Physical Cosmology
Homework 2 - FLRW, age and distance

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1. Assuming the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ with convention $\eta_{00} = -1$, and a perfect fluid has an energy momentum tensor $T^{\mu\nu}$ taking the form

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where ρ and $P = w\rho$ are the density and pressure of the fluid. For non-relativistic matter, radiation and dark energy, $w = 0$, $1/3$, -1, respectively. Please calculate $T \equiv g_{\mu\nu}T^{\mu\nu}$ and $S_{00} = T_{00} - 1/2 g_{00}\, T$ for the three species.

2. Assuming a flat space geometry (K=0), the FLRW metric takes the form: $\mathrm{d}\tau^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. Please calculate all the components of the Christoffel connection $\Gamma^{\alpha}_{\mu\nu}$, show that: $\Gamma^{0}_{ij} = a\dot{a}\delta_{ij}$, $\Gamma^{i}_{0j} = \dot{a}/a\delta_{ij}$, $\Gamma^{i}_{jl} = 0$, and $R_{00} = 3\ddot{a}/a$, $R_{ij} = -(a\ddot{a} + 2\dot{a}^2)\delta_{ij}$.

3. The Friedmann equation derived from the GR field equation ultimately determines the evolution of the scale factor $a(t)$. Now let us assume the universe is dominated by a single component i below in a flat space geometry (K=0). Through the Friedmann equation $(\dot{a}/a)^2 = 8\pi G/3\rho_i(a)$ and local energy conservation $\dot{\rho}_i + 3\dot{a}/a(1 + w_i)\rho_i = 0$, work out the time dependence of a(t), the particle horizon distance $D_{\mathrm{h}}(t)$ at $a(t)$, and the age of the Universe today t_0, show that:
(1) Non-relativistic matter: $a(t) \propto t^{2/3}$, $D^{\mathrm{m}}_{\mathrm{h}}(t) = 2c/H = 3ct$, $t_0 = 2/(3H_0)$;
(2) Radiation: $a(t) \propto t^{1/2}$, $D^{\mathrm{rad}}_{\mathrm{h}}(t) = c/H = 2ct$, $t_0 = 1/(2H_0)$;
(3) Dark energy (cosmological constant): $a(t) \propto \exp(H_0 t)$;
where c is the speed of light, H_0 is the Hubble constant today.

4. Assuming $\Omega_{\text{rad},0} = 8 \times 10^{-5}$, $\Omega_{\text{m},0} = 0.25 - \Omega_{\text{rad},0}$, $\Omega_{\Lambda,0} = 1 - \Omega_{\text{m},0}$, $H_0 = 70\,\text{km/s/Mpc}$, using numerical integration to calculate:
(1) Redshift and age of the universe when $\rho_{\text{matter}} = \rho_{\text{rad}}$.
(2) Redshift and age of the universe when $\rho_{\text{matter}} = \rho_{\Lambda}$.
(3.1) Age of the universe when $z = 30$ and $z = 20$ (first stars form);
(3.2) Age of the universe when $z = 6$ (time about reionization);
(3.3) Age of the universe when $z = 2$ ("cosmic noon");
(3.4) Age of the universe when $z = 1$, was the Sun born by then?
(3.5) Age of the universe when $z = 0.1$, had dinosaurs come to exist?