Integral

1. Find the following definite integral

(1) $\int_0^1 x^2(2-x^2)dx$

Solution: $\int_0^1 x^2(2-x^2)dx = \int_0^1 (2x^2 - x^4)\,dx = \left(\frac{2}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{7}{15}$

(2) $\int_1^2 \frac{(x-1)(x^2-x+1)}{2x^2}dx$

Solution: $\int_1^2 \frac{(x-1)(x^2-x+1)}{2x^2}dx = \int_1^2 \left(\frac{1}{2}x - 1 + \frac{1}{x} - \frac{1}{2x^2}\right)dx$

$$= \left(\frac{1}{4}x^2 - x + \ln|x| + \frac{1}{2x}\right)\Big|_1^2$$

$$= \left(\ln 2 - \frac{3}{4}\right) - \left(-\frac{1}{4}\right)$$

$$= \ln 2 - \frac{1}{2}$$

(3) $\int_0^2 (2^x + 3^x)^2 dx$

Solution: $\int_0^2 (2^x + 3^x)^2 dx = \int_0^2 (4^x + 2\cdot 6^x + 9^x)\,dx$

$$= \left(\frac{4^x}{2\ln 2} + \frac{2\cdot 6^x}{\ln 6} + \frac{9^x}{2\ln 3}\right)\Big|_0^2$$

$$= \frac{15}{2\ln 2} + \frac{70}{\ln 6} + \frac{40}{\ln 3}$$

(4) $\int_0^{\frac{1}{2}} x(1-4x^2)^{10}\,dx$

Solution: $\int_0^{\frac{1}{2}} x(1-4x^2)^{10}\,dx = \int_0^{\frac{1}{2}} -\frac{1}{8}(1-4x^2)^{10}\,d(1-4x^2)$

$$= \frac{1}{8}\int_0^1 u^{10}\,du$$

$$= \frac{1}{8}\cdot\left(\frac{1}{11}u^{11}\right)\Big|_0^1$$

$$= \frac{1}{88}$$

(5) $\int_{-1}^1 \frac{(x+1)dx}{(x^2+2x+5)^2}$

Solution: $\int_{-1}^1 \frac{(x+1)dx}{(x^2+2x+5)^2} = \int_{-1}^1 \frac{\frac{1}{2}d(x^2+2x+5)}{(x^2+2x+5)^2}$ $(u = x^2 + 2x + 5)$

$$= \frac{1}{2}\int_4^8 \frac{du}{u^2}$$

$$= \frac{1}{2}\cdot\left(-\frac{1}{u}\right)\Big|_4^8$$

$$= \frac{1}{16}$$

(6) $\int_0^1 \arcsin x\, dx$

Solution: $\int_0^1 \arcsin x\, dx = (x \arcsin x)|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx = \frac{\pi}{2} - \left(\sqrt{1-x^2}\right)|_0^1 = \frac{\pi}{2} - 1$

(7) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx$

Solution: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x \sec^2 x\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x\, d(\tan x)$

$$= (x \tan x)|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\, dx = (\ln|\cos x|)|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 0$$

(8) $\int_0^{\frac{\pi}{4}} x \tan^2 x\, dx$

Solution: $\int_0^{\frac{\pi}{4}} x \tan^2 x\, dx = \int_0^{\frac{\pi}{4}} x\, (\sec^2 x - 1) dx$

$$= \int_0^{\frac{\pi}{4}} x \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} x\, dx$$

$$= \int_0^{\frac{\pi}{4}} x\, d(\tan x) - \left(\frac{1}{2}x^2\right)\Big|_0^{\frac{\pi}{4}}$$

$$= (x \tan x)|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\, dx - \frac{1}{32}\pi^2$$

$$= \frac{\pi}{4} + (\ln|\cos x|)|_0^{\frac{\pi}{4}} - \frac{1}{32}\pi^2$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2 - \frac{1}{32}\pi^2$$

(9) $\int_0^{\frac{\pi}{2}} e^x \sin^2 x\, dx$

Solution: $\int_0^{\frac{\pi}{2}} e^x \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2} e^x\, (1 - \cos 2x) dx$

$$\int_0^{\frac{\pi}{2}} e^x \cos 2x\, dx$$

$$= (e^x \cos 2x)|_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^x \sin 2x\, dx$$

$$= \left(-e^{\frac{\pi}{2}} - 1\right) + 2(e^x \sin 2x)|_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^x \cos 2x\, dx$$

$$= \left(-e^{\frac{\pi}{2}} - 1\right) - 4\int_0^{\frac{\pi}{2}} e^x \cos 2x\, dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^x \cos 2x\, dx = -\frac{1}{5}\left(e^{\frac{\pi}{2}} + 1\right)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} e^x \sin^2 x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x\, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \cos 2x\, dx$$

$$= \frac{1}{2}(e^x)|_0^{\frac{\pi}{2}} - \frac{1}{2} \cdot -\frac{1}{5}\left(e^{\frac{\pi}{2}} + 1\right)$$

$$= \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{1}{10}\left(e^{\frac{\pi}{2}} + 1\right)$$

$$= \frac{1}{5}(3e^{\frac{\pi}{2}} - 2)$$

(10) $\int_1^e \sin(\ln x)\,dx$

Solution: $\int_1^e \sin(\ln x)\,dx \quad (t = \ln x\,, x = e^t, dx = e^t dt)$

$$= \int_0^1 e^t \sin t\,dt$$

$$= (e^t \sin t)|_0^1 - \int_0^1 e^t \cos t\,dt$$

$$= e \sin 1 - (e^t \cos t)|_0^1 - \int_0^1 e^t \sin t\,dt$$

$$= e(\sin 1 - \cos 1) + 1 - \int_0^1 e^t \sin t\,dt$$

$$\Rightarrow \int_0^1 e^t \sin t\,dt = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$$

$$\Rightarrow \int_1^e \sin(\ln x)\,dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$$

(11) $\int_0^1 x^2 \arctan x\,dx$

Solution: $\int_0^1 x^2 \arctan x\,dx$

$$= \frac{1}{3}\int_0^1 \arctan x\,d(x^3)$$

$$= \frac{1}{3}(x^3 \arctan x)|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\,dx$$

$$= \frac{\pi}{12} - \frac{1}{3}\int_0^1 \left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{\pi}{12} - \frac{1}{3}\left(\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1)\right)|_0^1$$

$$= \frac{\pi}{12} - \frac{1}{6}(1 - \ln 2)$$

(12) $\int_1^{e+1} x^2 \ln(x-1)\,dx$

Solution: $\int x^2 \ln(x-1)\,dx$

$$= \frac{1}{3}\int \ln(x-1)\,d(x^3)$$

$$= \frac{1}{3}x^3 \ln(x-1) - \frac{1}{3}\int \frac{x^3}{x-1}\,dx$$

$$= \frac{1}{3}x^3 \ln(x-1) - \frac{1}{3}\int \left(x^2 + x + 1 + \frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}(x^3 - 1)\ln(x-1) - \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{1}{3}x + C$$

$$\int_1^{e+1} x^2 \ln(x-1)\,dx$$

$$= \left(\frac{1}{3}(x^3-1)\ln(x-1) - \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{1}{3}x\right)\Big|_1^{e+1}$$

$$= \frac{2}{9}e^3 + \frac{1}{2}e^2$$

(13) $\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2}\ dx$

Solution: $\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2}\ dx = \frac{1}{2}\int_0^{\sqrt{\ln 2}} x^2 e^{-x^2}\ d(x^2)\ \ (u = x^2)$

$$= \frac{1}{2}\int_0^{\ln 2} ue^{-u}\ du$$

$$= -\frac{1}{2}(ue^{-u})\Big|_0^{\ln 2} + \frac{1}{2}\int_0^{\ln 2} e^{-u}\ du$$

$$= -\frac{1}{4}\ln 2 - \frac{1}{2}(e^{-u})\Big|_0^{\ln 2}$$

$$= \frac{1}{4}(1 - \ln 2)$$

(14) $\int_0^1 e^{2\sqrt{x+1}}\, dx$

Solution: $\int_0^1 e^{2\sqrt{x+1}}\, dx\ \ \left(t = \sqrt{x+1}, x = t^2 - 1, dx = 2tdt\right)$

$$= \int_1^{\sqrt{2}} 2te^{2t}\, dt$$

$$= (te^{2t})\Big|_1^{\sqrt{2}} - \int_1^{\sqrt{2}} e^{2t}\, dt$$

$$= \sqrt{2}e^{2\sqrt{2}} - e^2 - \frac{1}{2}(e^{2t})\Big|_1^{\sqrt{2}}$$

$$= \sqrt{2}e^{2\sqrt{2}} - e^2 - \frac{1}{2}e^{2\sqrt{2}} + \frac{1}{2}e^2$$

$$= \left(\sqrt{2} - \frac{1}{2}\right)e^{2\sqrt{2}} - \frac{1}{2}e^2$$

(15) $\int_0^1 \frac{dx}{\sqrt{1+e^{2x}}}$

Solution: $\int_0^1 \frac{dx}{\sqrt{1+e^{2x}}} = \int_0^1 \frac{dx}{e^x\sqrt{1+e^{-2x}}}$

$$= \int_0^1 \frac{-d(e^{-x})}{\sqrt{1+(e^{-x})^2}}\quad (u = e^{-x})$$

$$= \int_{e^{-1}}^1 \frac{du}{\sqrt{1+u^2}}\quad (u = \tan t\,, du = \sec^2 t\, dt)$$

$$= \int_{\arctan e^{-1}}^{\frac{\pi}{4}} \frac{\sec^2 t dt}{\sec t}$$

$$= (\ln|\sec t + \tan t|)\Big|_{\arctan e^{-1}}^{\frac{\pi}{4}}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln\left(e^{-1} + \sqrt{1+e^{-2}}\right)$$

$$= \ln\frac{e(\sqrt{2}+1)}{1+\sqrt{e^2+1}}$$

(16) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)^3}}$$

Solution: $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)^3}} \quad (x = \sin t)$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos t dt}{\cos^3 t}$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sec^2 t\, dt$$

$$= (\tan t)|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{2\sqrt{3}}{3}$$

(17) $$\int_0^1 \left(\frac{x-1}{x+1}\right)^4 dx$$

Solution: $$\int_0^1 \left(\frac{x-1}{x+1}\right)^4 dx \quad \left(t = \frac{x-1}{x+1}, x = -\frac{t+1}{t-1}, dx = \frac{2}{(t-1)^2} dt\right)$$

$$= \int_{-1}^{0} t^4 \cdot \frac{2}{(t-1)^2} dt$$

$$= 2\int_{-1}^{0} \left(t^2 + 2t + 3 + \frac{4}{t-1} + \frac{1}{(t-1)^2}\right) dt$$

$$= 2\left(\frac{1}{3}t^3 + t^2 + 3t + 4\ln|t-1| - \frac{1}{t-1}\right)|_{-1}^{0}$$

$$= 2\left[1 - \left(4\ln 2 - \frac{11}{6}\right)\right]$$

$$= \frac{17}{3} - 8\ln 2$$

(18) $$\int_0^1 \frac{x^2+1}{x^4+1} dx$$

Solution: $$\int \frac{x^2+1}{x^4+1} dx = \int \frac{1+x^{-2}}{x^2+x^{-2}} dx$$

$$= \int \frac{d(x-x^{-1})}{(x-x^{-1})^2+2} \quad (u = x - x^{-1})$$

$$= \int \frac{du}{u^2+2}$$

$$= \frac{1}{\sqrt{2}} \arctan \frac{u}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \arctan \frac{x-x^{-1}}{\sqrt{2}} + C$$

$$\int_0^1 \frac{x^2+1}{x^4+1} dx = \left(\frac{1}{\sqrt{2}} \arctan \frac{x-x^{-1}}{\sqrt{2}}\right)|_0^1 = 0 - \frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{2}\pi}{4}$$

(19) $$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{1+x^2}}$$

Solution: $$\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{1+x^2}} \quad (x = \tan t, dx = \sec^2 t\, dt)$$

$$= \int_{\frac{\pi}{4}}^{\arctan\sqrt{2}} \frac{\sec^2 t dt}{\tan t \cdot \sec t}$$

$$= \int_{\frac{\pi}{4}}^{\arctan\sqrt{2}} \csc t\, dt$$

$$= (\ln|\csc t - \cot t|)|_{\frac{\pi}{4}}^{\arctan\sqrt{2}}$$

$$= \ln\left|\sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}}\right| - \ln\left|\sqrt{2} - 1\right|$$

$$= \ln\ |\frac{2+\sqrt{2}}{\sqrt{3}+1}|$$

(20) $\int_0^1 x\sqrt{\frac{x}{2-x}}\, dx$

Solution: $\int_0^1 x\sqrt{\frac{x}{2-x}}\, dx \quad \left(t = \sqrt{\frac{x}{2-x}}, x = \frac{2t^2}{1+t^2}, dx = \frac{4t}{(1+t^2)^2} dt\right)$

$$= \int_0^1 \frac{2t^2}{1+t^2} t \frac{4t}{(1+t^2)^2} dt$$

$$= \int_0^1 \frac{8t^4}{(1+t^2)^3} dt \quad (t = \tan u, dt = \sec^2 u\, du)$$

$$= \int_0^{\frac{\pi}{4}} \frac{8\tan^4 u}{\sec^6 u} \sec^2 u\, du$$

$$= 8 \cdot \int_0^{\frac{\pi}{4}} \sin^4 u\, du$$

$$= 8 \cdot \int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1 - \cos 2u)\right]^2 du$$

$$= 2 \cdot \int_0^{\frac{\pi}{4}} (1 - 2\cos 2u + \cos^2 2u)\, du$$

$$= 2 \cdot \int_0^{\frac{\pi}{4}} \left[1 - 2\cos 2u + \frac{1}{2}(1 + \cos 4u)\right] du$$

$$= \left(3u - 2\sin 2u + \frac{1}{4}\sin 4u\right)|_0^{\frac{\pi}{4}}$$

$$= \frac{3\pi}{4} - 2$$

2. Find the following limits

(1) Find $\lim_{n\to\infty}(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n-1}{n^2})$

Solution: $\lim_{n\to\infty}(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n-1}{n^2})$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i-1}{n} \cdot \frac{1}{n}$$

$$= \int_0^1 x\, dx$$

$$= \left(\frac{1}{2}x^2\right)|_0^1$$

$$= \frac{1}{2}$$

(2) Find $\lim_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}\quad(p>0)$

Solution: $\lim_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p\cdot\frac{1}{n}$

$=\int_0^1 x^p\,dx$

$=\left(\frac{1}{p+1}x^{p+1}\right)|_0^1$

$=\frac{1}{p+1}$

(3) Find $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)$

Solution: $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)$

$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{i-1}{n}\pi\right)\cdot\frac{1}{n}$

$=\int_0^1\sin\pi x\,dx$

$=-\frac{1}{\pi}(\cos\pi x)|_0^1$

$=\frac{2}{\pi}$

3. Find the following limits

(1) Find $\lim_{x\to 0}\frac{\int_0^x\cos t^2dt}{x}$

Solution: $\lim_{x\to 0}\frac{\int_0^x\cos t^2dt}{x}=\lim_{x\to 0}\frac{\cos x^2}{1}=1$

(2) Find $\lim_{x\to 0}\frac{x^2}{\int_{\cos x}^{1}\mathrm{e}^{-\omega^2}d\omega}$

Solution: $\lim_{x\to 0}\frac{x^2}{\int_{\cos x}^{1}\mathrm{e}^{-\omega^2}d\omega}=\lim_{x\to 0}\frac{2x}{-e^{-\cos^2 x}\cdot(-\sin x)}=2e$

(3) Find $\lim_{x\to+\infty}\frac{\int_0^x(\arctan v)^2dv}{\sqrt{1+x^2}}$

Solution: $\lim_{x\to+\infty}\frac{\int_0^x(\arctan v)^2dv}{\sqrt{1+x^2}}=\lim_{x\to+\infty}\frac{\int_0^x(\arctan v)^2dv}{x}=\lim_{x\to+\infty}\frac{(\arctan x)^2}{1}=\frac{\pi^2}{4}$

(4) Find $\lim_{x\to+\infty}\frac{\left(\int_0^x e^{u^2}du\right)^2}{\int_0^x e^{2u^2}du}$

Solution: $\lim_{x\to+\infty}\frac{\left(\int_0^x e^{u^2}du\right)^2}{\int_0^x e^{2u^2}du}=\lim_{x\to+\infty}\frac{2\left(\int_0^x e^{u^2}du\right)e^{x^2}}{e^{2x^2}}=\lim_{x\to+\infty}\frac{2\left(\int_0^x e^{u^2}du\right)}{e^{x^2}}=\lim_{x\to+\infty}\frac{2e^{x^2}}{2xe^{x^2}}=0$

4. Find $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx$

Solution: $I_0=\int_0^{\frac{\pi}{2}}\sin^0 x\,dx=\frac{\pi}{2},I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx=1$

$$\forall n \geq 2, \quad I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$= \int_0^{\frac{\pi}{2}} -\sin^{n-1} x\, d(\cos x)$$

$$= (-\sin^{n-1} x \cos x)\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x\, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)dx$$

$$= (n-1)(I_{n-2} - I_n)$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \begin{cases} \frac{(n-1)!!}{n!!} \cdot \frac{\pi}{2} & n = 2k \\ \frac{(n-1)!!}{n!!} & n = 2k-1 \end{cases}$$

5. Find $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x\, dx$

Solution: $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x\, dx = \int_0^{\frac{\pi}{2}} \sin^3 x\,(1-\sin^2 x)^2\, dx$

$$= \int_0^{\frac{\pi}{2}} (\sin^3 x - 2\sin^5 x + \sin^7 x)\, dx$$

$$= \frac{2!!}{3!!} - 2 \cdot \frac{4!!}{5!!} + \frac{6!!}{7!!}$$

$$= \frac{2}{3} - 2 \cdot \frac{4 \cdot 2}{5 \cdot 3} + \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3}$$

$$= \frac{2}{35}$$

6. Find $\int_0^{\pi} \sin^n x\, dx, \int_0^{\frac{\pi}{2}} \cos^n x\, dx, \int_0^{\pi} \cos^n x\, dx, \int_{-\pi}^{\pi} \sin^n x\, dx$

Solution: $\int_0^{\pi} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x\, dx + \int_{\frac{\pi}{2}}^{\pi} \sin^n x\, dx \quad (x = \pi - t)$

$$= \int_0^{\frac{\pi}{2}} \sin^n x\, dx + \int_{\frac{\pi}{2}}^{0} \sin^n(\pi - t)\, d(\pi - t)$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x\, dx + \int_0^{\frac{\pi}{2}} \sin^n t\, dt$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\, dx \quad \left(x = \frac{\pi}{2} - t\right)$$

$$= \int_{\frac{\pi}{2}}^{0} \cos^n\left(\frac{\pi}{2} - t\right) d\left(\frac{\pi}{2} - t\right)$$

$$= \int_0^{\frac{\pi}{2}} \sin^n t\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

$$\int_0^{\pi} \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx + \int_{\frac{\pi}{2}}^{\pi} \cos^n x\, dx \quad (x = \pi - t)$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x\, dx + \int_{\frac{\pi}{2}}^{0} \cos^n(\pi - t)\, d(\pi - t)$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x\, dx + (-1)^n \cdot \int_0^{\frac{\pi}{2}} \cos^n t\, dt$$

$$= \begin{cases} 2\int_0^{\frac{\pi}{2}} \cos^n x\, dx & n = 2k \\ 0 & n = 2k-1 \end{cases}$$

$$= \begin{cases} 2\int_0^{\frac{\pi}{2}} \sin^n x\, dx & n = 2k \\ 0 & n = 2k-1 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin^n x\, dx = \begin{cases} 2\int_0^{\pi} \sin^n x\, dx & n = 2k \\ 0 & n = 2k-1 \end{cases}$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin^n x\, dx = \begin{cases} 4\int_0^{\frac{\pi}{2}} \sin^n x\, dx & n = 2k \\ 0 & n = 2k-1 \end{cases}$$

7. Prove: (1)$\int_0^{\frac{\pi}{2}} f(\cos x)\, dx = \int_0^{\frac{\pi}{2}} f(\sin x)\, dx$

(2)$\int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$

Proof: (1) $\int_0^{\frac{\pi}{2}} f(\cos x)\, dx \quad \left(x = \frac{\pi}{2} - t\right)$

$$= \int_{\frac{\pi}{2}}^{0} f\left(\cos\left(\frac{\pi}{2} - t\right)\right) d\left(\frac{\pi}{2} - t\right)$$

$$= \int_0^{\frac{\pi}{2}} f(\sin t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} f(\sin x)\, dx$$

(2) $\int_0^{\pi} x f(\sin x)\, dx \quad (x = \pi - t)$

$$= \int_{\pi}^{0} (\pi - t) f(\sin(\pi - t))\, d(\pi - t)$$

$$= \int_0^{\pi} (\pi - t) f(\sin t)\, dt$$

$$= \int_0^{\pi} (\pi - x) f(\sin x)\, dx$$

$$= \pi \cdot \int_0^{\pi} f(\sin x)\, dx - \int_0^{\pi} x f(\sin x)\, dx$$

$$\Rightarrow \int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$$

8. Find the following definite integral

(1)$\int_0^{\pi} x \sin^4 x\, dx$

Solution: $\int_0^{\pi} x \sin^4 x\, dx = \frac{\pi}{2}\int_0^{\pi} \sin^4 x\, dx = \pi \int_0^{\frac{\pi}{2}} \sin^4 x\, dx = \pi \cdot \frac{3!!}{4!!} \cdot \frac{\pi}{2} = \frac{3}{16}\pi^2$

(2)$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

Solution: $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{x\sin x}{2-\sin^2 x}dx$

$$= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\sin^2 x}dx$$

$$= \frac{\pi}{2}\int_0^\pi \frac{-d(\cos x)}{1+\cos^2 x} \quad (u = \cos x)$$

$$= \frac{\pi}{2}\int_1^{-1} \frac{-du}{1+u^2}$$

$$= \frac{\pi}{2}\cdot (\arctan u)|_{-1}^{1}$$

$$= \frac{1}{4}\pi^2$$

(3)$\int_0^\pi \frac{x}{1+\sin^2 x}dx$

Solution: $\int_0^\pi \frac{x}{1+\sin^2 x}dx = \frac{\pi}{2}\int_0^\pi \frac{1}{1+\sin^2 x}dx = \pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}dx$

$$\int \frac{1}{1+\sin^2 x}dx = \int \frac{\csc^2 x}{\csc^2 x+1}dx$$

$$= \int \frac{-d(\cot x)}{\cot^2 x+2} \quad (u = \cot x)$$

$$= \int \frac{-du}{u^2+2}$$

$$= -\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C$$

$$= -\frac{1}{\sqrt{2}}\arctan\frac{\cot x}{\sqrt{2}} + C$$

$$\Rightarrow \int_0^\pi \frac{x}{1+\sin^2 x}dx = \pi\left(-\frac{1}{\sqrt{2}}\arctan\frac{\cot x}{\sqrt{2}}\right)\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{4}\pi^2$$

9. $f(x) = \begin{cases} \sin\frac{x}{2} & x \geq 0, \\ x\arctan x & x < 0. \end{cases}$ $find\ \ I = \int_0^{\pi+1} f(x-1)\,dx$

Solution: $I = \int_0^{\pi+1} f(x-1)\,dx \quad (u = x-1)$

$$= \int_{-1}^{\pi} f(u)\,du$$

$$= \int_{-1}^{0} f(u)\,du + \int_0^\pi f(u)\,du$$

$$= \int_{-1}^{0} u\arctan u\,du + \int_0^\pi \sin\frac{u}{2}\,du$$

$$= \left(\frac{1}{2}u^2\arctan u\right)\Big|_{-1}^{0} - \frac{1}{2}\int_{-1}^{0}\frac{u^2}{1+u^2}du + \left(-2\cos\frac{u}{2}\right)\Big|_0^\pi$$

$$= \frac{\pi}{8} - \frac{1}{2}(u - \arctan u)|_{-1}^{0} + 2$$

$$= \frac{\pi}{4} + \frac{3}{2}$$

10. Find $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

Solution: $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx \quad (x = \tan t\,, dx = \sec^2 t\, dt)$

$= \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan t)}{1+\tan^2 t} \sec^2 t\, dt$

$= \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\, dt$

$= \int_0^{\frac{\pi}{4}} \ln \frac{\sin t + \cos t}{\cos t} dt$

$= \int_0^{\frac{\pi}{4}} \ln \frac{\sqrt{2}\cos\left(\frac{\pi}{4} - t\right)}{\cos t} dt$

$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2}\, dt + \int_0^{\frac{\pi}{4}} \ln \cos\left(\frac{\pi}{4} - t\right) dt - \int_0^{\frac{\pi}{4}} \ln \cos t\, dt$

$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2}\, dt + \int_{\frac{\pi}{4}}^0 \ln \cos u\, d\left(\frac{\pi}{4} - u\right) - \int_0^{\frac{\pi}{4}} \ln \cos t\, dt$

$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2}\, dt + \int_0^{\frac{\pi}{4}} \ln \cos u\, du - \int_0^{\frac{\pi}{4}} \ln \cos t\, dt$

$= \int_0^{\frac{\pi}{4}} \ln \sqrt{2}\, dt$

$= \frac{\pi}{8} \ln 2$

11. Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b), $\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\, dx = f(b)$. Prove: $\exists \xi \in (a, b), s.t. f'(\xi) = 0$

Proof: $By\ \ integral\ mean\ value\ theorem, we\ have$

$\exists \eta \in \left[a, \frac{a+b}{2}\right], s.t. f(\eta) = \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\, dx = f(b)$

$By\ Rolle\ theorem, \exists \xi \in (\eta, b) \subset (a, b), s.t. f'(\xi) = 0$

12. Let $f(x)$ be continuous on $[a, b]$ and monotone decreasing. Prove: $\forall \alpha \in [0,1], \int_0^\alpha f(x)\, dx \geq \alpha \int_0^1 f(x)\, dx$

Proof: $\Leftarrow \forall \alpha \in [0,1], \int_0^\alpha f(x)\, dx \geq \alpha \int_0^1 f(x)\, dx$

$\Leftarrow \forall \alpha \in [0,1], (1-\alpha)\int_0^\alpha f(x)\, dx \geq \alpha \int_\alpha^1 f(x)\, dx$

$By\ \ integral\ mean\ value\ theorem, we\ have$

$\exists x_1 \in [0, \alpha], s.t. (1-\alpha)\int_0^\alpha f(x)\, dx = \alpha(1-\alpha) f(x_1)$

$\exists x_2 \in [\alpha, 1], s.t. \alpha \int_\alpha^1 f(x)\, dx = \alpha(1-\alpha) f(x_2)$

$Since\ f\ be$ monotone decreasing $\Rightarrow f(x_1) \geq f(x_2)$

$\Rightarrow (1-\alpha)\int_0^\alpha f(x)\, dx \geq \alpha \int_\alpha^1 f(x)\, dx$

$$\Rightarrow \forall\alpha \in [0,1], \int_0^\alpha f(x)\,dx \geq \alpha \int_0^1 f(x)\,dx$$

13. Find the following limits using the mean value theorem

(1) $\lim\limits_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\,dx$

(2) $\lim\limits_{n\to\infty} \int_n^{n+p} \frac{\sin x}{x}\,dx \ (p \in N_+)$

Solution: (1) $By\ \ integral\ mean\ value\ theorem, we\ have$

$$\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x}\,dx = \lim_{n\to\infty} \frac{1}{1+\xi} \int_0^1 x^n\,dx \quad (0 \leq \xi \leq 1)$$

$$= \lim_{n\to\infty} \tfrac{1}{1+\xi}\tfrac{1}{n+1}$$

$$= 0$$

(2) $By\ \ integral\ mean\ value\ theorem, we\ have$

$$\exists\xi \in [n, n+p], s.t. \left|\int_n^{n+p} \frac{\sin x}{x}\,dx\right| = \left|\frac{\sin\xi}{\xi}p\right| \leq \frac{p}{n}$$

$$\Rightarrow \lim_{n\to\infty} \int_n^{n+p} \frac{\sin x}{x}\,dx = 0$$

14. Find the following generalized integral

(1) $I_n = \int_0^{+\infty} e^{-x}x^n\,dx$

Solution: $I_0 = \int_0^{+\infty} e^{-x}\,dx = 1$

$When\, n \geq 1,\ I_n = \int_0^{+\infty} e^{-x}x^n\,dx$

$$= (-e^{-x}x^n)|_0^{+\infty} + n\int_0^{+\infty} e^{-x}x^{n-1}\,dx$$

$$= n\int_0^{+\infty} e^{-x}x^{n-1}\,dx$$

$$= nI_{n-1}$$

$$\Rightarrow I_n = n!\ \ (n \geq 1)$$

(2) $\int_0^1 \ln x\,dx$

Solution: $\int_0^1 \ln x\,dx = (x\ln x)|_0^1 - \int_0^1 dx = -1$

(3) $I = \int_0^{\frac{\pi}{2}} \ln\sin x\,dx$

Solution: $I = \int_0^{\frac{\pi}{2}} \ln\sin x\,dx \quad (x = 2t, dx = 2dt)$

$$= 2\int_0^{\frac{\pi}{4}} \ln\sin 2t\,dt$$

$$= 2\int_0^{\frac{\pi}{4}} \ln 2\sin t\cos t\,dt$$

$$= 2\int_0^{\frac{\pi}{4}} \ln 2\, dt + 2\int_0^{\frac{\pi}{4}} \ln\sin t\, dt + 2\int_0^{\frac{\pi}{4}} \ln\cos t\, dt$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\, dt + 2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\cos\left(\frac{\pi}{2} - u\right) d\left(\frac{\pi}{2} - u\right)$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\, dt + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin u\, du$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{2}} \ln\sin t\, dt$$

$$= \frac{\pi}{2}\ln 2 + 2I$$

$$\Rightarrow I = -\frac{\pi}{2}\ln 2$$

(4) $\displaystyle\int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)}$ $(\alpha \in R)$

Solution: $\int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)} = \int_0^1 \frac{dx}{(1+x^2)(1+x^\alpha)} + \int_1^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)}$

$$\int_1^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)} \quad \left(x = \frac{1}{t}, dx = -\frac{1}{t^2}dt\right)$$

$$= \int_1^0 \frac{-\frac{1}{t^2}dt}{\left(1+\frac{1}{t^2}\right)\left(1+\frac{1}{t^\alpha}\right)}$$

$$= \int_0^1 \frac{t^\alpha}{(1+t^2)(1+t^\alpha)} dt$$

$$= \int_0^1 \frac{x^\alpha}{(1+x^2)(1+x^\alpha)} dx$$

$$\int_0^{+\infty} \frac{dx}{(1+x^2)(1+x^\alpha)} = \int_0^1 \frac{dx}{(1+x^2)(1+x^\alpha)} + \int_0^1 \frac{x^\alpha}{(1+x^2)(1+x^\alpha)} dx$$

$$= \int_0^1 \frac{(1+x^\alpha)dx}{(1+x^2)(1+x^\alpha)}$$

$$= \int_0^1 \frac{1}{1+x^2} dx$$

$$= (\arctan x)\big|_0^1$$

$$= \frac{\pi}{4}$$