1. Flatness Problem

Friedmann equation: $H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3} \rho_{tot}$

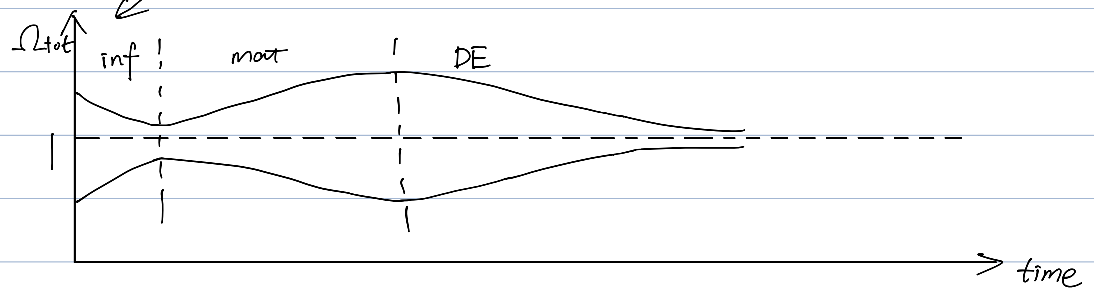
we define $\rho_{cr} = \frac{3H^2}{8\pi G}$, $\Omega_{tot} = \frac{\rho_{tot}}{\rho_{cr}}$, substitution into Friedmann equation,

$\Rightarrow \Omega_{tot} - 1 = \frac{kc^2}{a^2H^2} \propto \frac{1}{a^2H^2}$ ①

$\Rightarrow \frac{d}{dt}|\Omega_{tot} - 1| = |k|c^2 \frac{d}{dt}\frac{1}{a^2H^2} = |k|c^2 \frac{-2}{\dot{a}^3}\ddot{a}$ ②

Assuming the universe is matter dominated, $a \propto t^{\frac{2}{3}}$, $H = \frac{2}{3t} \Rightarrow \frac{1}{a^2H^2} \propto t^{\frac{2}{3}}$, use ①, the deviation of total density from critical density at 1s after the Big Bang is $|\Omega_{tot}(1s) - 1| = |\Omega_{tot}(t_{now}) - 1|(1s/t_{now})^{\frac{2}{3}} \lesssim 2\times10^{-12}$, which is very small.

An accelerating universe can help solve the flatness problem, because when $\ddot{a} > 0$, from eq. ②, $|\Omega_{tot} - 1|$ decrease with time and can be quite small at 1s.



2. Horizon Problem

Comoving particle horizon $D^c_{ph} = \int_0^t \frac{c\,dt'}{a(t')} = c\int_0^a \frac{da'}{a'}\frac{dt'}{da'} = c\int_0^a \frac{da'}{a'^2H(a')}$

Assuming the universe is matter dominated, $a = (H_0 t)^{\frac{2}{3}}$, $H = \frac{2}{3t} \Rightarrow H = \frac{2}{3}H_0 a^{-\frac{3}{2}}$ ①

$\Rightarrow D^c_{ph} = \frac{3c}{2H_0}\int_0^a \frac{da'}{\sqrt{a'}}$ ②

$\Rightarrow D^c_{ph,today} / D^c_{ph,CMB} = \int_0^1 \frac{da'}{\sqrt{a'}} \Big/ \int_0^{\frac{1}{1+1100}} \frac{da'}{\sqrt{a'}} = 1/\sqrt{\frac{1}{1101}} \approx 33.18$

The CMB indicates the horizon problem, because the CMB sky is smooth, homogeneous and isotropic to level of 10^{-5} over the entire sky, which means the CMB from different directions have come to thermal equilibrium. But the comoving particle horizon at $z=1100$ is many times smaller than today, which is just the horizon problem.

From eq. ②, we know that $D^c_{ph} \propto \sqrt{a} \xrightarrow{①} D^c_{ph} \propto \frac{1}{aH} = \frac{1}{\dot{a}}$

an accelerating universe means $\ddot{a} > 0$, so $\frac{d}{dt}D^c_{ph} = -\frac{1}{\dot{a}^2}\ddot{a} < 0$, D^c_{ph} decrease with time, which means there is a much bigger comoving particle horizon before when CMB communicate with each other.

3.

(3.1) $N(t_{start}) \equiv \ln[a(t_{end})/a(t_{start})]$

$$= \ln\left(\frac{a_{end}}{a_0}\frac{a_0}{a_{start}}\right) = \ln\left(\frac{a_{end}}{a_0}\right) + \ln\left(\frac{a_0}{a_{start}}\right) = \ln\left(\frac{a_{end}}{a_0}\right) + \ln\left(\frac{H_{start}}{k_{start}}\right)$$

$$= \ln\left(\frac{a_{end}}{a_0}\right) + \ln\left(\frac{H_{start}}{H_0}\frac{H_0}{k_{start}}\right) = \ln\left(\frac{a_{end}}{a_0}\right) + \ln\left(\frac{H_{start}}{H_0}\right) + \ln\left(\frac{H_0}{k_{start}}\right) \quad ①$$

(3.2) $H_{start}^2 \propto V_{SR}$, $H_0^2 \propto \rho_{cr,0}$ $\Rightarrow \left(\frac{H_{start}}{H_0}\right) = \sqrt{\frac{V_{SR}}{\rho_{cr,0}}}$

We require a cosmic inflation at the energy level of $(10^{16}GeV)^4$ to solve the horizon problem, so we could use 10^{16} GeV to normalize $\Rightarrow$

$$\left(\frac{H_{start}}{H_0}\right) = \left(\frac{V_{SR}^{\frac{1}{4}}}{10^{16}GeV}\right)\left(\frac{10^{16}GeV\, V_{SR}^{\frac{1}{4}}}{\rho_{cr,0}^{\frac{1}{2}}}\right) \quad ②$$

(3.3) $\ln\left(\frac{a_{end}}{a_0}\right) = \ln\left(\frac{a_{eq}}{a_0}\frac{a_{reh}}{a_{eq}}\frac{a_{end}}{a_{reh}}\right) \quad ③$

$$\begin{cases} a_{eq}^4\rho_{eq} \approx a_0^4\rho_{r,0} \\ a_{eq}^4\rho_{eq} = a_{reh}^4\rho_{reh} \\ a_{reh}^3\rho_{reh} = a_{end}^3\rho_{end} \end{cases}$$

$\Rightarrow$ substitution into ③, $\ln\left(\frac{a_{end}}{a_0}\right) = \ln\left(\frac{\rho_{r,0}}{\rho_{eq}}\right)^{\frac{1}{4}} + \ln\left(\frac{\rho_{eq}}{\rho_{reh}}\right)^{\frac{1}{4}} + \ln\left(\frac{\rho_{reh}}{\rho_{end}}\right)^{\frac{1}{3}}$

$$= \ln(\rho_{r,0})^{\frac{1}{4}} + \frac{1}{3}\ln\left(\frac{\rho_{reh}}{\rho_{end}}\right)^{\frac{1}{4}} + \ln\left(\frac{1}{\rho_{end}}\right)^{\frac{1}{4}} \quad ④$$

substitute ②.④ into ①, $N(t_{start}) = \ln\left(\frac{H_0}{k_{start}}\right) + \ln\left(\frac{10^{16}GeV\,\rho_{r,0}^{\frac{1}{4}}}{\rho_{cr,0}^{\frac{1}{2}}}\right)$

$$+ \frac{1}{3}\ln\left(\frac{\rho_{reh}}{\rho_{end}}\right)^{\frac{1}{4}} + \ln\left(\frac{V_{SR}}{\rho_{end}}\right)^{\frac{1}{4}} + \ln\left(\frac{V_{SR}^{\frac{1}{4}}}{10^{16}GeV}\right) \quad ⑤$$

(3.4) $\rho_{r,0} = (2.5\times10^{-13}GeV)^4$, $\rho_{cr,0} = 3\times10^{-12}GeV)^4h^2$, $h\approx0.7$, $H_0 \gtrsim k_{start}$

Ignore the last 3 terms of eq.⑤, $N(t_{start}) = \ln\left[\frac{10^{16}GeV\ 2.5\times10^{-13}GeV}{(3\times10^{-12}GeV)^2\times0.7}\right] + \ln\left(\frac{H_0}{k_{start}}\right)$

$$\gtrsim \ln(0.4\times10^{27}) \approx \ln(e^{61})$$

$$\Rightarrow N(t_{start}) \gtrsim 61$$

4. Friedmann equation $\begin{cases} 3H\dot\phi = -V'(\phi) \quad ① \\ H^2 = \frac{V(\phi)}{3M_{pl}^2} \quad ② \end{cases}$

slow-roll condition $\begin{cases} \frac{1}{2}\dot\phi^2 \ll V(\phi) \quad ③ \\ |\ddot\phi| \ll |3H\dot\phi| \quad ④ \end{cases}$

(4.1) (1) $\epsilon \equiv \frac{1}{2}M_{pl}^2\left(\frac{V'}{V}\right)^2 \xRightarrow{①.②} \epsilon = 3\frac{\frac{1}{2}\dot\phi^2}{V} \quad ⑤$

we know $\frac{1}{2}\dot\phi^2 \ll V(\phi)$ from ③ $\Longrightarrow \epsilon \ll 1$

(2) Take derivative of eq.② $\Rightarrow 2H\dot H = \frac{V'\dot\phi}{3M_{pl}^2} \quad \Rightarrow H\dot H = \frac{V'\phi}{6M_{pl}^2} \quad ⑥$

⑥ over ② $\Rightarrow \frac{\dot H}{H} = \frac{V'\dot\phi}{2V}$, and over ① $\Rightarrow -\frac{\dot H}{H^2} = \frac{3}{2}\frac{\dot\phi^2}{V}$, i.e. ⑤

$\Rightarrow \epsilon = -\frac{\dot H}{H^2}$

(3) $\epsilon \ll 1 \Rightarrow -\dot H \ll H^2 \Rightarrow H^2 + \dot H \gg 0$

since $H = \frac{\dot a}{a}$, so $\dot H = \frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = \frac{\ddot a}{a} - H^2 \Rightarrow \frac{\ddot a}{a} = \dot H + H^2 \gg 0 \quad ⑦$

$\Rightarrow \ddot a \gg 0$, i.e. an accelerating universe.

(4.2) End condition: $\epsilon=1$

$\epsilon=-\frac{\dot H}{H^2}=1 \Rightarrow \dot H+H^2=0 \overset{⑦}{\Rightarrow} \frac{\ddot a}{a}=\dot H+H^2=0$

when $V(\phi)=\frac{1}{2}m^2\phi^2$, $\epsilon\equiv\frac{1}{2}M_{pl}^2\left(\frac{V'}{V}\right)^2=\frac{1}{2}M_{pl}^2\left(\frac{2}{\phi_{end}}\right)^2=1 \Rightarrow \phi_{end}=\sqrt{2}M_{pl}$

(4.3) $N(t)\equiv\ln\frac{a(t_{end})}{a(t)}=\ln a(t_{end})-\ln a(t)=\int_{a(t)}^{a_{end}}d(\ln a)=\int_{a(t)}^{a_{end}}\frac{da}{a}=\int_t^{t_{end}}H\,dt=\int_{\phi(t)}^{\phi_{end}}\frac{H\,d\phi}{\dot\phi}$ ⑧

② over ① $\Rightarrow \frac{H}{\dot\phi}=-\frac{V}{V'}\frac{1}{M_{pl}^2}$, substitution into ⑧ $\Rightarrow N(t)=\frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi(t)}\frac{V(\phi)}{V'(\phi)}d\phi$

(4.4) (1) When $V(\phi)=\frac{1}{2}m^2\phi^2$, $N(t)=\frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi(t)}\frac{1}{2}\phi\,d\phi$

$N(t_{start})=\frac{1}{2M_{pl}^2}\left[\frac{1}{2}\phi^2(t_{start})-\frac{1}{2}\phi^2(t_{end})\right]=\frac{1}{4M_{pl}^2}\left[\phi^2(t_{start})-2M_{pl}^2\right]\geqslant 60$

$\phi^2(t_{start})-2M_{pl}^2\geqslant 240M_{pl}^2$

$\phi(t_{start})\geqslant\sqrt{242}\,M_{pl}$

(2) $\epsilon=\frac{1}{2}M_{pl}^2\left(\frac{V'}{V}\right)^2=\frac{1}{2}M_{pl}^2\left(\frac{2}{\phi}\right)^2=2M_{pl}^2/\phi^2$, when $\phi=\phi(t_{start})$, $\epsilon\leqslant 2/242=\frac{1}{121}$,

which satisfy $\epsilon<<1$ for slow-roll inflation.