Indefinite integral III

1. Find the following indefinite integral

(1) $\int \frac{dx}{(x-1)(x+1)^2}$

Solution: $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\Rightarrow 1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

$Let \quad x = 1, \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$

$x = -1, \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$

$Consider\ x^2, \Rightarrow 0 = A + B \Rightarrow B = -\frac{1}{4}$

$\Rightarrow \int \frac{dx}{(x-1)(x+1)^2} = \int \left(\frac{\frac{1}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-\frac{1}{2}}{(x+1)^2} \right) dx$

$= \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{2(x+1)} + C$

(2) $\int \frac{2x+3}{(x^2-1)(x^2+1)} dx$

Solution: $\frac{2x+3}{(x^2-1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$\Rightarrow 2x + 3 = A(x+1)(x^2+1) + B(x-1)(x^2+1) +$

$(Cx + D)(x-1)(x+1)$

$Let \quad x = 1, \Rightarrow 5 = 4A \Rightarrow A = \frac{5}{4}$

$x = -1, \Rightarrow 1 = -4B \Rightarrow B = -\frac{1}{4}$

$x^2 = -1 (x = i) \Rightarrow 2i + 3 = -2(Ci + D)$

$\Rightarrow \begin{cases} -2C = 2 \\ -2D = 3 \end{cases} \Rightarrow \begin{cases} C = -1 \\ D = -\frac{3}{2} \end{cases}$

$\Rightarrow \int \frac{2x+3}{(x^2-1)(x^2+1)} dx$

$= \int \left(\frac{\frac{5}{4}}{x-1} + \frac{-\frac{1}{4}}{x+1} + \frac{-x-\frac{3}{2}}{x^2+1} \right) dx$

$$= \frac{5}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \int \frac{x}{x^2+1}dx - \int \frac{\frac{3}{2}}{x^2+1}dx$$
$$= \frac{5}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\ln(x^2+1) - \frac{3}{2}\arctan x + C$$

(3) $\int \frac{xdx}{(x+1)(x+2)^2(x+3)^3}$

Solution: $\frac{x}{(x+1)(x+2)^2(x+3)^3} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{x+3} + \frac{E}{(x+3)^2} + \frac{F}{(x+3)^3}$

$$\Rightarrow x = A(x+2)^2(x+3)^3 + B(x+1)(x+2)(x+3)^3 +$$
$$C(x+1)(x+3)^3 + D(x+1)(x+2)^2(x+3)^2 +$$
$$E(x+1)(x+2)^2(x+3) + F(x+1)(x+2)^2$$

$Let\quad x = -1, \Rightarrow -1 = 8A \Rightarrow A = -\frac{1}{8}$

$x = -2, \Rightarrow -2 = -C \Rightarrow C = 2$

$x = -3, \Rightarrow -3 = -2F \Rightarrow F = \frac{3}{2}$

$x = 0, \Rightarrow 0 = -\frac{27}{2} + 54B + 54 + 36D + 12E + 6$

$Consider\ x^5, \Rightarrow 0 = -\frac{1}{8} + B + D$

$Consider\ x^4, \Rightarrow 0 = -\frac{13}{8} + 12B + 2 + 11D + E$

$$\Rightarrow \begin{cases} 0 = -\frac{27}{2} + 54B + 54 + 36D + 12E + 6 \\ 0 = -\frac{1}{8} + B + D \\ 0 = -\frac{13}{8} + 12B + 2 + 11D + E \end{cases}$$

$$\Rightarrow \begin{cases} B = -5 \\ D = \frac{41}{8} \\ E = \frac{13}{4} \end{cases}$$

$$\Rightarrow \int \frac{xdx}{(x+1)(x+2)^2(x+3)^3}$$

$$= \int \left(\frac{-\frac{1}{8}}{x+1} + \frac{-5}{x+2} + \frac{2}{(x+2)^2} + \frac{\frac{41}{8}}{x+3} + \frac{\frac{13}{4}}{(x+3)^2} + \frac{\frac{3}{2}}{(x+3)^3}\right)dx$$

$$= -\frac{1}{8}\ln|x+1| - 5\ln|x+2| - \frac{2}{x+2} + \frac{41}{8}\ln|x+3| -$$

$\frac{13}{4(x+3)} - \frac{3}{4(x+3)^2} + C$

(4) $\int \frac{dx}{(x^2+4x+4)(x^2+4x+5)^2}$

Solution: $\int \frac{dx}{(x^2+4x+4)(x^2+4x+5)^2}$

$$= \int (\frac{1}{(x^2+4x+4)(x^2+4x+5)} - \frac{1}{(x^2+4x+5)^2})dx$$

$$= \int (\frac{1}{x^2+4x+4} - \frac{1}{x^2+4x+5} - \frac{1}{(x^2+4x+5)^2})dx$$

$$= \int (\frac{1}{(x+2)^2} - \frac{1}{(x+2)^2+1} - \frac{1}{(x^2+4x+5)^2})dx$$

$$= -\frac{1}{x+2} - \arctan(x+2) - \frac{1}{2}\arctan(x+2) - \frac{x+2}{2(x^2+4x+5)} + C$$

$$= -\frac{1}{x+2} - \frac{3}{2}\arctan(x+2) - \frac{x+2}{2(x^2+4x+5)} + C$$

$$(\int \frac{1}{(x^2+4x+5)^2} dx = \int \frac{d(x+2)}{[(x+2)^2+1]^2} \quad (u = x+2)$$

$$= \int \frac{du}{(u^2+1)^2}$$

$$= \int \frac{u^2+1-u^2}{(u^2+1)^2} du$$

$$= \int \frac{1}{u^2+1} du + \int \frac{1}{2} u \, d\left(\frac{1}{u^2+1}\right)$$

$$= \arctan u + \frac{1}{2}\frac{u}{u^2+1} - \frac{1}{2}\int \frac{1}{u^2+1} du$$

$$= \frac{1}{2}\arctan u + \frac{u}{2(u^2+1)} + C$$

$$= \frac{1}{2}\arctan(x+2) + \frac{x+2}{2(x^2+4x+5)} + C)$$

(5) $\int \frac{3}{x^3+1} dx$

Solution: $\int \frac{3}{x^3+1} dx = \int \frac{3}{(x+1)(x^2-x+1)} dx$

$$\frac{3}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\Rightarrow 3 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$Let \quad x = -1, \Rightarrow 3 = 3A \Rightarrow A = 1$$

$$Consider\ x^2, \Rightarrow 0 = 1 + B \Rightarrow B = -1$$

$$Consider\ constant\ term, \Rightarrow 3 = 1 + C \Rightarrow C = 2$$

$$\Rightarrow \int \frac{3}{x^3+1} dx$$

$$= \int (\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}) dx$$

$$= \ln|x+1| + \int \frac{-x+\frac{1}{2}}{x^2-x+1} dx + \int \frac{\frac{3}{2}}{x^2-x+1} dx$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \int \frac{\frac{3}{2} d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \quad (u = x - \frac{1}{2})$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \int \frac{\frac{3}{2} \cdot \frac{\sqrt{3}}{2} d\left(\frac{2}{\sqrt{3}}u\right)}{\frac{3}{4}[1+\left(\frac{2}{\sqrt{3}}u\right)^2]}$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}u + C$$

$$= \ln|x+1| - \frac{1}{2}\ln(x^2-x+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}(x - \frac{1}{2}) + C$$

(6) $\int \frac{dx}{x^4+x^2+1}$

Solution: $\int \frac{dx}{x^4+x^2+1} = \int \frac{1}{(x^2+1)^2-x^2} dx = \int \frac{1}{(x^2+1-x)(x^2+1+x)} dx$

$$\frac{1}{(x^2+1-x)(x^2+1+x)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1}$$

$$\Rightarrow 1 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1)$$

$$\Rightarrow \begin{cases} A+C=0 \\ -A+B+C+D=0 \\ A-B+C+D=0 \\ B+D=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{2} \\ C=-\frac{1}{2} \\ D=\frac{1}{2} \end{cases}$$

$$\Rightarrow \int \frac{dx}{x^4+x^2+1}$$

$$= \int \left(\frac{\frac{1}{2}x+\frac{1}{2}}{x^2+x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2-x+1}\right) dx$$

$$= \frac{1}{2}\int \left(\frac{x+\frac{1}{2}}{x^2+x+1} + \frac{\frac{1}{2}}{x^2+x+1} + \frac{-x+\frac{1}{2}}{x^2-x+1} + \frac{\frac{1}{2}}{x^2-x+1}\right) dx$$

$$= \frac{1}{4}[\ln(x^2+x+1) - \ln(x^2-x+1) + \int (\frac{1}{x^2+x+1} + \frac{1}{x^2-x+1}) dx]$$

$$= \frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\left(\arctan\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + \arctan\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + C$$

$$\int\left(\frac{1}{x^2+x+1}+\frac{1}{x^2-x+1}\right)dx$$

$$=\int\left[\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right]dx$$

$$=\int\frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}+\int\frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\quad\left(u=x+\frac{1}{2},v=x-\frac{1}{2}\right)$$

$$=\int\frac{\frac{\sqrt{3}}{2}d\left(\frac{2}{\sqrt{3}}u\right)}{\frac{3}{4}\left[1+\left(\frac{2}{\sqrt{3}}u\right)^2\right]}+\int\frac{\frac{\sqrt{3}}{2}d\left(\frac{2}{\sqrt{3}}v\right)}{\frac{3}{4}\left[1+\left(\frac{2}{\sqrt{3}}v\right)^2\right]}$$

$$=\frac{2}{\sqrt{3}}\left(\arctan\frac{2}{\sqrt{3}}u+\arctan\frac{2}{\sqrt{3}}v\right)+C$$

$$=\frac{2}{\sqrt{3}}\left[\arctan\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)+\arctan\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right]+C$$

(7) $\int\frac{x^4+5x+4}{x^2+5x+4}dx$

Solution: $\int\frac{x^4+5x+4}{x^2+5x+4}dx=\int\left(x^2-5x+21-80\cdot\frac{x+1}{x^2+5x+4}\right)dx$

$$=\frac{1}{3}x^3-\frac{5}{2}x^2+21x-80\ln|x+4|+C$$

(8) $\int\frac{x^3+1}{x^3+5x+6}dx$

Solution: $\int\frac{x^3+1}{x^3+5x+6}dx=\int\left(1-\frac{5x+5}{(x+1)(x^2-x+6)}\right)dx$

$$=x-\int\frac{5}{x^2-x+6}dx$$

$$=x-\int\frac{5}{\left(x-\frac{1}{2}\right)^2+\frac{23}{4}}dx$$

$$=x-\int\frac{5\cdot\frac{\sqrt{23}}{2}d\left(\frac{2}{\sqrt{23}}\left(x-\frac{1}{2}\right)\right)}{\frac{23}{4}\left[1+\left(\frac{2}{\sqrt{23}}\left(x-\frac{1}{2}\right)\right)^2\right]}$$

$$=x-\frac{10}{\sqrt{23}}\arctan\frac{2}{\sqrt{23}}\left(x-\frac{1}{2}\right)+C$$

(9) $\int\frac{x^2+2}{1-x^4}dx$

Solution: $\int\frac{x^2+2}{1-x^4}dx=\int\frac{x^2}{1-x^4}dx+\int\frac{2}{1-x^4}dx$

$$=\frac{1}{2}\int\left(\frac{1}{1-x^2}-\frac{1}{1+x^2}\right)dx+\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx$$

$$=\int\left(\frac{3}{2}\cdot\frac{1}{1-x^2}+\frac{1}{2}\cdot\frac{1}{1+x^2}\right)dx$$

$$= \int \frac{3}{4}\left(\frac{1}{1-x} + \frac{1}{1+x}\right) dx + \frac{1}{2}\arctan x$$

$$= \frac{3}{4}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}\arctan x + C$$

(10) $\int \frac{dx}{x^4+1}$

Solution: $\int \frac{dx}{x^4+1} = \int \frac{dx}{(x^2+1)^2-(\sqrt{2}x)^2} = \int \frac{dx}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}$

$$\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)} = \frac{Ax+B}{x^2+\sqrt{2}x+1} + \frac{Cx+D}{x^2-\sqrt{2}x+1}$$

$$\Rightarrow 1 = (Ax+B)\left(x^2-\sqrt{2}x+1\right) + (Cx+D)(x^2+\sqrt{2}x+1)$$

$$\Rightarrow \begin{cases} A+C=0 \\ -\sqrt{2}A+B+\sqrt{2}C+D=0 \\ A-\sqrt{2}B+C+\sqrt{2}D=0 \\ B+D=1 \end{cases} \Rightarrow \begin{cases} A=\frac{\sqrt{2}}{4} \\ B=\frac{1}{2} \\ C=-\frac{\sqrt{2}}{4} \\ D=\frac{1}{2} \end{cases}$$

$$\Rightarrow \int \frac{dx}{x^4+1}$$

$$= \int\left(\frac{\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{\sqrt{2}}{4}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}\right)dx$$

$$= \frac{1}{4}\int\left(\frac{\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2+\sqrt{2}x+1} + \frac{-\sqrt{2}x+1}{x^2-\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1}\right)dx$$

$$= \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right| +$$

$$\frac{\sqrt{2}}{4}\left[\arctan\sqrt{2}\left(x+\frac{\sqrt{2}}{2}\right) + \arctan\sqrt{2}\left(x-\frac{\sqrt{2}}{2}\right)\right] + C$$

2. Find the following indefinite integral

(1) $\int \frac{x}{\sqrt{2+4x}}dx$

Solution: $\int \frac{x}{\sqrt{2+4x}}dx \quad \left(t=\sqrt{2+4x}, x=\frac{t^2-2}{4}, dx=\frac{1}{2}tdt\right)$

$$= \int \frac{\frac{t^2-2}{4}}{t}\frac{1}{2}tdt$$

$$= \frac{1}{8}\int (t^2-2)\,dt$$

$$= \frac{1}{24}t^3 - \frac{1}{4}t + C$$

$$= \frac{1}{24}\sqrt{(2+4x)^3} - \frac{1}{4}\sqrt{2+4x} + C$$

$$= \frac{1}{6}(x-1)\sqrt{2+4x} + C$$

(2) $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution: $\int \frac{dx}{\sqrt{(x-a)(b-x)}} \quad (b > a)$

$$= \int \frac{1}{x-a} \cdot \sqrt{\frac{x-a}{b-x}}\, dx \quad \left(t = \sqrt{\frac{x-a}{b-x}}, x = \frac{bt^2+a}{t^2+1}, dx = \frac{2t(b-a)}{(t^2+1)^2} dt\right.$$

$$= \int \frac{1}{\frac{bt^2+a}{t^2+1} - a} \cdot t \frac{2t(b-a)}{(t^2+1)^2} dt$$

$$= \int \frac{2}{t^2+1} dt$$

$$= 2\arctan t + C$$

$$= 2\arctan\sqrt{\frac{x-a}{b-x}} + C$$

$$If\ a > b, \int \frac{dx}{\sqrt{(x-a)(b-x)}} = \int \frac{dx}{\sqrt{(a-x)(x-b)}} = 2\arctan\sqrt{\frac{x-b}{a-x}} + C$$

(3) $\int \frac{x^2}{\sqrt{1+x-x^2}} dx$

Solution: $\int \frac{x^2}{\sqrt{1+x-x^2}} dx$

$$= \int \frac{x^2-x-1}{\sqrt{1+x-x^2}} dx + \int \frac{\frac{1}{2}(2x-1)}{\sqrt{1+x-x^2}} dx + \int \frac{\frac{3}{2}}{\sqrt{1+x-x^2}} dx$$

$$= \int -\sqrt{1+x-x^2}\, dx + \int -\frac{1}{2}\frac{d(1+x-x^2)}{\sqrt{1+x-x^2}} + \int \frac{\frac{3}{2}}{\sqrt{1+x-x^2}} dx$$

$$= -\frac{5}{8}\arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) - \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} -$$

$$\sqrt{1+x-x^2} + \frac{3}{2}\arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) + C$$

$$= \frac{7}{8}\arcsin\frac{2x-1}{\sqrt{5}} - \frac{1}{4}(2x+3)\sqrt{1+x-x^2} + C$$

Supplement: $\int -\sqrt{1+x-x^2}$

$$= \int -\sqrt{\frac{5}{4} - \left(x-\frac{1}{2}\right)^2}\, dx \quad \left(x - \frac{1}{2} = \frac{\sqrt{5}}{2}\sin t, dx = \frac{\sqrt{5}}{2}\cos t\, dt\right)$$

$$= \int -\frac{\sqrt{5}}{2}\cos t \cdot \frac{\sqrt{5}}{2}\cos t\, dt$$
$$= -\frac{5}{4}\int \cos^2 t\, dt$$
$$= -\frac{5}{8}\int (1+\cos 2t)\, dt$$
$$= -\frac{5}{8}t - \frac{5}{16}\sin 2t + C$$
$$\left(\sin t = \frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right), \cos t = \frac{2}{\sqrt{5}}\sqrt{1+x-x^2}, t = \arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right)\right)$$
$$= -\frac{5}{8}\arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) - \frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} + C$$
$$\int \frac{\frac{3}{2}}{\sqrt{1+x-x^2}}\, dx = \int \frac{\frac{3}{2}}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}}\, d\left(x-\frac{1}{2}\right) \quad \left(u = x-\frac{1}{2}\right)$$
$$= \int \frac{\frac{3}{2}}{\sqrt{\frac{5}{4}-u^2}}\, du$$
$$= \int \frac{\frac{\sqrt{5}}{2}\cdot\frac{3}{2} d\left(\frac{2}{\sqrt{5}}u\right)}{\frac{\sqrt{5}}{2}\sqrt{1-\left(\frac{2}{\sqrt{5}}u\right)^2}}$$
$$= \frac{3}{2}\arcsin\frac{2}{\sqrt{5}}u + C$$
$$= \frac{3}{2}\arcsin\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) + C$$

(4) $\int \frac{x^2+1}{x\sqrt{x^4+1}}\, dx$

Solution: $\int \frac{x^2+1}{x\sqrt{x^4+1}}\, dx \quad (x>0)$

$$= \int \frac{x^2+1}{x^2\sqrt{x^2+x^{-2}}}\, dx$$
$$= \int \frac{1+x^{-2}}{\sqrt{x^2+x^{-2}}}\, dx$$
$$= \int \frac{d(x-x^{-1})}{\sqrt{(x-x^{-1})^2+2}} \quad (t = x-x^{-1})$$
$$= \int \frac{dt}{\sqrt{t^2+2}} \quad \left(t = \sqrt{2}\tan u, dt = \sqrt{2}\sec^2 u\, du\right)$$
$$= \int \frac{\sqrt{2}\sec^2 u du}{\sqrt{2}\sec u}$$
$$= \int \sec u\, du$$
$$= \ln|\sec u + \tan u| + C \quad \left(\tan u = \frac{x-x^{-1}}{\sqrt{2}}, \sec u = \frac{\sqrt{x^2+x^{-2}}}{\sqrt{2}}\right)$$

$$= \ln\left|\frac{\sqrt{x^2+x^{-2}}}{\sqrt{2}} + \frac{x-x^{-1}}{\sqrt{2}}\right| + C$$

$$= \ln\left|\frac{\sqrt{x^4+1}+x^2-1}{x}\right| + C$$

$$Check\ \left(\ln\left|\frac{\sqrt{x^4+1}+x^2-1}{x}\right|\right)' = \frac{x^2+1}{x\sqrt{x^4+1}}$$

$$\Rightarrow \int \frac{x^2+1}{x\sqrt{x^4+1}}dx = \ln\left|\frac{\sqrt{x^4+1}+x^2-1}{x}\right| + C$$

(5) $\int \frac{dx}{\sqrt{x(x+1)}}$

Solution: $\int \frac{dx}{\sqrt{x(x+1)}} \quad (x > 0)$

$$= \int \frac{1}{x}\sqrt{\frac{x}{x+1}}dx \quad \left(t = \sqrt{\frac{x}{x+1}}, x = -\frac{t^2}{t^2-1}, dx = \frac{2t}{(t^2-1)^2}dt\right)$$

$$= \int \frac{1}{-\frac{t^2}{t^2-1}} \cdot t \cdot \frac{2t}{(t^2-1)^2}dt$$

$$= \int -\frac{2}{t^2-1}dt$$

$$= \int \left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt$$

$$= \ln\left|\frac{t+1}{t-1}\right| + C$$

$$= \ln\left|\frac{\sqrt{\frac{x}{x+1}}+1}{\sqrt{\frac{x}{x+1}}-1}\right| + C$$

$$= 2\ln\left|\sqrt{x+1} + \sqrt{x}\right| + C$$

$$Check\ \left(2\ln\left|\sqrt{x+1} + \sqrt{x}\right|\right)' = \frac{1}{\sqrt{x(x+1)}}$$

$$\Rightarrow \int \frac{dx}{\sqrt{x(x+1)}} = 2\ln\left|\sqrt{x+1} + \sqrt{x}\right| + C$$

(6) $\int \frac{dx}{x^4\sqrt{1+x^2}}$

Solution: $\int \frac{dx}{x^4\sqrt{1+x^2}} \quad (x = \tan t, dx = \sec^2 t\, dt)$

$$= \int \frac{\sec^2 t dt}{\tan^4 t \cdot \sec t}$$

$$= \int \frac{\cos^3 t}{\sin^4 t}dt$$

$$= \int \frac{1-\sin^2 t}{\sin^4 t} d(\sin t) \quad (u = \sin t)$$

$$= \int \frac{1-u^2}{u^4} du$$

$$= -\frac{1}{3u^3} + \frac{1}{u} + C \quad \left(u = \frac{x}{\sqrt{1+x^2}}\right)$$

$$= -\frac{\sqrt{(1+x^2)^3}}{3x^3} + \frac{\sqrt{1+x^2}}{x} + C$$

$$= \frac{2x^2-1}{3x^3}\sqrt{1+x^2} + C$$

(7) $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

Solution: $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}} \quad (x = t^4, dx = 4t^3 dt)$

$$= \int \frac{4t^3 dt}{t^2+t}$$

$$= 4\int \left(t - 1 + \frac{1}{t+1}\right) dt$$

$$= 2t^2 - 4t + 4\ln|t+1| + C$$

$$= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(\sqrt[4]{x} + 1) + C$$

(8) $\int \frac{dx}{x\sqrt[4]{1+x^4}}$

Solution: $\int \frac{dx}{x\sqrt[4]{1+x^4}} = \int \frac{x^3 dx}{x^4 \cdot \sqrt[4]{1+x^4}}$

$$= \int \frac{\frac{1}{4}d(1+x^4)}{x^4 \cdot \sqrt[4]{1+x^4}} \quad (u = 1 + x^4)$$

$$= \frac{1}{4}\int \frac{du}{(u-1)\sqrt[4]{u}} \quad (u = t^4, du = 4t^3 dt)$$

$$= \frac{1}{4}\int \frac{4t^3 dt}{(t^4-1)t}$$

$$= \int \frac{t^2}{t^4-1} dt$$

$$= \frac{1}{2}\int \left(\frac{1}{t^2-1} - \frac{1}{t^2+1}\right) dt$$

$$= \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{2}\arctan t + C \quad \left(t = \sqrt[4]{1+x^4}\right)$$

$$= \frac{1}{4}\ln\left|\frac{\sqrt[4]{1+x^4}-1}{\sqrt[4]{1+x^4}+1}\right| + \frac{1}{2}\arctan\sqrt[4]{1+x^4} + C$$

3. Find the following indefinite integral

(1) $\int \frac{dx}{4+5\cos x}$

Solution: $\int \frac{dx}{4+5\cos x}$

$$\left(t=\tan\frac{x}{2}, \sin x=\frac{2t}{1+t^2}, \cos x=\frac{1-t^2}{1+t^2}, dx=\frac{2}{1+t^2}dt\right)$$

$$=\int \frac{\frac{2}{1+t^2}dt}{4+5\cdot\frac{1-t^2}{1+t^2}}$$

$$=\int \frac{2}{9-t^2}dt$$

$$=\frac{1}{3}\int\left(\frac{1}{3-t}+\frac{1}{3+t}\right)dt$$

$$=\frac{1}{3}\ln\left|\frac{3+t}{3-t}\right|+C$$

$$=\frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right|+C$$

(2) $\int \frac{dx}{2+\sin x}$

Solution: $\int \frac{dx}{2+\sin x}=\int \frac{\frac{2}{1+t^2}dt}{2+\frac{2t}{1+t^2}}$

$$=\int \frac{dt}{t^2+t+1}$$

$$=\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\int \frac{\frac{2}{\sqrt{3}}d\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)}{\left[\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right]^2+1}$$

$$=\frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)+C$$

$$=\frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(\tan\frac{x}{2}+\frac{1}{2}\right)+C$$

(3) $\int \frac{dx}{3+\sin^2 x}$

Solution: $\int \frac{dx}{3+\sin^2 x}=\int \frac{\csc^2 x\,dx}{3\csc^2 x+1}$

$$=\int \frac{-d(\cot x)}{3\cot^2 x+4} \quad (u=\cot x)$$

$$=\int -\frac{du}{3u^2+4}$$

$$= \int -\frac{\frac{2}{\sqrt{3}}d\left(\frac{\sqrt{3}}{2}u\right)}{4\left[\left(\frac{\sqrt{3}}{2}u\right)^2+1\right]}$$

$$= -\frac{1}{2\sqrt{3}}\arctan\frac{\sqrt{3}}{2}u + C$$

$$= -\frac{1}{2\sqrt{3}}\arctan(\frac{\sqrt{3}}{2}\cot x) + C$$

(4) $\int \frac{dx}{1+\cos x+\sin x}$

Solution: $\int \frac{dx}{1+\cos x+\sin x} = \int \frac{\frac{2}{1+t^2}dt}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$

$$= \int \frac{dt}{1+t}$$

$$= \ln|1+t| + C$$

$$= \ln\left|1+\tan\frac{x}{2}\right| + C$$

(5) $\int \frac{dx}{2\sin x-\cos x+5}$

Solution: $\int \frac{dx}{2\sin x-\cos x+5} = \int \frac{\frac{2}{1+t^2}dt}{2\cdot\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}+5}$

$$= \int \frac{dt}{3t^2+2t+2}$$

$$= \int \frac{dt}{3\left(t+\frac{1}{3}\right)^2+\frac{5}{3}}$$

$$= \int \frac{\frac{1}{\sqrt{5}}d\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{1+\left[\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right]^2}$$

$$= \frac{1}{\sqrt{5}}\arctan\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right) + C$$

$$= \frac{1}{\sqrt{5}}\arctan\frac{3}{\sqrt{5}}\left(\tan\frac{x}{2}+\frac{1}{3}\right) + C$$

(6) $\int \frac{dx}{(2+\cos x)\sin x}$

Solution: $\int \frac{dx}{(2+\cos x)\sin x} = \int \frac{\frac{2}{1+t^2}dt}{\left(2+\frac{1-t^2}{1+t^2}\right)\frac{2t}{1+t^2}}$

$$= \int \frac{t^2+1}{t(t^2+3)}dt$$

$$= \int \frac{1}{3} \frac{d(t^3+3t)}{t^3+3t}$$

$$= \frac{1}{3} \ln|t^3 + 3t| + C$$

$$= \frac{1}{3} \ln \left|\tan^3 \frac{x}{2} + 3 \tan \frac{x}{2}\right| + C$$

(7) $\int \frac{dx}{\tan x+\sin x}$

Solution: $\int \frac{dx}{\tan x+\sin x} = \int \frac{\cos x dx}{\sin x(1+\cos x)}$

$$= \int \frac{\frac{1-t^2}{1+t^2} \cdot \frac{2}{1+t^2} dt}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{1-t^2}{2t} dt$$

$$= \int (\frac{1}{2t} - \frac{1}{2} t) dt$$

$$= \frac{1}{2} \ln|t| - \frac{1}{4} t^2 + C$$

$$= \frac{1}{2} \ln \left|\tan \frac{x}{2}\right| - \frac{1}{4} \tan^2 \frac{x}{2} + C$$

(8) $\int \frac{dx}{\sin(x+a) \cos\ (x+b)}$

Solution: $\int \frac{dx}{\sin(x+a) \cos\ (x+b)}$

$$= \frac{1}{\cos(a-b)} \int \frac{\cos[(x+a)-(x+b)]}{\sin(x+a) \cos\ (x+b)} dx$$

$$= \frac{1}{\cos(a-b)} \int \left[\frac{\cos(x+a)}{\sin(x+a)} + \frac{\sin(x+b)}{\cos(x+b)}\right] dx$$

$$= \frac{1}{\cos(a-b)} \int [\cot(x + a) + \tan(x + b)]\, dx$$

$$= \frac{1}{\cos(a-b)} \ln \left|\frac{\sin(x+a)}{\cos(x+b)}\right| + C$$

(9) $\int \frac{dx}{\sin^2 x \cos^2 x}$

Solution: $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int (\sec^2 x + \csc^2 x)\, dx$$

$$= \tan x - \cot x + C$$

(10) $\int \frac{\sin^2 x}{1+\sin^2 x} dx$

Solution:

$$\int \frac{\sin^2 x}{1+\sin^2 x} dx = \int \left(1 - \frac{1}{1+\sin^2 x}\right) dx$$

$$= x - \int \frac{\csc^2 x}{\csc^2 x+1} dx$$

$$= x + \int \frac{d(\cot x)}{\cot^2 x+2} \quad (u = \cot x)$$

$$= x + \int \frac{du}{u^2+2}$$

$$= x + \frac{1}{\sqrt{2}} \arctan \frac{u}{\sqrt{2}} + C$$

$$= x + \frac{1}{\sqrt{2}} \arctan(\frac{1}{\sqrt{2}} \cot x) + C$$