30 Monday, April 25, 2011

Axioms for Cohomology

(q. cohomology sheaves)

- $H^0(X, M) = M(X)$
- $H^0, H^1, \ldots$ is cohomology functor(?)
- $Y \xrightarrow{f} X$ is the inclusion of affine oben subset N q. cohomology on Y, $\implies H^q(X, f_N) = 0$ for all $q > 0$

Y open $\implies H^q(X, f_*N) \approx H^q(Y, N)$

Uniqueness: Choose an affine open cover of X: $U = \{U^\nu\}$, $U \xrightarrow{j} X$, $M \xrightarrow{q} j_*j^*M$ …(SOME MATERIAL NOT INCLUDED)

Define $R^0_M = j_*j^*M$

$$0 \to M \to R^0_M \to M^q \to 0$$

exact.

$H^q(R^0_M) = 0$ for $q > 0$.

$\therefore$

$$0 \to H^0(M) \to H^0(R^0_M) \to H^0(M^1) \to H^1(M) \to 0 \to H^1(M^1) \to H^2(M) \to 0 \cdots$$

H^0 is the identity by axiom 1, $\therefore H^1(M) = H^0(M^1)/\operatorname{im}(H^0(R^0_M))$

$\therefore H^1(M)$ is unique for all M. Then $H^1(M^1) \approx H^2(M)$, so $H^2(M)$ is unique for all M.

$$0 \to M^1 \to R^1_M \to M^2 \to 0$$

So we can repeat the construction, replacing M^1 in $0 \to M \to R^0_M \to M^1 \to 0$ with M^1 to get an *acyclic resolution* of M (resolution means exact, acyclic means that $H^q(R^i_M) = 0$ for $q > 0$).

$$0 \to M \to R^0_M \to R^1_M \to R^2_M \to \cdots$$

$R^1_M = j_*j^*M^1$

Look at the complex

$$0 \to R^0_M \to R^1_M \to \cdots$$

Take $H^0(R^{\cdot}_M) = R^{\cdot}_M(X)$ global section ($\cdot$ is a "variable"). Not exact. But it's a complex.[1] Define $H^q(X, M) = \mathcal{H}^q(R^{\cdot}_M(X)) = (\ker / \operatorname{im})$ in the complex $R^{\cdot}_M(X)$.

We know

$$0 \to M(X) \to R^0_M(X) \to R^1_M(X)$$

is exact. $\therefore M(X) \approx \mathcal{H}^0(R^{\cdot}_M(X))$ (First axiom ✓)

Second axiom: If $0 \to M \to M' \to M'' \to 0$ is exact, want long cohomological sequence. $R^0_M = j_*j^*M$. j^* is the trivial restriction which is exact. Since $U^\nu \xrightarrow{j^\nu} X$ is the inclusion of an affine open, so j^ν_* is exact. $\therefore j_*$ is exact.

$N \to N' \to N''$, quasi-coherent on U, is exact if for all affine open sets $V \subset U$, $N(V) \to N(V') \to N(V'')$ is exact.

[1] Definition of *complex*: if you compose two maps, you get zero.

?W is an affine open in X?
?$j_*N(W) \to j_*N'(W) \to j_*N''(W)$ exact?

$$j_*N(W) = N(U^\nu \cap W)$$

affine because U^ν and W are both affine (therefore $U^\nu \cap W$ is affine). So yes.

$$* = N(U \cap W) \to N'(U \cap W) \to N''(U \cap W)$$

If $0 \to M \to M' \to M'' \to 0$ is exact, then $0 \to R^0_M \to R^0_{M'} \to R^{M''} \to 0$ is exact.

$$\begin{aligned} R^0_M &= j_*j^*M \\ R^0_M(X) &= j^*M(U) \qquad U \text{ affine} \\ &= M(U) \end{aligned}$$

If $0 \to M \to M' \to M'' \to 0$ is exact, then $0 \to M(U) \to M'(U) \to M''(U) \to 0$ is exact (because U is affine).

Therefore

$$0 \to R^0_M(X) \to R^0_{M'}(X) \to R^0_{M''}(X) \to 0$$

is exact.

Applying this to each degree separately,

$$0 \to R^\cdot_M(X) \to R^\cdot_{M'}(X) \to R^\cdot_{M''}(X) \to 0$$

is an exact sequence of complexes. (So $\mathcal{H}^q$ is a cohomological functor.) (Second axiom ✓)

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "j'"] & U \arrow[d, "j"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

with $V^\nu = Y \cap U^\nu$ an affine open cover of Y. We now suppress indices.

Let N be quasicoherent on Y.

We want to show $H^q(X, f_*N) = 0$ for $q > 0$. We can show that $H^q(X, f_*N) \approx H^q(Y, N)$ whether or not Y is affine.

On Y: say $S^0_N = j'_*j'^*N$ We get an acyclic resolution

$$0 \to N \to S^0_N \to S^1_N \to \cdots$$

On the other hand, take $j_*j^*(f_*N) = R^0_{f_*N}$. We get an acyclic resolution

$$0 \to f_*N \to R^0_{f_*N} \to R^1_{f_*N} \to \cdots$$

$$\begin{aligned} H^q(Y, N) &\approx \mathcal{H}^q(S^\cdot_N(Y)) \\ H^q(X, f_*N) &\approx \mathcal{H}^q(R^\cdot_{f_*N} * (X)) \end{aligned}$$

Plan: Show that $R^\cdot_{f_*N} \approx f_*S^\cdot_N$. Then $R^\cdot_{f_*N}(X) \approx [f_*S^\cdot_N](X) = S^\cdot_N(Y)$ (by the definition of f_*). Then we get isomorphisms(?).

$R^0_{f_*N} = j_* j^* f_* N$

$$j^* f_* N \approx f'_* j'^* N$$

Then

$$R^0_{f_*N} = j_* j^* f_* N \approx j_* f'_* j'^* N$$

By commutativity of the diagram above,

$$j_* f'_* = f_* j'_*$$

Then

$$R^0_{f_*N} = j_* j^* f_* N \approx j_* f'_* j'^* N \approx f_* j'_* j'^* N = f_* S^0_N$$

(Third axiom ✓(almost; we need to check compatibility))

$$0 \to M \to R^0_M \to M^1 \to 0$$

$$0 \to H^{q-1}(M^1) \to H^q(M) \to 0$$