Physical Cosmology Homework - CMB

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1. Let us show a *scale-invariant* primordial power spectrum of early Universe curvature perturbation (also referred to as the "Harrison-Zel'dovich power spectrum") $\mathcal{P}_{\mathrm{R}}(k) = A^2 = \mathrm{const}$, will produce CMB angular power spectrum $l(l+1)\,C_l = (2\pi/25)A^2$.

(1.1) Let us assume that we have a real field $f(\mathbf{x})$ representing some early Universe scalar perturbation, which can be expressed as a Fourier series that $f(\mathbf{x}) = \Sigma_{\mathbf{k}} f_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$ with the Fourier transform $f_{\mathbf{k}}$ given by $\frac{1}{V}\int_{\mathrm{V_{full}}} \mathrm{d}^3x\, f(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}}$. Now we evaluate this real function at the last scatter screen (*LSS*), which is located at a comoving distance of x_{LSS} from us. The function becoming $f(x_{\mathrm{LSS}}\mathbf{e_x})$ can then be expressed as a series of spherical harmonics as $f(x_{\mathrm{LSS}}\mathbf{e_x}) = \Sigma_{lm} a_{lm} Y_{lm}(\mathbf{e_x})$, with $a_{lm} = \int \mathrm{d}\Omega Y^*_{lm}(\mathbf{e_x}) f(x_{\mathrm{LSS}}\mathbf{e_x})$, where Y_{lm} and Y^*_{lm} are spherical harmonic functions and their complex conjugates, and $\mathbf{e_x}$ is a unit vector in the direction of $\mathbf{x}$. Note that a plane wave $\exp(i\mathbf{k}\cdot\mathbf{x})$ can also be written as $e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\,\Sigma_{lm}\, i^l\, j_l(kx) Y^*_{lm}(\mathbf{e_k}) Y_{lm}(\mathbf{e_x})$, where $j_l(kx)$ is the spherical Bessel function, and $\mathbf{e_k}$ is a unit vector in the direction of $\mathbf{k}$. Using all above, show that:

$$a_{lm} = 4\pi\, i^l\, \Sigma_{\mathbf{k}}\, f_{\mathbf{k}}\, j_l(kx_{\mathrm{LSS}}) Y^*_{lm}(\mathbf{e_k}) \tag{1}$$

Note that due to the nature of the spherical Bessel function, for which the first peak is reached when $kx_{\mathrm{LSS}} = l$, the Fourier mode k that satisfies $kx_{\mathrm{LSS}} = l$ will contribute the most to the summation and thus also contributes the most to the power in the multipole moment l.

(1.2) The CMB temperature anisotropy $\left(\frac{\delta T}{T_0}\right)$ on super-horizon scales are dominated by the Sachs-Wolfe effect of the Primordial scalar perturbation, such that $\left(\frac{\delta T}{T_0}\right)(\mathbf{e_x}) = -1/5\, R(x_{\mathrm{LSS}}\mathbf{e_x})$, where $R(\mathbf{x})$ is the field of primordial curvature perturbation originated during the inflation era. Now let us consider the real function defined in (1.1) to be $f(\mathbf{x}) \equiv R(\mathbf{x}) = \Sigma_{\mathbf{k}} R_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$. When evaluated at *LSS*, the spherical harmonic coefficient a_{lm} is given by Eq. (1). If we write $a_{lm} = \Sigma_{\mathbf{k}} b_{\mathbf{k}} R_{\mathbf{k}}$, then show that:

$$b_{\mathbf{k}} = -(4\pi/5)\, i^l\, j_l(k x_{\mathrm{LSS}}) Y^*_{lm}(\mathbf{e}_{\mathbf{k}}). \tag{2}$$

(1.3) Let us assume that the primordial power spectrum (per $\ln k$ interval) of the curvature perturbation $\mathcal{P}_{\mathrm{R}}(k) = k^3\, V/(2\pi^2)\langle |R_{\mathbf{k}}|^2\rangle$ is isotropic and follows the Harrison-Zel'dovich form, i.e., scale invariant, such that $\mathcal{P}_{\mathrm{R}}(k) = A^2 = \text{const}$. This essentially determines angular power spectrum of the CMB temperature anisotropy. The theoretical angular power spectrum is given by $C_l = \frac{1}{2l+1}\Sigma_m \langle |a_{lm}|^2\rangle$. Show that:

$$C_l = \frac{A^2}{25}\frac{2\pi}{l(l+1)}. \tag{3}$$

Hints:

(1) $R(\mathbf{x})$ is a Gaussian random field such that $\langle R_{\mathbf{k}} R^*_{\mathbf{k}'}\rangle = \delta_{\mathbf{k}\mathbf{k}'}\langle |R_{\mathbf{k}}|^2\rangle$;

(2) $\Sigma_m |Y_{lm}(\mathbf{e}_{\mathbf{k}})|^2 = \frac{2l+1}{4\pi}$;

(3) $\frac{(2\pi)^3}{V}\,\Sigma_{\mathbf{k}} = \int \mathrm{d}^3k = \int_0^\infty 4\pi k^2\,\mathrm{d}k$;

(4) $\int_0^\infty \frac{\mathrm{d}k}{k} j_l^2(k x_{\mathrm{LSS}}) = \frac{1}{2l(l+1))}$.

Note that the observed CMB angular power spectrum put constraints on the primordial power spectrum $\mathcal{P}_{\mathrm{R}}(k)$ of scalar perturbation, which has been found to be nearly scale invariant. Note that because of Eq.3, the angular power spectrum is customarily plotted as $l(l+1)C_l/(2\pi)$, as it makes the ordinary Sachs-Wolfe part of the spectrum look flat for a scale-invariant primordial power spectrum.

2. Calculate the comoving sound horizon $D^{\mathrm{c}}_{\mathrm{SH}}$ at the time of photo decouple at $z_{\mathrm{dec}} = 1100$. It is given by $D^{\mathrm{c}}_{\mathrm{SH}} = a_0 \int_0^{t_{\mathrm{LSS}}(z_{\mathrm{dec}})} \frac{C_{\mathrm{s}}(t')\mathrm{d}t'}{a(t')}$, where C_{s} is the sound speed, determined by the ratio of the pressure perturbation $\delta P_{\mathrm{b}\gamma}$ and density perturbation $\delta\rho_{\mathrm{b}\gamma}$ of the baryon-photon (bγ) plasma, such that $C_{\mathrm{s}}^2 = \frac{\delta P_{\mathrm{b}\gamma}}{\delta\rho_{\mathrm{b}\gamma}}$. Note that the baryon-photon plasma is coupled through Thomson scatter and Coulomb interaction until photon decouple at the *LSS*.

(2.1) When perturbation of a given scale re-enters the horizon during the radiation era, the relation between the baryonic over-density $\delta_{\rm b}(\equiv \rho_{\rm b}/\bar{\rho}_{\rm b} - 1)$ and radiation over-density $\delta_{\rm r}(\equiv \rho_{\rm r}/\bar{\rho}_{\rm r} - 1)$ is fixed for the strongly coupled baryon-photon plasma, such that $\frac{\delta_{\rm b}}{3} = \frac{\delta_\gamma}{4}$, show that:

$$c_{\rm s}^2 = \frac{c^2}{3(1+R)}, \tag{4}$$

where $R \equiv \frac{\delta\rho_{\rm b}}{\delta\rho_\gamma} = \frac{3}{4}\frac{\bar{\rho}_{\rm b}}{\bar{\rho}_\gamma}$, and c is the speed of light. Hints: density perturbation $\delta\rho_{\rm b\gamma} = \delta\rho_{\rm b} + \delta\rho_\gamma = \delta_{\rm b}\,\bar{\rho}_{\rm b} + \delta_\gamma\,\bar{\rho}_\gamma$; while $\delta P_{\rm b\gamma} \approx \delta P_\gamma = \delta\rho_\gamma/3$, i.e., dominated by the photon pressure perturbation.

(2.2) Through numerical integration, calculate the comoving sound horizon at $z_{\rm dec}$, find out the corresponding $\theta_{\rm SH} = \frac{D^{\rm c}_{\rm SH}(z_{\rm dec})}{D^{\rm c}_{\rm A}(z_{\rm dec})}$, and $l_{\rm SH} = \frac{\pi}{\theta_{\rm SH}}$, where $D^{\rm c}_{\rm A} = a_0/aD_{\rm A}$ is the comoving angular diameter distance. For this calculation, let us assume a flat Universe with $\Omega_{\rm b} = 0.03$, $\Omega_{\rm rad} = 8\times10^{-5}$, $\Omega_{\rm m} = 0.3$, $H_0 = 70\,{\rm km/s/Mpc}$. Note that this sound horizon scale essentially marks the first acoustic peak in the CMB angular power spectrum and also corresponds to the scale of the *baryonic acoustic oscillation* (BAO).