12 Wednesday, March 2, 2011

S a multiplicative system

$1 \in S$

$0 \in S$

$S_1, S_2 \in S \implies S_1 S_2 \in S$.

Ring of fractions A_S localized ring

$A \hookrightarrow A_S$

$(J^c)^e = J$

$(A \cap J) A_S$

Localizing prime ideal (s...?)
I ideal of A, $I \cap S \neq \emptyset \implies I^e =$ unit ideal of A_S

Proposition. *P prime ideal of A. $P \cap S \neq \emptyset$. Then*

- $(P^e)^c = P$
- *P^e ($= P_S$) is a prime ideal of A_S*

$P^e = PA_S = \left\{ s^{-1}x \,\middle|\, x \in P \right\}$

Proof. For any ideal P, $(P^e)^c \supset P$.

We want to show $\subset$. Let $z \in (P^e)^c$. Then $z = s^{-1}x$ for some $x \in P$, and $z \in A$. Then $sz = ss^{-1}x = x \in P$. Since P is prime, and $s \notin P$, $z \in P$, and so $(P^e)^c \subset P$.

Now we show that P^e is prime:
We have that $z_1 z_2 \in P^e$ for $z_i \in A_S$. Then $z_1 = s_1^{-1}a_1$, $z_2 = s_2^{-1}a_2$. Then $z_1 z_2 = (s_1 s_2)^{-1}(a_1 a_2) \in P^e$. Therefore $(s_1 s_2)(z_1 z_2) = a_1 a_2 \in P^e$. Since $a_1 a_2 \in A$, this is also in $(P^e)^c = P$. Since $a_1 a_2 \in P$ and P prime, either $a_1 \in P$ or $a_2 \in P$, $s_1^{-1}a_1 \in P^e$ or $s_2^{-1}a_2 \in P^e$.[1] $\square$

$$P \operatorname{Spec} A_S \longleftrightarrow \text{subset of } P \operatorname{Spec} A = \{P \mid P \cap S \neq \emptyset\}$$

Back to the case where P is a prime ideal of A and $S = A - P = \{s \in A \mid s \notin P\}$.
Write A_P for A_S. If I is an ideal of A, $I_P = I_S$ extended ideal.

Proposition. *P_P is a maximal ideal of A_P and it is the only maximal ideal of A_P.*

Lemma. *For a ring R, the following are equivalent:*

(1) R has a unique maximal ideal $\mathfrak{M}$

(2) The elements of R that are not invertible form an ideal

Proof. (2) $\implies$ (1) Suppose that the non-units form an ideal I. Then R/I is a field because every element is the residue of a unit, and therefore invertible. Thus I is a maximal ideal. Since any other element is a unit, we cannot include any other element without turning the ideal into the entire ring. Thus, this is maximal.

(1) $\implies$ (2) Suppose there exists a unique maximal ideal $\mathfrak{M}$. Let $u \in R$. Then $(u) = R$ if and only if u is a unit. If u is not a unit, then $(u) < R$, and so $(u) \subset$ some maximal ideal.[2] Then $(u) \subset \mathfrak{M}$.

[1] Sorry if this proof is unclear. I was trailing behind Prof. Artin, and so wasn't understanding the proof well.
[2] If R is not noetherian, this requires Zorn's Lemma/The Axiom of Choice.

Then $\mathfrak{M}$ contains all the non-invertible elements, and so the non-invertible elements of R form an ideal (in particular $\mathfrak{M}$).

$\square$

Proposition above. $s^{-1}a \in A_P$, $s \notin P$.
If $a \in P$, then $s^{-1}a \in P_P$. If $a \notin P$, then $s^{-1}a$ is invertible, and so $a^{-1}s \in A_S$. $\square$

Definition. A (noetherian) ring R is *local* if it has a unique maximal ideal $\mathfrak{M}$. (Note that $R/\mathfrak{M}$ is a field.)

Example. $A = \mathbb{C}[x, y]$. *The prime ideals are*

- (0)
- $(f(x, y))$ *for* f *irreducible*
- *maximal ideal* $\mathfrak{M}_{(a,b)} = (x - a, y - b) \longleftrightarrow (a, b) \in \mathbb{C}^2$

  $A_{(0)}$*: fraction field* $\mathbb{C}(x, y)$ *of* $\mathbb{C}[x, y]$

  $A_{\mathfrak{M}_{(a,b)}}$*: a local ring. The prime ideals* $\operatorname{PSpec} A_{\mathfrak{M}} = \{P \mid P \cap S \neq \emptyset\} = \{P \mid P \subset \mathfrak{M}\} = \begin{cases} (0) \\ P = (f) \mid f(a, b) = 0 \\ \mathfrak{M}_{(a,b)} \end{cases}$

Lemma. *Suppose* I *is an ideal of the ring* A *and* M *is a finite* A*-module such that* $M = IM$*. Then there exists a* $z \in I$ *such that* $(1 - z)M = 0$.

Proof. Say $(x_1, \ldots, x_r$ generate M. We can write x_i as a combination of $\{x_1, \ldots, x_r\}$ with coefficients in I:

$$\begin{aligned} x_i &= \sum_j p_{ij} x_j && p_{ij} \in I \\ X &= PX && P \text{ matrix } (p_{ij}) \\ (\Bbbk - P)X &= 0 \\ Q(\Bbbk - P) &= \delta \Bbbk \end{aligned}$$

where Q is the cofactor matrix for $\Bbbk - P$ with entries in A, and $\delta = \det(\Bbbk - P)$.

$$\begin{aligned} Q(\Bbbk - P)X &= 0 \\ \therefore \delta X &= 0 \\ \Bbbk - P &= \begin{pmatrix} 1 - p_{11} & \cdots & \\ & \ddots & \\ & & 1 - p_{nn} \end{pmatrix} \\ \delta &= 1 - z \end{aligned}$$

Since the $p_{ij} \in I$, we have $z \in I$. Then $(1 - z)X = 0$, so $(1 - z)$ kills M. $\square$

Lemma (Nakayama Lemma)**.** *Let* A *be a local ring with a maximal ideal* $\mathfrak{M}$*, and let* M *be a finite* A*-module. If* $M = \mathfrak{M}M$*, then* $M = 0$.

Proof. Take $z \in \mathfrak{M}$. We have a z with $(1 - z)M = 0$. Since $1 - z \notin \mathfrak{M}$, $1 - z$ is invertible, and so $M = 0$ (since we can multiply by $(1 - z)^{-1}$). $\square$