# Physical Cosmology
# - Cosmological distances and cosmic times

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## 1 Cosmological age and look-back time

Recall that the Hubble parameter $H(a)$ is given by:

$$H^2(a) = H_0^2 \left( \Omega_{\mathrm{m},0} \left( \frac{a}{a_0} \right)^{-3} + \Omega_{\mathrm{rad},0} \left( \frac{a}{a_0} \right)^{-4} + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0} \left( \frac{a}{a_0} \right)^{-2} \right). \tag{1}$$

Now let us define $x \equiv a/a_0 = 1/(1+z)$. Therefore $H \equiv \dot{a}/a = \mathrm{d}a/(a\mathrm{d}t) = \mathrm{d}x/(x\mathrm{d}t) = -\mathrm{d}z/((1+z)\mathrm{d}t)$. We then have:

$$\mathrm{d}t = \frac{\mathrm{d}x}{xH(x)} = \frac{\mathrm{d}x}{H_0 x E(x)} \tag{2}$$

$$= -\frac{\mathrm{d}z}{(1+z)H(z)} = -\frac{\mathrm{d}z}{H_0(1+z)E(x)}, \tag{3}$$

where $E(x) \equiv \sqrt{\Omega_{\mathrm{m},0} x^{-3} + \Omega_{\mathrm{rad},0} x^{-4} + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0} x^{-2}}$, and $E(z) \equiv \sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0}(1+z)^2}$.

With this, the age of the Universe for an event at redshift $z$ is given by:

$$t(z) = \int_z^\infty \frac{\mathrm{d}z}{(1+z)H(z)} = \int_0^x \frac{\mathrm{d}x}{xH(x)}. \tag{4}$$

While the *look-back time* of this event is given by:

$$t_{\mathrm{LB}}(z) = \int_0^z \frac{\mathrm{d}z}{(1+z)H(z)} = \int_x^1 \frac{\mathrm{d}x}{xH(x)}. \tag{5}$$

## 2 Cosmological distances

For light, $d\tau^2 = 0$. A *radially* moving light ray thus satisfies $c^2 dt^2 - a^2(t) d\chi^2 = 0$ or $c^2 dt^2 - a^2(t)(1 - Kr^2)^{-1} dr^2 = 0$, where $r = S_K(\chi)$. In particular, for a light ray which travels towards us, $-d\chi = -dr/\sqrt{1 - Kr^2} = c/a(t) dt$. Therefore a comoving coordinate distance $\chi_e$ (and $r_e$) between us and a light beam which emits at time $t_e$ and arrives at us at time $t_0$ is then give by:

$$\chi_e = \int_0^{\chi_e} d\chi' = \int_0^{r_e} \frac{dr'}{\sqrt{1 - Kr'^2}} = \int_{t_e}^{t_0} \frac{c\, dt}{a(t)}. \tag{6}$$

Several important cosmological distances are defined based on the comoving coordinate distances $\chi_e$ and $r_e = S_K(\chi_e)$.

### 2.1 Line-of-sight comoving distance

When the comoving *coordinate* distance $\chi(z)$ is rescaled by today's scale factor $a_0$, this gives the *line-of-sight comoving* distance $D_{\rm com}^{\rm los}(z)$:

$$D_{\rm com}^{\rm los}(z) \equiv a_0 \chi(z) = a_0 \int_{t(z)}^{t_0} \frac{c\, dt'}{a(t')} = \frac{c}{H_0} \int_0^z \frac{dz'}{E(z')} = \frac{c}{H_0} \int_x^1 \frac{dx'}{x'^2 E(x')}. \tag{7}$$

### 2.2 Transverse comoving distance

When the comoving *coordinate* distance $r(z) = S_K(\chi(z))$ is rescaled by $a_0$, this gives the *transverse comoving* distance $D_{\rm com}^{\rm trans}(z)$:

$$D_{\rm com}^{\rm trans}(z) \equiv a_0 r(z) = a_0 S_K\left(\frac{c}{a_0 H_0} \int_0^z \frac{dz'}{E(z')}\right). \tag{8}$$

Using definition of $\Omega_{\rm K,0} = -Kc^2/(a_0 H_0)^2$ to re-write $a_0$, Eq. (8) is given by:

$$= \frac{c}{H_0} \begin{cases} \sqrt{\Omega_{K,0}}^{-1} \sinh\left[\sqrt{\Omega_{K,0}} \int_0^z \frac{dz'}{E(z')}\right] & \text{if } K = -1 \\ \int_0^z \frac{dz'}{E(z')} & \text{if } K = 0 \\ \sqrt{|\Omega_{K,0}|}^{-1} \sin\left[\sqrt{|\Omega_{K,0}|} \int_0^z \frac{dz'}{E(z')}\right] & \text{if } K = 1 \end{cases}$$

One can further use the fact that $\sinh(ix) = i \sin(x)$ to write:

$$D_{\rm com}^{\rm trans}(z) = \frac{c}{H_0 \sqrt{\Omega_{K,0}}} \sinh\left[\sqrt{\Omega_{K,0}} \int_0^z \frac{dz'}{E(z')}\right]. \tag{9}$$

### 2.3 Proper distance

An object that emits a light ray at redshift $z$ towards us and arrives at us today (i.e., the world line of this object intercepts our today's past light cone at $z$) has a proper distance $D_{\rm prop}(z)$, which is the distance measured at the past time $t(z)$ corresponding to $z$, and given by:

$$D_{\rm prop}(z) \equiv a(z)\chi(z) = a(t)\int_{t(z)}^{t_0} \frac{c\,{\rm d}t'}{a(t')} \tag{10}$$

$$= \frac{c}{H_0(1+z)}\int_0^z \frac{{\rm d}z'}{E(z')} \tag{11}$$

$$\left( = \frac{c\,x}{H_0}\int_x^1 \frac{{\rm d}x'}{x'^2 E(x')} \right). \tag{12}$$

We shall note that the proper distance is the comoving *coordinate* distance $\chi(z)$ rescaled by $a(z)$ at time of light emission instead of today's scale factor $a_0$. In particular, in the case of a flat space geometry (K=0), we have:

$$D_{\rm prop}(z) = \frac{D_{\rm com}^{\rm los}(z)}{1+z} = \frac{D_{\rm com}^{\rm trans}(z)}{1+z} = \frac{c}{H_0(1+z)}\int_0^z \frac{{\rm d}z'}{E(z')}. \tag{13}$$

### 2.4 Angular diameter distance

An object at redshift $z$ has an angular diameter $D_{\rm AD}(z)$, which is a distance defined using angular size argument ($\theta = \Delta X/X$) assuming an Euclidean geometry (i.e., a local Universe approximation). $D_{\rm AD}(z)$ is essentially the comoving *coordinate* distance $r(z)$ rescaled by $a(z)$ at time of light emission (instead of today's scale factor $a_0$), and given by:

$$D_{\rm AD}(z) \equiv a(z)r(z) = \frac{a_0 r(z)}{1+z} = \frac{D_{\rm com}^{\rm trans}(z)}{1+z}, \tag{14}$$

which in a flat universe (K=0) equals to proper distance $D_{\rm prop}(z)$ (Eq. 13). Note (1) this distance is used when measurements involve angular sizes, such as gravitational lensing; (2) both $D_{\rm DA}$ and $D_{\rm prop}$ can be non-increasing (or even decreasing) with $z$ increasing under certain cosmological parameters, essentially reflecting the fact that the universe was smaller at earlier times.

### 2.5 Luminosity distance

An object at redshift $z$ has a luminosity distance $D_{\rm lum}(z)$, which is a distance defined using surface brightness argument ($s = L/(4\pi D^2)$) assuming an Euclidean geometry (i.e., a local Universe approximation). It is given by:

$$D_{\rm lum}(z) \equiv (1+z) a_0 r(z) = (1+z) D_{\rm com}^{\rm trans}. \tag{15}$$

Note (1) this distance is used when measurements essentially involve luminosities, such as SNIa distances; (2) it is important to see that:

$$D_{\rm AD}(z) < D_{\rm com}^{\rm trans}(z) < D_{\rm lum}(z). \tag{16}$$

## 2.6 Particle horizon

At any given time $t$, a particle/event/influence can maximally travel to a distance given by:

$$a(t) \int_0^t \frac{c\,\mathrm{d}t'}{a(t')} = \frac{c}{H_0(1+z)} \int_z^\infty \frac{\mathrm{d}z'}{E(z')} = \frac{c\,x}{H_0} \int_0^x \frac{\mathrm{d}x'}{x'^2 E(x')}. \tag{17}$$

This distance is the *particle horizon* at time $t$. The particle horizon today is $\sim 14\,\mathrm{Gyr}$, which is the maximal distance a particle can travel since $t = 0$.

## 2.7 Event horizon

The *event horizon* at any given time $t$ is a proper distance $D_{\rm EH}(t)$ (to us) of an object , the photons which emit then will arrive at us at time $t \to \infty$ (i.e., this object is on our light cone joining us at $t \to \infty$). This means that objects at and beyond $D_{\rm EH}(t)$ (at given time $t$) will never be seen by us. This distance is given by $D_{\rm EH}(t) = a(t) \int_t^\infty \frac{c\,\mathrm{d}t'}{a(t')}$. One can show that if a universe is dominated by a cosmological constant with energy density $\rho_\Lambda > 0$, then the event horizon has a finite value proportional to $1/\sqrt{\rho_\Lambda}$. The smaller $\rho_\Lambda$ is, the bigger the event horizon is.

## 2.8 Hubble sphere

At any given time $t$, there is a proper distance $D_{\rm HS}(t)$ (to us) where galaxies recede from us (at that time) with the speed of light, i.e., $c = H(t) D_{\rm HS}(t)$. These distances (at all times) compose the *Hubble sphere*. Note that all light cones that join us in the past, present or future satisfy $\dot{D}(t) = H(t)D(t) - c$, where $D(t) = a(t)\chi(t)$ is the proper distance of photons which are at comoving coordinate distance $\chi(t)$ from us at time $t$. Any one of our light cones always first increases with time (i.e., $\dot{D}(t) > 0$, meaning the Hubble flow is moving faster than the speed of light), until reaching a "turn-around" point where $\dot{D}(t) = 0$ (i.e., $H(t)D(t) = c$), and then starts decreasing (i.e.,

$\dot{D}(t) < 0$, meaning the Hubble flow is moving slower than the speed of light), and eventually reaching us at $t > 0$. We can see that the Hubble sphere is essentially composed of all light-cone distances $D(t)$ at time of "turn-around". In a universe for which the cosmological constant $\Lambda > 0$, the Hubble sphere converges to the event horizon at $t \to \infty$. Question: *can we see galaxies that recede from us with the Hubble flow at speed of light?*