

1. Since $\begin{bmatrix}-2&-8&-7\\2&5&2\\3&2&6\end{bmatrix}\sim\begin{bmatrix}2&5&2\\0&3&5\\0&-\frac{1}{2}&3\end{bmatrix}\sim\begin{bmatrix}2&5&2\\0&3&5\\0&-1&6\end{bmatrix}\sim\begin{bmatrix}2&5&2\\0&-1&6\\0&0&23\end{bmatrix}$, the matrix

$\begin{bmatrix}-2&-8&-7\\2&5&2\\3&2&6\end{bmatrix}$ has a pivot position on each column. And thus $\{\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3}\}$ is a basis of $\mathbb{R}^3$.

2. Since $\mathfrak{B}$=$\{\overrightarrow{v_1},\overrightarrow{v_2},\overrightarrow{v_3}\}$, the change-of coordinates matrix

$$\mathrm{P}_{\mathcal{E}\leftarrow\mathfrak{B}}=[[\overrightarrow{v_1}]_{\mathcal{E}}\quad[\overrightarrow{v_2},]_{\mathcal{E}}\quad[\overrightarrow{v_3}]_{\mathcal{E}}]=[\overrightarrow{v_1}\quad\overrightarrow{v_2}\quad\overrightarrow{v_3}]=\begin{bmatrix}-2&-8&-7\\2&5&2\\3&2&6\end{bmatrix}.$$

3. Observe that $\mathrm{P}_{\mathcal{E}\leftarrow\mathcal{D}}=[[\overrightarrow{u_1}]_{\mathcal{E}}\quad[\overrightarrow{u_2},]_{\mathcal{E}}\quad[\overrightarrow{u_3}]_{\mathcal{E}}]=[\overrightarrow{u_1}\quad\overrightarrow{u_2}\quad\overrightarrow{u_3}]$. In order to determine $\{\overrightarrow{u_1},\overrightarrow{u_2},\overrightarrow{u_3}\}$, we only need to find $\mathrm{P}_{\mathcal{E}\leftarrow\mathcal{D}}$.

$$\mathrm{P}_{\mathcal{E}\leftarrow\mathcal{D}}=\mathrm{P}_{\mathcal{E}\leftarrow\mathfrak{B}}\mathrm{P}_{\mathfrak{B}\leftarrow\mathcal{D}}=\begin{bmatrix}-2&-8&-7\\2&5&2\\3&2&6\end{bmatrix}P=\begin{bmatrix}-2&-8&-7\\2&5&2\\3&2&6\end{bmatrix}\begin{bmatrix}1&2&-1\\-3&-5&0\\4&6&1\end{bmatrix}$$

$$=\begin{bmatrix}-6&-6&-5\\-5&-9&0\\21&32&3\end{bmatrix}$$

IV. Solution:

1. $$\det(A-\lambda I)=\begin{vmatrix}-\lambda&-4&-6\\-1&-\lambda&-3\\1&2&5-\lambda\end{vmatrix}=-\begin{vmatrix}1&2&5-\lambda\\-1&-\lambda&-3\\-\lambda&-4&-6\end{vmatrix}$$

$$=-\begin{vmatrix}1&2&5-\lambda\\0&2-\lambda&2-\lambda\\0&2(\lambda-2)&-(\lambda-2)(\lambda-3)\end{vmatrix}=(\lambda-2)^2\begin{vmatrix}1&2&5-\lambda\\0&1&1\\0&2&3-\lambda\end{vmatrix}$$

$$=(\lambda-2)^2\begin{vmatrix}1&2&5-\lambda\\0&1&1\\0&0&1-\lambda\end{vmatrix}=(\lambda-2)^2(1-\lambda).$$

So the eigenvalues of A are 1 and 2.

2. $$A-I=\begin{bmatrix}-1&-4&-6\\-1&-1&-3\\1&2&4\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&1&1\\0&-2&-2\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&1&1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\end{bmatrix}.$$

Each solution of $(A-I)\vec{x}=\vec{0}$ has the form $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}-2\\-1\\1\end{bmatrix}.$

So the Eigenspace corresponding to $\lambda = 1$ has a set of basis $\left\{\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}\right\}$.

$$A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Each solution of $(A - 2I)\vec{x} = \vec{0}$ has the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

So the Eigenspace corresponding to $\lambda = 2$ has a set of basis $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The matrix A has 3 linearly independent eigenvectors, and thus it is diagonalizable.

Let $P = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $A = PDP^{-1}$.

V. Solution:

1. Observe that $A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$, $A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda_2^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A^2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = A^2\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 18 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 26 \\ 22 \end{bmatrix}.$$

2. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$

VI. Solution:

Since $\vec{u_1}$ and $\vec{u_2}$ are orthogonal, $\vec{u_1} \cdot \vec{u_2} = 0$.

Given that $\vec{u_1} = \vec{u_2} + a\vec{u_3}$, $(\vec{u_2} + a\vec{u_3}) \cdot \vec{u_2} = \vec{u_2} \cdot \vec{u_2} + a\vec{u_3} \cdot \vec{u_2} = 0$.

Observe that $\vec{u_2} \cdot \vec{u_2} = \|\vec{u_2}\|^2 = 4^2 = 16$, $\vec{u_3} \cdot \vec{u_2} = \vec{u_2}^T \vec{u_3} = 7$.

$a = -16/7$.