11 Monday, February 28, 2011

B finite type, G operator(?)

$A = B^G$

Showed A finite type

$Y = \operatorname{Spec} B$

$X = \operatorname{Spec} A$

$X \leftrightarrow \{G \text{ orbits in } Y\}$

$G \times B \to B$

$\sigma, b \rightsquigarrow \sigma(b)$

(left) $\sigma\tau(b)$: first τ, then σ

Then G operates on the *right* on Y.

$q \in Y$, σ sends $q \to q^\sigma$

$q^{\sigma\tau}$: first σ, then τ

View

$$\begin{aligned} y &\leftrightarrow \{\text{homomorphisms } B \to \mathbb{C}\} \\ q &\leftrightarrow \pi_q : B \to \mathbb{C} \\ &\leftrightarrow \{\text{max ideals of } B\} \\ q &\leftrightarrow m_q \end{aligned}$$

Operation on Y:

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[dr, "\pi\circ\sigma"'] & B \arrow[d, "\pi"] \\ & \mathbb{C} \end{tikzcd}$$

$\pi_q \circ \sigma(b) = \pi_q(\sigma b)$

Define $q^\sigma =$ that point such that $\pi_{q^\sigma} = \pi_q \circ \sigma$.

Operation on max ideals:

$\mathfrak{M}_{q^\sigma} = \sigma^{-1}\mathfrak{M}_q$

$Y \to X$?

For any $p \in X$,

$$\begin{tikzcd} B \arrow[r, "\pi\circ p"] & \mathbb{C} \\ A \arrow[u, hook] \arrow[ur, dashed, "\pi_p"'] & \end{tikzcd}$$

$Y \to X$ sends $q \rightsquigarrow r$

$$\begin{tikzcd} B \arrow[r, "\sigma"'] \arrow[rr, bend left, "\pi"] & B \arrow[r, "\pi_q"'] & \mathbb{C} \\ A \arrow[u, hook] \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "\pi_p"'] & A \arrow[u, hook] \arrow[r, "\pi_p"] & \mathbb{C} \arrow[u, equal] \end{tikzcd}$$

Therefore, G-orbits in Y map to points of X.

We want to show that different orbits $\{q_1, \ldots, q_r\} \neq \{q'_1, \ldots, q'_s\}$ in Y map to different points p, p' in X.

Proof. Plan: Find an element $a \in A$ such that $a = 0$ on orbit $\{q_i\}$, $\pi_q(a) = 0$, $a \neq 0$ on orbit $\{q'_j\}$. Then $\pi_{q'_j}(a) \neq 0$. This would give us that $a \in \mathfrak{M}_{q_i}$ (same as $\in \mathfrak{M}_p$) and $a \notin \mathfrak{M}_{q'_j}$ (same as $\notin \mathfrak{M}_{p'}$).

In B, choose $b \in \mathfrak{M}_{q_1}$ (then) such that $b \notin \mathfrak{M}_{q'_j}$ for all $j = 1, \ldots, s$. (Note: $b(q) \coloneqq \pi_q(b)$.)

(Diversion: Suppose $B = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$. b represented by the polynomial $p(x_1, \ldots, x_n)$. $\operatorname{Spec} B \approx V(f_1, \ldots, f_k)$ in $\mathbb{C}^n$.
$\operatorname{Spec} B =$ (max ideals) $=$ (homomorphisms $B \to \mathbb{C}$) $= (V(I)$ if $B = \mathbb{C}[x]/I)$)

We can do this. (Think about choosing (hyper-?)planes that do not pass through finite sets of points.)

Let $a = \prod_{\sigma \in G} \sigma(b)$. a is invariant. $a = 0$ on q_1 because b divides a in B (since some $\sigma \in G$ is the identity). Therefore $a = 0$ on the orbit (q_1). $a \neq 0$ on q'_1.

$\sigma(b)$ evaluated at q'_1 is $\pi_{q'_1}(\sigma b) = \pi_{{q'}_1^\sigma}(b) = b$ evaluated at ${q'}_1^\sigma$.

Therefore, $a \neq 0$ on the orbit. $\square$

11.1 Localization

Note: we always assume that the rings are domains and assume (whenever possible) that they're finite type algebras.

Definition. A *multiplicative system* S in a domain A is a subset of A satisfying

- $1 \in S$
- $0 \notin S$
- if $s, t \in S$, then $st \in S$

Definition. The elements of S serve as denominators in the *ring of fractions*

$$A_S \coloneqq \left\{ \frac{a}{s} \,\middle|\, s \in S, a \in A \right\} \Big/ \sim$$

where $\frac{a}{s} \sim \frac{b}{t}$ if $at = bs$

$$\begin{aligned} A &\hookrightarrow A_S \\ a &\rightsquigarrow a/1 \end{aligned}$$

Example. $S = \{1, s, s^2, \ldots\}$, $s \neq 0$. $A_S = A[s^{-1}] = A[y]/(sy - 1)$

Example. $S = A - \{0\}$, $A_S =$ fraction field

Example. P a prime ideal of A, $S = A - P = \{s \mid s \notin P\}$. Then $s \notin P$, $t \notin P \implies st \notin P$. Then A_S is the *localization of A at P*. This is (perversely) denoted A_P.

If $A \subset B$ a subring, then we can relate ideals of A and B:
Extended ideal: I^e

I ideal of A

$IB =$ ideal of B generated by $\{I\}$

The elements are

$$\sum_{\text{finite}} x_i b_i \qquad , \; x_i \in I, \; b_i \in B$$

Contracted ideal: J^c: For J an ideal of B, $(J \cap A) =$ ideal of A

$(I^e)^c \supset I$

$(J^c)^e \subset J$

For $A \subset B = A_S$:

$I^e = IA_s = \{x/s \mid x \in I, \; s \in S\} / \sim$

$J^c = J \cap A$. If $y/s \in J$, then $y \in J \cap A = J^c$. Therefore, $y/s \in (J^c)^e$. Thus, $J \subset (J^c)^e$, so $J = (J^c)^e$.

Corollary. *If A noetherian, then A_S noetherian*

Proof. Take an increasing sequence $J_1 \subset J_2 \subset \cdots$ of ideals in A_S. Let $I_\nu = J_\nu \cap A$. Then $I_1 \subset I_2 \subset \cdots$. Since A is noetherian, this is eventually constant. Therefore $I_\nu^e = (J_\nu^c)^e$ eventually constant. Thus A_S is noetherian. $\square$