

## 15 Wednesday, March 9, 2011

**Theorem.** Let  $A$  be a finite type algebra and a domain, and let  $K$  be the field of fractions. Then the integral closure of  $A$  is a finite extension  $L/K$ , and is a finite  $A$ -module.

*Proof.* Had the trace pairing  $L \times L \rightarrow K$ ,  $x, y \rightsquigarrow \text{tr}(xy)$ . It's non-degenerate because:

$$yx \neq 0, \text{ put } y = x^{-1}$$

$$\langle x, y \rangle = t(1) = [L : K]$$

Reduce to the case of  $A$  integrally closed by Noether Normalization.

$$k[y] \subset A, A \text{ a finite } k[y]\text{-module}$$

$$k[y] \subset K \subset L.$$

Replace  $A$  by  $k[y]$ .

$\therefore$ , we may assume that  $A$  is integrally closed. Then if  $\alpha \in L$  is integral over  $A$ ,  $\text{tr}(\alpha) \in A$ .

Therefore, if  $B$  is any subring of  $L$ , and a finite  $A$ -module, then all elements are integral over  $A$ .

$A$ . Therefore,  $B \times B \rightarrow A$ ,  $x, y \rightsquigarrow \langle x, y \rangle = \text{tr}(xy)$ .

We want to show that there is a maximal such  $B$ . Then  $B$  is the integral closure in  $L$ .

Start with one,  $B_0$ , big enough so that it contains a basis for  $L/K$ . (We can do this because for any  $\gamma \in L$ ,  $\gamma$  is algebraic in  $K$ ;  $\gamma^n - a_1\gamma^{n-1} + \cdots \pm a_n = 0$  with  $a_i \in K$ . Since  $K$  is the field of fractions, we can clear denominators, getting  $d\gamma^n - a'_1\gamma^{n-1} + \cdots \pm a'_n = 0$ , with  $d, a'_i \in A$ . Then  $d\gamma$  is integral over  $A$ ; multiply everything by  $d^{n-1}$ .) Denote the basis by  $(v_1, \dots, v_n)$ ,  $v_i \in B_0$ ,  $n = [L : K]$ .

Investigate some larger algebra (which is still a finite  $A$ -module)  $B$ . Then  $A \subset B_0 \subset B$ .

$$B \times B \rightarrow A$$

$$\beta, v_i \rightsquigarrow b_i := \langle \beta, v_i \rangle \in A$$

map

$$\beta \rightsquigarrow (b_1, \dots, b_n) \in A^n$$

$$B \xrightarrow{\Phi} A^n$$

This is  $A$ -linear (homomorphism of  $A$ -modules)

$\Phi$  is injective:  $\Phi(\beta) = 0$  means  $\langle \beta, v_i \rangle = 0$  for all  $i$ . Since  $\{v_i\}$  is a basis for  $L/K$ ,  $\langle \beta, y \rangle = 0$  for all  $y \in L$ . Thus,  $\beta = 0$ .

Then we can identify  $B$  with  $\Phi(B)$  as a submodule of  $A^n$ . Since  $A$  is noetherian, submodules have the ascending chain condition.  $\square$

(DIGRESSION ABOUT GALOIS GROUPS AND  $G$ -ORBITS NOT INCLUDED)

Let  $A \subset B$  be a domain,  $A$  finite-type and integrally closed, and  $B$  a finite  $A$ -module.

What about prime ideals in  $A$  and  $B$ ?

extended ideal of  $P \subset A$  is  $P^e = PB$  ideal of  $B$

contracted ideal of  $Q \subset B$  is  $Q^c = A \cap Q$  ideal of  $A$

**Fact** (General Fact). If  $Q$  is a prime ideal of  $B$ , then  $Q^c$  is a prime ideal of  $A$ .

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \searrow \text{kernel } Q^c & & \swarrow \text{kernel } Q \\ & B/Q \text{ a domain } (Q \text{ prime}) & \end{array}$$

The image is a domain, so  $Q^c$  is a prime ideal.

Back to the case above,

$$\begin{array}{ccc} A \hookrightarrow B \\ P' & Q' & \text{we mean } P' = A \cap Q' \\ P & Q & P = A \cap Q, \quad Q' \supset Q \end{array}$$

**Fact 15.1** (Lying Over). Given  $P$  a prime ideal of  $A$ , there exists a  $A$  prime ideal of  $B$ , with  $A \cap Q = P$ . (The map  $\text{PSpec } B \rightarrow \text{PSpec } A$  is injective.)

**Fact 15.2** (Going Up). Given

$$\begin{array}{ccc} A \hookrightarrow B \\ P' & \textcircled{Q'} & \longleftarrow \text{exists} \\ P & Q & \end{array}$$

**Fact 15.3** (Going Down). If  $A$  is integrally closed, then given

$$\begin{array}{ccc} A \hookrightarrow B \\ P' & Q' \\ P & \textcircled{Q} & \longleftarrow \text{exists} \end{array}$$

**Lemma.** *Given*

$$\begin{array}{ccc} A \hookrightarrow B \\ P' & Q' \\ P & Q \end{array}$$

*If  $P' = P$ , then  $Q' = Q$ .*

*Proof.* Case 1:  $P = 0$ . We need to show that if  $P' = 0$ , then  $Q' = 0$ . Take  $\alpha \in Q' \subset B$ . Then  $\alpha$  is integral over  $A$  (because it's in  $B$ ). We have  $\alpha^r - a_1\alpha^{r-1} + \cdots \pm a_r = 0$ . Since  $a_r \in Q'$  and  $a_r \in A$ , we have  $a_r \in P' = \{0\}$ . Thus  $a_r = 0$ . If  $\alpha \neq 0$ , then cancel  $\alpha$  from the relation, and repeat until  $r = 1$ . Then  $\alpha^1 - a_1 = 0$ . Since  $a_1 \in A$  and  $a_1 \in Q'$ ,  $a_1 \in P'$ . Then  $a_1 = 0$  and  $\alpha = 0$ , which is a contradiction.

Case 2: (general case). Go to  $\bar{A} = A/P \subset \bar{B} = B/Q$ .

$$\begin{array}{ccc} \bar{A} \hookrightarrow \bar{B} \\ \bar{P}' & \bar{Q}' \\ (0) & (0) \end{array}$$

Then case 1 says that  $\bar{P}' = (0)$  implies that  $\bar{Q}' = (0)$ . Then  $\bar{P}' = (0) \iff P' = P$  and  $\bar{Q}' = (0) \iff Q' = Q$ .

□

**Lemma.** *If  $Q$  is a maximal ideal of  $B$ , then  $Q^c = P$  is maximal in  $A$ .*

*Proof.*  $Q$  is maximal in  $B$  if and only if  $\bar{B} = B/Q$  is a field. Then  $A/P = \bar{A} \subset \bar{B}$  is a field. Then  $\bar{B}$  is a finite  $\bar{A}$ -module.

**Lemma.** *If  $A \subset B$  is a field and  $B$  is a finite  $A$ -module, then  $A$  is a field.*

*Proof.* Take a non-zero element in  $\alpha \in A$  (we want to show that it's invertible). Then  $\alpha$  is invertible in  $B$ . Since  $B$  is a finite  $A$ -module, so  $u = \alpha^{-1}$  is integral over  $A$ . Then  $u^r - a_1 u^{r-1} + \cdots \pm a_r = 0$  with  $a_i \in A$ . Multiply by  $\alpha^{r-1}$ . Then we get  $u - a_1 + a_2 \alpha - \cdots \pm a_r \alpha^r = 0$ , with all of these elements of  $A$ . □

(To be finished next time?) □