Derivative

1. The derivative of implicit function. (The equation in question below can determine the implicit function $y = y(x)$)

(1) $(\cos x)^y = (\sin y)^x$, $y'(x) =$ ____.

Solution: $\ln(\cos x)^y = \ln(\sin y)^x \Rightarrow y \ln\cos x = x \ln\sin y$

$$\Rightarrow y' \ln\cos x + y \cdot \frac{-\sin x}{\cos x} = \ln\sin y + x \cdot \frac{\cos y}{\sin y} \cdot y'$$

$$\Rightarrow y' = \frac{\ln\sin y + y\tan x}{\ln\cos x - x\cot y}$$

(2) $\ln\sqrt{x^2 + y^2} = \arctan\frac{y}{x}$, $dy =$ ____.

Solution: $d\left(\ln\sqrt{x^2 + y^2}\right) = d\left(\arctan\frac{y}{x}\right)$

$$\Rightarrow \frac{1}{2} \cdot \frac{2xdx + 2ydy}{x^2 + y^2} = \frac{-\frac{y}{x^2}dx + \frac{1}{x}dy}{1 + \left(\frac{y}{x}\right)^2}$$

$$\Rightarrow xdx + ydy = -ydx + xdy$$

$$\Rightarrow dy = \frac{x+y}{x-y}dx$$

(3) $ln(x^2 + y + 1) = x^3 y + sin\, x$.

a) Find the tangent equation of the curve $y = y(x) at \left(0, y(0)\right)$.

b) Find $\lim\limits_{n\to\infty} ny(\frac{2}{n})$.

Solution: $a)\ Let\ x = 0 \Rightarrow \ln(y(0) + 1) = 0 \Rightarrow y(0) = 0$

$$[\ln(x^2 + y + 1)]' = (x^3 y + \sin x)'$$

$$\Rightarrow \frac{2x + y'}{x^2 + y + 1} = 3x^2 y + x^3 y' + \cos x$$

$$\Rightarrow y' = \frac{(3x^2 y + \cos x)(x^2 + y + 1) - 2x}{1 - x^3(x^2 + y + 1)}$$

$$\Rightarrow y'(0) = 1 \Rightarrow The\ equation\ is\ y = x$$

$$b)\ \lim_{n\to\infty} ny(\frac{2}{n}) = \lim_{n\to\infty} 2 \cdot \frac{y\left(\frac{2}{n}\right) - y(0)}{\frac{2}{n} - 0} = 2y'(0) = 2$$

(4) $sin(xy) - ln\frac{x+1}{y} = 1$, $y'(0) =$ ____.

Solution: $Let\ x = 0 \Rightarrow 0 + \ln y(0) = 1 \Rightarrow y(0) = e$

$$[\sin(xy) - \ln\frac{x+1}{y}]' = 1' \Rightarrow (y + xy')\cos xy - \frac{1}{x+1} + \frac{y'}{y} = 0$$

$$\Rightarrow y' = \frac{\frac{1}{x+1} - y\cos xy}{x\cos xy + \frac{1}{y}} \Rightarrow y'(0) = e - e^2$$

(5) $e^{2x+y} - cos(xy) = e - 1$, $y'(0) =$ ____.

Solution: $Let\ x = 0 \Rightarrow e^{y(0)} - 1 = e - 1 \Rightarrow y(0) = 1$

$$[e^{2x+y} - \cos(xy)]' = 0 \Rightarrow (2 + y')e^{2x+y} + (y + xy')\sin(xy) = 0$$

$$\Rightarrow y' = -\frac{2e^{2x+y} + y\sin(xy)}{e^{2x+y} + x\sin(xy)} \Rightarrow y'(0) = -2$$

(6) Assume $f(x)$ be continuous, $\exists\delta>0, s.t. \forall x\in(-\delta,\delta): f(1+sin\,x)-3f(1-sin\,x)=8x+o(x)$. $f(x)$ is derivable at $x=1$. Find the tangent equation of the curve $y=y(x) at \left(0,y(0)\right)$.

Solution: $Let\ x=0 \Rightarrow f(1)-3f(1)=0 \Rightarrow f(1)=0$

$Since\ f(1+\sin x)-3f(1-\sin x)=8x+o(x)$

$$\Rightarrow \lim_{x\to 0}\frac{f(1+\sin x)-3f(1-\sin x)}{x}=8$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{f(1+\sin x)-f(1)}{x}+3\cdot\frac{f(1-\sin x)-f(1)}{-x}\right)=8$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{f(1+\sin x)-f(1)}{\sin x}\cdot\frac{\sin x}{x}+3\cdot\frac{f(1-\sin x)-f(1)}{-\sin x}\cdot\frac{\sin x}{x}\right)=8$$

$$\Rightarrow f'(1)+3f'(1)=8$$

$$\Rightarrow f'(1)=2 \Rightarrow The\ equation\ is\ y=2(x-1)$$

(7) $e^y+6xy+x^2-1=0$. $y''(0)=$_____.

Solution: $Let\ x=0 \Rightarrow e^{y(0)}-1=0 \Rightarrow y(0)=0$

$$(e^y+6xy+x^2-1)'=0 \Rightarrow y'e^y+6y+6xy'+2x=0$$

$$\Rightarrow y'=-\frac{2x+6y}{6x+e^y} \Rightarrow y'(0)=0$$

$Since\ \ (y'e^y+6y+6xy'+2x)'=0$

$$\Rightarrow (y')^2e^y+y''e^y+6y'+6y'+6xy''+2=0$$

$$\Rightarrow y''=-\frac{2+12y'+(y')^2e^y}{6x+e^y} \Rightarrow y''(0)=-2$$

2. Derivative of parameter

(8) $\begin{cases} x=ln(1+t^2) \\ y=t-arctan\,t \end{cases}, \left(\frac{d^2y}{dx^2}\right)_{t=1}=$ ______.

Solution: $$\frac{dy}{dx}=\frac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\frac{t^2}{2t}=\frac{t}{2}$$

$$\begin{cases} x=\ln(1+t^2) \\ \frac{dy}{dx}=\frac{t}{2} \end{cases}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{1}{2}}{\frac{2t}{1+t^2}}=\frac{1+t^2}{4t} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{t=1}=\frac{1}{2}$$

(9) $\begin{cases} x=ln(sin\,t) \\ y=cos\,t+t\,sin\,t \end{cases}, \frac{dy}{dx}=$ ___, $\frac{d^2y}{dx^2}=$ ______.

Solution: $$\frac{dy}{dx}=\frac{-\sin t+\sin t+t\cos t}{\frac{\cos t}{\sin t}}=t\sin t$$

$$\begin{cases} x=\ln(\sin t) \\ \frac{dy}{dx}=t\sin t \end{cases}$$

$$\frac{d^2y}{dx^2}=\frac{\sin t+t\cos t}{\frac{\cos t}{\sin t}}=(t+\tan t)\sin t$$

(10) $\begin{cases} x=t-ln(1+t) \\ y=t^2+t^3 \end{cases}, \frac{d^2y}{dx^2}=$ ____.

Solution: $$\frac{dy}{dx}=\frac{2t+3t^2}{1-\frac{1}{1+t}}=(2+3t)(1+t)=3t^2+5t+2$$

$$\begin{cases} x = t - \ln(1+t) \\ \frac{dy}{dx} = 3t^2 + 5t + 2 \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{6t+5}{1-\frac{1}{1+t}} = \frac{(1+t)(6t+5)}{t} = 6t + 11 + \frac{5}{t}$$

(11) $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$ $\quad \frac{dy}{dx} =$ ______, $\frac{d^2y}{dx^2} =$ ____.

Solution: $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$

$$\begin{cases} x = t - \sin t \\ \frac{dy}{dx} = \frac{\sin t}{1-\cos t} \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{\cos t(1-\cos t)-\sin^2 t}{(1-\cos t)^2}}{1-\cos t} = \frac{\cos t - 1}{(1-\cos t)^3} = -\frac{1}{(1-\cos t)^2}$$

3. Derivative of higher order

(12) $f(x) = arctan\, x\,, f^{(2019)}(0) =$ ______.

Solution: $y = \arctan x \Rightarrow y' = \frac{1}{1+x^2} \Rightarrow y'(0) = 1$

$$y' = \frac{1}{1+x^2} \Rightarrow (1+x^2)y' = 1 \Rightarrow [(1+x^2)y']^{(n)} = 1^{(n)} = 0$$

$$\Rightarrow (1+x^2)y^{(n+1)} + 2nxy^{(n)} + n(n-1)y^{(n-1)} = 0$$
$$Let\ x = 0 \Rightarrow y^{(n+1)}(0) + n(n-1)y^{(n-1)}(0) = 0$$
$$\Rightarrow y^{(n+1)}(0) = -n(n-1)y^{(n-1)}(0)$$
$$\Rightarrow y^{(2n+1)}(0) = -(2n)(2n-1)y^{(2n-1)}(0)$$
$$= (-1)^2(2n)(2n-1)(2n-2)(2n-3)y^{(2n-3)}(0)$$
$$= \cdots\cdots$$
$$= (-1)^n(2n)(2n-1)(2n-2)(2n-3)\cdots\cdot 2\cdot 1y'(0)$$
$$= (-1)^n \cdot (2n)!$$
$$\Rightarrow f^{(2019)}(0) = (-1)^{1009}\cdot(2018)! = -2018!$$

(13) $f(x) = x^2\, ln(1+x)\,, f^{(2018)}(0) =$ ______.

Solution: $We\ have\ [\ln(1+x)]^{(n)} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$

$$f^{(n)}(x) = [x^2\ln(1+x)]^{(n)} = x^2[\ln(1+x)]^{(n)} + 2nx[\ln(1+x)]^{(n-1)} + n(n-1)[\ln(1+x)]^{(n-2)}$$

$$\Rightarrow f^{(2018)}(0) = 2018\cdot 2017\cdot[\ln(1+x)]^{(2016)}$$

$$= 2018\cdot 2017\cdot(-1)^{2015}\frac{2015!}{(1+0)^{2016}}$$

$$= -2018\cdot 2017\cdot 2015!$$

$$= -\frac{2018!}{2016}$$

(14) $f(x) = x^2\, sin\, x\, cos\, x\,, f^{(2015)}(0) =$ ______.

Solution: $f(x) = x^2\sin x\cos x = \frac{1}{2}x^2\sin 2x$

$$We\ have\ (\sin 2x)^{(n)} = 2^n\sin\left(2x + \frac{n}{2}\pi\right)$$

$$f^{(n)}(x) = \frac{1}{2}(x^2\sin 2x)^{(n)} = \frac{1}{2}[x^2(\sin 2x)^{(n)} +$$

$$2nx(\sin 2x)^{(n-1)} + n(n-1)(\sin 2x)^{(n-2)}]$$

$$f^{(2015)}(0) = \frac{1}{2} \cdot 2015 \cdot 2014 \cdot (\sin 2x)^{(2013)}$$

$$= \frac{1}{2} \cdot 2015 \cdot 2014 \cdot 2^{2013} \sin\left(0 + \frac{2013}{2}\pi\right)$$

$$= 2^{2012} \cdot 2014 \cdot 2015$$

(15) $y = cos\,2x\,, y^{(n)}(x) =$______.

Solution: $y^{(n)}(x) = 2^n \cos(2x + \frac{n}{2}\pi)$

(16) $f(x) = e^{2x}\,sin\,x\,, f^{(4)}(0) =$ ______.

Solution: $f^{(n)}(x) = \sum_{k=0}^{n} C_n^k\,(e^{2x})^{(n-k)}(\sin x)^{(k)}$

$$= \sum_{k=0}^{n} C_n^k\, 2^{n-k} e^{2x} \sin\left(x + \frac{k}{2}\pi\right)$$

$$\Rightarrow f^{(4)}(0) = \sum_{k=0}^{4} C_4^k\, 2^{4-k} \sin\left(0 + \frac{k}{2}\pi\right) = 24$$

(17) $y = ln\,x$, find $y^{(n)}(x)$.

Solution: $y^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$

4. Comprehensive application

(18) Assume $f''(x)$ exist in the neighborhood of $x = 0$, and $\lim\limits_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{\frac{1}{x}} = e^3$. find $f(0), f'(0), f''(0)$.

Solution: $e^3 = \lim\limits_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{\frac{1}{x}} = \lim\limits_{x\to 0} e^{\frac{1}{x}\ln\left(1+x+\frac{f(x)}{x}\right)}$

$$\Rightarrow \lim_{x\to 0}\frac{\ln\left(1+x+\frac{f(x)}{x}\right)}{x} = 3 \Rightarrow \lim_{x\to 0}\frac{f(x)}{x} = 0 \Rightarrow \lim_{x\to 0} f(x) = 0$$

$$Since\ f\ is\ continuous \Rightarrow f(0) = 0$$

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = 0$$

$$3 = \lim_{x\to 0}\frac{\ln\left(1+x+\frac{f(x)}{x}\right)}{x} = \lim_{x\to 0}\frac{x+\frac{f(x)}{x}}{x} \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$$

$$According\ to\ L'Hospital, since\ f''(0)\ exist,$$

$$we\ have\ \lim_{x\to 0}\frac{f(x)}{x^2} = \lim_{x\to 0}\frac{f'(x)}{2x} \Rightarrow \lim_{x\to 0}\frac{f'(x)}{x} = 4$$

$$\Rightarrow f''(0) = \lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0}\frac{f'(x)}{x} = 4$$

(19) $f(x) = \begin{cases}\frac{\varphi(x)-cos\,x}{x}, x \neq 0\\ a, \quad\quad x = 0\end{cases}$ is continuous at $x = 0$, $\varphi''(x)$ exist in the neighborhood of $x = 0$. $\varphi(0) = \varphi'(0) = \varphi''(0) = 1$, find $a\,, f'(0)$.

Solution: $a = f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\frac{\varphi(x)-\cos x}{x} = \lim\limits_{x\to 0}(\,\varphi'(x) + \sin x) = \varphi'(0) = 1$

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{\frac{\varphi(x)-\cos x}{x}-1}{x} = \lim_{x\to 0}\frac{\varphi(x)-\cos x-x}{x^2} = \lim_{x\to 0}\frac{\varphi'(x)+\sin x-1}{2x}$$

$$= \lim_{x\to 0}\frac{\varphi'(x)-\varphi'(0)}{2x} + \frac{1}{2} = \frac{1}{2}\varphi''(0) + \frac{1}{2} = 1$$

(20) $f(x) = e^{x^2}, f[\varphi(x)] = 1 - 3x,\ \varphi(x) \geq 0,\ \varphi(x) =$______.

Solution: $1 - 3x = f[\varphi(x)] = e^{\varphi^2(x)}$

$\Rightarrow \varphi^2(x) = \ln(1 - 3x)$

$Since\ \varphi(x) \geq 0 \Rightarrow \varphi(x) = \sqrt{\ln(1 - 3x)}$

(21) Find the extreme values of following functions.

1) $y = 2x^3 - 3x^2 - 12x + 1$.

2) $y = x + \sin x$.

3) $y = 2e^x + e^{-x}$.

4) $y = \frac{1+3x}{\sqrt{4+5x^2}}$.

Solution:

1) $y' = 6x^2 - 6x - 12. Let\ y' = 0 \Rightarrow x = -1,2$

According to the sign of the first derivative, we have f monotonically increasing in the $(-\infty, -1]$ and $[2, +\infty)$, monotonically decreasing in the $[-1,2]$.

$\Rightarrow x = -1$ is local maximum point, $x = 2$ is local minimum point.

2) $y' = 1 + \cos x \geq 0. f$ monotonically increasing in the $(-\infty, +\infty)$. There is no extreme point.

3) $y' = 2e^x - e^{-x}. Let\ y' = 0 \Rightarrow x = -\frac{1}{2}\ln 2$

According to the sign of the first derivative, we have f monotonically increasing in the $\left[-\frac{1}{2}\ln 2, +\infty\right)$, monotonically decreasing in the $(-\infty, -\frac{1}{2}\ln 2]$.

$\Rightarrow x = -\frac{1}{2}\ln 2$ is local minimum point.

4) $y' = \frac{12 - 5x}{(4 + 5x^2)^{\frac{3}{2}}}. Let\ y' = 0 \Rightarrow x = \frac{12}{5}$

According to the sign of the first derivative, we have f monotonically increasing in the$(-\infty, \frac{12}{5}]$, monotonically decreasing in the $\left[\frac{12}{5}, +\infty\right)$.

$\Rightarrow x = \frac{12}{5}$ is local maximum point.

(22) Prove inequality:

1) $3 - \frac{1}{x} < 2\sqrt{x}, x > 1$.

Proof: $Let\ f(x) = 2\sqrt{x} - \left(3 - \frac{1}{x}\right) \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} > 0, x > 1$

$\Rightarrow$ f strictly monotonically increasing in the $[1, +\infty)$

$\Rightarrow f(x) > 0 \Rightarrow 3 - \frac{1}{x} < 2\sqrt{x}, x > 1.$

2) $\tan x + 2\sin x > 3x, x \in \left(0, \frac{\pi}{2}\right)$.

Proof: $Let\ f(x) = \tan x + 2\sin x - 3x \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x + 2\cos x - 3 \geq 3\sqrt[3]{\sec^2 x \cos x \cos x} - 3 = 0, \forall x \in [0, \frac{\pi}{2})$

$\Rightarrow$ f strictly monotonically increasing in the $\left(0, \frac{\pi}{2}\right)$.

$\Rightarrow f(x) > 0 \Rightarrow \tan x + 2\sin x > 3x, x \in \left(0, \frac{\pi}{2}\right)$.

(23) Find the rectangle with the largest area connected to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with sides parallel to the axis of the ellipse.

Solution: Let the length and width of the rectangle be $2x$ and $2y$ respectively

$\Rightarrow$The area of the rectangle $S = 4xy = \frac{4bx\sqrt{a^2-x^2}}{a}$

$\Rightarrow \frac{dS}{dx} = \frac{4b}{a}\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2-x^2}}\right) = 0$

$\Rightarrow x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}$

So when the sides of the rectangle are $\sqrt{2}a$ and $\sqrt{2}b$, the inner rectangle has the largest area.

5. Differential mean value theorem

(24) Assume $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).

Prove: $\exists\, c \in (a, b), s.t\ 2c[f(b) - f(a)] = (b^2 - a^2)f'(c)$.

Proof: $\Leftarrow \exists\, c \in (a, b), s.t\ 2c[f(b) - f(a)] = (b^2 - a^2)f'(c)$

$\Leftarrow \exists\, c \in (a, b), s.t\ \frac{f(b)-f(a)}{b^2-a^2} = \frac{f'(c)}{2c}$

$$By\ \ Cauchy\ mean\ value\ theorem \Rightarrow\ \exists\, c \in (a, b), s.t\ \frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(c)}{2c}$$

(25) Assume $a, b > 0$, prove: $\exists c \in (a, b), s.t\ \ ae^b - be^a = (1 - c)e^c(a - b)$.

Proof: $\Leftarrow \exists c \in (a, b), s.t\ \ ae^b - be^a = (1 - c)e^c(a - b)$

$\Leftarrow \exists c \in (a, b), s.t\ \ \frac{ae^b-be^a}{a-b} = (1 - c)e^c$

$\Leftarrow \exists c \in (a, b), s.t\ \ \frac{\frac{ae^b-be^a}{ab}}{\frac{a-b}{ab}} = (1 - c)e^c$

$\Leftarrow \exists c \in (a, b), s.t\ \ \frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}} = (1 - c)e^c$

$$By\ \ Cauchy\ mean\ value\ theorem \Rightarrow\ \exists\, c \in (a, b), s.t\ \frac{\frac{e^b}{b} - \frac{e^a}{a}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{e^c(c-1)}{c^2}}{-\frac{1}{c^2}} = e^c(1 - c)$$