Physics Cosmology
Assignment IV - Early Universe

22 Points in total

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1. (共 9 分)

Ans: IV.1

(1.1)

At $T > 175$ TeV, the degrees of freedom are calculated as follows:

Fermions:

- Quarks: $6\,\text{species} \times 2\,(\text{antiparticles}) \times 3\,(\text{color}) \times 2\,(\text{spin}) = 72$
- Charged leptons: $3\,\text{species} \times 2\,(\text{antiparticles}) \times 2\,(\text{spin}) = 12$
- Uncharged leptons (neutrinos): $3\,\text{species} \times 2\,(\text{antiparticles}) = 6$

Bosons:

- Gluons: $8\,\text{species} \times 2\,(\text{spin}) = 16$
- Photon (γ): $1\,\text{species} \times 2\,(\text{spin}) = 2$
- Weak bosons ($W^{\pm}, Z^0$): $3\,\text{species} \times 3\,(\text{spin}) = 9$
- Higgs boson: 1

Thus, the total degrees of freedom are:

$$g_*(T) = g_B + \frac{7}{8} g_F = 28 + \frac{7}{8} \times 90 = 106.75$$

1 分

(1.2)

At $T \sim 10$ GeV:

- Fermions: 1 quark species out $(90 - 12 = 78)$.

- Bosons: Weak gauge bosons and Higgs boson out $(28-9-1=18)$.

Thus,

$$g_*(T) = 86.25$$

1 分

(1.3)

At $T \sim 150$ MeV:

- Fermions: 2 quark species out, lepton τ out $(78-24-4=50)$.
- Bosons: Remains unchanged before QCD.

Thus,

$$g_*(T) = 61.75$$

1 分

The relativistic sea includes:

Quarks: s, d, u and their antiparticles

Leptons: $e^{\pm}, \mu^{\pm}$ and neutrinos

1 分

(1.4)

After QCD (quark confinement):

- Fermions: All quarks confined, leaving $50-36=14$ degrees of freedom.
- Bosons: Gluons confined. QCD produces 3 mesons $(18-16+3=5)$.

Thus,

$$g_*(T) = 17.25$$

1 分

(1.5)

At $T < 100$ MeV:

- Fermions: lepton μ out $(14-4=10)$
- Bosons: Mesons out $(5-3=2)$.

Thus,

$$g_*(T) = 10.75$$

1 分

The relativistic sea includes electrons, neutrinos and their antiparticles along with photons. 1 分

(1.6)

(1) After neutrinos decoupled (before $T = 1$ eV), neutrinos were no longer in thermal equilibrium with photons. Since they decoupled from other species, they could move freely without interactions.

Neutrinos were still relativistic because the total mass of neutrinos ($\sum m_\nu$) is considered less than 1 eV (Planck 2018). Neutrinos themselves were assumed to be in kinematic equilibrium, so their temperature satisfies:

$$T_\nu = T_\gamma > 1\,\text{eV}, \quad T_\nu \propto a^{-1}.$$

1 分

(2) After e^+e^- annihilation (before $3000\,\text{K}$), the remaining free electrons were still in equilibrium with photons.

Since the interaction rate of electrons with photons ($\Gamma_e \propto n_e\langle\sigma_T\rangle$) was much larger than the expansion rate of the universe (H), frequent interactions between free electrons and photons ensured thermal equilibrium. The photon-to-electron ratio ($\eta \sim 10^9$) indicates each electron was surrounded by a vast number of photons.

The free electrons were non-relativistic because their mass ($m_ec^2 = 0.511\,\text{MeV}$) was much greater than the photon temperature (T_γ) which indicates the kinetic energy of electrons $k_BT_\gamma \ll m_ec^2$.

1 分

2. (共 6 分)

Ans: IV.2

(2.1)

Before e^+e^- annihilation, the relativistic species include:

- Fermions: e^+e^- contribute 4, neutrinos contribute 6.

- Bosons: Photons (γ) contribute 2.

Thus, the total degrees of freedom are:

$$g_{*,0} = 2 + \frac{7}{8} \times (4 + 6) = 10.75.$$

0.5 分

The entropy density is given by:

$$s(T_0) = \frac{2\pi^2}{45} g_{*,0} T_0^3.$$

0.5 分

(2.2)

After e^+e^- annihilation: Set $T \equiv T_{\gamma,1}$ (chose as the equilibrium temperature), the degrees of freedom are:

$$g_{*s,1} = 2 + \frac{7}{8} \times 6 \times \left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^3 = 2 + 5.25 \left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^3$$

0.5 分

The entropy density after annihilation is:

$$s(T_{\gamma,1}) = \frac{2\pi^2}{45} g_{*s,1} T_{\gamma,1}^3.$$

0.5 分

(2.3)

Using entropy conservation:

$$g_{*,0} T_0^3 a_0^3 = g_{*s,1} T_{\gamma,1}^3 a_1^3,$$

and the relation between neutrino temperature and scale factor:

$$\frac{a_0}{a_1} = \frac{T_{\nu,1}}{T_0}.$$

Combined 2 equations above, we can get:

$$\frac{T_{\nu,1}}{T_{\gamma,1}} = \left(\frac{g_{*s,1}}{g_{*,0}}\right)^{1/3}$$

1 分

At the same time, from (2.2):

$$\frac{g_{*s,1}}{g_{*,0}} = \frac{2 + 5.25\left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^3}{10.75} = \left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^3 \quad \Rightarrow \quad \frac{T_{\nu,1}}{T_{\gamma,1}} = \left(\frac{4}{11}\right)^{1/3}.$$

1 分

(2.4)

Before annihilation:

$$g_{*,0}(T_0) = g_{*s,0} = 10.75.$$

0.5 分

After annihilation:

$$g_{*s,1}(T) = 2 + 5.25 \times \left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^4 = 3.363.$$

0.5 分

(2.5)

The ratio of neutrino-to-photon densities after annihilation is:

$$\frac{\rho_{\nu,1}}{\rho_{\gamma,1}} = \frac{g_{*\nu,1}}{g_{*\gamma,1}}\left(\frac{T_{\nu,1}}{T_{\gamma,1}}\right)^4 = \frac{5.25}{2}\left(\frac{4}{11}\right)^{4/3} \approx 0.681.$$

1 分

3. (共 7 分)

Ans: IV.3

(3.1)

In the radiation era, the energy density is:

$$\rho_r(T) = \frac{\pi^2}{30} g_*(T) T^4,$$

and the Hubble parameter is:

$$H(T) = \sqrt{\frac{8\pi G}{3}\rho_r} = \sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30} g_*(T) T^4} \propto T^2, \quad \alpha = 2.$$

In radiation era,

$$a \propto t^{1/2}, \quad H = \frac{\dot{a}}{a} = \frac{1}{2t}.$$

Thus:

$$H \propto T^2 \propto t^{-1}, \quad T \propto t^{-1/2}, \quad \beta = -1/2.$$

The time-temperature relation is:

$$t(T) = \frac{1}{2H(T)} = \sqrt{\frac{45}{16\pi^3 G g_*(T)}} T^{-2}.$$

2 分

(3.2)

At $T_{\rm ND} \sim 0.8\,\text{MeV}$ (before e^+e^- annihilation), $g_*(T_{\rm ND}) = 10.75$. The time at decoupling is:

$$t_{\rm ND} = \sqrt{\frac{45}{16\pi^3 G g_*(T_{\rm ND})}} T_{\rm ND}^{-2} = \frac{2.4}{\sqrt{g_*(T_{\rm ND})}} \left(\frac{T_{\rm ND}}{\text{MeV}}\right)^{-2} \approx 1.14\,\text{s}.$$

After e^+e^- annihilation at $T_d \sim 0.07\,\text{MeV}$, $g_*(T_d) = 3.363$. The time is:

$$t_d = \frac{2.4}{\sqrt{g_*(T_d)}} \left(\frac{T_d}{\text{MeV}}\right)^{-2} \approx 267.09\,\text{s}.$$

Thus, the time interval is:

$$t_d - t_{\rm ND} \approx 266.0\,\text{s}.$$

2 分

Since neutrinos decoupled at $T_\nu \sim 1\,\text{MeV}$, they gradually decayed into protons, resulting in:

$$X_n(t_d) = X_n(t_{\rm ND}) e^{-(t_d - t_{\rm ND})/\tau_n}.$$

Substituting with $t_d - t_{\rm ND}$, we find:

$$X_n(t_d) \approx 12.28\%, \quad X_4 \approx 2X_n(t_d) \approx 24.56\%.$$

1 分

(3.3) In the matter era:

$$a \propto t^{2/3}, \quad H \propto t^{-1}, \quad T \propto a^{-1}.$$

The thermal status is governed by relativistic species (γ, ν), and:

$$T_\gamma \propto a^{-1}.$$

Since $H \propto a^{-3/2},\ a \propto t^{2/3}$, we have:

$$H \propto T_\gamma^{3/2}, \quad T_\gamma \propto t^{-2/3}.$$

Thus, the power-law slopes are:

$$\alpha = 3/2, \quad \beta = -2/3.$$

2 分