If $A \xrightarrow{\varphi} B \xrightarrow{\pi} \mathbb{C}$

get $\pi\varphi$

Topics for Affine Algebraic Geometry

- localization (adjoining inverses)
- integral extensions (B a finite A-module)
- prime ideals
- dimension

Ex: $A = \mathbb{C}[x]$. $\operatorname{Spec} A = \mathbb{A}^1$

$B = A[g^{-1}]$ g some non-zero polynomial

Spec B?

$B = A[y]/(yg-1) = \mathbb{C}[x_1,\dots,x_n, y]/(yg(x)-1)$

$\operatorname{Spec} B$ = locus $yg=1$ in $\mathbb{A}^{n+1}_{x,y}$

Say $(x^0, y^0) \in \operatorname{Spec} B$. So $y^0 g(x^0) = 1$.

Given x^0 can solve uniquely for y^0, provided $g(x^0) \neq 0$

If $g(x^0) = 0$, no solution.

Cor: $\operatorname{Spec} \mathbb{C}[x][g^{-1}] \xleftrightarrow{\text{bij}}$ points of $\mathbb{A}^n_x$ where $g(x) \neq 0$

Outline:

I. playing with curves in $\mathbb{P}^2$

II. Affine algebraic geometry (structure of spec A)

III. Projective geometry

IV cohomology

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thm: Nullstellensatz:

Maximal ideals of $\mathbb{C}[x_1, \dots, x_n] \longleftrightarrow$ points in $\mathbb{A}^n (= \mathbb{C}^n)$

$(a_1, \dots, a_n) \in \mathbb{A}^n$ max ideal $\mathfrak{m}_a$ = kernel of hom. $\mathbb{C}[x] \to \mathbb{C}$, $f(x) \mapsto f(a)$

$\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$

$A = \mathbb{C}[x]/I$ quotient of $\mathbb{C}[x]$, say $I = (f_1, \dots, f_n)$, $f_i \in \mathbb{C}[x]$

cor: Then $\operatorname{Spec} A = \{\text{max ideals}\} \overset{\text{bij}}{\longleftrightarrow} V(I) =$ locus of zeros

Why? B/C

thm Correspondence Thm: ideals of $A \overset{\text{bij}}{\longleftrightarrow}$ ideals of $\mathbb{C}[x]$ that contain I

max ideals $\overset{\text{bij}}{\longleftrightarrow}$ max ideals containing I

If A is any finitely generated $\mathbb{C}$-algebra (ring that contains $\mathbb{C}$),

then $A \cong$ (isomorphic) $\mathbb{C}[x]/I$

$\operatorname{Spec} A \longleftrightarrow V(I)$ variety in $\mathbb{A}^n$

max ideals

Zariski topology: closed sets are $V(I)$, I an ideal

Affine Algebraic Geometry

$\updownarrow$ finitely generated $\mathbb{C}$-algebra

Say $A \xrightarrow{\varphi} B$ homomorphism of finitely generated algebras.

Then the map goes $\operatorname{Spec} A \longleftarrow \operatorname{Spec} B$

Equivalent Sets: A a fin.-gen. $\mathbb{C}$-alg.

- (max ideals of A)
- (homomorphisms $A \to \mathbb{C}$)
- ($V(I)$ = locus of zeros of I if $A = \mathbb{C}[x]/I$)