Continuity, Differentiation and Derivative

1. Piecewise limits and unilateral limits

(1) Find $\lim\limits_{x\to 0}(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|})$.

Solution: $\lim\limits_{x\to 0+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\lim\limits_{x\to 0+}\left(\frac{2e^{-\frac{1}{x}}+1}{e^{-\frac{1}{x}}+e^{\frac{3}{x}}}+\frac{\sin x}{x}\right)=1$

$\lim\limits_{x\to 0-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\lim\limits_{x\to 0-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{-x}\right)=2-1=1$

$Since\ \lim\limits_{x\to 0+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\lim\limits_{x\to 0-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=1$

$\Rightarrow \lim\limits_{x\to 0}(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|})=1$

(2) $f(x)=\begin{cases}\frac{\sqrt{ax+b}-2}{x-1} & x\neq 1\\ -1 & x=1\end{cases}$ is continuous at $x=1$,find the values of $a\ and\ b$.

Solution: f is continuous at $x=1\ \Rightarrow \lim\limits_{x\to 1}\frac{\sqrt{ax+b}-2}{x-1}=-1$

$\Rightarrow \lim\limits_{x\to 1}\sqrt{ax+b}-2=0\Rightarrow \sqrt{a+b}-2=0\Rightarrow a+b=4$

$\lim\limits_{x\to 1}\frac{\sqrt{ax+b}-2}{x-1}=\lim\limits_{x\to 1}\frac{ax+b-4}{(x-1)(\sqrt{ax+b}+2)}=\lim\limits_{x\to 1}\frac{ax-a}{(x-1)(\sqrt{a+b}+2)}=\frac{a}{4}$

$\Rightarrow \frac{a}{4}=-1\Rightarrow a=-4\Rightarrow b=8$

(3) $f(x)=\begin{cases}\frac{3\sin(x-1)}{x-1}, & x<1\\ e^{ax}+1, & x\geq 1\end{cases}$ is continuous on $(-\infty,+\infty)$, a=______.

Solution: $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^-}\frac{3\sin(x-1)}{x-1}=3$

$\mathrm{Lim}_{x\to 1^+}f(x)=\lim\limits_{x\to 1^+}e^{ax}+1=e^{a}+1$

$\Rightarrow e^{a}+1=3\Rightarrow a=\ln 2$

2. The discontinuous point and kinds

(4) Let $x=0$ and $x=\pm 1$ are discontinuous points of $f(x)=\frac{x^2-x}{|x-a|(x^2-b)}$, find the values of $a\ and\ b$ and determine the type of discontinuous point.

Solution: $a,\pm\sqrt{b}$ are discontinuous points of $f(x)\Rightarrow a=0,b=1$

$f(x)=\frac{x^2-x}{|x|(x^2-1)}=\frac{x(x-1)}{|x|(x-1)(x+1)}$

$when\ x=0,\ \lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}\frac{x(x-1)}{(-x)(x-1)(x+1)}=\lim\limits_{x\to 0^-}\frac{-1}{(x+1)}=-1$

$\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}\frac{x(x-1)}{x(x-1)(x+1)}=\lim\limits_{x\to 0^+}\frac{1}{x+1}=1$

$\lim\limits_{x\to 0^-}f(x)\neq\lim\limits_{x\to 0^+}f(x)\Rightarrow x=0\ \ jump\ discontinuity$

$when\ x=-1,\ \lim\limits_{x\to -1^-}f(x)=\lim\limits_{x\to -1^-}\frac{-1}{x+1}=-\infty$

$\Rightarrow x = 0 \; is \; discontinuity \; point \; of \; the \; second \; kind$

$when\; x = 1, \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x(x-1)}{x(x-1)(x+1)} = \lim_{x\to 1^-} \frac{1}{x+1} = \frac{1}{2}$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x(x-1)}{x(x-1)(x+1)} = \lim_{x\to 1^+} \frac{1}{x+1} = \frac{1}{2}$$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \frac{1}{2} \Rightarrow x = 1 \; removable \; discontinuity$

(5) Let $x = 1$ be the removable of the function $\frac{x^2+3x+b}{x^2-3x+a}$, find the values of $a \; and \; b$.

Solution: $since \; x = 1 \; removable \; discontinuity$

$\Rightarrow \lim_{x\to 1}(x^2 - 3x + a) = 0 \Rightarrow a = 2$

$\Rightarrow \lim_{x\to 1}(x^2 + 3x + b) = 0 \Rightarrow b = -4$

(6) Find the asymptote equation of the following curve

a) $y = \frac{x^2}{1+x}$.

Solution: $Since \lim_{x\to -1} \frac{x^2}{1+x} = \infty$

$\Rightarrow x = -1 \;\; vertical \; asymptote$

$a = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{x^2}{x(1+x)} = 1$

$b = \lim_{x\to\infty} \left(\frac{x^2}{1+x} - ax\right) = \lim_{x\to\infty} \left(\frac{x^2}{1+x} - x\right) = \lim_{x\to\infty} -\frac{x}{1+x} = -1$

$\Rightarrow y = x - 1 \; is \; an \; oplique \; asymptote$

b) $y = \frac{2x}{1+x^2}$.

Solution: $Since \lim_{x\to\infty} \frac{2x}{1+x^2} = 0$

$\Rightarrow y = 0 \;\; horizontal \; asymptote$

c) $y = \sqrt{6x^2 - 8x + 3}$.

Solution: $Since \lim_{x\to +\infty} \frac{\sqrt{6x^2-8x+3}}{x} = \sqrt{6}$

$\lim_{x\to +\infty} \left(\sqrt{6x^2 - 8x + 3} - \sqrt{6}x\right) = \lim_{x\to +\infty} \frac{-8x+3}{\sqrt{6x^2-8x+3}+\sqrt{6}x} = -\frac{2\sqrt{6}}{3}$

$\Rightarrow When \; x \to +\infty, the \; asymptote \; is \; y = \sqrt{6}x - \frac{2\sqrt{6}}{3}$

$\lim_{x\to -\infty} \frac{\sqrt{6x^2-8x+3}}{x} = -\sqrt{6}$

$\lim_{x\to -\infty} \left(\sqrt{6x^2 - 8x + 3} + \sqrt{6}x\right) = \lim_{x\to -\infty} \frac{-8x+3}{\sqrt{6x^2-8x+3}-\sqrt{6}x} = \frac{2\sqrt{6}}{3}$

$\Rightarrow When \; x \to -\infty, the \; asymptote \; is \; y = -\sqrt{6}x + \frac{2\sqrt{6}}{3}$

d) $y = (2 + x)e^{\frac{1}{x}}$.

Solution: $Since \lim_{x\to 0^+}(2+x)e^{\frac{1}{x}} = \infty$

$\Rightarrow x = 0 \ \ vertical\ asymptote$

$a = \lim_{x\to\infty}\frac{(2+x)e^{\frac{1}{x}}}{x} = \lim_{x\to\infty} e^{\frac{1}{x}} = 1$

$b = \lim_{x\to\infty}((2+x)e^{\frac{1}{x}} - x) \ \ (x = \frac{1}{t})$

$= \lim_{t\to 0}\left(2+\frac{1}{t}\right)e^t - \frac{1}{t}$

$= 2 + \lim_{t\to 0}\frac{e^t - 1}{t}$

$= 3$

$\Rightarrow y = x + 3 \ \ is\ an\ oplique\ asymptote$

3. Intermediate Value Theorem

(7) Prove equation $x = a\sin x + b(a, b > 0)$ at least exist one positive zero.

Solution: $Let\ f(x) = x - a\sin x - b, take\ A > a + b;$

$f(0) = -a - b < 0$

$f(A) = A - a\sin A - b > A - a - b > 0$

$Since\ f(0)\cdot f(A) < 0, f\ is\ continuous\ at\ (-\infty, +\infty)$

$By\ intermediate\ value\ theorem \Rightarrow \exists c \in (0, A), s.t. f(c) = 0$

(8) Let the function $f(x)$ be continuous on $[a,b]$. $a \le x_1 < x_2 < \cdots < x_n \le b$. Prove: $\exists c \in [a,b], s.t\ f(c) = \frac{1}{n}[f(x_1) + f(x_2) + \cdots f(x_n)]$.

Solution: $Let\ M = \max_{x\in[a,b]} f(x), m = \min_{x\in[a,b]} f(x)$

$m \le \frac{1}{n}[f(x_1) + f(x_2) + \cdots f(x_n)] \le M$

$\Rightarrow \exists c \in [a,b], s.t.\ f(c) = \frac{1}{n}[f(x_1) + f(x_2) + \cdots f(x_n)]$

4. Definition of derivative

(9) $f(x) + xf'(x) = \sin x. f(x)$ is differentiable on $(-\infty, +\infty)$, find $f(0)$ and $f'(0)$.

Solution: $Let\ x = 0 \Rightarrow f(0) + 0 = 0 \Rightarrow f(0) = 0$

$\sin x = f(x) + xf'(x) = \left(xf(x)\right)' \Rightarrow xf(x) = -\cos x + C$

$f(0) = 0 \Rightarrow 0 = -1 + C \Rightarrow C = 1$

$\Rightarrow f(x) = \frac{-\cos x + 1}{x}(x \ne 0)$

$f'(0) = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{1 - \cos x}{x^2} = \frac{1}{2}$

(10) The polar coordinate equation of the Archimedes spiral is known as $r = \theta$. Find the rectangular coordinate equation of the tangent line at the point $\theta = \pi$ on the curve.

Solution: $x = r\cos\theta = \theta\cos\theta, y = r\cos\theta = \theta\sin\theta$

$\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta, \frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$

$\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$

$when\ \theta = \pi \Rightarrow x = -\pi, y = 0, \frac{dy}{dx} = \pi$

$\Rightarrow The\ equation\ is\ y = \pi(x + \pi)$

(11) Let $f(x)$ differentiate at $x = 1, f(1) = f'(1) = 2.$ find $\lim_{x\to 0} \frac{f^3(1+x) - f^3(1)}{x}$.

Solution: $\lim_{x\to 0} \frac{f^3(1+x) - f^3(1)}{x} = \lim_{x\to 0} \frac{\big(f(1+x) - f(1)\big)\left(f^2(1+x) + f(1+x)f(1) + f(1)^2\right)}{x}$

$= \lim_{x\to 0} \frac{f(1+x) - f(1)}{x}\left(f^2(1) + f^2(1) + f^2(1)\right)$

$= f'(1) \cdot 3f^2(1) = 24$

(12) Let $f(x)$ differentiate at $x = x_0$. $\lim_{x\to x_0} \frac{f\left(\frac{x+x_0}{2}\right) - f(x_0)}{x - x_0} =$ _____.

Solution: $Let\ \frac{x+x_0}{2} = t \Rightarrow x = 2t - x_0; x \to x_0 \Rightarrow t \to x_0$

$\lim_{x\to x_0} \frac{f\left(\frac{x+x_0}{2}\right) - f(x_0)}{x - x_0} = \lim_{t\to x_0} \frac{f(t) - f(x_0)}{2(t - x_0)} = \frac{1}{2} f'(x_0)$

(13) $f(1) = f'(1) = 2$, find $\lim_{x\to 0} \frac{f^2(1+x) - f^2(1)}{x}$.

Solution: $\lim_{x\to 0} \frac{f^2(1+x) - f^2(1)}{x} = \lim_{x\to 0} \frac{f(1+x) - f(1)}{x}\left(f(1+x) + f(1)\right)$

$= f'(1) \cdot 2f(1)$

$= 8$

(14) Discuss the differentiability of the function $y = \begin{cases} x^2 & x > 0 \\ ax + b & x \le 0 \end{cases}$ at $x = 0$

Solution: $f\ is\ derivable \Rightarrow f\ is\ continuous$

$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow b = 0$

$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} \Rightarrow a = 0$

$When\ a = b = 0, f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \le 0 \end{cases}$

$\Rightarrow f\ is\ derivable\ at\ x = 0$

$\Rightarrow f\ is\ derivable\ at\ x = 0\ if\ and\ only\ if\ a = b = 0$

5. Derivative or differential of a compound function

(15) $y = x^4 \arctan x$, dy=_______.

Solution: $y' = 4x^3 \arctan x + \frac{x^4}{1+x^2}$

$\Rightarrow dy = \left(4x^3 \arctan x + \frac{x^4}{1+x^2}\right) dx$

(16) $y = \frac{1}{2}\arctan\left(\sqrt[4]{1 + x^4}\right) + \frac{1}{4}\ln\frac{\sqrt[4]{(1+x^4)} + 1}{\sqrt[4]{(1+x^4)} - 1}$, y'=________.

Solution: $Let\ u=(1+x^4)^{\frac{1}{4}}, u'(x)=x^3(1+x^4)^{-\frac{3}{4}}$

$$y=\frac{1}{2}\arctan u+\frac{1}{4}\ln(u+1)-\frac{1}{4}\ln(u-1)$$

$$y'=\frac{1}{2}\cdot\frac{1}{1+u^2}\cdot u'+\frac{1}{4}\cdot\frac{1}{u+1}u'-\frac{1}{4}\cdot\frac{1}{u-1}\cdot u'$$

$$=[\frac{1}{2(1+u^2)}+\frac{1}{2(1-u^2)}]u'$$

$$=\frac{1}{1-u^4}u'$$

$$=\frac{1}{-x^4}\cdot x^3(1+x^4)^{-\frac{3}{4}}$$

$$=-\frac{(1+x^4)^{-\frac{3}{4}}}{x}$$

(17) $y=f(\ln^2 x-e^{-x}), dy=$ _____.

Solution: $y'=\ f'(\ln^2 x-e^{-x})\left(\frac{2\ln x}{x}+e^{-x}\right)$

$$\Rightarrow dy=\left(\frac{2\ln x}{x}+e^{-x}\right)f'(\ln^2 x-e^{-x})dx$$

(18) Find the derivative of $y=xe^x\sqrt{\sin(x^2-1)}$.

Solution: $\ln y=\ln x+x+\frac{1}{2}\ln\sin(x^2-1)$

$$\frac{y'}{y}=\frac{1}{x}+1+\frac{1}{2}\cdot\frac{1}{\sin(x^2-1)}\cdot\cos(x^2-1)\cdot 2x$$

$$=\frac{1}{x}+1+x\cot(x^2-1)$$

$$\Rightarrow y'=\left[\frac{1}{x}+1+x\cot(x^2-1)\right]\cdot xe^x\sqrt{\sin(x^2-1)}$$

$$=[1+x+x^2\cot(x^2-1)]e^x\sqrt{\sin(x^2-1)}$$

(19) Let $f(x)$ be differentiable and find the derivative of the following function

a) $f(\sqrt[3]{x^2})$

b) $\arctan f(x)$

c) $f\left(\frac{1}{f(x)}\right)$

d) $\sin(f(\sin x))$

Solution: $a)\ [f(\sqrt[3]{x^2})]'=f'\left(x^{\frac{2}{3}}\right)\cdot\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3}x^{-\frac{1}{3}}f'\left(x^{\frac{2}{3}}\right)$

$$b)\ [\arctan f(x)]'=\frac{f'(x)}{1+f^2(x)}$$

$$c)\ \left[f\left(\frac{1}{f(x)}\right)\right]'=f'\left(\frac{1}{f(x)}\right)\cdot\left[\frac{1}{f(x)}\right]'=f'\left(\frac{1}{f(x)}\right)\cdot-\frac{f'(x)}{f^2(x)}=-\frac{f'(x)}{f^2(x)}f'\left(\frac{1}{f(x)}\right)$$

$$d)\ [\sin(f(\sin x))]'=\cos(f(\sin x))\cdot f'(\sin x)\cdot\cos x$$

(20) Find the derivative of the following function by logarithmic derivation

a) $y=x^x$

b) $y = \ln^x(2x+1)$

c) $y = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^3}}$

d) $y = \sin x^{\sqrt{x}}$

Solution: $a)\ \ln y = x\ln x \Rightarrow \frac{y'}{y} = \ln x + 1$

$$\Rightarrow y' = y(\ln x\ + 1)$$
$$\Rightarrow y' = x^x(\ln x\ + 1)$$

$b)\ \ln y = x \ln\ln(2x+1)$

$$\Rightarrow \frac{y'}{y} = \ln\ln(2x+1) + x\cdot\frac{2}{(2x+1)\ln(2x+1)}$$

$$\Rightarrow y' = \left[\ln\ln(2x+1) + x\cdot\frac{2}{(2x+1)\ln(2x+1)}\right]\ln^x(2x+1)$$

$c)\ \ln y = \ln x + \frac{1}{2}\ln(1-x^2) - \frac{1}{2}\ln(1+x^3)$

$$\Rightarrow \frac{y'}{y} = \frac{1}{x} - \frac{2x}{2(1-x^2)} - \frac{3x^2}{2(1+x^3)}$$

$$\Rightarrow y' = \left[\frac{1}{x} - \frac{x}{1-x^2} - \frac{3x^2}{2(1+x^3)}\right]\cdot\frac{x\sqrt{1-x^2}}{\sqrt{1+x^3}}$$

$d)\ Let\ u = x^{\sqrt{x}}, y = \sin u, \ln u = \sqrt{x}\ln x$

$$\frac{u'}{u} = \frac{1}{2\sqrt{x}}\ln x + \frac{1}{\sqrt{x}} \Rightarrow u' = \frac{\ln x + 2}{2\sqrt{x}}\cdot x^{\sqrt{x}}$$

$$y' = \frac{dy}{du}\cdot\frac{du}{dx} = \cos u\cdot u' = \frac{\ln x+2}{2\sqrt{x}}\cdot x^{\sqrt{x}}\cos x^{\sqrt{x}}$$