21 Friday, April 1, 2011

A (Zariski) closed set in $\mathbb{P}^n$ (projective variety) is the set of zeros of some homogeneous polynomial.

Segre embedding $\mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$, $(x_i), (y_j) \rightsquigarrow (u_{ij})$, $u_{ij} = x_i y_j$, $N = (m+1)(n+1) - 1$. The defining equations are $u_{ij}u_{kl} = u_{il}i_{kj}$. This mapping is bijective.

If $X \subset \mathbb{P}^N$ is a closed set, then the Zariski topology on X is the induced topology: a closed subset C of X is a subset that is closed in $\mathbb{P}$ or $C = Y \cap X$ with Y closed in $\mathbb{P}$.

So we can get the Zariski topology on $\mathbb{P}^m \times \mathbb{P}^n$ $(= X)$. What are the closed subsets? If C is closed in $\mathbb{P}^m \times \mathbb{P}^n$, then it is the intersection of the set of zeros of some homogeneous polynomials $f(u)$ with $\mathbb{P}^m \times P^n$. We may replace $f(u)$ with $f(x_i y_j)$ homogeneous in x and in y, with the same degree.

Proposition. *Closed subsets of $\mathbb{P}^m \times \mathbb{P}^n$ are zeros of some polynomials $f(x,y)$ homogeneous in x and homogeneous in y, not necessarily of the same degree.*

Proof. Say C is the set of zeros of $f(x,y)$ homogeneous of degree r in x and homogeneous of degree s in y. Say $r \le s$. Look at the zeros of $x_i^{s-r} f(x,y)$ (homogeneous of degree s in x), $i = 0, 1, \ldots, m$. □

Corollary.

$$\mathbb{P}^n \overset{\Delta}{\to} \mathbb{P}^n \times \mathbb{P}^n$$
$$(x) \rightsquigarrow (x), (x)$$

The diagonal is a closed subset.

Proof. Label the coordinates $(x_i), (x'_i) \in \mathbb{P}^n \times P^n$. We want $x_i = x'_i$. The (homogeneous polynomial) equation that defines this is $x_i x'_j - x_j x'_i$, so this is a closed set. □

What is the product topology on $\mathbb{P}^n \times \mathbb{P}^n$? The closed subsets are given by the basis $(C \times \mathbb{P}) \cap (\mathbb{P} \times C')$. This is bad, because we don't get curves.

Definition. A space X is *Hausdorff* if, for any p, q distinct points, there exist disjoint open subsets U, V with $p \in U$ and $q \in V$.

Proposition. *A space X is Hausdorff if and only if the diagonal is closed in $X \times X$ in the product topology.*

Proof. Exercise □

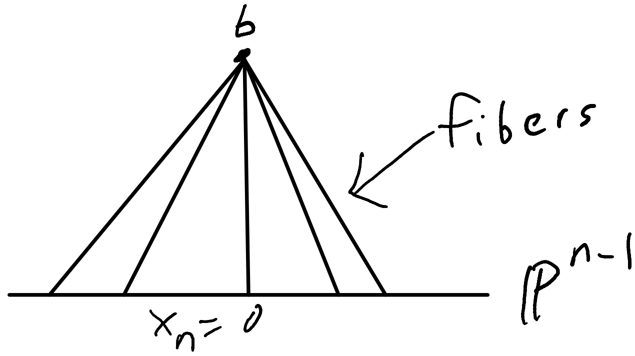
Theorem (Hrushovski-Zilber Theorem)**.** *A "Zariski topology" is a set X and, for all products $X \times X \times \cdots \times X$, a collection of "closed" subsets of X, satisfying the following axioms:*

- *Compatible with projection and inclusions of $X \times X \times \cdots \times p \times \cdots \times X$.*
- *Noetherian (descending chain condition)*
- *dimension 1*

Then there exists an algebraically closed field K and an algebraic curve X over K in the Zariski topology.[1]

[1] See `http://en.wikipedia.org/wiki/Zariski_geometry`. Wikipedia references the paper "Hrushovski, Ehud; Zilber, Boris (1996). "Zariski Geometries". *Journal of the American Mathematical Society* 9 (01): 156. doi:10.1090/S0894-0347-96-00180-4." (`http://www.ams.org/jams/1996-9-01/S0894-0347-96-00180-4/S0894-0347-96-00180-4.pdf`) This is a theorem of model theory, which Wikipedia defines as "the study of (classes of) mathematical structures such as groups, fields, graphs or even models of set theory using tools from mathematical logic."

Projection: $\mathbb{P}^n \xrightarrow{\pi} \mathbb{P}^{n-1}$, $(x_0, \ldots, x_n) \rightsquigarrow (x_0, \ldots, x_{n-1})$. This is undefined at the "center of the projection," $(0, 0, 0, \ldots, 0, 1) = p$.



In $\mathbb{P}^n \times P^{n-1}$ let Γ be the graph. $\bar{\Gamma}$ is the Zariski closure of Γ. Label the points $(x_i), (y_i) \in \mathbb{P}^n \times \mathbb{P}^{n-1}$. Γ: $x_i = y_i$ for $i = 0, \ldots, n-1$. The equations $x_i y_j - x_j y_i$ for $i, j = 0, \ldots, n-1$ define a Zariski closed set of $\bar{\Gamma}$. Look where $x_0 \neq 0$: Take $x_0 = 1$. Then $y_j = x_j y_0$. We can't have $y_0 = 0$: Take $y_0 = 1$. Then $y_j = x_j$, $j = 0, \ldots, n-1$, x_n arbitrary. Only the center $p = (0, 0, \ldots, 0, 1)$ escapes. In this case, all the equations are trivial. So no conditions on (y). The result is

$$\bar{\Gamma} = \{(x, \pi(x)) \,|\, x \neq p\} \cup \{(p, y) \,|\, y \text{ arbitrary}\}.$$

Grassmannians: $G(r, n)$ is the r-dimensional subspace of $\mathbb{C}^n$. For example, $\mathbb{P}^n = G(1, n+1)$. Look at $G(2, 4) =$ 2-dimensional subspaces of $\mathbb{C}^4$ or lines in $\mathbb{P}^3$.

V a vector space of dimension 4, basis (v_1, v_2, v_3, v_4). There is an exterior algebra $\bigwedge V$. The rule is $vw = -wv$. (Or $v \wedge w = -w \wedge v$.) Then $vv = 0$.
$\bigwedge^2 V$ has a basis $v_i v_j$ for $i = j$ dimension $\binom{4}{2} = 6$
$\bigwedge^3 V$ has a basis $v_i v_j v_k$ for $i < j < k$ dimension 4
$\bigwedge^4 V$ has a basis $v_1 v_2 v_3 v_4$ dimension 1
$\bigwedge^k V = 0$ for $k > 4$

Proposition. *The following are equivalent:*

- *There is a subspace $W \subset V$ of dimension 2*
- *Vectors w in $\bigwedge^2 V$, non-zero, and decomposable into $w = uv$, $u, v \in V$.*
- *w in $\bigwedge^2 V$, $ww = 0$ $/(scalar)$*
- *Let $w = \sum_{i<j} a_{ij} v_i v_j \longleftrightarrow (a_{ij})$ in $\mathbb{P}^5$. Then $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$.*

Proof. Partial sketch of one of the implications: $w = \sum a_{ij} u_i v_j$, $ww = \sum_{i<j,k<l} a_{ij} a_{kl} v_i v_j v_k v_l$. Plug in all the possible values. □