27 Friday, April 15, 2011

Sheaf on $\mathbb{P}^n$

Define on standard affine, $U_i = \operatorname{Spec} \mathbb{C}\left[\left\{\frac{x_j}{x_i}\right\}\right]$.

Other affine opens: take homogeneous polynomial in $f(x_0, \dots, x_n)$, degree r. (A function on $\mathbb{P}^n$: ratio $\frac{f}{g}$ of homogeneous polynomials of some degree $\frac{f(\lambda x)}{g(\lambda x)} = \frac{f(x)}{g(x)}$.)

The locus $\{f = 0\}$ exists in $\mathbb{P}^n$, and is a closed set. Let $U_{(f)} = \mathbb{P}^n - \{f = 0\}$. $U_{(f)}$ is an affine variety. Coordinate ring $R_{(f)} = \left\{\frac{p(x)}{f(x)^k} \,\middle|\, p \text{ has degree } kr\right\}$. $\frac{p}{f^k}$ is a well-defined function on $U_{(f)}$.

$R_{(f)}$ is generated by $\left\{\frac{m}{f} \,\middle|\, m \text{ is a monomial of degree } r\right\}$.

Example. $f = x_0^2 + x_1^2 + x_2^2$

$R_{(f)}$ is generated by $\frac{x_0^2}{f}$, $\frac{x_0 x_1}{f}$, ..., with many relations.

Proposition.

$$U_{(f)} \approx \operatorname{Spec} R_{(f)}$$

On $V_{(f)}$, this has the structure sheaf $\mathcal{O}_{V_{(f)}}$.

The definition says that the slobal struture sheaf(?) $\mathcal{O}_{V_{(f)}}(V_{(f)}) = R_{(f)}$, $\mathcal{O}_{V_{(f)}}(V_{(f)} - \text{zeros of } s) = R_{(f)}[s^{-1}]$.

We have defined the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$

(If we have $Y \xrightarrow{j} X$ (variety) is an inclusion of an open set and has structure sheaf $\mathcal{O}_X$, then Y is the topological space with the "indced topology": a subset $V \subset Y$ is open if V is open in X.
The structure sheaf $\mathcal{O}_Y :=$ restriction of the structure sheaf $\mathcal{O}_X$.
V open in Y, $\mathcal{O}_Y(V) = \mathcal{O}_X(V)$, $\mathcal{O}_Y = [j^* \mathcal{O}_X]$)

$\therefore$ We have defined the structure sheaf $\mathcal{O}_{U_{(f)}}$.

The statement of the proposition $U_{(f)} \approx V_{(f)}$ then means:

- homomorphism $U_{(f)} \to V_{(f)}$
- & $\mathcal{O}_{U_{(f)}} \xleftrightarrow{\approx} \mathcal{O}_{V_f}$

A morphism

$$(Y, \mathcal{O}_Y) \xrightarrow{f} (X, \mathcal{O}_X)$$

(a) *continuous* map $Y \to X$

(b) Map $\mathcal{O}_Y \xleftarrow{f^*} \mathcal{O}_X$

If U is open in X, then $f^{-1}U = V$ open in Y. Then we have a homomorphism of algebras $\mathcal{O}_Y(X) \xleftarrow{f^*} \mathcal{O}_X(U)$.

An *isomorphism* is an invertible morphism.

Proof. In Prof. Artin's notes. □

Suppose f and g are homomorphisms of degrees r, respectively. Then we have three finite-type algebras $R_{(f)}$, $R_{(g)}$, $R_{(fg)}$.

Proposition. *Let $w = \frac{g^r}{f^s} \in R_{(f)}$. Then*

$$R_{(f)}[w^{-1}] \approx R_{(fg)} \approx R_{(g)}[w]$$

Proof. $R_{(fg)}$ is generated by $\frac{M}{(fg)}$ (M is a monomial), M has degree $r+s$. We can write $M = m_1 m_2$, with m_1 of degree r and m_2 of degree s.

$$\frac{M}{fg} = \frac{m_1}{f}\frac{m_2}{g}$$

$$\frac{m}{g} = m\frac{f^s}{g^r}\frac{g^{r-1}}{f^s} = \frac{mg^{r-1}}{f^s}w$$

Let's not check the rest. □

$\{U_{(f)}\}$ = all affine opens in $\mathbb{P}^n$

Definition. A *quasi-coherent sheaf* $\mathcal{M}$ on an affine scheme $X = \operatorname{Spec} A$ corresponds to an A-module M given by the following rules:

- Given a quasi-coherent sheaf $\mathcal{M}$, $M = \mathcal{M}(X)$ =global sections.
- Gven an A-module M, define $\mathcal{M}$ by $\mathcal{M}(X_s) = M_s$.

Note $\mathcal{M}(X_s)$ (M_s) is a $\mathcal{O}_X(X_s)$-module (A_s).
$\mathcal{M}$ is a sheaf of $\mathcal{O}_x$-modules.[1]
A sheaf is *coherent* if M is finite-type.
If $(X, \mathcal{O}_X)$ is a variety, e.g., $\mathbb{P}^n$, not necesarily affine, then a *quasi-coherent sheaf* $\mathcal{M}$ on X is the sheaf of $\mathcal{O}_X$-modules, and when we restrict to an affine open, we get a quasi-coherent sheaf on that affine.

Example. The sheaf $\mathcal{O}(n)$ on $\mathbb{P}^n$:
Its section on an affine open $U_{(f)}$ case: (f homomorphism of degree r)

$$[\mathcal{O}_{\mathbb{P}}(n)]\,(U_{(f)}) = \left\{ \frac{p(x)}{f(x)^k} \,\middle|\, p \text{ has degree } rk + n \right\}$$

$$\mathcal{O}_{\mathbb{P}}(U_{(f)}) = R_{(f)} = \left\{ \frac{p}{f^k} \,\middle|\, p \text{ has degree } rk \right\}$$

The global sections $\mathcal{O}_{\mathbb{P}}(n)(\mathbb{P}^n) = H^0(\mathbb{P}^n, \mathcal{O}(n))$ are homogeneous polynomials of degree n.
Let h be a polynomial of degree m.

$$0 \to \mathcal{O}_{\mathbb{P}}(n) \xrightarrow{\text{multiplication by } h} \mathcal{O}_{\mathbb{P}}(n+m) \to \text{cokernel}$$

Let Y be the locus $h = 0$ in $\mathbb{P}^n$.

$$0 \to \mathcal{O}_{\mathbb{P}}(-m) \xrightarrow{h} \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_Y \to 0$$

is exact, and so is

$$0 \to \mathcal{O}_{\mathbb{P}}(n-m) \xrightarrow{h} \mathcal{O}_{\mathbb{P}}(n) \to \mathcal{O}_Y(n) \to 0$$

This is something analogous to $\mathcal{O}(n)$, but on Y.
($\mathcal{O}_{\mathbb{P}}(-m)$, $\mathcal{O}_{\mathbb{P}}$ are independent of h)

[1] Why not call it an $\mathcal{O}_x$-module? Don't know.

Example. $x_0 = h$

$$\mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

$Y =$ locus $\{x_0 = p\} = \mathbb{P}^{n-1}$ (hyperplane at ∞)

Next week: cohomology of coherent sheaves