4 Friday, February 11, 2011

Last time, we gave a proof that almost every plane curve of degree d is smooth parameter space $\mathbb{A}^N : N = \binom{d+2}{2}$.

Another proof, continuing from the middle of the last one:

Proof. The dimension of S (as defined last time) is $N + 2 - 3 = N - 1$ (the three from $F_x = F_y = F_z = 0$). So $\pi(S)$ is at most $N - 1$ dimensional, and so it's $\overline{\pi(S)}$. But $\dim Z = N$, so $\overline{\pi(S)} \neq Z$. □

Some words about topology $\mathbb{A}^N = \mathbb{C}^N$ is a complex variety of dimension N. As a real manifold, it's dimension is $2N$. In the complex topology, you can have closed disks, e.g. $|z| \leq 1$ (has positive measure). In the Zariski topology, closed subsets have no measure. e.g., in $\mathbb{C}$, the only closed subsets are finite point sets. In $\mathbb{C}^2$, $V(ax + by + c)$ has no measure (it's a complex plane (dimension 1)).

Proposition. *A smooth curve C of degree 3 in $\mathbb{P}^2$ contains exactly 9 flex points.*

Proof. Let f be a cubic defining C. The second partial derivatives of f are linear, so the determinant of the Hessian is a cubic polynomial which defines the Hessian curve H.

Theorem (Bézout's theorem)**.** *A curve of degree m in $\mathbb{P}^2$ intersects a curve of degree n in exactly mn points.*

By this theorem (not yet proved), the two cubics C and H intersect in 9 points. One can show that the multiplicities are one, and that C and H don't have a factor in common. Thus, we get exactly 9 flexes. □

Example. $y^2 = x^3 - x$
homogenization gives $y^2 z = x^3 - xz^2$
Then $\boxed{f = x^3 - xz^2 - y^2 z}$.
The Hessian matrix is

$$\begin{bmatrix} 6x & 0 & -2z \\ 0 & -2z & -2y \\ -2z & -2y & -2x \end{bmatrix}$$

Then $H(f) = 8(3zx^2 - 3y^2x + z^3)$.
The flexes: You can eliminate z from $f = H(f) = 0$. Then you get a homogeneous polynomial in x and y. You can solve for x/y, let y be 1, and then plug back in and solve for z. In this example, we get that one of the flex points is at $(x : y : z) = (0 : 1 : 0)$.

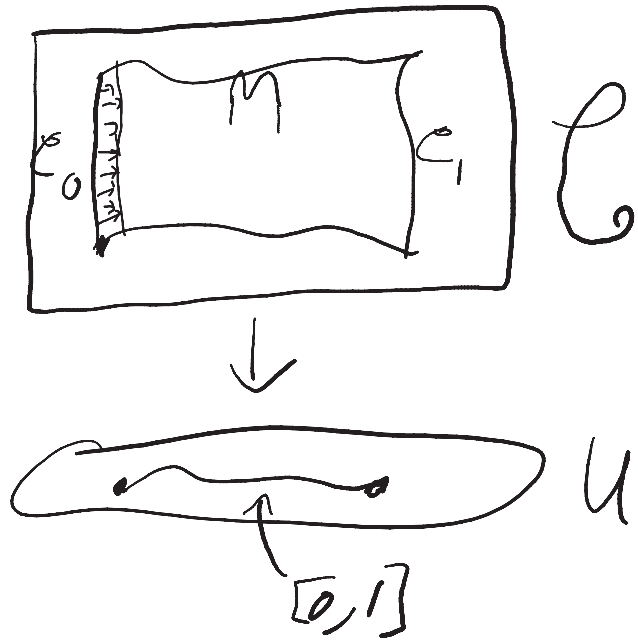
Genus and Euler characteristic

Goal: Want to understand the topological structure of smooth plane curves.

It's useful to put them in a family. Notation as above. Let $U = Z - \Sigma = Z - \pi(S)$. This is the parameter space for smooth plane curves of degree d. The smooth plane curves are the fibers of the projection $\mathcal{C} \subset \mathbb{P}^2 \times U$ to U.

Proposition. *All the smooth curves of degree d are homeomorphic to each other (as real manifolds of dimension 2).*

Proof. The problem set shows that U is path-connected (in the complex topology). Connect the two points in U (which correspond to curves in $\mathbb{P}^2 \times U$) by a path.



We have a function $f : M \to [0, 1]$. Define a diffeomorphism by taking the gradient of f, and look at the gradient flow. This tells us how to identify the fibers. □

Corollary. *Smooth plane curves are orientable, connected surfaces.*

Proof. Orientability is simple. To orient a smooth surface, we must give a continuously varying orientation to the tangent planes. But tangent plane is a $\mathbb{C}$-vector space (of dimension one, $\sum f_i(p)v_i = U$). So multiplying any tangent vector by i defines a counterclockwise rotation by $90°$, which orients the tangent plane.

We'll do connected next time. □