16 Friday, March 11, 2011

Finite group action on an integrally closed finite-type domain B, $A = B^G$, (has the?) invariants

$\max A \leftrightarrow G$-orbit in $\max B$

A is finite type and integrally closed

B is a finite A-module

Definition. A is *normal* if it's integrally closed in the field of fractions of A (i.e., it's own fraction field).

Theorem. *Let G be a finite group that acts on an integrally closed finite-type domain B, $A = B^G$. Then*

- $\operatorname{PSpec} A \leftrightarrow (\operatorname{PSpec} B)/G$
- *This preserves inclusions. More formally, If $P \leftrightarrow$ orbit $\{Q_j\}$ and $P' \leftrightarrow$ orbit $\{Q'_i\}$, then $P \subset P'$ if and only if $\forall j \exists i$ such that $Q_j \subset Q'_i$.*

  We have $Q \rightsquigarrow A \cap Q$, $\operatorname{Spec} B \to \operatorname{Spec} A$

Lemma. *Let $Q_1, \ldots, Q_m$; $Q'_1, \ldots, Q'_n$ be prime ideals of B. Suppose $Q_j \not\subset$ any Q'_i, $j = 1, \ldots, m$, $i = 1, \ldots, n$. Then there exists $\alpha \in B$, $\alpha \in Q_1, \ldots, Q_m$, $\alpha \notin Q'_1, \ldots, Q'_n$.*

Proof. Plan: Solve for a single Q_j. Find $\alpha_j \in Q_j$, $\notin Q'_1, \ldots, Q'_n$. Then the product $(\alpha_1 \cdots \alpha_m)$ works; $\alpha_1 \cdots \alpha_m \in Q'_i \implies$ some $\alpha_j \in Q'_i$. This would be a contradiction.

Take $Q_1 = Q$, $Q'_1, \ldots, Q'_n$, $\alpha \in Q$, $\notin Q'_i$.

$\alpha = \beta_i \in Q$, $\notin Q'_i$, $i = 1, \ldots$. $\beta_1 \cdots \beta_n \in Q$. $\ldots$(Proof not included) □

Proof. (of theorem)
In C:

$$\begin{tikzcd} B \arrow[r, "\sigma"] & B \\ A \arrow[u, hook] \arrow[r, "\mathrm{id}"] & A \end{tikzcd}$$

Since Q is a prime ideal of B and $P = A \cap Q$, we have that σQ is a prime ideal of B and $P = A \cap \sigma Q$. Then an orbit of Q corresponds to one point in $\operatorname{PSpec} A$. By the lying over theorem, $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective. Therefore $(\operatorname{Spec} B)/G \to \operatorname{Spec} A$ is also surjective.

Injective: $\{Q_1, \ldots, Q_m\}$, $\{Q'_1, \ldots, Q'_n\}$ distinct orbits. $A \cap Q_j = P$, $A \cap Q'_i = P'$, show $P \neq P'$.

Claim. *We can't have $Q_i \subset Q'_j$ for some i, j and also $Q'_k \subset Q_\ell$ for some k, ℓ.*

Proof. We can renumber, $i = j = 1$. $Q'_k \subset Q_\ell \implies \sigma Q'_k \subset \sigma Q_\ell$. We may then assume that $\ell = 1$; since σ runs through the whole orbit, we can choose an appropriate σ.

Then $Q_1 \subset Q'_1$, $Q'_k \subset Q_1$. Then $Q'_k \subset Q_1 \subset Q'_1$. Then $Q'_k \subset Q'_1$. Thus $k = 1$, because $Q'_k = \sigma Q'_1$ for some σ (and permutations can't take sets to proper subsets of themselves). Then $Q'_1 \subset Q_1 \subset Q'_1$, so $Q_1 = Q'_1$. This is a contradiction, since orbits are disjoint. Thus, we may suppose that $Q_j \not\subset Q'_i$ for any i, j. □

There exists an $\alpha \in Q_1 \cap \cdots \cap Q_m$, $\notin Q'_i$. Take $\gamma = \prod_\sigma \sigma\alpha$, $\gamma \in Q_j$, $\notin Q'_i$, and $\gamma \in A$. Thus, $\gamma \in P$, $\notin P'$, so $P \neq P'$.

The proof of the second point is skipped (See notes that Prof. Artin posted). □

Let B be a finite A-module for a normal A.

$$\begin{tikzcd} A \arrow[r, hook] & B \\ P' \arrow[u, hook] & Q' \arrow[u, hook] \\ P \arrow[u, hook] & Q \arrow[u, hook] \end{tikzcd}$$

$P' = A \cap Q'$, $P = A \cap Q$

Theorem (Going Up)**.** *In the diagram above, given P, P', Q, there exists a Q'.*

Theorem (Going Down)**.** *In the diagram above, given P, P', Q', there exists a Q.*

Proof. Let K be the fraction field of A and L be the fraction field of B.

Case 1: L/K is Galois, has Galois group $G(?)$. B is normal.

σ acts on B: The elements of B are integral of A. If β is integral, so is $\sigma\beta$. Then, since B is normal, B is the integral closure of A in L. Thus, $\sigma\beta \in B$.

$A = B^G$. (We know that B is a finite B^G-module.): $A \subset B^G$ and A is normal. Since B is integral over A, B^G is integral over A, so $A = B^G$.

Then the theorem follows from the previous theorem.

Case 2: (general case): L/G not Galois, and/or B not normal: Put $L \subset F$ a Galois extension of K with Galois group G. Let C be the integral closure of A in F. This is a finite A-module. Then G operates on C. Then $A = C^G$ (by the same reasoning as in case 1).

$$\begin{array}{ccc} A \hookrightarrow & B \hookrightarrow & C \\ P' & Q' & R' \\ P & & R \end{array}$$

By lying over, there exists a prime ideal R', $B \cap R' = Q'$. Case 1 says that there exists an R. Then $A \cap R = P$. Put $Q = B \cap R$.

□