Physical Cosmology
- CMB basics I

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1 Gaussian perturbation

If a real field $g(\mathbf{x})$ with zero mean (such as over-density $\delta \equiv \rho/\bar{\rho} - 1$ or temperature fluctuation $\delta T \equiv T/\overline{T} - 1$) is generated by an *isotropic* Gaussian random process (GRP, e.g., inflation), then this field can be simply summarized by merely two statistical properties, i.e., $\langle g(\mathbf{x}) \rangle = 0$ and $\langle g(\mathbf{x})^2 \rangle = S^2$, where S is a number. $\langle \rangle$ is taking average over all realizations of such a GRP.

We can expand $g(\mathbf{x})$ with Fourier series as $g(\mathbf{x}) = \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, with $g_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ being the Fourier coefficient for wave vector $\mathbf{k} = (k_x, k_y, k_z)$, in each dimension $k_n = 2\pi n/L$, $n = 0, \pm 1, \pm 2, ..$, with L being the size of the field. Each realization of the GRP will generate a specific field of $g(\mathbf{x})$ with its corresponding set of Fourier coefficient $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Being a product of an isotropic GRP means that statistically the probability density $p(g_{\mathbf{k}})$ of $\{g_{\mathbf{k}}\}$ follows a Gaussian distribution:

$$\begin{aligned} p(g_{\mathbf{k}}) &= \frac{1}{2\pi S_{\mathbf{k}}^2} \exp\left(-\frac{|g_{\mathbf{k}}^2|}{2S_{\mathbf{k}}^2} \right) \\ &= \frac{1}{\sqrt{2\pi} S_{\mathbf{k}}} \exp\left(-\frac{a_{\mathbf{k}}^2}{2S_{\mathbf{k}}^2} \right) \frac{1}{\sqrt{2\pi} S_{\mathbf{k}}} \exp\left(-\frac{b_{\mathbf{k}}^2}{2S_{\mathbf{k}}^2} \right), \end{aligned} \tag{1}$$

where under the isotropic condition $S_{\mathbf{k}}^2 = S_k^2$ is the variance of the population, where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

One can show that over the many realizations we must have the following:

- $\langle g_{\mathbf{k}} \rangle = \int p(g_{\mathbf{k}})\, g_{\mathbf{k}}\, \mathrm{d}a_{\mathbf{k}} \mathrm{d}b_{\mathbf{k}} = 0$
- $\langle |g_{\mathbf{k}}|^2 \rangle = \int p(g_{\mathbf{k}})\, |g_{\mathbf{k}}|^2\, \mathrm{d}a_{\mathbf{k}} \mathrm{d}b_{\mathbf{k}} = 2S_k^2$
- $\langle g_{\mathbf{k}} g_{\mathbf{k}'}^* \rangle = 2\delta_{\mathbf{k}\mathbf{k}'} S_k^2$ (where $g_{\mathbf{k}}^*$ is the complex conjugate of $g_{\mathbf{k}}$).

One can also show that over the many realizations we must also have:

- $\langle g(\mathbf{x})\rangle = \langle \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\rangle = \Sigma_{\mathbf{k}}\langle g_{\mathbf{k}}\rangle e^{i\mathbf{k}\cdot\mathbf{x}} = 0$
- $\langle g(\mathbf{x})^2\rangle = 2\Sigma_{\mathbf{k}} S_k^2$ (using the reality condition $g_{-\mathbf{k}} = g_{\mathbf{k}}^*$)
- $\langle g(\mathbf{x}) g(\mathbf{x}+\mathbf{r})\rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}}$.

2 Theoretical power spectrum and two-point correlation function

In the case where we can expand the field from a finite size of L^3 to $\pm\infty$, we can then write the Fourier series in the form of Fourier integral with $\Sigma_{\mathbf{k}}(\frac{2\pi}{L})^3 = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{k}$ when $L \to \infty$, we then obtain:

$$g(\mathbf{x}) = \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \approx \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{k}, \tag{2}$$

with $g(\mathbf{k}) = \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{x} = L^3 g_{\mathbf{k}}$ connecting the continuous and discrete Fourier coefficients.

Let us be more specific without loss generality, we take $g(\mathbf{x})$ to be the over-density field $\delta(\mathbf{x})$. By design, $\langle\delta(\mathbf{x})\rangle = 0$. We define power spectrum $P(\mathbf{k})$, $\mathcal{P}(k)$ and two-point correlation function $\xi(\mathbf{r})$ in the following way:

- Power spectrum $P(\mathbf{k})$ has unit of volume L^3, it is defined as:

$$P(\mathbf{k}) \equiv L^3\langle|\delta_{\mathbf{k}}|^2\rangle = 2L^3 S_k^2 \tag{3}$$

$$\langle\delta(\mathbf{x})^2\rangle = \Sigma_{\mathbf{k}} P(\mathbf{k})/L^3 = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} P(\mathbf{k})\,\mathrm{d}^3\mathbf{k} \tag{4}$$

- Define dimensionless power spectrum $\mathcal{P}(k)$ per $\ln k$ interval:

$$\mathcal{P}(k) \equiv k^3 P(k)/(2\pi^2) \tag{5}$$

$$\langle\delta(\mathbf{x})^2\rangle = \int_{k=0}^{k=\infty} \mathcal{P}(k)\,\mathrm{d}\ln k \tag{6}$$

- Correlation function $\xi(\mathbf{r})$ and power spectrum $P(\mathbf{k})$ are connected through the Fourier transformation:

$$\xi(\mathbf{r}) \equiv \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} P(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\,\mathrm{d}^3\mathbf{k} \tag{7}$$

3 Theoretical angular power spectrum and angular correlation function

Now instead of having a $3d$ fluctuation field $\delta(\mathbf{x})$, we have a spherical $2d$ fluctuation field, such as the CMB temperature fluctuation $\delta T(\hat{n} = \{\theta, \phi\})$ in the sky. In this case, instead of having power spectrum and correlation function of the over-density field in $3d$ through the help of plane wave functions and Fourier transform, we will have *angular* power spectrum and *angular* correlation function all defined on a $2d$ sphere with the help of Legendre transform and spherical harmonic transform.

3.1 Legendre polynomials, angular power spectrum and angular correlation function

Legendre polynomials $P_l(x)$ are a complete and orthogonal set of basis functions, defined on $x \in [-1, 1]$ through (Rodriguez formula):

$$P_l(x) = \frac{1}{2^l\, l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}[(x^2-1)^l]. \tag{8}$$

Being complete and orthogonal means:

$$\int_{-1}^{1} P_n(x)P_l(x)\mathrm{d}x = \frac{2}{2l+1}\delta_{nl}. \tag{9}$$

With two unit vectors $\hat{n}$ and $\hat{n}'$ denoting two directions in the sky, where the temperature fluctuations are $\delta T(\hat{n})$ and $\delta T(\hat{n}')$, respectively, we can define an angular correlation function $C(\bar{\theta}) \equiv \langle \delta T(\hat{n})\delta T(\hat{n}')\rangle$, with $\cos\bar{\theta} = \hat{n}\cdot\hat{n}'$. $C(\bar{\theta})$ can be expanded with Legendre polynomials:

$$C(\bar{\theta}) \equiv \langle \delta T(\hat{n})\delta T(\hat{n}')\rangle = \Sigma_l\, \mathcal{C}_l\big(\frac{2l+1}{4\pi}\big)P_l(\hat{n}\cdot\hat{n}'), \tag{10}$$

The coefficient $\mathcal{C}_l$ can be obtained through the inverse transformation:

$$\mathcal{C}_l = \frac{1}{4\pi}\int \mathrm{d}^2\hat{n}\int \mathrm{d}^2\hat{n}' P_l(\hat{n}\cdot\hat{n}')\langle \delta T(\hat{n})\delta T(\hat{n}')\rangle. \tag{11}$$

We can compare this with Eq. (7). Just like the power spectrum $P(\mathbf{k})$ being the Fourier transform of the two-point correlation function $\xi(\mathbf{r})$ of a $3d$ over-density field $\delta(\mathbf{x})$, here $\mathcal{C}_l$ is the angular power spectrum of the temperature fluctuation field $\delta T(\hat{n})$ on a $2d$ sphere. It is connected to the angular correlation function $C(\bar{\theta})$ through the Legendre transformation.

3.2 Spherical Harmonics

Spherical harmonics $Y_{lm}(\theta,\,\phi)$ are a complete and orthogonal set of basis functions defined for $\theta \in [0,\pi)$ and $\phi \in [0, 2\pi)$. We can write out a given function, e.g., $\delta T(\hat{n} = \{\theta, \phi\})$ using a series of spherical harmonics via:

$$\delta T(\theta,\phi) = \Sigma_{lm} a_{lm} Y_{lm}(\theta,\phi). \tag{12}$$

The function $Y_{lm}(\theta,\phi)$ has the form:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{13}$$

with $P_l^m(\cos\theta)$ being the associated Legendre polynomial, defined as:

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}[(x^2-1)^l],\ x = \cos\theta, \tag{14}$$

with $m = 0, \pm 1, \pm 2, ... \pm l$.

Being complete and orthogonal means:

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) = \delta_{mm'}\delta_{ll'}. \tag{15}$$

With this, the coefficient a_{lm} of the full-sky temperature fluctuation field $\delta T(\theta,\phi)$ can be obtained through the inverse transformation:

$$a_{lm} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \delta T(\theta,\phi) Y^*_{lm}(\theta,\phi). \tag{16}$$

We will see that the spherical harmonic coefficients a_{lm} are connected to the Legendre coefficients $\mathcal{C}_l$ in the next section.

- Reality condition: if $\delta T(\theta,\phi)$ is a real function, then $Y_{l\,-m} = (-1)^m\, Y^*_{lm}$, therefore $a_{l\,-m} = (-1)^m\, a^*_{lm}$.
- Closure relation: $\Sigma_m |Y_{lm}(\hat{u})|^2 = (2l+1)/(4\pi)$, independent of m.
- Addition relation: $\Sigma_m Y_{lm}(\hat{u}) Y^*_{lm}(\hat{v}) = (2l+1)/(4\pi)\, P_l(\hat{u}\cdot\hat{v})$.
- Plane wave in spherical harmonic expansion:

  $$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\, \Sigma_{lm}\, i^l\, j_l(kx)\, Y_{lm}(\hat{e}_{\mathbf{k}}) Y^*_{lm}(\hat{e}_{\mathbf{x}}),\ \mathbf{k} = k\hat{e}_{\mathbf{k}},\ \mathbf{x} = x\hat{e}_{\mathbf{x}}, \tag{17}$$

  where $j_l(kx)$ is the spherical Bessel function given by:

  $$j_l(kx) = (-1)^l (kx)^l \left[\frac{\mathrm{d}}{kx\,\mathrm{d}(kx)}\right]^l \frac{\sin(kx)}{kx} \tag{18}$$

- Plane wave in Legendre expansion:

  $$e^{i\mathbf{k}\cdot\mathbf{x}} = \Sigma_l (2l+1)\, i^l\, j_l(kx)\, P_l(\hat{e}_{\mathbf{k}}\cdot\hat{e}_{\mathbf{x}}). \tag{19}$$

3.3 A comparison

In Section 1 we have seen that a $3d$ over-density field $\delta(\mathbf{x})$ generated by an isotropic GRP has following statistical properties in the frequency domain:

- $\langle \delta_{\mathbf{k}} \rangle = 0$, where $\delta_{\mathbf{k}}$ is the discrete Fourier coefficient of $\delta(\mathbf{x})$
- $\langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle = 2\delta_{\mathbf{k}\mathbf{k}'} S_k^2$, where $\delta_{\mathbf{k}\mathbf{k}'}$ is the Kronecker delta, and $S_k^2 \equiv 1/2\langle |\delta_{\mathbf{k}}|^2 \rangle$, is the variance of the Gaussian distribution of $\{\delta_{\mathbf{k}}\}$,

and correspondingly statistical properties in real space:

- $\langle \delta(\mathbf{x}) \rangle = 0$
- $\xi(\mathbf{r}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} P(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}^3\mathbf{k}$ (Eq. 7), i.e., the two-point correlation function $\xi(\mathbf{r})$, or equivalently the power spectrum $P(\mathbf{k})$, contains all the statistical information about the Gaussian perturbation field, and the two quantities are connected through the Fourier transformation.

For a temperature fluctuation field $\delta T(\hat{n} = \{\theta, \phi\})$ in the sky which is generated through an isotropic GRP, one can also show that this field has following statistical properties in the frequency domain:

- $\langle a_{lm} \rangle = 0$, where a_{lm} is the spherical harmonic coefficient of $\delta T(\theta, \phi)$ (see Eq.16). Note that $\langle\rangle$ is again taking average over all realizations from the GRP.
- $\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C_l$, where C_l under the isotropic condition (meaning independent of m) is given by:

$$C_l \equiv \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \Sigma_m \langle |a_{lm}|^2 \rangle. \tag{20}$$

Correspondingly in real space, one can also show that this Gaussian perturbation field has the following statistical properties:

- $\langle \delta T(\theta, \phi) \rangle = 0$
- $C(\bar{\theta}) \equiv \langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle = \Sigma_l C_l \big(\frac{2l+1}{4\pi}\big) P_l(\hat{n} \cdot \hat{n}')$, with $\cos\bar{\theta} = \hat{n} \cdot \hat{n}'$. Now we see that the angular power spectrum $\mathcal{C}_l$ in Eq. (11) is equal to C_l here, related to a_{lm} through Eq. (20). Again, the angular two-point correlation function $C(\bar{\theta})$, or equivalently the angular power spectrum $\mathcal{C}_l$, contains all the statistical information about the Gaussian perturbation field in the $2d$ sphere, and the two quantities are connected through the Legendre transformation.