

## 5 Monday, February 14, 2011

### 5.1 Plane curves

monomials  $m_1, \dots, m_N$ , coefficients  $z_1, \dots, z_N$

$Z$  = the space of all homogeneous polynomials of degree  $d$  in  $x, y$ , and  $z$   
 = affine space with coordinates  $z_\nu$

(We have  $f(x, y, z) = z_1 x^d + z_2 x^{d-1} + \dots$ )

$U$  = open subset in  $Z$  corresponding to smooth plane curves

$\mathcal{C} \subset \mathbb{P}^2 \times Z$

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathbb{P}^2 \times Z \\ & \searrow & \downarrow \\ & & Z \end{array} \qquad \begin{array}{ccc} \mathcal{C}|_U = \mathcal{C}_U & \subset & \mathbb{P}^2 \times U \\ & \searrow & \downarrow \\ & & U \end{array}$$

**Proposition.** *Smooth plane curves are orientable and connected surfaces, and compact.*

*Proof.* Oreintability was done last time.

To check connectedness, we just need to check one smooth curve of degree  $d$  is connected.  
 $\mathcal{C} : \{x^d + y^d - z^d = 0\}$ .

Look at the line  $y = z$ . On  $U_2$ , taking  $z = 1$  we have  $x^d + y^d = 1$ . Since  $y = z$ ,  $y = 1$ , and then  $x^d = 0$ . Since this is a root of order  $d$ ,  $C$  meets this line in only one point. This means that it's connected. (WHY?)  $\square$

Given a connected, orientable, compact surface, it's topologically characterized by  $g$  = the genus = the # of holes.

**Definition.** The *Euler characteristic* of  $C$  is  $2 - 2g$ .

The Euler characteristic can be computed used an arbitrary triangulation, and then  $E = \# \text{ vertices} - \# \text{ edges} + \# \text{ faces}$ .

A sphere is, topologically, a tetrahedron, we have that the Euler characteristic is  $4 - 6 + 4 = 2$ . We can do a similar thing for a torus.

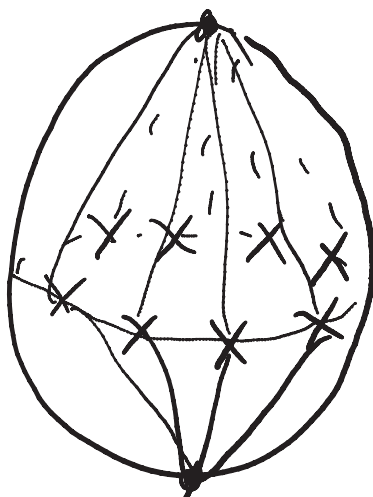
What is the Euler characteristic and genus of a smooth plane curve of degree  $d$ ?

Let's represent a smooth plane curve as a branched cover of  $\mathbb{P}^1$ .

Method I Start with the Fermat curve, and do an explicit calculation.  $C : \{x^d + y^d - z^d = 0\}$   
 Taking  $z = 1$ ,  $U_2 \simeq \mathbb{A}^2 \simeq \mathbb{C}^2$ . Then we have  $x^d + y^d = 1$ . Drop  $y$  by projection. Fix a value  $x_0$  for  $x$ . Then the line  $y = x_0$  intersects the curve in  $\leq d$  points. Typically, we get  $d$  values for  $y$ .

Case I  $x_0^d \neq 1$ . Solve  $y^d = 1 - x_0^d$ . There are  $d$  solution s (if  $y$  is a solution, then so is  $yr^{2\pi ij/d}$  for  $0 \leq j \leq d$ )

Case II  $x_0^d = 1$ . Solve  $y^d = 0$ . The only solution is  $y = 0$ . Now we look at  $x_0$ .  
 Triangulate  $\mathbb{P}^1$ , which is a sphere, as follows. There are  $d$  values for  $x_0$ ,  $x_i^d = 1$ ,  $x_i = e^{2\pi ij/d}$ .



These points distribute themselves along the equator of  $\mathbb{P}^1$ . Adding points at the poles, there are  $2 + d$  vertices,  $d + d + d = 3d$  edges, and  $2d$  faces, which gives us 2. This is what happens “downstairs” (in the projected curve onto  $X = \mathbb{P}^1$ ).

Upstairs, there is an induced triangulation:

- vertices:  $d + d + d = 3d$
- edges:  $3d^2$
- faces:  $2d^2$

Then the Euler characteristic is  $E = 3d - 3d^2 + 2d^2 = 3d - d^2 = d(3 - d)$ .

Then  $g = \frac{1}{2}(d - 1)(d - 2)$ .

Method II Let  $C$  be a smooth curve of degree  $d$ . Assume the coefficient of  $z^d$  is not zero. Divide by that coefficient, giving  $f(x, y, z) = z^d - a_1(x, y)z^{d-1} + \cdots \pm a_d(x, y)$ . Homogenous polynomial  $\implies a_i(x, y)$  is degree  $i$  in  $x$  and  $y$ . Then drop  $z$  by projection onto  $\mathbb{P}^1(x, y)$ .

Fix  $(x_0, y_0)$ . View  $z^d - a_1(x_0, y_0)z^{d-1} + \cdots \pm a_d(x_0, y_0) = 0$  as a polynomial of degree  $d$  in  $z$ . Typically this has  $d$  roots, but for some values of  $(x, y)$ , there are  $d - 1$  roots.

There is a polynomial  $\Delta$ , discriminant, degree  $d(d - 1)$  in  $x$  and  $y$ .  $\Delta = 0$  iff there are less than  $d$  roots. (The discriminant for a quadratic is  $b^2 - 4ac$ . It tells you whether or not the polynomial has double roots.) (Note, if the discriminant has only simple roots, then the claim above (that there are only  $d$  or  $d - 1$  roots at any point) is intuitively/geometrically true.)

Triangulate  $X = \mathbb{P}^1$  by putting vertices at these  $d(d - 1)$  points. For  $\mathbb{P}^1$ , the Euler characteristic is 2. Pulling the triangulation up, the Euler characteristic is approximately  $2d$  (everything gets multiplied by  $d$ ). However, we placed the vertices at the  $d(d - 2)$  points where there are  $d - 1$  roots. Then the Euler characteristic is  $2d - d(d - 1) = 3d - d^2$ .