I. (2 questions, 5 points each, 10 points in total)

(1) Six students, three boys and three girls, lineup in a random order for a photograph. What is the probability that the boys and girls alternate?

Solution: $\frac{2\cdot 3!\cdot 3!}{6!} = \frac{1}{10}$.

(2) 12 different toys are to be divided among 3 children so that each one gets 4 toys. How many ways can this be done?

Solution: $\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{12!}{4!4!4!}$.

II. (10 points) John and Bob take turns in flipping a fair coin. The first one to get a Heads wins. John starts the game. What is the probability that he wins?

Solution: $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots = \frac{2}{3}$.

III. (2 questions, 5 points each, 10 points in total)

(1) Suppose that A, B, and D are any three events such that $P(A \mid D) \geq P(B \mid D)$ and $P(A \mid D') \geq P(B \mid D')$. Show that $P(A) \geq P(B)$.

Solution: $P(A \mid D) \geq P(B \mid D) \Rightarrow \frac{P(AD)}{P(D)} \geq \frac{P(BD)}{P(D)} \Rightarrow P(AD) \geq P(BD)$,

$P(A \mid D') \geq P(B \mid D') \Rightarrow \frac{P(AD')}{P(D')} \geq \frac{P(BD')}{P(D')} \Rightarrow P(AD') \geq P(BD')$,

So $P(A) = P(AD) + P(AD') \geq P(BD) + P(BD') = P(B)$.

(2) Suppose that A, B, and D are events such that A and B are independent, $P(A \cap B \cap D) = 0.01, P(D \mid A \cap B) = 0.25$, and $P(B) = 4P(A)$. Find $P(A \cup B)$.

Solution: $P(D \mid A \cap B) = \frac{P(D\cap A\cap B)}{P(A\cap B)} = \frac{0.01}{P(A\cap B)} = \frac{1}{4} \Rightarrow P(A \cap B) = 0.04$,

Because A and B are independent, $P(A \cap B) = P(A)P(B)$.

And $P(B) = 4P(A)$, so $4P^2(A) = 0.04 \Rightarrow P(A) = 0.1, P(B) = 0.4$.

Then $P(A \cup B) = P(A) + P(B) - P(AB) = 0.46$.

IV. (10 points) Suppose that when a machine is adjusted properly, 50 percent of the items produced by it are of high quality and the other 50 percent are of medium quality. Suppose. However, that the machine is improperly adjusted during 10 percent of the time and that, under these conditions, 25 percent of the items produced by it are of high quality and 75 percent are of medium quality. Suppose that three items produced by the machine at a certain time are selected at random and inspected.

1. What is the probability that at least one of these items is of high quality?
2. Suppose at least one of these items is of high quality, what is the probability that the machine was adjusted properly at that time?

Solution: Let $X = \{$ the number of high quality in the selected three items$\}$,

$A = \{$ at least one of selected three items is of high quality $\}$,

$B = \{$ the machine is adjusted properly $\}$.

Then $X \mid B \sim \text{Bin}(3, 0.5)$, $X \mid B' \sim \text{Bin}(3, 0.25)$ and $P(A) = P(X \geq 1)$.

(1) By the law of total probability

$$P(A) = P(AB) + P(AB') = P(A \mid B)\cdot P(B) + P(A \mid B')\cdot P(B') = (1-0.5^3)\times 0.9 + (1-0.75^3)\times 0.1 = 0.8453.$$

(2) The probability is

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A)} = \frac{(1-0.5^3)\times 0.9}{0.8453} = 0.9316.$$

V. (10 points) The n candidates for a job have been ranked $1,2,3,\ldots,n$. Let $X =$ the rank of a randomly selected candidate, so that X has p.m.f.

$$p(x) = \begin{cases} 1/n, & x = 1,2,3,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

(this is called the discrete uniform distribution). Compute $E(X)$ and $V(X)$.

Solution: By the definition of $E(X)$,

$$E(X) = \sum_{n=1}^{n} x\cdot\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{n=1}^{n} x = \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}.$$

Similarly,

$$E\left(X^2\right) = \sum_{n=1}^{n} x^2\cdot\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}.$$

So

$$V(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}.$$

VI. (10 points) Let X be a discrete random variable. Let a and b be two real numbers. Prove that $E(aX+b) = aE(X)+b$ and $V(aX+b) = a^2\cdot V(x)$.

Solution: By the definition of $E[g(X)]$,

$$E(aX+b) = \sum(ax+b)p(x) = a\sum xp(x) + b\sum p(x) = aE(X)+b,$$

and

$$V(aX+b) = E\left[((aX+b)-E(aX+b))^2\right] = \sum[ax+b-E(aX+b)]^2 p(x)$$

$$= \sum[ax+b-(a\mu+b)]^2 p(x) = \sum[ax-a\mu]^2 p(x) = a^2\sum(x-\mu)^2 p(x) = a^2 V(X).$$

VII. (10 points) Let X be a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} A\left(1-x^2\right), & -1\le x\le 1, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find the value of A.

2. What is the probability $P(X \le 0.5)$?

3. Compute $E(X)$ and $V(X)$.

Solution:

1. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} A(1-x^2)dx = 2A(x-\frac{x^3}{3})\Big|_0^1 = \frac{4A}{3} \Rightarrow A = \frac{3}{4}$.

2. $P(X \le 0.5) = \int_{-\infty}^{0.5} f(x)dx = \frac{3}{4}\int_{-1}^{0.5}(1-x^2)dx = 0.5 + \frac{3}{4}\int_0^{0.5}(1-x^2)dx = \frac{27}{32}$.

3. $E(X) = \int_{-\infty}^{\infty} xf(x)dx = 0$. ($xf(x)$ is an odd function)

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)dx = \frac{3}{2}(\frac{x^3}{3}-\frac{x^5}{5})\Big|_0^1 = \frac{1}{5}$

$\Rightarrow V(X) = E(X^2) - E(X)^2 = \frac{1}{5}$.

VIII. (10 points) Suppose that X have an exponential distribution with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

1. What is the probability $P(X > E(X))$?

2. Let $h(X) = \begin{cases} e, & X > E(X), \\ 0, & X \le E(X). \end{cases}$ Compute $E(h(X))$.

Solution:

1. $E(X) = \frac{1}{\lambda} \Rightarrow P(X > E(X)) = \int_{\frac{1}{\lambda}}^{+\infty} f(x)dx = -e^{-\lambda x}\Big|_{\frac{1}{\lambda}}^{+\infty} = \frac{1}{e}$.

2. $E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x)dx = \int_{\frac{1}{\lambda}}^{+\infty} ef(x)dx = 1$.

IX. (10 points) 40% of seeds from maize (modern-day corn) ears carry single spikelets, and the other 60% carry paired spikelets. A seed with single spikelets will produce an ear with single spikelets 29% of the time, whereas a seed with paired spikelets will produce an ear with single spikelets 26% of the time. Consider randomly selecting ten seeds.

1. What is the probability that exactly five of these seeds carry a single spikelet and produce an ear with a single spikelet?

2. What is the probability that exactly five of the ears produced by these seeds have single spikelets? What is the probability that at most five ears have single spikelets?

Solution:

1. Let event A = seed carries single spikelets, and event B = seed produces ears with single spikelets. Then $P(A\cap B) = P(A)\cdot P(B \mid A) = (.40)(.29) = .116$.

Next, let X = the number of seeds out of the 10 selected that meet the condition $A \cap B$. Then $X \sim \text{Bin}(10, .116)$. So,

$$P(X = 5) = \binom{10}{5}(.116)^5(.884)^5 = .002857.$$

2. For any one seed, the event of interest is B = seed produces ears with single spikelets. Using the law of total probability, $P(B) = P(A\cap B) + P(A'\cap B) = (.40)(.29) + (.60)(.26) = .272$.

Next, let Y = the number out of the 10 seeds that meet condition B. Then $Y \sim \text{Bin}(10, .272)$. So

$$P(Y=5)=\binom{10}{5}(.272)^5(1-.272)^5=.0767,$$

while

$$P(Y\leq 5)=\sum_{y=0}^{5}\binom{10}{y}(.272)^y(1-.272)^{10-y}=.041813+\ldots+.076719=.97024.$$

X. (10 points) Suppose a particular state allows individuals filing tax returns to itemize deductions only if the total of all itemized deductions is at least \$5000. Let X (in 1000s of dollars) be the total of itemized deductions on a randomly chosen form. Assume that X has the p.d.f.

$$f(x)=\begin{cases} k/x^3, & x\geq 5, \\ 0, & \text{otherwise.}\end{cases}$$

1. Find the value of k, the c.d.f. of X and $E(X)$?
2. What is the c.d.f. of $\ln(X/5)$?

Solution:

1. Since $\int_5^\infty k/x^3dx=1$, then $k=50$. By the definition of cdf, we know that

$$F(x)=\begin{cases} \int_5^x 50/t^3dt=1-25/x^2 & x\geq 5 \\ 0 & \text{otherwise}\end{cases}$$

The mean value of X is given by $E(x)=\int_5^\infty 50/x^2dx=10$.

2. By the definition of cdf $Y=\ln(X/5)$, we know that

$$F_Y(y)=P(Y\leq y)=P(\ln(X/5)\leq y)=P(X\leq 5\exp(y))=\int_5^{5\exp(y)}50/t^3dt=1-\exp(-2y)$$

for $y\geq 0$; for $y<0$, we have $F_Y(y)=0$.