

# 1 Friday, February 4, 2011

Geometry of solutions to sets of polynomial equations.

e.g.,  $x^2 + y^2 + 1 = 0 \rightarrow$  Set of solutions (over  $\mathbb{C}$ ) is really a sphere (with two points removed)

e.g.,  $y^2 = x^3 - x = x(x-1)(x+1)$  Over  $\mathbb{C}$ , the set of solutions is a torus (with one point removed)

Lots of applications to number theory, representation theory, etc.

We'll work over the field  $\mathbb{C}$ .

Recall (background reading, §10-7, 10-8 in Artin's *Algebra*):

**Theorem** (Hilbert's Nullstellensatz (weak)). *The maximal ideals of  $\mathbb{C}[x_1, \dots, x_n]$  are exactly those of the form  $(x_1 - a_1, \dots, x_n - a_n)$  corresponding to points  $(a_1, \dots, a_n) \in \mathbb{C}^n$ .*

This means we can consider  $\mathbb{C}^n$  as a purely algebraic object. It's called affine  $n$ -space,  $\mathbb{A}_{\mathbb{C}}^n$  or  $\mathbb{A}^n$  for short.

We want to define a nice topology on this space. One choice is to take the Euclidean (complex) topology: define open balls by

$$B_r(x) = \{y \in \mathbb{C}^n \mid |y - x| < r\}$$

and take these to be a basis.

But this is too many open sets (closed sets), e.g.  $\{(x, y) \in \mathbb{C}^n \mid y = e^x\}$  is closed in the Euclidean topology. But we only care about polynomials, so we'll use a coarser topology (fewer open/closed sets).

We'll use the smallest topology such that polynomial functions are continuous. This is called the Zariski topology. Defined by: for a polynomial function  $f \in \mathbb{C}[x_1, \dots, x_n]$ , define  $D(f) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid f(a_1, \dots, a_n) \neq 0\}$  and declare all  $D(f)$  to be open. (Note:  $D$  stands for distinguished.) As  $f$  varies over all polynomials, these  $D(f)$  are taken to be a basis.

We have, e.g.  $D(0) = \emptyset$ ,  $D(1) = \mathbb{C}^n$ ,  $D(fg) = D(f) \cap D(g)$ .

Alternatively, let's see what the closed sets are. For every ideal  $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ , the vanishing locus of  $I$  is  $V(I) = \{(a_1, \dots, a_n) \in \mathbb{A}^n \mid f(a_1, \dots, a_n) = 0 \forall f \in I\}$ . As  $I$  varies, these describe all the closed sets in the Zariski topology.

Note that  $\mathbb{C}^n \setminus D(f) = V(f)$ .

NOTE:

1. Because  $\mathbb{C}[x_1, \dots, x_n]$  is noetherian<sup>1</sup>, any  $I$  can be written as  $(f_1, \dots, f_k)$  for some  $f_1, \dots, f_k$ . So  $V(I) = V(\{f_1, \dots, f_k\})$ .
2. The maximal ideals (the smallest non-empty closed sets) exactly correspond to the points of  $\mathbb{C}^n$ . (weak Nullstellensatz)

e.g.

- 1)  $\mathbb{A}^1$ : the closed sets are  $\emptyset = V(1)$ ,  $\mathbb{A}^1 = V(0)$ , and sets of zeros of polynomials, that is, all finite sets of points. (also called the cofinite topology)
- 2)  $\mathbb{A}^2$ : the closed sets are  $\emptyset = V(1)$ ,  $\mathbb{A}^1 = V(0)$ , finite sets of points, but also union of  $V(f)$  with a finite point set for some polynomial  $f \in \mathbb{C}[x, y]$ .

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<sup>1</sup>Every ascending chain of ideals stabilizes: given  $I_1 \subseteq I_2 \subseteq \dots \subseteq R$ ,  $\exists k$  such that  $I_k = I_{k+1} = \dots$ . Equivalent to that every ideal is finitely generated

Why don't we include  $V(f, g)$  for  $f, g \in \mathbb{C}[x, y]$ ? It's because the locus of points where both  $f$  and  $g$  vanish (assuming they have no common factor) is a finite set of points. Note also that  $V(f_1) \cup V(f_2) = V(f_1 f_2)$ .

$\mathbb{C}[x_1, \dots, x_n]$  is called the affine coordinate ring of  $\mathbb{A}^n$ . (think of it as set of functions on the space  $\mathbb{A}^n$ )

## 1.1 Projective Space

Claim:  $\mathbb{A}^2$  is somewhat defective.

Not all lines intersect. In particular, rotations of one line can cause it to not intersect another line.

To fix this, we add points at infinity to get  $\mathbb{P}^2$ .

Nice way of doing this: Let

$$\mathbb{P}^2 = \{(x, y, z) \in \mathbb{A}^3 \setminus \{0, 0, 0\}\} / (x, y, z) \sim (\lambda x, \lambda y, \lambda z) \quad (\lambda \neq 0)$$

$\mathbb{A}^2$  is contained in the set of points for which  $z \neq 0$ : If  $z \neq 0$ , then

$$(x, y, z) \stackrel{\mathbb{P}^2}{=} \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

We have  $(x, y) \in \mathbb{A}^2 \longrightarrow (x, y, 1)$ .

$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1 = \mathbb{A}^2 \coprod \mathbb{A}^1 \coprod \text{point}(= \mathbb{A}^0)$$

Elements of  $\mathbb{P}^2$  are written as  $(x : y : z)$ .

Define a topology on  $\mathbb{P}^2$ . Most natural way is to take a quotient topology from the Zariski topology on  $\mathbb{A}^3 \setminus \{0, 0, 0\} \subseteq \mathbb{A}^3$ .

Take a polynomial function  $f \in \mathbb{C}[x, y, z]$ . We want to say  $V(f) = \{(x : y : z) \in \mathbb{P}^2 \mid f(x, y, z) = 0\}$  is closed. But  $f$  gives different values on equivalent points.

We can write  $f = f_0 + f_1 + \dots + f_d$ ,  $f_i$  homogeneous of degree  $i$ . Then  $f(\lambda x, \lambda y, \lambda z) = f_0(x, y, z) + \lambda f_1(x, y, z) + \dots + \lambda^d f_d(x, y, z)$ . We need to take homogeneous polynomials ( $f = f_i$  for some  $i$ ). Now  $f = 0$  and  $f \neq 0$  makes sense. Then  $V(f_1, \dots, f_k)$  describe the closed sets.