

12 Wednesday, March 2, 2011

S a multiplicative system

$$1 \in S$$

$$0 \in S$$

$$S_1, S_2 \in S \implies S_1 S_2 \in S.$$

Ring of fractions A_S localized ring

$$A \hookrightarrow A_S$$

$$(J^c)^e = J$$

$$(A \cap J)A_S$$

Localizing prime ideal (s...?)

$$I \text{ ideal of } A, I \cap S \neq \emptyset \implies I^e = \text{unit ideal of } A_S$$

Proposition. P prime ideal of A . $P \cap S \neq \emptyset$. Then

- $(P^e)^c = P$
- $P^e (= P_S)$ is a prime ideal of A_S

$$P^e = PA_S = \{s^{-1}x \mid x \in P\}$$

Proof. For any ideal P , $(P^e)^c \supset P$.

We want to show \subset . Let $z \in (P^e)^c$. Then $z = s^{-1}x$ for some $x \in P$, and $z \in A$. Then $sz = ss^{-1}x = x \in P$. Since P is prime, and $s \notin P$, $z \in P$, and so $(P^e)^c \subset P$.

Now we show that P^e is prime:

We have that $z_1 z_2 \in P^e$ for $z_i \in A_S$. Then $z_1 = s_1^{-1}a_1$, $z_2 = s_2^{-1}a_2$. Then $z_1 z_2 = (s_1 s_2)^{-1}(a_1 a_2) \in P^e$. Therefore $(s_1 s_2)(z_1 z_2) = a_1 a_2 \in P^e$. Since $a_1 a_2 \in A$, this is also in $(P^e)^c = P$. Since $a_1 a_2 \in P$ and P prime, either $a_1 \in P$ or $a_2 \in P$, $s_1^{-1}a_1 \in P^e$ or $s_2^{-1}a_2 \in P^e$.¹ \square

$$P \text{ Spec } A_S \longleftrightarrow \text{subset of } P \text{ Spec } A = \{P \mid P \cap S \neq \emptyset\}$$

Back to the case where P is a prime ideal of A and $S = A - P = \{s \in A \mid s \notin P\}$.

Write A_P for A_S . If I is an ideal of A , $I_P = I_S$ extended ideal.

Proposition. P_P is a maximal ideal of A_P and it is the only maximal ideal of A_P .

Lemma. For a ring R , the following are equivalent:

- (1) R has a unique maximal ideal \mathfrak{M}
- (2) The elements of R that are not invertible form an ideal

~~(2) \implies~~ (1) Suppose that the non-units form an ideal I . Then R/I is a field because every element is the residue of a unit, and therefore invertible. Thus I is a maximal ideal. Since any other element is a unit, we cannot include any other element without turning the ideal into the entire ring. Thus, this is maximal.

- (1) \implies (2) Suppose there exists a unique maximal ideal \mathfrak{M} . Let $u \in R$. Then $(u) = R$ if and only if u is a unit. If u is not a unit, then $(u) < R$, and so $(u) \subset$ some maximal ideal.² Then $(u) \subset \mathfrak{M}$.

¹Sorry if this proof is unclear. I was trailing behind Prof. Artin, and so wasn't understanding the proof well.

²If R is not noetherian, this requires Zorn's Lemma/The Axiom of Choice.

Then \mathfrak{M} contains all the non-invertible elements, and so the non-invertible elements of R form an ideal (in particular \mathfrak{M}). □

Proposition above. $s^{-1}a \in A_P$, $s \notin P$.

If $a \in P$, then $s^{-1}a \in P_P$. If $a \notin P$, then $s^{-1}a$ is invertible, and so $a^{-1}s \in A_S$. □

Definition. A (noetherian) ring R is *local* if it has a unique maximal ideal \mathfrak{M} . (Note that R/\mathfrak{M} is a field.)

Example. $A = \mathbb{C}[x, y]$. The prime ideals are

- (0)
- $(f(x, y))$ for f irreducible
- maximal ideal $\mathfrak{M}_{(a,b)} = (x - a, y - b) \longleftrightarrow (a, b) \in \mathbb{C}^2$

$A_{(0)}$: fraction field $\mathbb{C}(x, y)$ of $\mathbb{C}[x, y]$

$A_{\mathfrak{M}_{(a,b)}}$: a local ring. The prime ideals $\text{PSpec } A_{\mathfrak{M}} = \{P \mid P \cap S \neq \emptyset\} = \{P \mid P \subset \mathfrak{M}\} =$

$$\begin{cases} (0) \\ P = (f) \mid f(a, b) = 0 \\ \mathfrak{M}_{(a,b)} \end{cases}$$

Lemma. Suppose I is an ideal of the ring A and M is a finite A -module such that $M = IM$. Then there exists a $z \in I$ such that $(1 - z)M = 0$.

Proof. Say x_1, \dots, x_r generate M . We can write x_i as a combination of $\{x_1, \dots, x_r\}$ with coefficients in I :

$$\begin{aligned} x_i &= \sum_j p_{ij} x_j & p_{ij} &\in I \\ X &= PX & P &\text{matrix } (p_{ij}) \\ (\mathbb{K} - P)X &= 0 \\ Q(\mathbb{K} - P) &= \delta \mathbb{K} \end{aligned}$$

where Q is the cofactor matrix for $\mathbb{K} - P$ with entries in A , and $\delta = \det(\mathbb{K} - P)$.

$$\begin{aligned} Q(\mathbb{K} - P)X &= 0 \\ \therefore \delta X &= 0 \end{aligned}$$

$$\begin{aligned} \mathbb{K} - P &= \begin{pmatrix} 1 - p_{11} & \cdots & \\ & \ddots & \\ & & 1 - p_{nn} \end{pmatrix} \\ \delta &= 1 - z \end{aligned}$$

Since the $p_{ij} \in I$, we have $z \in I$. Then $(1 - z)X = 0$, so $(1 - z)$ kills M . □

Lemma (Nakayama Lemma). Let A be a local ring with a maximal ideal \mathfrak{M} , and let M be a finite A -module. If $M = \mathfrak{M}M$, then $M = 0$.

Proof. Take $z \in \mathfrak{M}$. We have a z with $(1 - z)M = 0$. Since $1 - z \notin \mathfrak{M}$, $1 - z$ is invertible, and so $M = 0$ (since we can multiply by $(1 - z)^{-1}$). □