Midterm-Exam for《Calculus I》

2022-2023 1st Semester

Class: ________________Name:_______________Sit NO.: _____________

Note:1.Please clearly write down the private information needed before exam.
2.Please clearly write down the answers on the paper or answer sheet.
3.Full mark: 100
4.Duration of Exam: 100 minutes

Question Number	1-6	7-11	12-17	Total score
Score				

I. Please fill the correct answers in the following blanks. (1-6: $4' \times 6 = 24'$)

1. $\lim_{n\to\infty} n(\sqrt[n]{n}-1) =$ ____ $+\infty$ ____,

2. Suppose that $f'(c)$ exists, then $\lim_{x\to 0}\frac{f(c+3x)-f(c)}{5\tan x} = \frac{3}{5} f'(c)$ ______________.

3. Let $y=\sqrt[3]{\frac{x-3}{\sqrt{x^2+3}}}x^2\sin x$, $\frac{dy}{dx}=\sqrt[3]{\frac{x-3}{\sqrt{x^2+3}}}x^2\sin x[\frac{1}{3(x-1)}-\frac{x}{3(x^2+3)}+\frac{2}{x}+\cot x]$.

4. The equation $\cos(xy^2)+\ln\frac{x}{y}=\sin 1$ defines y as an implicit function of x,

$dy=\frac{y-xy^3\sin(xy^2)}{x+2x^2y^2\sin(xy^2)}dx$ ______________.

5. Let $f(x)=\begin{cases} e^{2x}-k, x\le 0 \\ \arcsin(cx), x>0 \end{cases}$ be continuous and differentiable at point $x=0$, then

$c=$ ___ 2 ___, $k=$ ___ 1 ___.

6. If the inflection point of the curve $y=ax^3+bx^2$ is（1, 3）, then constants $a=$ _ $-\frac{3}{2}$ _____, $b=$ $\frac{9}{2}$ ______.

II. Evaluate the following limits. (7-11: $6' \times 5 = 30'$)

7. $\lim_{x\to 0}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{3}{x}}}+\frac{\sin 2x}{|x|}\right)$.

Solution:

$\lim_{x\to 0^+} e^{\frac{1}{x}} = +\infty, \lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

$\lim_{x\to 0^-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{3}{x}}}+\frac{\sin 2x}{|x|}\right)=2-2=0,\ \lim_{x\to 0^+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{3}{x}}}+\frac{\sin 2x}{|x|}\right)=0+2=2.$

So the limit doesn't exist.

8. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{x}$.

Solution:

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{x}=\lim_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x}=\lim_{x\to 0}\frac{\sin x - x\cos x}{x^3}$$

$$=\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{3x}=\frac{1}{3}$$

9. $\lim\limits_{x\to+\infty}\left(a_0+a_1^x+a_2^x+a_3^x\right)^{\frac{1}{x}}, (a_0, a_1, a_{2,}a_3>0)$.

Solution:Let a_3 be the largest number among a_1, a_2 and a_3.

For case $a_3\le 1,$ the limit is 1.

For case $a_3>1,$ the type of this limit is $[\infty^0]$,

$$\lim_{x\to+\infty}\left(a_0+a_1^x+a_2^x+a_3^x\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to+\infty}\frac{\ln(a_0+a_1^x+a_2^x+a_3^x)}{x}}$$

$$=e^{\lim\limits_{x\to+\infty}\frac{a_1^x\ln a_1+a_2^x\ln a_2+a_3^x\ln a_3}{a_0+a_1^x+a_2^x+a_3^x}}=e^{\lim\limits_{x\to+\infty}\frac{a_1^x(\ln a_1)^2+a_2^x(\ln a_2)^2+a_3^x(\ln a_3)^2}{a_1^x\ln a_1+a_2^x\ln a_2+a_3^x\ln a_3}}$$

$$=\begin{cases} a_3, \text{ if } a_1<a_3 \text{ and } a_2<a_3 \\ e^{\frac{(\ln a_1)^2+(\ln a_3)^2}{\ln a_1+\ln a_3}}=a_3, \text{if } a_1=a_3 \text{ and } a_2<a_3 \\ e^{\frac{(\ln a_2)^2+(\ln a_3)^2}{\ln a_2+\ln a_3}}=a_3, \text{if } a_2=a_3 \text{ and } a_1<a_3 \\ a_3, \text{if } a_1=a_2=a_3 \end{cases}$$

10. $\lim\limits_{x\to 0}\left(\frac{\sqrt{1+2x^4}-\sqrt[3]{1-x^2}}{4x^2-x^3}\right)$.

Solution :

$$\sqrt{1+2x^4}-1\sim x^4,\quad 1-\sqrt[3]{1-x^2}\sim\frac{1}{3}x^2,\quad \sqrt{1+2x^4}-\sqrt[3]{1-x^2}\sim\frac{1}{3}x^2$$

$$4x^2-x^3\sim 4x^2$$

$$\lim_{x\to 0}\left(\frac{\sqrt{1+2x^4}-\sqrt[3]{1-x^2}}{4x^2-x^3}\right)=\lim_{x\to 0}\frac{\frac{1}{3}x^2}{4x^2}=\frac{1}{12}$$

11. $\lim\limits_{x\to+\infty}[(2x-1)e^{\frac{1}{x}}-2x]$.

Solution :

$$\lim_{x\to+\infty}[(2x-1)e^{\frac{1}{x}}-2x]\overset{t=\frac{1}{x}}{=}\lim_{t\to0^+}[(\frac{2}{t}-1)e^t-\frac{2}{t}]$$

$$=\lim_{t\to0^+}[\frac{2e^t-2}{t}-e^t]=1$$

III. **Finish the following questions. (12-16: $8'\times5=40'$; $17:6'$)**

12. If $f(x)=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq0\\ 0, & x=0\end{cases}$, find its derivative $f'(x)$. Is $f'(x)$ continuous at x=0? Why?

Solution：

$$x\neq0,\ f'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}};$$

$$x=0,\ f'(0)=\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to0}\frac{\frac{1}{x}}{e^{\frac{1}{x^2}}}=\lim_{x\to0}\frac{x}{2e^{\frac{1}{x^2}}}=0$$

$$\lim_{x\to0}f'(x)=\lim_{x\to0}\frac{2}{x^3}e^{-\frac{1}{x^2}}=\lim_{x\to0}\frac{\frac{2}{x^3}}{e^{\frac{1}{x^2}}}=\lim_{x\to0}\frac{\frac{3}{x}}{e^{\frac{1}{x^2}}}=0=f'(0)$$

$\therefore$ $f'(x)$ is continuous at $x=0$.

13. If $f(x)=\lim\limits_{n\to\infty}\dfrac{x^{2n+1}+(2k-1)x^n-1}{x^{2n}-kx^n-1}$ is continuous on $(0,+\infty)$, please find the constant k.

Solution :

$$f(x)=\lim_{n\to\infty}\frac{x^{2n+1}+(2k-1)x^n-1}{x^{2n}-kx^n-1}=\begin{cases}1, & 0<x<1\\ \frac{2k-1}{-k}, & x=1\\ x, & x>1\end{cases}$$

For $f(x)$ is continuous at $x=1$, we can get $\dfrac{2k-1}{-k}=1\Rightarrow k=\dfrac{1}{3}$

14. If $Q(x)=\int_1^{x-1}[\int_1^t f(z)\mathrm{d}z]\mathrm{d}t-\int_1^x e^x f(t)\mathrm{d}t$, Find $\dfrac{\mathrm{d}Q}{\mathrm{d}x}$.

Solution : Let $F(t)=\int_1^t f(z)\mathrm{d}z$

$$Q(x)=\int_1^{x-1}[\int_1^t f(z)\mathrm{d}z]\mathrm{d}t-\int_1^x e^x f(t)\mathrm{d}t=\int_1^{x-1}F(t)\mathrm{d}t-e^x\int_1^x f(t)\mathrm{d}t$$

$$\frac{dQ}{dx}=F(x-1)-e^x\int_1^x f(t)\mathrm{d}t-e^x f(x)$$

$$=\int_1^{x-1}f(z)\mathrm{d}z-e^x\int_1^x f(t)\mathrm{d}t-e^x f(x)$$

15. Let $f(x)=x^3\ln(1+x)$, please find $f^{(2022)}(0)$.

Solution:

$$\ln^{(n)}(1+x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$$

$$f^{(2022)}(x)=C_{2022}^0x^3[\ln(1+x)]^{(2022)}+C_{2022}^1 3x^2[\ln(1+x)]^{(2021)}+C_{2022}^2 6x[\ln(1+x)]^{(2020)}$$

$$+C_{2022}^3\times 6\times[\ln(1+x)]^{(2019)},$$

$$f^{(2022)}(0)=2022\times 2021\times 2020\times(-1)^{2018}\times 2018!=2022\times 2021\times 2020\times 2018!$$

16. Where is the function $g(x)=6\sqrt{x}-4x$ increasing, decreasing, concave up, and concave down? Find, if possible, the (global) maximum and minimum values of this function on $[0,+\infty)$.

Solution:

$$g'(x)=\frac{3-4\sqrt{x}}{\sqrt{x}}$$

$0<x<\frac{9}{16}, g'(x)>0, g(x)$ is increasing on $[0,\frac{9}{16}]$;

$x>\frac{9}{16}, g'(x)<0, g(x)$ is decreasing on $[\frac{9}{16},+\infty)$.

$$g''(x)=-\frac{3}{2}x^{\frac{-3}{2}}<0, x\in(0,+\infty),$$

the curve of g(x) is concave down on $[0,+\infty)$.

The global maximum is $g(\frac{9}{16})=\frac{9}{4}$.

17. If $f(x)\in C[0,1]$, and $f(x)$ is differentiable on (0,1). $f(0)=f(1), |f'(x)|<1$. Try to prove that for any $x_1,x_2\in(0,1)$, we have $|f(x_1)-f(x_2)|<\frac{1}{2}$.

Proof: (1) $0\le x_2-x_1<\frac{1}{2}$, by Mean Value Theorem,

$$|f(x_1)-f(x_2)|=|f'(c)(x_1-x_2)|<|x_1-x_2|<\frac{1}{2},\ c\in(x_1,x_2)$$

(2) $\frac{1}{2}\le x_2-x_1<1$, since $f(0)=f(1)$,

$$|f(x_1)-f(x_2)|=|f(x_1)-f(0)+f(1)-f(x_2)|$$

$$\le|f'(c_1)|x_1+|f'(c_2)|(1-x_2)<1-(x_2-x_1)<\frac{1}{2}\qquad c_1\in(0,x_1),\ c_2\in(x_2,1)$$