24 Friday, April 8, 2011

Category Theory

Definition. A *category* is a collection of objects, together with morphisms (maps) between objects satisfying the following axioms:

- Composition of morphisms is defined.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "f \circ g \text{ is defined}"'] & y \arrow[r, "g"] & z \end{tikzcd}$$

- Associative law: If we have
$$f \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w,$$
then $(h \circ g) \circ f = h \circ (g \circ f)$

- For all objects y, there exists an identity $y \xrightarrow{i_y} y$ such that $g \circ i_y = g$, $i_y \circ f = f$ for all f and g with the appropriate domains and ranges.

Examples of categories include the category of: sets; topological spaces, algebras.

Definition. Let C and D be categories. Then a *functor* F is a map $C \xrightarrow{F} D$ which maps $\text{Obj}(C) \to \text{Obj}(D)$ and maps $\text{Morph}(C) \to \text{Morph}(D)$ and which (1) is compatible with composition and (2) maps identity morphisms to identity morphisms.

Example. (top) $\to$ (sets), $x \rightsquigarrow$ underlying set

Example. (groups) $\to$ (abelian groups), $G \to G/(aba^{-1}b^{-1} = 1)$

Example. (top) $\xrightarrow{1-1_q}$ abelian groups, $x \rightsquigarrow H_q(x)$ homology

Example. pointed path connected topologies $\to$ groups, $x \rightsquigarrow \pi_1(x)$ (homology group)

Definition. The *dual category* C° of a category C is defined by:

$$\text{Obj}(C^\circ) = \text{Obj}(C)$$

$$\begin{array}{ccc} \text{A morphism} & & \text{A morphism} \\ x^\circ \xleftarrow{f^\circ} y^\circ & = & x \xrightarrow{f} y \\ \text{in } C^\circ & & \text{in } C \end{array}$$

Definition. A *contravariant functor* is a functor F on C from $C^\circ \to D$.

Example. (vect) = category of complex vector spaces. $(\text{vect})^\circ \to (\text{vect})$, $v \rightsquigarrow v^*$. This functor takes a vector space to its dual space.

Example. (top) $\to$ (abelian groups), $x \rightsquigarrow H^q(x)$ cohomology

Example. Let X be a topological space. Then let (opens) denote the category where the objects are open sets in X, and the morphisms are inclusions $U \subset V$; if $U \subset V$, $\exists! U \to V$, and if $U \not\subset V$, $\nexists U \to V$.

Definition. The *sheaf of functions* F on X is the contravariant functor on (opens): $(\text{opens})^\circ \xrightarrow{F} (\text{algebras})$, $U \rightsquigarrow F(U) =$ complex valued functions with domain U (an algebra).

If $V \to U$ (i.e., $V \subset U$), then we can restrict a function on U to V; we get $F(U) \xleftarrow{\text{rest}_V} F(U)$, $f|_V \leftrightsquigarrow f$.

Sheaf Axiom for F: Functions on U can be defined "locally." More formally, suppose U is open in X and U^i are open subsets of U that together cover U. Let U^{ij} denote $U^i \cap U^j$.

$$U \leftarrow \{U^i\} \leftleftarrows \{U^{ij}\}$$

$$\begin{tikzcd} U^i & \\ & U^{ij} \arrow[hook', ul] \arrow[hook, dl] \\ U^j & \end{tikzcd}$$

Then, given functions f^i on U^i such that the restriction of f^i and f^j to U^{ij} are equal for all i, j, the sheaf axiom requires that there exists a unique function f on U such that the restriction of f to U^i is f^i.

Example. Let $X = U$ be the real line. Let $U^1 = (-\infty, 1)$ and let $U^2 = (0, \infty)$. Then $U = U^1 \cup U^2$. Let $U^1 \cap U^2 = U^{12}$ $(= (0, 1))$. Then if f^1 and f^2 are functions on U^1 and U^2 and if the restriction of f^1 to U^{12} is equal to the restriction of f^2 to U^{12}, then $\exists!$ f on U, such that its restriction to $U^i = f^i$.

Definition. A *sheaf on a topological space* X is a contravariant functor $(\text{opens})^\circ \xrightarrow{M} C$ (some category), $U \rightsquigarrow M(U)$ which satisfies the sheaf axiom:

Suppose $\{U^i\}$ covers U and let $U^{ij} = U^i \cap U^j$. Given an element $\alpha^i \in M(U^i)$, if the restriction[1] of α^i and α^j to $M(U^{ij})$ are equal for all i, j, then $\exists! \alpha \in M(U)$ whose restriction to U^i is α^i for all i.

We want to (eventually) write the sheaf axiom more compactly. First, we rewrite it in terms of $M(U)$. Recall

$$\begin{tikzcd} U^i & \\ & U^{ij} \arrow[hook', ul] \arrow[hook, dl] \\ U^j & \end{tikzcd}$$

Then

$$0 \to M(U) \to \prod_i M(U^i) \underset{d_1^*}{\overset{d_0^*}{\rightrightarrows}} \prod_{i,j} M(U^{ij})$$

$$U \leftarrow \{U^i\} \underset{d_1}{\overset{d_0}{\leftleftarrows}} \{U^{ij}\}$$

Then the sheaf axiom says that the above sequence is exact[2] if we replace $\rightrightarrows$ by the difference $d_0^* - d_1^*$.[3]

Think about for next time: The structure sheaf on $X = \mathbb{A}^1$ (with the Zariski topology): U open $= X - S$ (with S a finite set) together with $U = \emptyset$. If $S = (s_1, \ldots, s_k)$. Let $f = (x_1 - s_1) \cdots (x - s_k)$, $\mathcal{O}(U) = \mathbb{C}[x][f^{-1}]$. Check the sheaf axiom.

[1] If $V \to U$, then $M(V) \xleftarrow{\text{"restriction"}} M(U)$.

[2] A sequence is exact if the kernel of each map is equal to the image of the preceding map.

[3] We're assuming that the category we're mapping into has a $-$ (e.g., that of abelian groups), i.e., that each $M(U)$ has a zero element and subtraction.