6 Wednesday, February 16, 2011

6.1 Plane curves

monomials $m_1, \ldots, m_N$, coefficients $z_1, \ldots, z_N$

$$\begin{aligned} Z &= \text{the space of all homogeneous polynomials of degree } d \text{ in } x,\ y, \text{ and } z \\ &= \text{affine space with coordinates } z_\nu \end{aligned}$$

(We have $f(x, y, z) = z_1 x^d + z_2 x^{d-1} + \cdots$)
U = open subset in Z corresponding to smooth plane curves
$\mathcal{C} \subset \mathbb{P}^2 \times Z$

$$\begin{tikzcd} \mathcal{C} \arrow[drr] & \subset & \mathbb{P}^2 \times Z \arrow[d] \\ & & Z \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}|_U = \mathcal{C}_U \arrow[drr] & \subset & \mathbb{P}^2 \times U \arrow[d] \\ & & U \end{tikzcd}$$

Proposition. *Smooth plane curves are orientable and connected surfaces, and compact.*

Proof. Oreintability was done last time.

To check connectedness, we just need to check one smooth curve of degree d is connected. $\mathcal{C} : \{x^d + y^d - z^d = 0\}$.

Look at the line $y = z$. On U_2, taking $z = 1$ we have $x^d + y^d = 1$. Since $y = z$, $y = 1$, and then $x^d = 0$. Since this is a root of order d, C meets this line in only one point. This means that it's connected. (WHY?) □

Given a connected, orientable, compact surface, it's topologically characterized by g = the genus = the # of holes.

Definition 1. The *Euler characteristic* of C is $2 - 2g$.

The Euler characteristic can be computed used an arbitrary triangulation, and then $E = \#\text{ vertices} - \#\text{ edges} + \#\text{ faces}$.

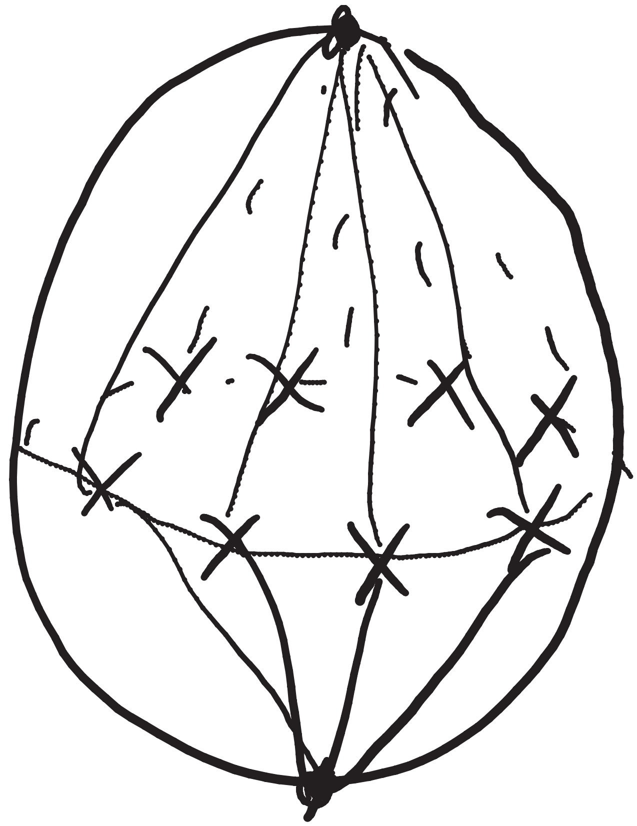
A sphere is, topologically, a tetrahedron, we have that the Euler characteristic is $4 - 6 + 4 = 2$. We can do a similar thing for a torus.

What is the Euler characteristic and genus of a smooth plane curve of degree d?

Let's represent a smooth plane curve as a branched cover of $\mathbb{P}^1$.

Method I Start with the Fermat curve, and do an explicit calculation. $C : \{x^d + y^d - z^d = 0\}$ Taking $z = 1$, $U_2 \simeq \mathbb{A}^2 \simeq \mathbb{C}^2$. Then we have $x^d + y^d = 1$. Drop y by projection. Fix a value x_0 for x. Then the line $y = x_0$ intersects the curve in $\leq d$ points. Typically, we get d values for y.

- Case I $x_0^d \neq 1$. Solve $y^d = 1 - x_0^d$. There are d solution s (if y is a solution, then so is $yr^{2\pi ij/d}$ for $0 \leq j \leq d$)
- Case II $x_0^d = 1$. Solve $y^d = 0$. The only solution is $y = 0$. Now we look at x_0. Triangulate $\mathbb{P}^1$, which is a sphere, as follows. There are d values for x_0, $x_i^d = 1$, $x_i = e^{2\pi ij/d}$.



These points distribute themselves along the equator of $\mathbb{P}^1$. Adding points at the poles, there are $2 + d$ vertices, $d + d + d = 3d$ edges, and $2d$ faces, which gives us 2. This is what happens "downstairs" (in the projected curve onto $X = \mathbb{P}^1$).

Upstairs, there is an induced triangulation:

- vertices: $d + d + d = 3d$
- edges: $3d^2$
- faces: $2d^2$

Then the Euler characteristic is $E = 3d - 3d^2 + 2d^2 = 3d - d^2 = d(3 - d)$. Then $g = \frac{1}{2}(d - 1)(d - 2)$.

Method II Let C be a smooth curve of degree d. Assume the coefficient of z^d is not zero. Divide by that coefficient, giving $f(x, y, z) = z^d - a_1(x, y)z^{d-1} + \cdots \pm a_d(x, y)$. Homogenous polynomial $\implies$ $a_i(x, y)$ is degree i in x and y. Then drop z by projection onto $\mathbb{P}^1(x, y)$.

Fix (x_0, y_0). View $z^d - a_1(x_0, y_0)z^{d-1} + \cdots \pm a_d(x_0, y_0) = 0$ as a polynomial of degree d in z. Typically this has d roots, but for some values of (x, y), there are $d - 1$ roots.

There is a polynomial Δ, discriminant, degree $d(d-1)$ in x and y. $\Delta = 0$ iff there are less than d roots. (The discriminant for a quadratic is $b^2 - 4ac$. It tells you whether or not the polynomial has double roots.) (Note, if the discriminant has only simple roots, then the claim above (that there are only d or $d - 1$ roots at any point) is intuitively/geometrically true.)

Triangulate $X = \mathbb{P}^1$ by putting vertices at these $d(d - 1)$ points. For $\mathbb{P}^1$, the Euler characteristic is 2. Pulling the triangulation up, the Euler characteristic is approximately $2d$ (everything gets multiplied by d). However, we placed the vertices at the $d(d-2)$ points where there are $d-1$ roots. Then the Euler characteristic is $2d - d(d-1) = 3d - d^2$.