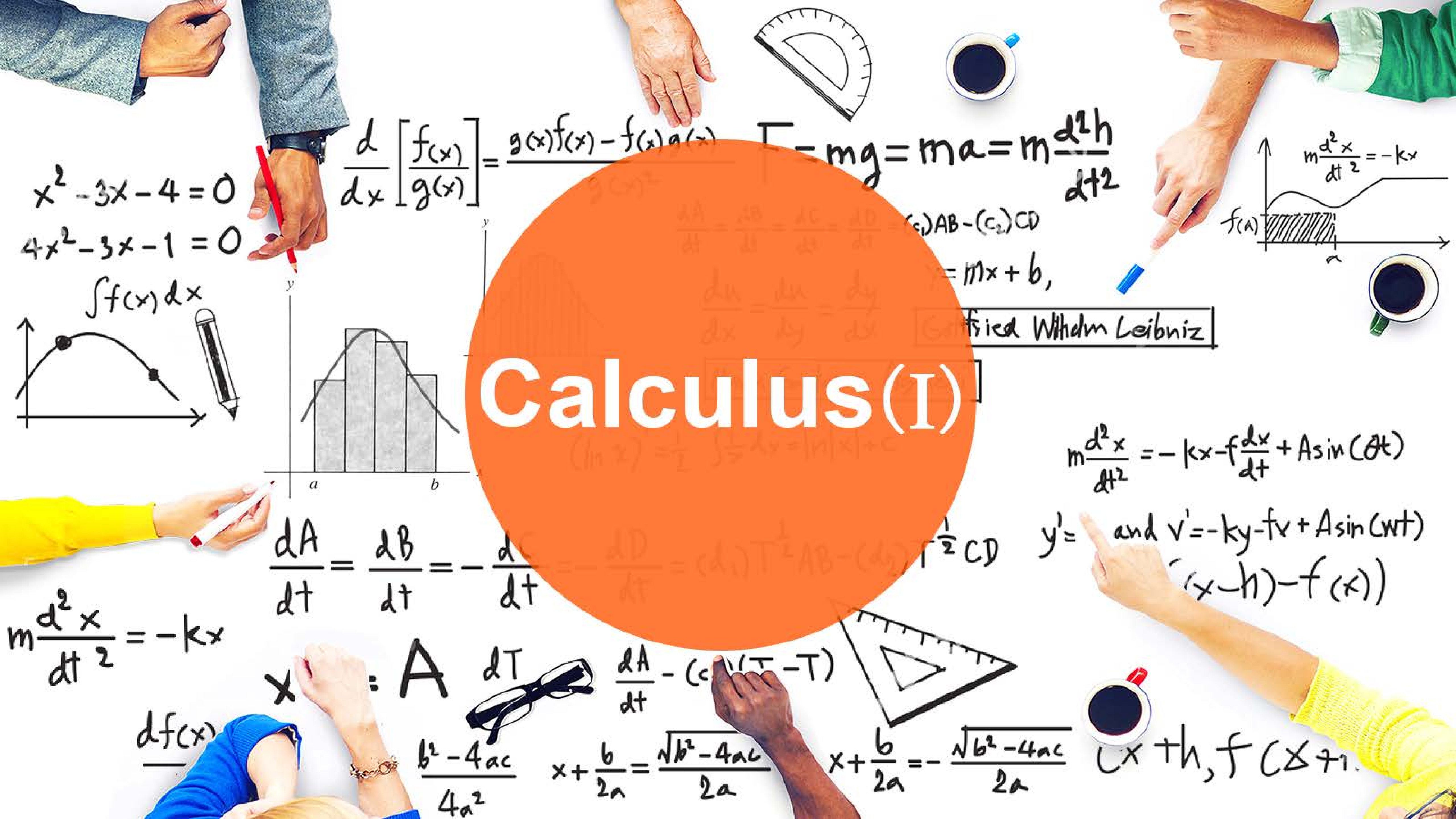

Calculus(Ⅰ)

Summary

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If $f(x)=\int_{\boxed{\sin x}}^{2}\frac{1}{1+t^2}dt$, find $f'(x)$.

$$f(x)=-\int_{2}^{\sin x}\frac{1}{1+t^2}\mathrm{d}t \xlongequal{u=\sin x} -\int_{2}^{u}\frac{1}{1+t^2}\mathrm{d}t$$

$$f'(x)=\frac{\mathrm{d}}{\mathrm{d}u}\left(-\int_{2}^{u}\frac{1}{1+t^2}\mathrm{d}t\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$=-\frac{1}{1+u^2}\cdot\cos x$$

$$=\frac{-\cos x}{1+\sin^2 x}.$$

1

If $f(x)=\int_{\sin x}^{2}\frac{1}{1+t^2}\mathrm{d}t$, find $f'(x)$. By Th A

(1) $$f(x)=-\int_{2}^{\sin x}\frac{1}{1+t^2}\mathrm{d}t \overset{u=\sin x}{=\!=\!=} -\int_{2}^{u}\frac{1}{1+t^2}\mathrm{d}t$$

$$f'(x)=\frac{\mathrm{d}}{\mathrm{d}u}\left(-\int_{2}^{u}\frac{1}{1+t^2}\mathrm{d}t\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$=-\frac{1}{1+u^2}\cdot\cos x=\boxed{\frac{-\cos x}{1+\sin^2 x}}$$

(2)

$$\boxed{f'(x)=-\frac{1}{1+\sin^2 x}\cdot\cos x=\frac{-\cos x}{1+\sin^2 x}.}$$

$$\lim_{x\to 0}\frac{\int_{\cos x}^{1} e^{-t^2}\,\mathrm{d}t}{x^2}$$

(Use the L'Hopital's Rule)

Analyze This is an indeterminate form of type $\frac{0}{0}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\cos x}^{1} e^{-t^2}\,\mathrm{d}t$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\int_{1}^{\cos x} e^{-t^2}\,\mathrm{d}t$$

$$= -e^{-\cos^2 x}\cdot(\cos x)'$$

$$= \sin x\cdot e^{-\cos^2 x}$$

$$\lim_{x\to 0}\frac{\int_{\cos x}^{1} e^{-t^2}\,\mathrm{d}t}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin x\cdot e^{-\cos^2 x}}{2x}$$

$$= \frac{1}{2e}.$$

If $f(x) \in C_{[0,1]}$, and $\boxed{f(x) < 1.}$ Prove: $2x - \int_0^x f(t)\mathrm{d}t = 1$ on the interval [0,1], there is only one solution.

Let $F(x) = 2x - \int_0^x f(t)\mathrm{d}t - 1$ then, $F'(x) = 2 - f(x) > 0$

$F(x)$ is a monotonically increasing function on the interval [0,1],

$F(0) = -1 < 0$

$F(1) = 1 - \int_0^1 f(t)\mathrm{d}t$ or $= \int_0^1 1\,\mathrm{d}t - \int_0^1 f(t)\mathrm{d}t$

$= 1 - f(\xi) > 0$ $= \int_0^1 [1 - f(t)]\mathrm{d}t > 0$

So, there is only one solution on the interval [0,1].

If $f(x) \in C_{[0,+\infty)}$, and $\boxed{f(x) > 0.}$ Prove: $F(x) = \dfrac{\int_0^x tf(t)\mathrm{d}t}{\int_0^x f(t)\mathrm{d}t}$ increasing in $[0,+\infty)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x tf(t)\mathrm{d}t = xf(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t = f(x)$$

$$F'(x) = \frac{xf(x)\cdot\int_0^x f(t)\mathrm{d}t - f(x)\cdot\int_0^x tf(t)\mathrm{d}t}{\left(\int_0^x f(t)\mathrm{d}t\right)^2}$$

$$\boxed{f(x) > 0}$$

$$F'(x) = \frac{x f(x)\cdot\int_0^x f(t)\mathrm{d}t - f(x)\cdot\int_0^x t f(t)\mathrm{d}t}{\left(\int_0^x f(t)\mathrm{d}t\right)^2}$$

$$F'(x) = \frac{\boxed{f(x)\int_0^x (x-t) f(t)\mathrm{d}t}}{\boxed{\left(\int_0^x f(t)\mathrm{d}t\right)^2}} > 0$$

$$\int_0^x f(\cancel{x}\,t)\mathrm{d}\cancel{x}\,t > 0 \quad (x > 0)$$

$$\int_0^x (x-t) f(t)\mathrm{d}t > 0$$

$F(x)$ is increasing in $(0, +\infty)$.

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Find $\lim\limits_{x\to 0}\dfrac{\int_0^x\left[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t\right]\mathrm{d}u}{\boxed{x(1-\cos x)}}$ $\left(\dfrac{0}{0}\right)$

$$\text{The original limit} = 2\lim_{x\to 0}\frac{\int_0^x\left[\int_0^{u^2}\arctan(1+t)\,\mathrm{d}t\right]\mathrm{d}u}{x^3}$$

$$\boxed{1-\cos x \sim \frac{x^2}{2}\quad (x\to 0)}$$

$$=2\lim_{x\to 0}\frac{\int_0^{x^2}\arctan(1+t)\,\mathrm{d}t}{3x^2}\quad\left(\frac{0}{0}\right)$$

$$=\frac{2}{3}\lim_{x\to 0}\frac{\arctan(1+x^2)\cdot 2x}{2x}$$

$$=\frac{2}{3}\cdot\frac{\pi}{4}=\frac{\pi}{6}.$$

14. If $Q(x) = \int_1^{x-1} [\int_1^t f(z)\mathrm{d}z]\mathrm{d}t - \int_1^x e^x f(t)\mathrm{d}t$, Find $\frac{\mathrm{d}Q}{\mathrm{d}x}$.

Solution : Let $F(t) = \int_1^t f(z)\mathrm{d}z$

$$Q(x) = \int_1^{x-1} [\int_1^t f(z)\mathrm{d}z]\mathrm{d}t - \int_1^x e^x f(t)\mathrm{d}t = \int_1^{x-1} F(t)\mathrm{d}t - e^x \int_1^x f(t)\mathrm{d}t$$

$$\frac{dQ}{dx} = F(x-1) - e^x \int_1^x f(t)\mathrm{d}t - e^x f(x)$$

$$= \int_1^{x-1} f(z)\mathrm{d}z - e^x \int_1^x f(t)\mathrm{d}t - e^x f(x)$$

(7) find the length of the curve $y=\int_1^x \sqrt{u^3-1}\,du,\ \ 1\le x\le 2$

Solution:

$$y'=\sqrt{x^3-1}$$

$$length=\int_1^2 \sqrt{1+(y')^2}\,dx=\int_1^2 \sqrt{x^3}\,dx=\frac{2}{5}(4\sqrt{2}-1)$$

(6) Find $G'(x), if\ G(x)=\int_{\cos x}^{\sin x}\frac{x\,du}{\sqrt{u^2+c^2}}$

$$G(x)=\int_{\cos x}^{\sin x}\frac{x\,du}{\sqrt{u^2+c^2}}=x\int_{\cos x}^{\sin x}\frac{1}{\sqrt{u^2+c^2}}du$$

$$G'(x)=\int_{\cos x}^{\sin x}\frac{1}{\sqrt{u^2+c^2}}du+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$$=(\ln\left|\sqrt{u^2+c^2}+u\right|)\Big|_{\cos x}^{\sin x}+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$$=\ln\left|\frac{\sqrt{\sin^2 x+c^2}+\sin x}{\sqrt{\cos^2 x+c^2}+\cos x}\right|+x(\cos x\frac{1}{\sqrt{\cos^2 x+c^2}}+\sin x\frac{1}{\sqrt{\sin^2 x+c^2}})$$

$(Note:\int\frac{1}{\sqrt{x^2+a^2}}dx=\ln\left|\sqrt{x^2+a^2}+x\right|+C,$ You can prove it by supposing x=a * tan (t))

6. Find $\lim\limits_{x\to 0}\dfrac{x-\int_0^x \frac{\sin t}{t}\,dt}{x-\sin x}$.

$$\text{Solu:} L=\lim_{x\to 0}\frac{x-\int_0^x \frac{\sin t}{t}\,dt}{x-\sin x}$$

$$=\lim_{x\to 0}\frac{1-\frac{\sin x}{x}}{1-\cos x}=\lim_{x\to 0}\frac{\frac{x-\sin x}{x}}{\frac{1}{2}x^2}=2\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=2\lim_{x\to 0}\frac{1-\cos x}{3x^2}=2\lim_{x\to 0}\frac{\frac{1}{2}x^2}{3x^2}=\frac{1}{3}.$$

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