1 Friday, February 4, 2011

Geometry of solutions to sets of polynomial equations.

e.g., $x^2 + y^2 + 1 = 0 \longrightarrow$ Set of solutions (over $\mathbb{C}$) is really a sphere (with two points removed)

e.g., $y^2 = x^3 - x = x(x-1)(x+1)$ Over $\mathbb{C}$, the set of solutions is a torus (with one point removed)

Lots of applications to number theory, representation theory, etc.

We'll work over the field $\mathbb{C}$.

Recall (background reading, §10-7, 10-8 in Artin's *Algebra*):

Theorem (Hilbert's Nullstellensatz (weak))**.** *The maximal ideals of* $\mathbb{C}[x_1, \ldots, x_n]$ *are exactly those of the form* $(x_1 - a_1, \ldots, x_n - a_n)$ *corresponding to points* $(a_1, \ldots, a_n) \in \mathbb{C}^n$*.*

This means we can consider $\mathbb{C}^n$ as a purely algebraic object. It's called affine n-space, $\mathbb{A}^n_{\mathbb{C}}$ or $\mathbb{A}^n$ for short.

We want to define a nice topology on this space. One choice is to take the Euclidean (complex) topology: define open balls by

$$B_r(x) = \{y \in \mathbb{C}^n \mid |y - x| < r\}$$

and take these to be a basis.

But this is too many open sets (closed sets), e.g. $\{(x, y) \in \mathbb{C}^n \mid y = e^x\}$ is closed in the Euclidean topology. But we only care about polynomials, so we'll use a coarser topology (fewer open/closed sets).

We'll use the smallest topology such that polynomial functions are continuous. This is called the Zariski topology. Defined by: for a polynomial function $f \in \mathbb{C}[x_1, \ldots, x_n]$, define $D(f) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f(a_1, \ldots, a_n) \neq 0\}$ and declare all $D(f)$ to be open. (Note: D stands for distinguished.) As f varies over all polynomials, these $D(f)$ are taken to be a basis.

We have, e.g, $D(0) = \emptyset$, $D(1) = \mathbb{C}^n$, $D(fg) = D(f) \cap D(g)$.

Alternatively, let's see what the closed sets are. For every ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, the vanishing locus of I is $V(I) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0\ \forall f \in I\}$. As I varies, these describe all the closed sets in the Zariski topology.

Note that $\mathbb{C}^n \setminus D(f) = V(f)$.

NOTE:

1. Because $\mathbb{C}[x_1, \ldots, x_n]$ is noetherian[1], any I can be written as $(f_1, \ldots, f_k)$ for some $f_1, \ldots, f_k$. So $V(I) = V(\{f_1, \ldots, f_k\})$.

2. The maximal ideals (the smallest non-empty closed sets) exactly correspond to the points of $\mathbb{C}^n$. (weak Nullstellensatz)

e.g.

1) $\mathbb{A}^1$: the closed sets are $\emptyset = V(1)$, $\mathbb{A}^1 = V(0)$, and sets of zeros of polynomials, that is, all finite sets of points. (also called the cofinite topology)

2) $\mathbb{A}^2$: the closed sets are $\emptyset = V(1)$, $\mathbb{A}^1 = V(0)$, finite sets of points, but also union of $V(f)$ with a finite point set for some polynomial $f \in \mathbb{C}[x, y]$.

[1] Every ascending chain of ideals stabilizes: given $I_1 \subseteq I_2 \subseteq \cdots \subseteq R$, $\exists k$ such that $I_k = I_{k+1} = \cdots$. Equivalent to that every ideal is finitely generated

Why don't we include $V(f,g)$ for $f,g \in \mathbb{C}[x,y]$? It's because the locus of points where both f and g vanish (assuming they have no common factor) is a finite set of points. Note also that $V(f_1) \cup V(f_2) = V(f_1 f_2)$.

$\mathbb{C}[x_1,\ldots,x_n]$ is called the affine coordinate ring of $\mathbb{A}^n$. (think of it as set of functions on the space $\mathbb{A}^n$)

1.1 Projective Space

Claim: $\mathbb{A}^2$ is somewhat defective.

Not all lines intersect. In particular, rotations of one line can cause it to not intersect another line.

To fix this, we add points at infinity to get $\mathbb{P}^2$.

Nice way of doing this: Let

$$\mathbb{P}^2 = \left\{(x,y,z) \in \mathbb{A}^3 \setminus \{0,0,0\}\right\} \big/ (x,y,z) \sim (\lambda x, \lambda y, \lambda z) \qquad (\lambda \neq 0)$$

$\mathbb{A}^2$ is contained in the set of points for which $z \neq 0$: If $z \neq 0$, then

$$(x,y,z) \overset{\mathbb{P}^2}{=} \left(\frac{x}{z}, \frac{y}{z}, 1\right)$$

We have $(x,y) \in \mathbb{A}^2 \longrightarrow (x,y,1)$.

$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1 = \mathbb{A}^2 \coprod \mathbb{A}^1 \coprod \text{point}(= \mathbb{A}^0)$$

Elements of $\mathbb{P}^2$ are written as $(x:y:z)$.

Define a topology on $\mathbb{P}^2$. Most natural way is to take a quotient topology from the Zariski topology on $\mathbb{A}^3 \setminus \{0,0,0\} \subseteq \mathbb{A}^3$.

Take a polynomial function $f \in \mathbb{C}[x,y,z]$. We want to say $V(f) = \left\{(x:y:z) \in \mathbb{P}^2 \,\middle|\, f(x,y,z) = 0\right\}$ is closed. But f gives different values on equivalent points.

We can write $f = f_0 + f_1 + \cdots + f_d$, f_i homogeneous of degree i. Then $f(\lambda x, \lambda y, \lambda z) = f_0(x,y,z) + \lambda f_1(x,y,z) + \cdots + \lambda^d f_d(x,y,z)$. We need to take homogeneous polynomials ($f = f_i$ for some i). Now $f = 0$ and $f \neq 0$ makes sense. Then $V(f_1,\ldots,f_k)$ describe the closed sets.