Seat No.________ Major/Class________ School________ Student ID | | | | | | | | | | | | Name________

(DON NOT WRITE YOUR ANSWER IN THIS AREA)

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WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Midterm Exam

***Linear algebra and analytic geometry* 2023-2024 1st term Midterm Exam Paper A**

Notice:
1. **Make sure that you have filled the form on the left side of the seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 100 minutes.**

Question No.	**I**	**II**	**III**	**IV**	**V**	**VI**	**Sum**
Score							

Score

I. (20 points) Find the values of k for which the system of equations

$$\begin{cases} x+ky=1 \\ kx+y=1 \end{cases}$$

has

1. no solution.
2. exactly one solution.
3. infinitely many solutions.
4. When there is exactly one solution, what is it?

1. $k=-1$
2. $k\neq -1,1$
3. $k=1$
4. $\frac{1}{k+1}, \frac{1}{k+1}$.

II. (15 points)

1. Write down the formula for $(I-A)^{-1}$ if $A^k = 0$.

2. If B is nilpotent, that is, $B^k = 0$ for some k, show that B cannot be invertible.

1.

$$(1-A)^{-1} = 1 + A + A^2 + A^3 + \ldots. + A^{(k-1)}$$

Score

2. We start by assuming that B is invertible and obtain a statement that cannot possibly be true, and so our assumption has to be wrong . Suppose that B is nilpotent AND B is invertible. That is, $B^k = 0$ for some integer k_1 AND B^{-1} exists. Multiply both sides of $B^k = 0$ by B^{-1} to get:

$$\begin{aligned} B^{-1} \cdot B^k &= B^{-1} \cdot 0 \\ B^{-1} B B^{(k-1)} &= 0 \qquad \Big(\text{since } B^k = B^1 B^{(k-1)} \end{aligned}$$

That is, $\mathrm{B}^{(\mathrm{k}-1)} = 0$ Now multiply both sides by B^{-1} again to get:

$$\mathrm{B}^{(\mathrm{k}-2)} = 0$$

Repeating this, in total k times, gives

$$\mathrm{B}^{-1}\,\mathrm{B} = 0$$

That is, $1 = 0$ which is clearly a contradiction so our assumption that B is invertible is false. Hence if B is nilpotent then B cannot be invertible.

III. (15 points) Let M be the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a formula for M^k for any positive integer power k. Try some simple examples like $k = 2, 3$ if confused. This is a block matrix problem. Notice the that matrix M is really just $M = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, where I and 0 are the 3×3 identity zero matrices, respectively. But

Score

$$M^2 = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 2I \\ 0 & I \end{pmatrix}$$

and

$$M^3 = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 2I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 3I \\ 0 & I \end{pmatrix}$$

so, $M^k = \begin{pmatrix} I & kI \\ 0 & I \end{pmatrix}$, or explicitly

$$M^k = \begin{pmatrix} 1 & 0 & 0 & k & 0 & 0 \\ 0 & 1 & 0 & 0 & k & 0 \\ 0 & 0 & 1 & 0 & 0 & k \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

IV. (10 points) Find the rank of the following matrix

Score

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, we see, the reduced row-echelon form has 3 leading 1's, therefore, the rank is 3 .

V. (20 points) Find an LU decomposition for the matrix

$$\begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & -3 & -4 & 6 \\ 0 & 4 & 7 & -2 \end{pmatrix}$$

Use your result to solve the system

$$\begin{cases} x+y-z+2w=7 \\ x+3y+2z+2w=6 \\ -x-3y-4z+6w=12 \\ 4y+7z-2w=-7 \end{cases}$$

$$LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 2 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Score

To solve $MX = V$ using $M = LU$ we first solve $LW = V$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 1 & 1 & 0 & 0 & 6 \\ -1 & -1 & 1 & 0 & 12 \\ 0 & 2 & -\frac{1}{2} & 1 & -7 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 2 & -\frac{1}{2} & 1 & -7 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 18 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right),$$

from which we can read off W. Now we compute X by solving $UX = W$ with the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & -2 & 8 & 18 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 7 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

So $x = 1, y = 1, z = -1$ and $w = 2$.

VI. (20 points) Solve the following system of equations using Cramer's rule:

$$x+y+z=6$$
$$y+3z=11$$
$$x+z=2y$$

$$D=|A|=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix}=1(1+6)-1(0-3)+1(0-1)=7+3-1=9$$

Score

$D\neq 0$ so the given system of equations has a unique solution. Also,

$$D_x=\begin{vmatrix} 6 & 1 & 1 \\ 11 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix}=6(1+6)-1(11-0)+1(-22-0)=42-11-22=9$$

$$D_y=\begin{vmatrix} 1 & 6 & 1 \\ 0 & 11 & 3 \\ 1 & 0 & 1 \end{vmatrix}=1(11-0)-6(0-3)+1(0-11)=11+18-11=18$$

$$D_z=\begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 11 \\ 1 & -2 & 0 \end{vmatrix}=1(0+22)-1(0-11)+6(0-1)=22+11-6=27$$

Thus,

$$x=D_x/D=9/9=1$$
$$y=D_y/D=18/9=2$$
$$z=D_z/D=27/9=3$$