19 Friday, March 18, 2011

GK-dimension A (finite-type algebra, domain)
Choose V a one-dimensional subspace which

- generates A
- contains 1 (that is, $1 \in V$)

V^n is the span of products of length n of elements of V.
$V \subset V^2 \subset \cdots$
$\bigcup V^n = A$
The generating function g is $g(n) = \dim V^n$
$GK\text{-}\dim V = d$ means that $g(n) \le cn^d$ and $\not\le cn^{d-\epsilon}$

Lemma.

- *This is independent of the choice of V.*
- $\mathrm{gk}(k[x_1, \ldots, x_n]) = d$ *(?)*

Take V = polynomials of degree ≤ 1. Then $\dim V^n = \binom{n+d}{d}$.
Trivial fact: If $\bar{A} = A/I$ then $\mathrm{gk}(\bar{A}) \le \mathrm{gk}(A)$.

Lemma. *Suppose $A \subset B$, B a finite A-module. Then* $\mathrm{gk}(B) = \mathrm{gk}(A)$.

Proof. Choose a finite dimensional subspace U of B that generates B as an A-module. Then $U^2 \subset B = AU$. Now choose a (one-dimensional) generating space V for A large enough so that $U^2 \subset VU$. Take $W = V + U$ to generate B as an algebra. Then we have

$$W^n = V^n + V^{n-1}U + V^{n-2}U^2 + \cdots + U^n.$$

Since $V^{n-2}U^2 \subset V^{n-1}U$, by induction, $U^n \subset V^{n-1}U$. Then $W^n \subset V^n + V^{n-1}U$. Since $W = V + U$, we also have that $V^n \subset W^n$.

Let $h(n) = \dim W^n$. Then

$$g(n) \le h(n) \le g(n) + rg(n-1) \le (r+1)g(n)$$

with $r = \dim U$. □

Let K be the fraction field of A. Define $\operatorname{tr deg} A = \operatorname{tr deg} K/\mathbb{C}$.[1]

Corollary. $\mathrm{gk}(A) = \operatorname{tr deg} K/\mathbb{C}$.

Proof. Noether Normalization says that A is a finite module over $k[x_1, \ldots, x_k]$, $k[x] \subset A$. Then $k = \operatorname{tr deg} K/\mathbb{C}$. □

Corollaries:

- $\mathrm{gk}(A) \in \mathbb{N}$
- $s \ne 0, s \in A \implies \mathrm{gk}(A_s) = \mathrm{gk}(A)$ (same field of fractions, therefore same degree)

[1] tr deg is the transcendence degree.

Let A be a finite type domain, P a prime ideal of codimension 1 ($(0) < P$ and no primes in between).

Proposition. $\operatorname{gk}(A/P) = \operatorname{gk}(A) - 1$ *or* $\operatorname{tr}\deg A/P = \operatorname{tr}\deg A - 1$

Proof. Replace A by its normalization A', a finite A-module.

$$\begin{array}{ccc} A & \hookrightarrow & B \\ P & & Q \quad \text{also codim } 1\ (P = A \cap Q) \\ (0) & & (0) \end{array}$$

$$\operatorname{tr}\deg A = \operatorname{tr}\deg B$$

B/Q is a finite A/P-module, $\operatorname{tr}\deg B/Q = \operatorname{tr}\deg A/P$.

Assume A is normal. Then the local ring A_P is a DVR (discrete valuation ring). The maximal ideal $P_P = tA_P$ for some $t \in A_P$. It's a principal ideal. t is a fraction $s^{-1}a$, $s \notin P$.

Localize: Replace A by A_s, P by P_s. (It still has codimension 1 because the prime ideals in the localized ring are subsets of the original prime ideals.(?)) Now $t \in A$, but it might not generate P. Say $P = (u_1, \ldots, u_r)$. In the local ring, $t \mid u_i$. Then, we can get a common denominator, so $u_i = t(s_1^{-1}b_i)$, $b_i \in A$, $s \notin P$. Now replace A by A_{s_1}. Now t generates P.

We've reduced to the case that $P = tA$ (A normal). Extend t to a transcendence basis[2] for the fraction field of A, $(t, x_1, \ldots, x_k)$, $x_i \in A$.

Now $k[t, x_1, \ldots, x_k] \subset A$ and the elements of A are algebraic over $k[t, x]$. Look at $k[x_1, \ldots, x_k] \to \bar{A} = A/P = A/tA$. What's the kernel? $f(x) \rightsquigarrow 0$ means that $f(x) = t\alpha$, $\alpha \in A$.

...We seem to not have a proof. (To be posted online. We'll assume the proposition is true.) $\square$

Theorem. $\operatorname{gk}(A) = \operatorname{tr}\deg A = $ *Krull* $\dim A$. *More precisely, every maximal chin of prime ideals* $(0) < P_1 < P_2 < \cdots < P_d$ *has length* $d = \operatorname{gk}(A) = \operatorname{tr}\deg A$.

Proof. We have already done $\operatorname{gk}(A) = \operatorname{tr}\deg A$.

We induct on d. We prove the statement: If a maximal chain has length d, then $d = \operatorname{gk}(A)$. Look at $\bar{A} = A/P_1$. A maximal chain in $\bar{A}$ is $(0) = \bar{P}_1 < \bar{P}_2 < \cdots < \bar{P}_d$ (by the correspondence theorem) has length $d - 1$. Then the Krull dimension of $\bar{A}$ is $\operatorname{gk}(A) = d - 1$. Therefore, $\operatorname{gk}(A) = d$.

(Where is Krull's Theorem?!) $\square$

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[2] *Transcendence basis*: Maximally algebraically independent set in the fraction field.