22 Monday, April 4, 2011

Review

$G(2,4)$ = planes in V^4, lines in $\mathbb{P}^3$

V has basis u_1, u_2, u_3, u_4

$\bigwedge^2 V$ has basis $u_1u_2, \dots, u_3u_4$ (u_iu_j for $i < j$), dimension 6

W a 2-dimensional subspace of V with basis (u, v), then $w = uv \in \bigwedge^2 V$ determines W, and conversely W determines uv. $(u', v') = (u, v)P \implies u'v' = uv \det P$.

Proposition. *2-dimensional subspaces of V correspond to decomposable non-zero elements $w = uv$ of $\bigwedge^2 V$ modulo scalars.*

$w \in \bigwedge^2 V$ is decomposable $\iff ww = 0 \iff w = \sum_{i<j} a_{ij}u_iu_j$ and $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$.

The grassmanian $G(2,4)$ is the locus of $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$ in $\mathbb{P}^5_{(a_{ij})}$ (dimension = 4).

We can show directly that $\dim G(2,4) = 4$. W is a subspace of dimension 2 with basis (u, v). ($u = \sum a_iu_i$, $v = \sum b_iu_i$)

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \overset{\text{choose a basis}}{\longrightarrow} \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

Ask for the locus of pairs (p, ℓ) with $p \in \mathbb{P}^3$ a point and ℓ a line in $\mathbb{P}^3$. Let $\Gamma = \text{locus}\,\{(p, \ell) \,|\, p \in \ell\}$. We can embed this in $\mathbb{P}^3 \times \mathbb{G}$ where $\mathbb{G} = G(2,4)$.

$$\Gamma \subset \mathbb{P}^3 \times G \subset \mathbb{P}^3 \times \mathbb{P}^5 \overset{\text{Segre}}{\to} \mathbb{P}^{23}$$

We can show that Γ is a closed set in $\mathbb{P}^3 \times \mathbb{G} \subset \mathbb{P}^3 \times \mathbb{P}^5$ (with coordinates $(x) \times (a_{ij})$): Find defining homogeneous equations. We need $f(x, a)$ homogeneous in x_i and in a_{ij}. If ℓ is the line through (u, v) (basis for $W \longleftrightarrow \ell$), $(x) \in \ell$ means $x = su + tv$ for some s, t. This is the case if and only if x, u, v dependent, which is true if and only if $xuv = 0$ in $\bigwedge^3 V$.

$$uv = \sum_{i<j} a_{ij}u_iu_j \longleftrightarrow (a_{ij}) \in \mathbb{P}^5$$

Then $(x) = \sum x_iu_i$ and so $xuv = \sum x_ia_{ijk}u_iu_ju_k$. Expand this and plug in the relation $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$.

Surfaces of degree d in $\mathbb{P}^3$

Given as $f = 0$, f homogeneous of degree d in (x_1, x_2, x_3, x_4). The number of monomials of degree d is $\binom{d+3}{d}$.

$$\{\text{surfaces of degree } d\} \longleftrightarrow \mathbb{S} := \text{points in } \mathbb{P}^N$$

with $N = \binom{d+3}{d} - 1$.

We ask in $\mathbb{G} \times \mathbb{S}$ for $\Theta := \{(\ell, S) \,|\, \ell \subset S\}$.

The following isn't quite right. (Something is wrong with the degree of f?)

We can show that Θ is closed in $\mathbb{G} \times \mathbb{S}$. What are its defining equations? Say ℓ is a line through u, v, and say that S is the surface given by $f = 0$. Then $\ell \subset S \iff f(su + tv) = 0$ for all s, t. Expand $f(su + tv)$:

$$f(su + tv) = s^d f_0(u) + s^{d-1}tf_1(u, v) + \cdots + st^{d-1}f_{d-1}(u, v) + t^d f_d(v)$$

Then $f(su+tv)=0$ for all s, $t \iff f_i(u,v)=0$ for all i.

Example.

$$f = x_i x_j x_k$$

$$f(su+tv) = s^3u_iu_ju_k + s^2t(u_iu_jv_k+u_iv_ju_k+v_iu_ju_k) + st^2(u_iv_jv_k+v_iu_jv_k+v_iv_ju_k) + t^3(v_iv_jv_k)$$

Lemma. *f_i depends only on f, not on u and v.*

That is, if we change basis to $(u',v')=(u,v)P$, the f_i don't change.

Check: We check this for P elementary; say $u=u'+\varepsilon v'$, $v=v'$. Then $f(su+tv)=f(su'+s\varepsilon v'+tv')=f(su'+(s\varepsilon+t)v')$. We can say $t'=s\varepsilon+t$. Then the expansion doesn't change, except for the transformation of $t\mapsto t'$ (which doesn't change the f_i).

The above isn't quite right. (Something is wrong with the degree of f?)

What's the dimension of Θ?

Plan: We can carry any line in $\mathbb{P}^3$ to any other line by a change of coordinates.

Take a particular line ℓ_0. Suppose we determine $\dim\{S \,|\, \ell_0\subset S\}=r$. Then $\dim\Theta=\dim\mathbb{G}+r=r+4$.

Suppose we take ℓ_0 to be the line $\{(x_1,x_2,0,0)\}$. Is $f(x_1,x_2,0,0)$ identically zero? When we substitute in, we get a homogeneous polynomial of degree d in x_1, x_2:

$$f(x_1,x_2,0,0)=\alpha_0x_1^d+\alpha_1x_1^{d-1}x_2+\cdots+\alpha_dx_2^d$$

The $d+1$ coefficients of f must then be zero, so $\dim\{S \,|\, \ell_0\subset S\}=\dim\mathbb{S}-(d+1)$. So $r=\binom{d+3}{d}-1-(d+1)$. Then $\dim\Theta=\binom{d+3}{d}-d+2$.

For $d=1$, $\dim\Theta=5$. $\dim\mathbb{S}=3$. We expect the set of lines in a given S to have dimension 2, so this is correct.

For $d=2$, $\dim\Theta=10$. $\dim\mathbb{S}=9$ (10 coefficients, but it's projective space). We expect a particular S to have a 1-dimensional family of lines.

For $d=3$, $\dim\Theta=19$. We expect a finite number of liens in a cubic. The number of lines contained in a generic cubic surface is 27.