

11 Monday, February 28, 2011

B finite type, G operator(?)

$$A = B^G$$

Showed A finite type

$$Y = \operatorname{Spec} B$$

$$X = \operatorname{Spec} A$$

$$X \leftrightarrow \{G \text{ orbits in } Y\}$$

$$G \times B \rightarrow B$$

$$\sigma, b \leadsto \sigma(b)$$

(left) $\sigma\tau(b)$: first τ , then σ

Then G operates on the *right* on Y .

$$q \in Y, \sigma \text{ sends } q \rightarrow q^\sigma$$

$$q^{\sigma\tau}: \text{first } \sigma, \text{ then } \tau$$

View

$$y \leftrightarrow \{\text{homomorphisms } B \rightarrow \mathbb{C}\}$$

$$q \leftrightarrow \pi_q : B \rightarrow \mathbb{C}$$

$$\leftrightarrow \{\text{max ideals of } B\}$$

$$q \leftrightarrow m_q$$

Operation on Y :

$$\begin{array}{ccc} B & \xrightarrow{\sigma} & B \\ & \searrow \pi \circ \sigma & \downarrow \pi \\ & & \mathbb{C} \end{array}$$

$$\pi_q \circ \sigma(b) = \pi_q(\sigma b)$$

Define q^σ = that point such that $\pi_{q^\sigma} = \pi_q \circ \sigma$.

Operation on *max ideals*:

$$\mathfrak{M}_{q^\sigma} = \sigma^{-1}\mathfrak{M}_q$$

$$Y \rightarrow X?$$

For any $p \in X$,

$$\begin{array}{ccc} B & \xrightarrow{\pi \circ p} & \mathbb{C} \\ \uparrow & \nearrow \pi_p & \\ A & & \end{array}$$

$Y \rightarrow X$ sends $q \rightsquigarrow r$

$$\begin{array}{ccccc}
 B & \xrightarrow{\sigma} & B & \xrightarrow{\pi_q} & C \\
 \uparrow & & \uparrow & & \parallel \\
 A & \xrightarrow{\text{id}} & A & \xrightarrow{\pi_p} & C \\
 & \searrow & \searrow & & \\
 & & & \pi_p &
 \end{array}$$

Therefore, G -orbits in Y map to points of X .

We want to show that different orbits $\{q_1, \dots, q_r\} \neq \{q'_1, \dots, q'_s\}$ in Y map to different points p, p' in X .

Proof. Plan: Find an element $a \in A$ such that $a = 0$ on orbit $\{q_i\}$, $\pi_q(a) = 0$, $a \neq 0$ on orbit $\{q'_j\}$. Then $\pi_{q'_j}(a) \neq 0$. This would give us that $a \in \mathfrak{M}_{q_i}$ (same as $\in \mathfrak{M}_p$) and $a \notin \mathfrak{M}_{q'_j}$ (same as $\notin \mathfrak{M}_{p'}$).

In B , choose $b \in \mathfrak{M}_{q_1}$ (then) such that $b \notin \mathfrak{M}_{q'_j}$ for all $j = 1, \dots, s$. (Note: $b(q) := \pi_q(b)$.)

(Diversion: Suppose $B = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_k)$. b represented by the polynomial $p(x_1, \dots, x_n)$. $\text{Spec } B \approx V(f_1, \dots, f_k)$ in \mathbb{C}^n .
 $\text{Spec } B = (\text{max ideals}) = (\text{homomorphisms } B \rightarrow \mathbb{C}) = (V(I) \text{ if } B = \mathbb{C}[x]/I)$)

We can do this. (Think about choosing (hyper-?)planes that do not pass through finite sets of points.)

Let $a = \prod_{\sigma \in G} \sigma(b)$. a is invariant. $a = 0$ on q_1 because b divides a in B (since some $\sigma \in G$ is the identity). Therefore $a = 0$ on the orbit (q_1) . $a \neq 0$ on q'_1 .

$\sigma(b)$ evaluated at q'_1 is $\pi_{q'_1}(\sigma b) = \pi_{q'_1 \sigma}(b) = b$ evaluated at $q'_1 \sigma$.

Therefore, $a \neq 0$ on the orbit. □

11.1 Localization

Note: we always assume that the rings are domains and assume (whenever possible) that they're finite type algebras.

Definition. A *multiplicative system* S in a domain A is a subset of A satisfying

- $1 \in S$
- $0 \notin S$
- if $s, t \in S$, then $st \in S$

Definition. The elements of S serve as denominators in the *ring of fractions*

$$A_S := \left\{ \frac{a}{s} \mid s \in S, a \in A \right\} / \sim$$

where $\frac{a}{s} \sim \frac{b}{t}$ if $at = bs$

$$\begin{aligned}
 A &\hookrightarrow A_S \\
 a &\rightsquigarrow a/1
 \end{aligned}$$

Example. $S = \{1, s, s^2, \dots\}$, $s \neq 0$. $A_S = A[s^{-1}] = A[y]/(sy - 1)$

Example. $S = A - \{0\}$, $A_S = \text{fraction field}$

Example. P a prime ideal of A , $S = A - P = \{s \mid s \notin P\}$. Then $s \notin P, t \notin P \implies st \notin P$. Then A_S is the localization of A at P . This is (perversely) denoted A_P .

If $A \subset B$ a subring, then we can relate ideals of A and B :

Extended ideal: I^e

I ideal of A

$IB = \text{ideal of } B \text{ generated by } \{I\}$

The elements are

$$\sum_{\text{finite}} x_i b_i, \quad x_i \in I, b_i \in B$$

Contracted ideal: J^c : For J an ideal of B , $(J \cap A) = \text{ideal of } A$

$$(I^e)^c \supset I$$

$$(J^c)^e \subset J$$

For $A \subset B = A_S$:

$$I^e = IA_s = \{x/s \mid x \in I, s \in S\} / \sim$$

$J^c = J \cap A$. If $y/s \in J$, then $y \in J \cap A = J^c$. Therefore, $y/s \in (J^c)^e$. Thus, $J \subset (J^c)^e$, so $J = (J^c)^e$.

Corollary. If A noetherian, then A_S noetherian

Proof. Take an increasing sequence $J_1 \subset J_2 \subset \dots$ of ideals in A_S . Let $I_\nu = J_\nu \cap A$. Then $I_1 \subset I_2 \subset \dots$. Since A is noetherian, this is eventually constant. Therefore $I_\nu^e = (J_\nu^c)^e$ eventually constant. Thus A_S is noetherian. \square