Solutions to Problems 5

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5.1

Cosmological N-body simulations predict that the CDM subhalo populations in galaxies may cause a maximum probability of 25% for strong lensing galaxies to exhibit "flux-ratio anomalies". Current observations have in total observed 5 out of 8 multiply lensed quasar systems with clear evidence of "flux-ratio anomalies" (note, observers know exactly which 5 out of 8 behave so). Using (1) maximum likelihood estimation (assuming Gaussian approximation) and (2) Bayesian analysis assuming a flat prior for your model, can you rule out the proposed theoretical explanation at a confidence level of $\alpha = 5\%$?

Solution: Let:

- $n = 8$ be the total number of observed quasar systems.
- $k = 5$ be the number of systems exhibiting "flux-ratio anomalies".
- p be the true probability that a system exhibits a "flux-ratio anomaly".
- $p_0 = 25\% = 0.25$ be the maximum probability predicted by the theoretical model.

5.1.1

The MLE of p is:

$$\hat{p} = \frac{k}{n} = \frac{5}{8} = 0.625 \tag{5.1.1}$$

We test the null hypothesis $H_0 : p = p_0 = 0.25$ against the alternative hypothesis $H_1 : p > p_0 = 0.25$. Under the null hypothesis, the standard error of $\hat{p}$ is:

$$\sigma_{\hat{p}} = \sqrt{p_0(1-p_0)} = \sqrt{0.25 \times (1-0.25)} \approx 0.433 \tag{5.1.2}$$

We compute the z-score as:

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}/\sqrt{n}} \approx \frac{0.625 - 0.25}{0.433/\sqrt{8}} \approx 2.45 \tag{5.1.3}$$

Using the standard normal distribution, the p-value for a one-tailed test is:

$$\begin{aligned} p\text{−value} &= 1 - \Phi(z) = 1 - \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x \\ &\approx 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2.45} e^{-x^2/2} \approx 0.00715 \end{aligned} \tag{5.1.4}$$

where $\Phi(z)$ is the CDF of the standard normal distribution. Since the p-value is less than $\alpha = 5\% = 0.05$ ($0.00715 < 0.05$), we reject the null hypothesis at the 5% significance level. Therefore, we can **rule out** the proposed theoretical explanation based on MLE.

5.1.2

We assume a flat (uniform) prior for p over the interval [0,1], reflecting no initial preference for any value of p:

$$P_{\text{prior}}(p) = \begin{cases} 1, & 0 \leqslant p \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1.5}$$

The likelihood of observing the data given p is determined by the binomial distribution:

$$\mathcal{L}(k|p,n) = C_n^k p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \tag{5.1.6}$$

Using Bayes' theorem, we combine the prior and the likelihood to obtain the posterior distribution, which reflects our updated beliefs about p after observing the data:

$$\begin{aligned} P_{\text{posterior}}(p|k,n) &= \frac{\mathcal{L}(k|p,n) P_{\text{prior}}(p)}{\displaystyle\int_0^1 \mathcal{L}(k|p,n) P_{\text{prior}}(p) \mathrm{d}p} = \frac{\mathcal{L}(k|p,n)}{\displaystyle\int_0^1 \mathcal{L}(k|p,n) \mathrm{d}p} \\ &= \frac{p^k (1-p)^{n-k}}{\displaystyle\int_0^1 p^k (1-p)^{n-k} \mathrm{d}p} = \frac{p^k (1-p)^{n-k}}{\mathrm{B}(k+1, n-k+1)} \end{aligned} \tag{5.1.7}$$

where $\mathrm{B}(m,n)$ is the Beta function. Our goal is to compute the posterior probability that the true probability p is less than or equal to the maximum theoretical value p_0:

$$\begin{aligned} P(p \leqslant p_0|k,n) &= \int_0^{p_0} P_{\text{posterior}}(p|k,n) \mathrm{d}p = \int_0^{p_0} \frac{p^k (1-p)^{n-k}}{\mathrm{B}(k+1, n-k+1)} \mathrm{d}p \\ &= \int_0^{0.25} \frac{p^5 (1-p)^3}{\mathrm{B}(6,4)} \mathrm{d}p = \frac{9!}{5!3!} \int_0^{0.25} p^5 (1-p)^3 \mathrm{d}p \approx 0.0099945 \end{aligned} \tag{5.1.8}$$

Since the posterior probability that $p \leqslant 0.25$ is less than the significance level $\alpha = 5\% = 0.05$ ($0.0099945 < 0.05$), we conclude that it is unlikely that the true probability p is less than or equal to 25%. Therefore, we have strong evidence to **rule out** the proposed theoretical explanation at the 5% significance level based on Bayesian analysis with a flat prior.

5.2

We want to generate random numbers from the distribution:

$$p(x) \propto e^{-(2x+3\cos^2 x)^2} \tag{5.2.1}$$

Implement this by using a stochastic process constructed with the Metropolis–Hastings algorithm:

1. Start with some random guess x_0 for which $p(x_0)$ is not zero.

2. Make a proposal for x_i in your chain by *adding* a random number drawn uniformly from the interval $[-1,1]$ to x_{i-1}.

3. Accept the proposal with probability:

$$r = \min\left(1, \frac{p(x_i)}{p(x_{i-1})}\right) \tag{5.1.2}$$

   i.e., in the case of acceptance make it the entry x_i in your Monte Carlo chain. Otherwise, adopt the unmodified x_{i-1} as your element x_i. Then proceed with the next element $i+1$.

4. Produce a chain with $N = 10^6$ elements, and make a histogram with bin size $\Delta x = 0.02$ of the entries. In order to verify that they correctly sample the distribution, overplot the given distribution. How many unique points are in your chain?

Solution: Out of $N = 10^6$ samples, approximately 2.36×10^5 are unique points. The histogram is shown in Figure 1.

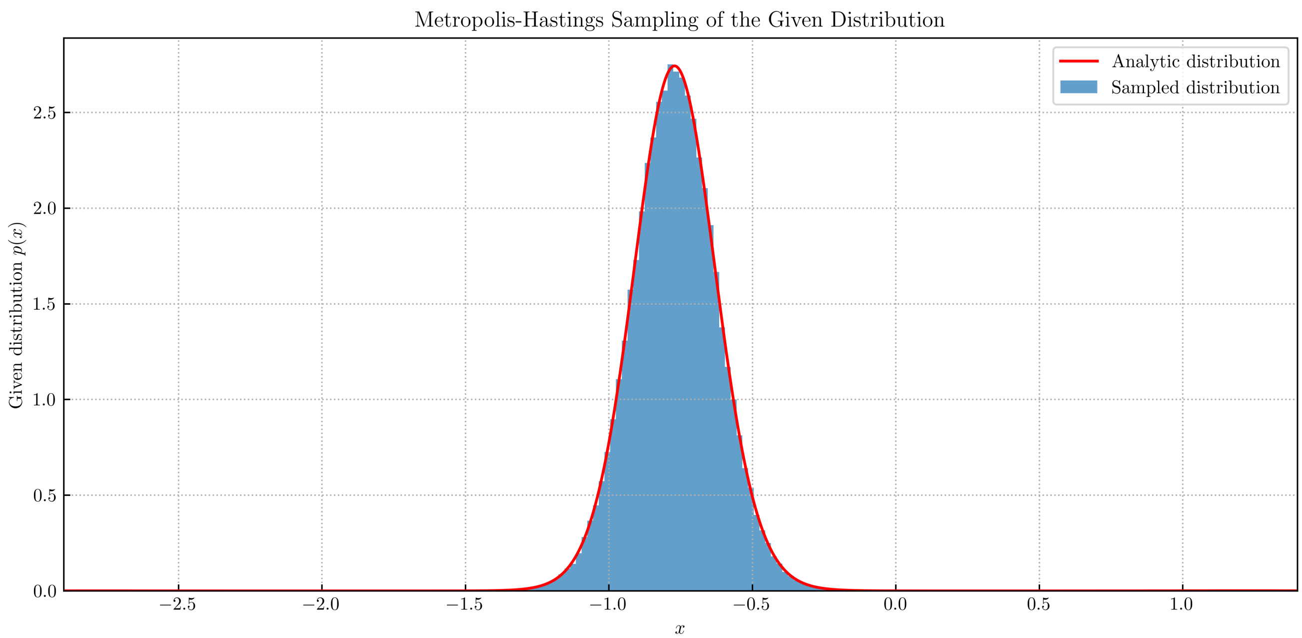

Figure 1: Metropolis-Hastings Sampling of the Given Distribution.

5.3

N-body simulations of cosmic structure formation under the standard CDM cosmology predict that the radial density distribution of a spherical dark matter halo is given by the NFW profile:

$$\rho_\Lambda(r) = \rho_c \frac{\delta_c}{\dfrac{r}{r_s}\left(1 + \dfrac{r}{r_s}\right)^2} = \rho_c \frac{\delta_c r_s^3}{r(r + r_s)^2} \tag{5.3.1}$$

where r_s is a scale radius, δ_c is characteristic (dimensionless) density and $\rho_c = 3H^2/(8\pi G)$ is the critical density of the Universe, determined by the Hubble parameter H, both are functions of the cosmic time.

We introduce a radius r_{200}, within which the mean density is $200\rho_c$, i.e., the mass enclosed within r_{200} is given by $M_{200} = 200\rho_c(4\pi/3)r_{200}^3$. With the help of r_{200}, we can further define a concentration parameter $C = r_{200}/r_s$ and a normalized radius $x = r/r_{200}$. The cumulative mass distribution $M_\Lambda(r)$ can be written as:

$$M_\Lambda(r) = 4\pi\rho_c\delta_c r_s^3 \left[\ln(1 + Cx) + \frac{1}{1 + Cx} - 1\right] \tag{5.3.2}$$

For any given C and r_s, the density normalization is constrained by $M_\Lambda(r_{200}) = M_{200}$. This yields δ_c

given in terms of C:

$$\delta_c = \frac{200}{3} \frac{C^3}{\ln(1+C) + \dfrac{1}{1+C} - 1} \tag{5.3.3}$$

Now use Monte Carlo method to generate $N = 10^6$ particles that follow a NFW distribution inside a radius of 30 kpc, for a dark matter halo with $r_s = 20$ kpc and $C = 10$, living in the present day, for which the Hubble constant is $H_0 = 70$ km s^{-1} Mpc^{-1}. Plot the logarithmic radial density (lg $\left[\rho_\Lambda(r)/(M_\odot \text{ kpc}^{-3})\right]$) distribution of the particle realization as a function of the logarithmic radius (lg$[r/\text{kpc}]$) and compare it to Equation 5.3.1.

Solution: First, we need to calculate δ_c using Equation 5.3.3 for $C = 10$:

$$\begin{aligned}\delta_c &= \frac{200}{3} \frac{C^3}{\ln(1+C) + \dfrac{1}{1+C} - 1} \\ &= \frac{200}{3} \frac{10^3}{\ln(1+10) + \dfrac{1}{1+10} - 1} \approx 44778.66\end{aligned} \tag{5.3.4}$$

Next, we'll convert H_0 and G to appropriate units to express ρ_c in units of $M_\odot$ kpc^{-3}. Convert H_0 to km s^{-1} kpc^{-1}:

$$H_0 = 70 \ \frac{\text{km/s}}{\text{Mpc}} = 70 \ \frac{\text{km/s}}{10^3\text{kpc}} = 0.07 \ \frac{\text{km/s}}{\text{kpc}} \tag{5.3.5}$$

Convert gravitational constant G to kpc (km/s)^2 $M_\odot^{-1}$:

$$\begin{aligned}G &= 6.6743015 \times 10^{-11} \text{m}^3 \text{ kg}^{-1} \text{ s}^{-2} \\ &= 6.6743015 \times 10^{-11} \text{m} \frac{\text{kpc}}{\text{kpc}} (\text{m/s})^2 \frac{(\text{km/s})^2}{(\text{km/s})^2} \frac{1}{\text{kg}} \frac{M_\odot}{M_\odot} \\ &= 6.6743015 \times 10^{-11} \frac{1}{3.0856775814913673 \times 10^{19}} \frac{1}{(10^3)^2} \frac{1.988475 \times 10^{30}}{1} \text{kpc (km/s)}^2 \ M_\odot^{-1} \\ &= 4.301059 \times 10^{-6} \text{ kpc (km/s)}^2 \ M_\odot^{-1}\end{aligned} \tag{5.3.6}$$

The critical density ρ_c is given by:

$$\rho_c = \frac{3H_0^2}{8\pi G} \approx \frac{3 \times (0.07)^2}{8\pi \times 4.301059 \times 10^{-6}} M_\odot \text{ kpc}^{-3} \approx 135.98846 \ M_\odot \text{ kpc}^{-3} \tag{5.3.7}$$

We use the simple inverse transform sampling method. The Monte Carlo sampling via simple inversion method involves the following key steps:

1. Normalize the cumulative mass function $M_\Lambda(r)$ given by Equation 5.3.2 up to $r_{max} = 30$ kpc to obtain the CDF $F(r)$:

$$\begin{aligned}F(r) &= \frac{M_\Lambda(r)}{M_\Lambda(r_{max})} = \frac{\dfrac{1}{1+Cx} - 1 + \ln(1+Cx)}{\dfrac{1}{1+Cx_{max}} - 1 + \ln(1+Cx_{max})} \\ &= \frac{\dfrac{1}{1+r/r_s} - 1 + \ln(1+r/r_s)}{\dfrac{1}{1+r_{max}/r_s} - 1 + \ln(1+r_{max}/r_s)} \approx 3.161648 \left[\frac{r_s}{r+r_s} - 1 + \ln(1+\frac{r}{r_s})\right]\end{aligned} \tag{5.3.8}$$

where $Cx = \dfrac{r_{200}}{r_\mathrm{s}}\dfrac{r}{r_{200}} = \dfrac{r}{r_\mathrm{s}}$. This CDF $F(r)$ maps $r \in [0, 30\ \mathrm{kpc}]$ to $F(r) \in [0, 1]$.

2. Invert the CDF. Since the CDF does not have an analytical inverse, we perform numerical inversion. This involves creating an interpolation function that maps values of $F(r)$ back to r.

3. By generating $N = 10^6$ uniformly distributed random numbers $u \in [0, 1]$ and applying the inverse CDF, we obtain the radii r of the particles that follow the desired NFW distribution.

4. Bin the sampled radii and compute the density in each shell (at each radius):

$$\begin{aligned} &\mathrm{d}M_\Lambda(r) = 4\pi\rho_\Lambda(r)r^2\mathrm{d}r \\ \Longrightarrow\ &\rho_\Lambda(r) = \frac{\mathrm{d}M_\Lambda(r)}{\frac{4}{3}\pi\mathrm{d}(r^3)} = \frac{M_\Lambda(r_\mathrm{max})\mathrm{d}F(r)}{\frac{4}{3}\pi\mathrm{d}(r^3)} \end{aligned} \tag{5.3.9}$$

The Figure 2 is generated.

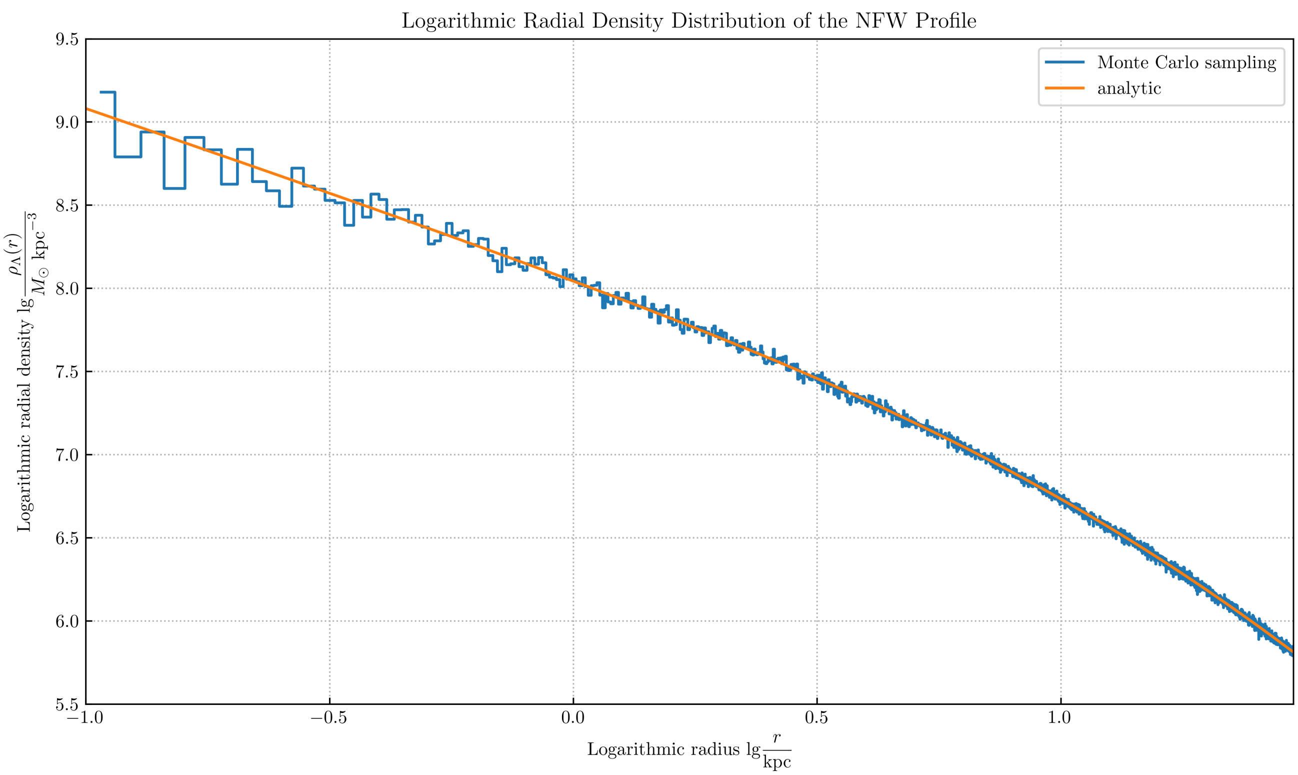

Figure 2: Logarithmic Radial Density Distribution of the NFW Profile