15 Wednesday, March 9, 2011

Theorem. *Let A be a finite type algebra and a domain, and let K be the field of fractions. Then the integral closure of A is a finite extension L/K, and is a finite A-module.*

Proof. Had the trace pairing $L \times L \to K$, $x, y \rightsquigarrow tr(xy)$. It's non-degenerate because:

$yx \neq 0$, put $y = x^{-1}$

$\langle x, y\rangle = t(1) = [L : K]$

Reduce to the case of A integrally closed by Noether Normalization.

$k[y] \subset A$, A a finite $k[y]$-module

$k[y] \subset K \subset L$.

Replace A by $k[y]$.

$\therefore$, we may assume that A is integrally closed. Then if $\alpha \in L$ is integral over A, $tr(\alpha) \in A$.

Therefore, if B is any subring of L, and a finite A-module, then all elements are integral over A. Therefore, $B \times B \to A$, $x, y \rightsquigarrow \langle x, y\rangle = tr(xy)$.

We want to show that there is a maximal such B. Then B is the integral closure in L.

Start with one, B_0, big enough so that it contains a basis for L/K. (We can do this because for any $\gamma \in L$, γ is algebraic in K; $\gamma^n - a_1\gamma^{n-1} + \cdots \pm a_n = 0$ with $a_i \in K$. Since K is the field of fractions, we can clear denominators, getting $d\gamma^n - a'_1\gamma^{n-1} + \cdots \pm a'_n = 0$, with $d, a'_i \in A$. Then $d\gamma$ is integral over A; multiply everything by d^{n-1}.) Denote the basis by $(v_1, \ldots, v_n)$, $v_i \in B_0$, $n = [L : K]$.

Investigate some larger algebra (which is still a finite A-module) B. Then $A \subset B_0 \subset B$.

$$\begin{aligned} B \times B &\to A \\ \beta, v_i &\rightsquigarrow b_i := \langle \beta, v_i\rangle \in A \end{aligned}$$

map

$$\begin{aligned} \beta &\rightsquigarrow (b_1, \ldots, b_n) \in A^n \\ B &\xrightarrow{\Phi} A^n \end{aligned}$$

This is A-linear (homomorphism of A-modules)

Φ is injective: $\Phi(\beta) = 0$ means $\langle \beta, v_i\rangle = 0$ for all i. Since $\{v_i\}$ is a basis for L/K, $\langle \beta, y\rangle = 0$ for all $y \in L$. Thus, $\beta = 0$.

Then we can identify B with $\Phi(B)$ as a submodule of A^n. Since A is noetherian, submodules have the ascending chain condition. □

(DIGRESSION ABOUT GALOIS GROUPS AND G-ORIBTS NOT INCLUDED)

Let $A \subset B$ be a domain, A finite-type and integrally closed, and B a finite A-module.

What about prime ideals in A and B?

extended ideal of $P \subset A$ is $P^e = PB$ ideal of B

contracted ideal of $Q \subset B$ is $Q^c = A \cap Q$ ideal of A

Fact (General Fact)**.** If Q is a prime ideal of B, then Q^c is a prime ideal of A.

$$\begin{tikzcd} A \arrow[rr, hook] \arrow[dr, "\text{kernel } Q^c"'] & & B \arrow[dl, "\text{kernel } Q"] \\ & B/Q\text{a domain } (Q \text{ prime}) & \end{tikzcd}$$

The image is a domain, so Q^c is a prime ideal.

Back to the case above,

$$\begin{array}{lll} A \hookrightarrow B & & \\ P' \quad\quad Q' & & \text{we mean } P' = A \cap Q' \\ P \quad\quad\; Q & & P = A \cap Q, \quad Q' \supset Q \end{array}$$

Fact 15.1 (Lying Over)**.** Given P a prime ideal of A, there exists a A prime ideal of B, with $A \cap Q = P$. (The map $\operatorname{PSpec} B \to \operatorname{PSpec} A$ is injective.)

Fact 15.2 (Going Up)**.** Given

$$\begin{tikzcd} A \arrow[r, hook] & B & \\ P' & Q' & \text{exists} \arrow[l] \\ P & Q & \end{tikzcd}$$

Fact 15.3 (Going Down)**.** If A is integrally closed, then given

$$\begin{tikzcd} A \arrow[r, hook] & B & \\ P' & Q' & \\ P & Q & \text{exists} \arrow[l] \end{tikzcd}$$

Lemma. *Given*

$$\begin{tikzcd} A \arrow[r, hook] & B \\ P' & Q' \\ P & Q \end{tikzcd}$$

If $P' = P$, then $Q' = Q$.

Proof. Case 1: $P = 0$. We need to show that if $P' = 0$, then $Q' = 0$. Take $\alpha \in Q' \subset B$. Then α is integral over A (because it's in B?). We have $\alpha^r - a_1\alpha^{r-1} + \cdots \pm a_r = 0$. Since $a_r \in Q'$ and $a_r \in A$, we have $a_r \in P' = \{0\}$. Thus $a_r = 0$. If $\alpha \neq 0$, then cancel α from the relation, and repeat until $r = 1$. Then $\alpha^1 - a_1 = 0$. Since $a_1 \in A$ and $a_1 \in Q'$, $a_1 \in P'$. Then $a_1 = 0$ and $\alpha = 0$, which is a contradiction.

Case 2: (general case). Go to $\bar{A} = A/P \subset \bar{B} = B/Q$.

$$\begin{tikzcd} \bar{A} \arrow[r, hook] & \bar{B} \\ \bar{P}' & \bar{Q}' \\ (0) & (0) \end{tikzcd}$$

Then case 1 says that $\bar{P}' = (0)$ implies that $\bar{Q}' = (0)$. Then $\bar{P}' = (0) \iff P' = P$ and $\bar{Q}' = (0) \iff Q' = Q$.

□

Lemma. *If Q is a maximal ideal of B, then $Q^c = P$ is maximal in A.*

Proof. Q is maximal in B if and only if $\bar{B} = B/Q$ is a field. Then $A/P = \bar{A} \subset \bar{B}$ is a field. Then $\bar{B}$ is a finite $\bar{A}$-module.

Lemma. *If $A \subset B$ is a field and B is a finite A-module, then A is a field.*

Proof. Take a non-zero element in $\alpha \in A$ (we want to show that it's invertible). Then α is invertible in B. Since B is a finite A-module, so $u = \alpha^{-1}$ is integral over A. Then $u^r - a_1 u^{r-1} + \cdots \pm a_r = 0$ with $a_i \in A$. Multiply by α^{r-1}. Then we get $u - a_1 + a_2\alpha - \cdots \pm a_r\alpha^r = 0$, with all of these elements of A. □

(To be finished next time?) □