## 22 Monday, April 4, 2011

## Review

 $G(2,4) = \text{planes in } V^4, \text{ lines in } \mathbb{P}^3$ 

V has basis  $u_1, u_2, u_3, u_4$ 

 $\bigwedge^2 V$  has basis  $u_1 u_2, \dots, u_3 u_4$  ( $u_i u_j$  for i < j), dimension 6

W a 2-dimensional subspace of V with basis (u, v), then  $w = uv \in \bigwedge^2 V$  determines W, and conversely W determines uv.  $(u', v') = (u, v)P \implies u'v' = uv \det P$ .

**Proposition.** 2-dimensional subspaces of V correspond to decomposable non-zero elements w = uv of  $\bigwedge^2 V$  modulo scalars.

 $w \in \bigwedge^2 V$  is decomposable  $\iff ww = 0 \iff w = \sum_{i < j} a_{ij} u_i u_j$  and  $a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0$ .

The grassmanian G(2,4) is the locus of  $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$  in  $\mathbb{P}^5_{(a_{ij})}$  (dimension = 4).

We can show directly that dim G(2,4)=4. W is a subspace of dimension 2 with basis (u,v).  $(u=\sum a_iu_i,\ v=\sum b_iu_i)$ 

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \xrightarrow{\text{choose a basis}} \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

Ask for the locus of pairs  $(p, \ell)$  with  $p \in \mathbb{P}^3$  a point and  $\ell$  a line in  $\mathbb{P}^3$ . Let  $\Gamma = \text{locus } \{(p, \ell) \mid p \in \ell\}$ . We can embed this in  $\mathbb{P}^3 \times \mathbb{G}$  where  $\mathbb{G} = G(2, 4)$ .

$$\Gamma \subset \mathbb{P}^3 \times G \subset \mathbb{P}^3 \times \mathbb{P}^5 \stackrel{\text{Segre}}{\Rightarrow} \mathbb{P}^{23}$$

We can show that  $\Gamma$  is a closed set in  $\mathbb{P}^3 \times \mathbb{G} \subset \mathbb{P}^3 \times \mathbb{P}^5$  (with coordinates  $(x) \times (a_{ij})$ ): Find defining homogeneous equations. We need f(x,a) homogeneous in  $x_i$  and in  $a_{ij}$ . If  $\ell$  is the line through (u,v) (basis for  $W \longleftrightarrow \ell$ ),  $(x) \in \ell$  means x = su + tv for some s, t. This is the case if and only if x, u, v dependent, which is true if and only if xuv = 0 in  $\bigwedge^3 V$ .

$$uv = \sum_{i < j} a_{ij} u_i u_j \longleftrightarrow (a_{ij}) \in \mathbb{P}^5$$

Then  $(x) = \sum x_i u_i$  and so  $xuv = \sum x_i a_{ijk} u_i u_j u_k$ . Expand this and plug in the relation  $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$ .

## Surfaces of degree d in $\mathbb{P}^3$

Given as f = 0, f homogeneous of degree d in  $(x_1, x_2, x_3, x_4)$ . The number of monomials of degree d is  $\binom{d+3}{d}$ .

$$\{\text{surfaces of degree }d\} \longleftrightarrow \mathbb{S} \coloneqq \text{points in }\mathbb{P}^N$$

with  $N = {d+3 \choose d} - 1$ .

We ask in  $\mathbb{G} \times \mathbb{S}$  for  $\Theta := \{(\ell, S) | \ell \subset S\}$ .

The following isn't quite right. (Something is wrong with the degree of f?)

We can show that  $\Theta$  is closed in  $\mathbb{G} \times \mathbb{S}$ . What are its defining equations? Say  $\ell$  is a line through u, v, and say that S is the surface given by f = 0. Then  $\ell \subset S \iff f(su + tv) = 0$  for all s, t. Expand f(su + tv):

$$f(su + tv) = s^{d} f_{0}(u) + s^{d-1} t f_{1}(u, v) + \dots + st^{d-1} f_{d-1}(u, v) + t^{d} f_{d}(v)$$

Then f(su + tv) = 0 for all  $s, t \iff f_i(u, v) = 0$  for all i.

$$\left( \begin{array}{c} \textbf{Example.} \\ f = x_i x_j x_k \\ f(su+tv) = s^3 u_i u_j u_k + s^2 t (u_i u_j v_k + u_i v_j u_k + v_i u_j u_k) + st^2 (u_i v_j v_k + v_i u_j v_k + v_i v_j u_k) + t^3 (v_i v_j v_k) \end{array} \right)$$

**Lemma.**  $f_i$  depends only on f, not on u and v.

That is, if we change basis to (u', v') = (u, v)P, the  $f_i$  don't change.

Check: We check this for P elementary; say  $u = u' + \varepsilon v'$ , v = v'. Then f(su + tv) = f(su' + t) $s\varepsilon v' + tv' = f(su' + (s\varepsilon + t)v')$ . We can say  $t' = s\varepsilon + t$ . Then the expansion doesn't change, except for the transformation of  $t \mapsto t'$  (which doesn't change the  $f_i$ ).

The above isn't quite right. (Something is wrong with the degree of f?)

What's the dimension of  $\Theta$ ?

*Plan*: We can carry any line in  $\mathbb{P}^3$  to any other line by a change of coordinates.

Take a particular line  $\ell_0$ . Suppose we determine dim  $\{S | \ell_0 \subset S\} = r$ . Then dim  $\Theta = \dim \mathbb{G} + r$ r = r + 4.

Suppose we take  $\ell_0$  to be the line  $\{(x_1, x_2, 0, 0)\}$ . Is  $f(x_1, x_2, 0, 0)$  identically zero? When we substitute in, we get a homogeneous polynomial of degree d in  $x_1, x_2$ :

$$f(x_1, x_2, 0, 0) = \alpha_0 x_1^d + \alpha_1 x_1^{d-1} x_2 + \dots + \alpha_d x_2^d$$

The d+1 coefficients of f must then be zero, so  $\dim\{S \mid \ell_0 \subset S\} = \dim \mathbb{S} - (d+1)$ . So  $r = \binom{d+3}{d} - 1 - (d+1)$ . Then  $\dim \Theta = \binom{d+3}{d} - d + 2$ . For d=1,  $\dim \Theta = 5$ .  $\dim \mathbb{S} = 3$ . We expect the set of lines in a given S to have dimension 2,

so this is correct.

For d=2, dim  $\Theta=10$ . dim S=9 (10 coefficients, but it's projective space). We expect a particular S to have a 1-dimensional family of lines.

For d=3, dim  $\Theta=19$ . We expect a finite number of liens in a cubic. The number of lines contained in a generic cubic surface is 27.