9 Wednesday, February 23, 2011

Hilbert Basis Theorem

A ring A is Noetherian if the ideals are finitely generated.

Theorem (Hilbert Basis Theorem)**.** *If R is Noetherian, then $R[x]$ is Noetherian*

Corollary. *$\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian*
Any finite-type (finitely generated as an algebra (everything is a polynomial in finitely many things)) $\mathbb{C}$-algebra is Noetherian. ($A \cong \mathbb{C}[x]/I$)

Equivalent conditions on A:

1. A is noetherian (ideals are finitely generated)
2. Every infinite increasing family $I_1 \subset I_2 \subset \cdots$ of ideals becomes constant eventually

   ($I_1 < I_2 < \cdots$ chain is finite)
3. Every non-empty set S of ideals contains maximal elements ($\exists I \in S$ such that $I \not\subset J$ for any $J \in S$, $J \neq I$)

Corollary. *If A is noetherian,I is an ideal of A, and $I < A$, then I is contained in a maximal ideal. (The maximal ideal is a maximum element in the set of ideals $< A$.)*

Corollary. *If A contains no maximal ideal, then A is the zero ring.*
$\operatorname{spec} A \neq \emptyset \iff A = \{0\}$

Adjoining inverses to $A = \mathbb{C}[x]$ $(x = x_1, \ldots, x_n)$, $B = A[g^{-1}] = \mathbb{C}[x, y]/(yg - 1)$.
Then $\operatorname{spec} A = \mathbb{A}^n$, and $\operatorname{spec} B \approx \mathbb{A}^n - V(g)$.

Theorem (Strong Nullstellensatz)**.** *Let I be an ideal of $\mathbb{C}[x]$, $g \in \mathbb{C}[x]$. Suppose g vanishes identically on $V(I)$. Then $g^N \in I$ for some $N \gg 0$.*

Proof. Idea: Find a ring with no maximal ideal. It is therefore the zero ring. Play with this fact.

Say $I = (f_1, \ldots, f_r)$, $f_i \in \mathbb{C}[x]$ $(x = x_1, \ldots, x_n)$. Let's inspect the locus of zeros in $\mathbb{A}^{n+1}_{x,y}$, $V = V(f_1, \ldots, f_r; yg - 1)$.

If $(x^0, y^0) \in V$, then $x^0 \in V(I) = V(f_1, \ldots, f_r) \subset \mathbb{A}^n_x$. Therefore $g(x^0) = 0$ (by hypothesis). Then there is no y^0 such that $y^0 g(x_0) = 1$.

Therefore, $V = \emptyset$.

We also have that $V = \operatorname{spec} \mathbb{C}[x, y]/(f_1, \ldots, f_r, yg - 1)$. Then $\mathbb{C}[x, y]/(f, yg - 1) = \{0\}$. Therefore, $(g, yg - 1)$ is the unit ideal in $\mathbb{C}[x, y]$. This means that we can write 1 as a polynomial combination of f and $yg - 1$. Say

$$1 = p_1(x, y)f_1(x) + \cdots + f_r(x, y)f_r(x) + q(x, y)(yg - 1).$$

Now work in the ring $B = \mathbb{C}[x][g^{-1}] = \mathbb{C}[x, y]/(yg - 1)$. In B $yg - 1 = 0$ and $y = g^{-1}$. Then

$$1 = p_1(x, g^{-1})f_1(x) + \cdots + p_r(x, g^{-1})f_r(x) + 0.$$

Multiply by g^N to clear denominators. Then, since $g = g(x)$,

$$g^N = \tilde{p}_1(x)f_1(x) + \cdots + \tilde{p}_r(x)f_r(x).$$

Therefore, $g^N \in I$. □

NOTE: If $I \subset J$ are ideals in $\mathbb{C}[x]$, then $V(I) \supseteq V(J)$. But $V(x_1) = V(x_1^2)$.
Let I be an ideal. Then $\operatorname{rad} I =$ radical of $I = \{g \mid g^n \in I, \text{ some } n > 0\}$.

Theorem.

$$V(I) \supset V(J) \iff I \subset \operatorname{rad} J$$

$$V(I) = V(J) \iff \operatorname{rad} I = \operatorname{rad} J$$

Proof. Say $V(I) \supset V(J)$. Take $g \in I$. Then $g = 0$ on $V(J)$. Then $g^N \in J$ for some N by the strong Nullstellensatz, and so $g \in \operatorname{rad} J$.

The other direction is left as an exercise. □

Definition 1. Let X be a topological space. Then a closed subset C is *irreducible* if you can't write $C = C_1 \cup C_2$ where C_i closed, $C_i < C$.

A a finite type algebra is noetherian, satisfies the ascending chain condition on ideals. Then $\operatorname{spec} A$ has the descending chain condition on ideals.
Prime ideals: Given a polynomial ring R: (equivalent conditions)

- R/P is a domain
- $P < R$, $ab \in P \implies a \in P$ or $b \in P$
- A, B ideals of R, $AB \subset P \implies A \subset P$ or $B \subset P$. (Recall that the product ideal $AB = \{\text{finite sums } \sum a_i b_i \mid a_i \in A, b_i \in B\}$.)

Proof. (2) $\implies$ (3)
Say $AB \subset P$, but $A \subset P$.
$\exists a \in A$, $a \notin P$.
$AB \subset P \implies B \subset P$
$\forall b \in B$, $ab \in P$, $\therefore b \in P$, so $B \subset P$. □