Physics Cosmology
Assignment III - Inflation

19 Points in total

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1. (共 3 分)

Ans: III.1

$$|\Omega_{\text{tot}} - 1| = \frac{|K|}{a^2H^2} = |\Omega_k|$$

For matter dominated universe, assuming $H_0 = 70$ km/s/Mpc:

$$a \propto t^{\frac{2}{3}}, \quad H = \frac{\dot{a}}{a} \propto t^{-1}, \quad \text{Present day cosmic age: } t_0 = \frac{2}{3H_0} = 2.94 \times 10^{17}\text{s}$$

$$|\Omega_k| = \frac{|K|}{a^2H^2} \propto t^{\frac{2}{3}} \Rightarrow \frac{|\Omega_{k,1\text{s}}|}{|\Omega_{k,0}|} = \left(\frac{1\text{s}}{t_0}\right)^{\frac{2}{3}}$$

1 分

So the deviation of total matter density from the critical density at 1s:

$$|\Omega_{k,1\text{s}}| = |\Omega_{\text{tot},1\text{s}} - 1| = |\Omega_{k,0}| \left(\frac{1\text{s}}{t_0}\right)^{\frac{2}{3}} \leq 2.26 \times 10^{-12}$$

1 分

Consider the time evolution of $|\Omega_{\text{tot}} - 1|$:

$$\frac{d}{dt}|\Omega_{\text{tot}} - 1| = |K|\frac{d}{dt}\left(\frac{1}{a^2H^2}\right) = |K|\frac{d}{dt}\left(\frac{1}{\dot{a}^2}\right) = -\frac{2|K|}{\dot{a}^3}\ddot{a}$$

Since for an expanding universe, $\dot{a} > 0$, if we want $|\Omega_{\text{tot}} - 1|$ to decrease, we need $\ddot{a} > 0$, which indicates an accelerating expansion of the universe. Then $|\Omega_{\text{tot},1\text{s}} - 1|$ don't need to be such an extremely small value. 1 分

2. (共 3 分)

Ans: III.2

For a matter diminated universe, the comoving (particle) horizon distance is:

$$D_h^c = \int_0^t \frac{cdt'}{a(t')} = \frac{c}{H_0}\int_z^\infty \frac{dz'}{E(z')} = \frac{c}{H_0}\int_z^\infty \frac{dz'}{(1+z')^{3/2}} = \frac{2c}{H_0\sqrt{1+z}}$$

So ratio between the comoving horizon today and that at $z = 1100$ is:

$$\frac{D_{h,0}^c}{D_{h,1100}^c} = \frac{\sqrt{1+1100}}{\sqrt{1+0}} \approx 33.2$$

1 分

This indicates that at $z = 1100$, The region with causal connections is much smaller than the scale of the uniform CMB we see across the entire sky today.

1 分

(Alternative solution, *supplementary material for particle horizon estimates)

In order to research into the behaviour of particle horizon integral, we can set up a starting point of time t_i:

$$I(t) = \int_t^{t_0} \frac{cdt'}{a(t')} \to D_h^c,\ \text{When } t_i \to 0$$

The integral can be rewritten in terms of scale factor:

$$I(a) = \int_{t_i}^{t_0} \frac{cdt'}{a(t')} = \int_{a_i}^{a} \frac{cda}{a\dot{a}} = \int_{\ln a_i}^{\ln a} c(aH)^{-1}\, d\ln a = \int_{\ln a_i}^{\ln a} l_H^c\, d\ln a$$

where $l_H^c = c(aH)^{-1}$ is the comoving Hubble distance. For ordinary matter sources(including radiation), the comoving Hubble radius is a monotonically increasing function of time (or scale factor), and the integral is dominated by the contributions from late times. So the integral can be well approximated by l_H^c at the late time:

$$D_h^c = I(a) \approx l_H^c(a) = \frac{c}{aH}$$

The ratio between the comoving horizon today and that at $z = 1100$ can be estimated by:

$$\frac{D_{h,0}^c}{D_{h,1100}^c} \approx \frac{a_{1100}H_{1100}}{a_0 H_0} = \frac{H_0\sqrt{1+1100}}{H_0} \approx 33.2$$

So the increasing behaviour of Hubble distance will inevitably lead to the increasing behaviour of the particle horizon distance. Then the horizon problem comes up. In order to solve this problem, we need the Hubble distance to decrease in the early universe:

$$\frac{dl_H^c}{dt} = -\frac{c\ddot{a}}{\dot{a}^2} < 0 \Rightarrow \ddot{a} > 0$$

1 分

This is the motivation for adding up the inflationary scenario which makes the particle horizon distance dominated by the contributions from early times:

$$D^c_{h,\,\exists\text{ inflation}} \approx l^c_H(a_i) = c/(a_i H_i)$$

where a_i stands for the scale factor when the universe started to inflat. To solve the horizon problem, we just need the Hubble distance to decrease in the early universe and let $D^c_{h,\,\exists\text{ inflation}} \geq D_{h,\,\nexists\text{ inflation}} \approx c/(a_0 H_0)$.

3. (共 5 分)

Ans: III.3

(1)

$$N(t_{\text{start}}) = \ln\left(\frac{a_{\text{end}}}{a_{\text{start}}}\right) = \ln\left(\frac{a_{\text{end}}}{a_0}\cdot\frac{H_{\text{start}}}{H_0}\cdot\frac{H_0}{k_{\text{start}}}\right) \tag{1}$$

$$= \ln\left(\frac{a_{\text{end}}}{a_0}\right) + \ln\left(\frac{H_{\text{start}}}{H_0}\right) + \ln\left(\frac{H_0}{k_{\text{start}}}\right) \tag{2}$$

1 分

(2) Friedmann equation: $H^2 = \frac{8\pi G}{3}\rho$. During slow roll inflation, $\rho \approx V_{\text{SR}}$, so we have:

$$\frac{H_{\text{start}}}{H_0} = \left(\frac{V_{\text{SR}}}{\rho_{\text{cr},0}}\right)^{1/2} = \left(\frac{V_{\text{SR}}^{1/4}}{10^{16}\,\text{GeV}}\right)\left(\frac{10^{16}\,\text{GeV}\,V_{\text{SR}}^{1/4}}{\rho_{\text{cr},0}^{1/2}}\right)$$

1 分

(3) Expand the first term in (1) use the scale factor at different cosmic time as "bridges". Firstly from the assumptions provided by the problem we have:

$$\frac{\rho_{\text{end}}}{\rho_{\text{reh}}} = \left(\frac{a_{\text{reh}}}{a_{\text{end}}}\right)^3, \frac{\rho_{\text{reh}}}{\rho_{r,0}} = \left(\frac{a_0}{a_{\text{reh}}}\right)^4$$

1 分

Then we can replace the scale factor with the energy density:

$$\begin{aligned}\ln\left(\frac{a_{\text{end}}}{a_0}\right) &= \ln\left(\frac{a_{\text{end}}}{a_{\text{reh}}}\right) + \ln\left(\frac{a_{\text{reh}}}{a_0}\right) \\ &= \frac{1}{3}\ln\left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}}\right) + \frac{1}{4}\ln\left(\frac{\rho_{r,0}}{\rho_{\text{reh}}}\right) \\ &= \frac{1}{12}\ln\left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}}\right) + \frac{1}{4}\ln\left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}}\cdot\frac{\rho_{r,0}}{\rho_{\text{reh}}}\right) \\ &= \frac{1}{3}\ln\left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}}\right)^{1/4} + \ln\left(\frac{\rho_{r,0}}{\rho_{\text{end}}}\right)^{1/4}\end{aligned}$$

At last, combine the results from (1), (2) and (3), exchange the position of $\rho_{\text{r},0}^{1/4}$ and $V_{\text{SR}}^{1/4}$ in term $\left(\frac{10^{16}\,\text{GeV}\,V_{\text{SR}}^{1/4}}{\rho_{\text{cr},0}^{1/2}}\right)$ to get the final answer. 1 分

(4) Since it is required to solve the horizon problem, we have the relation:

$$N(t_{\text{start}}) - \left(\frac{10^{16}\,\text{GeV}\,\rho_{\text{r},0}^{1/4}}{\rho_{\text{cr},0}^{1/2}}\right) \approx \ln\left(\frac{H_0}{k_{\text{start}}}\right) \geq 0$$

$$N(t_{\text{start}}) \geq \left(\frac{10^{16}\,\text{GeV}\,\rho_{\text{r},0}^{1/4}}{\rho_{\text{cr},0}^{1/2}}\right) \approx 61\ (h = 0.7)$$

So at least 61 e-folds are required to solve the horizon problem. 1 分

4. (共 8 分)

Ans: III.4

4.1

(1)

$$\epsilon = \frac{M_{\text{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \approx \frac{M_{\text{pl}}^2}{2}\frac{9H^2\dot{\phi}^2}{9H^4M_{\text{pl}}^4} = \frac{\dot{\phi}^2}{2H^2M_{\text{pl}}^2} = \frac{\dot{\phi}^2}{2V(\phi)/3} = \frac{3\dot{\phi}^2}{2V(\phi)}$$

Under slow-roll condition, $\frac{\dot{\phi}^2}{2} \ll V(\phi)$. So we have:$\epsilon \ll 1$ 1 分

(2) Under the slow-roll condition:

$$H^2 = \frac{V(\phi)}{3M_{\text{pl}}^2} \Rightarrow 2H\dot{H} = \frac{V'\dot{\phi}}{3M_{\text{pl}}^2}$$

$$3H\dot{\phi} = -V'(\phi)$$

$$-\frac{\dot{H}}{H^2} = -\frac{V'\dot{\phi}}{6H^3M_{\text{pl}}^2} = -\frac{V'\dot{\phi}}{2HV} = -\frac{V'\dot{\phi}H}{2H^2V} = \frac{V'^2/3}{2V^2/3M_{\text{pl}}^2} = \frac{M_{\text{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 = \epsilon$$

1 分

(3)

$$H = \frac{\dot{a}}{a} \quad \dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2\left(\frac{\dot{H}}{H^2} + 1\right) = H^2(1 - \epsilon)$$

Since $\epsilon \ll 1$, $\frac{\ddot{a}}{a} > 0$, indicating an accelerating universe. 1 分

4.2

$$V(\phi) = \frac{m^2}{2}\phi^2 \quad V'(\phi) = m^2\phi$$

$$\epsilon = \frac{M_{\text{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 = 1 \Rightarrow \frac{M_{\text{pl}}^2}{2}\left(\frac{2}{\phi}\right)^2 = 1$$

$$\phi_{\text{end}} = \sqrt{2}M_{\text{pl}}$$

1 分

4.3

$$N(t) = \ln\frac{a(t_{\text{end}})}{a(t)} = \int_{a(t)}^{a(t_{\text{end}})} \frac{da}{a} = \int_t^{t_{\text{end}}} H(t)\,dt = \int_{\phi(t)}^{\phi_{\text{end}}} \frac{H}{\dot{\phi}}\,d\phi$$

$$\text{Use} \quad \begin{cases} 3H\dot{\phi} = -V' \\ H^2 = \frac{V}{3M_{\text{pl}}^2} \end{cases} \quad \Rightarrow \frac{\dot{\phi}}{H} = -\frac{V'}{V}\cdot M_{\text{pl}}^2$$

$$\text{So} \quad N(t) = \int_{\phi(t)}^{\phi_{\text{end}}} \left(-\frac{V}{V'}\right)\frac{1}{M_{\text{pl}}^2}\,d\phi = \frac{1}{M_{\text{pl}}^2}\int_{\phi_{\text{end}}}^{\phi(t)} \frac{V(\phi)}{V'(\phi)}\,d\phi$$

1 分

4.4

$$\frac{V}{V'} = \frac{1}{2}\phi$$

$$N(t) = \frac{1}{M_{\text{pl}}^2}\int_{\phi_{\text{end}}}^{\phi(t)} \frac{1}{2}\phi\,d\phi = \frac{1}{4M_{\text{pl}}^2}\left(\phi^2(t) - \phi_{\text{end}}^2\right)$$

1 分

(1)

$$\phi_{\text{end}} = \sqrt{2}M_{\text{pl}}, \quad N(t_{\text{start}}) = \frac{\phi_{\text{start}}^2}{4M_{\text{pl}}^2} - \frac{1}{2} \geq 60$$

$$\Rightarrow \phi(t_{\text{start}}) \geq 11\sqrt{2}M_{\text{pl}} \approx 15.6M_{\text{pl}}$$

1 分

(2)

$$\epsilon = \frac{2M_{\text{pl}}^2}{\phi(t_{\text{start}})^2} \leq \frac{1}{121} \ll 1 \quad \text{which satisfy the slow-roll condition.}$$

1 分