prob 4.1

(1) $g_{fermion} = 90$, $g_{boson} = 28 \Rightarrow g_*(>175\text{ GeV}) = 28 + \frac{7}{8}\times 90 = 106.75$

(2) $g_{fermion} = 90 - 2\times 6 = 78$, $g_{boson} = 16+2 = 18$

$\Rightarrow g_*(\approx 10\text{ GeV}) = 18 + \frac{7}{8}\times 78 = 86.25$

(3) $s, \bar{s}, d, \bar{d}, u, \bar{u}, \mu^-, \mu^+, e^-, e^+, \nu_\tau, \bar{\nu}_\tau, \nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e$

$g_{fermion} = 6\times 6 + 4\times 2 + 6 = 50$, $g_{boson} = 18$

$\Rightarrow g_*(150\text{ MeV} < T \leq 1\text{ GeV}) = 18 + \frac{7}{8}\times 50 = 61.75$

(4) $g_{fermion} = 4\times 2 + 6 = 14$, $g_{boson} = 2+3 = 5$

$\Rightarrow g_*(\approx 150\text{ MeV}) = 5 + \frac{7}{8}\times 14 = 17.25$

(5) $e^-, e^+, \nu_\tau, \bar{\nu}_\tau, \nu_\mu, \bar{\nu}_\mu, \nu_e, \bar{\nu}_e, \gamma$

$g_{fermion} = 2\times2+6 = 10, \; g_{boson} = 2$

$\Rightarrow g_*(< 100\,\text{MeV}) = 2 + \frac{7}{8}\times 10 = 10.75$

(6) After decoupling, neutrinos were no longer in thermal equilibrium with photons. They continued to cool independently and stayed relativistic until much later in the Universe's evolution given their small masses.

Free electrons remained in thermal equilibrium with photons due to Thomson scattering. Electrons were non-relativistic because their ress mass energy (≈ 511 keV) was much larger than the thermal energy at temperatures below 3000K.

prob 4.2

(1) $g_*(T_0) = 10.75$

$$s(T_0) = \frac{2\pi^2}{45} g_*(T_0) T_0^3 = \frac{43\pi^2}{90} T_0^3$$

(2) $T = T_\gamma$

$$g_s^*(T) = 2 + \frac{7}{8} \times 6 \left(\frac{T_\nu}{T}\right)^3 = 2 + 5.25 \frac{T_\nu^3}{T^3}$$

$$s(T) = \frac{2\pi^2}{45} g_s^*(T) T^3 = \frac{2\pi^2}{45} (2T^3 + 5.25 T_\nu^3)$$

(3) $s(T_0) a_0^3 = s(T_\gamma) a_1^3 \Rightarrow g^*(T_0) T_0^3 a_0^3 = g_s^*(T_\gamma) T_\gamma^3 a_1^3$

$T_\nu \propto a^{-1} \Rightarrow T_0 a_0 = T_\nu a_1$

$$\Rightarrow g^*(T_0) T_\nu^3 a_1^3 = g_s^*(T_\gamma) T_\gamma^3 a_1^3 \Rightarrow \frac{T_\nu}{T_\gamma} = \sqrt[3]{\frac{g_s^*(T_\gamma)}{g^*(T_0)}} \text{ and } 10.75 = 2\frac{T_\gamma^3}{T_\nu^3} + 5.25$$

$$\Rightarrow \frac{T_\nu}{T_\gamma} = \sqrt[3]{\frac{4}{11}} = \sqrt[3]{\frac{g_s^*(T_\gamma)}{g^*(T_0)}}$$

(4) before: $g^*(T_0) = 10.75$

after: $g^*(T) = 2 + 5.25\frac{T_\nu^4}{T^4} \approx 3.36$

(5)
$$\left.\begin{aligned} \rho_\nu &= 3\cdot\frac{7}{8}\,6\,T_\nu^4 \\ \rho_\gamma &= 6\,T_\gamma^4 \end{aligned}\right\} \Rightarrow \frac{\rho_\nu}{\rho_\gamma} = \frac{21}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 \approx 0.681$$

prob 4.3

(1) in the radiation era: $T_\gamma = T$

$$\left.\begin{aligned} g^*(T) &= \frac{30\rho(T)}{\pi^2 T^4} \\ \text{Friedmann eq. } H^2(T) &= \frac{8\pi G}{3}\rho(T) \end{aligned}\right\} \Rightarrow H^2(T) = \frac{8\pi G}{3}\frac{\pi^2}{30} g^*(T) T^4$$

$$\Rightarrow H(T) = \sqrt{\frac{4\pi^3 G}{45} g^*(T)}\, T^2 \propto T^2 \Rightarrow \alpha = 2$$

Since $g^*(T)$ changes slowly for most of the time, we can approximate $g^*(T) = \text{Constant}$.

$$\left.\begin{aligned} T &\propto a^{-1} \\ H &\propto T^2 \end{aligned}\right\} \Rightarrow H \propto a^{-2} \Rightarrow H(t) = \frac{H(t_i) a_i^2}{a^2} = \frac{da/dt}{a}$$

$$\Rightarrow dt = \frac{1}{H(t_i) a_i^2}\, a\, da \Rightarrow t = \frac{1}{H(t)} \int_0^a \frac{x\,dx}{a^2} = \frac{1}{2H(t)} = \sqrt{\frac{45}{16\pi^3 G g^*(T)}}\, T^{-2}$$

$$\Rightarrow T(t) \propto t^{-\frac{1}{2}} \Rightarrow \beta = -\frac{1}{2}$$

(2) $g^*(T) \simeq 3.36$

$$t_{ND} = \frac{2.4}{\sqrt{g^*(T)}} \left(\frac{T_{ND}}{\mathrm{MeV}}\right)^{-2} s \simeq 2.046\, s$$

$$t_d = \frac{2.4}{\sqrt{g^*(T)}} \left(\frac{T_d}{\mathrm{MeV}}\right)^{-2} s \simeq 267.2\, s$$

$$X_n(t_d) = X_n(t_{ND})\, e^{-(t_d - t_{ND})/\tau_n} \simeq 0.1229$$

$$X_4 \simeq 2 X_n(t_d) \simeq 0.2458$$

(3) in the matter era, $\rho \propto a^{-3}$

Friedmann eq. $H^2 = \frac{8\pi G}{3}\rho$

$$\left.\begin{array}{l} \rho \propto a^{-3} \\ H^2 = \frac{8\pi G}{3}\rho \end{array}\right\} \Rightarrow \left.\begin{array}{l} H \propto a^{-\frac{3}{2}} \\ T \propto a^{-1} \end{array}\right\} \Rightarrow H \propto T^{\frac{3}{2}}, \alpha = \frac{3}{2}$$

$$H \propto a^{-\frac{3}{2}} \Rightarrow t = \frac{1}{H(t)} \int_0^a \frac{x^{\frac{1}{2}} dx}{a^{\frac{3}{2}}} = \frac{2}{3H(t)} \propto T^{-\frac{3}{2}}$$

$$\Rightarrow T \propto t^{-\frac{2}{3}}, \ \beta = -\frac{2}{3}$$

A higher baryon-to-photon ratio η leads to a higher primordial helium abundance X_4, because nucleosynthesis proceeds more efficiently and more neutrons are captured into helium nuclei before they decay.