2 Monday, February 7, 2011

2.1 Affine varieties

Definition 1. An *affine variety* is the set of solutions to a system of polynomial equations

$$f_1(x) = f_2(x) = \cdots = f_r(x) = 0$$

for $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$. ($x$ is shorthand for $(x_1, x_2, \dots, x_n)$.)

Alternatively, it's $V(I)$ for some $I = (f_1, \dots, f_r) \subset \mathbb{C}[x_1, \dots, x_n]$.
e.g., point: $(x_0, y_0) = (x - x_0, y - y_0)$
line: all (x, y) for which $ax + by + c = 0$
conic: locus of complex zeros of a quadratic equation in two variables $q(x, y) = 0$
cubic: locus of complex zeros of a cubic polynomial

Classification of these depends on what coordinate changes one allows. If we allow arbitrary invertible linear operators and translations, any line can be converted to $x = 0$. Any conic can be converted to either $x^2 - y^2 - 1 = 0$ or $x^2 - y = 0$, by completing the square.

Zariski topology on an affine variety $(V(I_0)) = X \subset \mathbb{A}^n$ is just subspace topology induced from Zariski topology on $\mathbb{A}^n$ (i.e., closed sets of X are just all the $y = V(I)$ which are contained in X). Note: $V(0) = \mathbb{A}^n$ so $V(0) \cap X = X$.
Check:

$$\begin{aligned} V(1) &= \emptyset \\ \bigcap_{\alpha \in S} V(I_\alpha) &= V\left(\bigcup_{\alpha \in S} I_\alpha\right) \\ &= V\left(\sum_{\alpha \in S} I_\alpha\right) \end{aligned}$$

(arbitrary intersection of closed sets is closed)

$$V(I) \cup V(J) = V(I \cap J)$$

(To see, consider a point $p \in V(I \cap J)$, $p \notin V(I)$, show that $p \in V(J)$)

2.2 Projective Plane $\mathbb{P}^2$

Recall

$$\mathbb{P}^2 = \left\{(x : y : z) \in \mathbb{A}^3 \setminus \{0, 0, 0\}\right\} / (x : y : z) \sim (\lambda x : \lambda y : \lambda : z) \qquad (\lambda \neq 0)$$

(lines through origin in $\mathbb{A}^3$ or $\mathbb{C}^3$)

A line in $\mathbb{P}^2$ is given by an equation $ax + by + cz = 0$ as long as $(a, b, c) \neq (0, 0, 0)$. Note that (a, b, c) and $(\lambda a, \lambda b, \lambda c$ give the same line for $\lambda \neq 0$. So the set of lines forms another projective plane (dual projective plane $\check{\mathbb{P}}$). This equation $ax + by + cz = 0$ exhibits the duality between points and lines.

Lemma. *A pair of distinct lines contains exactly one point in common, and a pair of distinct points lie on exactly one line.*

Recall that we had $\mathbb{A}^2 \to \mathbb{P}^2$, $(x, y) \mapsto (x : y : 1)$. Bijection between $\mathbb{A}^2$ and $U_z \coloneqq \mathbb{P}^2 \setminus \{z = 0\}$ (since $(x : y : z) \mapsto \left(\frac{x}{z}, \frac{y}{z}\right)$ if $z \neq 0$). Similarly we have U_x and U_x also in bijection with $\mathbb{A}^2$. Then $U_x \cup U_y \cup U_z = \mathbb{P}^2$. This is a cover, and is called the standard affine open covering of $\mathbb{P}^2$.

Note that $U_x \cap U_y \subseteq U_x$ (this is $\{(x : y : z) \,|\, x \neq 0, y \neq 0\} \subset \{(x : y : z) \,|\, x \neq 0\}$) is the set $\{y \neq 0\} = D(y)$. So its open in the Zariski topology. $U_x \cap U_y \subseteq U_y$ is also open: $V \subset \mathbb{P}^2$ is open iff all its intersections with the standard affines (standard affine open covers) are open (in the standard affines).

This is the same topology as the other version of the Zariski topology on $\mathbb{P}^2$ by taking $V(f)$, f homogeneous polynomial in $\mathbb{C}[x, y, z]$. to be a closed set in $\mathbb{P}^2$, and then take arbitrary intersections and finite unions. (same as quotient topology on $\mathbb{A}^3 \setminus \{0\}$)

Note: Since Zariski topology $\subseteq$ classical/complex/Euclidean topology (all open sets in Zariski are open in classical)
This means we can define a classical topology as well on $\mathbb{P}^2$. It makes $\mathbb{P}^2$ into a compact Hausdorff space.

Proof. Compact: Let $C_z \subseteq U_z$ be the set of $(u, v, 1)$ such that $|u| \leq 1$, $|v| \leq 1$ and similarly C_x and C_y. It's clear that C_x, C_y, C_z are compact (also closed subspaces of $\mathbb{P}^2$ in the complex topology). Since $\mathbb{P}^2 = C_x \cup C_y \cup C_z$, $\mathbb{P}^2$ is compact. □

2.3 Change of coordinates in $\mathbb{P}^2$

Four special points determine coordinates in $\mathbb{P}^2$:

$$\begin{aligned} e_1 &= (1 : 0 : 0) & e_2 &= (0 : 1 : 0) \\ e_3 &= (0 : 0 : 1) & \epsilon &= (1 : 1 : 1) \end{aligned}$$

Think of these as column vectors.

Change of coordinates is described by a 3×3 invertible matrix P.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

(x is old, x' is new)

if the matrix is a scalar (diagonal) matrix, then it doesn't affect any coordinate change. Similarly, the coordinate changes corresponding to P and sP are the same.

If $A = (a, b, c)$ and ℓ is the line $ax + by + cz = 0$, $AX = 0$, then $A(PX') = 0$ or $(AP)X' = 0$, so the equation for ℓ in the new coordinates is AP.

Proposition. *Let p_1, p_2, q_3, q be four points in $\mathbb{P}^2$, now three collinear. Then $\exists!$ change of coordinates $PX' = X$ such that $X = p_1, p_2, p_3, q$ because $X' = e_1, e_2, e_3, \epsilon$.*

Proof. p_1, p_2, p_3 are linearly independent vectors in $\mathbb{C}^3$. So $\exists$ transformation P such that $Pp_i = e_i$. Now q is non-collinear with p_1, p_2, p_3; all of its coordinates are non-zero. Then scale each coordinate to take q to ϵ (modifies P). This doesn't affect e_1, e_2, e_3. □

Conics we had $x^2 - y^2 - 1 = 0$ and $x^2 - y = 0$ in $\mathbb{A}^2$. In $\mathbb{P}^2$, these become $x^2 - y^2 - z^2 = 0$ and $x^2 - yz = 0$. We can transform the first into the second by doing $x^2 - y^2 = z^2$, $(x - y)(x + y) = z^2$, coordinate change to $x'y' = z'^2$.