17 Monday, March 14, 2011

Definition. The *Krull dimension* of a ring A is the length of the longest chain of prime ideals $P_0 < P_1 < \cdots < P_d$.

In a dimension zero ring, every prime ideal is maximal and also minimal. Therefore, there are a finite number of prime/minimal ideals.

If A is a domain of dimension 1, then (0) is a prime ideal and all the other prime ideals are maximal.

Definition. The *codimension* of a prime ideal P is the length of the longest chain $P_0 < P_1 < \cdots < P_d = P$.

If A is a domain, then P has codimension 0 if $P = (0)$, and P has codimension 1 if it's not zero and there does not exist P' with $(0) < P' < P$.

Theorem (Knull's Principal Ideal Theorem)**.** *Let A be a domain. Let $x \in A$ with $x \neq 0$, and let P be a prime ideal. If $x \in P$ and P is is the minimal prime ideal containing x, then P has codimension 1.*

($P/x \in P \longleftrightarrow$ prime ideals $\bar{B}$ of $\bar{A} = A/(x)$.

Therefore, the Krull dimension of $\bar{A}$ is the Krull dimension of A, - 1.)

Discrete valuations

Generalize order of vanishing of $f(x)$ at $x = a$. For $f(x_1, \ldots, x_n)$, talking about order of vanishing at $(a_1, \ldots, a_n)$ doesn't make much sense. But we can define the order of vanishing along a subvariety of codimension 1.

Let K be a field. A discrete valuation(?) v is a group homomorphism $K^\times \xrightarrow{v} \mathbb{Z}^+$ such that $v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\}$ and $v(\alpha\beta) = v(\alpha) + v(\beta)$. "$v(\alpha) = k$ means α vanishes to order k (has a zero of order k)". $v(\alpha) = -k$, "α has a *pole* of order k.

A homomorphism $K^\times \xrightarrow{v} \Gamma^+$ (an ordered group with $+$) is also a valuation. This is unimportant!

Definition. The (discrete) valuation ring R associated to a discrete valuation v is $R = \{\alpha \in K^\times \,|\, v(\alpha) \geq 0\} \cup \{0\}$ ("no pole").

Remorse: You always have to add $\{0\}$, or define $v(0) = \infty$ (but this is artificial)

Definition. A *fractional ideal* is a non-zero finitely generated submodule of K.

Properties of DVR (discrete valuation ring):

- subring of K and local domain (local ring that's a domain, ring with one maximal ideal)
- $\mathfrak{M} = \{\alpha \in K \,|\, v(\alpha) > 0\} \cup \{0\}$
- $\mathfrak{M}$ is a principle ideal generated by any $t \in K^\times$ with $v(t) = 1$.
- ideals of R are $\mathfrak{M}^k = (t^k)$ and zero ideal $(K \geq 0)$ $(\mathfrak{M}^k = \{\alpha \,|\, v(\alpha) \geq k\})$
- R is normal
- The fractional ideals of R are (t^k), $k \in \mathbb{Z}$.

Write $(t^k) = \mathfrak{M}^k$ also for $k < 0$. Then $\mathfrak{M}^{k+l} = \mathfrak{M}^k\mathfrak{M}^l$, $k, l \in \mathbb{Z}$.

Lemma. $v(1) = 0$, $v(-1) = 0$, $v(a^{-1}) = -v(a)$

Take $\alpha \in K^\times$, $v(\alpha) = k$. Also, $v(t^k) = k$. $v(t^{-k}\alpha) = 0$. Then $t^{-k}\alpha = u$ is a unit in R, so $\alpha = ut^k$.

Proposition. *The following are equivalent conditions on a local noetherian domain A:*

(1) A is a DVR

(2) A is normal and has dimension 1

(3) A is normal and there exists an $x \in A$ such that $\mathfrak{M}$ is the minimal prime ideal containing x

(4) $\mathfrak{M}$ is principle

Proof.

(1) $\implies$ (2) We're not doing it (it follows from the properties of a DVR)

(2) $\implies$ (3) There exist only two prime ideals in A: $(0) \neq \mathfrak{M}$. $x \in \mathfrak{M}$, $x \neq 0$ also works.

(3) $\implies$ (4) Take x as in (3). Then $\bar{A} = A/(x)$ has only one prime ideal $\bar{\mathfrak{M}}$ both maximal and minimal. The intersection of the minimal prime ideals is the nilradical of $\bar{A}$. Therefore $\bar{\mathfrak{M}}$ is the nilradical. Therefore, $\bar{\mathfrak{M}}^N = (0)$, $N \gg 0$. Thus $\mathfrak{M}^N < (x)$.

Choose r such that $\mathfrak{M}^{r-1} \not\subset (x)$ but $\mathfrak{M}^r \subset (x)$. Take $y \in \mathfrak{M}^{r-1}$, $y \notin (x)$. We want to show that $w = x/y$ generates $\mathfrak{M}$. Let $z = w^{-1} = y/x$. Since $y \notin (x)$, $z \notin A$. Now consider $z\mathfrak{M}$.

Lemma. *Let A be a normal noetherian domain, I a non-zero ideal, $\gamma \in K = \mathrm{Fract}(A)$. If $\gamma I \subset I$, then $\gamma \in A$.*

Proof. $\gamma I \subset I$ means that I is an $A[\gamma]$-module. I $(\gamma I?)$ is faithful, and a finite A-module. Then γ is integral over A, and thus an element of A. □

$z\mathfrak{M} = \frac{y}{x}\mathfrak{M} \subset \frac{\mathfrak{M}^r}{x} \subset A$. We stop here (continue next time).

□