CHAPTER 1
INTRODUCTION

There are numerous ways to analyze and solve current technical problems, all of which can be categorized into two sections, *quantitative analysis* and *qualitative analysis*. Both these types of analysis are equally crucial and play an important role in understanding the overall behavior of any physical system. The results obtained from one supplement the results obtained from the other. Thus, a combined analysis involving both of them would yield a complete picture of the behavior of the system. However, we focus exclusively and extensively on qualitative analysis and elaborate on it.

1.1 QUALITATIVE ANALYSIS

Qualitative analysis is a methodology which is used to describe a situation with no pre-determined response, and gain insights by using particular technical practices. Unlike quantitative analysis, there isn't a lot of number crunching to be done in qualitative analysis. Our focus is more on how to use the in-depth explanatory data from a small sample (theoretical), and draw out patterns from concepts and insights. The obtained results are mostly illustrative explanations and corresponding responses. One key differentiator between qualitative and quantitative analysis is that the research question is *fixed/focused* in quantitative analysis, whereas it is *broader, contextual and flexible* in qualitative analysis [1].

1.2 PHYSICAL SYSTEMS

All physical systems are divided into two types, *linear systems* and *nonlinear systems.* Very few systems are truly linear in nature, and the rest of them all exhibit nonlinear behavior. There are many systems for which the nonlinearities are important and therefore cannot be ignored. For these systems, nonlinear analysis and design techniques exist and can be used.

A linear system is a system which satisfies the properties of *superposition* and *homogeneity*. Any system that does not satisfy these properties is nonlinear in nature. In general, linear systems have one equilibrium point (a point where the slope is equal to zero) at the origin. Nonlinear systems may have many equilibrium points.

The principle of superposition states that for two different inputs, *x* and *y*, in the domain of the function *f*,

$$f(x + y) = f(x) + f(y)$$

The property of homogeneity states that for a given input, *x*, in the domain of the function *f*, and for any real number *k*,

$$f(kx) = k.f(x)$$

The systems considered are written in the following general form, where *x* is the state of the system, *u* is the control input, *w* is a disturbance, and *f* is a nonlinear function.

$$\frac{dx}{dt} = f\ (t, x, u, w)$$

The systems considered are dynamical systems that are modeled by a finite number of coupled, first-order ordinary differential equations. In many cases we do not consider the disturbance explicitly in the system analysis. Furthermore, in some cases the properties of the system are observed when *f* does not depend explicitly on *u*, that is, $\frac{dx}{dt} = f\ (t, x)$. This is called the *unforced response* of the system. When *f* does not explicitly depend on *t*, that is, if $\frac{dx}{dt} = f(x)$, the system is said to be *autonomous or time invariant***.** We call *x* the *state variables* of the system. The state variables represent the minimum amount of information that needs to be retained at any time *t* in order to determine the future behavior of the system. Although the number of state variables is unique, for a given system, the choice of state variables is not [2].

1.3 LINEAR ANALYSIS OF PHYSICAL SYSTEMS

The linear analysis approach starts with considering the general nonlinear form for a dynamical system, and seeking to transform this system into a linear system for the purposes of analysis. This transformation is called *linearization* and is possible at a selected operating point of the system.

Equilibrium points are an important class of solutions of a differential equation. They are defined as the point x_e such that:

$$\frac{dx}{dt} = 0$$

The study of a nonlinear system starts by finding its equilibrium points. Linearization is often performed about the equilibrium points of the system. They allow one to characterize the behavior of the solutions in the neighborhood of the equilibrium point.

1.4 DYNAMICAL SYSTEMS

There are two main types of dynamical systems: *differential equations* and *iterated maps* (also known as difference equations). Differential equations describe the evolution of systems in continuous time whereas iterated maps arise in problem where time is discrete. Confining our attention to differential equations, the main distinction is between ordinary and partial differential equations. An equilibrium point when considering system dynamics is a point x_0 in the state space of the autonomous system $\frac{dx}{dt} = Ax$ if when the state reaches x_0, it stays at $x_0$for all future time.

A very general framework for ordinary differential equations is provided by the system:

$$\dot{x}_1 = f_1(x_1, \dots, x_n)$$

.

.

$$\dot{x}_n = f_n(x_1, \dots, x_n)$$

Here the over dots denote differentiation with respect to *t*. Thus $\dot{x}_i = \frac{dx_i}{dt}$

The variables $x_1, \dots, x_n$ might represent concentrations of chemicals in a reactor, populations of different species in an ecosystem or the positions and velocities of the planets in the solar system. The functions $f_1, \dots, f_n$ are determined by the problem at hand.

For example, the equation of a damped harmonic oscillator is given as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

The above equation can be rewritten in the form of the above framework, thanks to the following trick: we introduce new variables $x_1 = x$ and $x_2 = \dot{x}$. Then $\dot{x}_1 = x_2$, from the definitions and

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -(\frac{b}{m})x_2 - (\frac{k}{m})x_1$$

This system is said to be *linear*, because all the x_i on the right-hand side appear to the first power only. Otherwise the system would be *nonlinear*. Typical examples of nonlinear terms are products, powers, and functions of the x_i, such as $x_1x_2, (x_1)^3$, or $cosx_2$. For example, the equation of a swinging pendulum: $\ddot{x} + (g/L)sinx = 0$ [3].

CHAPTER 2

LITERATURE REVIEW

2.1 FLOWS ON A LINE

The solutions of the general system of equations (mentioned earlier) could be visualized as trajectories flowing through an n-dimensional phase space with coordinates $(x_1, \ldots, x_n)$. Taking $n = 1$, a simple case, we get a single equation of the form

$$\dot{x} = f(t)$$

Here, $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x. Such equations are called *one-dimensional* or *first-order systems*.

NOTE: We do not allow f to depend explicitly on time. Time-dependent or "non-autonomous" equations are more complicated as now we have two state variables to deal with.

2.2 GEOMETRIC WAY OF THINKING

One of the most basic techniques for analyzing nonlinear system dynamics is *"interpreting a differential equation as a vector field"*.

Consider the following nonlinear differential equation:

$$\dot{x} = sin\, x$$

Upon separating the variables and integrating, we can obtain an exact solution to the equation, but it is very difficult to interpret. In contrast, a graphical analysis is clear and simple. We think of t as time x as the position of an imaginary particle moving along a real line, and $\dot{x}$ as the velocity of that particle.

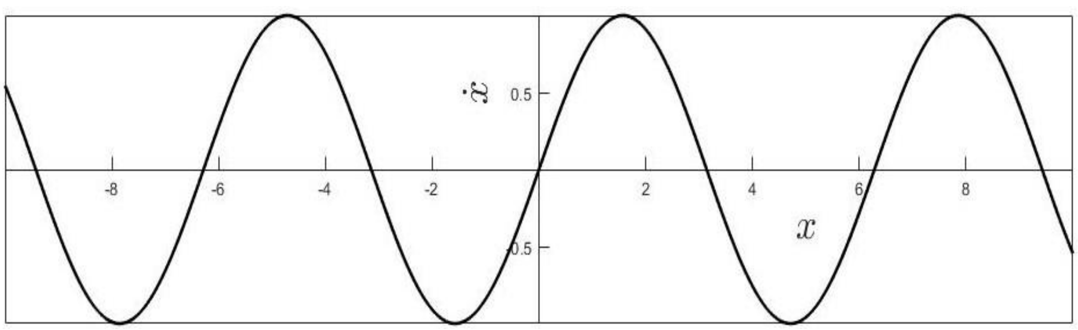

Figure 2.2.1: $\dot{x} = sin\, x$

The differential equation $\dot{x} = \sin x$ represents a *vector field* on the line: it indicates the velocity vector $\dot{x}$ at each x. To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the $x - axis$ to indicate the corresponding velocity vector at each x. The arrow points to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$ [4]. This is because as $\dot{x}$ becomes positive, the rate of change of roll angle keeps on increasing, thus increasing the roll angle and vice versa.

2.3 FIXED POINTS AND STABILITY

A more physical way to think about the vector field is to imagine a fluid flowing steadily along the $x - axis$ with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. The flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At the points where $\dot{x} = 0$, there is no flow; such points are therefore called *fixed points*. There are two kinds of fixed points. The solid black dots represent *stable* fixed points (often called *attractors* or *sinks*, because the flow is towards them) and open circles represent *unstable* fixed points (also known as *repellers* or *sources*).

Figure 2.3.1 shows that a particle starting at say, $x_0 = \frac{\pi}{4}$ moves to the right faster and faster until it crosses $x = \frac{\pi}{2}$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus the qualitative form of the solution is as shown in figure 2.3.1.

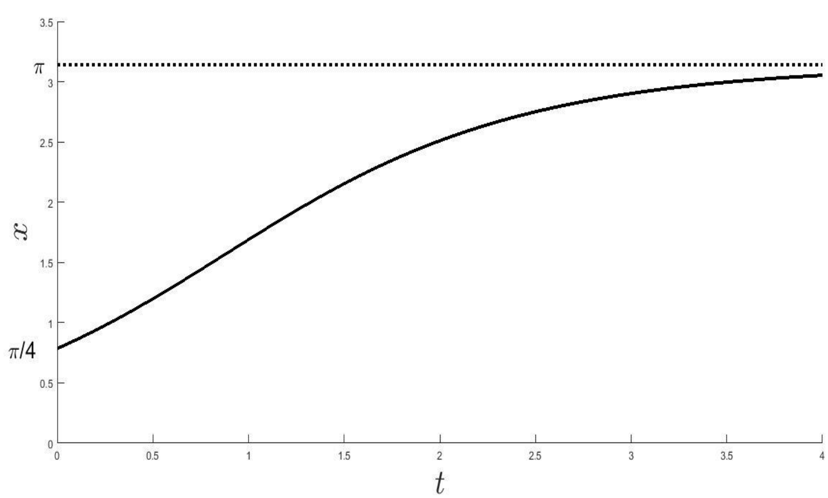

Figure 2.3.1: Qualitative Solution

The qualitative form of the solution for any initial condition is shown in figure 2.3.2. **NOTE**: These qualitative sketches cannot tell us certain quantitative things: for instance, we don't know the time at which the speed $|\dot{x}|$ is greatest.

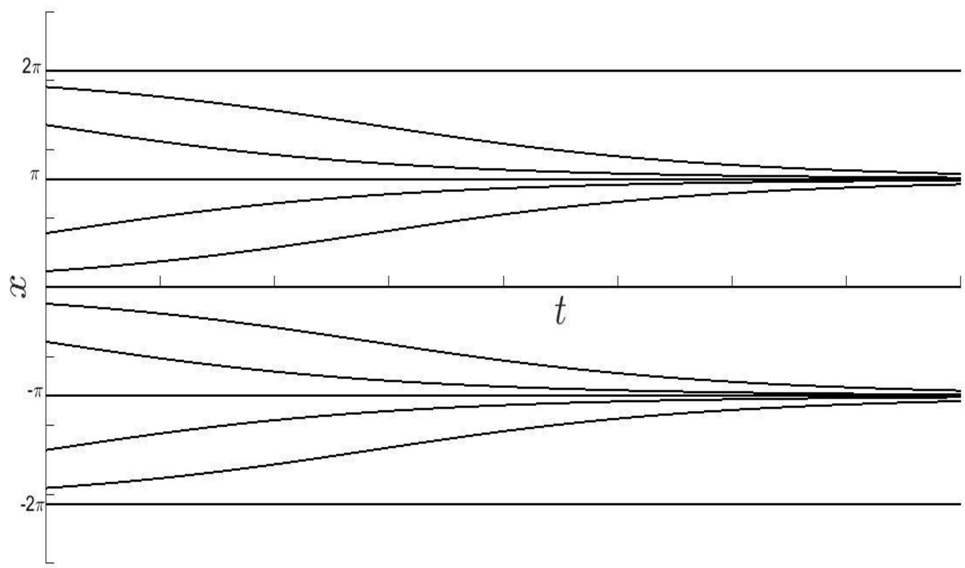

Figure 2.3.2: Qualitative Solution at any Initial Condition

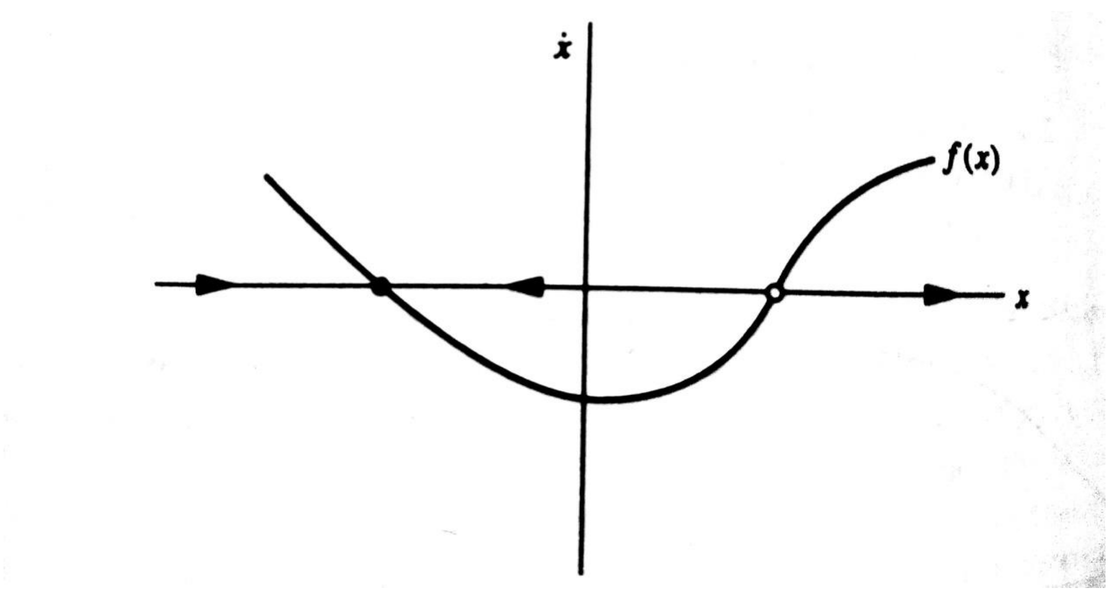

Figure 2.3.3: A Vector Field

The ideas developed in the last section can be extended to any one-dimensional system $\dot{x} = f(x)$ we just need to draw the graph of $f(x)$ and then use it to sketch the vector field on the real line. As before, we imagine that a fluid is flowing along the real line with a local velocity $f(x)$ this imaginary fluid is called the *phase fluid*, and the real line is the *phase space*. The flow is to the right where $f(x) > 0$ and to the left where $f(x) < 0$.

To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0, we place an imaginary particle (known as a *phase point*) at x_0 and watch how it is carried along by the flow. As time goes on, the phase point moves along the $x - axis$ according to some function $x(t)$. This function is called the *trajectory* based at x_0, and it represents the solution of the differential equation starting from the initial condition x_0.

A figure like 2.3.3, which shows all the qualitatively different trajectories of the system, is called a *phase portrait*. The fixed points x^*, defined by $f(x^*) = 0$, correspond to the stagnation points of the flow. The solid black dot is a stable fixed point (the local flow is towards it) and the open dot is an unstable fixed point (the local flow is away from it) [5].

2.4 BIFURCATIONS

The dynamics of vector fields on the line is very limited: all solutions either settle down to equilibrium or head out to $\pm\infty$. However what's interesting in one-dimensional systems is their *dependence on parameters.* The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called *bifurcations,* and the parameter values at which they occur are called *bifurcation points*.

Bifurcations are important scientifically: they provide models of transitions and instabilities as some *control parameter* is varied. For example, consider the buckling of a beam. If a small weight is placed on top of the beam in figure 2.4.1, the beam can support the load and remain vertical. But if the slenderness ratio becomes too high, the vertical position becomes unstable, and the beam may buckle. Here the weight plays the role of the control parameter, and the deflection of the beam from vertical plays the role of the dynamic variable x [6].

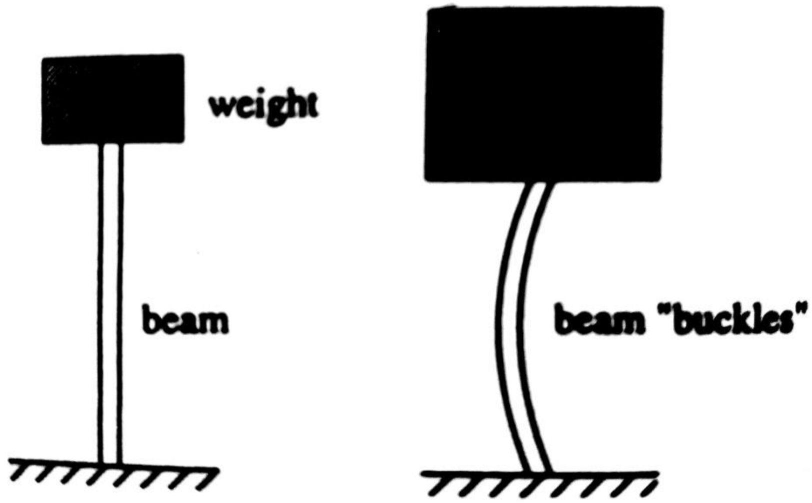

Figure 2.4.1: Beam Buckling under Weight

2.4.1 SADDLE NODE BIFURCATION

The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed*. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate. The prototypical example of a saddle-node bifurcation is given by the first-order system.

$$\dot{x} = r + x^2$$

where *r* is a parameter, which may be positive, negative or zero. When *r* is negative, there are two fixed points, one stable and one unstable (Figure 2.4.1.1a).

When *r* approaches 0 from below, the parabola moves up and the two fixed points move toward each other.

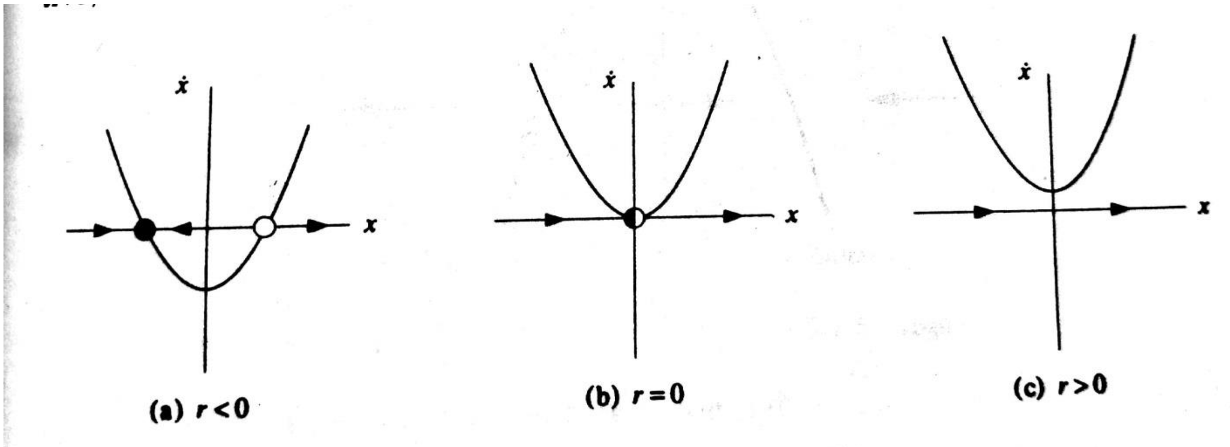

Figure 2.4.1.1: Saddle-node Bifurcation

When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x* = 0$. This type of fixed point is extremely delicate. It vanishes as soon as $r > 0$ and now there are no fixed points. In this example, we say that a *bifurcation* occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

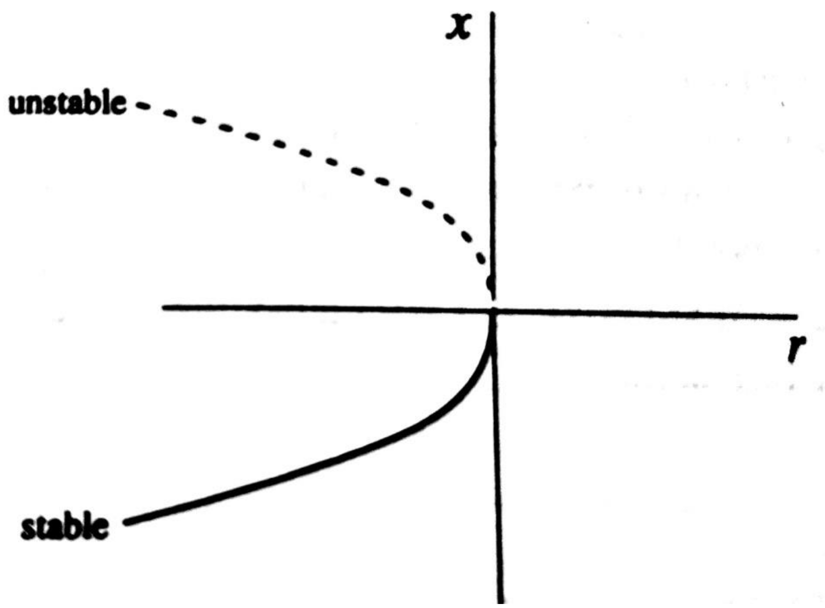

Figure 2.4.1.2: Graphical Notation of Saddle-node Bifurcation

The above representation emphasizes the dependence of the fixed points on r. The curve shown is $r = -x^2$, i.e., $\dot{x} = 0$, which gives the fixed points for different r. To distinguish between stable and unstable fixed points, we use a solid line for stable points and a broken line for unstable ones [7].

2.4.2 TRANSCRITICAL BIFURCATION

There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. However, such a fixed point may *change its stability* as the parameter is varied. The transcritical bifurcation is the standard mechanism of such changes in stability. The normal form for a transcritical bifurcation is

$\dot{x} = r.x - x^2$ Figure 2.4.2.1 shows the vector field as r varies. Note that there is a fixed point at $x* = 0$ for *all* values of r.

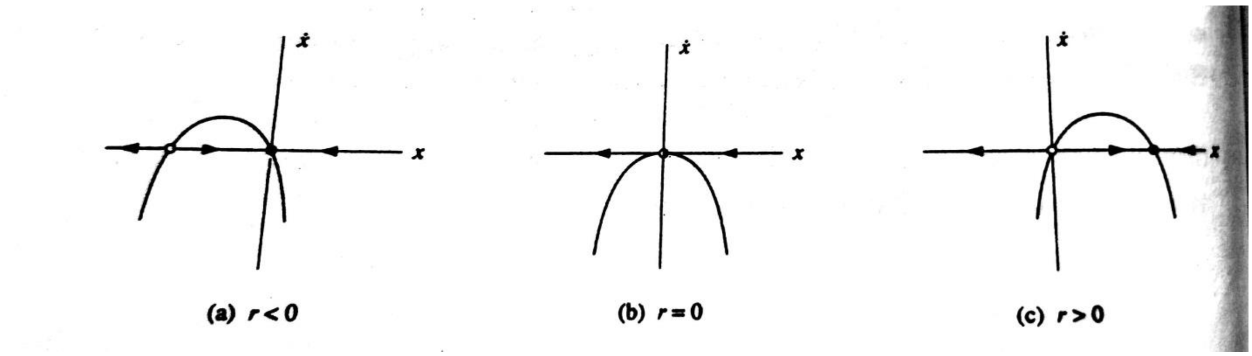

Figure 2.4.2.1: Transcritical Bifurcation

For $r<0$, there is an unstable fixed point at $x^* = r$ and a stable fixed point at $x^* = 0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r = 0$. Finally, when $r > 0$, the origin has become unstable, and $x^* = r$ is now stable. It is said that an *exchange of stabilities* has taken place between the two fixed points [8].

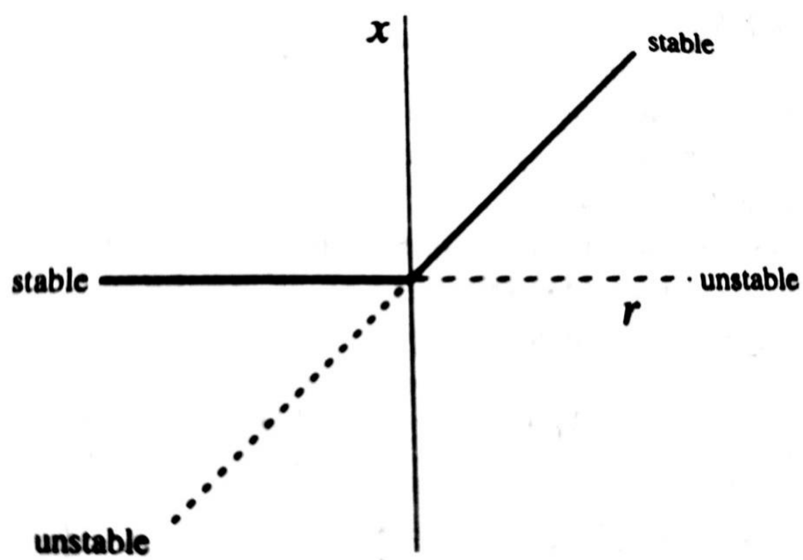

Figure 2.4.2.2: Transcritical Bifurcation Diagram

2.4.3 PITCHFORK BIFURCATION

The pitchfork bifurcation is common in physical *problems* that have *symmetry*. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. There are two very different types of pitchfork bifurcations. The simpler type is called *supercritical bifurcation*. The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = r.x - x^3$$

Note that this equation is *invariant* under the change of variables $x \rightarrow -x$.

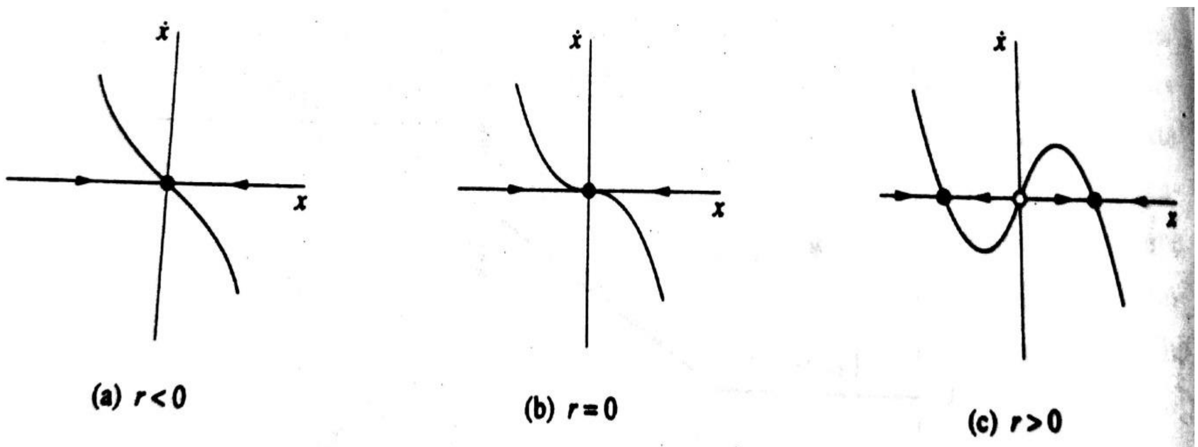

Figure 2.4.3.1: Supercritical Pitchfork Bifurcation

Figure 2.4.3.1. shows the vector field for different values of r. When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast-instead the decay is a much slower algebraic function of time. Finally when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.

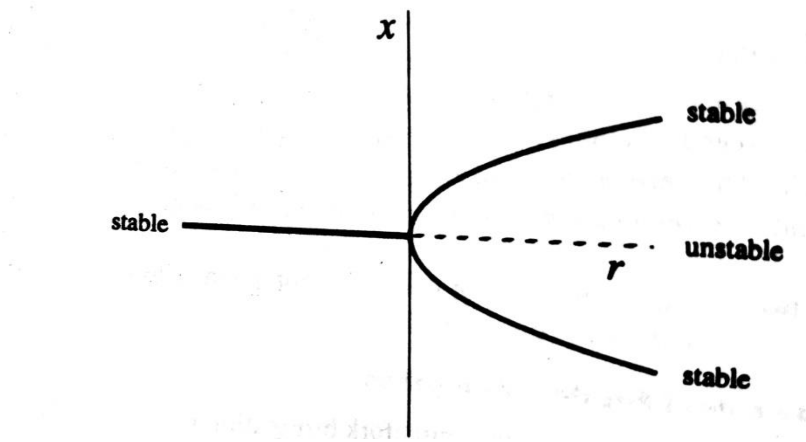

Figure 2.4.3.2: Supercritical Pitchfork Bifurcation Diagram

CHAPTER 3
METHODOLOGY USED

There are many methods to solve differential equations with nonlinearities. To name a few are the above mentioned geometric way of thinking, linear stability analysis and the potential approach. However, these are only valid for first-order nonlinear systems. For higher order systems, certain specific techniques are used. *Phase Plane Analysis* is one such technique which is exclusively limited for second-order systems. For second order systems, solution trajectories can be represented by curves in the plane, which allows for visualization of the qualitative behavior of the system. In particular, it is of interest to consider the behavior of systems around equilibrium points. Under certain conditions, stability information can be inferred from this approach.

3.1 PHASE PLANE ANALYSIS:

Phase plane analysis is a technique for the analysis of the qualitative behavior of second-order systems. It provides physical insights regarding the behavior of these systems. Let us consider a second-order system described by the following equations:

$$\dot{x}_1 = p(x_1, x_2)$$
$$\dot{x}_2 = q(x_1, x_2)$$

Where, $x_1\ and\ x_2$ are the states of the system, and $p\ and\ q$ are two nonlinear functions of the states. Now, a *phase plane* is defined as a plane having $x_1\ and\ x_2$ as coordinates. This is primarily done to get rid of time.

$$\frac{dx_2}{dx_1} = \frac{q(x_1, x_2)}{p(x_1, x_2)}$$

The above equation can be solved analytically by the method of integration. However, the solutions are implicit functions in $x\ and\ y$, and are difficult to interpret. Hence, we follow a different approach. We look for the equilibrium points of the system (also called singular points), i.e. points at which:

$$p\left(x_{1_e}, x_{2_e}\right) = 0$$
$$q\left(x_{1_e}, x_{2_e}\right) = 0$$

After obtaining the equilibrium points, we investigate the linear behavior of the system about a singular point:

$$x_1 = x_{1_e} + \delta x_1$$

$$x_2 = x_{2_e} + \delta x_2$$

$$\delta \dot{x}_1 = \left.\frac{\partial p}{\partial x_1}\right|_e \delta x_1 + \left.\frac{\partial p}{\partial x_2}\right|_e \delta x_2$$

$$\delta \dot{x}_2 = \left.\frac{\partial q}{\partial x_1}\right|_e \delta x_1 + \left.\frac{\partial q}{\partial x_2}\right|_e \delta x_2$$

Set

$$a = \left.\frac{\partial p}{\partial x_1}\right|_e, b = \left.\frac{\partial p}{\partial x_2}\right|_e, c = \left.\frac{\partial q}{\partial x_1}\right|_e, d = \left.\frac{\partial q}{\partial x_2}\right|_e$$

Then

$$\delta \dot{x}_1 = a\delta x_1 + b\delta x_2$$

$$\delta \dot{x}_2 = c\delta x_1 + d\delta x_2$$

This can be written in matrix form as:

$$\dot{X} = AX, with\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Here, A is called the *Jacobian matrix*. The above form is the general form of a second-order linear system. Such a system is linear in nature in the sense that if $x_1 and\ x_2$ are solutions, then so is any linear combination $c_1x_1 + c_2x_2$. Notice that $\dot{x} = 0\ when\ x = 0$, so the origin is always an equilibrium point for any choice of A. The solutions of $\dot{X} = AX$ can be visualized as trajectories moving on the (x_1, x_2) plane, in this context called the phase plane [9].

The next step is to obtain the characteristic equation:

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

which gives:

$$(\lambda - a)(\lambda - d) - bc = 0$$

This equation admits the roots:

$$\lambda_{1,2} = \frac{a+d}{2} \pm \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Thus, we obtain two Eigen values, λ_1 & λ_2

$\boldsymbol{\lambda_1}$& $\boldsymbol{\lambda_2}$	**Nature of solutions**
Real and negative	Stable node
Real and positive	Unstable node
Real and have opposite signs	Saddle point
Complex and negative real parts	Stable focus
Complex and positive real parts	Unstable focus
Complex and zero real parts	Center

Table 3.1: Nature of Eigen Values and Corresponding Solutions

CHAPTER 4
SHIP DYNAMICS UNDER PURE ROLL MOTION

4.1 INTRODUCTION

A ship motion is defined by the six degrees of freedom (6DOF) that it can experience. The six degrees of freedom of a ship are **surge, heave** and **sway** (translational motions) and **roll, yaw** and **pitch** (rotational motions). Instability in any one or more of these motions may lead to marine accidents. Most current design rules for ships do not consider their dynamic behavior in irregular seas. Large roll motions can cause major damages to cargo and can lead to capsize in the worst case. Hence, risk assessment of ships in random seas is very important for the development of new intact stability criteria and also for the first design stage, where many ship designs have to be compared [10].

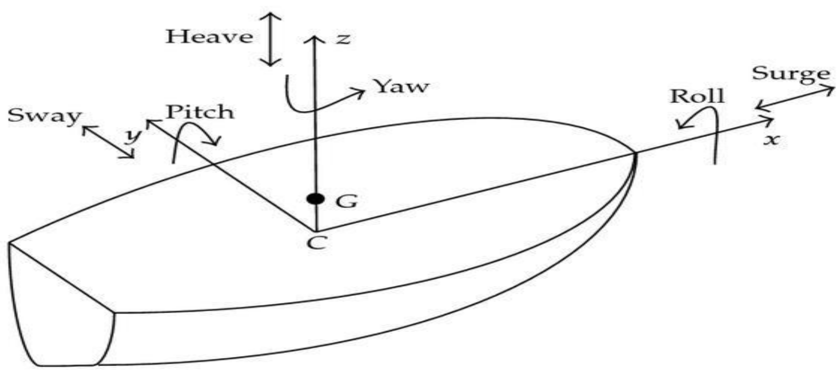

Figure 4.1.1: Degrees of Motion: Ship [11]

4.2 DESCRIPTION OF OCEAN WAVES

The irregular wave surface of an ocean can be modeled by a superposition of infinitely many harmonic waves with frequencies corresponding to a spectral density and having a random phase shift. The irregular wave surface is approximated by an effective wave

$$Z_{eff} = \eta_c \cos\left(\frac{2\pi}{L}x\right) + \eta_s \sin\left(\frac{2\pi}{L}x\right) = \eta cos\left(\frac{2\pi}{L}x + \psi\right)$$

It consists of two harmonic components with random amplitudes η_c and η_s which can be transformed into a cosine wave with random amplitude η and random phase ψ. The wavelength chosen is equal to the ship length L.

4.3 ROLL MOTION IN RANDOM SEAS

The rolling behavior of a ship can be represented by the following equation [12]

$$[I_{xx} + A_{xx}(\omega_n)]\ddot{\phi}(t) + b_1\dot{\phi}(t) + b_3\dot{\phi}(t)^3 + g\Delta GZ_{app}\big(\phi(t), \psi(t), \eta(t)\big) = M(t),$$

where ϕ is the roll angle, I_{xx} is the roll moment of inertia, $A_{xx}(\omega_n)$ is the hydrodynamic added mass (added mass or virtual mass is the inertia added to a system because an accelerating or decelerating body must move or deflect some volume of surrounding fluid as it moves through it) evaluated at the natural frequency ω_n, b_1 and b_3 are linear and cubic damping coefficients, g is the acceleration due to gravity, Δ is the displacement, and M is an additive excitation moment, which is small, if the ship travels in about the same direction as the waves. With the scaled time $\tau = \omega_n t$ we get the non dimensional roll equation

$$\ddot{y} + y = \varepsilon(-\beta_1\dot{y} - \beta_3\dot{y}^3 + \alpha_3 y^3) + \sqrt{\varepsilon}(\gamma_0 + \gamma_1 y)\xi \tag{4.1}$$

Here, $\xi = \eta\cos(\psi)$. The positive coefficient α_3 results in a softening spring. Therefore, at a certain roll angle the restoring moment is zero and can lead to capsize. We define $\gamma_0 = \tilde{\gamma}_0 sin\vartheta, \gamma_1 = \tilde{\gamma}_1 cos\vartheta$ and introduce the angle ϑ, which is the angle between wave direction and ship direction.

CHAPTER 5

RESULTS AND DISCUSSION

5.1 RESULTS:

The equation (4.1) is the non-dimensional roll equation, on which phase plane analysis is performed and corresponding ship dynamics is observed. The second order differential equation is converted into two first order differential equation.

$$y = x_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \epsilon\,(-\beta_1\, x_2 - \beta_3\, x_2^3 + \alpha_3\, x_1^3) + \sqrt{\epsilon}\,(\gamma_0 + \gamma_1\, x_1)\epsilon_1 - x_1$$

The two first order differential equations are equated to zero to get the fixed points. For the constants that depend on ship geometry, we have considered the values of a vessel named 173m Ro Ro passenger ferry [13].

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = 0 \Rightarrow \epsilon\,\alpha_3\; x_1^3 + \left(\sqrt{\epsilon}\,\gamma_1\;\xi\right)x_1 + \sqrt{\epsilon}\,\gamma_0\,\xi_1 = 0$$

$$\epsilon = 0.001, \beta_1 = 5, \beta_3 = 95, \alpha_3 = 44, \hat{\gamma}_0 = \hat{\gamma}_1 = 1, \vartheta = 0$$

$$\Rightarrow\; 0.44\, x_1^3 + (0.1\,\xi\, cos\vartheta - 1)x_1 + 0.1\xi\, sin\vartheta = 0$$

$$\Rightarrow\; 0.44\, x_1^3 + (0.1\,\xi - 1\, x_1) = 0$$

$$x_1[\,0.44\, x_1^2 + (0.1\,\xi - 1)\,] = 0$$

The equilibrium points depend on the ξ value.

$$x_1 = 0, \pm\sqrt{\frac{(1-0.1)\xi}{0.44}}$$

Number of equilibrium points for $\xi < 10 = 3$

Number of equilibrium points for $\xi > 10 = 1$

The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ 3\,\epsilon\;\alpha_3\;x_1^3 + \xi\sqrt{\epsilon} - 1 & -\epsilon\beta_1 - 3\epsilon\beta_3 x_2^2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ 1.32x_1^2 + 0.1\xi - 1 & -0.05 - 2.85x_2^2 \end{bmatrix}$$

At equilibrium points

$$(0,0), \left(\pm\left(\sqrt{\frac{1-0.1\xi}{0.44}}, 0\right)\right)$$

Eigen values,

$$\lambda = \frac{-0.05 \pm \sqrt{(0.05^2 + 4(1.32x_1^2 + 0.1\xi - 1)}}{2}$$

For, $\xi > 10$

$$\lambda = \frac{-0.05 \pm \sqrt{0.05^2 + 0.1\xi - 1}}{2} \Rightarrow \ Saddle\ point$$

For, $\xi = 10 \Rightarrow (0,0), (0,0), (0,0) \Rightarrow \ \lambda = -0.05, 0$

For, $\xi < 10$

At (0, 0),

$$\xi > 9.99375 \Rightarrow \lambda = -ve(real) \Rightarrow Stable\ node$$

$$\xi = 9.99375 \Rightarrow \lambda = 0.025, -0.025 \Rightarrow \ Degenerate\ node$$

$$\xi < 9.99375 \Rightarrow \lambda = -0.025 \pm i \Rightarrow Stable\ focus$$

At $\pm\left(1 - \frac{0.1\xi}{0.44}, 0\right)$,

$$\lambda = \frac{-0.05 \pm \sqrt{(0.05)^2 + 8(1 - 0.1\xi)}}{2} \Rightarrow Saddle\ point$$

The following are the simulated plots, using MATLAB, corresponding to the same

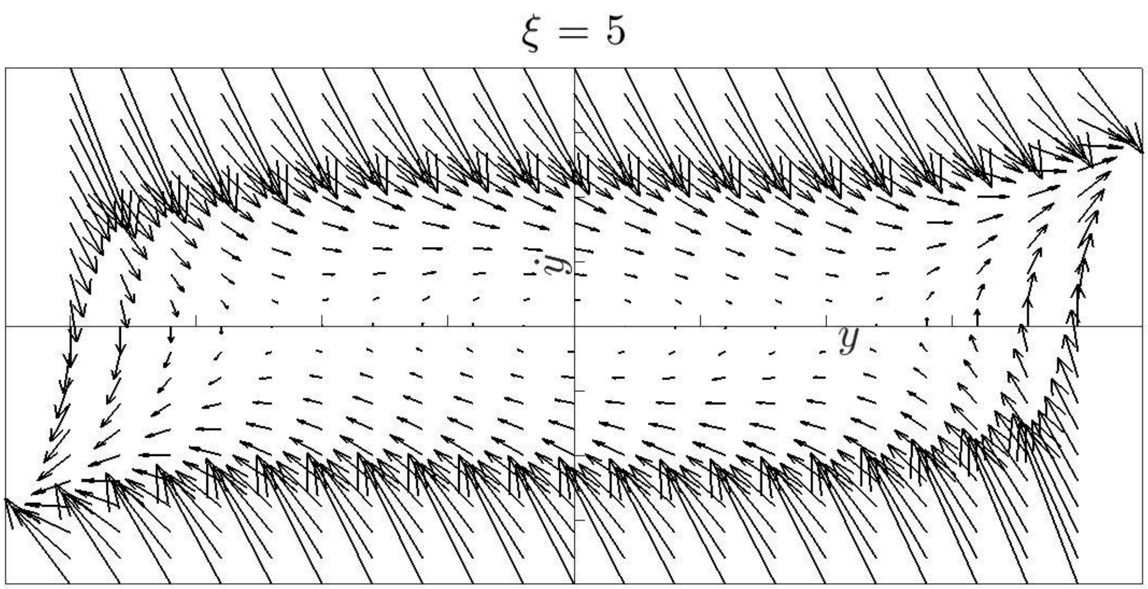

Figure 5.1.1: Phase Portrait for $\xi = 5$

The above plot shows the results for $\xi = 5$. From phase plane analysis, we obtained that there are three fixed points for any $\xi < 9.99375$, one *stable focus* and two *saddle points*. Clearly, we can observe the stable focus at the origin and the two saddle points at either sides of it.

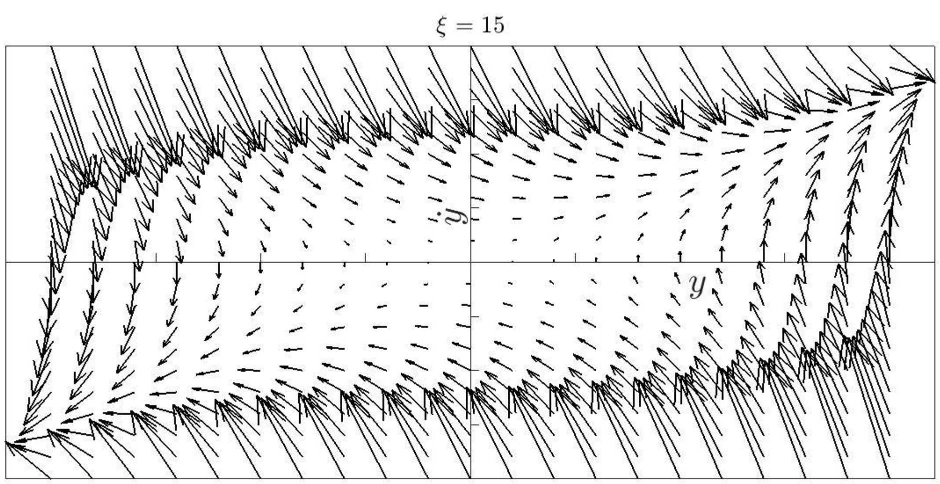

Figure 5.1.2: Phase Portrait for $\xi = 15$

The above plot shows the results for $\xi = 15$. Again, from phase plane analysis, for all $\xi > 10$, we obtain only one fixed point, which is a saddle point.

NOTE: Since ξ has very close tolerance with respect to the nature of fixed points, it is difficult to compare the simulations in MATLAB with hand calculations.

5.2 DISCUSSION

The following conclusions on dynamics of ship are drawn by considering various initial conditions of non-dimensional roll angle and its rate of change. They are presented as the behavior of the roll angle and its rate of change with time, and the resulting plots depict the stability of ship motion.

CASE 1: $\xi = 0, y = 1, \dot{y} = 0$

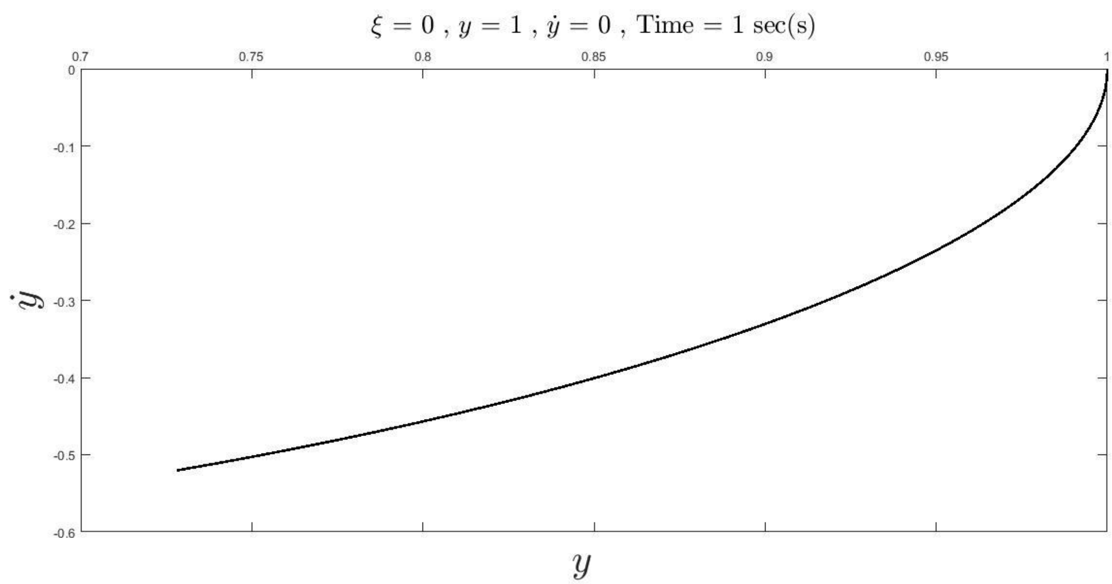

Figure 5.2.1(a): Plot for Initial Conditions: $y = 1$, $\dot{y} = 0$, time = 1 sec(s)

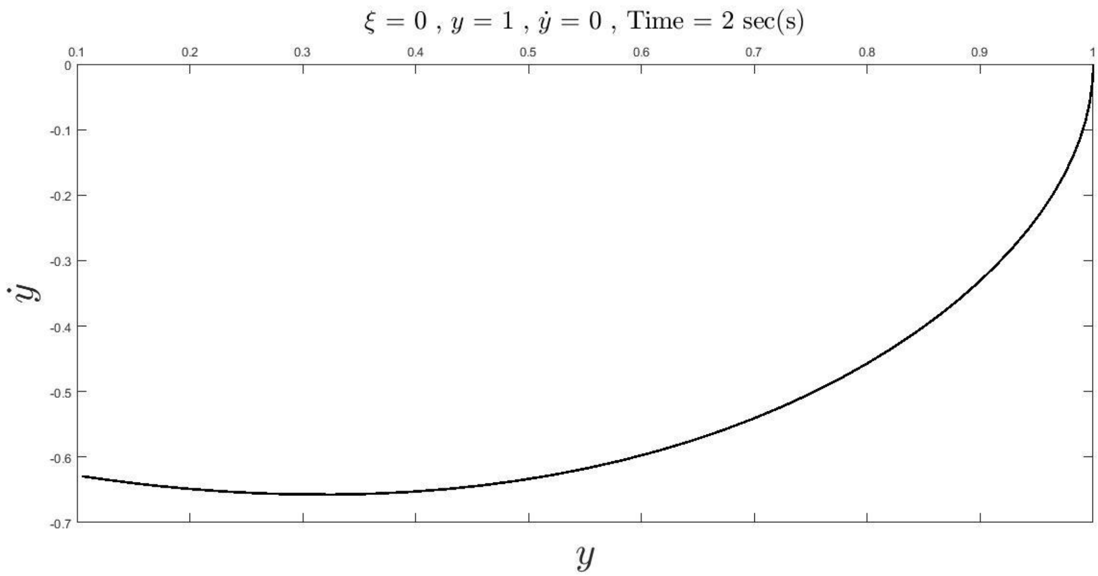

Figure 5.2.1(b): Time = 2 sec(s)

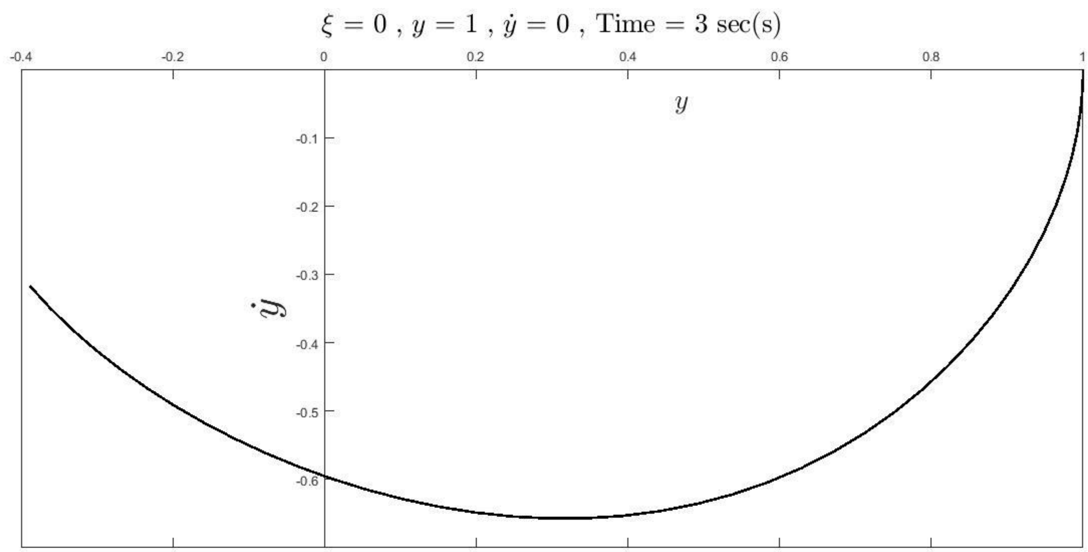

Figure 5.2.1(c): Time = 3 sec(s)

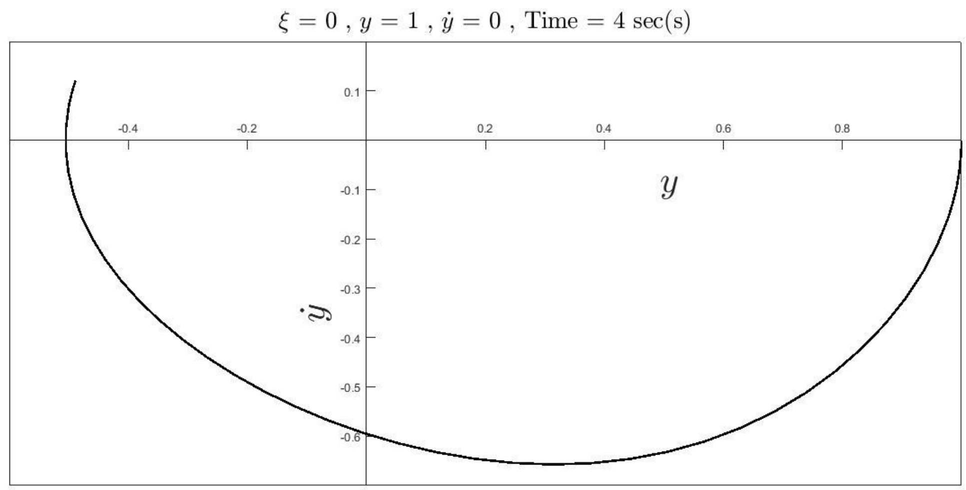

Figure 5.2.1(d): Time = 4 sec(s)

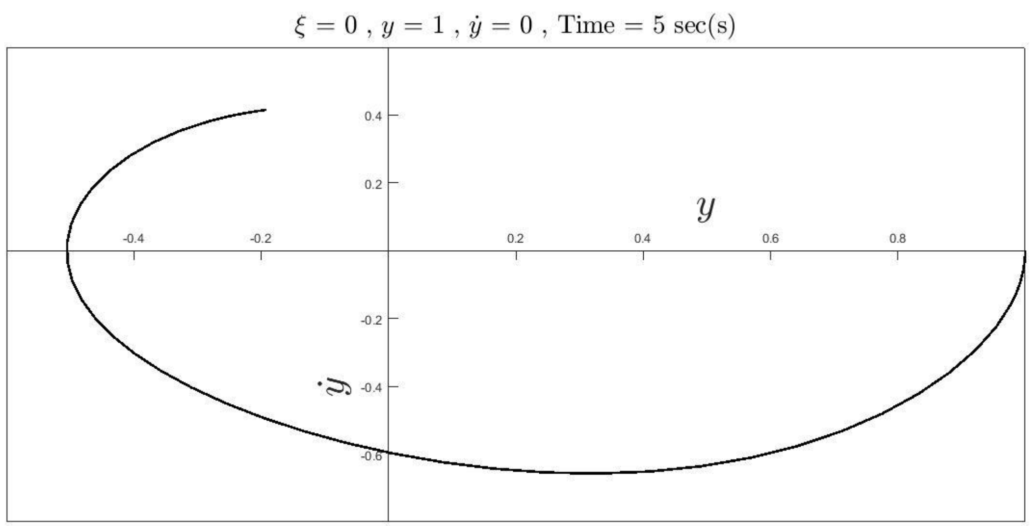

Figure 5.2.1(e): Time = 5 sec(s)

Since for any $\xi < 10$, we have three equilibrium points, we concentrate on the behavior of roll angle in the neighborhood of these points. For the initial conditions $y = 1, \dot{y} = 0$, we can see that roll angle tends to approach the stable focus at 0 with each second (each colored portion indicates the variation of roll angle in 1 second). We can observe in the next plot that it reaches a stable solution in finite time.

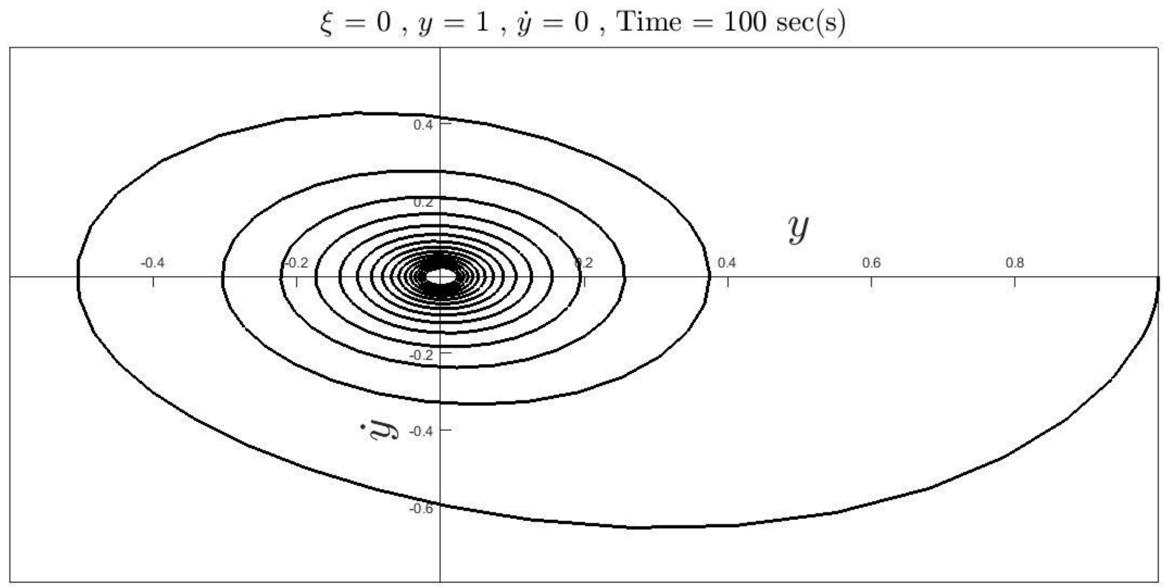

Figure 5.2.2: Behavior of Roll over Time, for $y = 1, \dot{y} = 0$

CASE 2: $\xi = 0, y = 2, \dot{y} = 0$

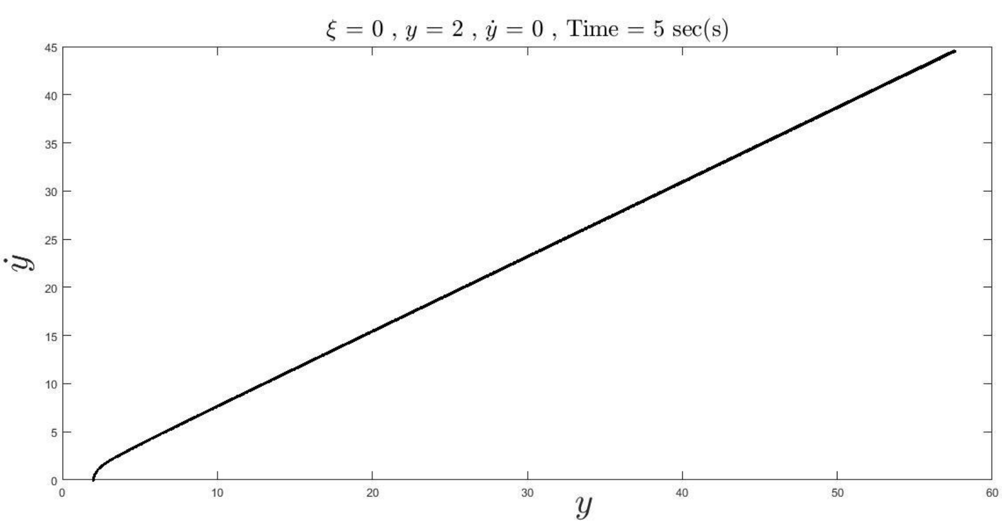

Figure 5.2.3: Plot for Initial Conditions: $y = 2, \dot{y} = 0$

Through manual calculations we have observed that the two saddle points lie close to $y = 1.5$. So we have considered initial conditions $y = 2, \dot{y} = 0$, corresponding to a situation beyond the saddle point. We can observe from the above graph that the roll angle tends to increase rapidly into infinite space. What this actually reflects in real time is that the ship capsizes in finite time under the specified conditions.

CASE 3: $\xi = 0, y = 0, \dot{y} = 1$

The present condition is observed by giving input to the rate of change of roll angle, $\dot{y} = 1$. The behavior of the roll angle, as seen in the plot below, is quite interesting. The roll angle tends to decrease rapidly as its rate of change reaches 0, and then it stabilizes with time. This behavior is observed for all initial conditions of roll angle which lie within the saddle point. This behavior is sustained as long as rate of change of roll angle reaches 6. Once it crosses 6, the behavior changes and is discussed in the next case.

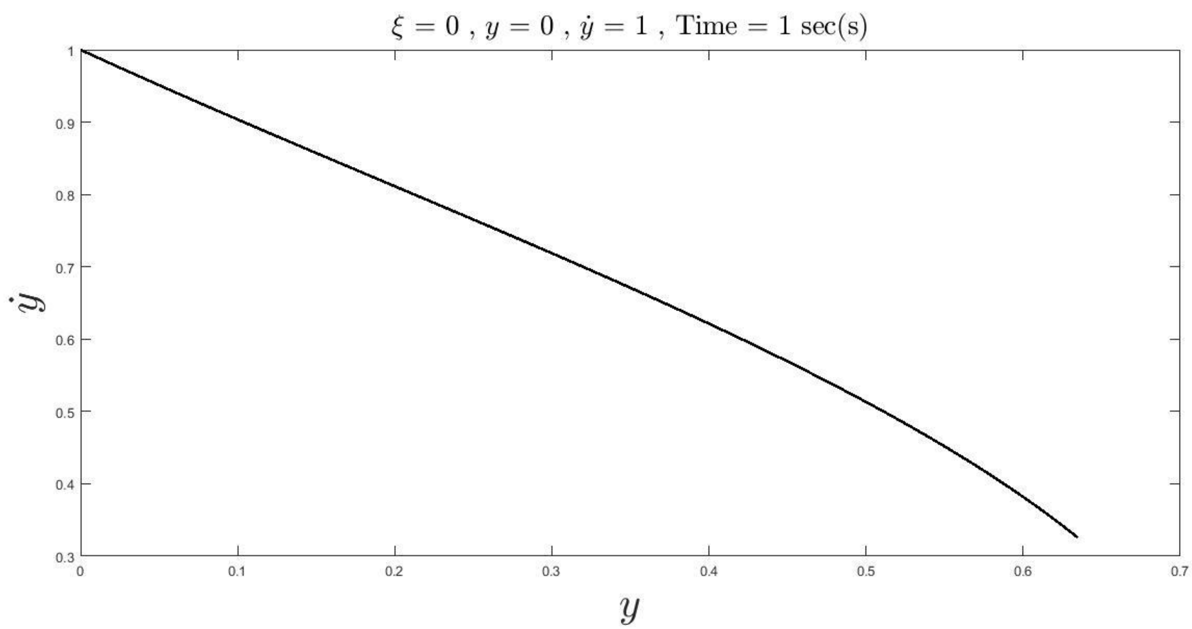

Figure 5.2.4(a): Plot for $y = 0, \dot{y} = 1$, time = 1 sec(s)

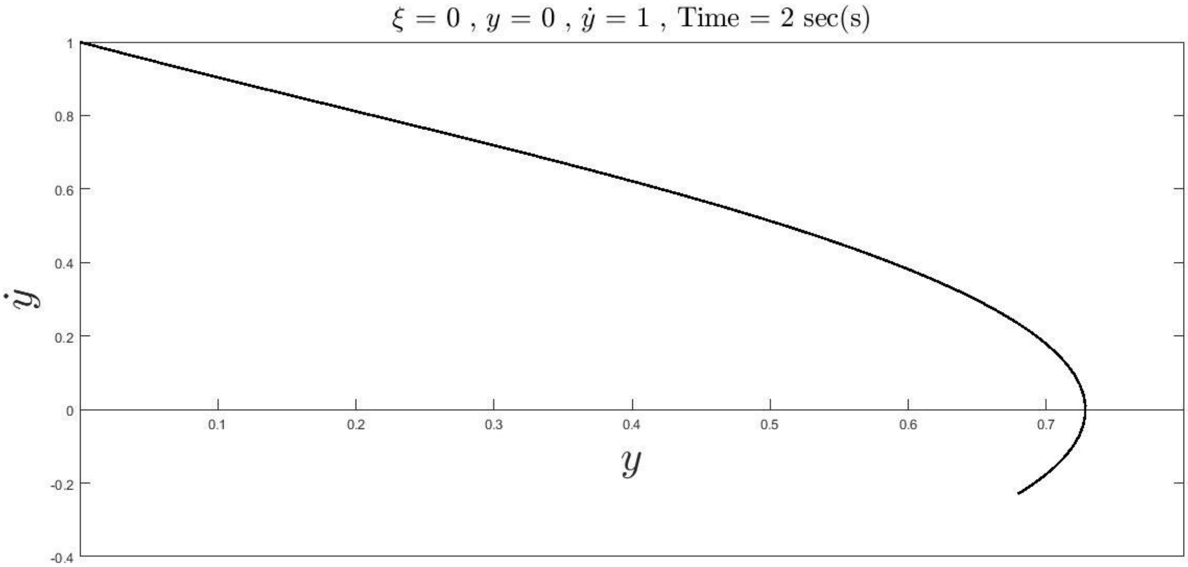

Figure 5.2.4(b): Time = 2 sec(s)

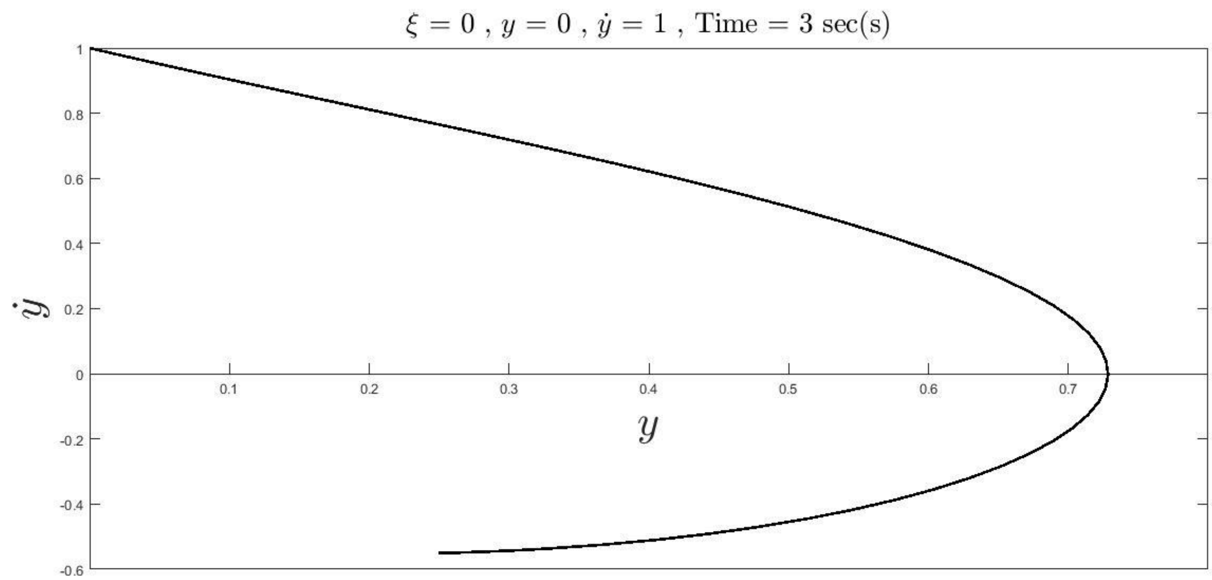

Figure 5.2.4(c): Time = 3 sec(s)

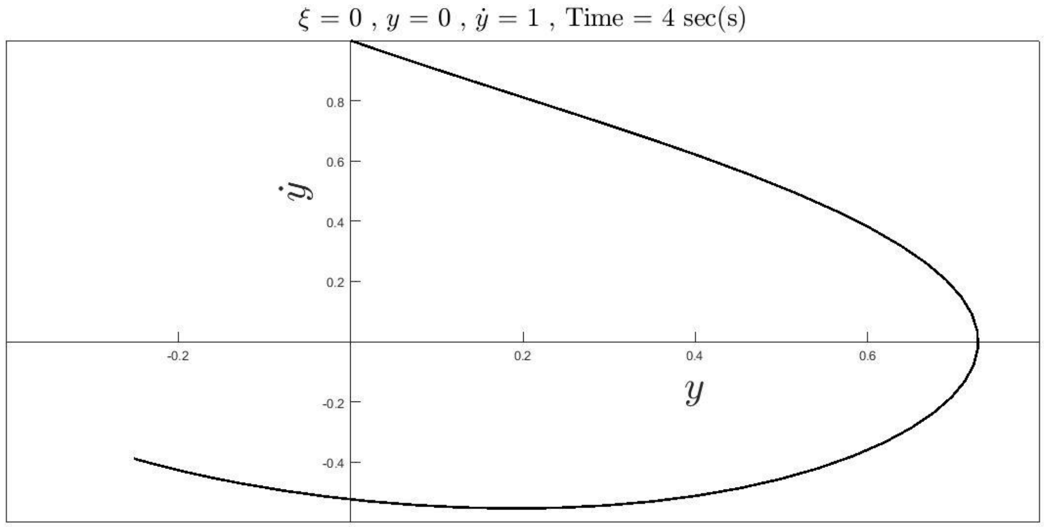

Figure 5.2.4(d): Time = 4 sec(s)

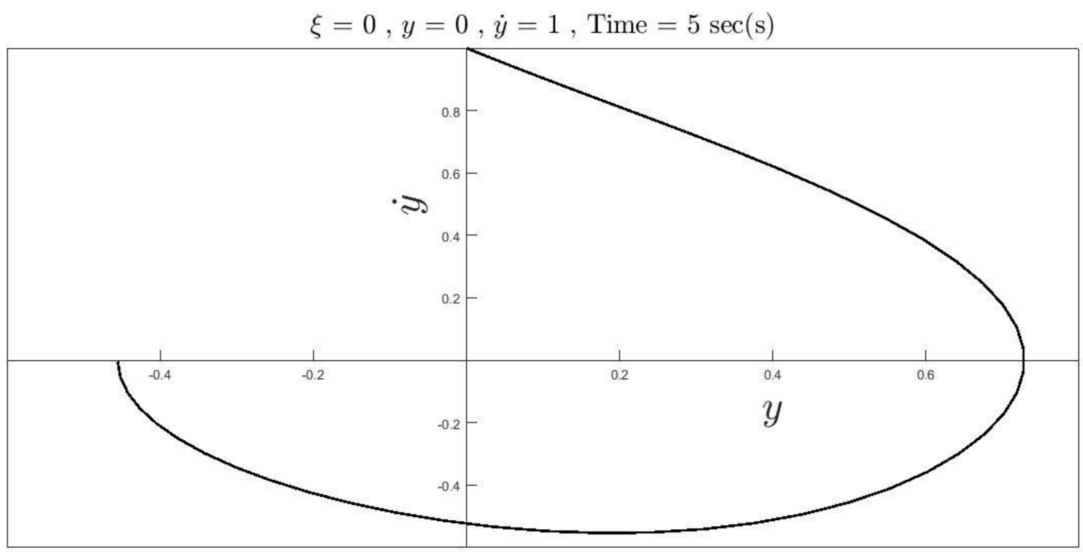

Figure 5.2.4(e): Time = 5 sec(s)

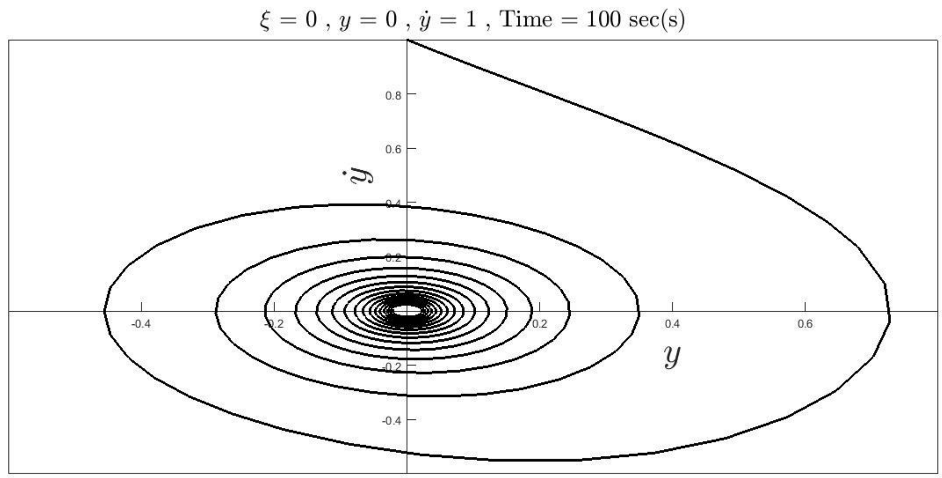

Figure 5.2.5: Behavior of Roll over Time, for $y = 0, \dot{y} = 1$

CASE 4: $\xi = 0, y = 0, \dot{y} = 7$

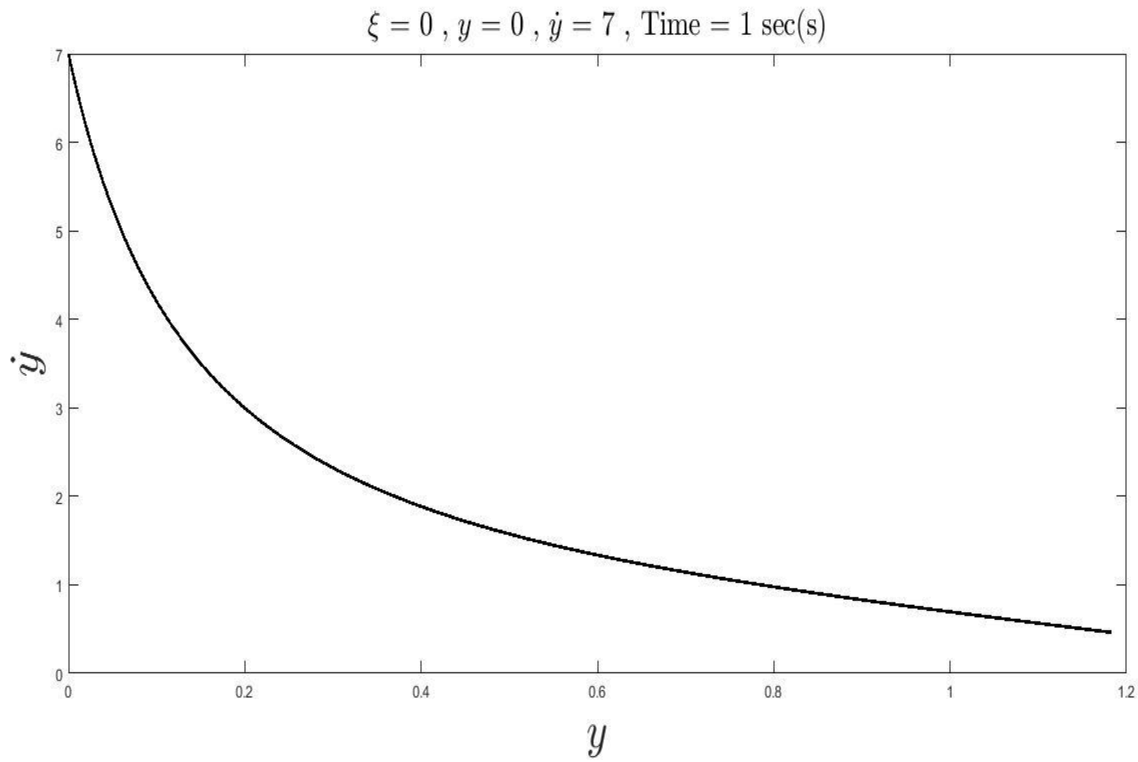

Figure 5.2.6(a): Plot for $\xi = 0, y = 0, \dot{y} = 7$, time = 1 sec(s)

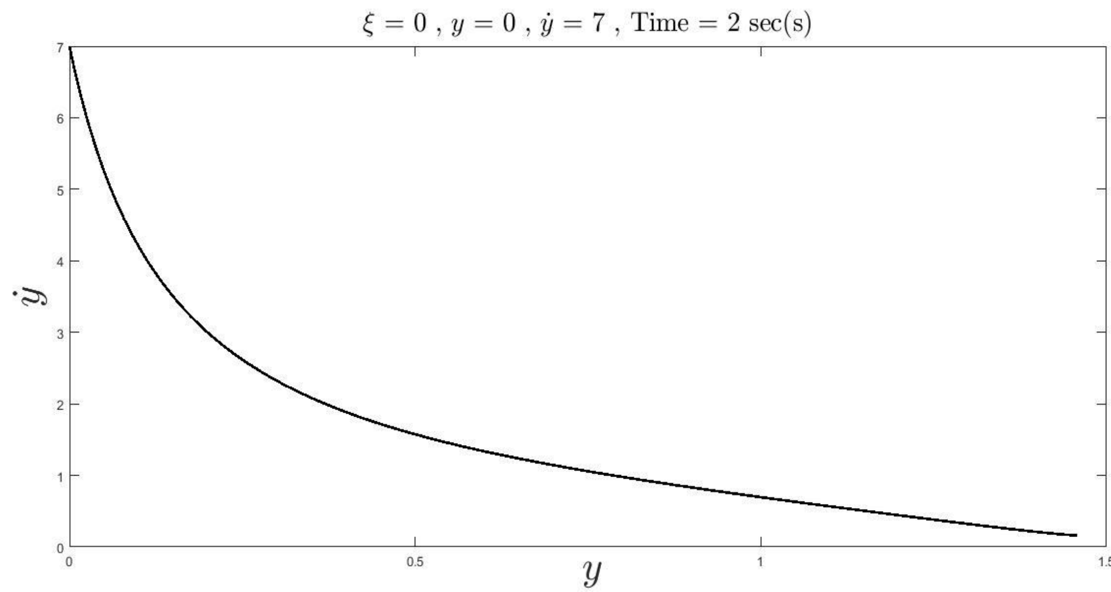

Figure 5.2.6(b): Time = 2 sec(s)

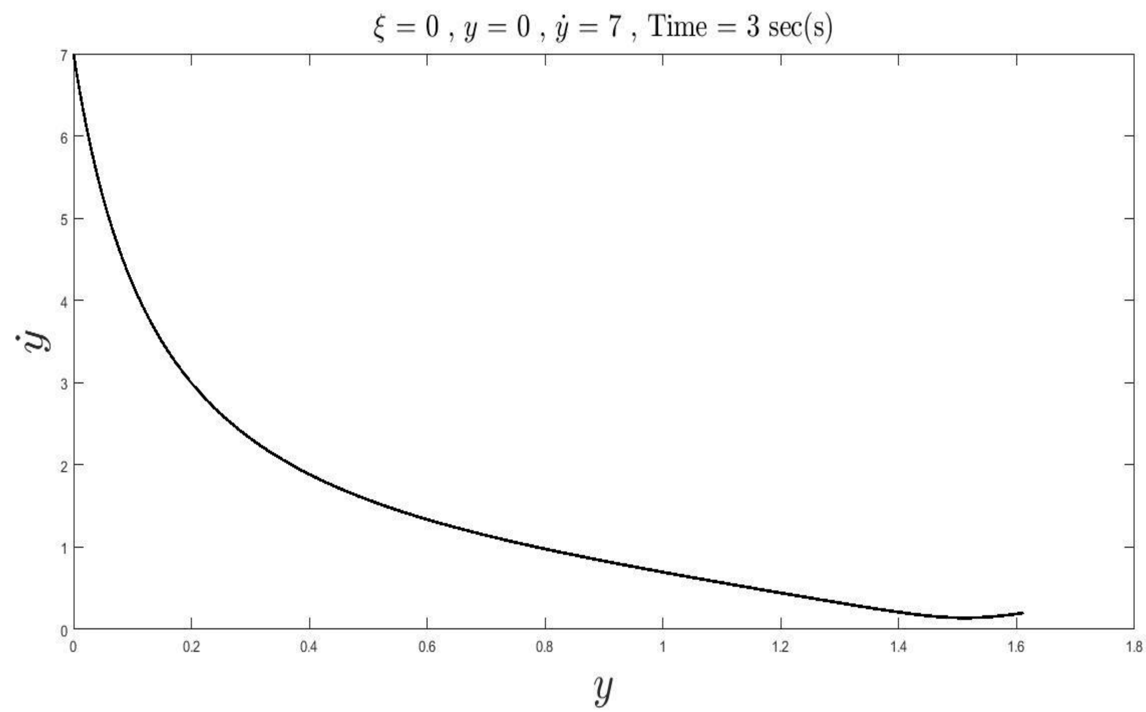

Figure 5.2.6(c): Time = 3 sec(s)

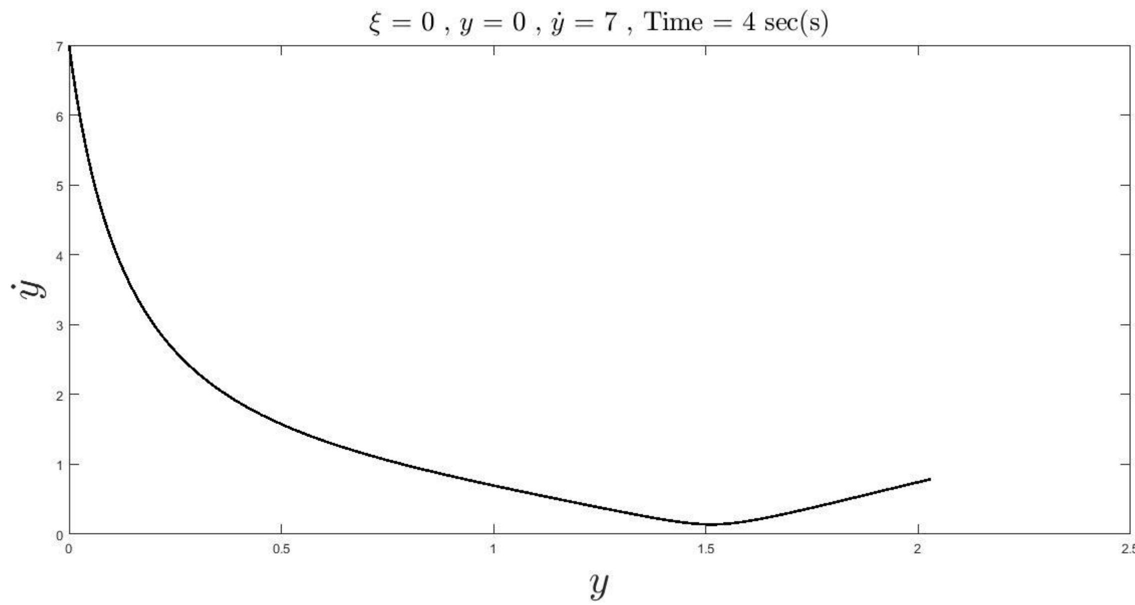

Figure 5.2.6(d): Time = 4 sec(s)

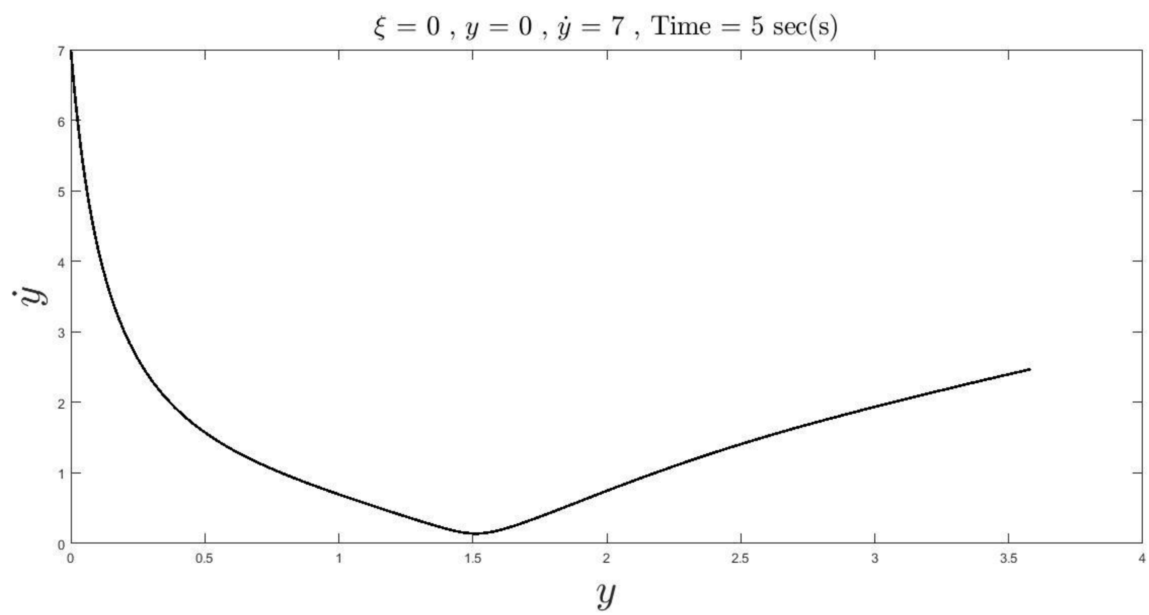

Figure 5.2.6(e): Time = 5 sec(s)

When the rate of change of roll angle, $\dot{y} = 7$, the behavior is quite different from that of the previous case. The roll angle crosses the saddle point after 2 seconds, and then it shoots off to infinity. In real time conditions, this implies that the ship stabilizes for a brief period of time before gaining instability and it ultimately capsizes.

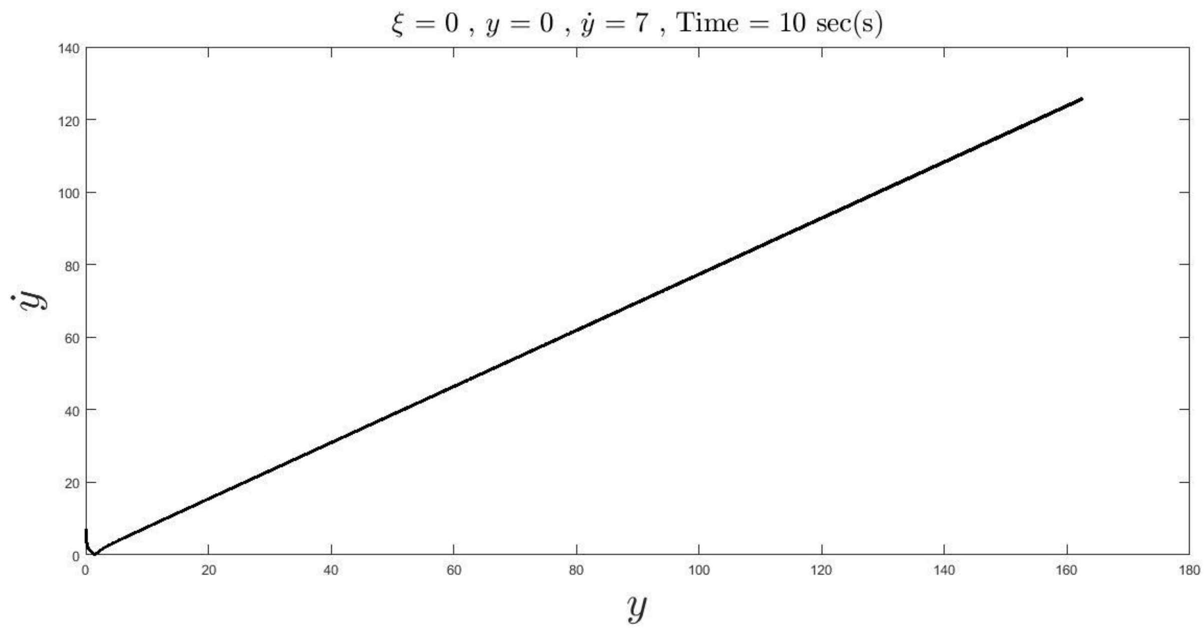

Figure 5.2.6(f): Behavior of Roll over time, for $y = 0, \dot{y} = 7$

CASE 5: $\xi = 0, y = 0.5, \dot{y} = 0.5$

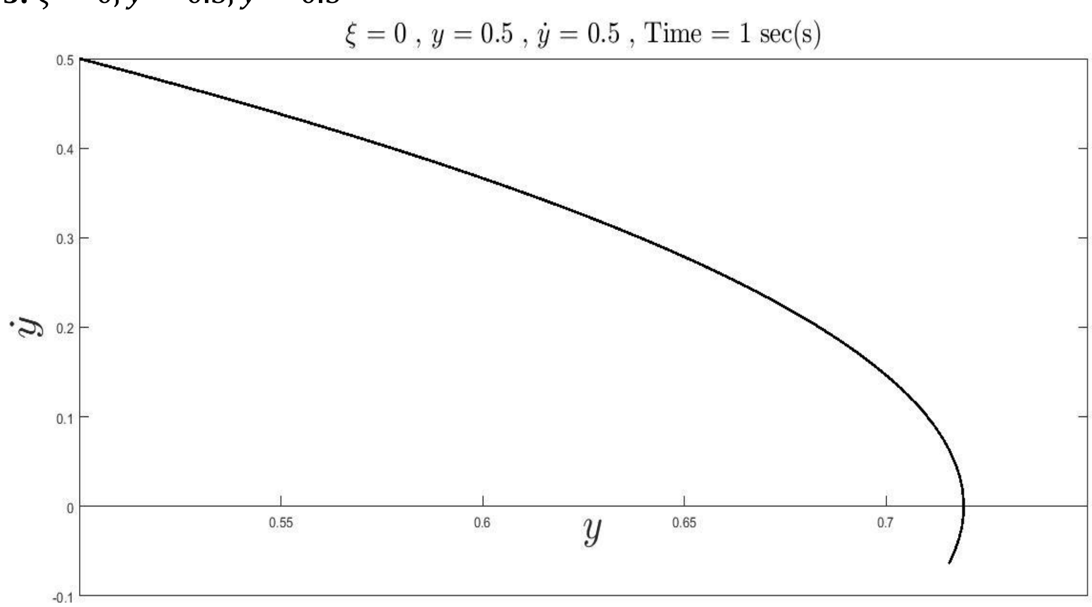

Figure 5.2.7(a): Plot for $\xi = 0, y = 0.5, \dot{y} = 0.5$, time = 1 sec(s)

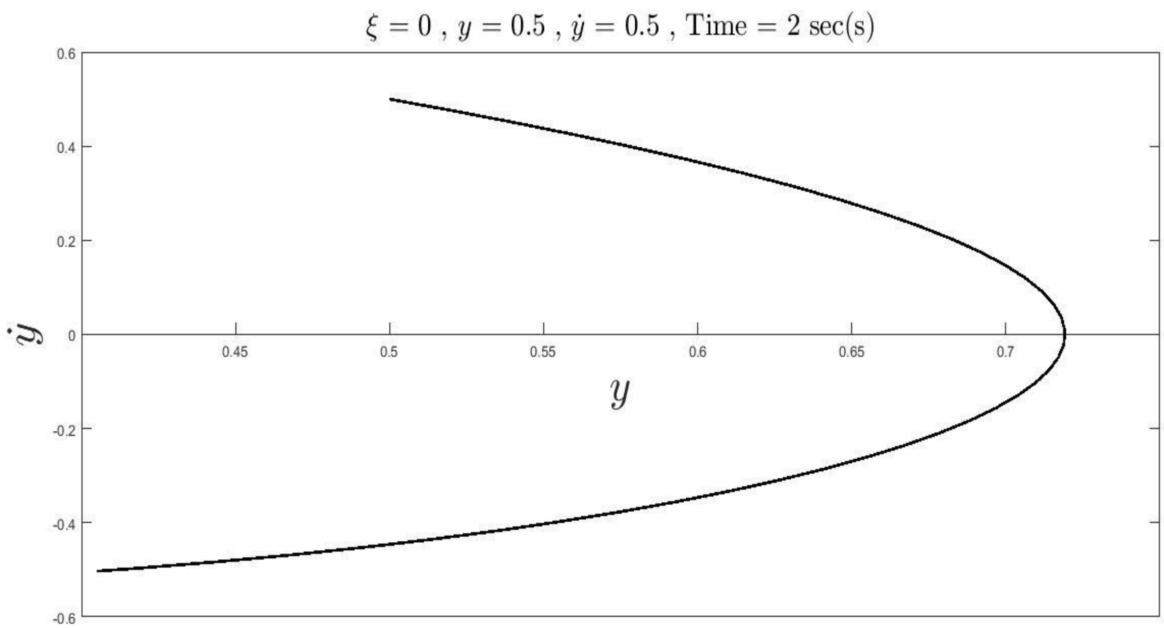

Figure 5.2.7(b): Time = 2 sec(s)

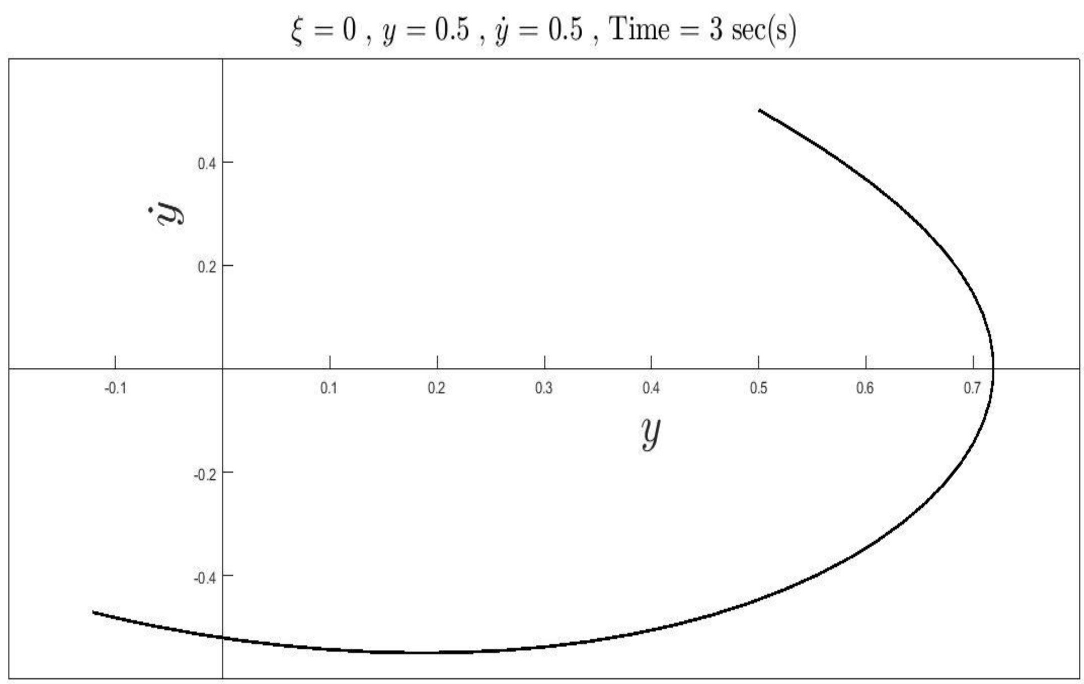

Figure 5.2.7(c): Time = 3 sec(s)

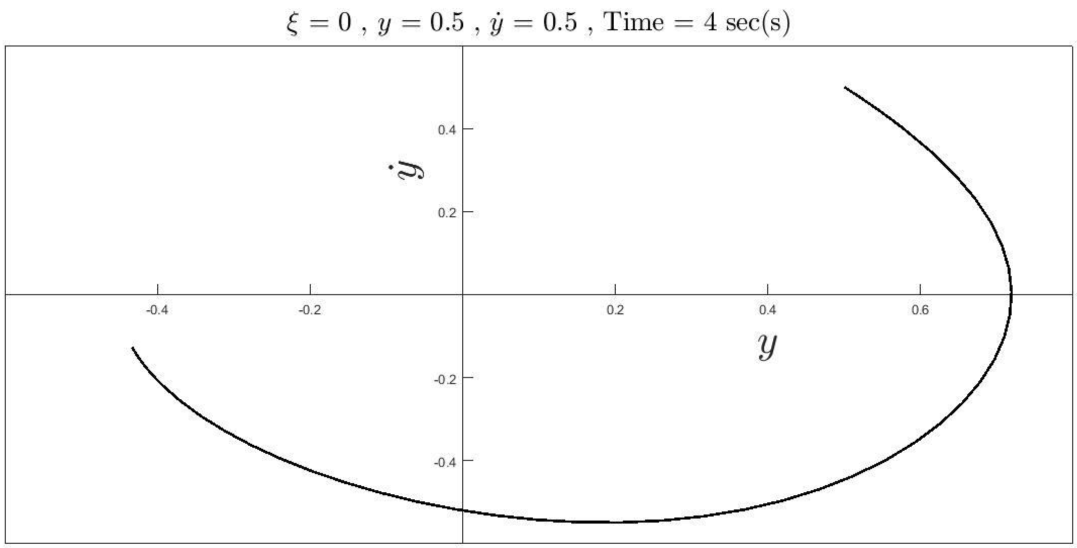

Figure 5.2.7(d): Time = 4 sec(s)

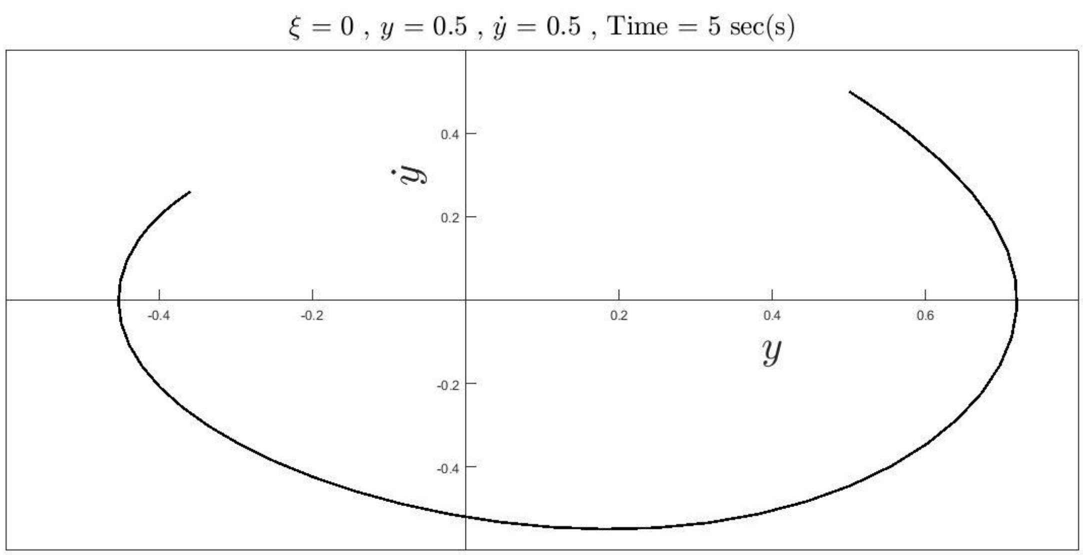

Figure 5.2.7(e): Time = 5 sec(s)

In this case, we have considered certain values of initial conditions for both parameters, y & $\dot{y}$. The behavior is slightly different from that of previous cases. We can observe that the roll angle takes longer time to reach the stable focus. However, it reaches the stable focus in finite time and the ship stabilizes.

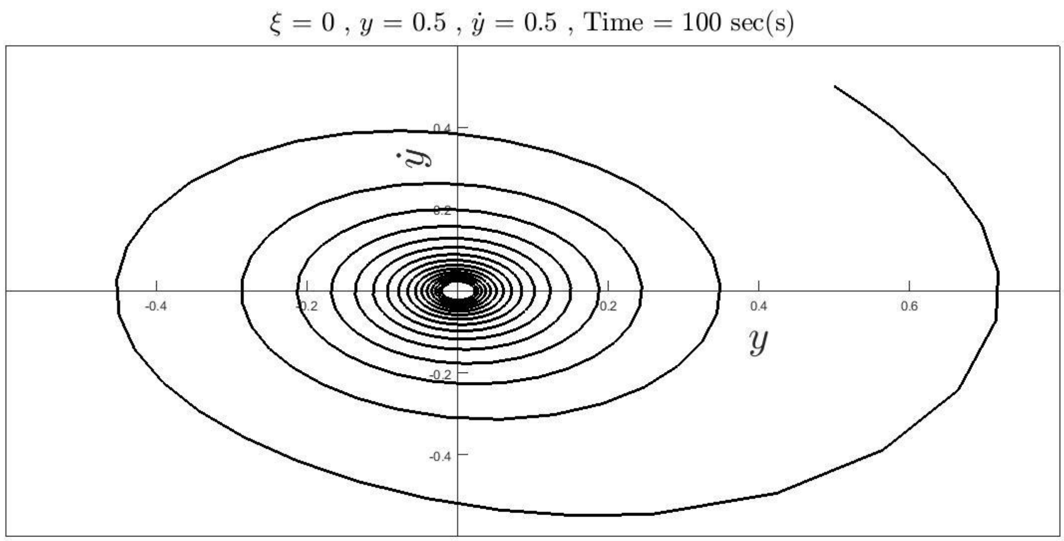

Figure 5.2.8: Behavior of Roll over Time, for $y = 0.5, \dot{y} = 0.5$

CASE 6: $\xi = 0, y = 1, \dot{y} = 1$

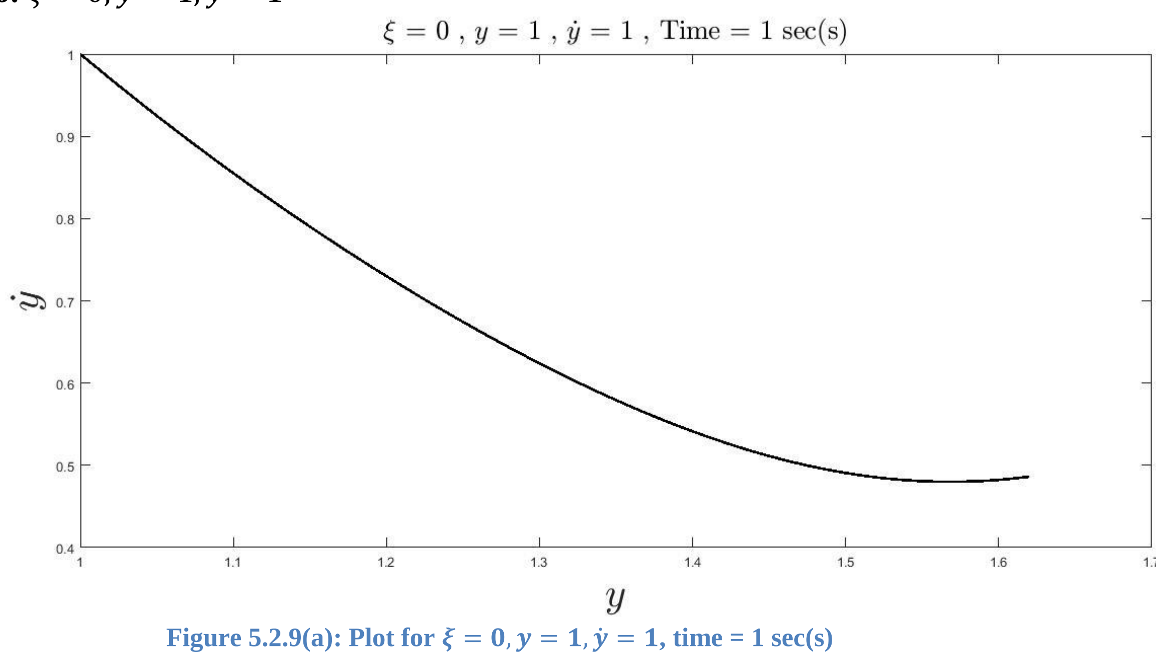

Figure 5.2.9(a): Plot for $\xi = 0, y = 1, \dot{y} = 1$, time = 1 sec(s)

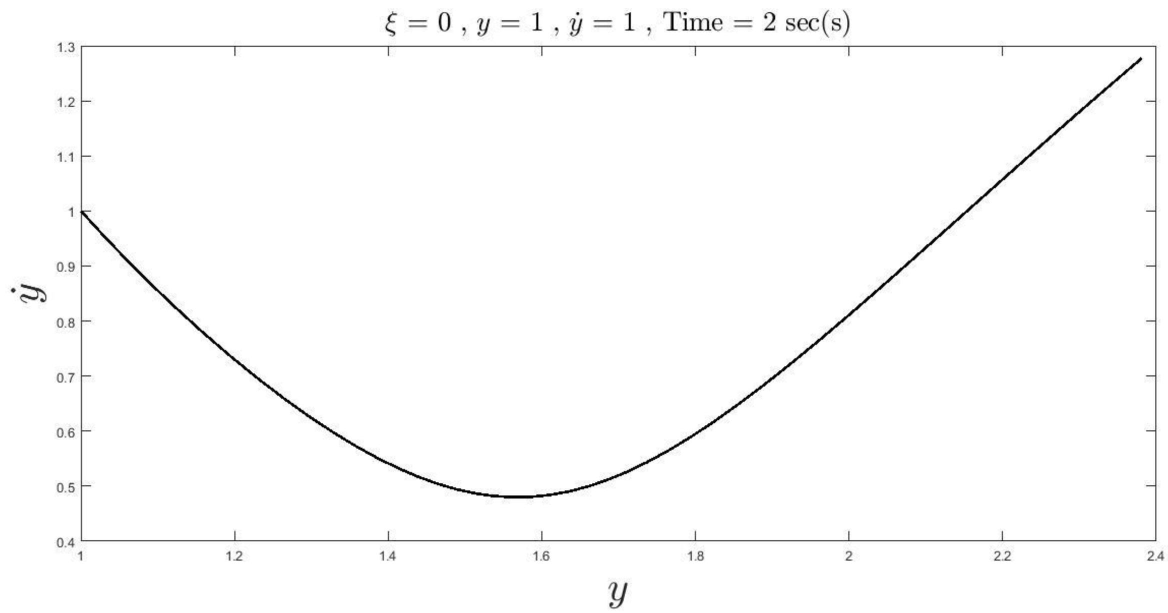

Figure 5.2.9(b): Time = 2 sec(s)

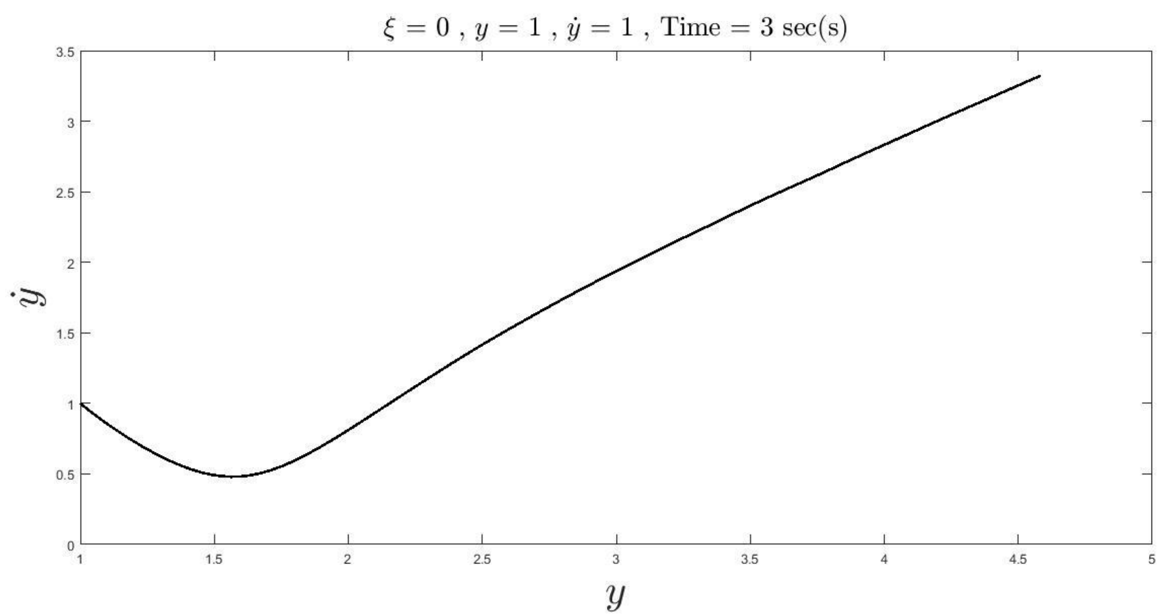

Figure 5.2.9(c): Time = 3 sec(s)

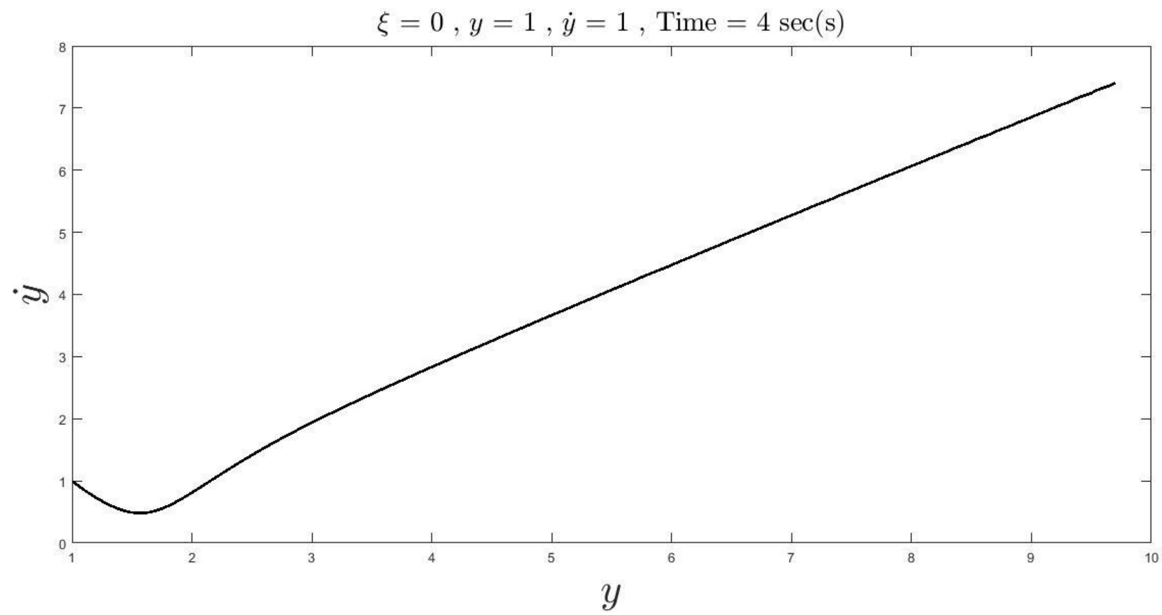

Figure 5.2.9(d): Time = 4 sec(s)

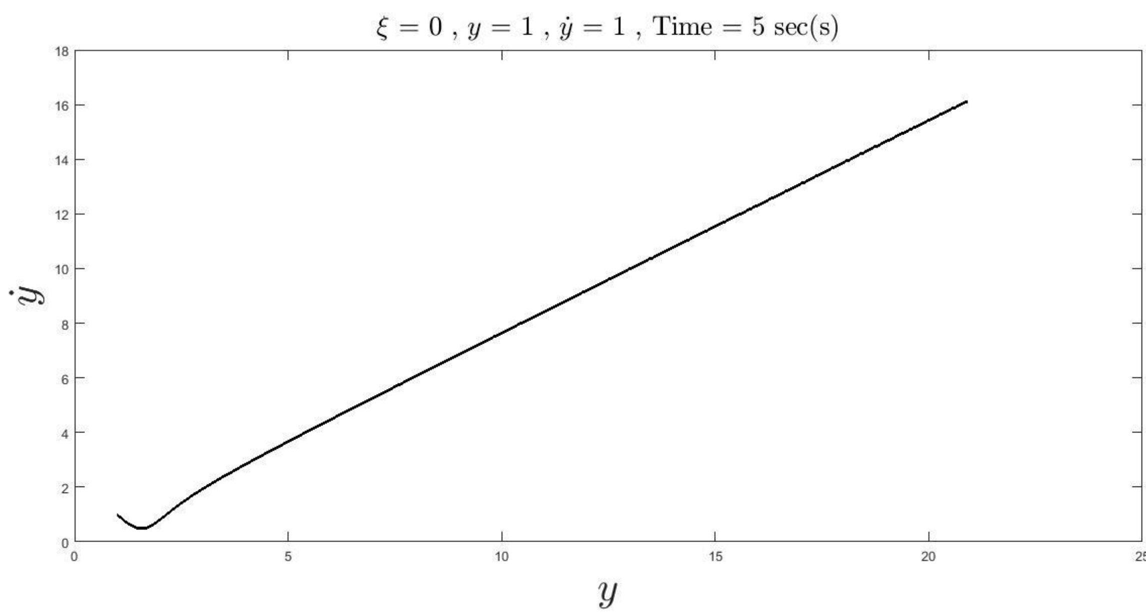

Figure 5.2.9(e): Time = 5 sec(s)

Now we have taken a slightly different approach. We considered a roll angle close to the saddle point with certain rate of change correspondingly. We can observe that upon reaching a certain value of $\dot{y}$, the roll angle slowly passes the saddle point and then tends to infinity. This in real time implies that the ship capsizes as soon as the roll angle crosses the saddle point value.

CASE 7: $\xi = 5, y = 1, \dot{y} = 0$

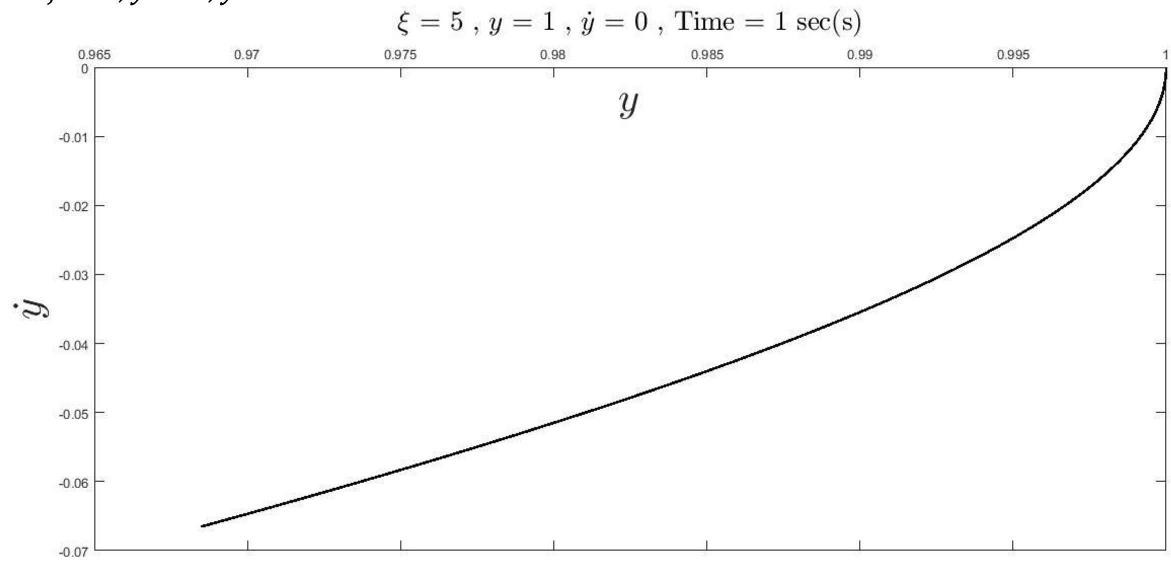

Figure 5.2.10(a): Plot for $\xi = 5, y = 1, \dot{y} = 0$, time = 1 sec(s)

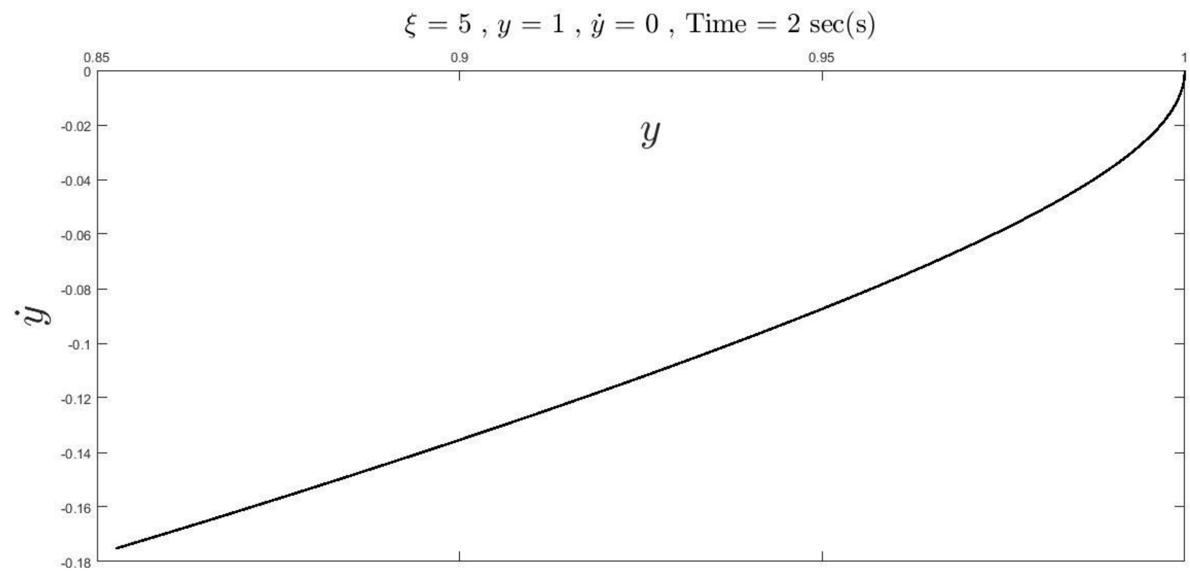

Figure 5.2.10(b): Time = 2 sec(s)

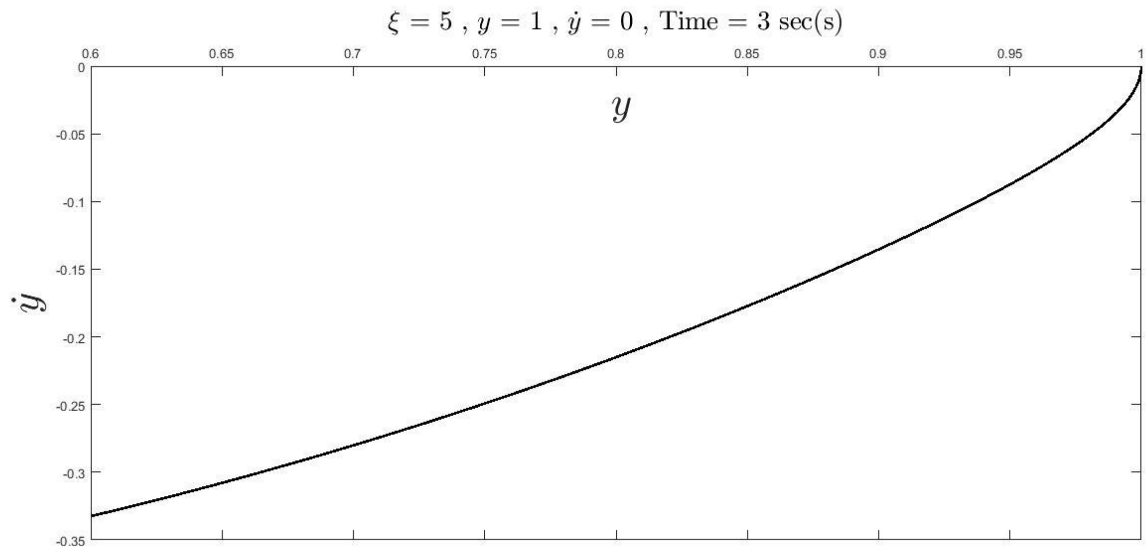

Figure 5.2.10(c): Time = 3 sec(s)

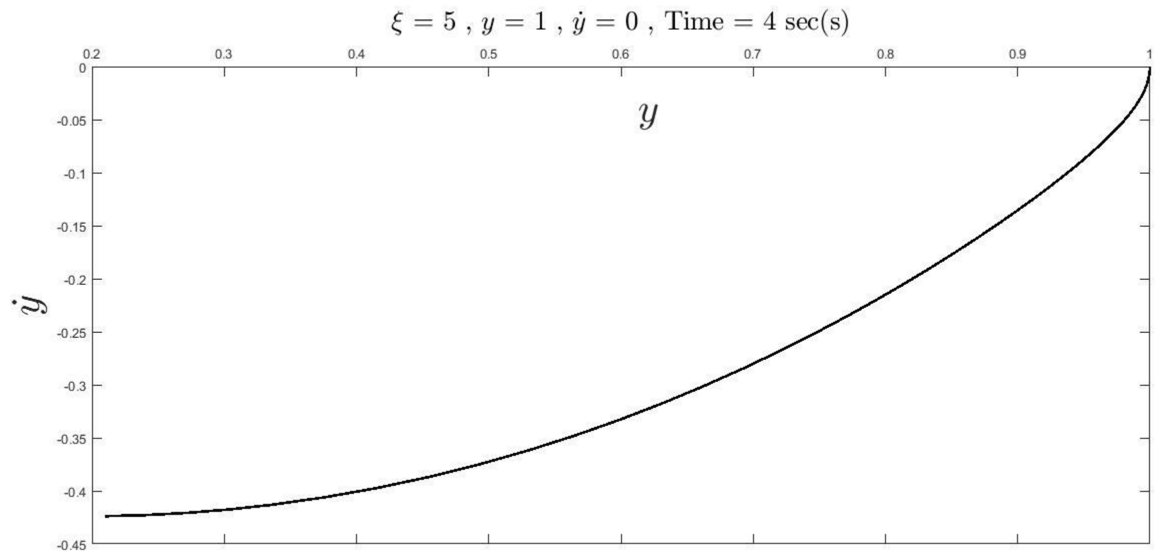

Figure 5.2.10(d): Time = 4 sec(s)

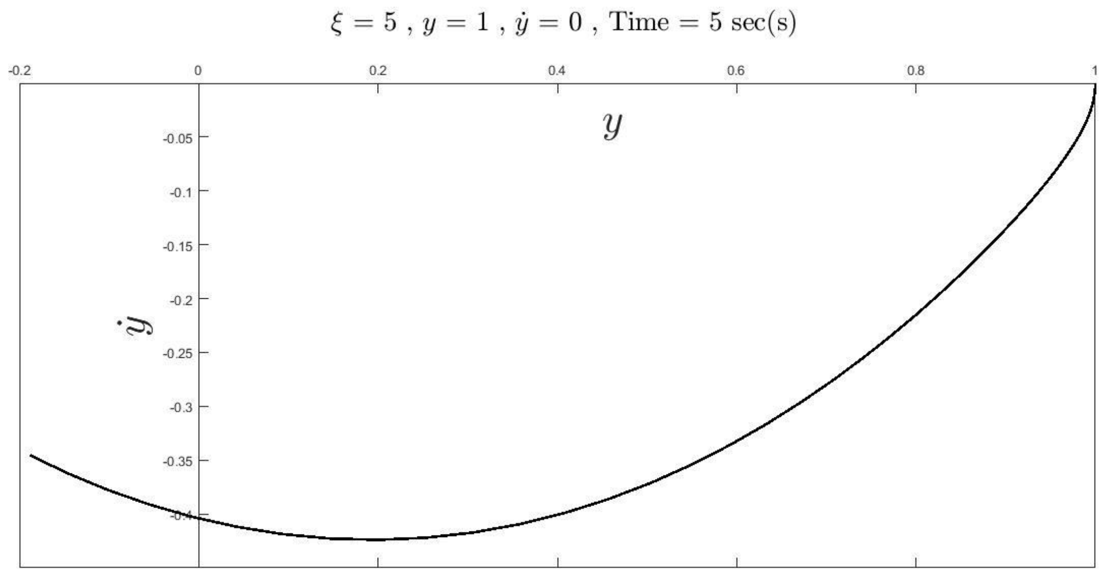

Figure 5.2.10(e): Time = 5 sec(s)

Now, as we started increasing the value of $\xi(here, 5)$, the behavior of the roll angle become a bit more critical. With no rate of change of roll angle, small roll angles tend to move towards the stable focus slowly. It takes more time to reach the stable equilibrium than that of $\xi = 0$ condition.

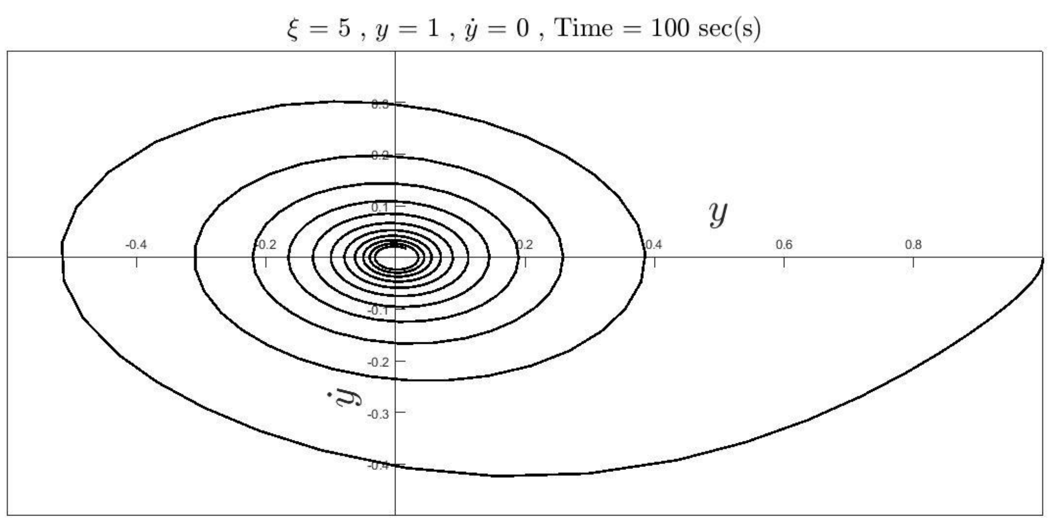

Figure 5.2.11: Behavior of Roll for $y = 1, \dot{y} = 0$

CASE 8: $\xi = 5, y = 2, \dot{y} = 0$

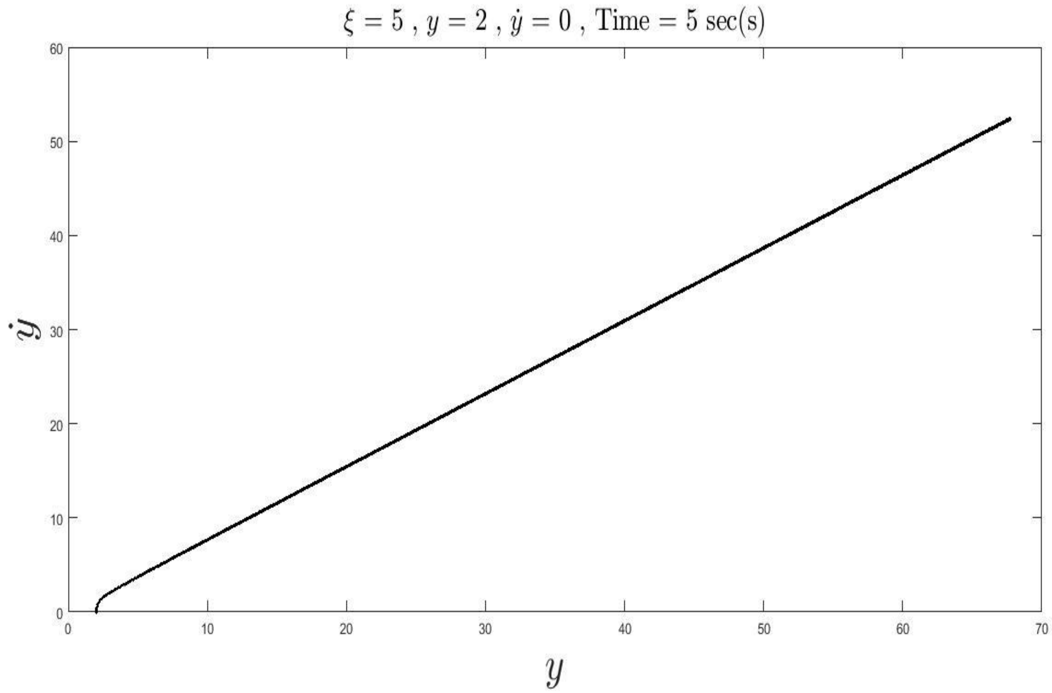

Figure 5.2.12: Plot for $\xi = 5, y = 2, \dot{y} = 0$

As we increase the roll angle in irregular seas ($\xi > 0$), we can observe that the plots become steeper and steeper. This indicates that the ship reaches instability way sooner in random seas than in calm seas ($\xi = 0$). The ship ultimately capsizes quicker.

CASE 9: $\xi = 5, y = 0, \dot{y} = 1$

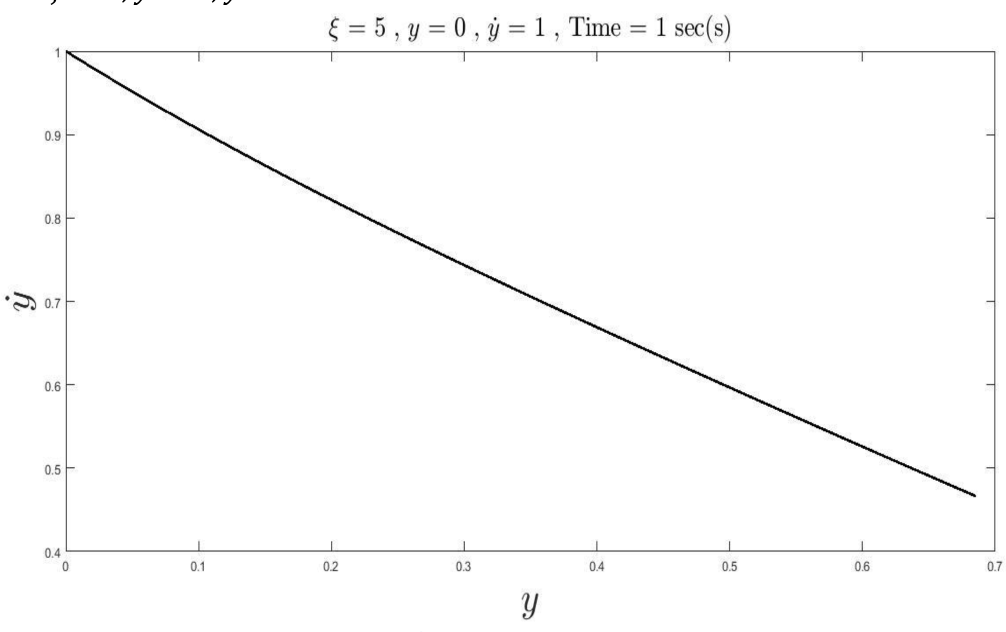

Figure 5.2.13(a): Plot for $\xi = 5, y = 0, \dot{y} = 1$, time = 1 sec(s)

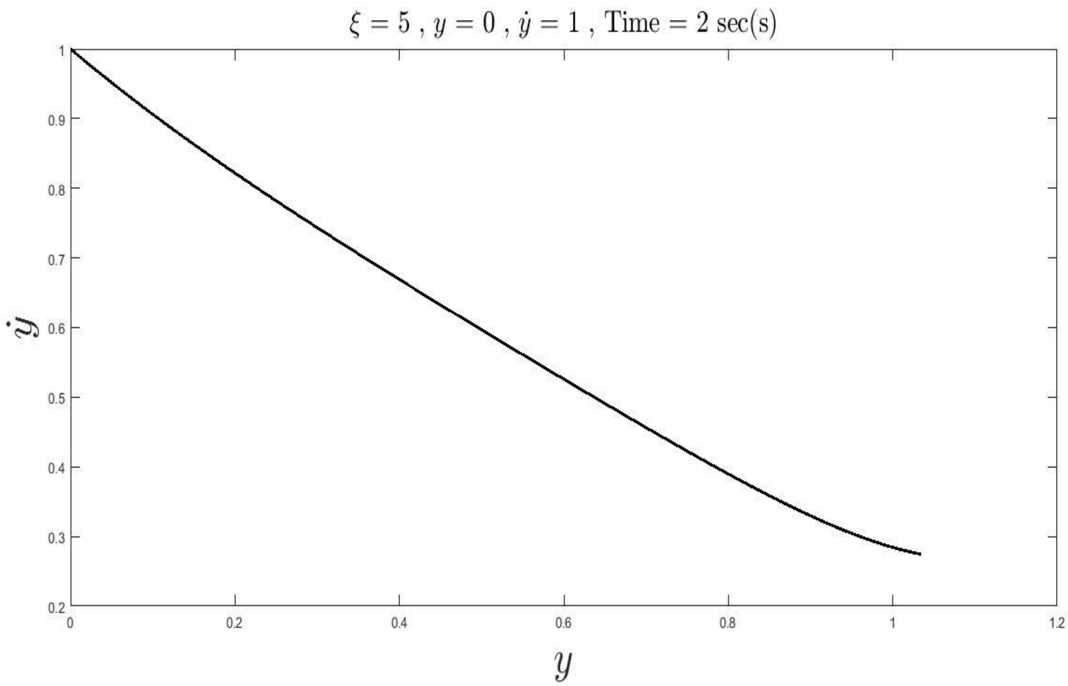

Figure 5.2.13(b): Time = 2 sec(s)

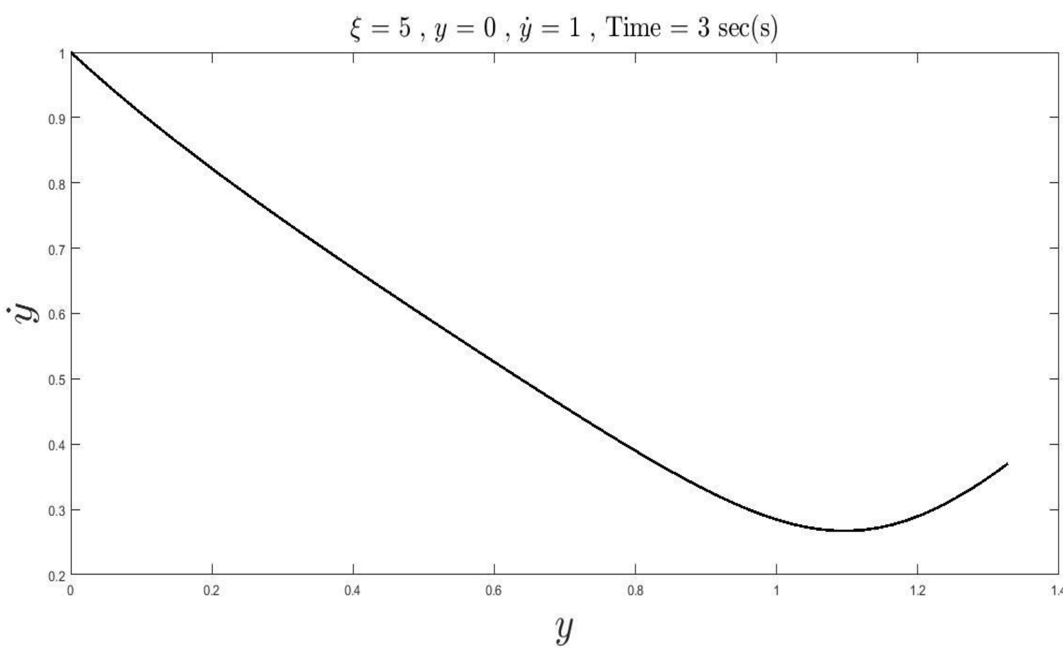

Figure 5.2.13(c): Time = 3 sec(s)

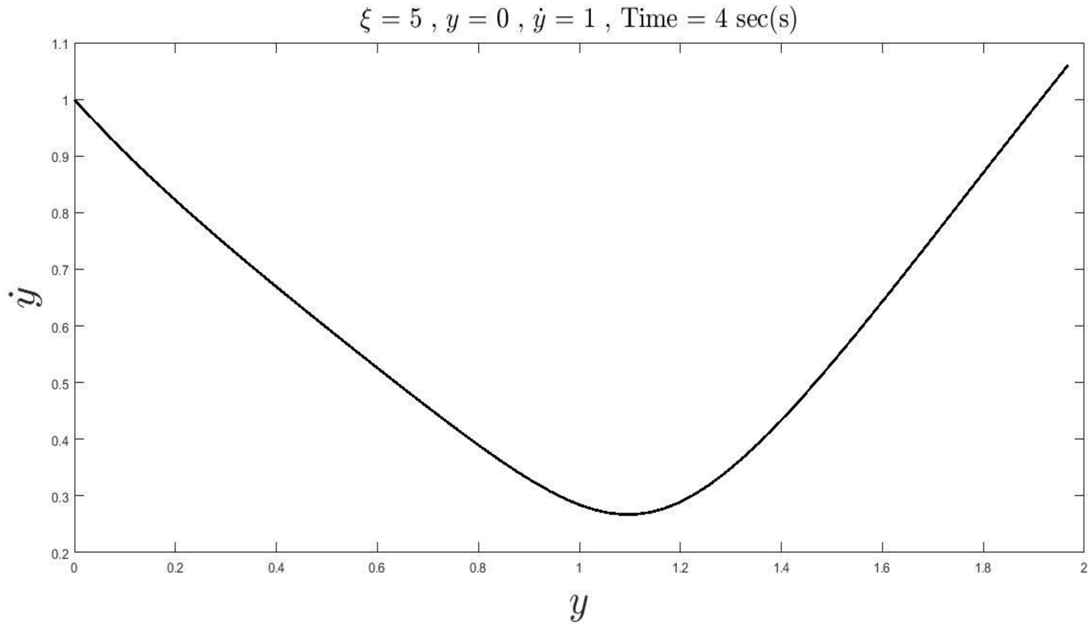

Figure 5.2.13(d): Time = 4 sec(s)

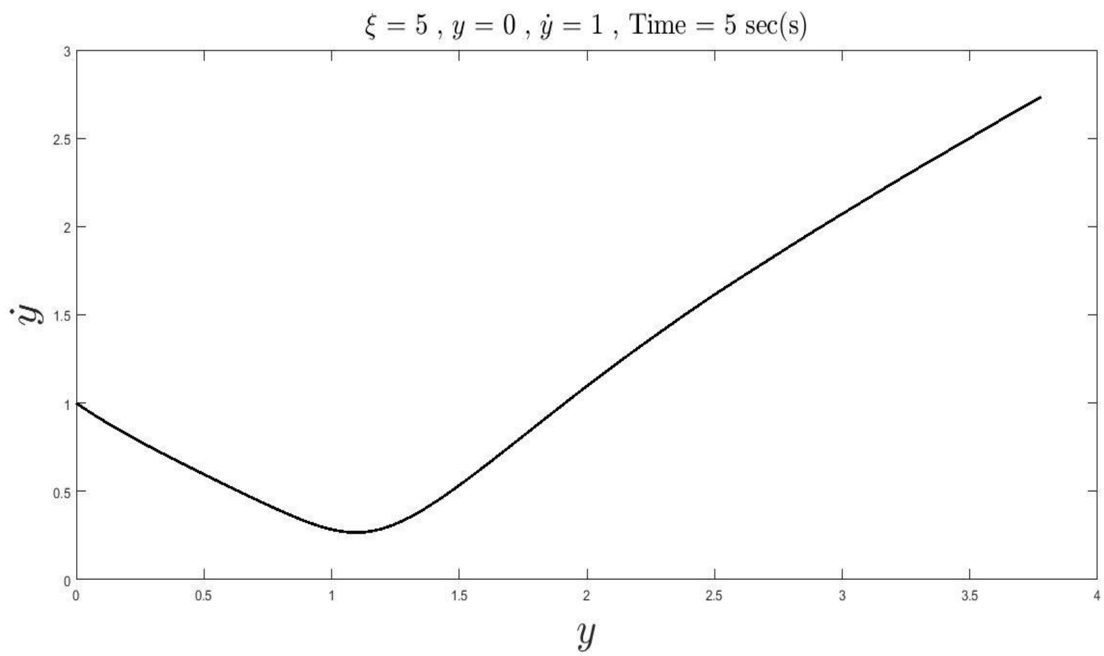

Figure 5.2.13(e): Time = 5 sec(s)

In the above condition, we have considered certain rate of change to a zero roll angle as the initial condition. We can observe that the behavior is much more drastic in irregular seas. The roll angle steeply crosses the saddle point quickly and stabilizes momentarily before shooting off to infinity again, causing the ship to capsize in finite time.

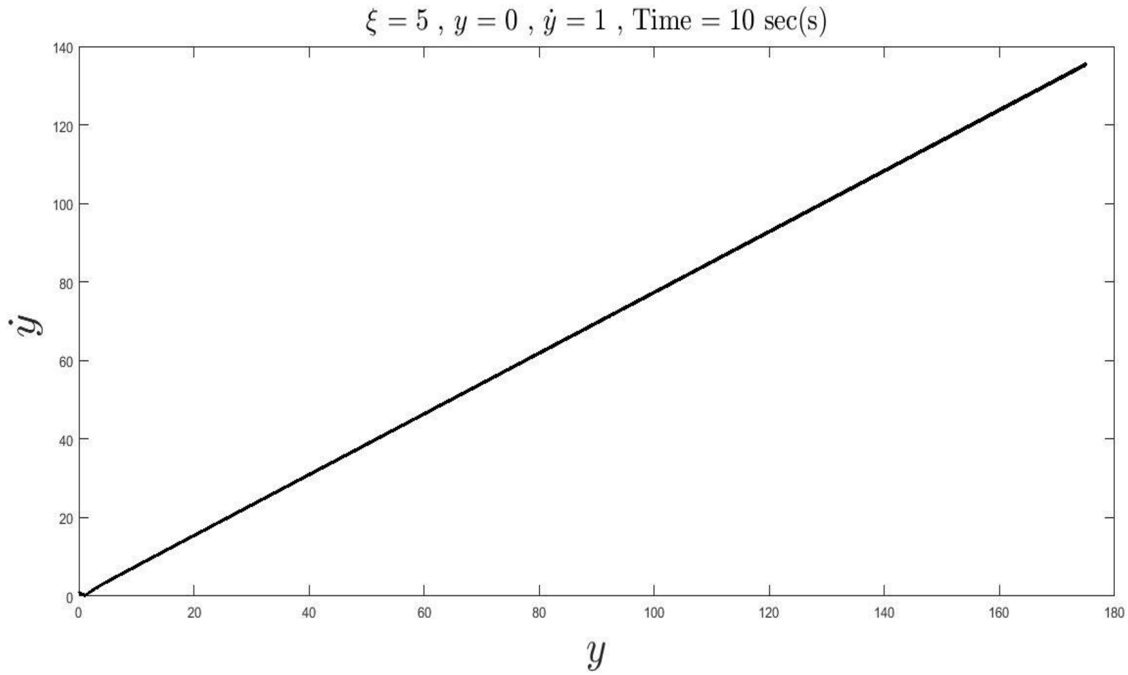

Figure 5.2.13(f): Time = 10 sec(s)

CASE 10: $\xi = 5, y = 0, \dot{y} = 7$

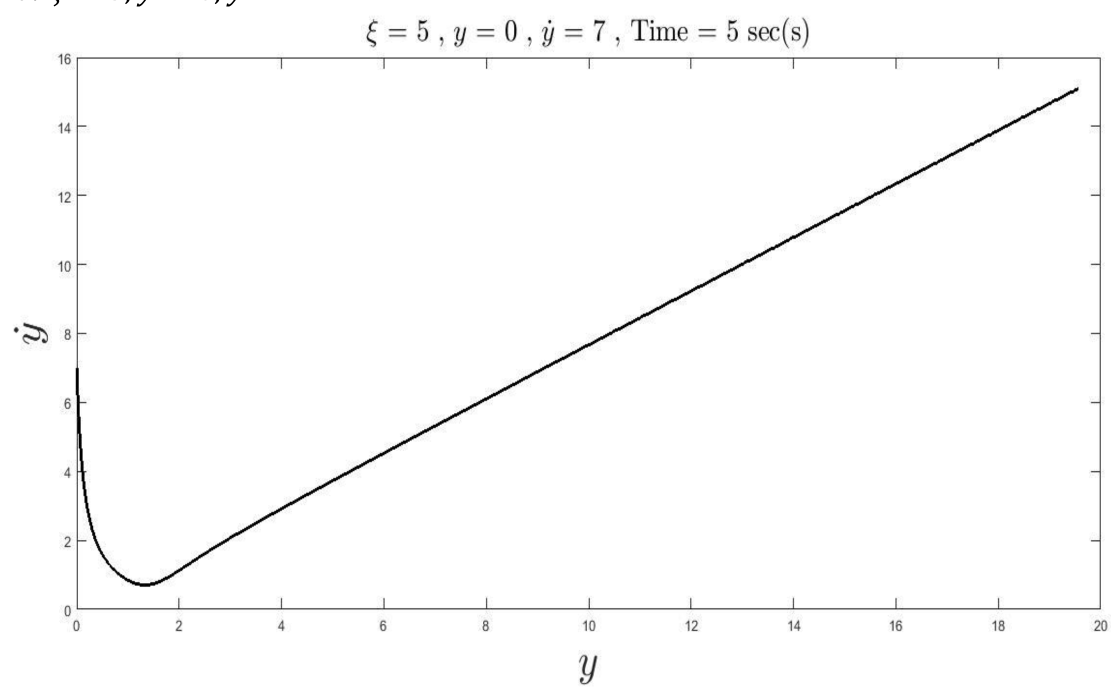

Figure 5.2.14 Plot for $\xi = 5, y = 0, \dot{y} = 7$

Upon considering a larger rate of change of roll angle value as initial condition, we can notice the change in the degree of steepness from the previous case. The roll angle reaches the saddle point in no time, and equally fast, it shoots off to infinity, triggering capsize.

CASE 11: $\xi = 5, y = 0.5, \dot{y} = 0.5$

When we consider a more practical condition, wherein the roll angle and its rate of change has certain value initially, we observe that the roll angle tends to infinity even for small valued combination of y & $\dot{y}$.

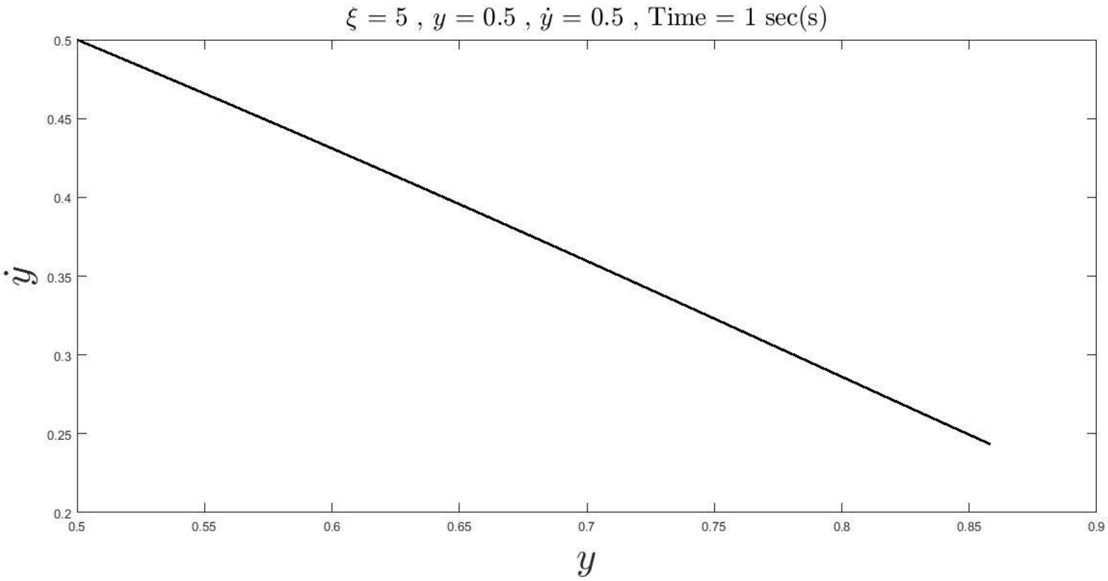

Figure 5.2.15(a): Plot for $\xi = 5, y = 0.5, \dot{y} = 0.5$, time = 1 sec(s)

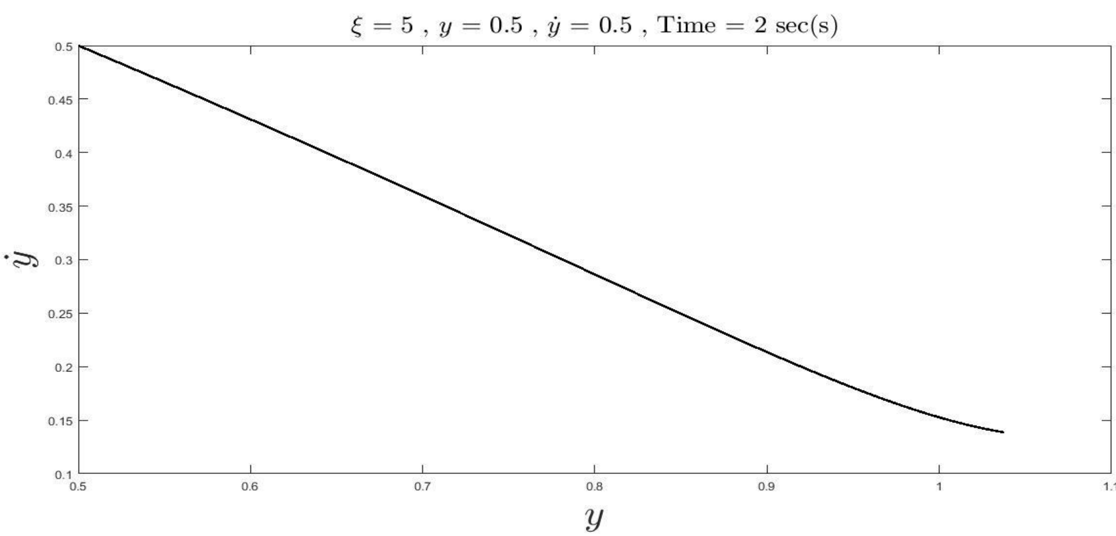

Figure 5.2.15(b): Time = 2 sec(s)

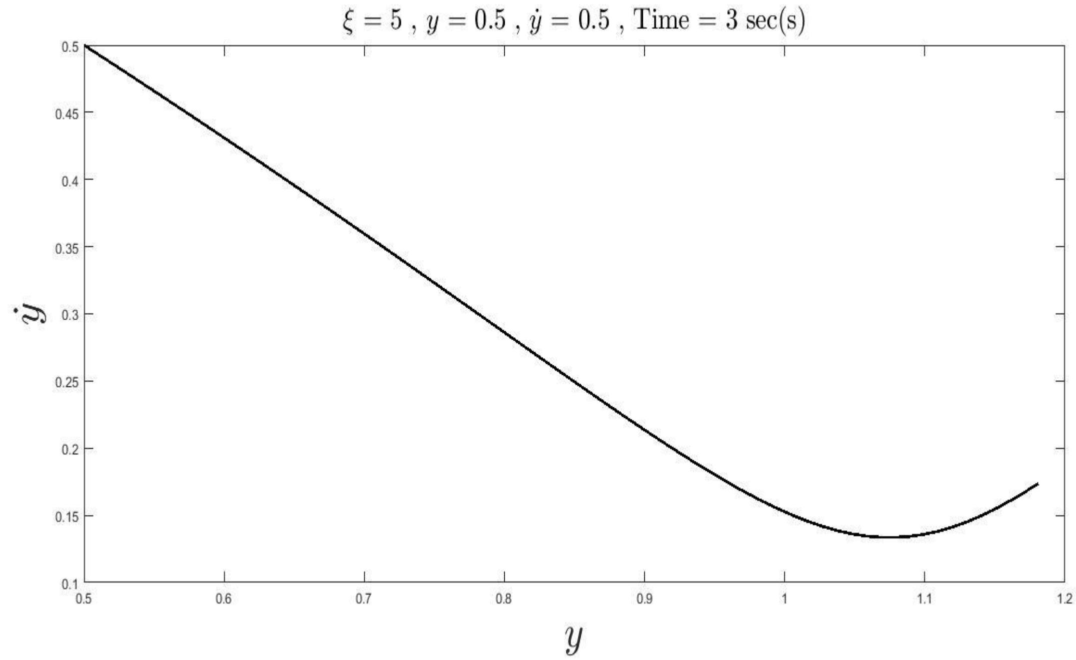

Figure 5.2.15(c): Time = 3 sec(s)

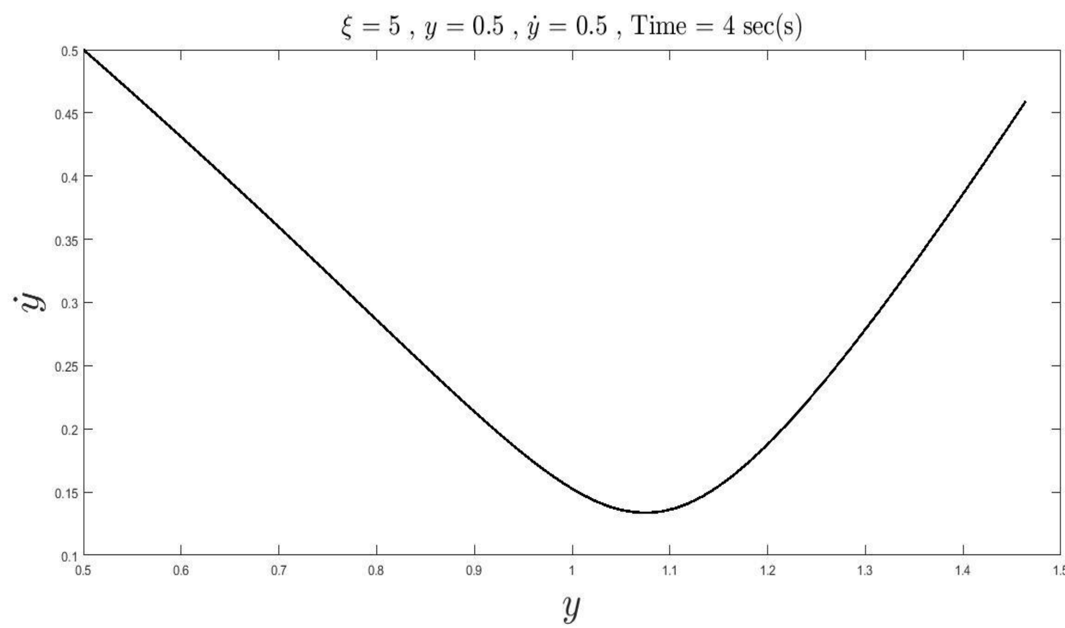

Figure 5.2.15(d): Time = 4 sec(s)

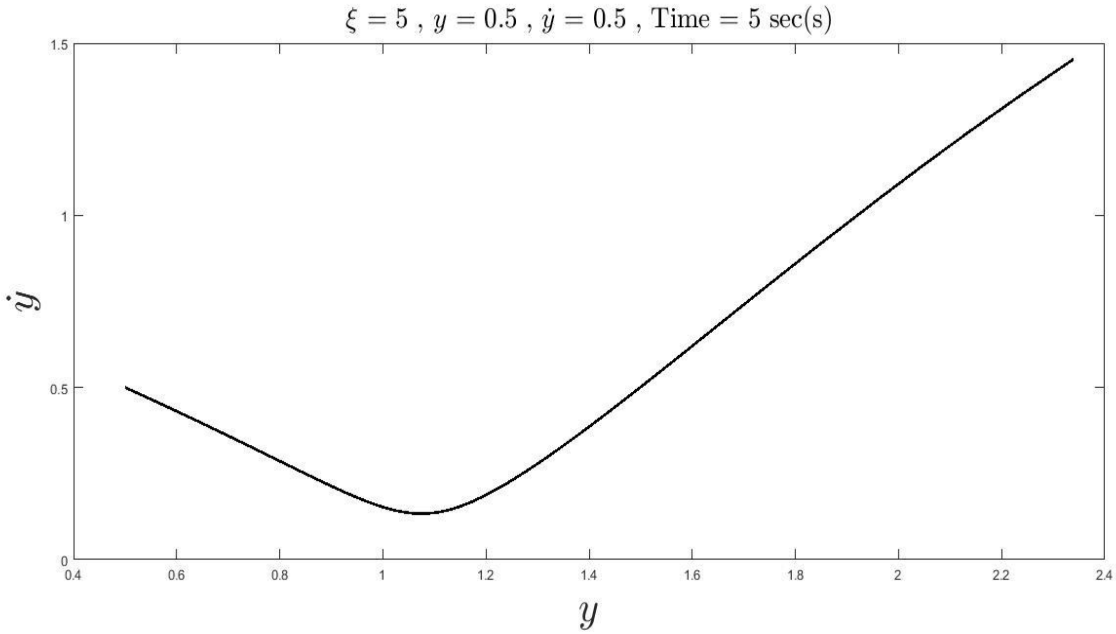

Figure 5.2.15(e): Time = 5 sec(s)

CASE 12: $\xi = 5, y = 1, \dot{y} = 1$

As we go on increase the values for initial conditions, the chance of capsizing increases

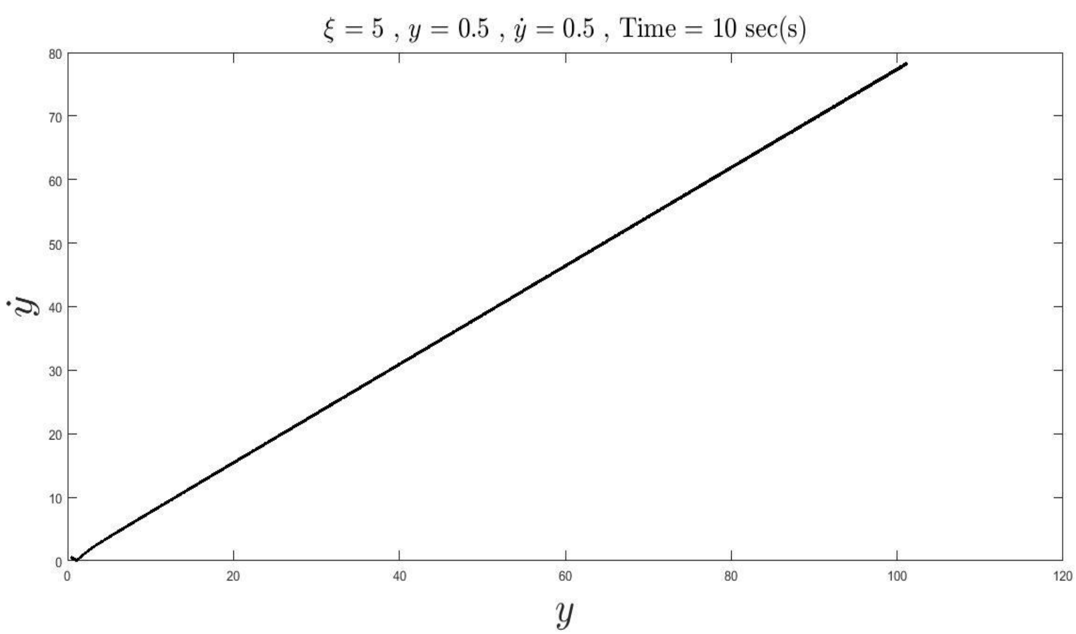

Figure 5.2.16: Plot for $\xi = 5, y = 1, \dot{y} = 1$

CASE 13: $\xi = 15, y = 1, \dot{y} = 1$

If the ξ values is more than 10, there is only one fixed point, which is a saddle point, and the ship will always capsize.

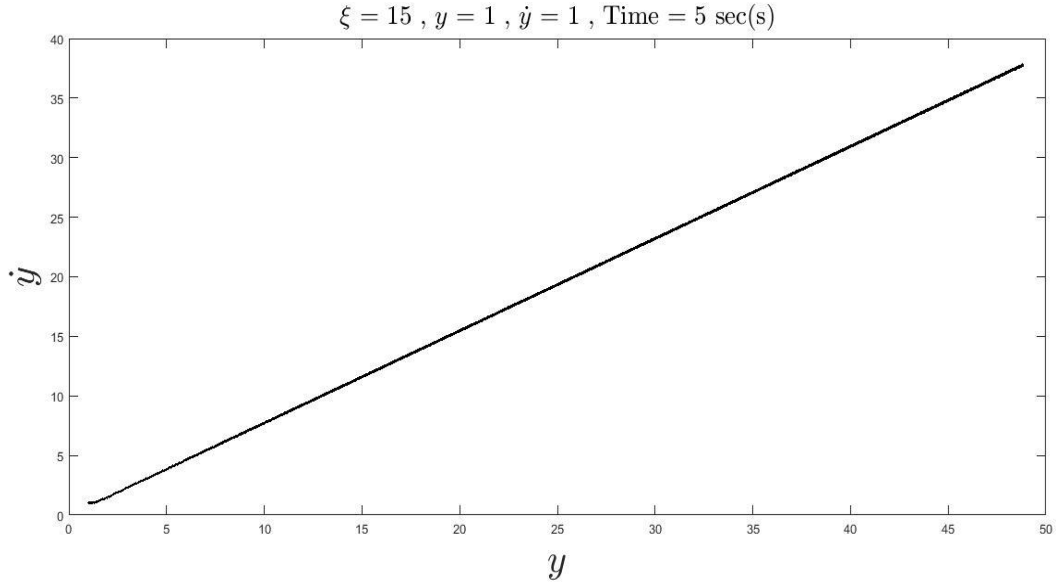

Figure 5.2.17: Plot for $\xi = 15, y = 1, \dot{y} = 1$

Thus from the above conclusions, we could notice that there is a pattern of bifurcation in the case $\vartheta = 0$. The pattern is similar to that of a pitchfork bifurcation.

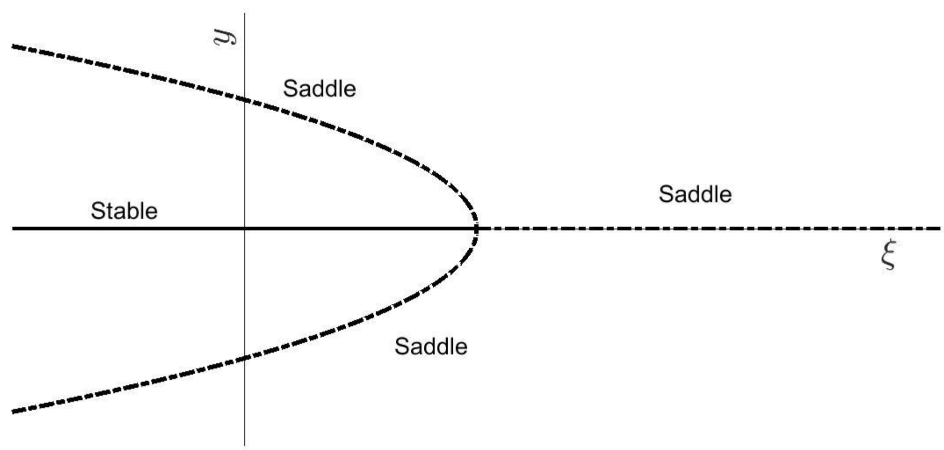

Figure 5.2.18: Bifurcation pattern

CHAPTER 6

FUTURE SCOPE

The effects of pure roll motion on the ship dynamics under various conditions have been explained in the previous chapters. However, in practicality, the motion of any ship on an irregular sea is more likely to be of a combination of two or more types of ship motions. We can get a more accurate picture of the ship motion when coupled motions, say pitch and roll, are considered. Therefore, this scenario of coupled motion gives us a bigger opportunity to dig deeper into the study of coupled motion and to explain their effects on the ship dynamics. Furthermore, for the sake of analysis purposes, we selected a particular type of ship vessel (Ro Ro ferry) and used its corresponding values to attain the necessary conclusions. A similar approach can be adopted for various models of ships and then we perform a thorough analysis so as to determine which dimensional aspects of a ship actually contribute the most for the stability of the ship in random seas. What we would like to do with these inferences is that we'd suggest a few optimal ranges for these parameters so that they can be incorporated in the first design stage.

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