Name______ Student ID______ School______ Major/Class______ Seat No.______

(DONNOT WRITE YOUR ANSWER IN THIS AREA)

Seal Line …… Seal Line …… Seal Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam Answer Sheet

Linear algebra and analytic geometry **Exam Paper B (2022-2023-1)**

Notice:

1. Make sure that you have filled the form on the left side of seal line.
2. Write your answers on the answer sheet NOT the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	I	II	III	IV	V	VI	Sum
Score							

Answer to Question __I__

Score

(1)

(a) $10\cdot 9\cdot \cdots \cdot 9 = 10\cdot 9^{n-1}$

(b) $\binom{n}{i} 9^{n-i}$

(2)

$$\frac{\dfrac{n!}{n_1!\,n_2!\cdots n_6!}}{6^n}$$

(3)

$$\frac{\binom{7}{5}\cdot 5!}{7^5} = \frac{360}{2401}$$

(4) A_1: in the mathematics class, A_2: in the history class

A_3: in the music class. $P(A_1 \cap A_2 \cap A_3) = 18/200$

$P(A_1) = 137/200$, $P(A_2) = 50/200$, $P(A_3) = 124/200$

$P(A_1 \cap A_2) = 33/200$, $P(A_2 \cap A_3) = 29/200$, $P(A_1 \cap A_3) = 92/200$

$$\Rightarrow P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2 \cup A_3) - P((A_1 \cap A_2) \cup (A_1 \cap A_3))$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_2 \cap A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= 7/8.$$

(5)

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th letter is in the correct envelop,} \\ 0, & \text{otherwise} \end{cases}$$

$$P(X_i = 1) = \frac{1}{n}, \quad \forall i$$

$$E(X) = E(X_1 + \cdots + X_n) = \sum_{i=1}^{n} E(X_i) = 1.$$

Score

Answer to Question __II__

1) A_i: A wins in his/her i-th drawn.

$$P(A \text{ wins}) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \sum_{i=1}^{\infty} \left(\frac{8}{12}\cdot\frac{8}{12}\cdot\frac{8}{12}\right)^{i-1} \frac{4}{12} = \frac{9}{19}$$

$$P(B \text{ wins}) = \sum_{i=1}^{\infty} P(B \text{ wins in his/her } i\text{-th drawn}) = \sum_{i=1}^{\infty} \left(\frac{8}{27}\right)^{i-1} \frac{2}{3}\cdot\frac{1}{3} = \frac{6}{19}$$

$$P(C \text{ wins}) = \sum_{i=1}^{\infty} \left(\frac{8}{27}\right)^{i-1} \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} = \frac{4}{19}$$

2) If A wins, then A must win within 3 drawn. ABC ABC ABC ABC

$$P(A \text{ wins}) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\cdot\frac{4}{9} + \frac{8\cdot 7\cdots 3}{12\cdot 11\cdots 7}\cdot\frac{4}{6} = \frac{7}{15}$$

$$P(B \text{ wins}) = \frac{8}{12}\cdot\frac{4}{11} + \frac{8}{12}\cdot\frac{7}{11}\cdot\frac{6}{10}\cdot\frac{5}{9}\cdot\frac{4}{8} + \frac{8\cdot 7\cdots 2}{12\cdot 11\cdots 6}\cdot\frac{4}{5} = \frac{68}{165}$$

$$P(C \text{ wins}) = 1 - \frac{7}{15} - \frac{68}{165} = \frac{4}{33}$$

Answer to Question __III__

Score

1) $P(\text{head}) = \sum_{i=1}^{5} P(\text{head} \mid i\text{-th coin}) \cdot P(i\text{-th coin})$

$$= \frac{1}{5}\left(0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right) = \frac{1}{2}.$$

$$P(i\text{-th coin} \mid \text{head}) = \frac{P(i\text{th coin}, \text{head})}{P(\text{head})} = \frac{\frac{1}{5} \cdot p_i}{1/2} = \frac{2}{5} p_i,$$

2) A: first head on the fourth toss. Answer: 0, 0.1, 0.2, 0.3, 0.4

$$P(A) = \sum_{i=1}^{5} P(A \mid i\text{-th coin}) \cdot P(i\text{-th coin})$$

$$= \frac{1}{5} \sum_{i=1}^{5} (1-p_i)^3 p_i = \frac{1}{5}\left(0 + \frac{27}{64} + \frac{1}{16} + \frac{3}{64} + 0\right) = \frac{17}{160}$$

$$P(i\text{-th coin} \mid A) = \frac{P(A \mid i\text{-th coin}) P(i\text{-th coin})}{P(A)} = \frac{32}{17}(1-p_i)^3 p_i$$

Answer: 0, 0.5870, 0.3478, 0.0652, 0.

Answer to Question __IV__

Score

1) $F_Y(y) = P(aZ + b \le y) = P\left(Z \le \frac{y-b}{a}\right) \quad (a>0)$

$$= \Phi\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{d}{dy}\Phi\left(\frac{y-b}{a}\right) = \frac{1}{a} \cdot \frac{1}{2\pi} e^{-\frac{(y-b)^2}{2a^2}}$$

$$\Rightarrow Y \sim N(b, a^2)$$

2) Let $X = \frac{Y-\mu}{\sigma}$. Then

$$F_X(x) = P(X \le x) = P\left(\frac{Y-\mu}{\sigma} \le x\right) = P(Y \le \sigma x + \mu)$$

$$= F_Y(\sigma x + \mu)$$

$$\Rightarrow f_X(x) = \frac{d}{dx} F_Y(\sigma x + \mu) = \sigma f_Y(\sigma x + \mu)$$

$$= \sigma \cdot \frac{1}{2\pi \cdot \sigma} e^{-\frac{(\sigma x + \mu - \mu)^2}{2\sigma^2}} = \frac{1}{2\pi} e^{-\frac{x^2}{2}} \quad \Rightarrow X \sim N(0,1).$$

Score

Answer to Question __V__

1). $\frac{1}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{n(n+2)} = \left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$\Rightarrow 1 = \sum_{n\geq 1} P(X=n) = c\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)} = \frac{c}{4} \Rightarrow c = 4.$

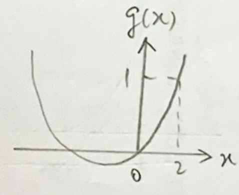
2) $E(X) = \sum_{n\geq 1} nP(X=n) = \sum_{n\geq 1} \frac{4}{(n+1)(n+2)} = 2$

$E(X^2) = \sum_{n\geq 1} n^2 P(X=n) = 4\sum_{n\geq 1} \frac{n}{(n+1)(n+2)} \geq 2\sum_{n\geq 1} \frac{1}{n+2} = \infty.$

VI. 1) $Y = \pi X^2$. $F_Y(y) = P(\pi X^2 \leq y) = \begin{cases} 0, & y<0 \\ P(X \leq \sqrt{\frac{y}{\pi}}), & y\geq 0 \end{cases} = \begin{cases} 0, & y<0 \\ F_X(\sqrt{\frac{y}{\pi}}), & y\geq 0. \end{cases}$

$\Rightarrow f_Y(y) = \frac{d}{dy} F_X(\sqrt{\frac{y}{\pi}}) = f_X(\sqrt{\frac{y}{\pi}}) \cdot \frac{1}{2\sqrt{\pi y}} = \frac{1}{16}\left(\frac{3}{\pi} + \frac{1}{\sqrt{\pi y}}\right), \quad 0<y<4\pi.$

2). $F_X(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 2 \\ \int_0^x \frac{1}{8}(3t+1)\,dt, & 0<t<2 \end{cases} = \begin{cases} 0, & x<0 \\ \frac{1}{8}(3x^2+x), & 0<x<2 \\ 1, & x\geq 2. \end{cases}$



Score

Answer to Question __VI__

Let $g(x) = F_X(x)$ and $Y = g(X)$. Then

$F_Y(y) = P(Y\leq y) = P(g(X) \leq y) = \begin{cases} 0, & y\leq 0 \\ 1, & y\geq 1 \\ P(X \leq g^{-1}(y)), & 0<y<1. \end{cases}$

$\Rightarrow f_Y(y) = \begin{cases} 0, & y\leq 0 \text{ or } y\geq 1 \\ \frac{d}{dy} F_X(g^{-1}(y)) = f_X(g^{-1}(y)) \big/ g'(g^{-1}(y)) = 1, & 0<y<1. \end{cases}$

$\Rightarrow Y = F_X(X)$ is uniform on $(0,1)$.

VII. 1) $F_T(t) = P(T\leq t) = P(\frac{1}{Z} \leq t) = P(Z \geq \frac{1}{t}) = 1 - F_Z(\frac{1}{t}), \quad t>0.$

$\Rightarrow f_T(t) = \frac{d}{dt}(1 - F_Z(\frac{1}{t})) = -f_Z(\frac{1}{t}) \cdot (-\frac{1}{t^2}) = 2\frac{1}{t^2} e^{-2/t}, \quad t>0.$

2). $F_T(t) = \begin{cases} e^{-2/t}, & t>0 \\ 0, & t\leq 0 \end{cases}$. Let $Y = F(T)$.

$F_Y(y) = P(e^{-2/T} \leq y) = P(T \leq -\frac{2}{\ln y}) = F_T(-\frac{2}{\ln y}) = y \quad (0\leq y\leq 1)$

$\Rightarrow f_Y(y) = \begin{cases} 1, & y\in(0,1) \\ 0, & \text{otherwise}. \end{cases}$