

10 Friday, February 25, 2011

Recall:

If I is an ideal of A , then $\text{rad } I = \{x \in A \mid x^n \in I, \text{ some } n > 0\}$

$$V(I) = V(\text{rad } I)$$

$$V(I) \supset V(J) \iff I \subset \text{rad } J$$

irreducible closed set C : $C \neq C_1 \cup C_2$, $C_i < C$.

Proposition. I an ideal, $V(I)$ irreducible iff $\text{rad } I$ is prime ideal

Theorem. Let A be a finite-type noetherian ring, I an ideal. Then $\text{rad } I$ is the intersection of a finite number of prime ideals $\text{rad } I = P_1 \cap \cdots \cap P_k$. Then we can organize $P_i \not\subset P_j$ for $i \neq j$. Then we have that $V(I) = V(P_1) \cup \cdots \cup V(P_k)$; $V(I)$ is a finite union of irreducible closed sets.

$$\text{Spec } A = \{\text{maximal ideals}\}$$

$$V(I) = \{\mathfrak{M} \text{ that contains } I\}$$

I an ideal. Then $\bar{A} = A/I$.

$$\begin{aligned} A &\rightarrow \bar{A} \\ I &\sim (\bar{0}) \\ \text{rad } I &= \text{rad}(\bar{0}) \\ &= \text{nilradical} \\ &= \{x \mid x^n = 0\} \end{aligned}$$

The following are equivalent:

- $\text{rad } I = P_1 \cap \cdots \cap P_k$
- $\text{rad}(\bar{0}) = \bar{P}_1 \cap \cdots \cap \bar{P}_k$
- (in \bar{A}) also prime ideals

Proof. Suppose the theorem is false for some A and some I . Let $S = \{\text{ideals } J \text{ of } A \text{ such that } \text{rad } J \text{ is not a finite intersection of prime ideals}\}$. By hypothesis, $S \neq \emptyset$. Then there exists a maximal element I in S . Then $\text{rad } I$ is not a finite intersection of prime ideals, but every larger ideal is a finite intersection of prime ideals.

Note:

- $I = \text{rad } I$
- I is not a prime ideal

Then there exist ideals K, L , $K \not\subset I$, $L \not\subset I$, but $KL \subset I$. Replace K by $K + I$, L by $L + I$. Then we have

$$(K + I)(L + I) \subset KL + KI + IL + II \subset I.$$

(so it's ok to do these replacements).

Now $K, L > I$. Replace K and L with their radicals.

$$(\text{rad } K)(\text{rad } L) \subset \text{rad}(KL) \subset \text{rad } I = I$$

(so it's ok to these replacements.)

We have

$$\begin{aligned}\mathrm{rad} K &= P_1 \cap \cdots \cap P_r \\ \mathrm{rad} L &= Q_1 \cap \cdots \cap Q_s\end{aligned}$$

...

We made a mistake somewhere. (Supposed to replace I by a zero ideal?) The version on the web is probably correct. Let's skip this for now. □

Finite Group Action

Let B be a finite type domain, G the finite group of automorphisms of B . Let $A = B^G$.

Theorem. • A is a finite type domain

- G operates on $\mathrm{Spec} B = Y$
- Y maps to $\mathrm{Spec} A = X$. $Y \rightarrow X$ is surjective, and the fibers are the G -orbits in Y

Example. $B = \mathbb{C}[x, y]$, $\sigma(x) = -x$, $\sigma(y) = -y$. Then $\langle \sigma \rangle$ is a group of order 2.

The invariant functions are $u = x^2$, $v = y^2$, and $w = xy$. It's not hard to see that every invariant function can be written as some combination of these.

$$A = B^G = \mathbb{C}[u, v, w]/(w^2 - uv)$$

Then $\mathrm{Spec} A = X = \text{locus of } w^2 = uv \text{ in } \mathbb{A}_{u,v,w}^3$. This is a double cone in 3-space.

Proof. A finite type domain:

Take $\beta \in B$, orbit $\{b_1 = \beta, \dots, b_r\}$. Let $p(x) = (x - b_1) \cdots (x - b_r) = x^r - s_1(b)x^{r-1} + \cdots \pm s_r(b)$. We have that β is a root of $p(x)$. Since the $s_i(b)$ are symmetric functions, $s_i(b) \in B^G = A$.

Since B is a finite type domain, say B is generated as an algebra by β_1, \dots, β_m . Then each β_i is a root of the polynomial, coefficients in A .

Let $A_0 = \mathbb{C}$ -algebra generated by these roots. Then A_0 is a finite-type domain contained in B . Every element in B is a polynomial in β_1, \dots, β_m .

If a polynomial with β_i as a root has degree d_i , then we only need monomials in β_i with degree $\leq d_i$. There are only a finite number of monomials in β_i of degree $\leq d_i$. Then B is spanned as an A_0 -module by these monomials. Thus, B is a finite A_0 -module.

$A_0 \subset A \subset B$. Since A_0 is a finite-type domain, A_0 is noetherian. B is a finite-type A_0 -module. A is a submodule. Therefore, A is a finite-type A_0 module. Thus, A is a finite-type algebra. □