Seat NO. ______ Major ______ Student ID □□□□□□□□□□□□ Chinese Name ______

（DONOT WRITE YOUR ANSWER IN THIS AREA）

Seal Line

**WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.**

# SCUT Final Exam

## 2022-2023-1《Calculus I》 Exam Paper A

**Notice:**
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam paper.**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

| Question No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | Sum |
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| Score | | | | | | | | | | | |

1. Evaluate the following limits.(20 points)

(1) $\lim\limits_{x\to 0}\dfrac{\sin x + x^2\sin\frac{1}{x}}{(1-\sin x)\ln(1+x)}$

Solution:

$$\text{The limit} = \lim_{x\to 0}\frac{\sin x + x^2\sin\frac{1}{x}}{\ln(1+x)} = \lim_{x\to 0}\frac{\sin x}{\ln(1+x)} + \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\ln(1+x)} = 1 + 0 = 1$$

(2) $\lim\limits_{x\to 0}\left(\dfrac{1}{1-\cos x}\right)^{x^3}$

Solution:

$$\text{The limit} = e^{\lim\limits_{x\to 0} x^3[-\ln(1-\cos x)]} = e^{\lim\limits_{x\to 0}\frac{-\ln(1-\cos x)}{x^{-3}}} = e^{\lim\limits_{x\to 0}\frac{\frac{\sin x}{1-\cos x}}{3x^{-4}}} = e^{\lim\limits_{x\to 0}\frac{x^4\sin x}{3(1-\cos x)}} = 1$$

(3) $\lim\limits_{x\to 1}(2e^{\frac{x-1}{x}} - 1)^{\frac{1}{x-1}}$

Solution :

$$\text{The limit} = \lim_{x\to 1}[1+(2e^{\frac{x-1}{x}}-2)]^{\frac{1}{2e^{\frac{x-1}{x}}-2}\frac{2e^{\frac{x-1}{x}}-2}{x-1}} = \lim_{x\to 1} e^{\frac{2(e^{\frac{x-1}{x}}-1)}{x-1}}$$

$$= \lim_{x\to 1} e^{\frac{2(\frac{x-1}{x})}{x-1}} = e^2$$

(4) $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(1+\frac{2i}{n})^2\frac{2}{n}$

Solution:

Method 1 The limit $= \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(\frac{2}{n}+\frac{4i}{n}\frac{2}{n}+\frac{4i^2}{n^2}\frac{2}{n})$

$$= \lim_{n\to\infty}[2+\frac{8}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}] = 2+4+\frac{8}{3} = \frac{26}{3}$$

Method 2 The limit $= \int_0^2 (1+x)^2 dx = [\frac{(1+x)^3}{3}]_0^2 = \frac{26}{3}$

2. Evaluate the following problems. (20 points)

(1) Let $y = \ln(x+\sqrt{1+x^2})$, find $y^{(2022)}(0)$.

Solution:

$y = \ln(x+\sqrt{1+x^2})$, $y' = \dfrac{1}{\sqrt{1+x^2}}$, $(1+x^2)(y')^2 = 1$.

Differentiating both sides, we get $(1+x^2)2y'y''+2x(y')^2 = 0$,

For $y' \neq 0$, it follows that $(1+x^2)y''+xy' = 0$ (*).

Next, find the $(n-2)$ - th derivatives of both sides of the above equation (*),

$C_{n-2}^0(1+x^2)y^{(n)}+C_{n-2}^1 2x\cdot y^{(n-1)}+C_{n-2}^2 2y^{(n-2)}+C_{n-2}^0 xy^{(n-1)}+C_{n-2}^1 y^{(n-2)} = 0$

Let $x = 0$,

$y^{(n)}(0)+(n-2)(n-3)y^{(n-2)}(0)+(n-2)y^{(n-2)}(0) = 0$

$y^{(n)}(0) = -(n-2)^2 y^{(n-2)}(0)$, $y^{(0)}(0) = y(0) = 0$,

$\therefore\ y^{(2022)}(0) = 0$

(2) Let $f(x) = \begin{cases} \dfrac{\int_0^x (e^{t^2} - 1)\mathrm{d}t}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, find $f'(0)$.

Solution:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\dfrac{\int_0^x (e^{t^2} - 1)\mathrm{d}t}{x^2}}{x}$$

$$= \lim_{x\to 0} \frac{\int_0^x (e^{t^2} - 1)\mathrm{d}t}{x^3} \overset{\text{L'Hospital}}{=} \lim_{x\to 0} \frac{e^{x^2} - 1}{3x^2} = \frac{1}{3}$$

(3) If parametric equation $\begin{cases} x = t - \arctan t \\ e^{ty} + \cos(\mathrm{t} + y) = y \end{cases}$ defines $y$ as a differentiable function of $x$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$$

$$e^{ty}(y + t\frac{\mathrm{d}y}{\mathrm{d}t}) - \sin(t+y)(1+\frac{\mathrm{d}y}{\mathrm{d}t}) = \frac{\mathrm{d}y}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\sin(t+y) - ye^{ty}}{te^{ty} - \sin(t+y) - 1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}} = \frac{(1+t^2)[\sin(t+y) - ye^{ty}]}{t^2[te^{ty} - \sin(t+y) - 1]}$$

(4) The equation $x^y - 2x + y = 0$ defines y as an implicit function of $x$, find d$y$.

Solution: $x^y = e^{y\ln x}$

Find the differentials of both sides of the above equation,

$x^y d(y\ln x) - 2dx + dy = 0$

Follows that $x^y(\ln x dy + \frac{y}{x}dx) - 2dx + dy = 0$

So $\mathrm{dy} = \frac{2 - yx^{y-1}}{x^y \ln x + 1}dx$

3. Evaluate the following integrals. (25 points)

(1) $\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} \quad (a < b)$

Solution:

Method 1

$$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = 2\int \frac{\mathrm{d}\sqrt{x-a}}{\sqrt{(b-a)-(x-a)}} \qquad (a < x < \mathrm{b})$$

$$2\int \frac{\mathrm{d}\sqrt{\frac{x-a}{b-a}}}{\sqrt{1-(\sqrt{\frac{x-a}{b-a}})^2}} = 2\arcsin\sqrt{\frac{x-a}{b-a}} + C$$

Method 2

$$\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = \int \frac{1}{x-a}\sqrt{\frac{x-a}{b-x}}\mathrm{d}x \qquad (a < x < \mathrm{b})$$

$$t = \sqrt{\frac{x-a}{b-x}},\ x = \frac{a+bt^2}{1+t^2},\ \mathrm{d}x = \frac{2bt-2at}{(1+t^2)^2}\mathrm{d}t$$

$$= \int \frac{1+t^2}{(b-a)t^2}\cdot t\cdot\frac{2bt-2at}{(1+t^2)^2}\mathrm{d}t = \int\frac{2}{1+t^2}\mathrm{d}t = 2\arctan t + C = 2\arctan\sqrt{\frac{x-a}{b-x}} + C$$

Method 3

$$(x-a)(b-x)=\frac{(a-b)^2}{4}-(x-\frac{a+b}{2})^2$$

Let $x-\dfrac{a+b}{2}=\dfrac{b-a}{2}\sin t$,

……

Method 4

Let $x-a=(b-a)\sin^2 t$

$b-x=(b-a)\cos^2 t$

$t\in(0,\dfrac{\pi}{2})$

……

(2) $\displaystyle\int\frac{\mathrm{d}x}{e^{2x}+e^{-2x}}$

Solution:

$$\int\frac{\mathrm{d}x}{e^{2x}+e^{-2x}}=\int\frac{e^{2x}\mathrm{d}x}{e^{4x}+1}=\frac{1}{2}\int\frac{\mathrm{d}e^{2x}}{1+(e^{2x})^2}=\frac{1}{2}\arctan(e^{2x})+C$$

(3) $\displaystyle\int_{-2}^{3}\left|x^2+2|x|-3\right|\mathrm{d}x$

Solution:

$$\int_{-2}^{3}\left|x^2+2|x|-3\right|\mathrm{d}x=2\int_{0}^{2}\left|x^2+2x-3\right|\mathrm{d}x+\int_{2}^{3}\left|x^2+2x-3\right|\mathrm{d}x$$

$$=-2\int_{0}^{1}(x^2+2x-3)\mathrm{d}x+2\int_{1}^{2}(x^2+2x-3)\mathrm{d}x+\int_{2}^{3}(x^2+2x-3)\mathrm{d}x$$

$$=\frac{49}{3}$$

(4) $\displaystyle\int_{1}^{4}\arctan\sqrt{x-1}\,\mathrm{d}x$

Solution:

$let\ t = \sqrt{x-1}, x = t^2+1, \mathrm{d}x = 2t\mathrm{d}t$

$\int_1^4 \arctan\sqrt{x-1}\mathrm{d}x = \int_0^{\sqrt{3}} \arctan t\ \mathrm{d}(t^2+1)$

$= [(t^2+1)\arctan t]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} (t^2+1)\ \mathrm{d}(\arctan t)$

$= \frac{4\pi}{3} - \int_0^{\sqrt{3}} (t^2+1)\frac{1}{t^2+1}\mathrm{d}t$

$= \frac{4\pi}{3} - \sqrt{3}$

(5) $\int_0^1 \ln x\mathrm{d}x$

Solution:It is an improper integral.

$\int_0^1 \ln x\mathrm{d}x = [x\ln x]_{0^+}^1 - \int_0^1 x\mathrm{d}\ln x = 0 - \lim_{x\to 0^+} x\ln x - 1 = -1$

4.(5 points)Prove that $\lim_{x\to -1}(x^2-2x-1) = 2$ by using $\varepsilon$ - $\delta$ definition.

Proof:

$\forall\varepsilon > 0, bound \mid x-(-1) \mid < 1$, which implies $3 < \mid x-3 \mid < 5$.

In order to $\mid (x^2-2x-1)-2 \mid = \mid x-3 \mid \mid x-(-1) \mid < 5 \mid x-(-1) \mid < \varepsilon$,

we need $\mid x-(-1) \mid < \frac{\varepsilon}{5}$. So choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$,

when $\mid x-(-1) \mid < \delta$, we have $\mid (x^2-2x-1)-2 \mid < \varepsilon$.

$\therefore \lim_{x\to -1}(x^2-2x-1) = 2$

5.(5 points)Prove that $x^a - ax \le 1-a\ (x > 0, 0 < a < 1)$.

Proof:

Let $f(x) = x^a - ax - (1 - a)$, $f'(x) = a(x^{a-1} - 1)$.

$0 < x < 1$, $f'(x) > 0$; $x > 1$, $f'(x) < 0$.

So $f(1) = 0$ is the maximum value of $f(x)$ on $[0, +\infty)$.

That is $f(x) \le f(1) = 0$, i.e. $x^a - ax \le 1 - a \ (x > 0)$.

6.(5 points)If $f$ is periodic with period $p$, prove $\int_a^{a+p} f(x)\mathrm{d}x = \int_0^p f(x)\mathrm{d}x$.

Proof:

Method 1

Let $F(a) = \int_a^{a+p} f(x)\mathrm{d}x$, $D_aF(a) = f(a + p) - f(a) = 0$,

$$\int_a^{a+p} f(x)\mathrm{d}x = F(a) \equiv C = F(0) = \int_0^p f(x)\mathrm{d}x$$

Method 2

$$\int_a^{a+p} f(x)\mathrm{d}x = \int_a^0 f(x)\mathrm{d}x + \int_0^p f(x)\mathrm{d}x + \int_p^{a+p} f(x)\mathrm{d}x,$$

$$\int_p^{a+p} f(x)\mathrm{d}x \overset{u=x-p}{=} \int_0^a f(u + p)\mathrm{d}u = \int_0^a f(u)\mathrm{d}u = -\int_a^0 f(x)\mathrm{d}x$$

$$\therefore \int_a^{a+p} f(x)\mathrm{d}x = \int_0^p f(x)\mathrm{d}x$$

7.(5 points)Show that the graph of a concave up function $f$ is always above its tangent line; that is , show that $f(x) > f(c) + f'(c)(x - c)$, $x \ne c$.

Proof:

If $x < c$, $f(x) - f(c) = f'(c_1)(x - c)$, $x < c_1 < c$.

for $f(x)$ is a concave up function, $f''(x) > 0$,

which implies $f'(c_1) < f'(c)$, so

$f(x) - f(c) > f'(c)(x - c)$,

i.e. $f(x) > f(c) + f'(c)(x - c)$.

If $x > c$, $f(x) - f(c) = f'(c_2)(x - c)$, $c < c_2 < x$.

for $f''(x) > 0$,

which implies $f'(c_2) > f'(c)$, so

$f(x) - f(c) > f'(c)(x - c)$,

i.e. $f(x) > f(c) + f'(c)(x - c)$.

So the graph of a concave up function $f$ is always above its tangent line.

8.(5 points)Find the oblique asymptote(s) for $y = (x - 1)e^{\frac{\pi}{2} + \arctan x}$.

Solution:

$$\lim_{x \to +\infty} \frac{(x - 1)e^{\frac{\pi}{2} + \arctan x}}{x} = e^{\pi}$$

$$\lim_{x \to +\infty} \left[(x - 1)e^{\frac{\pi}{2} + \arctan x} - e^{\pi} x\right] = \lim_{x \to +\infty} \left[x(e^{\frac{\pi}{2} + \arctan x} - e^{\pi}) - e^{\frac{\pi}{2} + \arctan x}\right]$$

$$= \lim_{x \to +\infty} \frac{e^{\pi}(e^{\arctan x - \frac{\pi}{2}} - 1)}{\frac{1}{x}} - e^{\pi} = \lim_{x \to +\infty} \frac{e^{\pi}(\arctan x - \frac{\pi}{2})}{\frac{1}{x}} - e^{\pi}$$

$$= \lim_{x \to +\infty} \frac{e^{\pi} \frac{1}{1 + x^2}}{-\frac{1}{x^2}} - e^{\pi} = -2e^{\pi}$$

one oblique asymptote is $y = e^{\pi} x - 2e^{\pi}$.

$$\lim_{x\to-\infty}\frac{(x-1)e^{\frac{\pi}{2}+\arctan x}}{x}=1$$

$$\lim_{x\to-\infty}[(x-1)e^{\frac{\pi}{2}+\arctan x}-x]=\lim_{x\to-\infty}[x(e^{\frac{\pi}{2}+\arctan x}-1)-e^{\frac{\pi}{2}+\arctan x}]$$

$$=\lim_{x\to-\infty}\frac{(e^{\arctan x+\frac{\pi}{2}}-1)}{\frac{1}{x}}-1=\lim_{x\to-\infty}\frac{(\arctan x+\frac{\pi}{2})}{\frac{1}{x}}-1$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}-1=-2$$

The other oblique asymptote is $y=x-2$.

9.(5 points)Find the area of the region trapped between $y=\sin x$ and $y=\frac{1}{2}$, $0\le x\le\frac{13}{6}\pi$.

Solution:

Solve $\sin x=\frac{1}{2}$ for $0\le x\le\frac{13\pi}{6}$, $x=\frac{\pi}{6}$, $x=\frac{5\pi}{6}$, $x=\frac{13\pi}{6}$

The area of the trapped region is

$$\int_0^{\frac{\pi}{6}}(\frac{1}{2}-\sin x)dx+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(\sin x-\frac{1}{2})dx+\int_{\frac{5\pi}{6}}^{\frac{13\pi}{6}}(\frac{1}{2}-\sin x)dx$$

$$=(\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1)+(\sqrt{3}-\frac{\pi}{3})+(\sqrt{3}+\frac{2\pi}{3})$$

$$=\frac{5\pi}{12}+\frac{5\sqrt{3}}{2}-1$$

10.(5 points)Find the volume of the solid generated by revolving about the $x$-axis the region bounded by the line $y = 6x$ and the parabola $y = 6x^2$.

Solution:

$$V = \int_0^1 [\pi(6x)^2 - \pi(6x^2)^2]\mathrm{d}x = \frac{24\pi}{5}$$