积分复习1

2023年12月26日 11:23

•原函数与不定积分的概念

原函数：弄清谁是谁的原函数，谁是谁的导数！

不定积分：原函数F（x)+C

$F'(x)=f(x)$

F：f 的原函数

f：F 的导函数

$\int f(x)dx = F(x)+C$

$f(x)$ 的导函数是 $\sin x$，则它的原函数是 $\sin x + ax + b$.

$f'(x)=\sin x$，$f(x) = -\cos x + a$

$\int(-\cos x + a)dx = -\sin x + ax + b$

•基本积分方法：

1、分项积分法

$\int (kf(x)\pm lg(x))dx = k\int f(x)dx + l\int g(x)dx$

关键：熟练掌握基本积分公式（凑微分的基础）

(I)

$\int kdx = kx + C$

$\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1}+C \quad (\mu\neq -1)$

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{a^x}{\ln a}+C$

$\int \frac{1}{x}dx = \ln|x|+C$

$\int \cos x\, dx = \sin x + C$ 　$d(\sin x)$

$\int \sin x\, dx = -\cos x + C$ 　$-d(\cos x)$

$\int \sec^2 x\, dx = \int \frac{1}{\cos^2 x}dx = \tan x + C$ 　$d(\tan x)$

$\int \csc^2 x\, dx = \int \frac{1}{\sin^2 x}dx = -\cot x + C$

$\int \sec x \tan x\, dx = \sec x + C$ 　$d(\sec x)$

$\int \csc x \cot x\, dx = -\csc x + C$

$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C = -\arccos x + C$

$\int \frac{1}{1+x^2}dx = \arctan x + C = -\text{arccot}\, x + C$

$\int \sin^m x \cos^n x\, dx$

① m, n 有一个奇，用它凑微分

② m, n 都偶 → 降次

$\int_0^{\frac{\pi}{2}} \sin^{2k} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{2k} x\, dx$ （公式）

$\int \sin^2 x \cos x\, dx$ 　$d(\sin x)$

$\int \frac{\sin^2 x}{\cos x}dx = \int \sin^2 x \cdot \frac{\cos x dx}{\cos^2 x}$

$= \int \frac{\sin^2 x}{1-\sin^2 x}d(\sin x)$

$= \int \frac{u^{2}}{1-u^2}du$

$\int R(\sin x, \cos x)dx \doteq \int R\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right)\frac{2}{1+u^2}du$

$u = \tan\frac{x}{2}$，$\sin x = \frac{2u}{1+u^2}$，$\cos x = \frac{1-u^2}{1+u^2}$

→ $x = 2\arctan u$

⇒ $\arctan u = \frac{x}{2}$

$\sqrt{1+u^2}$　u　1

$\sin\frac{x}{2}, \cos\frac{x}{2}, \cdots$

$\int x \tan x \sec^2 x\, dx$ 　$(\tan x)$　$d(\tan x)$

$= \int x \cdot \tan x\, d(\tan x)$ 　$u=\tan x$　$\int u du = \frac{1}{2}\int d(u^2)$

$= \frac{1}{2}\int x\, d(\tan^2 x)$

$= \frac{1}{2}\left[x\tan^2 x - \int \tan^2 x\, dx\right]$

$= \frac{1}{2}\left[\cdots - \int(\sec^2 x - 1)dx\right]$ 　$\tan x$

(II)

$\int \tan x\, dx = -\ln|\cos x| + C$

$\int \cot x\, dx = \ln|\sin x| + C$

$\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$\int \csc x\, dx = \ln|\csc x - \cot x| + C$

$\int \sin^m x \cos^n x\, dx$　m, n 有一个奇

$\int \frac{\sin x\, dx}{\cos x} = -$

$\int \ln x dx = x\ln x - x + C$ （不必记公式，但需掌握方法-分部积分法）

$\int \arcsin x dx, \int \arccos x dx, \int \arctan x dx, \int \text{arccot}\, x dx$ （不必记公式，但需掌握方法-分部积分法）

(III)

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$ 　$(\Delta<0)$

$\int \frac{1}{x^2+2x+5}dx = \int \frac{1}{(x+1)^2+2^2}d(x+1)$ 　$u=x+1$

$= \int \frac{1}{2^2+u^2}du = \cdots$

$\int \frac{1}{(a^2+x^2)^n}dx$ （分部+递推）

$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$ 　$(\Delta>0)$

$\int \frac{1}{x^2+2x-3}dx = \frac{1}{4}\left[\int \frac{d(x-1)}{x-1} - \int \frac{d(x+3)}{x+3}\right]$

更一般地，有理函数的积分

$\int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right|+C$ （推广：根号下二次三项式）

$\int \frac{1}{\sqrt{x^2+2x+5}}dx = \int \frac{d(x+1)}{\sqrt{(x+1)^2+2^2}} \overset{u=x+1}{=} \int \frac{du}{\sqrt{u^2+a^2}} = \ln\left|u+\sqrt{u^2+a^2}\right|+C = \cdots$

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a}+C$

$\int \frac{1}{\sqrt{x^2+2x-3}}dx = \int \frac{d(x+1)}{\sqrt{(x+1)^2-2^2}} \overset{u=x+1}{=} \int \frac{du}{\sqrt{u^2-2^2}} = \ln\left|u+\sqrt{u^2-2^2}\right|+C = \cdots$

$\int \frac{1}{\sqrt{3+2x-x^2}}dx$

x^2+2x-3

$=(x+3)(x-1)$

$$\int \frac{1}{\sqrt{3+2x-x^2}}dx$$

$$\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = \frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+3}\right)$$

$$\int \frac{1}{\sqrt{x^2+2x-3}}dx = \int \frac{du}{\sqrt{(x+1)^2-2^2}} = \int \frac{du}{\sqrt{u^2-2^2}} = \cdots$$

2、凑微分法（第一换元法）

$$\int g(x)\,dx = \int f[\varphi(x)]\cdot\varphi'(x)\,dx \overset{u=\varphi(x)}{=\!=} \int f(u)\,du = F(u)+C = F(\varphi(x))+C$$

$$\int_a^b g(x)\,dx = \int_a^b f[\varphi(x)]\,\varphi'(x)\,dx \overset{u=\varphi(x)}{=\!=} \int_{u_1}^{u_2} f(u)\,du = F(u)\Big|_{u_1}^{u_2}$$

何时换元，何时分部？

$$\int g(x)\,dx = \int u(x)\cdot v'(x)\,dx = \int u\,dv = \begin{cases} \int f(v)\,dv = F(v)+C = \cdots \\ \int u\,dv = uv - \int v\,du \end{cases}$$

$$\int \frac{1}{x}\cdot\frac{1}{1+\ln^2 x}dx = \int \frac{1}{1+\ln^2 x}\,d(\ln x) \overset{u=\ln x}{=\!=} \int \frac{1}{1+u^2}du = \cdots$$

$$\int x\sec^2 x\,dx = \int x\cdot d(\tan x) = x\tan x - \int \tan x\,dx$$
$$= x\tan x - (-\ln|\cos x|) + C$$

☆ $\int x f(ax^2+b)\,dx$ $\int x^{\frac{1}{3}} f(ax^2+b)\,dx$ // $\int e^x f(e^x)\,dx$ // $\int \frac{1}{x} f(\ln x)\,dx$, …

（20级）

11. 求不定积分 $\int \frac{1-x}{\sqrt{9-4x^2}}dx$.

$$\int \frac{du}{\sqrt{1-u^2}} \to \text{arc}\cdots$$

$$\text{原式} = \int \frac{dx}{\sqrt{9-4x^2}} - \boxed{\int \frac{x\,dx}{\sqrt{9-4x^2}}}$$

$$\frac{1}{2}\int \frac{d(2x)}{\sqrt{3^2-(2x)^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$$

$$= \frac{1}{2}\arcsin\frac{2}{3}x + \frac{1}{4}\sqrt{9-4x^2} + C$$

$$\int x\cdot(9-4x^2)^{-\frac{1}{2}}dx = -\frac{1}{8}\int (9-4x^2)^{-\frac{1}{2}}\,d(9-4x^2) \qquad \frac{d(9-4x^2)}{-8x\,dx}$$

$$= -\frac{1}{8}\cdot\frac{1}{-\frac{1}{2}+1}(9-4x^2)^{\frac{1}{2}} + C$$

$$\int \frac{x^3}{\sqrt{9-4x^2}}dx \qquad \sqrt{9-4x^2} = t,\quad 9-4x^2 = t^2,\quad -4x\,dx = t\,dt$$

$$= \int \frac{x^2\cdot x\,dx}{\sqrt{9-4x^2}} = \int \frac{(9-t^2)/4}{t}\left(-\frac{1}{4}t\,dt\right) = -\frac{1}{16}\int (9-t^2)\,dt = \cdots$$

$$\int \frac{x}{x^2+2x-3}dx = \frac{1}{2}\int \frac{2x+2-2}{x^2+2x-3}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x-3}dx - \int \frac{1}{(x+3)(x-1)}dx =$$

$$(x^2+2x-3)' = 2x+2$$

$$= \frac{1}{2}\ln|x^2+2x-3| - \frac{1}{4}\int \left(\frac{1}{x-1}-\frac{1}{x+3}\right)dx = \cdots$$

$$\int \frac{x}{\sqrt{x^2+2x+5}}dx = \frac{1}{2}\int \frac{2x+2-2}{\sqrt{x^2+2x+5}}dx = \int \frac{2x+2}{\sqrt{x^2+2x+5}}dx - \int \frac{1}{\sqrt{x^2+2x+5}}dx$$

$$(x^2+2x+5)' = 2x+2$$

$$= \int (x^2+2x+5)^{-\frac{1}{2}}\,d(x^2+2x+5) - \int \frac{d(x+1)}{\sqrt{(x+1)^2+2^2}}$$

$$\int u^{-\frac{1}{2}}du \ \checkmark$$

$$\int \frac{e^x}{1+e^x}dx = \int \frac{d(e^x+1)}{1+e^x} = \ln|1+e^x| + C$$

$$\int \frac{1}{\cdots}dx = \int \frac{e^x}{\cdots} \quad \int \frac{1}{\cdots}d(e^x) \overset{u=e^x}{\longrightarrow} \int \frac{du}{\cdots} \quad \cdots$$

$\int \frac{}{1+e^x} \quad \int \frac{}{1+e^x} \quad = \ln|1+e^x|+C$

$$\int \frac{1}{1+e^x}dx = \int \frac{e^x}{e^x(1+e^x)}dx = \int \frac{1}{e^x(1+e^x)}d(e^x) \overset{u=e^x}{=\!=} \int \frac{du}{u(1+u)} \quad \Delta>0 \text{ 裂项}$$

3、第二换元法

- $\sqrt{a^2-x^2}$ ✓ $x=a\sin t \ (-\frac{\pi}{2},\frac{\pi}{2})$ $dx = a\cos t\, dt$; $x = a\cos t \ (0,\pi)$ ……
- $\sqrt{x^2+a^2}$ ✓ $x = a\tan t \ (-\frac{\pi}{2},\frac{\pi}{2})$. $dx = a\sec^2 t\, dt$; or $x = a\cot t$
- $\sqrt{x^2-a^2}$, $x>a$ or $x<-a$: $x = a\sec t$ $\begin{cases} t\in(0,\frac{\pi}{2}) \to x>a \\ t\in(\frac{\pi}{2},\pi) \to x<-a \end{cases}$

注：x 奇 $\sqrt{a^2\pm x^2}$ 用凑微分法

- $\sqrt{\cdots} = t$ $\begin{cases} \sqrt[n]{\frac{ax+b}{cx+d}} = t \Rightarrow \frac{ax+b}{cx+d} = t^n \Rightarrow \begin{cases} x = f(t) \\ dx = \cdots dt \end{cases} \text{ 不会积分} \\ \sqrt{A\ldots} = t \\ x\text{奇}\sqrt{a^2\pm x^2} \to \sqrt{a^2\pm x^2} = t \text{（凑微分法）} \end{cases}$

- $\frac{1}{x} = t$：分母次数高

$$\int \frac{1}{x(1+x^6)}dx \xrightarrow[x=\frac{1}{t},\ dx=-\frac{1}{t^2}dt]{\frac{1}{x}=t} \int t\cdot \frac{1}{1+(\frac{1}{t})^6}\boxed{(-\frac{1}{t^2})dt} = -\int \frac{t^5}{t^6+1}dt$$

$$\int \frac{1}{x^5\cdot\sqrt{1-x^2}}dx \xrightarrow{x=\frac{1}{t},\ dx=-\frac{1}{t^2}dt} \int t^5 \cdot \frac{1}{\sqrt{1-\frac{1}{t^2}}}\cdot(-\frac{1}{t^2})dt$$

$x = \sin t$

$$= \int \frac{t^4}{\sqrt{t^2-1}}dt \quad \underset{t=\sec u,}{\text{再代换}}$$

(20级)

12. 求不定积分 $\int \frac{1}{x+\sqrt{1-x^2}}dx$.

$x = \sin t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$ $dx = \cos t\, dt$

原式 $= \int \frac{\cos t\, dt}{\sin t + \cos t}$ $\quad (\sin t + \cos t)' = \cos t - \sin t$

$= \frac{1}{2}\int \frac{\cos t + \sin t + \cos t - \sin t}{\cos t + \sin t}dt = \frac{1}{2}(t + \ln|\sin t + \cos t|) + C$

$= \frac{1}{2}(\arcsin x + \ln|x+\sqrt{1-x^2}|) + C$ ……

$u = \tan\frac{x}{2}$

$x = 2\arctan u$

$dx = 2\cdot\frac{1}{1+u^2}du$

方法2：$u = \tan\frac{x}{2}$, $I = \int \frac{\frac{1-u^2}{\cancel{1+u^2}}}{\frac{2u}{\cancel{1+u^2}}+\frac{1-u^2}{\cancel{1+u^2}}}\cdot 2\cdot\frac{1}{1+u^2}du$

$= \int \frac{1-u^2}{2u+1-u^2}\cdot\frac{2}{1+u^2}du$

$= \cdots$

19级 **10.** 求不定积分 $\int \frac{1}{(x^2+1)^2}dx$.

解：令 $x = \tan t$，得 $dx = \sec^2 t\, dt$

原式 $= \int \cos^2 t\, dt = \frac{1}{2}\int (1+\cos 2t)dt$

$= \frac{1}{2}(t + \sin t\cos t) + C$

$= \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} + C$

$\int \frac{1}{(\sec^2 t)^2}\cdot \sec^2 t\, dt =$

(22 级 a)

$$\int_0^2 \sqrt{1-\frac{1}{4}x^2}\,dx = \pi/2$$

方法1 $x = 2\sin t$. $dx = 2\cos t\, dt$ $\quad = 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$I = \int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{4}\cdot 4\sin^2 t}\cdot 2\cos t\, dt = 2\int_0^{\frac{\pi}{2}} \cos^2 t\, dt$

方法2：$= \frac{1}{2}\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\cdot\frac{1}{4}\pi\cdot 2^2$

$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx;$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & n \text{ positive even} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\cdot\frac{4}{5}\cdot\frac{2}{3}, & n \text{ positive odd} \end{cases}$$

$\int_0^1 x^4\sqrt{1-x^2}\,dx$