Outline.

I. playing with curves in $\mathbb{P}^2$

II Affine algebraic geometry (structure of Spec A)

III. Projective geometry

IV cohomology

thm: Nullstellensatz:

Maximal ideals of $\mathbb{C}[x_1, \ldots, x_n] \longleftrightarrow$ points in $\mathbb{A}^n$ ($=\mathbb{C}^n$)

$(a_1, \ldots, a_n) \in \mathbb{A}^n$ max ideal $\mathfrak{m}_a$ = kernel of hom. $\mathbb{C}[x] \to \mathbb{C}$, $f(x) \mapsto f(a)$

$\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$

$A = \mathbb{C}[x]/I$ quotient of $\mathbb{C}[x]$, say $I = (f_1, \ldots, f_n)$, $f_i \in \mathbb{C}[x]$

cor: Then Spec A = {max ideals} $\overset{\text{bij}}{\longleftrightarrow} V(I)$ = locus of zeros

Why? b/c

Correspondence Thm: ideals of $A \overset{\text{bij}}{\longleftrightarrow}$ ideals of $\mathbb{C}[x]$ that contain I

max ideals $\overset{\text{bij}}{\longleftrightarrow}$ max ideals containing I

If A is any finitely generated $\mathbb{C}$-algebra (ring that contains $\mathbb{C}$), then $A \cong \mathbb{C}[x]/I$ (isomorphic)

Spec A (max ideals) $\longleftrightarrow V(I)$ variety in $\mathbb{A}^n$

Zariski topology: closed sets are $V(I)$, I an ideal

Affine Algebraic Geometry

$\Leftrightarrow$ finitely generated $\mathbb{C}$ algebra

Say $A \xrightarrow{\varphi} B$ homomorphism of finitely generated algebras.

Then the map goes Spec $A \longleftarrow$ Spec B

Equivalent Sets: A a fin. gen. $\mathbb{C}$-alg.

- {max ideals of A}
- {homomorphisms $A \to \mathbb{C}$}
- {$V(I)$ = locus of zeros of I if $A = \mathbb{C}[x]/I$}

If $A \xrightarrow{\varphi} B \xrightarrow{\pi} \mathbb{C}$
get $\pi\varphi$

Topics for <u>Affine Algebraic Geometry</u>

- localization (adjoining inverses)
- integral extensions (B a finite A-module)
- prime ideals
- dimension

Ex: $A = \mathbb{C}[x]$. $\operatorname{Spec} A = \mathbb{A}^n$
$B = A[g^{-1}]$ g some non-zero polynomial

Spec B:

$$B = A[y]/(yg-1) = \mathbb{C}[\overset{x_1,\ldots,x_n}{x}, y]/(yg(x)-1)$$

$\operatorname{Spec} B =$ locus $yg=1$ in $\mathbb{A}^{n+1}_{x,y}$

Say $(x^0, y^0) \in \operatorname{Spec} B$. So $y g(x^0) = 1$.
Given x^0 can solve uniquely for y^0, provided $g(x^0) \neq 0$
If $g(x^0)=0$, no solution.

Cor: $\operatorname{spec} \mathbb{C}[x][g^{-1}] \xleftrightarrow{bij}$ points of $\mathbb{A}^n_x$ where $g(x) \neq 0$