Solutions to Problems 2

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2.1

Assume that you have both emission and absorption spectra for the 21 cm line in a particular direction, and can therefore determine τ_ν across the line profile.

2.1.1

Show that the measured excitation temperature T_e in any line of sight is actually the harmonic mean of all components with various excitation temperatures in that direction. That is, if there are M separate clouds along the line of sight, each with column density N_i and excitation temperature T_i and total column density $N_{\text{tot}} = \sum_{i=1}^{M} N_i$, then

$$\frac{1}{T_e} = \frac{1}{N_{\text{tot}}} \sum_{i=1}^{M} \frac{N_i}{T_i} \tag{2.1.1}$$

Solution: For each cloud i, if the excitation temperature T_i is constant along the line of sight, the optical depth at frequency ν is given by:

$$\tau_{\nu,i} = \frac{g_u}{g_l} \sigma_{ul} (1 - e^{-h\nu_{ul}/k_B T_i}) N_i \tag{2.1.2}$$

where σ_{ul} is the cross section for stimulated emission. When looking at the 21 cm line, we are in the regime where $k_B T_i$ is much greater than $h\nu_{ul} = 5.87$ μeV. With $k_B T_i \gg h\nu_{ul}$,

$$\tau_{\nu,i} \approx \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_i} N_i \tag{2.1.3}$$

Similarly, the total optical depth along the line of sight at frequency ν is given by:

$$\tau_\nu \approx \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_e} N_{\text{tot}} \tag{2.1.4}$$

The total optical depth is also the sum of the optical depths of individual clouds:

$$\tau_\nu = \sum_{i=1}^{M} \tau_{\nu,i} \approx \sum_{i=1}^{M} \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_i} N_i = \frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \sum_{i=1}^{M} \frac{N_i}{T_i} \tag{2.1.5}$$

Compare Equation 2.1.4 and Equation 2.1.5, we get:

$$\frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \frac{N_{\text{tot}}}{T_e} = \frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \sum_{i=1}^{M} \frac{N_i}{T_i}$$

$$\implies \frac{1}{T_e} = \frac{1}{N_{\text{tot}}} \sum_{i=1}^{M} \frac{N_i}{T_i} \tag{2.1.6}$$

2.1.2

Typical measured excitation temperatures for lines of sight near the galactic plane are $T_e \approx 150$ K. Assume that all emission is optically thin, and all the H I is either in cold neutral clouds or in a warm neutral phase (this is the 2-phase model of the ISM). What is the fraction of the H I in each phase, given $T_e = 150$ K?

Solution: Since all the H I gas is either in cold neutral clouds or in a warm neutral phase, utilize Equation 2.1.1:

$$\begin{cases} N_{\text{tot}} = N_c + N_w \\ \dfrac{1}{T_e} = \dfrac{1}{N_{\text{tot}}} \left(\dfrac{N_c}{T_c} + \dfrac{N_w}{T_w} \right) \end{cases} \tag{2.1.7}$$

where subscript $_c$ refers to the variables of H I in the cold phase; subscript $_w$ refers to the variables of H I in the warm phase. Typical temperature in CNM is $T_c \approx 80$ K and the one in WNM is $T_w \approx 8000$ K. Now, substitute these values into Equation 2.1.7 and solve this linear system:

$$\begin{cases} \dfrac{N_c}{N_{\text{tot}}} = \dfrac{T_c(T_w - T_e)}{T_e(T_w - T_c)} \approx 52.86\% \\[4mm] \dfrac{N_w}{N_{\text{tot}}} = \dfrac{T_w(T_c - T_e)}{T_e(T_c - T_w)} \approx 47.14\% \end{cases} \tag{2.1.8}$$

2.2

Consider a region containing only partially-ionized hydrogen. Let ζ be the ionization rate per H atom and α be the recombination coefficient.

2.2.1

Determine the steady-state ionization fraction x_{ss} in terms of $n_H = n(\text{HI}) + n(\text{HII})$, ζ and α.

Solution: The ionization fraction $x(t)$ is defined as:

$$x(t) = \frac{n(\text{HII})}{n_H} \tag{2.2.1}$$

Then the neutral fraction is:

$$1 - x(t) = \frac{n(\text{HI})}{n_H} \tag{2.2.2}$$

The ionization rate per unit volume is $\zeta n(\text{HI}) = \zeta n_H (1 - x)$ and the recombination rate per unit volume is $\alpha n_e n(\text{HII}) = \alpha n^2(\text{HII}) = \alpha n_H^2 x^2$. Notice that we have used $n_e = n(\text{HII})$ since each ionized hydrogen atom contributes 1 electron. The rate of change of the ionized hydrogen density is given by the balance between ionization and recombination:

$$\frac{dn(\text{HII})}{dt} = \zeta n_H (1 - x) - \alpha n_H^2 x^2 = n_H \frac{dx}{dt} \tag{2.2.3}$$

where n_H is constant. At steady state:

$$\frac{dx}{dt} = 0$$
$$\implies 0 = \zeta(1 - x_{\text{ss}}) - \alpha n_H x_{\text{ss}}^2 \tag{2.2.4}$$

This is a quadratic equation. Use the quadratic formula:

$$x_{\text{ss}} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4\alpha n_{\text{H}}(-\zeta)}}{2\alpha n_{\text{H}}} = \frac{\zeta}{2\alpha n_{\text{H}}}\left(-1 \pm \sqrt{1 + \frac{4\alpha n_{\text{H}}}{\zeta}}\right) \tag{2.2.5}$$

Since ionization fractions must be between 0 and 1, we discard the negative root. Thus, the steady-state ionization fraction is:

$$x_{\text{ss}} = \frac{\zeta}{2\alpha n_{\text{H}}}\left(\sqrt{1 + \frac{4\alpha n_{\text{H}}}{\zeta}} - 1\right) \tag{2.2.6}$$

2.2.2

Suppose that the fractional ionization at time $t = 0$ is given by $x(0) = x_{\text{ss}} + \delta(0)$. If $|\delta(0)| \ll x_{\text{ss}}$, determine the solution $x(t)$ by assuming n_{H}, ζ, and α to be constant.

Solution: Start from the rate equation, Equation 2.2.3:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \zeta(1 - x) - \alpha n_{\text{H}} x^2 \tag{2.2.7}$$

We can write $x(t)$ as a perturbation around the steady state:

$$x(t) = x_{\text{ss}} + \delta(t) \tag{2.2.8}$$

Given that $|\delta| \approx |\delta(0)| \ll x_{\text{ss}}$, we can linearize the rate equation by expanding to 1st order in δ. Substitute $x(t)$ into Equation 2.2.7:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}\delta}{\mathrm{d}t} &= \zeta(1 - x_{\text{ss}} - \delta) - \alpha n_{\text{H}}(x_{\text{ss}} + \delta)^2 \\
&\approx \zeta(1 - x_{\text{ss}}) - \zeta\delta - \alpha n_{\text{H}}(x_{\text{ss}}^2 + 2x_{\text{ss}}\delta) \\
&= \zeta(1 - x_{\text{ss}}) - \alpha n_{\text{H}} x_{\text{ss}}^2 - (\zeta + 2\alpha n_{\text{H}} x_{\text{ss}})\,\delta \\
&= -(\zeta + 2\alpha n_{\text{H}} x_{\text{ss}})\,\delta
\end{aligned} \tag{2.2.9}$$

Notice that $\zeta(1 - x_{\text{ss}}) - \alpha n_{\text{H}} x_{\text{ss}}^2 = 0$ due to the steady-state condition (Equation 2.2.4). This is a 1st-order linear differential equation with solution:

$$\begin{aligned}
\delta(t) &= \delta(0)e^{-(\zeta + 2\alpha n_{\text{H}} x_{\text{ss}})t} \\
&= \delta(0)e^{-\sqrt{\zeta^2 + 4\alpha n_{\text{H}}\zeta}\,t}
\end{aligned} \tag{2.2.10}$$

Substitute back into Equation 2.2.8:

$$x(t) = x_{\text{ss}} + \delta(0)e^{-\sqrt{\zeta^2 + 4\alpha n_{\text{H}}\zeta}\,t} \tag{2.2.11}$$

2.3

We have worked out the methodology for the determination of the electron temperature in a photoionized medium using the 4363 Å and 5007 Å lines of [O III], because the electron density dependence is cancelled out in this ratio.

2.3.1

If we add the 4959 Å in the diagnostics, derive the expression for the $j(4959\text{Å} + 5007\text{Å})/j(4363\text{Å})$ by also including the term for collisional de-excitation of the $^1\text{D}_2$ level. Solve for the relation between the observed emissivity ratio and the temperature. [Hint: in this solution, the temperature-dependent is again through the collisional rate, but there will be an additional term that includes the electron density.]

Solution: Consider the relevant energy levels of the [O III] ion:

- ground state (level 0, 1, 2): $^3\text{P}_J$ ($J = 0, 1, 2$)

- 1st excited state (level 3): $^1\text{D}_2$

- 2nd excited state (level 4): $^1\text{S}_0$

The relevant transitions and their associated wavelengths are:

- auroral line (level 4 → 3): $^1\text{S}_0 \to {}^1\text{D}_2$ (4363Å)

- nebular line (level 3 → 1): $^1\text{D}_2 \to {}^3\text{P}_1$ (4959Å)

- nebular line (level 3 → 2): $^1\text{D}_2 \to {}^3\text{P}_2$ (5007Å)

We assume that the radiative de-excitation will be primarily spontaneous rather than stimulated emission. In excitation equilibrium, the rate of collisional excitation from the ground state to level 4 will be balanced by radiative de-excitation from level 4.

$$n_0 n_e C_{04} = n_4 (A_{41} + A_{43}) \tag{2.3.1}$$

where n_i is the population of level i; n_e is the electron number density; C_{lu} is the collisional excitation rate coefficient from lower level l to upper level u; A_{ul} is the spontaneous emission coefficient from upper level u to lower level l. The emissivity of the $4 \to 3$ transition is:

$$j(4 \to 3) = \frac{1}{4\pi} n_4 A_{43} h\nu_{43} = \frac{1}{4\pi} n_0 n_e C_{04} \frac{A_{43}}{A_{41} + A_{43}} h\nu_{43} \tag{2.3.2}$$

where $h\nu_{ul}$ is the energy difference between upper level u and lower level l. Level 3 can be populated in 2 ways: by collisional excitation directly from the ground state, or by collisional excitation from the ground state to level 4, followed by radiative de-excitation to level 3. We also include the term for collisional de-excitation of level 3.

$$n_0 n_e C_{03} + n_0 n_e C_{04} \frac{A_{43}}{A_{41} + A_{43}} = n_3 (A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}) \tag{2.3.3}$$

The emissivity of the $3 \to 2$ transition is

$$
\begin{aligned}
j(3 \to 2) &= \frac{1}{4\pi} n_3 A_{32} h\nu_{32} \\
&= \frac{1}{4\pi} n_0 n_e \left(C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}} h\nu_{32}
\end{aligned} \tag{2.3.4}
$$

Similarly, the emissivity of the $3 \to 1$ transition is

$$
\begin{aligned}
j(3 \to 1) &= \frac{1}{4\pi} n_3 A_{31} h\nu_{31} \\
&= \frac{1}{4\pi} n_0 n_e \left(C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{31}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}} h\nu_{31}
\end{aligned} \tag{2.3.5}
$$

The observed emissivity ratio is given by:

$$\frac{j(4959\text{Å}) + j(5007\text{Å})}{j(4363\text{Å})} = \frac{j(3 \to 1) + j(3 \to 2)}{j(4 \to 3)}$$

$$= \frac{\frac{1}{4\pi}n_0 n_e \left(C_{03} + C_{04}\frac{A_{43}}{A_{41} + A_{43}}\right)\frac{A_{31}h\nu_{31} + A_{32}h\nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}}{\frac{1}{4\pi}n_0 n_e C_{04}\frac{A_{43}}{A_{41} + A_{43}}h\nu_{43}}$$

$$= \frac{\left(C_{03} + C_{04}\frac{A_{43}}{A_{41} + A_{43}}\right)\frac{A_{31}\nu_{31} + A_{32}\nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}}{C_{04}\frac{A_{43}\nu_{43}}{A_{41} + A_{43}}}$$

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$$= \left(\frac{C_{03}}{C_{04}}\frac{A_{41} + A_{43}}{A_{43}} + 1\right)\frac{A_{31}\nu_{31} + A_{32}\nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}$$

miu_43

$$(2.3.6)$$

The collisional excitation and de-excitation rate coefficients are given by:

$$C_{lu} = \sqrt{\frac{2\pi\hbar^4}{k_B m_e^3}}\frac{\Omega_{ul}}{g_l}\frac{e^{-h\nu_{ul}/k_B T}}{\sqrt{T}} \tag{2.3.7}$$

$$C_{ul} = \sqrt{\frac{2\pi\hbar^4}{k_B m_e^3}}\frac{\Omega_{ul}}{g_u}\frac{1}{\sqrt{T}} \tag{2.3.8}$$

where Ω_{ul} is the dimensionless collision strength for the transition from lower level l to upper level u; g_i is the degeneracy statistical weight) of level i. The ratio of the collisional excitation rates is then:

$$\frac{C_{03}}{C_{04}} = \frac{\Omega_{30}}{\Omega_{40}}e^{-h(\nu_{30}-\nu_{40})/k_B T}$$

$$= \frac{\Omega_{30}}{\Omega_{40}}e^{-h\nu_{43}/k_B T} \tag{2.3.9}$$

Substitute Equation 2.3.9 into Equation 2.3.7, we get the relation:

$$\frac{j(4959\text{Å}) + j(5007\text{Å})}{j(4363\text{Å})} = \left(\frac{\Omega_{30}}{\Omega_{40}}\frac{A_{41} + A_{43}}{A_{43}}e^{-h\nu_{43}/k_B T} + 1\right)\frac{A_{31}\nu_{31} + A_{32}\nu_{32}}{A_{31} + A_{32} + n_e\sqrt{\frac{2\pi\hbar^4}{k_B m_e^3}}\frac{\Omega_{30} + \Omega_{31} + \Omega_{32}}{g_3}\frac{1}{\sqrt{T}}}$$

$$(2.3.10)$$

2.3.2

Make a plot of the predicted observed ratio as a function of temperature for electron density of 10^3, 10^4 and 10^5 cm^{-3}.

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2.3.3

Solution: We use following numerical values for the parameters:

$$\begin{cases} \nu_{31} = c/4959\text{Å} \\ \nu_{32} = c/5007\text{Å} \\ \nu_{43} = c/4363\text{Å} \\ A_{31} = 6.951 \times 10^{-3} \\ A_{32} = 2.029 \times 10^{-2} \\ A_{43} = 1.685 \\ \Omega_{30} = 0.243 \\ \Omega_{31} = 0.243 \times 3 \\ \Omega_{32} = 0.243 \times 5 \\ \Omega_{40} = 0.0321 \end{cases} \tag{2.3.11}$$
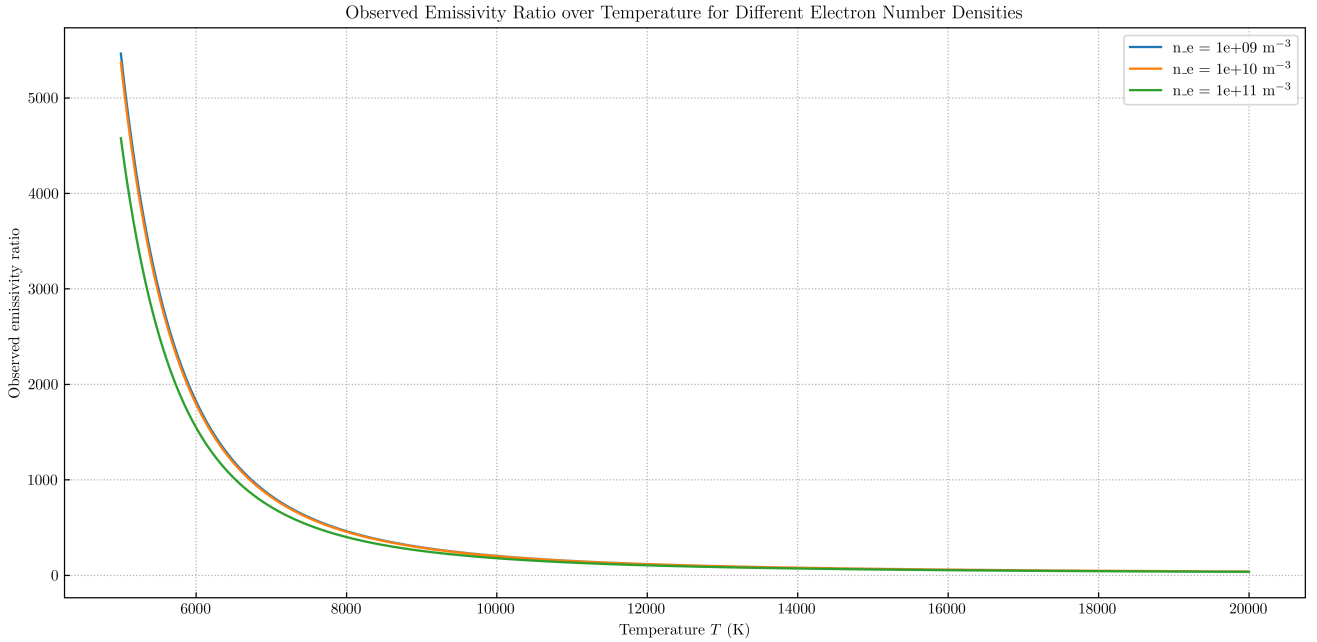
The plot is shown in Figure 1.

Figure 1: Observed Emissivity Ratio over Temperature for Different Electron Number Densities

2.4

Derive the Rankine-Hugoniot jump condition for the density, pressure and temperature ratios across the shock front.

Solution: Consider a steady, 1D shock wave moving through a stationary medium. The subscripts $_1$ and $_2$ denote the upstream (pre-shock) and downstream (post-shock) conditions, respectively. Mass, momentum and energy conservation give:

$$\rho_1 v_1 = \rho_2 v_2 \tag{2.4.1}$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \tag{2.4.2}$$

$$\frac{1}{2}v_1^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}v_2^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} \tag{2.4.3}$$

where ρ is the density, v is the fluid velocity, p is the pressure and γ is the adiabatic index. The Mach number is defined as:

$$\mathcal{M}_1 = \sqrt{\frac{\rho_1 v_1^2}{\gamma p_1}} \tag{2.4.4}$$

Rewrite momentum conservation equation (Equation 2.4.2) using mass conservation equation (Equation 2.4.1):

$$p_1 - p_2 = \rho_2 v_2^2 - \rho_1 v_1^2 = \rho_2 \left(\frac{\rho_1}{\rho_2} v_1\right)^2 - \rho_1 v_1^2 = \rho_1 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right) \tag{2.4.5}$$

Similarly, rewrite energy conservation equation (Equation 2.4.3) using Equation 2.4.1:

$$\left(\frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1}\right) \rho_1 \rho_2 = \left(\frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2}\right) \rho_1 \rho_2$$

$$\implies \frac{1}{2} \rho_1 \rho_2 v_1^2 + \frac{\gamma}{\gamma - 1} p_1 \rho_2 = \frac{1}{2} \rho_1 \rho_2 \left(\frac{\rho_1}{\rho_2} v_1\right)^2 + \frac{\gamma}{\gamma - 1} p_2 \rho_1$$

$$\implies \frac{\gamma}{\gamma - 1} (p_1 \rho_2 - p_2 \rho_1) = \frac{1}{2} \frac{\rho_1^3}{\rho_2} v_1^2 - \frac{1}{2} \rho_1 \rho_2 v_1^2$$

$$\implies \frac{\gamma}{\gamma - 1} (p_1 \rho_2 - p_2 \rho_1) = \frac{1}{2} \rho_1 \rho_2 v_1^2 \left(\frac{\rho_1^2}{\rho_2^2} - 1\right) \tag{2.4.6}$$

Rewrite the left-hand side using Equation 2.4.4 and Equation 2.4.5:

$$p_1 \rho_2 - p_2 \rho_1 = p_1 \rho_2 - \left[p_1 - \rho_1 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right)\right] \rho_1$$

$$= (\rho_2 - \rho_1) p_1 + \rho_1^2 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right)$$

$$= (\rho_2 - \rho_1) \frac{\rho_1 v_1^2}{\gamma \mathcal{M}_1^2} + \rho_1^2 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right) \tag{2.4.7}$$

Substitute this into Equation 2.4.6:

$$\frac{\gamma}{\gamma - 1} \left[(\rho_2 - \rho_1) \frac{\rho_1 v_1^2}{\gamma \mathcal{M}_1^2} + \rho_1^2 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right)\right] = \frac{1}{2} \rho_1 \rho_2 v_1^2 \left(\frac{\rho_1^2}{\rho_2^2} - 1\right)$$

$$\implies (\rho_2 - \rho_1) \frac{\rho_1}{\mathcal{M}_1^2} + \gamma \rho_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right) = \frac{1}{2} (\gamma - 1) \rho_1 \rho_2 \left(\frac{\rho_1^2}{\rho_2^2} - 1\right)$$

$$\implies \left(\frac{\rho_2}{\rho_1} - 1\right) \frac{1}{\mathcal{M}_1^2} + \gamma \left(\frac{\rho_1}{\rho_2} - 1\right) = \frac{1}{2} (\gamma - 1) \frac{\rho_2}{\rho_1} \left(\frac{\rho_1^2}{\rho_2^2} - 1\right)$$

$$\implies \left(\frac{\rho_2}{\rho_1} - 1\right) \frac{1}{\mathcal{M}_1^2} + \frac{1}{2} (\gamma - 1) \frac{\rho_2}{\rho_1} = \frac{1}{2} (\gamma - 1) \frac{\rho_1}{\rho_2} - \gamma \left(\frac{\rho_1}{\rho_2} - 1\right)$$

$$\implies \left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma - 1}{2}\right) \frac{\rho_2}{\rho_1} - \frac{1}{\mathcal{M}_1^2} = -\frac{\gamma + 1}{2} \frac{\rho_1}{\rho_2} + \gamma$$

$$\implies \left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma - 1}{2}\right) \frac{\rho_2^2}{\rho_1^2} - \left(\frac{1}{\mathcal{M}_1^2} + \gamma\right) \frac{\rho_2}{\rho_1} + \frac{\gamma + 1}{2} = 0 \tag{2.4.8}$$

This is a quadratic equation with 1 unknown. Use the corresponding formula:

$$\frac{\rho_2}{\rho_1} = \frac{\left(\frac{1}{\mathcal{M}_1^2} + \gamma\right) \pm \sqrt{\left(\frac{1}{\mathcal{M}_1^2} + \gamma\right)^2 - 4\left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{2}\right)\frac{\gamma+1}{2}}}{2\left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{2}\right)}$$

$$= \frac{1 + \gamma\mathcal{M}_1^2 \pm \mathcal{M}_1^2\sqrt{\frac{1}{\mathcal{M}_1^4} + \frac{2\gamma}{\mathcal{M}_1^2} + \gamma^2 - \frac{2(\gamma+1)}{\mathcal{M}_1^2} - (\gamma-1)(\gamma+1)}}{2 + (\gamma-1)\mathcal{M}_1^2}$$

$$= \frac{1 + \gamma\mathcal{M}_1^2 \pm \mathcal{M}_1^2\sqrt{\frac{1}{\mathcal{M}_1^4} - \frac{2}{\mathcal{M}_1^2} + 1}}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

$$= \frac{1 + \gamma\mathcal{M}_1^2 \pm (1 - \mathcal{M}_1^2)}{(\gamma-1)\mathcal{M}_1^2 + 2}$$

$$= 1 \text{ (discarded) or } \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2} \tag{2.4.9}$$

This is the density ratio across the shock front. From Equation 2.4.4, Equation 2.4.5 and the density ratio (Equation 2.4.9):

$$\frac{p_2}{p_1} = \frac{p_1 - \rho_1 v_1^2\left(\frac{\rho_1}{\rho_2} - 1\right)}{p_1}$$

$$= 1 - \gamma\mathcal{M}_1^2\left[\frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} - 1\right]$$

$$= 1 - \gamma\mathcal{M}_1^2\frac{(\gamma-1)\mathcal{M}_1^2 + 2 - (\gamma+1)\mathcal{M}_1^2}{(\gamma+1)\mathcal{M}_1^2}$$

$$= 1 - \frac{\gamma}{\gamma+1}2(1 - \mathcal{M}_1^2)$$

$$= \frac{\gamma + 1 - 2\gamma(1 - \mathcal{M}_1^2)}{\gamma+1}$$

$$= \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1} \tag{2.4.10}$$

Using the ideal gas law $p\mu = \rho RT$, the temperature ratio is:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

$$= \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1}\frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2}$$

$$= \frac{[2\gamma\mathcal{M}_1^2 - (\gamma-1)][(\gamma-1)\mathcal{M}_1^2 + 2]}{(\gamma+1)^2\mathcal{M}_1^2} \tag{2.4.11}$$

2.5

The evolution of supernova remnant in the Sedov-Taylor phase can be derived by simple dimensional analysis:

$$R_\mathrm{s} = AE^{1/5}\rho_0^{-1/5}t^{2/5} \tag{2.5.1}$$

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2.5.1

Obtain an estimate of the dimensionless factor A by assuming that 50% of the total energy will be in ordered kinetic energy $Mv_{\rm s}^2/2$, where M is the swept-up mass.

Solution: Mathematically, the assumption can be expressed as:

$$\frac{1}{2}E = \frac{1}{2}Mv_{\rm s}^2 \tag{2.5.2}$$

We first determine the speed of the spherical shock front $v_{\rm s}$ by differentiating the radius of the spherical shock front $R_{\rm s}$ with respect to time t:

$$v_{\rm s} = \frac{{\rm d}R_{\rm s}}{{\rm d}t} = \frac{2}{5}AE^{1/5}\rho_0^{-1/5}t^{-3/5} = \frac{2}{5}\frac{R_{\rm s}}{t} \tag{2.5.3}$$

Next, the swept-up mass M is related to the ambient density ρ_0 and $R_{\rm s}$ by:

$$M = \frac{4}{3}\pi R_{\rm s}^3 \rho_0 \tag{2.5.4}$$

Substituting M (Equation 2.5.4), $v_{\rm s}$ (Equation 2.5.3) and the Sedov-Taylor solution (Equation 2.5.1) into the energy equation (Equation 2.5.2):

$$\begin{aligned}
\frac{1}{2}E &= \frac{1}{2}\left(\frac{4}{3}\pi R_{\rm s}^3 \rho_0\right)\left(\frac{2}{5}\frac{R_{\rm s}}{t}\right)^2 \\
&= \frac{8\pi}{75}R_{\rm s}^5 \rho_0 t^{-2} \\
&= \frac{8\pi}{75}(AE^{1/5}\rho_0^{-1/5}t^{2/5})^5 \rho_0 t^{-2} \\
&= \frac{8\pi}{75}A^5 E \\
\implies A &= \left(\frac{75}{16\pi}\right)^{1/5} \approx 1.083
\end{aligned} \tag{2.5.5}$$

2.5.2

Above we considered the case of uniform ambient density ρ and constant total energy E. Suppose that we instead assume that the ambient density decreases as

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^\delta \tag{2.5.6}$$

and energy is increasing with time as a power law:

$$E(t) = E_0 \left(\frac{t}{t_0}\right)^\varepsilon \tag{2.5.7}$$

where $\delta > -3$ and $\varepsilon \geqslant 0$. Find γ in the time evolution of the radius of the blast wave $R_{\rm s} \propto t^\gamma$ as a function of δ and ε.

Solution: Blast wave radius $R_{\rm s}$ satisfy:

$$R_{\rm s}(t) \propto t^\gamma \tag{2.5.8}$$

The velocity of the spherical shock front v_s is the time derivative of the blast wave radius:

$$v_s = \frac{dR_s}{dt} \propto t^{\gamma-1} \tag{2.5.9}$$

The mass M swept up by the blast wave up to time t is given by:

$$M = \int_0^{R_s(t)} 4\pi r^2 \rho(r) dr \propto \int_0^{R_s(t)} r^{\delta+2} dr \propto R_s(t)^{\delta+3} \propto t^{\gamma(\delta+3)} \tag{2.5.10}$$

The total energy $E(t)$ is assumed to be proportional to the kinetic energy of the swept-up mass:

$$E(t) = E_0 \left(\frac{t}{t_0}\right)^\varepsilon \propto \frac{1}{2} M v_s^2 \propto t^{\gamma(\delta+3)} t^{2(\gamma-1)} = t^{(\delta+5)\gamma-2} \tag{2.5.11}$$

The exponents of t on both sides must be equal. Therefore, we equate the powers of t:

$$\varepsilon = (\delta+5)\gamma - 2$$
$$\implies \gamma = \frac{\varepsilon+2}{\delta+5} \tag{2.5.12}$$

2.5.3

If $R_s \propto t^\gamma$, how does the shock temperature T_s vary with time?

Solution: The shock temperature T_s is proportional to the squared velocity of the spherical shock front:

$$T_s \propto v_s^2 \propto t^{2(\gamma-1)} = t^{2\gamma-2} \tag{2.5.13}$$

2.5.4

Suppose that the density profile in the ambient medium is $\rho \propto r^{-2}$, as would apply to a constant-velocity steady stellar wind pre-process the gas before supernova explosion. Suppose that there is a sudden explosion depositing an energy $E_0 = $ Constant. What will be γ for this case?

Solution: Since $\delta = -2$ and $\varepsilon = 0$, we can calculate γ using Equation 2.5.12:

$$\gamma = \frac{\varepsilon+2}{\delta+5} = \frac{2}{3} \tag{2.5.14}$$