9 Wednesday, February 23, 2011

Hilbert Basis Theorem

A ring A is Noetherian if the ideals are finitely generated.

Theorem (Hilbert Basis Theorem). If R is Noetherian, then R[x] is Noetherian

Corollary. $\mathbb{C}[x_1,\ldots,x_n]$ is Noetherian

Any finite-type (finitely generated as an algebra (everything is a polynomial in finitely many things)) \mathbb{C} -algebra is Noetherian. $(A \cong \mathbb{C}[x]/I)$

Equivalent conditions on A:

- 1. A is noetherian (ideals are finitely generated)
- 2. Every infinite increasing family $I_1 \subset I_2 \subset \cdots$ of ideals becomes constant eventually $(I_1 < I_2 < \cdots \text{ chain is finite})$
- 3. Every non-empty set S of ideals contains maximal elements $(\exists I \in S \text{ such that } I \not\subset J \text{ for any } J \in S, J \neq I)$

Corollary. If A is noetherian, I is an ideal of A, and I < A, then I is contained in a maximal ideal. (The maximal ideal is a maximum element in the set of ideals < A.)

Corollary. If A contains no maximal ideal, then A is the zero ring.

$$\operatorname{Spec} A \neq \emptyset \iff A = \{0\}$$

Adjoining inverses to $A = \mathbb{C}[x]$ $(x = x_1, \dots, x_n), B = A[g^{-1}] = \mathbb{C}[x, y]/(yg - 1).$ Then Spec $A = \mathbb{A}^n$, and Spec $B \approx \mathbb{A}^n - V(g)$.

Theorem (Strong Nullstellensatz). Let I be an ideal of $\mathbb{C}[x]$, $g \in \mathbb{C}[x]$. Suppose g vanishes identically on V(I). Then $g^N \in I$ for some $N \gg 0$.

Proof. Idea: Find a ring with no maximal ideal. It is therefore the zero ring. Play with this fact. Say $I = (f_1, \ldots, f_r)$, $f_i \in \mathbb{C}[x]$ $(x = x_1, \ldots, x_n)$. Let's inspect the locus of zeros in $\mathbb{A}^{n+1}_{x,y}$, $V = V(f_1, \ldots, f_r; yg - 1)$.

If $(x^0, y^0) \in V$, then $x^0 \in V(I) = V(f_1, \dots, f_r) \subset \mathbb{A}_x^n$. Therefore $g(x^0) = 0$ (by hypothesis). Then there is no y^0 such that $y^0g(x_0) = 1$.

Therefore, $V = \emptyset$.

We also have that $V = \operatorname{Spec} \mathbb{C}[x,y]/(f_1,\ldots,f_r,yg-1)$. Then $\mathbb{C}[x,y]/(f,yg-1) = \{0\}$. Therefore, (g,yg-1) is the unit ideal in $\mathbb{C}[x,y]$. This means that we can write 1 as a polynomial combination of f and yg-1. Say

$$1 = p_1(x, y)f_1(x) + \dots + f_r(x, y)f_r(x) + q(x, y)(yg - 1).$$

Now work in the ring $B = \mathbb{C}[x][g^{-1}] = \mathbb{C}[x,y]/(yg-1)$. In Byg-1=0 and $y=g^{-1}$. Then

$$1 = p_1(x, g^{-1})f_1(x) + \dots + p_r(x, g^{-1})f_r(x) + 0.$$

Multiply by g^N to clear denominators. Then, since g = g(x),

$$g^N = \tilde{p}_1(x)f_1(x) + \dots + \tilde{p}_r(x)f_r(x).$$

Therefore, $g^N \in I$.

NOTE: If $I \subset J$ are ideals in $\mathbb{C}[x]$, then $V(I) \supseteq V(J)$. But $V(x_1) = V(x_1^2)$. Let I be an ideal. Then rad I = radical of $I = \{g \mid g^n \in I, \text{ some } n > 0\}$.

Theorem.

$$V(I) \supset V(J) \iff I \subset \operatorname{rad} J$$

 $V(I) = V(J) \iff \operatorname{rad} I = \operatorname{rad} J$

Proof. Say $V(I) \supset V(J)$. Take $g \in I$. Then g = 0 on V(J). Then $g^N \in J$ for some N by the strong Nullstellensatz, and so $g \in \operatorname{rad} J$.

The other direction is left as an exercise.

Definition. Let X be a topological space. Then a closed subset C is *irreducible* if you can't write $C = C_1 \cup C_2$ where C_i closed, $C_i < C$.

A a finite type algebra is noetherian, satisfies the ascending chain condition on ideals. Then Spec A has the descending chain condition on ideals.

Prime ideals: Given a polynomial ring R: (equivalent conditions)

- R/P is a domain
- P < R, $ab \in P \implies a \in P$ or $b \in P$
- A, B ideals of R, $AB \subset P \implies A \subset P$ or $B \subset P$. (Recall that the product ideal $AB = \{\text{finite sums } \sum a_ib_i \mid a_i \in A, b_i \in B\}$.)

Proof. (2) \Longrightarrow (3) Say $AB \subset P$, but $A \subset P$. $\exists a \in A, a \notin P$. $AB \subset P \Longrightarrow B \subset P$ $\forall b \in B, ab \in P, \therefore b \in P$, so $B \subset P$.