Answers to Calculus Review-integral

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(1) Find the integral $\int \frac{e^x}{4+9e^{2x}}dx$

$$\int \frac{e^x}{4+9e^{2x}}dx$$

$$= \int \frac{1}{2^2+(3e^x)^2}\,d(e^x) \qquad u=3e^x$$

$$= \int \frac{1}{2^2+u^2}\,du$$

$$= \frac{1}{2}\arctan\frac{u}{2}+C$$

$$= \frac{1}{2}\arctan\frac{3e^x}{2}+C$$

(2) Find the integral $\int_1^{16} \arctan\sqrt{\sqrt{x}-1}\,dx$

$$\int_1^{16} \arctan\sqrt{\sqrt{x}-1}\,dx$$

$$= \int_0^{\sqrt{3}} \arctan u \cdot d\left((u^2+1)^2\right) \qquad u=\sqrt{\sqrt{x}-1},\quad x=(u^2+1)^2$$

$$= \left[\arctan u\cdot(u^2+1)^2\right]\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} (u^2+1)^2\cdot\frac{1}{1+u^2}\,du$$

$$= \frac{\pi}{3}\cdot 16 - \int_0^{\sqrt{3}} (u^2+1)\,du$$

$$= \frac{16\pi}{3} - \frac{1}{3}(3\sqrt{3}-0) - (\sqrt{3}-0)$$

$$= \frac{16}{3}\pi - 2\sqrt{3}.$$

(3) Find the integral $I=\int_0^{\infty} e^{-3x}\sin x\,dx$;

Sol. $I = -\frac{1}{3}\int_0^{\infty} \sin x\, d(e^{-3x})$

$$= -\frac{1}{3}\left[\underbrace{(\sin x\cdot e^{-3x})\Big|_0^{\infty}} - \int_0^{\infty} e^{-3x}\cos x\,dx\right] \rightarrow \lim_{x\to\infty}\sin x\cdot e^{-3x} = \lim_{x\to\infty}\frac{\sin x}{e^{3x}} = 0$$

$$= -\frac{1}{3}\left(0-0-\int_0^{\infty} e^{-3x}\cos x\,dx\right)$$

$$= \frac{1}{3}\int_0^{\infty} e^{-3x}\cos x\,dx$$

$$= \frac{1}{3}\cdot\left(-\frac{1}{3}\right)\int_0^{\infty}\cos x\,d(e^{-3x})$$

$$= \frac{1}{3}\cdot\left(-\frac{1}{3}\right)\int_0^{\infty} \cos x \, d\left(e^{-3x}\right)$$

$$= -\frac{1}{9}\left[\left(\cos x \cdot e^{-3x}\right)\Big|_0^{\infty} - \int_0^{\infty} e^{-3x}(-\sin x)\,dx\right] \qquad \lim_{x\to\infty} \cos x \cdot e^{-3x} = 0.$$

$$= -\frac{1}{9}\left(0 - 1 + \int_0^{\infty} e^{-3x} \sin x \, dx\right)$$

$$= \frac{1}{9} - \frac{1}{9} I$$

i.e. $I = \frac{1}{9} - \frac{1}{9} I \Rightarrow I = \frac{1}{10}$. $\qquad \frac{10}{9} I = \frac{1}{9}$

(4) $f(x)$ is continuous, try to prove that $\int_1^a f\left(x^2 + \frac{a^2}{x^2}\right)\frac{dx}{x} = \int_1^a f\left(x + \frac{a^2}{x}\right)\frac{dx}{x}$.

Proof. Let $t = x^2$, $x = \sqrt{t}$, $dx = \frac{dt}{2\sqrt{t}}$

$$\text{Left side} = \int_1^{a^2} f\left(t + \frac{a^2}{t}\right)\frac{1}{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_1^{a^2} f\left(t + \frac{a^2}{t}\right)\frac{dt}{t}$$

$$= \frac{1}{2}\int_1^{a} f\left(t + \frac{a^2}{t}\right)\frac{dt}{t} + \frac{1}{2}\int_a^{a^2} f\left(t + \frac{a^2}{t}\right)dt$$

$$= \frac{1}{2}\int_1^{a} f\left(t + \frac{a^2}{t}\right)\frac{dt}{t} + \frac{1}{2}\int_a^{1} f\left(\frac{a^2}{s} + s\right)\frac{s}{a^2}\cdot\left(-\frac{a^2}{s^2}ds\right) \qquad \left(s = \frac{a^2}{t},\ t = \frac{a^2}{s}\right)$$

$$= \frac{1}{2}\int_1^{a} f\left(t + \frac{a^2}{t}\right)\frac{dt}{t} + \frac{1}{2}\int_1^{a} f\left(s + \frac{a^2}{s}\right)\frac{ds}{s}$$

$$= \frac{1}{2}\int_1^{a} f\left(x + \frac{a^2}{x}\right)\frac{dx}{x} + \frac{1}{2}\int_1^{a} f\left(x + \frac{a^2}{x}\right)\frac{dx}{x} = \text{Right side}.$$

(5) Find the integral $\int \frac{\arctan x}{x^2(1+x^2)}dx$

$$= \int \arctan x \cdot \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)dx$$

$$= \int \arctan x \; d\left(-\frac{1}{x} - \arctan x\right) \qquad \frac{1}{x(1+x^2)} = \frac{1+x^2-x^2}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

$$= -\int \arctan x \; d\left(\frac{1}{x} + \arctan x\right)$$

$$= -\left[\arctan x \cdot \left(\frac{1}{x} + \arctan x\right) - \int\left(\frac{1}{x} + \arctan x\right)d(\arctan x)\right]$$

$$= -\frac{1}{x}\arctan x - (\arctan x)^2 + \int \frac{1}{x}\,d(\arctan x) + \int \arctan x \; d(\arctan x)$$

$$= -\frac{1}{x}\arctan x - (\arctan x)^2 + \int \frac{1}{x}\cdot\frac{1}{1+x^2}\,dx + \frac{1}{2}(\arctan x)^2$$

$$= -\frac{1}{x}\arctan x - \frac{1}{2}(\arctan x)^2 + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$

$$= -\frac{1}{x}\arctan x - \frac{1}{2}(\arctan x)^2 + \ln|x| - \frac{1}{2}\ln(1+x^2) + C.$$

(6) Find the integral $\int_a^{2a} \frac{\sqrt{x^2 - a^2}}{x^4}dx, (a > 0)$

(6) Find the integral $\int_a^{2a} \frac{\sqrt{x^2-a^2}}{x^4}\,dx, (a>0)$

$$\int_{\underline{a}}^{2a} f(x)\,d\underline{x} \Rightarrow a \le x \le 2a$$

$x = a\sec t.\quad t\in(0,\frac{\pi}{2})\ ,\ dx = a\sec t\tan t$

$$= \int_0^{\frac{\pi}{6}} \frac{\cancel{a}\tan t}{a^4\sec^4 t}\cdot \cancel{a}\sec t\tan t\,dt$$

$$=\frac{1}{a^2}\int_0^{\frac{\pi}{6}} \frac{\tan^2 t}{\sec^3 t}\,dt$$

$$=\frac{1}{a^2}\int_0^{\frac{\pi}{6}} \frac{\sin^2 t}{\cos^2 t}\cdot\cos^3 t\,dt \qquad \int \sin^m t\cos^n t\,dt$$

$$=\frac{1}{a^2}\int_0^{\frac{\pi}{6}} \underline{\sin^2 t}\cos t\,dt$$

$$=\frac{1}{a^2}\int_0^{\frac{\pi}{6}} \sin^2 t\,d(\sin t)$$

$$=\frac{1}{a^2}\cdot\frac{1}{3}\sin^3 t\Big|_0^{\frac{\pi}{6}}$$

$$=\frac{1}{a^2}\cdot\frac{1}{3}\cdot\left(\left(\frac{1}{2}\right)^3-0^3\right)$$

$$=\frac{1}{24a^2}.$$

(7) Find the integral $\int\sqrt{e^x+1}\,dx$

$t=\sqrt{e^x+1},\quad x=\ln(t^2-1).$

$dx=\frac{2t}{t^2-1}dt$

$$\int\sqrt{e^x+1}\,dx=\int t\cdot\frac{2t}{t^2-1}\,dt$$

$$=2\int\frac{t^2-1+1}{t^2-1}\,dt$$

$$=2\int\left(1+\frac{1}{t^2-1}\right)dt$$

$$=2\left[t+\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\right]$$

$$=2t+\ln\left|\frac{t-1}{t+1}\right|+C$$

$$=2\sqrt{e^x+1}+\ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|+C$$

$$=2\sqrt{e^x+1}+\ln\left|\frac{e^x+2-2\sqrt{e^x+1}}{e^x}\right|+C$$

$$=2\sqrt{e^x+1}+\ln\left|e^x+2-2\sqrt{e^x+1}\right|-x+C.$$

(8) Find the integral $\int \frac{x^2}{(1+x^2)^{\frac{3}{2}}} \mathrm{d}x$

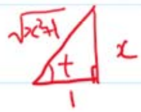
$x = \tan t$
$dx = \sec^2 t \cdot dt$

$= \int \frac{\tan^2 t}{\sec^3 t} \cdot \sec^2 t \, dt$

$= \int \tan^2 t \cdot \cos t \, dt \quad \longrightarrow \sin^m x \cdot \cos^n x$

$= \int \frac{\sin^2 t}{\cos t} dt$

$= \int \frac{\sin^2 t \cdot \cos t}{\cos^2 t} dt$



$= \int \frac{\sin^2 t}{1-\sin^2 t} d(\sin t)$

$= \int \frac{u^2}{1-u^2} du \qquad u = \sin t = \frac{x}{\sqrt{x^2+1}}$

$= \int \frac{u^2-1+1}{1-u^2} du$

$= -u + \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$

$= -u + \frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| + C$

$= -\frac{x}{\sqrt{x^2+1}} + \frac{1}{2} \ln \left| \frac{\frac{x}{\sqrt{x^2+1}} + 1}{\frac{x}{\sqrt{x^2+1}} - 1} \right| + C$

$= -\frac{x}{\sqrt{x^2+1}} + \frac{1}{2} \ln \left| \frac{x+\sqrt{x^2+1}}{x-\sqrt{x^2+1}} \right| + C$

$= -\frac{x}{\sqrt{x^2+1}} + \ln |x+\sqrt{x^2+1}| + C.$

(9) Find the integral $\int (\arcsin x)^2 \mathrm{d}x$

$= x(\arcsin x)^2 - \int x \cdot 2 \arcsin x \cdot \frac{1}{\sqrt{1-x^2}} dx$

$= x(\arcsin x)^2 + 2\int \arcsin x \, d\sqrt{1-x^2}$

$= x(\arcsin x)^2 + 2\left[(\arcsin x) \cdot \sqrt{1-x^2} - \int \sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} dx\right]$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2} \arcsin x - 2x + C.$

(10) Find the integral $\int \frac{1}{(x+1)^2 \sqrt{x^2+2x+2}} \mathrm{d}x$

$= \int \frac{1}{(x+1)^2 \sqrt{(x+1)^2+1}} d(x+1) \quad u = x+1$

$= \int \frac{1}{\quad} du \longrightarrow u^{\text{even}}: f(u^2 \pm a^2)$

$$= \int \frac{1}{(x+1)^2\sqrt{(x+1)^2+1}}\, d(x+1)$$

$$= \int \frac{1}{u^2\sqrt{u^2+1}}\, du \longrightarrow u^{\text{even}}\ f(u^2 \pm a^2)$$

$$= \int \frac{1}{\tan^2 t \cdot \sec t} \sec^2 t\, dt \quad \left(\begin{matrix} u = \tan t \\ du = \sec^2 t\, dt \end{matrix}\right)$$

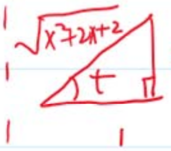
$$= \int \frac{\cos t}{\sin^2 t}\, dt$$

$$= \int \frac{1}{\sin^2 t}\, d(\sin t)$$

$$= -\frac{1}{\sin t} + C$$

$$= -\frac{\sqrt{x^2+2x+2}}{x+1} + C.$$

$x+1 = u = \tan t$



$\sin t = \frac{x+1}{\sqrt{x^2+2x+2}}$

(11) If $f(x)$ is continue and 2 second differentiable on $[0,\ 2]$ and

$$\lim_{x\to 0^-} \frac{\ln(1+\frac{f(x)}{x})}{\sin x} = 3, \quad \int_1^2 f(x)dx = 0$$

Find $f'(0)$ and there exist $\xi \in (0,\ 2)$, $f'(\xi) + f''(\xi) = 0$

Sol. $\lim\limits_{x\to 0^+} \ln(1+\frac{f(x)}{x}) = \lim\limits_{x\to 0^+} \frac{\ln(1+\frac{f(x)}{x})}{\sin x} \cdot \sin x = 3\times 0 = 0.$

$\therefore \lim\limits_{x\to 0^+} \frac{f(x)}{x} = 0.$

$\because f(0) = \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{f(x)}{x}\cdot x = \left(\lim\limits_{x\to 0^+}\frac{f(x)}{x}\right)\cdot\left(\lim\limits_{x\to 0^+} x\right) = 0\times 0 = 0$

$\therefore f'(0) = \lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+}\frac{f(x)}{x} = 0.$

by the integral intermediate value theorm,

$\exists\ \xi_1 \in [1,2]$, s.t.

$0 = \int_1^2 f(x)dx = f(\xi_1)(2-1) = f(\xi_1)$,

Let $g(x) = e^x f(x)$, $x \in [0, \xi_1]$

$\because g(x) \in C[0, \xi_1]$

$g(0) = e^0 f(0) = f(0) = 0$, $g(\xi_1) = e^{\xi_1} f(\xi_1) = 0$

$\therefore \exists\ \xi_2 \in (0, \xi_1) \subset (0,2)$, s.t.

$g'(\xi_2) = 0$,

i.e. $e^{\xi_2} f(\xi_2) + e^{\xi_2} f'(\xi_2) = 0$

$\therefore \quad f(\xi_2)+f'(\xi_2)=0$

Let $\quad h(x)=f(x)+f'(x),\ x\in[0,\xi_2]$,

then $\quad h(x)\in C[0,\xi_2]$ and

$h(0)=f(0)+f'(0)=0,\quad h(\xi_2)=f(\xi_2)+f'(\xi_2)=0$

by Rolle theorem, $\exists\,\xi\in(0,\xi_2)\subset(0,2)$, s.t.

$h'(\xi)=0$,

i.e. $\quad f'(\xi)+f''(\xi)=0$.

If $f(x)$ is continue on $[0,\ 1]$, show $\int_0^1\left[\int_{x^2}^{\sqrt{x}} f(t)dt\right]dx=\int_0^1[\sqrt{x}-x^2]f(x)dx$.

Proof.

Let $\varphi(x)=\int_{x^2}^{\sqrt{x}} f(t)dt$, then $\varphi'(x)=f(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}-f(x^2)\cdot 2x$

$$\begin{aligned}
L&=\int_0^1 \varphi(x)\,dx\\
&=x\varphi(x)\Big|_0^1-\int_0^1 x\cdot\varphi'(x)dx\\
&=(1\cdot\underbrace{\varphi(1)}_{=0}-0\cdot\varphi(0))-\int_0^1 x\cdot\left[f(\sqrt{x})\cdot\tfrac{1}{2\sqrt{x}}-f(x^2)\cdot 2x\right]dx\\
&=\underbrace{\int_0^1 f(x^2)\cdot 2x^2\,dx}_{I_1}-\underbrace{\int_0^1 \tfrac{1}{2}\sqrt{x}\,f(\sqrt{x})\,dx}_{I_2}\\
&=I_1-I_2,
\end{aligned}$$

$I_1=\int_0^1 f(x^2)\cdot 2x^2\,dx \qquad t=x^2 \quad x=\sqrt{t}$

$\quad=\int_0^1 f(t)\cdot 2t\cdot\frac{1}{2\sqrt{t}}\,dt \qquad dx=\frac{1}{2\sqrt{t}}dt$

$I_1=\int_0^1 \sqrt{t}f(t)\,dt=\int_0^1\sqrt{x}f(x)\,dx$

$I_2=\int_0^1\frac{1}{2}\sqrt{x}\,f(\sqrt{x})\,dx \qquad s=\sqrt{x},\ x=s^2,\ dx=2s\,ds$

$\quad=\int_0^1 \frac{1}{2}s\cdot f(s)\cdot 2s\,ds$

$\quad=\int_0^1 s^2 f(s)\,ds$

$\quad=\int_0^1 x^2 f(x)\,dx$

$L=I_1-I_2=\int_0^1\sqrt{x}f(x)\,dx-\int_0^1 x^2f(x)\,dx$

$\quad=\int_0^1(\sqrt{x}-x^2)\,f(x)\,dx.$

Find the integral $\int \frac{x\cos x}{\sin^3 x}dx$

Note $\frac{\cos x}{\sin^3 x}dx = \frac{\cos x}{\sin x}\cdot \csc^2 x\,dx = -\cot x\,d(\cot x)$

$= -\frac{1}{2}d(\cot^2 x)$

or $\frac{\cos x}{\sin^3 x}dx = \frac{1}{\sin^3 x}d(\sin x) \xlongequal{u=\sin x} u^{-3}du = -\frac{1}{2}d(u^{-2})$

$= -\frac{1}{2}d(\csc^2 x)$

$\therefore \int \frac{x\cos x}{\sin^3 x}dx = -\frac{1}{2}\int x\,d(\cot^2 x)$

$= -\frac{1}{2}\left[x\cot^2 x - \int \cot^2 x\,dx\right]$

$= -\frac{1}{2}x\cot^2 x + \frac{1}{2}\int (\csc^2 x - 1)\,dx$

$= -\frac{1}{2}x\cot^2 x - \frac{1}{2}\cot x - \frac{1}{2}x + C$.

(16) If $f(x)$ is differentiable on $[0,\ 1]$ and $2\int_0^{\frac{1}{2}} x^2 f(x)dx = 1$, $f(1)=1$.

there exist $\xi \in (0,\ 1)$, $f(\xi) = -\frac{\xi}{2}f'(\xi)$

Analysis: $f(\xi) = -\frac{\xi}{2}f'(\xi) \Leftrightarrow 2f(\xi) + \xi f'(\xi) = 0$

$\Leftrightarrow 2\xi f(\xi) + \xi^2 f'(\xi) = 0 \quad (\xi \in (0,1))$

$\Leftrightarrow (x^2 f(x))'_{x=\xi} = 0$.

Consider $\varphi(x) = x^2 f(x)$, $x \in [a, b]$

Note $f(1) = 1 \Rightarrow \varphi(1) = 1^2 \cdot f(1) = 1$.

$2\int_0^{\frac{1}{2}} x^2 f(x)dx = 1 \Rightarrow 2\cdot c^2 f(c)\cdot(\frac{1}{2} - 0) = 1$, $c \in [0, \frac{1}{2}]$

Integral intermediate value theorem $\Rightarrow c^2 f(c) = 1$, i.e $\varphi(c) = 1 = \varphi(1)$

So let $[a,b] = [c, 1]$,

Proof. By integral intermediate value theorem. $\exists c \in [0, \frac{1}{2}]$

Proof. By integral intermediate value theorem, $\exists c \in [0, \frac{1}{2}]$.

s.t. $1 = 2\int_0^{\frac{1}{2}} x^2 f(x)\,dx = 2\, c^2 f(c) \cdot (\frac{1}{2} - 0) = c^2 f(c)$.

Let $\varphi(x) = x^2 f(x)$, $x \in [c, 1] \subset [0, 1]$

then $\varphi(x)$ is continuous on $[c, 1]$,

differentiable in $(c, 1)$

and $\varphi(c) = 1 = f(1) = \varphi(1)$

by Rolle theorem, $\exists \xi \in (c, 1) \subset (0, 1)$, s.t.

$$\varphi'(\xi) = 0$$

i.e. $(x^2 f(x))'_{x=\xi} = 2\xi f(\xi) + \xi^2 f'(\xi) = 0$

$$\Rightarrow \quad f(\xi) = -\frac{\xi}{2} f'(\xi) \quad (\because \xi \in (0, 1),\ \xi \neq 0).$$