10 Friday, February 25, 2011

Recall:
If I is an ideal of A, then $\operatorname{rad} I = \{x \in A \,|\, x^n \in I, \text{ some } n > 0\}$
$V(I) = V(\operatorname{rad} I)$
$V(I) \supset V(J) \iff I \subset \operatorname{rad} J$
irreducible closed set C: $C \neq C_1 \cup C_2$, $C_i < C$.

Proposition. *I an ideal, $V(I)$ irreducible iff $\operatorname{rad} I$ is prime ideal*

Theorem. *Let A be a finite-type noetherian ring, I an ideal. Then $\operatorname{rad} I$ is the intersection of a finite number of prime ideals $\operatorname{rad} I = P_1 \cap \cdots \cap P_k$. Then we can organize $P_i \not\subset P_j$ for $i \neq j$. Then we have that $V(I) = V(P_1) \cup \cdots \cup V(P_k)$; $V(I)$ is a finite union of irreducible closed sets.*

$\operatorname{Spec} A = \{\text{maximal ideals}\}$
$V(I) = \{\mathfrak{M} \text{ that contains } I\}$

I an ideal. Then $\bar{A} = A/I$.

$$\begin{aligned} A &\to \bar{A} \\ I &\rightsquigarrow (\bar{0}) \\ \operatorname{rad} I \quad &\operatorname{rad}(\bar{0}) \\ &= \textit{nilradical} \\ &= \{x \,|\, x^n = 0\} \end{aligned}$$

The following are equivalent:

- $\operatorname{rad} I = P_1 \cap \cdots \cap P_k$
- $\operatorname{rad}(\bar{0}) = \bar{P}_1 \cap \cdots \cap \bar{P}_k$

  (in $\bar{A}$) also prime ideals

Proof. Suppose the theorem is false for some A and some I. Let $S = \{\text{ideals } J \text{ of } A \text{ such that } \operatorname{rad} J$ is not a finite intersection of prime ideals$\}$. By hypothesis, $S \neq \emptyset$. Then there exists a maximal element I in S. Then $\operatorname{rad} I$ not a finite intersection of prime ideals, but every larger ideal is a finite intersection of prime ideals.
Note:

- $I = \operatorname{rad} I$
- I is not a prime ideal

Then there exist ideals K, L, $K \not\subset I$, $L \not\subset I$, but $KL \subset I$. Replace K by $K + I$, L by $L + I$. Then we have

$$(K + I)(L + I) \subset KL + KI + IL + II \subset I.$$

(so it's ok to do these replacements).

Now $K, L > I$. Replace K and L with their radicals.

$$(\operatorname{rad} K)(\operatorname{rad} L) \subset \operatorname{rad}(KL) \subset \operatorname{rad} I = I$$

(so it's ok to these replacements.)

We have

$$\operatorname{rad} K = P_1 \cap \cdots \cap P_r$$
$$\operatorname{rad} L = Q_1 \cap \cdots \cap Q_s$$

...

We made a mistake somewhere. (Supposed to replace I by a zero ideal?) The version on the web is probably correct. Let's skip this for now.

$\square$

Finite Group Action

Let B be a finite type domain, G the finite group of automorphisms of B. Let $A = B^G$.

Theorem.

- *A is a finite type domain*
- *G operates on* $\operatorname{Spec} B = Y$
- *Y maps to* $\operatorname{Spec} A = X$. *$Y \to X$ is surjective, and the fibers are the G-orbits in Y*

Example. $B = \mathbb{C}[x, y]$, $\sigma(x) = -x$, $\sigma(y) = -y$. Then $\langle\sigma\rangle$ is a group of order 2.

The invariant functions are $u = x^2$, $v = y^2$, and $w = xy$. It's not hard to see that every invariant function can be written as some combination of these.

$A = B^G = \mathbb{C}[u, v, w]/(w^2 - uv)$

Then $\operatorname{Spec} A = X =$ locus of $w^2 = uv$ in $\mathbb{A}^3_{u,v,w}$. This is a double cone in 3-space.

Proof. *A finite type domain*:

Take $\beta \in B$, orbit $\{b_1 = \beta, \dots, b_r\}$. Let $p(x) = (x - b_1) \cdots (x - b_r) = x^r - s_1(b)x^{r-1} + \cdots \pm s_r(b)$. We have that β is a root of $p(x)$. Since the $s_i(b)$ are symmetric functions, $s_i(b) \in B^G = A$.

Since B is a finite type domain, say B is generated as an algebra by $\beta_1, \dots, \beta_m$. Then each β_i is a root of the polynomial, coefficients in A.

Let $A_0 = \mathbb{C}$-algebra generated by these roots. Then A_0 is a finite-type domain contained in B. Every element in B is a polynomial in $\beta_1, \dots, \beta_m$.

If a polynomial with β_i as a root has degree d_i, then we only need monomials in β_i with degree $\leq d_i$. There are only a finite number of monomials in β_i of degree $\leq d_i$. Then B is spanned as an A_0-module by these monomials. Thus, B is a finite A_0-module.

$A_0 \subset A \subset B$. Since A_0 is a finite-type domain, A_0 is noetherian. B is a finite-type A_0-module. A is a submodule. Therefore, A is a finite-type A_0 module. Thus, A is a finite-type algebra. $\square$