5.1 $\vec{e}_x = \hat{x}$

(1) $$a_{lm} = \int d\Omega \tilde{Y}^*_{lm}(\hat{x}) f(x_{LSS}\hat{x})$$

$$= \sum_{\vec{k}} \int d\Omega \tilde{Y}^*_{lm}(\hat{x}) f_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_{LSS}}$$

$$= 4\pi \sum_{\vec{k}} \sum_{l'm'} \int d\Omega \tilde{Y}^*_{lm}(\hat{x}) f_{\vec{k}} i^{l'} j_{l'}(kx_{LSS}) \tilde{Y}^*_{l'm'}(\hat{k}) \tilde{Y}_{l'm'}(\hat{x})$$

notice the orthonormality of the spherical harmonics:

$$\int d\Omega \tilde{Y}_{lm}(\theta,\varphi) \tilde{Y}^*_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'} \Rightarrow \int d\Omega \tilde{Y}^*_{lm}(\hat{x}) \tilde{Y}_{l'm'}(\hat{x}) = \delta_{ll'}\delta_{mm'}$$

$$\therefore a_{lm} = 4\pi \sum_{\vec{k}} f_{\vec{k}} i^l j_l(kx_{LSS}) \tilde{Y}^*_{lm}(\hat{k}) = 4\pi i^l \sum_{\vec{k}} f_{\vec{k}} j_l(kx_{LSS}) \tilde{Y}^*_{lm}(\hat{k})$$

(2) $$f(x_{LSS}\hat{x}) = \frac{\delta T}{T_0}(\hat{x}) = -\frac{1}{5} R(x_{LSS}\hat{x}) \Rightarrow f_{\vec{k}} = -\frac{1}{5} R_{\vec{k}}$$

$$\therefore a_{\ell m} = 4\pi i^{\ell} \sum_{\vec{k}} \left(-\frac{1}{5} R_{\vec{k}}\right) j_\ell(k x_{LSS}) \tilde{Y}^*_{\ell m}(\hat{k}) = \sum_{\vec{k}} b_{\vec{k}} R_{\vec{k}}$$

$$\Rightarrow b_{\vec{k}} = -\frac{4}{5}\pi i^{\ell} j_\ell(k x_{LSS}) \tilde{Y}^*_{\ell m}(\hat{k})$$

(3) $$\tilde{a}_{\ell m} = \sum_{k} \tilde{b}_{\vec{k}} \tilde{R}_{\vec{k}}, \quad \tilde{b}_{\vec{k}} = -\frac{4}{5}\pi i^{\ell} j_\ell(k x_{LSS}) \tilde{Y}^*_{\ell m}(\hat{k})$$

$$\Rightarrow \langle |\tilde{a}_{\ell m}|^2 \rangle = \langle |\sum_{\vec{k}} b_{\vec{k}} R_{\vec{k}}|^2 \rangle = \sum_{\vec{k}} \sum_{\vec{k}'} \tilde{b}_{\vec{k}} \tilde{b}^*_{\vec{k}'} \langle \tilde{R}_{\vec{k}} \tilde{R}^*_{\vec{k}'} \rangle = \sum_{\vec{k}} \sum_{\vec{k}'} \tilde{b}_{\vec{k}} \tilde{b}^*_{\vec{k}'} \delta_{\vec{k}\vec{k}'} \langle |\tilde{R}_{\vec{k}}|^2 \rangle$$

$$= \sum_{\vec{k}} |\tilde{b}_{\vec{k}}|^2 \langle |\tilde{R}_{\vec{k}}|^2 \rangle = \frac{2\pi^2}{V} \sum_{\vec{k}} \frac{1}{k^3} P_R(k) |\tilde{b}_{\vec{k}}|^2$$

$$\Rightarrow C_\ell = \frac{1}{2\ell+1} \sum_m \langle |a_{\ell m}|^2 \rangle$$

$$= \frac{16\pi^2}{25} \frac{1}{2\ell+1} \sum_m \frac{2\pi^2}{V} \sum_{\vec{k}} \frac{1}{k^3} P_R(k) j_\ell^2(k x_{LSS}) |\tilde{Y}^*_{\ell m}(\hat{k})|^2$$

$$= \frac{8\pi^3}{25V} \sum_{\vec{k}} \frac{1}{k^3} P_R(k) j_\ell^2(k x_{LSS})$$

Replacing the sum with an integral, we get

$$C_\ell = \frac{1}{25}\int \frac{d^3k}{k^3} P_R(k) j_\ell^2(k\chi_{LSS}) = \frac{4\pi}{25}\int_0^{+\infty} \frac{dk}{k} P_R(k) j_\ell^2(k\chi_{LSS})$$

$$= A^2 \frac{4\pi}{25}\int_0^{+\infty} \frac{dk}{k} j_\ell^2(k\chi_{LSS}) = \frac{A^2}{25}\frac{2\pi}{\ell(\ell+1)}$$

5.2 (1) $$C_s^2 = \frac{\delta P_{b\gamma}}{\delta \rho_{b\gamma}} \approx \frac{\delta P_\gamma}{\delta \rho_{b\gamma}} = \frac{1}{3}\frac{\delta\rho_\gamma}{\delta\rho_\gamma + \delta\rho_b} = \frac{1}{3}\frac{1}{1+\frac{\delta\rho_b}{\delta\rho_\gamma}} = \frac{1}{3(1+R)} = \frac{c^2}{3(1+R)}$$

(2) $$R = \frac{3\bar\rho_b}{4\bar\rho_\gamma} = \frac{3\Omega_b}{4\Omega_\gamma} = \frac{3\Omega_{b0}a^{-3}}{4\Omega_{\gamma 0}a^{-4}} = \frac{3\Omega_{b0}}{4\Omega_{\gamma 0}}\frac{1}{1+z} = \frac{281.25}{1+z}$$

$$D_{SH}^c = a_0\int_0^{t_{LSS}(z_{dec})} \frac{c_s\,dt}{a} = \int_0^{t_{LSS}(z_{dec})} c_s \frac{dt}{da}\frac{da}{a} = \int_0^{\frac{1}{1+z_{dec}}} \frac{c_s}{\dot a}\frac{da}{a}$$

$$= \int_{z_{dec}}^{+\infty} \frac{c_s}{H} dz = \int_{1100}^{+\infty} \frac{c}{\sqrt{3(1+R)}} \frac{1}{H_0 \sqrt{\Omega_\gamma (1+z)^4 + \Omega_m (1+z)^3 + (1-\Omega_\gamma - \Omega_m)}} dz$$

$$\approx 1.539 \times 10^5 \text{ Mpc}$$

$$D_A^c(z_{dec}) = \int_0^{z_{dec}} \frac{dz}{H(z)} = \int_0^{1100} \frac{c\,dz}{H_0 \sqrt{\Omega_\gamma (1+z)^4 + \Omega_m (1+z)^3 + (1-\Omega_\gamma - \Omega_m)}}$$

$$\approx 1.362 \times 10^7 \text{ Mpc}$$

$$\theta_{SH} = \frac{D_{SH}^c(z_{dec})}{D_A^c(z_{dec})} \approx 0.0113$$

$$l_{SH} = \frac{\pi}{\theta_{SH}} \approx 278.1$$