3 Wednesday, February 9, 2011

3.1 Curves in $\mathbb{P}^2$

Curves in $\mathbb{P}^2$ are defined by homogeneous irreducible polynomials f: $C = V(f)$.

e.g., the line containing a pair of points $(p, q) \in \mathbb{P}^2$ is the set of points $up + vq$ for $(u, v) \neq (0, 0)$. It's equation $Ax = 0$ is obtained by solving $Ap =$, $Aq = 0$. (Think of $A = (a, b, c)$, $p = (x_1, y_1, z_1)$, $q = (x_2, y_2, z_2)$. Then $ax_1 + by_1 + cz_1 = 0$, $ax_2 + by_2 + cz_2 = 0$. Then $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. i.e., $[a, b, c]$ is the kernel of the 2×3 matrix, which has rank 2.)

The restriction of a homogeneous polynomial $f(x, y, z)$ to a line $\ell = \{up + vq\}$ is obtained by substitution $f(up + vq)$. This is a homogeneous polynomial in u, v of degree $= \deg(f)$. Over $\mathbb{C}$, any such polynomial can be factored into linear factors $(up_i + vq_i)$. These are the points of ℓ (not necessarily distinct) that lie on $V(f)$. Thus, a plane polynomial curve of degree d meets a line in d points, counted with multiplicity.

Let f be a homogeneous polynomial of degree d in x_1, x_2, x_2, and let $C = V(f)$. Let f_i denote $\frac{\partial f}{\partial x_i}$ and let $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Then the Hessian Matrix is the 3×3 symmetric matrix

$$H(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{ij}.$$

Proposition (Euler's Formula)**.** *Let f, $d = \deg(f)$, f_i be as above. Then $\sum_{i=1}^3 f_i x_i = f \deg(f)$.*

Proof. Check it for monomials, since it's additive. (CHECK NOT INCLUDED) □

This works for polynomials in n variables.

Now consider the Taylor expansion of the restriction of f to $\ell = \{up + vq\}$.
Setting $u = 1$ ($v = 0$ is p, so looking near the point p):

$$f(p + vq) = f(p) + \left(\sum f_i(p) q_i \right) v + \frac{1}{2} \left(\sum f_{ij} q_i q_j \right) v^2 + \mathcal{O}(v^3)$$

with $q = (q_1, q_2, \ldots)$.

Proposition.

1. *If p is a point of C, then $f(p) = 0$.*
2. *Suppose $p \in C$, and $f_i(p)$ are not all 0. Then the equation of the tangent line T to C a t p is $\sum f_i(p) q_i = 0$.*
3. *Let h be the Hessian of f at p. Then $\det h = 0$ iff p is a* flex *point C (i.e., a restriction of f to the tangent line at p has a zero of order ≥ 3 at p*

Proof.

1. By definition.
2. Tangent line: if the restriction of f to T has at least a second order 0 (by definition). So looking at the coefficient of v, this is clear.
3. Exercise: Check that the restriction of the quadratic term to the tangent line is 0 iff $\det h = 0$.

□

Definition 1. If all the f_i vanish at p, then p is called a *singular* point of $C = V(f)$. Otherwise, say that C is *non-singluar* at p. Say that C is a *non-singular curve* if it has no singular points.

3.1.1 Nonsingular curves

e.g. 1, an irreducible conic is always non-singular

Proof. Convert to $x^2 - yz = 0$. $f_x = 2x$, $f_y = -z$, $f_z = -y$. Since $(x, y, z) \in \mathbb{P}^2$, not all these can be zero, so it's nonsinular $\square$

e.g. 2, An irreducible plane cubic can have at most one singular point (exercise)

e.g. 3, The curve $x^d + y^d + z^d = 0$ is non-singular (smooth) for $d \geq 1$. (Fermat polynomial of degree d).

The partial derivatives are dx^{d-1}, dy^{d-1}, dz^{d-1}, not all zero (in $\mathbb{P}^2$)

e.g. 4, The curve $x^3 + y^2 - xyz = 0$ is singular at the point $(0 : 0 : 1)$.

Proposition. *For most values of the coefficients of a polynomial of degree d, the curve $C = V(f) \subseteq \mathbb{P}^2$ is smooth.*

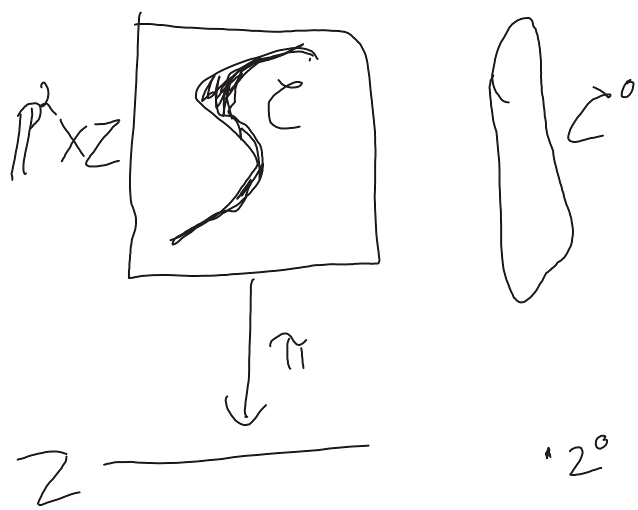
Proof. See two proofs, each of which depends on some theorem which will be proved later.

Setup: Order the monomials of degree d in x, y, z arbitrarily m_1, m_2, $\ldots$, m_N. (Note: $N = \binom{d+k-1}{k-1}$ for k variables.)

An arbitrary polynomial of degree d is a linear combination of the monomials m_ν with some coefficients z_ν. Think of z_ν as variables and let

$$F = \sum_{\nu=1}^{N} z_\nu m_\nu \in \mathbb{C}[x, y, z, \{z_\nu\}].$$

Then $F = 0$ defines a subvariety $\mathcal{C}$ of the product $\mathbb{P}^2 \times Z$, where Z is $\mathbb{A}^N$ with coordinates z_ν.



The *fiber* $\mathcal{C}^0 = \pi^{-1}(z^0)$ of $\mathcal{C}$ over a point $z^0 \in Z$ is the curve whose equation is the polynomial obtained.

by substituting z^0_ν for z_ν. The 3 partial derivatives F_x, F_y, F_z are polynomials in x, y, z, $\{z_\nu\}$ linear in z_ν and homogeneous of degree $d - 1$ in x, y, z. They define some subvariety of $\mathbb{P}^2 \times Z$. Let S be the variety $\{F_1 = F_2 = F_3 = 0\}$. Note that $S \subset \mathcal{C}$ (by Euler).

The fiber $\mathcal{C}^0$ over a point z^0 of Z is smooth if and only if $\mathcal{C}^0$ doesn't meet S.

We can construct $\Sigma = \pi(S)$ the image of S via a polynomial $\mathbb{P}^2 \times Z \to Z$. Later we'll prove that the image of the projection of any Zariski closed subvariety of $\mathbb{P}^2 \times Z$ to Z is also Zariski closed.

So the set Σ is closed in the affine space Z. But Σ is not all of Z (because the Fermat curve is smooth). So $\Sigma \subset Z$ is a proper closed subvariety. So the set of z^0 for which $\mathcal{C}^0$ is smooth is a Zariski open subset of $\mathbb{A}^N$. $\square$