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Geometry of solutions to sets of polynomial equations.

e.g., $x^2 + y^2 + 1 = 0 \longrightarrow$ Set of solutions (over \mathbb{C}) is really a sphere (with two points removed)

e.g., $y^2 = x^3 - x = x(x-1)(x+1)$ Over \mathbb{C} , the set of solutions is a torus (with one point removed)

Lots of applications to number theory, representation theory, etc.

We'll work over the field \mathbb{C} .

Recall (background reading, §10-7, 10-8 in Artin's *Algebra*):

Theorem (Hilbert's Nullstellensatz (weak)). *The maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$ are exactly those of the form $(x_1 - a_1, \dots, x_n - a_n)$ corresponding to points $(a_1, \dots, a_n) \in \mathbb{C}^n$.*

This means we can consider \mathbb{C}^n as a purely algebraic object. It's called affine n -space, $\mathbb{A}_{\mathbb{C}}^n$ or \mathbb{A}^n for short.

We want to define a nice topology on this space. One choice is to take the Euclidean (complex) topology: define open balls by

$$B_r(x) = \{y \in \mathbb{C}^n \mid |y - x| < r\}$$

and take these to be a basis.

But this is too many open sets (closed sets), e.g. $\{(x, y) \in \mathbb{C}^n \mid y = e^x\}$ is closed in the Euclidean topology. But we only care about polynomials, so we'll use a coarser topology (fewer open/closed sets).

We'll use the smallest topology such that polynomial functions are continuous. This is called the Zariski topology. Defined by: for a polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$, define $D(f) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid f(a_1, \dots, a_n) \neq 0\}$ and declare all $D(f)$ to be open. (Note: D stands for distinguished.) As f varies over all polynomials, these $D(f)$ are taken to be a basis.

We have, e.g. $D(0) = \emptyset$, $D(1) = \mathbb{C}^n$, $D(fg) = D(f) \cap D(g)$.

Alternatively, let's see what the closed sets are. For every ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$, the vanishing locus of I is $V(I) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \forall f \in I\}$. As I varies, these describe all the closed sets in the Zariski topology.

Note that $\mathbb{C}^n \setminus D(f) = V(f)$.

NOTE:

1. Because $\mathbb{C}[x_1, \dots, x_n]$ is noetherian¹, any I can be written as (f_1, \dots, f_k) for some f_1, \dots, f_k . So $V(I) = V(\{f_1, \dots, f_k\})$.
2. The maximal ideals (the smallest non-empty closed sets) exactly correspond to the points of \mathbb{C}^n . (weak Nullstellensatz)

e.g.

- 1) \mathbb{A}^1 : the closed sets are $\emptyset = V(1)$, $\mathbb{A}^1 = V(0)$, and sets of zeros of polynomials, that is, all finite sets of points. (also called the cofinite topology)
- 2) \mathbb{A}^2 : the closed sets are $\emptyset = V(1)$, $\mathbb{A}^1 = V(0)$, finite sets of points, but also union of $V(f)$ with a finite point set for some polynomial $f \in \mathbb{C}[x, y]$.

¹Every ascending chain of ideals stabilizes: given $I_1 \subseteq I_2 \subseteq \dots \subseteq R$, $\exists k$ such that $I_k = I_{k+1} = \dots$. Equivalent to that every ideal is finitely generated

Why don't we include $V(f, g)$ for $f, g \in \mathbb{C}[x, y]$? It's because the locus of points where both f and g vanish (assuming they have no common factor) is a finite set of points. Note also that $V(f_1) \cup V(f_2) = V(f_1 f_2)$.

$\mathbb{C}[x_1, \dots, x_n]$ is called the affine coordinate ring of \mathbb{A}^n . (think of it as set of functions on the space \mathbb{A}^n)

1.1 Projective Space

Claim: \mathbb{A}^2 is somewhat defective.

Not all lines intersect. In particular, rotations of one line can cause it to not intersect another line.

To fix this, we add points at infinity to get \mathbb{P}^2 .

Nice way of doing this: Let

$$\mathbb{P}^2 = \{(x, y, z) \in \mathbb{A}^3 \setminus \{0, 0, 0\}\} / (x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad (\lambda \neq 0)$$

\mathbb{A}^2 is contained in the set of points for which $z \neq 0$: If $z \neq 0$, then

$$(x, y, z) \stackrel{\mathbb{P}^2}{=} \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

We have $(x, y) \in \mathbb{A}^2 \longrightarrow (x, y, 1)$.

$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1 = \mathbb{A}^2 \coprod \mathbb{A}^1 \coprod \text{point}(= \mathbb{A}^0)$$

Elements of \mathbb{P}^2 are written as $(x : y : z)$.

Define a topology on \mathbb{P}^2 . Most natural way is to take a quotient topology from the Zariski topology on $\mathbb{A}^3 \setminus \{0, 0, 0\} \subseteq \mathbb{A}^3$.

Take a polynomial function $f \in \mathbb{C}[x, y, z]$. We want to say $V(f) = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, z) = 0\}$ is closed. But f gives different values on equivalent points.

We can write $f = f_0 + f_1 + \dots + f_d$, f_i homogeneous of degree i . Then $f(\lambda x, \lambda y, \lambda z) = f_0(x, y, z) + \lambda f_1(x, y, z) + \dots + \lambda^d f_d(x, y, z)$. We need to take homogeneous polynomials ($f = f_i$ for some i). Now $f = 0$ and $f \neq 0$ makes sense. Then $V(f_1, \dots, f_k)$ describe the closed sets.