Seat No.______ Major/Class______ School______ Student ID______ Name______

(DONNOT WRITE YOUR ANSWER IN THIS AREA)

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WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

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Notice: **1. Make sure that you have filled the form on the left side of seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	I	II	III	IV	V	VI	VII	VIII	Sum
Score									

1. (12 points) Let

$$A=\begin{bmatrix}1&3&2&-7\\-2&-2&-8&6\\2&3&7&1\\3&4&11&-7\end{bmatrix}.$$

(1) Find a set of basis for col A.

$$A=\begin{bmatrix}1&3&2&-7\\-2&-2&-8&6\\2&3&7&1\\3&4&11&-7\end{bmatrix}\sim\begin{bmatrix}1&3&2&-7\\0&4&-4&-8\\0&-3&3&15\\0&-5&5&14\end{bmatrix}$$

$$\sim\begin{bmatrix}1&3&3&-9\\0&1&-1&-2\\0&-3&3&15\\0&-5&5&14\end{bmatrix}\sim\begin{bmatrix}1&3&3&-9\\0&1&-1&-2\\0&0&0&9\\0&0&0&4\end{bmatrix}$$

$$\sim\begin{bmatrix}1&3&3&-9\\0&1&-1&-2\\0&0&0&1\\0&0&0&0\end{bmatrix}.$$

The 1^{st} , 2^{nd} and 4^{th} columns in A are the pivot columns and thus form a set basis for col A.

(2) What's rank A?
Solution: rank $A = 3$.

(3) Is the matrix equation $A\mathbf{X}=\mathbf{b}$ consistent for all possible **b?**
Solution: A does not have a pivot position in the last row and thus $A\mathbf{X}=\mathbf{b}$ is not consistent for all possible **b.**

2. (10 points) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(x_1, x_2, x_3) = (x_1 - 3x_2 + 2x_3, -2x_1 + 7x_2 + x_3, -4x_1 + 6x_2 + hx_3)$ with a scalar h. Note that x_1, x_2 and x_3 are entries in a $\mathbb{R}^3$ vector.

(1) (4 points) Find the standard matrix of T.

Solution: $T = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix}$

(2) (6 points) For what values of h the linear transformation T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

Solution:

$$T = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & \\ 0 & 1 & 5 & v \\ 0 & -6 & 8+h & \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 38+h \end{bmatrix}$$

If $h \neq -38$, T maps $\mathbb{R}^3$ onto $\mathbb{R}^3$.

3. (10 points) Let $A = \begin{bmatrix} 0 & 4 & 5 & -6 \\ -3 & -6 & 2 & 3 \\ 3 & 10 & 0 & 1 \\ 3 & 14 & 6 & -8 \end{bmatrix}$

(1) (7 points) Find the determinant of A.

Solution: $\det A = 3\begin{vmatrix} 0 & 4 & 5 & -6 \\ -1 & -6 & 2 & 3 \\ 1 & 10 & 0 & 1 \\ 1 & 14 & 6 & -8 \end{vmatrix} = -3\begin{vmatrix} 1 & 14 & 6 & -8 \\ -1 & -6 & 2 & 3 \\ 1 & 10 & 0 & 1 \\ 0 & 4 & 5 & -6 \end{vmatrix}$

$$= -3\begin{vmatrix} 1 & 14 & 6 & -8 \\ 0 & 8 & 8 & -5 \\ 0 & -4 & -6 & 9 \\ 0 & 4 & 5 & -6 \end{vmatrix} = -6\begin{vmatrix} 1 & 7 & 6 & -8 \\ 0 & 4 & 8 & -5 \\ 0 & -2 & -6 & 9 \\ 0 & 2 & 5 & -6 \end{vmatrix} = -6\begin{vmatrix} 1 & 7 & 6 & -8 \\ 0 & 0 & -2 & 7 \\ 0 & 0 & -1 & 3 \\ 0 & 2 & 5 & -6 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 7 & 6 & -8 \\ 0 & 2 & 5 & -6 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -2 & 7 \end{vmatrix} = -6\begin{vmatrix} 1 & 7 & 6 & -8 \\ 0 & 2 & 5 & -6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -12$$

(2) (3points) Is $\operatorname{adj} A$ invertible?

Solution: $\det A \neq 0$, and thus A and A^{-1} are invertible. Since $\operatorname{adj} A = (\det A)A^{-1}$, adj A is invertible.

4. (20 points) Let $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}$ and $W = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

(1) (4 points) what's the distance between $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$?

Solution: The distance= $\sqrt{(1+4)^2 + (2+1)^2 + (3-2)^2} = \sqrt{35} = 5.9161$

(2) (4 points) Are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ orthogonal?

Solution: $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = -4 - 2 + 6 = 0$, so $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are orthogonal.

(3) (6 points) Find the distance from $\boldsymbol{y}$ to the subspace W.

Solution: The orthogonal projection of $\boldsymbol{y}$ onto W is

$$\hat{\boldsymbol{y}} = \frac{\boldsymbol{y}\cdot\boldsymbol{u}_1}{\boldsymbol{u}_1\cdot\boldsymbol{u}_1}\boldsymbol{u}_1 + \frac{\boldsymbol{y}\cdot\boldsymbol{u}_2}{\boldsymbol{u}_2\cdot\boldsymbol{u}_2}\boldsymbol{u}_2 = \frac{-14}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{-21}{21}\begin{bmatrix}-4\\-1\\2\end{bmatrix} = \begin{bmatrix}3\\-1\\-5\end{bmatrix}.$$

The distance from $\boldsymbol{y}$ to the subspace W is $|\boldsymbol{y} - \hat{\boldsymbol{y}}| = \sqrt{24} = 4.899$.

(4) (6 points) Find a set a basis for the orthogonal compliment $\mathrm{W}^{\perp}$ of W.

Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are orthogonal,

$\mathbf{y} - \hat{\boldsymbol{y}} = \begin{bmatrix}-2\\4\\-2\end{bmatrix} = -2\begin{bmatrix}1\\-2\\1\end{bmatrix}$, a basis for the orthogonal compliment $\mathrm{W}^{\perp}$ is $\begin{bmatrix}1\\-2\\1\end{bmatrix}$.

5. (15 points) Let $\boldsymbol{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}1\\2\\0\end{bmatrix}$, $\boldsymbol{b}_3 = \begin{bmatrix}1\\2\\4\end{bmatrix}$, $\boldsymbol{c}_1 = \begin{bmatrix}1\\-1\\2\end{bmatrix}$, $\boldsymbol{c}_2 = \begin{bmatrix}3\\-2\\7\end{bmatrix}$, $\boldsymbol{c}_3 = \begin{bmatrix}0\\3\\2\end{bmatrix}$ and consider the basis for $\mathbb{R}^3$ given by $\mathrm{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ and $\mathrm{C} = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$.

(1) (6 points) Find $[\boldsymbol{X}]_B$, the B-coordinate vector of $\mathbf{X} = \begin{bmatrix}1\\2\\6\end{bmatrix}$.

Solution: $\begin{bmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\2\\6\end{bmatrix}$

Solve the above equation, we can obtain $[\boldsymbol{X}]_B = \begin{bmatrix}0\\-0.5\\1.5\end{bmatrix}$

(2) (6points) Find the change-of-coordinate matrix from B to C: $P_{C\leftarrow B}$.

Solution: $P_{C\leftarrow B} = [\boldsymbol{c}_1 \quad \boldsymbol{c}_2 \quad \boldsymbol{c}_3]^{-1}[\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \boldsymbol{b}_3]$

$$= \begin{bmatrix}25 & 6 & -9\\-8 & -2 & 3\\3 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\0 & 2 & 2\\0 & 0 & 4\end{bmatrix} = \begin{bmatrix}25 & 37 & 1\\-8 & -12 & 0\\3 & 5 & 1\end{bmatrix}.$$

(3) (3points) Find $[\boldsymbol{X}]_C$, the C-coordinate vector of $\mathbf{X} = \begin{bmatrix}1\\2\\6\end{bmatrix}$.

Solution: $[\boldsymbol{X}]_C = P_{C\leftarrow B}[\boldsymbol{X}]_B = \begin{bmatrix}25 & 37 & 1\\-8 & -12 & 0\\3 & 5 & 1\end{bmatrix}\begin{bmatrix}0\\-0.5\\1.5\end{bmatrix} = \begin{bmatrix}-17\\6\\-1\end{bmatrix}$.

6. For the following quadratic form

$$\boldsymbol{x}^T A\boldsymbol{x} = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$$

(1) Give the matrix A of the quadratic form and indicate which type this quadratic form is? (For example, positive definite, negative definite or indefinite).

Solution: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

The eigenvalues are 4 and 1, and thus positive definite.

(2) Find an orthogonal matrix P such that the change of variable $\boldsymbol{x} = P\boldsymbol{y}$ transforms $\boldsymbol{x}^T A\boldsymbol{x}$ into a new quadratic form with no cross-product term.

Solution: A basis of the Eigenspace of $\lambda = 1$ is $\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

A basis of the Eigenspace of $\lambda = 4$ is $\boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (5 points)

(3) Let $\mathbf{z} = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1}\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$. (2 points)

(4) Then $\boldsymbol{v}_1, \mathbf{z}, \boldsymbol{v}_3$ are orthogonal eigenvectors of A. We can normalize them to obtain orthonormal eigenvectors of A:

(5) $\boldsymbol{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, (3 points)

7. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -5 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 4 & 3 \end{bmatrix}$

(1) Compute A^{-1} and B^{-1}.

Solution: $A^{-1} = \begin{bmatrix} -13 & -3 & -1 \\ 10 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

(2) Find a matrix X such that $AXB = C$.

Solution: $X = A^{-1}CB^{-1} = \begin{bmatrix} -11 & -4 \\ 9 & 2 \\ 7 & -1 \end{bmatrix}$

8. (8 points) Let A be an $n \times n$ matrix. Show that if A has an eigenvalue 0, so is A^2.

Proof.

If A has an eigenvalue 0, $\det A = 0$.

Since $\det A^2 = (\det A)^2 = 0$, A^2 has an eigenvalue 0.