13 Friday, March 4, 2011

Integral Extensions

 $A \subset B$ domains

Definition. $b \in B$ is integral over A if it's a root of a monic polynomial. $f(x) = x^n - a_1 x^{n-1} + \cdots \pm a_r$ coefficients in A

Proposition. The following are equivalent:

- (1) b is integral over A
- (2) A[b] is a finite A-module
- (3) There exists an A[b]-module M which:
 - (i) is faithful¹ as an A[b]-module
 - (ii) is a finite A-module

Proof.

- $(1) \implies (2)$ Clear
- $(2) \implies (3) \text{ Clear}$
- (3) \Longrightarrow (1) Take the generators for M as an A-module, (v_1, \ldots, v_n) . Then $bv_i = \sum a_{ij}v_j$ for $a_{ij} \in A$. We can write this as

$$(b\mathbb{I} - A)V = 0$$
 $V = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

Let C be the coefficient matrix of $(b\mathbb{I}-A)$. Then $C(b\mathbb{I}-A) = \det(b\mathbb{I}-A)\mathbb{I}$. Let $\delta := \det(b\mathbb{I}-A)$. Then $\delta V = 0$. Since M is faithful, $\delta = 0$.) Expanding δ gives $\delta = b^n - (\operatorname{tr} A)b^{n-1} + \cdots \pm \det A$ is a monic polynomial for b. The coefficients are in A, so A[b] is integral.

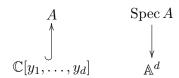
Proposition. Let A be noetherian.

- If B is generated as an A-algebra by elements integral over A, then every element of B is integral over A.
- If B is generated over A by a finite number of integral elements, then B is a finite A-module. (e.g., B integral over A and a finite type \mathbb{C} -algebra.)

Proof. Take $z \in B$. Then $z \in A[b_1, \ldots, b_k]$ for b_i integral. $A[b_1, \ldots, b_k]$ is a finite A-module. Therefore A[z] is contained in a finite module, so it itself is finite.

¹A module is faithful if for any $z \in A[b]$, $z \neq 0$, $zM \neq 0$.

Proposition (Noether Normalization). Let K be a field, and let A be a finite-type K-algebra and a domain. There exist $y_1, \ldots, y_d \in A$ algebraically independent², and A is a finite module over $K[y_1, \ldots, y_d]$. ³



Proof. Given A generated by x_1, \ldots, x_n (some finite set). Independent on n.(???) If x_1, \ldots, x_n dependent, there exists a polynomial relation $f(x_1, \ldots, x_n) = 0$ of degree d. Let h(x) be the degree d part of f. Then $h(0,0,0,\ldots,1)=$ coefficient of x_n^d in h and in f. If the coefficient of $x_n^d\neq 0$, then f(x) looks like $cx_n^d+g_{n-1}(x_1,\ldots,x_{n-1}^{d-1}+\cdots+g_0(x_1,\ldots,x_{n-1}))$. If $c\neq 0$, this is a monic polynomial of which x_n is a root, and its coefficients are in the ring $K[x_1,\ldots,x_{n-1}]$. Therefore, x_n is integral over $K[x_1, \ldots, x_{n-1}]$. By induction, we get a tower of integral extensions.

If c = 0, then make a change of variables $x_i \mapsto x_i + u_i x_n$ (for i < n), $x_n \mapsto u_n x_n$. Now the coefficient of x_n^d will be $h(x_1 + u_1 x_n, \dots, x_{n-1} + u_{n-1} x_{n-1}, u_n x_n)|_{x_1 = \dots = x_{n-1} = 0, x_n = 1} = h(u_1, \dots, u_n).$ This is non-zero for most choices of u_i .⁴

Corollary (A version of Nullstellensatz). Let K be a field, B a finite-type K-algebra that is also a field. Then B is a finite K-module.

It follows, without much trouble, from this that if $K = \mathbb{C}$, $B = \mathbb{C}$, too; if B is a finite field extension of \mathbb{C} , then $B = \mathbb{C}$.

Proof. Take $b \in B$. $\mathbb{C}[b]$ is a finite \mathbb{C} -module. Therefore, b is a root of an irreducible polynomial with coefficients in \mathbb{C} . Therefore, it is a root of a linear polynomial over \mathbb{C} , so $b \in \mathbb{C}$.

Proof. Noether Normalization says B a finite module over polynomial ring $A = K[y_1, \dots, y_d]$. If d=0, then we're done. If d>0, then $y_1\in A, B$, so $\frac{1}{y_1}\in B$. Thus B is a field. But $\frac{1}{y_1}$ is not integral over $K[y_1, \ldots, y_d]$.

If A is a domain with fraction field K, then A is integrally closed in K if every element of K which is integral over A is an element of A.

Example. $A = \mathbb{C}[x,y]/(y^2 - x^3)$. This is not integrally closed: Let $z = \frac{y}{x}$.

Then
$$z^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2} = x$$
.

$$z^3 = \frac{y^3}{r^3} = \frac{x^3y}{r^3} = y.$$

 $z^3 = \frac{y^3}{x^3} = \frac{x^3y}{x^3} = y.$ z is a root of the monic polynomial $z^2 - x$, and of $z^3 - y$.

Theorem. Let A be a finite-type algebra and a domain, and let K be the fraction field of A. Then the integral closure of A in K, the set of integral elements, is a finite A-module.

Preview:

Given an A-module M, then $M^* := \hom_A(M, A)$ is also an A-module.

If $M \subset N$, then $M^* \supset N^*$ (in good situations).

²There are no polynomial relations among them.

³I'm still confused by this statement.

⁴For fields with finite characteristic, you'll have to make a non-linear change of variables.