8 Tuesday, February 22, 2011

Outline:
I. playing with curves in $\mathbb{P}^2$

II. Affine algebraic geometry (structure of spec A)
III. Projective geometry
IV cohomology

---

thm: Nullstellensatz:
Maximal ideals of $\mathbb{C}[x_1, \dots, x_n] \longleftrightarrow$ points in $\mathbb{A}^n (= \mathbb{C}^n)$
$(a_1, \dots, a_n) \in \mathbb{A}^n$ max ideal $\mathfrak{m}_a$ = kernel of hom. $\mathbb{C}[x] \to \mathbb{C}$, $f(x) \mapsto f(a)$
$\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$

$A = \mathbb{C}[x]/I$ quotient of $\mathbb{C}[x]$, say $I = (f_1, \dots, f_n)$, $f_i \in \mathbb{C}[x]$
cor: Then Spec $A = \{\text{max ideals}\} \overset{\text{bij}}{\longleftrightarrow} V(I) = \text{locus of zeros}$
Why? B/C
thm Correspondence Thm: ideals of $A \overset{\text{bij}}{\longleftrightarrow}$ ideals of $\mathbb{C}[x]$ that contain I
max ideals $\overset{\text{bij}}{\longleftrightarrow}$ max ideals containing I
If A is a finitely generated $\mathbb{C}$-algebra (ring that contains $\mathbb{C}$),
then $A \overset{\text{isomorphic}}{\cong} \mathbb{C}[x]/I$
Spec $A \overset{\text{Max ideals}}{\longleftrightarrow} V(I)$ variety in $\mathbb{A}^n$

If $A \xrightarrow{\varphi} B \xrightarrow{\pi} \mathbb{C}$

get $\pi\varphi$

Topics for Affine Algebraic Geometry

- localization (adjoining inverses)
- integral extensions (B a finite A-module)
- prime ideals
- dimension

Ex: $A = \mathbb{C}[x]$ $\operatorname{spec} A = \mathbb{A}^1$

$B = A[g^{-1}]$ g some non-zero polynomial

Spec B? $x_1, \dots, x_n$

$B = A[y]/(yg-1) = \mathbb{C}[x, y]/(yg(x)-1)$

Spec B = locus $yg = 1$ in $\mathbb{A}^{n+1}_{x,y}$

Say $(x^0, y^0) \in \operatorname{Spec} B$. So $y^0 g(x^0) = 1$.

Given x^0, can solve uniquely for y^0, provided $g(x^0) \neq 0$

If $g(x^0) = 0$, no solution.

Cor: $\operatorname{spec} \mathbb{C}[x][g^{-1}] \overset{\text{bij}}{\longleftrightarrow}$ points of $\mathbb{A}^n_x$ where $g(x) \neq 0$