26 Wednesday, April 13, 2011

Structure Sheaf on Affine Variety

$X = \operatorname{Spec} A$

Define on basis $\mathcal{B}$ for topology $\mathcal{B} = \{\text{opens } X_x, s \in A\} = \{X - V(s)\}$

Put $\mathcal{O}_X(X_s) = A_s$.

Similarly, if M is an A-module, get sheaf $\mathcal{M}$, $\mathcal{M}(X_s)$ =localized module M_s.

$A_s = A[s^{-1}] = \{s^{-r}a\}$, $M_s = \{s^{-r}m \,|\, r \text{ integer}, m \in M\}$

$s^{-r}a = s^{-r'}a'$ if $s^{r'}a = s^r a'$

Eqivalence relation: $m/s^r = m'/s^{r'}$ if $s^n s^{r'} m = s^n s^r m'$ for some n

M_s is an A_s-module

Sheaf axiom on $\mathcal{B}$

$X_{s_1} \cup \cdots \cup X_{s_k}$ means $s_1 \cdots s_k$ generate a unit ideal. $\sum r_i s_i = 1$, $r_i \in A$.

Note s_1^n, $\ldots$, s_k^n also generate the unit ideal. So we can also write $\sum r_i s_i^n = 1$ (difference coefficients r_i. We can replace s_i by s_i^n.

Then the sheaf axiom (for X) says that

$$0 \to \mathcal{M}(X) \to \prod_i \mathcal{M}(X_{s_i}) \xrightarrow{\text{diff}} \prod_{i,j} \mathcal{M}(X_{(s_i s_j)}$$

is exact.

$$0 \to M \to \prod M_{s_i} \xrightarrow{\text{diff}} \prod_{i,j} M_{s_i s_j}$$

is exact.

We also need to check this for when we replace X by $U \in \mathcal{B}$ ($U = X_t$)

Suppose $m \in M$, $m \rightsquigarrow 0$ in M_{s_i} for all i. This means that $s_i^n m = s_i^n 0 = 0$ for $n \gg 0$.

Checking the exactness of the first $\to$: $m = 1 \cdot m = \sum r_i s_i^n m = 0$.

Given $\alpha_i \in M_{s_i}$ and $\alpha_i = \alpha_j m M_{s_i s_j}$, we want to find $w \in M$ with $w = \alpha_i$ in M_{s_i} for all i.

$$\alpha_i = s^{-n} m_i, \quad \alpha_j = s^{-n} m_j, \quad m_i \in M$$

$\alpha_i = \alpha_j$ in $M_{s_i s_j}$ means $(s_i s_j)^N s_j^n m_i = (s_i s_j)^N s_i^n m_j$. Let's absorb s_i^N into m_i, so that our new equation looks like

$$\boxed{s_j^\ell m_i = s_i^\ell m_j} \qquad (\ell = N + n)$$

Write $\sum r_i s_i^\ell = 1$.

$$\begin{aligned} m_j &= \sum r_i s_i^\ell m_j \\ &= \sum r_i s_j^\ell m_i \\ &= s_j^\ell w & \qquad w = \sum r_i m_i \\ \boxed{m_j = s_j^\ell w} & \quad \forall j \end{aligned}$$

bring s_js to the other side

$$s_j^{-\ell} m_j = w \in M$$

But we need n, not $\ell \ldots$

Structure sheaf on $\mathbb{P}^n$

Coordinates $(x_0, \ldots, x_n)$. Say we have covered $\mathbb{P}^n$ by the standard affine $U_i = \{x_i \neq 0\}$. We have a structure sheaf on $U_i \approx \mathbb{A}^n$.

Open subsets of U_i form a basis for the topology on $\mathbb{P}^n$. So we get a structure sheaf $\mathcal{O}_{\mathbb{P}}$ by descrbing it on each U_i.

We must check that if we restrict the structure sheaf on U_i and U_j to $U_i \cap U_j$, we get the same answer in both cases.

What's the benefit of describing the structure sheaf this way? If you give a variety by the topological space X and a sheaf $\mathcal{O}_X$ of algebras, then you "know" (in principle) $\mathcal{O}_X(U)$ for every U. The benefit is that we can define "morphism" easily.

Definition. A *regular function* on an open set U is an element of $\mathcal{O}_X(U)$. (Given a regular function f, and a point p, we can evaluate f at p to get a function.)

Definition. Given two varieties, we can define a *morphism*

$$(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$$

given by

- a continuous map $Y \xrightarrow{f} X$
- which carries regular functions to regular functions

$$(\text{Functions on } X) \xrightarrow{\circ f} (\text{Functions on } Y)$$

$$g \rightsquigarrow g \circ f$$

If U is an open set in X, then $V = f^{-1}U$ in Y.

$$(\text{Functions on } U) \xrightarrow{\circ f} (\text{Functions on } V)$$

$F_X =$ sheaf of functions on X

$$F_X(U) \xrightarrow{\circ f} F_Y(V)$$

$$F_X \xrightarrow{f^* = \circ f} F_Y$$

$$\begin{tikzcd} F_X(U) \arrow[r, "\circ f"] & F_Y(V) \\ \mathcal{O}_X(U) \arrow[r, "\circ f"] \arrow[u, hook] & \mathcal{O}_Y(V) \arrow[u, hook] \end{tikzcd}$$

$$V = f^{-1}(U)$$