Limit

1. Prove the following limits.

(1) $\lim\limits_{n\to\infty}\frac{n}{n+3}=1.$

Analysis: $\forall\varepsilon>0,\ \left|\frac{n}{n+3}-1\right|=\frac{3}{n+3}<\varepsilon\Leftrightarrow n+3>\frac{3}{\varepsilon}\Leftrightarrow n>\frac{3}{\varepsilon}-3$

Proof: $\forall 0<\varepsilon<1, take\ N=\left[\frac{3}{\varepsilon}\right]>\frac{3}{\varepsilon}-3, \forall n>N{:}\left|\frac{n}{n-3}-1\right|=\frac{3}{n+3}<\varepsilon$

(2) $\lim\limits_{x\to1}\frac{x+1}{x-2}=-2.$

Analysis: $\forall\varepsilon>0,\left|\frac{x+1}{x-2}-(-2)\right|=\frac{3|x-1|}{|x-2|}<\varepsilon.\ Let\ |x-1|<\frac{1}{2}\ \Rightarrow\frac{1}{2}<x<\frac{3}{2}\ \Rightarrow|x-2|>\frac{1}{2}$

$$\Rightarrow\frac{3|x-1|}{|x-2|}<6|x-1|\Leftrightarrow6|x-1|<\varepsilon\Leftrightarrow|x-1|<\frac{\varepsilon}{6}$$

Proof: $\forall\varepsilon>0, take\ \delta=\min\left\{\frac{1}{2},\frac{\varepsilon}{6}\right\}>0,\forall x(0<|x-1|<\delta){:}\left|\frac{x+1}{x-2}-(-2)\right|<\varepsilon$

(3) $\lim\limits_{x\to1}\sqrt{\frac{7}{16x^2-9}}=1.$

Analysis: $\left|\sqrt{\frac{7}{16x^2-9}}-1\right|=\left|\frac{\frac{7}{16x^2-9}-1}{\sqrt{\frac{7}{16x^2-9}}+1}\right|\le\left|\frac{16x^2-16}{16x^2-9}\right|=\frac{16|x+1||x-1|}{|4x+3||4x-3|}$

$$Let\ |x-1|<1\Rightarrow0<x<2\Rightarrow|x+1|<3,|4x+3|>3\Rightarrow\frac{16|x+1||x-1|}{|4x+3||4x-3|}<\frac{16|x-1|}{|4x-3|}$$

$$Let\ |x-1|<\frac{1}{8}\ \Rightarrow\frac{7}{8}<x<\frac{9}{8}\ \Rightarrow|4x-3|>\frac{1}{2}\Rightarrow\frac{16|x-1|}{|4x-3|}<32|x-1|$$

$$32|x-1|<\varepsilon\Leftrightarrow|x-1|<\frac{32}{\varepsilon}$$

Proof: $\forall\varepsilon>0, take\ \delta=\min\left\{\frac{1}{8},\frac{\varepsilon}{32}\right\}>0,\forall x(0<|x-1|<\delta){:}\left|\sqrt{\frac{7}{16x^2-9}}-1\right|<\varepsilon$

2. Monotonic Sequence Theorem.

(4) $x_1=1,x_2=\frac{x_1}{1+x_1},\cdots,x_{n+1}=\frac{x_n}{1+x_n}$. Prove the existence of $\lim\limits_{n\to\infty}x_n$ and find it.

Solution: $0<x_1\le1, 0<x_2=\frac{1}{2}<1$

$$If\ \ 0<x_n<1,x_{n+1}=\frac{x_n}{1+x_n}\Rightarrow\ 0<x_{n+1}<1.$$

$$\forall n\in N_+, we\ have\ 0<x_n\le1\Rightarrow\{x_n\}\ is\ bounded$$

$$x_{n+1}-x_n=\frac{x_n}{1+x_n}-x_n=-\frac{x_n^2}{1+x_n}<0\Rightarrow\{x_n\}\ is\ monotone\ decreasing$$

$$\Rightarrow\lim_{n\to\infty}x_n\ \ exist\ , Let\ \lim_{n\to\infty}x_n=a,\Rightarrow\ a=\frac{a}{1+a}\Rightarrow a=0.\Rightarrow\ \lim_{n\to\infty}x_n=0$$

(5) $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right),a>0,x_0>0$. Prove the existence of $\lim\limits_{n\to\infty}x_n$ and find it.

Solution: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \frac{1}{2}\cdot 2\sqrt{1+\frac{a}{x_n^2}} = \sqrt{a}$

$$\frac{x_{n+1}}{x_n} = \frac{1}{2}\left(1+\frac{a}{x_n^2}\right) \leq \frac{1}{2}\left(1+\frac{a}{a}\right) = 1 \Rightarrow \{x_n\}\ is\ monotone\ decreasing.$$

$$\Rightarrow \lim_{n\to\infty} x_n\ \ exist\ , Let \lim_{n\to\infty} x_n = x, \Rightarrow\ x = \frac{1}{2}\left(x+\frac{a}{x}\right) \Rightarrow x = \pm\sqrt{a}. \Rightarrow \lim_{n\to\infty} x_n = \sqrt{a}$$

(6) $x_n = \sum_{k=1}^{n}\left(\frac{1}{\sqrt{k}}\right) - 2\sqrt{n}$, prove the existence of $\lim_{n\to\infty} x_n$.

Proof: $\frac{1}{\sqrt{k}} > \frac{2}{\sqrt{k}+\sqrt{k+1}} = 2\left(\sqrt{k+1}-\sqrt{k}\right)$

$$\Rightarrow x_n = \sum_{k=1}^{n}\frac{1}{\sqrt{k}} - 2\sqrt{n} > 2\sum_{k=1}^{n}\left(\sqrt{k+1}-\sqrt{k}\right) - 2\sqrt{n} = 2\sqrt{n+1} - 2\sqrt{n} - 2\ > -2$$

$$\Rightarrow \{x_n\}\ is\ bounded\ below$$

$$x_{n+1} - x_n = \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n} = \frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+1}+\sqrt{n}} < 0$$

$$\Rightarrow \{x_n\}\ is\ monotone\ decreasing\ \Rightarrow \lim_{n\to\infty} x_n\ \ exist$$

3. Find the following limits.

$$(x\to 0, x \sim sinx \sim tanx \sim arcsinx \sim arctanx \sim ln(1+x) \sim e^x - 1 \sim \frac{a^x-1}{ln\,a} \sim \frac{(1+x)^b-1}{b})$$

(7) Find $\lim_{x\to 0}\frac{x(1-\cos x)}{(1-e^x)\sin x^2}$.

Solution: $\lim_{x\to 0}\frac{x(1-\cos x)}{(1-e^x)\sin x^2} = \lim_{x\to 0}\frac{x\cdot\frac{1}{2}x^2}{(-x)\cdot x^2} = -\frac{1}{2}$

(8) Find $\lim_{x\to 0}\cot x\,(\frac{1}{\sin x} - \frac{1}{x})$.

Solution: $\lim_{x\to 0}\cot x\,(\frac{1}{\sin x} - \frac{1}{x}) = \lim_{x\to 0}\frac{x-\sin x}{\tan x\cdot\sin x\cdot x} = \lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \frac{1}{6}$

(9) Let $e^{xcosx^2} - e^x$ and x^n be infinitesimals of the same order as $x\to 0$, then $n =$ ___.

Solution: $\lim_{x\to 0}\frac{e^{xcosx^2}-e^x}{x^n} = \lim_{x\to 0}\frac{e^x\left(e^{xcosx^2-x}-1\right)}{x^n} = \lim_{x\to 0}\frac{e^0(xcosx^2-x)}{x^n} = \lim_{x\to 0}\frac{x(cosx^2-1)}{x^n} = \lim_{x\to 0}\frac{-x\cdot 2\sin^2\frac{x^2}{2}}{x^n}$

$$= \lim_{x\to 0}\frac{-x\cdot 2\left(\frac{x^2}{2}\right)^2}{x^n} = -\frac{1}{2}\cdot\lim_{x\to 0}\frac{1}{x^{n-5}} = c(c\neq 0)\ \Rightarrow n-5=0 \Rightarrow n=5$$

(10) Let $\sqrt{4+3\sin x} - 2$ and x^k be infinitesimals of the same order as $x\to 0$, then $k =$ ___.

Solution: $\lim_{x\to 0}\frac{\sqrt{4+3\sin x}-2}{x^k} = \lim_{x\to 0}\frac{3\sin x}{x^k\left(\sqrt{4+3\sin x}+2\right)} = \frac{3}{4}\cdot\lim_{x\to 0}\frac{1}{x^{k-1}} = c(c\neq 0) \Rightarrow k-1=0 \Rightarrow k=1$

(11) $x_n = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cdots\cos\frac{x}{2^n}$, find $\lim_{n\to\infty} x_n$.

Solution: $If\ x=0 \Rightarrow x_n = 0 \Rightarrow \lim_{n\to\infty} x_n = 0.\ \ If\ x\neq 0$

$$\sin\frac{x}{2^n}\cdot x_n \quad = \left(\sin\frac{x}{2^n}\cos\frac{x}{2^n}\right)\cos\frac{x}{2^{n-1}}\cdots\cos\frac{x}{2^2}\cos\frac{x}{2} = \frac{1}{2}\left(\sin\frac{x}{2^{n-1}}\cos\frac{x}{2^{n-1}}\right)\cdots\cos\frac{x}{2^2}\cos\frac{x}{2}$$

$$= \frac{1}{2^2}\left(\sin\frac{x}{2^{n-2}}\cos\frac{x}{2^{n-2}}\right)\cdots\cos\frac{x}{2^2}\cos\frac{x}{2} = \cdots\cdots = \frac{1}{2^{n-1}}\sin\frac{x}{2}\cos\frac{x}{2} = \frac{1}{2^n}\sin x$$

$$\Rightarrow x_n = \frac{\sin x}{2^n \sin\frac{x}{2^n}} \Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\sin x}{2^n \sin\frac{x}{2^n}} = \lim_{n\to\infty} \frac{\sin x}{2^n \cdot \frac{x}{2^n}} = \frac{\sin x}{x}$$

(12) Find $\lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n$ $(a \geq 0, b \geq 0)$. $\left(\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e\right)$.

Solution: $\lim\limits_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1\right)^{\frac{1}{\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1}\cdot n\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1\right)} = e^{\lim\limits_{n\to\infty} n\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1\right)}$

$$Since \ \lim_{n\to\infty} n\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}-1\right) = \lim_{n\to\infty}\frac{1}{2}\left(\frac{a^{\frac{1}{n}}-1}{\frac{1}{n}}+\frac{b^{\frac{1}{n}}-1}{\frac{1}{n}}\right) = \frac{1}{2}(\ln a + \ln b) = \ln\sqrt{ab}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = e^{\ln\sqrt{ab}} = \sqrt{ab}$$

4. Squeeze theorem for limits of sequences.

(13) Find $\lim\limits_{n\to\infty}\sqrt[n]{1+e^n+\pi^n}$.

Solution: $\sqrt[n]{\pi^n} \leq \sqrt[n]{1+e^n+\pi^n} \leq \sqrt[n]{3\cdot\pi^n}$

$$\Rightarrow \pi \leq \sqrt[n]{1+e^n+\pi^n} \leq \sqrt[n]{3}\,\pi.$$

$$Since \ \lim_{n\to\infty}\sqrt[n]{3} = 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{1+e^n+\pi^n} = \pi$$

(14) $a_i > 0(i = 1,2,\cdots,n)$, find $\lim\limits_{p\to\infty}\left[\left(\sum\limits_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum\limits_{i=1}^n a_i^{-p}\right)^{\frac{1}{p}}\right]$.

Solution: $Let\ A = \max\limits_{1\leq i\leq n}\{a_i\}, a = \min\limits_{1\leq i\leq n}\{a_i\}$

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n a_i^{-p}\right)^{\frac{1}{p}} \leq (nA^p)^{\frac{1}{p}} + (na^{-p})^{\frac{1}{p}} = n^{\frac{1}{p}}A + n^{\frac{1}{p}}a^{-1};$$

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n a_i^{-p}\right)^{\frac{1}{p}} \geq A + a^{-1};$$

$$Since \ \lim_{p\to\infty} n^{\frac{1}{p}} = 1 \Rightarrow \lim_{p\to\infty}\left[\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n a_i^{-p}\right)^{\frac{1}{p}}\right] = A + a^{-1}$$

(15) $x_n = \sum\limits_{k=n^2}^{(n+1)^2}\frac{1}{\sqrt{k}}$, find $\lim\limits_{n\to\infty} x_n$.

Solution: $2 = \frac{2n+2}{n+1} \leq \sum\limits_{k=n^2}^{(n+1)^2}\frac{1}{\sqrt{k}} \leq \frac{2n+2}{n}$;

$$Since \ \lim_{n\to\infty}\frac{2n+2}{n} = 2 \Rightarrow \lim_{n\to\infty} x_n = 2$$

5. Undefined limit. (L'Hospital)

(16) Find $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{x}$.

Solution: $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{x} = \lim\limits_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x} = \lim\limits_{x\to 0}\frac{\sin x - x\cos x}{x^3} = \lim\limits_{x\to 0}\frac{\cos x - \cos x + x\sin x}{3x^2}$

$$= \lim_{x\to 0}\frac{x\sin x}{3x^2} = \frac{1}{3}$$

(17) Find $\lim\limits_{x\to 0+} x^{\sin x}$.

Solution: $\lim\limits_{x\to 0+} x^{\sin x} = e^{\lim\limits_{x\to 0+}\sin x\ln x}$;

$$\lim_{x\to 0+}\sin x\ln x = \lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0+}(-x) = 0$$

$$\Rightarrow \lim_{x\to 0+} x^{\sin x} = e^0 = 1$$

(18) Find $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$.

Solution: $\lim\limits_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} = \lim\limits_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x} = \lim\limits_{x\to 0}\frac{e\left(e^{\frac{1}{x}\ln(1+x)-1}-1\right)}{x} = e\cdot\lim\limits_{x\to 0}\frac{\frac{1}{x}\ln(1+x)-1}{x}$

$$= e\cdot\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2} = e\cdot\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x} = e\cdot\lim_{x\to 0}\frac{-x}{2x(1+x)} = -\frac{e}{2}$$

(19) Find $\lim\limits_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$.

Solution: $\lim\limits_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) = \lim\limits_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)} = \lim\limits_{x\to 0}\frac{x-\ln(1+x)}{x^2} = \lim\limits_{x\to 0}\frac{1-\frac{1}{1+x}}{2x} = \lim\limits_{x\to 0}\frac{x}{2x(1+x)} = \frac{1}{2}$

(20) Find $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$.

Solution: $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = \lim\limits_{x\to 0}\left(1+\frac{\sin x}{x}-1\right)^{\frac{1}{\frac{\sin x}{x}-1}\cdot\frac{\frac{\sin x}{x}-1}{1-\cos x}} = e^{\lim\limits_{x\to 0}\frac{\frac{\sin x}{x}-1}{1-\cos x}}$

$$\lim_{x\to 0}\frac{\frac{\sin x}{x}-1}{1-\cos x} = \lim_{x\to 0}\frac{\sin x - x}{x\cdot\left(\frac{1}{2}x^2\right)} = \lim_{x\to 0}\frac{\cos x-1}{\frac{3}{2}x^2} = \lim_{x\to 0}\frac{-\frac{1}{2}x^2}{\frac{3}{2}x^2} = -\frac{1}{3}$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}} = e^{-\frac{1}{3}}$$

(21) $f(x+1) = \lim\limits_{n\to\infty}\left(\frac{n+x}{n-2}\right)^n$,find $f(x)$.

Solution: $If\ x \neq -2, \lim\limits_{n\to\infty}\left(\frac{n+x}{n-2}\right)^n = \lim\limits_{n\to\infty}\left(\frac{n-2}{n-2}+\frac{x+2}{n-2}\right)^n$

$$= \lim_{n\to\infty}\left(1+\frac{x+2}{n-2}\right)^{\frac{n-2}{x+2}\cdot\frac{n(x+2)}{n-2}} = e^{\lim\limits_{n\to\infty}\frac{n(x+2)}{n-2}} = e^{x+2}$$

$$\Rightarrow f(x+1) = e^{x+2} \Rightarrow f(x) = e^{x+1};$$

$$If\ x=-2, f(x+1) = 1 = e^{-2+2}; So\ \forall x, f(x) = e^{x+1}$$

(22) Find $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{(x^2+x^3)\ln(1+x)}$.

Solution: $\lim\limits_{x\to 0}\frac{e^x-e^{\sin x}}{(x^2+x^3)\ln(1+x)} = \lim\limits_{x\to 0}\frac{(-e^x)\left(e^{\sin x - x}-1\right)}{x^2(1+x)x} = \lim\limits_{x\to 0}\frac{(-1)(\sin x - x)}{x^3}$

$$= \lim_{x\to 0}\frac{(-1)(\cos x - 1)}{3x^2} = \lim_{x\to 0}\frac{(-1)\left(-\frac{1}{2}x^2\right)}{3x^2} = \frac{1}{6}$$