Supplement

1. Find the area of the graph enclosed by the following curves

(1) $y=\frac{1}{x}\,, y=x\,, x=2;$

Solution: $S=\int_a^b\big(f(x)-g(x)\big)\,dx$

$$=\int_1^2\left(x-\frac{1}{x}\right)dx$$

$$=\left(\frac{1}{2}x^2-\ln x\right)\Big|_1^2$$

$$=(2-\ln 2)-\frac{1}{2}$$

$$=\frac{3}{2}-\ln 2$$

(2) $y^2=4(x+1), y^2=4(1-x);$

Solution: $S=2\int_c^d\big(x_1(y)-x_2(y)\big)\,dy$

$$=2\int_0^2\left[\left(1-\frac{y^2}{4}\right)-\left(\frac{y^2}{4}-1\right)\right]dy$$

$$=2\cdot\int_0^2\left(2-\frac{y^2}{2}\right)dy$$

$$=2\cdot\left(2y-\frac{1}{6}y^3\right)\Big|_0^2$$

$$=\frac{16}{3}$$

(3) $y=x\,, y=x+\sin^2 x\,, x=0, x=\pi;$

Solution: $S=\int_a^b\big(f(x)-g(x)\big)\,dx$

$$=\int_0^\pi(x+\sin^2 x-x)\,dx$$

$$=\int_0^\pi\sin^2 x\,dx$$

$$=2\int_0^{\frac{\pi}{2}}\sin^2 x\,dx$$

$$=2\cdot\frac{1!!}{2!!}\cdot\frac{\pi}{2}$$

$$=\frac{\pi}{2}$$

(4) $y=e^x, y=e^{-x}, x=1;$

Solution: $S=\int_a^b\big(f(x)-g(x)\big)\,dx$

$$=\int_0^1(e^x-e^{-x})\,dx$$

$= (e^x + e^{-x})|_0^1$
$= e + e^{-1} - 2$

(5) $y = |\ln x|, y = 0, x = 0.1, x = 10;$

Solution: $S = \int_a^b f(x)\,dx$

$= \int_{0.1}^{10} |\ln x|\,dx$

$= -\int_{0.1}^{1} \ln x\,dx + \int_{1}^{10} \ln x\,dx$

$= -(x\ln x - x)|_{0.1}^{1} + (x\ln x - x)|_1^{10}$
$= -(-1 - 0.1\ln 0.1 + 0.1) + (10\ln 10 - 10 + 1)$
$= 9.9\ln 10 - 8.1$

(6) $\begin{cases} x = 2t - t^2, \\ y = 2t^2 - t^3, \end{cases} 0 \le t \le 2;$

Solution: $S = \left|\int_{T_1}^{T_2} y(t)x'(t)\,dt\right|$

$= \left|\int_0^2 (2t^2 - t^3)(2 - 2t)\,dt\right|$

$= 2\cdot\left|\int_0^2 (t^4 - 3t^3 + 2t^2)\,dt\right|$

$= 2\cdot\left|\left(\frac{1}{5}t^5 - \frac{3}{4}t^4 + \frac{2}{3}t^3\right)|_0^2\right|$

$= 2\cdot\left|\frac{32}{5} - 12 + \frac{16}{3}\right|$

$= \frac{8}{15}$

(7) $\begin{cases} x = a\cos^3 t\,, \\ y = a\sin^3 t\,, \end{cases} 0 \le t \le 2\pi$

Solution: $S = \int_{T_1}^{T_2} |y(t)x'(t)|\,dt$

$= \int_0^{2\pi} |a\sin^3 t \cdot 3a\cos^2 t\,(-\sin t)|\,dt$

$= 3a^2 \int_0^{2\pi} \sin^4 t\cos^2 t\,dt$

$= 6a^2 \int_0^{\pi} \sin^4 t\cos^2 t\,dt$

$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$

$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t\,(1 - \sin^2 t)\,dt$

$= 12a^2\left(\frac{3!!}{4!!}\cdot\frac{\pi}{2} - \frac{5!!}{6!!}\cdot\frac{\pi}{2}\right)$

$= \frac{3}{8}\pi a^2$

(8) $r = a\theta, \theta = 0, \theta = 2\pi$

Solution: $S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)\, d\theta$

$$= \frac{1}{2}\int_{0}^{2\pi} a^2\theta^2\, d\theta$$

$$= \frac{1}{6}a^2 \cdot (\theta^3)|_0^{2\pi}$$

$$= \frac{4}{3}a^2\pi^3$$

(9) $r = ae^{\theta}, \theta = 0, \theta = 2\pi$

Solution: $S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)\, d\theta$

$$= \frac{1}{2}\int_{0}^{2\pi} a^2e^{2\theta}\, d\theta$$

$$= \frac{1}{4}a^2 \cdot \left(e^{2\theta}\right)|_0^{2\pi}$$

$$= \frac{a^2}{4}(e^{4\theta} - 1)$$

(10) $r = a\cos\theta + b \ \ (b \geq a > 0)$

Solution: $S = \frac{1}{2}\int_{\alpha}^{\beta} r^2(\theta)\, d\theta$

$$= \frac{1}{2}\int_{0}^{2\pi}(a\cos\theta + b)^2\, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi}(a^2\cos^2\theta + 2ab\cos\theta + b^2)\, d\theta$$

$$= \frac{1}{2}\int_{0}^{2\pi}\frac{a^2}{2}(1 + 2\cos 2\theta)\, d\theta + ab(\sin\theta)\,|_0^{2\pi} + \int_{0}^{2\pi}\frac{b^2}{2}d\theta$$

$$= \frac{a^2}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right)|_0^{2\pi} + 0 + b^2\pi$$

$$= \left(\frac{a^2}{2} + b^2\right)\pi$$

2. Find the volume of the rotating body enclosed by one rotation of the following curves about the specified axis:

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, About\ the\ x\ axis$

Solution: $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$$V = \pi\int_{a}^{b} y^2(x)\, dx$$

$$= \pi\int_{-a}^{a} b^2\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\pi b^2 \cdot \int_{0}^{a}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\pi b^2 \cdot \left(x - \frac{1}{3a^2}x^3\right)|_0^a$$

$$=\frac{4}{3}\pi ab^2$$

(2) $y = sin\,x\,, y = 0{,}0 \le x \le \pi,$

1) $About\ the\ x\ axis$

2) $About\ the\ y\ axis$

Solution: 1) $V = \pi \int_a^b y^2(x)\,dx$

$$= \pi \int_0^{\pi} \sin^2 x\,dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= 2\pi \cdot \frac{1!!}{2!!} \cdot \frac{\pi}{2}$$

$$= \frac{1}{2}\pi^2$$

2) $V = 2\pi \int_a^b xf(x)\,dx$

$$= 2\pi \int_0^{\pi} x \sin x\,dx$$

$$= 2\pi(-x\cos x)|_0^{\pi} + 2\pi \int_0^{\pi} \cos x\,dx$$

$$= 2\pi^2 + 2\pi(\sin x)|_0^{\pi}$$
$$= 2\pi^2$$

(3) $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t\,, \end{cases} 0 \le t \le \pi, About\ the\ x\ axis$

Solution: $V = \pi \int_{T_1}^{T_2} y^2(t)|x'(t)|\,dt$

$$= \pi \int_0^{\pi} a^2 \sin^6 t\,|3a\cos^2 t\,(-\sin t)|\,dt$$

$$= 3a^3\pi \int_0^{\pi} \sin^7 t \cos^2 t\,dt$$

$$= 6a^3\pi \int_0^{\frac{\pi}{2}} \sin^7 t\,(1 - \sin^2 t)\,dt$$

$$= 6a^3\pi \left(\frac{6!!}{7!!} - \frac{8!!}{9!!}\right)$$

$$= \frac{32}{105}\pi a^3$$

(4) $r = a(1 - \cos\theta),\ About\ the\ polar\ axis$

Solution: $V = \frac{2\pi}{3}\int_{\alpha}^{\beta} r^3(\theta)\sin\theta\,d\theta$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1-\cos\theta)^3 \sin\theta\,d\theta$$

$$= \frac{2\pi}{3}a^3 \int_0^{\pi} -(1-\cos\theta)^3\,d(\cos\theta) \quad (t = \cos\theta)$$

$$= \frac{2\pi}{3} a^3 \int_{-1}^{1} (1-t)^3 \, dt$$

$$= \frac{2\pi}{3} a^3 \cdot \left[-\frac{1}{4} (1-t)^4 \right] |_{-1}^{1}$$

$$= \frac{8}{3} \pi a^3$$

(5) $r = ae^{\theta}, 0 \le \theta \le \pi, About\ the\ polar\ axis$

Solution: $V = \frac{2\pi}{3} \int_{\alpha}^{\beta} r^3(\theta) \sin\theta \, d\theta$

$$= \frac{2\pi}{3} \int_0^{\pi} a^3 e^{3\theta} \sin\theta \, d\theta$$

$$= \frac{2\pi}{3} a^3 \cdot \int_0^{\pi} e^{3\theta} \sin\theta \, d\theta$$

$$\int_0^{\pi} e^{3\theta} \sin\theta \, d\theta$$

$$= \left(-e^{3\theta} \cos\theta \right) |_0^{\pi} + 3 \int_0^{\pi} e^{3\theta} \cos\theta \, d\theta$$

$$= (e^{3\pi} + 1) + 3\left(e^{3\theta} \sin\theta \right) |_0^{\pi} - 9 \int_0^{\pi} e^{3\theta} \sin\theta \, d\theta$$

$$= (e^{3\pi} + 1) - 9 \int_0^{\pi} e^{3\theta} \sin\theta \, d\theta$$

$$\Rightarrow \int_0^{\pi} e^{3\theta} \sin\theta \, d\theta = \frac{1}{10} (e^{3\pi} + 1)$$

$$\Rightarrow V = \frac{2\pi}{3} a^3 \cdot \frac{1}{10} (e^{3\pi} + 1) = \frac{1}{15} \pi a^3 (e^{3\pi} + 1)$$