## 3 Wednesday, February 9, 2011

## 3.1 Curves in $\mathbb{P}^2$

Curves in  $\mathbb{P}^2$  are defined by homogeneous irreducible polynomials f: C = V(f).

e.g., the line containing a pair of points  $(p,q) \in \mathbb{P}^2$  is the set of points up + vq for  $(u,v) \neq (0,0)$ . It's equation Ax = 0 is obtained by solving Ap = Aq = 0. (Think of A = (a,b,c),  $P = (x_1,y_1,z_1)$ ,

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$$q = (x_2, y_2, z_2)$$
. Then  $ax_1 + by_1 + cz_1 = 0$ ,  $ax_2 + by_2 + cz_2 = 0$ . Then  $\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . i.e.,

[a, b, c] is the kernel of the  $2 \times 3$  matrix, which has rank 2.)

The restriction of a homogeneous polynomial f(x, y, z) to a line  $\ell = \{up + vq\}$  is obtained by substitution f(up + vq). This is a homogeneous polynomial in u, v of degree  $= \deg(f)$ . Over  $\mathbb{C}$ , any such polynomial can be factored into linear factors  $(up_i + vq_i)$ . These are the points of  $\ell$  (not necessarily distinct) that lie on V(f). Thus, a plane polynomial curve of degree d meets a line in d points, counted with multiplicity.

Let f be a homogeneous polynomial of degree d in  $x_1, x_2, x_2$ , and let C = V(f). Let  $f_i$  denote  $\frac{\partial f}{\partial x_i}$  and let  $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ . Then the Hessian Matrix is the  $3 \times 3$  symmetric matrix

$$H(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}.$$

**Proposition** (Euler's Formula). Let f,  $d = \deg(f)$ ,  $f_i$  be as above. Then  $\sum_{i=1}^3 f_i x_i = f \deg(f)$ .

*Proof.* Check it for monomials, since it's additive. (CHECK NOT INCLUDED)

This works for polynomials in n variables.

Now consider the Taylor expansion of the restriction of f to  $\ell = \{up + vq\}$ . Setting u = 1 (v = 0 is p, so looking near the point p):

$$f(p+vq) = f(p) + \left(\sum f_i(p)q_i\right)v + \frac{1}{2}\left(\sum f_{ij}q_iq_j\right)v^2 + \mathcal{O}(v^3)$$

with  $q = (q_1, q_2, ...)$ .

**Proposition.** 1. If p is a point of C, then f(p) = 0.

- 2. Suppose  $p \in C$ , and  $f_i(p)$  are not all 0. Then the equation of the tangent line T to C a t p is  $\sum f_i(p)q_i = 0$ .
- 3. Let h be the Hessian of f at p. Then  $\det h = 0$  iff p is a flex point C (i.e., a restriction of f to the tangent line at p has a zero of order  $\geq 3$  at p

*Proof.* 1. By definition.

- 2. Tangent line: if the restriction of f to T has at least a second order 0 (by definition). So looking at the coefficient of v, this is clear.
- 3. Exercise: Check that the restriction of the quadratic term to the tangent line is 0 iff det h = 0.

**Definition 1.** If all the  $f_i$  vanish at p, then p is called a *singular* point of C = V(f). Otherwise, say that C is *non-singular* at p. Say that C is a *non-singular curve* if it has no singular points.

## 3.1.1 Nonsingular curves

e.g. 1, an irreducible conic is always non-singular

*Proof.* Convert to  $x^2 - yz = 0$ .  $f_x = 2x$ ,  $f_y = -z$ ,  $f_z = -y$ . Since  $(x, y, z) \in \mathbb{P}^2$ , not all these can be zero, so it's nonsinular

e.g. 2, An irreducible plane cubic can have at most one singular point (exercise)

e.g. 3, The curve  $x^d + y^d + z^d = 0$  is non-singular (smooth) for  $d \ge 1$ . (Fermat polynomial of degree d).

The partial derivatives are  $dx^{d-1}$ ,  $dy^{d-1}$ ,  $dz^{d-1}$ , not all zero (in  $\mathbb{P}^2$ )

e.g. 4, The curve  $x^3 + y^2 - xyz = 0$  is singular at the point (0:0:1).

**Proposition.** For most values of the coefficients of a polynomial of degree d, the curve  $C = V(f) \subseteq \mathbb{P}^2$  is smooth.

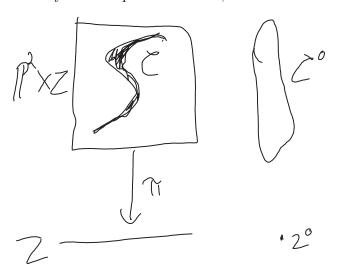
*Proof.* See two proofs, each of which depends on some theorem which will be proved later.

Setup: Order the monomials of degree d in x, y, z arbitrarily  $m_1, m_2, \ldots, m_N$ . (Note:  $N = \binom{d+k-1}{k-1}$  for k variables.)

An arbitrary polynomial of degree d is a linear combination of the monomials  $m_{\nu}$  with some coefficients  $z_{\nu}$ . Think of  $z_{\nu}$  as variables and let

$$F = \sum_{\nu=1}^{N} z_{\nu} m_{\nu} \in \mathbb{C}[x, y, z, \{z_{\nu}\}].$$

Then F = 0 defines a subvariety  $\mathcal{C}$  of the product  $\mathbb{P}^2 \times Z$ , where Z is  $\mathbb{A}^N$  with coordinates  $z_{\nu}$ .



The fiber  $C^0 = \pi^{-1}(z^0)$  of C over a point  $z^0 \in Z$  is the curve whose equation is the polynomial obtained.

by substituting  $z_{\nu}^{0}$  for  $z_{\nu}$ . The 3 partial derivatives  $F_{x}$ ,  $F_{y}$ ,  $F_{z}$  are polynomials in x, y, z,  $\{z_{\nu}\}$  linear in  $z_{\nu}$  and homogeneous of degree d-1 in x, y, z. They define some subvariety of  $\mathbb{P}^{2} \times Z$ . Let S be the variety  $\{F_{1} = F_{2} = F_{3} = 0\}$ . Note that  $S \subset \mathcal{C}$  (by Euler).

The fiber  $\mathcal{C}^0$  over a point  $z^0$  of Z is smooth if and only if  $\mathcal{C}^0$  doesn't meet S.

We can construct  $\Sigma = \pi(S)$  the image of S via a polynomial  $\mathbb{P}^2 \times Z \to Z$ . Later we'll prove that the image of the projection of any Zariski closed subvariety of  $\mathbb{P}^2 \times Z$  to Z is also Zariski closed.

So the set  $\Sigma$  is closed in the affine space Z. But  $\Sigma$  is not all of Z (because the Fermat curve is smooth). So  $\Sigma \subset Z$  is a proper closed subvariety. So the set of  $z^0$  for which  $\mathcal{C}^0$  is smooth is a Zariski open subset of  $\mathbb{A}^N$ .