

### 3 Wednesday, February 9, 2011

#### 3.1 Curves in $\mathbb{P}^2$

Curves in  $\mathbb{P}^2$  are defined by homogeneous irreducible polynomials  $f$ :  $C = V(f)$ .

e.g., the line containing a pair of points  $(p, q) \in \mathbb{P}^2$  is the set of points  $up + vq$  for  $(u, v) \neq (0, 0)$ . It's equation  $Ax = 0$  is obtained by solving  $Ap = 0$ ,  $Aq = 0$ . (Think of  $A = (a, b, c)$ ,  $p = (x_1, y_1, z_1)$ ,  $q = (x_2, y_2, z_2)$ . Then  $ax_1 + by_1 + cz_1 = 0$ ,  $ax_2 + by_2 + cz_2 = 0$ . Then  $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . i.e.,

$[a, b, c]$  is the kernel of the  $2 \times 3$  matrix, which has rank 2.)

The restriction of a homogeneous polynomial  $f(x, y, z)$  to a line  $\ell = \{up + vq\}$  is obtained by substitution  $f(up + vq)$ . This is a homogeneous polynomial in  $u, v$  of degree  $= \deg(f)$ . Over  $\mathbb{C}$ , any such polynomial can be factored into linear factors  $(up_i + vq_i)$ . These are the points of  $\ell$  (not necessarily distinct) that lie on  $V(f)$ . Thus, a plane polynomial curve of degree  $d$  meets a line in  $d$  points, counted with multiplicity.

Let  $f$  be a homogeneous polynomial of degree  $d$  in  $x_1, x_2, x_3$ , and let  $C = V(f)$ . Let  $f_i$  denote  $\frac{\partial f}{\partial x_i}$  and let  $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ . Then the Hessian Matrix is the  $3 \times 3$  symmetric matrix

$$H(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij}.$$

**Proposition** (Euler's Formula). *Let  $f$ ,  $d = \deg(f)$ ,  $f_i$  be as above. Then  $\sum_{i=1}^3 f_i x_i = f \deg(f)$ .*

*Proof.* Check it for monomials, since it's additive. (CHECK NOT INCLUDED) □

This works for polynomials in  $n$  variables.

Now consider the Taylor expansion of the restriction of  $f$  to  $\ell = \{up + vq\}$ . Setting  $u = 1$  ( $v = 0$  is  $p$ , so looking near the point  $p$ ):

$$f(p + vq) = f(p) + \left( \sum f_i(p) q_i \right) v + \frac{1}{2} \left( \sum f_{ij} q_i q_j \right) v^2 + \mathcal{O}(v^3)$$

with  $q = (q_1, q_2, \dots)$ .

**Proposition.** 1. *If  $p$  is a point of  $C$ , then  $f(p) = 0$ .*

2. *Suppose  $p \in C$ , and  $f_i(p)$  are not all 0. Then the equation of the tangent line  $T$  to  $C$  at  $p$  is  $\sum f_i(p) q_i = 0$ .*

3. *Let  $h$  be the Hessian of  $f$  at  $p$ . Then  $\det h = 0$  iff  $p$  is a flex point  $C$  (i.e., a restriction of  $f$  to the tangent line at  $p$  has a zero of order  $\geq 3$  at  $p$ ).*

*Proof.* 1. By definition.

2. Tangent line: if the restriction of  $f$  to  $T$  has at least a second order 0 (by definition). So looking at the coefficient of  $v$ , this is clear.

3. Exercise: Check that the restriction of the quadratic term to the tangent line is 0 iff  $\det h = 0$ . □

**Definition 1.** If all the  $f_i$  vanish at  $p$ , then  $p$  is called a *singular* point of  $C = V(f)$ . Otherwise, say that  $C$  is *non-singular* at  $p$ . Say that  $C$  is a *non-singular curve* if it has no singular points.

### 3.1.1 Nonsingular curves

e.g. 1, an irreducible conic is always non-singular

*Proof.* Convert to  $x^2 - yz = 0$ .  $f_x = 2x$ ,  $f_y = -z$ ,  $f_z = -y$ . Since  $(x, y, z) \in \mathbb{P}^2$ , not all these can be zero, so it's nonsingular  $\square$

e.g. 2, An irreducible plane cubic can have at most one singular point (exercise)

e.g. 3, The curve  $x^d + y^d + z^d = 0$  is non-singular (smooth) for  $d \geq 1$ . (Fermat polynomial of degree  $d$ ).

The partial derivatives are  $dx^{d-1}$ ,  $dy^{d-1}$ ,  $dz^{d-1}$ , not all zero (in  $\mathbb{P}^2$ )

e.g. 4, The curve  $x^3 + y^2 - xyz = 0$  is singular at the point  $(0 : 0 : 1)$ .

**Proposition.** For most values of the coefficients of a polynomial of degree  $d$ , the curve  $C = V(f) \subseteq \mathbb{P}^2$  is smooth.

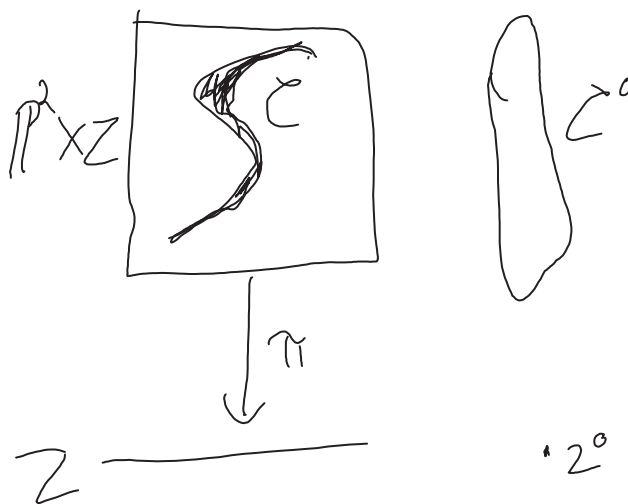
*Proof.* See two proofs, each of which depends on some theorem which will be proved later.

Setup: Order the monomials of degree  $d$  in  $x, y, z$  arbitrarily  $m_1, m_2, \dots, m_N$ . (Note:  $N = \binom{d+k-1}{k-1}$  for  $k$  variables.)

An arbitrary polynomial of degree  $d$  is a linear combination of the monomials  $m_\nu$  with some coefficients  $z_\nu$ . Think of  $z_\nu$  as variables and let

$$F = \sum_{\nu=1}^N z_\nu m_\nu \in \mathbb{C}[x, y, z, \{z_\nu\}].$$

Then  $F = 0$  defines a subvariety  $\mathcal{C}$  of the product  $\mathbb{P}^2 \times Z$ , where  $Z$  is  $\mathbb{A}^N$  with coordinates  $z_\nu$ .



The fiber  $\mathcal{C}^0 = \pi^{-1}(z^0)$  of  $\mathcal{C}$  over a point  $z^0 \in Z$  is the curve whose equation is the polynomial obtained.

by substituting  $z_\nu^0$  for  $z_\nu$ . The 3 partial derivatives  $F_x, F_y, F_z$  are polynomials in  $x, y, z, \{z_\nu\}$  linear in  $z_\nu$  and homogeneous of degree  $d-1$  in  $x, y, z$ . They define some subvariety of  $\mathbb{P}^2 \times Z$ . Let  $S$  be the variety  $\{F_1 = F_2 = F_3 = 0\}$ . Note that  $S \subset \mathcal{C}$  (by Euler).

The fiber  $\mathcal{C}^0$  over a point  $z^0$  of  $Z$  is smooth if and only if  $\mathcal{C}^0$  doesn't meet  $S$ .

We can construct  $\Sigma = \pi(S)$  the image of  $S$  via a polynomial  $\mathbb{P}^2 \times Z \rightarrow Z$ . Later we'll prove that the image of the projection of any Zariski closed subvariety of  $\mathbb{P}^2 \times Z$  to  $Z$  is also Zariski closed.

So the set  $\Sigma$  is closed in the affine space  $Z$ . But  $\Sigma$  is not all of  $Z$  (because the Fermat curve is smooth). So  $\Sigma \subset Z$  is a proper closed subvariety. So the set of  $z^0$  for which  $\mathcal{C}^0$  is smooth is a Zariski open subset of  $\mathbb{A}^N$ .  $\square$