# 23 Wednesday, April 6, 2011

## Double Planes

Affine case: $A = \mathbb{C}[x, y]$, $\operatorname{Spec} A = \mathbb{A}^2 = X$

$f(x, y) \in A$, square-free (no square factors, non-constant). Let $B = A[w]/(w^2 = f)$, $Y = \operatorname{Spec} B$.

$B$ is a domain, free $A$-module with basis $(1, w)$.

Automorphism $\sigma : B \to B$, $w \mapsto -w$. $A = B^{\langle\sigma\rangle}$

**Lemma.** *$B$ is normal.*

*Proof.* Let $\beta = a+bw$ in $\operatorname{fract}(B)$. It is integral over $A$, and not in $A$ $(b \neq 0)$, irreducible polynomial has coefficients in $A$. $t^2 - (\beta + \beta')t + \beta\beta'$, $\beta' = \sigma\beta$. Then $\beta + \beta' = 2a \in A$, $\beta\beta' = a^2 - b^2 f$ with $b^2 f \in A$. Since $f$ is square-free, $b \in A$, and so $\beta \in B$. $\square$

General theory says $\operatorname{PSpec} B/\langle\sigma\rangle = \operatorname{PSpec} A$.

Prime ideals of $A$: $(0)$; $(g)$ prime ideal, $g$ irreducible; $\mathfrak{M}_p$ maximal $\leftrightarrow p \in X$.

What prime ideals of $B$ lie over $P$?

$(0)$ lies over $(0)$.

Maximal ideals of $B \longleftrightarrow$ points of $Y = \operatorname{Spec} B$

inclusion $A \hookrightarrow B$ gives map $Y \to X$ $(q \rightsquigarrow p) \implies \mathfrak{M}_q$ lies over $\mathfrak{M}_p$. (usually 2 points of $Y$ $\rightsquigarrow$ 1 point of $X$)

Say $P = (g)$, $g$ irreducible in $A$.

What is a description of $Q$, the prime ideals lying over $P$ $(A \cap Q = P)$?

*Cases*:

- $P$ remains prime: $PB = Q$ is a prime ideal
- $f \in P$ ($g$ divides $f$) Then $P$ ramifies: there exists a unique prime $Q$ of $B$ over $P$, and $Q^2 = PB$.
- $P$ does not remain prime, and $f \notin P$. Then there exist two primes $Q$, $Q' = sigmaQ$ over $P$ and $PB = Q \cap Q'$.

(EXPLANATION OF FIRST BULLET NOT INCLUDED)

Second bullet: Say $f \in P$. What is $B/PB$? It's $A[w]/(w^2 - f, g)$. Let $\bar{A} = A/(g)$. Then we can write $B/PB$ as $\bar{A}[w]/(w^2)$. Then $P \longleftrightarrow (0) \in \bar{A}$, and $PB \longleftrightarrow (0) \in B/PB$. Then $w$ generates the prime ideal of $\bar{B}$, and the quotient is $\bar{A}$. This gives, using the correspondence theorem, a prime ideal $Q$ of $B$, $Q = (w, g)$.

$Q^2 = (w^2, wg, g^2) = (f, wg, g^2)$. Then $\gcd(f, g^2) = g = uf + vg^2$, so $g \in Q^2$.

Third bullet: $f \notin P$, $PB$ not prime. Choose $Q$ lying over $P$. Let $Q' = \sigma Q$. Since $Q$ lies over $P$, so does $Q'$ because $\sigma$ fixes $A$. Since $PB$ is not prime, $Q \neq B$.

**Lemma.** *$Q \cap Q' = PB$*

*$(Q \neq PB, \therefore Q \neq Q')$*

*Proof.* Take $\beta \in Q \cap Q'$, $\beta = a + bw$. Then $\beta' = \sigma\beta = a - bw$, so $\beta' \in Q \cap Q'$. Note that $\sigma$ fixes $Q \cap Q'$: $\sigma(Q \cap Q') = \sigma Q \cap \sigma Q' = Q' \cap Q$.

$$\begin{aligned} \beta + \beta' &= 2a \in Q \cap Q' \cap A = P \\ \beta\beta' &= a^2 - b^2 f \in Q \cap Q' \cap A = P \\ &\therefore b^2 f \in P \end{aligned}$$

$f \notin P$, $\therefore b \in P$, $ainP$, $\beta \in PB$. Thus, $PB = Q \cap Q'$. □

**Example.** $w^2 = f = x^2 + y^2 - 1$, $g = y$

Then $g \nmid f$. So we have $P = (g)$ remains prime.

Take $B/PB \approx \mathbb{C}[x, w]/(w^2 - x^2 + 1)$. This is a domain, so the 0-ideal is prime. Therefore, $P$ remains prime. Take $g = y - 1$. (This divides $y^2 - 1$.)

Then $B/P_2B \approx \mathbb{C}[x, w]/(w^2 - x^2)$, so $P_2B$ does not remain prime.

If we draw a picture, we see that $y = 0$ goes through the middle of the circle, but $y = 1$ is tangent.

*Show*: If $\Delta = \{f = 0\}$ (branch locus) and $C = \{g = 0\}$ (curve) intersect $\pitchfork$ (intersect transversely; the tangent lines are distinct) at some point $p$.

**Theorem.** *$C$ remains prime.*

*Proof.* Choose coordinates so that $p = (0, 0)$. Then $f = \sum a_{ij} x^i y^j$, $g = \sum b_{ij} x^i y^j$. $\Delta$ and $C$ meet at $p$, so $a_{00} = b_{00} = 0$.

Then $f = a_{10}x + a_{01}y + \cdots$. The tangent line is $a_{10}x + a_{01}y = 0$. We also have $g = b_{10}x + b_{01}y + \cdots$, with tangent line $b_{10}x + b_{01}y = 0$.

Let's make a linear change of coordinates: $f = x + u$, $g = y + u$, $u, v$ have all terms of degree $\geq 2$.

Now let's make an analytic change of coordinates. Set $x' = x+u$, $y' = y+v$. Then $\left(\frac{\partial(x',y')}{\partial(x,y)}\right)_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is invertible. The inverse function theorem says that this operator is invertible locally (analytically).

$\mathbb{C}^2_{x,y} \to \mathbb{C}^2_{x',y'}$: replace $x, y$ with $x', y'$. Then $w^2 = x'$ and $g = y'$. This doesn't split locally. □

( A power series $c_k x^k + c_{k+1} x^{k+1} + \cdots$, $c_k \neq 0$ is a square of a series $\iff$ $k$ is even. )

## Projective Double Plane

Start with an affine double plane $w'^2 = F(x', y')$. Say $F$ has degree $d = 2k$.

Make the substitution $\frac{x}{z} = x'$, $\frac{y}{z} = y'$, $\frac{w}{z^k} = w'$. Then $\left(\frac{w}{z^k}\right)^z = F\left(\frac{x}{z}, \frac{y}{z}\right) \to \boxed{w^2 = f(x, y, z)}$ (homogeneous of degree $d$), a double cover of $\mathbb{P}^2_{xyz}$.

To embed, we need weighted projective space, where $x$, $y$, $z$ have weight 1 and $w$ has weight $k$. In this space, $(w, x, y, z) = (\lambda^k w, \lambda x, \lambda y, \lambda z)$. (Note: "weighted projective spaces are a bit pathological.") A better way to do this is to treat this as a sheaf of algebras over $\mathbb{P}^2_{xyz}$.