

Problem 1.1

作业1

① 度规张量 $ds^2 = dr^2 + r^2 d\theta^2$ 描述了 _____ 的几何性质?

- a) 2D sphere
- b) 2D flat

• 请写出: $g_{rr}, g_{r\theta}, g_{\theta\theta}, g^{rr}, g^{r\theta}, g^{\theta\theta}$ 分别表达 (用 r, θ)

• 请写出 Γ_{bc}^a 的全部非零分量, 用 r, θ 表达. 计算求证 $\nabla_a g_{mn} = 0$

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(1) 2D flat /

(2) obviously $g_{rr} = 1, g_{r\theta} = 0, g_{\theta\theta} = r^2 \Rightarrow$ metric tensor $\begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

invert the metric tensor, we get $g^{rr} = 1, g^{r\theta} = 0, g^{\theta\theta} = \frac{1}{r^2}$

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(3) $\bar{\Gamma}_{bc}^a = \frac{1}{2} g^{ad} \left(\frac{\partial g_{cd}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \right)$

for the given metric, $\bar{I}_{bc}^a \neq 0$ only if $a=d$, such that

$$\bar{I}_{bc}^a = \frac{1}{2} g^{aa} \left(\frac{\partial g_{ac}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^a} \right)$$

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if $a=b$, then $\bar{I}_{ac}^a = \frac{1}{2} g^{aa} \frac{\partial g_{aa}}{\partial x^c} \Rightarrow \bar{I}_{\theta r}^\theta = \bar{I}_{r\theta}^\theta = \frac{1}{2r^2} 2r = \frac{1}{r}$

if $b=c$, then $\bar{I}_{bb}^a = \frac{1}{2} g^{aa} \left(- \frac{\partial g_{bb}}{\partial x^a} \right) \Rightarrow \bar{I}_{\theta\theta}^r = -\frac{1}{2} 2r = -r$

$$(4) g_{mn;a} = \frac{\partial g_{mn}}{\partial x^a} - \bar{I}_{am}^b g_{bn} - \bar{I}_{an}^b g_{bm}$$

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only if $m=n$ such that $g_{mn} \neq 0 \Rightarrow g_{mm;a} = \frac{\partial g_{mm}}{\partial x^a} - 2 \bar{I}_{am}^b g_{bm} = \frac{\partial g_{mm}}{\partial x^a} - 2 \bar{I}_{am}^m g_{mm}$

$$\Rightarrow \begin{cases} g_{rr;a} = -2 \bar{I}_{ar}^r g_{rr} = 0 \\ g_{\theta\theta;a} = \frac{\partial g_{\theta\theta}}{\partial x^a} - 2 \bar{I}_{a\theta}^\theta g_{\theta\theta} \Rightarrow g_{\theta\theta;r} = \frac{\partial g_{\theta\theta}}{\partial r} - 2 \bar{I}_{r\theta}^\theta g_{\theta\theta} = 2r - 2 \frac{1}{r} r^2 = 0 \end{cases} \quad \therefore g_{mn;a} = 0$$

Problem 1.2

② 在低速弱引力场条件下，时空近似平直。我们可以假设度规是

Minkowski 时空间度规上加一个微小的扰动项： $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\eta_{\mu\nu} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h_{\mu\nu} \text{ 为小量.} \quad \left(ds^2 = g_{\mu\nu} dx^\mu dx^\nu \right) \quad \left(dt = \sqrt{-ds^2} \right)$$

将测地线方程： $\frac{d^2 x^\mu}{dT^2} + \Gamma_{\sigma\nu}^\mu \frac{dx^\sigma}{dT} \frac{dx^\nu}{dT} = 0$ 在上述条件下近似。

通过与牛顿引力对比，证明：

(1) $h_{00} = -2\phi$, 这里中是牛顿引力势 ($\nabla^2 \phi = 4\pi G \rho$)

(2) 我们对引力场中静钟计时间隔 $dt \approx (1-\phi)dT$,
其中 dT 为固有时间间隔。

(1) Under the assumption of low velocities ($\beta = \frac{v}{c} \ll 1$) and weak fields,
we can make the following approximations:

① $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, neglect high-order terms in $h_{\mu\nu}$ and its derivatives;

② spatial components of velocity $\frac{dx^i}{dt}$ are much smaller than $c (=1)$;

③ use coordinate time t instead of proper time τ

With these assumptions, the geodesic eq simplifies to: $\frac{d^2x^\mu}{dt^2} + \bar{\Gamma}_{\nu\nu}^\mu = 0$

$$\text{here } \bar{\Gamma}_{\nu\nu}^\mu = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\nu\rho}}{\partial x^\nu} + \frac{\partial g_{\nu\rho}}{\partial x^\rho} - \frac{\partial g_{\nu\nu}}{\partial x^\rho} \right) \approx \frac{1}{2} \eta^{\mu\rho} \left(\frac{\partial h_{\nu\rho}}{\partial x^\nu} + \frac{\partial h_{\nu\rho}}{\partial x^\rho} - \frac{\partial h_{\nu\nu}}{\partial x^\rho} \right)$$

$$\Rightarrow \bar{\Gamma}_{00}^\mu = \frac{1}{2} \eta^{\mu\rho} \left(2 \frac{\partial h_{\rho 0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\rho} \right) = \eta^{\mu\rho} \left(\frac{\partial h_{\rho 0}}{\partial x^0} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^\rho} \right)$$

note that $\frac{\partial h_{\rho 0}}{\partial x^0} = \frac{\partial h_{\rho 0}}{\partial t} \frac{1}{c} \ll \frac{\partial h_{00}}{\partial t} \frac{dt}{dx^\rho} = \frac{\partial h_{00}}{\partial x^\rho}$, so:

$$\bar{\Gamma}_{00}^\mu \approx -\frac{1}{2} \eta^{\mu\rho} \frac{\partial h_{00}}{\partial x^\rho} \Rightarrow \begin{cases} \bar{\Gamma}_{00}^0 = -\frac{1}{2} \eta^{0\rho} \frac{\partial h_{00}}{\partial x^\rho} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^0} \\ \bar{\Gamma}_{00}^i = -\frac{1}{2} \eta^{i\rho} \frac{\partial h_{00}}{\partial x^\rho} = -\frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \end{cases}$$

$$\therefore \text{geodesic eq} \left\{ \begin{array}{l} \frac{d^2x^0}{dt^2} + \frac{1}{2} \frac{\partial h_{00}}{\partial x^0} = 0 \quad (\text{identical eq}) \\ \frac{d^2x^i}{dt^2} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \end{array} \right.$$

(2) In Newton grav, the eq of motion is: $\frac{d^2x^i}{dt} = -\frac{\partial \varphi}{\partial x^i}$

Compare this with geodesic eq, we get: $\frac{1}{2} \frac{\partial h_{00}}{\partial x^i} = -\frac{\partial \varphi}{\partial x^i} \Rightarrow h_{00} = -2\varphi + C$

Since h_{00} should vanish at inf where $\varphi \rightarrow 0$, the constant C is zero. So:

$$h_{00} = -2\varphi$$

(3) for a clock at rest in the grav field, $dx^i = 0$: ✓

$$ds^2 = g_{00} dx^0 dx^0 = (g_{00} + h_{00}) dt^2 = -(1+2\varphi) dt^2$$

$$\Rightarrow dt = \sqrt{-ds^2} = \sqrt{(1+2\varphi) dt^2} \approx (1+\varphi) dt \Leftrightarrow dt = (1-\varphi) d\tau$$

Problem 1.3

③ 利用①中极坐标表达考虑时空间度规 $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2$,
 $d\tau = \sqrt{-ds^2}$, 写出测地线方程所有非零项对应的表达.

另: 自由粒子平直空间中 Lagrangian: $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$, 在极坐标下

利用 Euler-Lagrange Equation, 写出自由粒子 Equation of motion.
 考察和理解二者等价.

$$(1) \text{ geodesic eq: } \frac{d^2x^\kappa}{d\tau^2} + \Gamma_{\nu\lambda}^\kappa \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$



non-zero Christoffel connection (from prob 1.1): $\Gamma_{\theta\theta}^r = -r$, $\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r}$

$$\Rightarrow \begin{cases} \frac{d^2r}{d\tau^2} - r \left(\frac{d\theta}{d\tau} \right)^2 = 0 \\ \frac{d^2\theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} = 0 \Rightarrow \frac{d}{d\tau} \left(r^2 \frac{d\theta}{d\tau} \right) = 0 \end{cases}$$

(2) Lagrange eq: $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$ and Lagrange func $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$$\Rightarrow \left\{ \begin{array}{l} q_1 = r, \quad 0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 \Rightarrow \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = 0 \\ q_2 = \theta, \quad 0 = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) \Rightarrow \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = 0 \end{array} \right.$$

Since there is no grav here, $dt = d\tau$, these eqs of motion are equivalent to the geodesic eqs.

$\sqrt{10} \frac{dr}{d\tau} - \frac{d\theta}{d\tau}$