

## 4 Friday, February 11, 2011

Last time, we gave a proof that almost every plane curve of degree  $d$  is smooth parameter space  $\mathbb{A}^N : N = \binom{d+2}{2}$ .

Another proof, continuing from the middle of the last one:

*Proof.* The dimension of  $S$  (as defined last time) is  $N + 2 - 3 = N - 1$  (the three from  $F_x = F_y = F_z = 0$ ). So  $\pi(S)$  is at most  $N - 1$  dimensional, and so it's  $\overline{\pi(S)}$ . But  $\dim Z = N$ , so  $\overline{\pi(S)} \neq Z$ .  $\square$

**Some words about topology**  $\mathbb{A}^N = \mathbb{C}^N$  is a complex variety of dimension  $N$ . As a real manifold, it's dimension is  $2N$ . In the complex topology, you can have closed disks, e.g.  $|z| \leq 1$  (has positive measure). In the Zariski topology, closed subsets have no measure. e.g., in  $\mathbb{C}$ , the only closed subsets are finite point sets. In  $\mathbb{C}^2$ ,  $V(ax + by + c)$  has no measure (it's a complex plane (dimension 1)).

**Proposition.** *A smooth curve  $C$  of degree 3 in  $\mathbb{P}^2$  contains exactly 9 flex points.*

*Proof.* Let  $f$  be a cubic defining  $C$ . The second partial derivatives of  $f$  are linear, so the determinant of the Hessian is a cubic polynomial which defines the Hessian curve  $H$ .

**Theorem** (Bézout's theorem). *A curve of degree  $m$  in  $\mathbb{P}^2$  intersects a curve of degree  $n$  in exactly  $mn$  points.*

By this theorem (not yet proved), the two cubics  $C$  and  $H$  intersect in 9 points. One can show that the multiplicities are one, and that  $C$  and  $H$  don't have a factor in common. Thus, we get exactly 9 flexes.  $\square$

**Example.**  $y^2 = x^3 - x$

homogenization gives  $y^2z = x^3 - xz^2$

Then  $f = x^3 - xz^2 - y^2z$ .

The Hessian matrix is

$$\begin{bmatrix} 6x & 0 & -2z \\ 0 & -2z & -2y \\ -2z & -2y & -2x \end{bmatrix}$$

Then  $H(f) = 8(3xz^2 - 3y^2x + z^3)$ .

The flexes: You can eliminate  $z$  from  $f = H(f) = 0$ . Then you get a homogeneous polynomial in  $x$  and  $y$ . You can solve for  $x/y$ , let  $y$  be 1, and then plug back in and solve for  $z$ . In this example, we get that one of the flex points is at  $(x : y : z) = (0 : 1 : 0)$ .

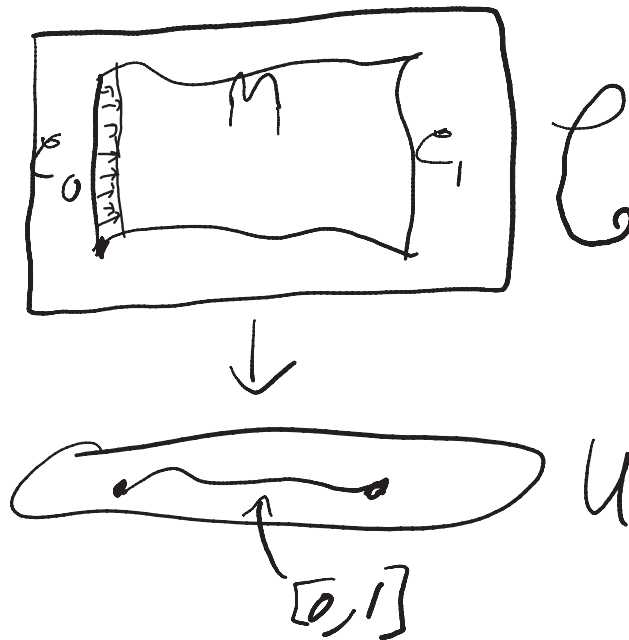
## Genus and Euler characteristic

Goal: Want to understand the topological structure of smooth plane curves.

It's useful to put them in a family. Notation as above. Let  $U = Z - \Sigma = Z - \pi(S)$ . This is the parameter space for smooth plane curves of degree  $d$ . The smooth plane curves are the fibers of the projection  $\mathcal{C} \subset \mathbb{P}^2 \times U$  to  $U$ .

**Proposition.** *All the smooth curves of degree  $d$  are homeomorphic to each other (as real manifolds of dimension 2).*

*Proof.* The problem set shows that  $U$  is path-connected (in the complex topology). Connect the two points in  $U$  (which correspond to curves in  $\mathbb{P}^2 \times U$ ) by a path.



We have a function  $f : M \rightarrow [0, 1]$ . Define a diffeomorphism by taking the gradient of  $f$ , and look at the gradient flow. This tells us how to identify the fibers.  $\square$

**Corollary.** *Smooth plane curves are orientable, connected surfaces.*

*Proof.* Orientability is simple. To orient a smooth surface, we must give a continuously varying orientation to the tangent planes. But tangent plane is a  $\mathbb{C}$ -vector space (of dimension one,  $\sum f_i(p)v_i = U$ ). So multiplying any tangent vector by  $i$  defines a counterclockwise rotation by  $90^\circ$ , which orients the tangent plane.

We'll do connected next time.  $\square$