Physics Cosmology
Assignment II

1. (共 3 分)

Ans: II.1

$$T = \eta_{\mu\nu} T^{\mu\nu} = (3\omega - 1)\rho$$

$$S_{00} = T_{00} - \frac{1}{2} g_{00} T = \frac{1}{2}(\rho + 3p)$$

ω	0	$1/3$	-1
T	$-\rho$	0	-4ρ
S_{00}	$\frac{1}{2}\rho$	ρ	$-\rho$

评分细则：共 3 分，每写对一种 species 的 S_{00} 和 T，给 1 分。

2. (共 5 分)

Ans: II.2

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}\right)$$

(a) $\Gamma^0_{ij} = \frac{1}{2} g^{0\sigma}(\partial_j g_{i\sigma} + \partial_i g_{\sigma j} - \partial_\sigma g_{ij}) = -\frac{1}{2} g^{00} \partial_0 g_{ij} = a\dot{a}\delta_{ij}$ 1 分

(b) $\Gamma^i_{0j} = \frac{1}{2} g^{i\sigma}(\partial_j g_{0\sigma} + \partial_0 g_{\sigma j} - \partial_\sigma g_{0j}) = \frac{1}{2} g^{i\sigma} \partial_0 g_{\sigma j} = \frac{1}{2}\frac{1}{a^2}\delta^{i\sigma}(2a\dot{a})\delta_{\sigma j} = \frac{\dot{a}}{a}\delta^i_j$ 1 分

(c) $\Gamma^i_{jl} = \frac{1}{2} g^{i\sigma}(\partial_l g_{j\sigma} + \partial_j g_{\sigma l} - \partial_\sigma g_{jl}) = 0$ 1 分

(If and only if $\sigma = i$, $g^{i\sigma} \neq 0$. However the spatial derivatives of $g_{\mu\nu}$ are equal to 0.)

(d) $\Gamma^0_{00} = \frac{1}{2} g^{0\sigma}(\partial_0 g_{0\sigma} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00}) = 0$

(e) $\Gamma^0_{0j} = \frac{1}{2} g^{0\sigma}(\partial_j g_{0\sigma} + \partial_0 g_{\sigma j} - \partial_\sigma g_{0j}) = 0$

(f) $\Gamma^j_{00} = \frac{1}{2} g^{j\sigma}(\partial_0 g_{0\sigma} + \partial_0 g_{\sigma 0} - \partial_\sigma g_{00}) = 0$

$$R^\lambda_{\rho\mu\nu} = \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\rho\mu} - \Gamma^\lambda_{\mu\alpha}\Gamma^\alpha_{\rho\nu}$$

(a) $R_{00} = \partial_0\Gamma^\lambda_{0\lambda} - \partial_\lambda\Gamma^\lambda_{00} + \Gamma^\lambda_{0\alpha}\Gamma^\alpha_{0\lambda} - \Gamma^\lambda_{\lambda\alpha}\Gamma^\alpha_{00} = \partial_0\Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{0\alpha}\Gamma^\alpha_{0\lambda} = \partial_t\left(\frac{3\dot{a}}{a}\right) + \frac{\dot{a}}{a}\delta^i_j\frac{\dot{a}}{a}\delta^j_i = \frac{3\ddot{a}}{a}$ 1 分

(b)

$$\begin{aligned} R_{ij} &= \partial_j\Gamma^\lambda_{i\lambda} - \partial_\lambda\Gamma^\lambda_{ij} + \Gamma^\lambda_{j\alpha}\Gamma^\alpha_{i\lambda} - \Gamma^\lambda_{\lambda\alpha}\Gamma^\alpha_{ij} \\ &= -\partial_t\Gamma^0_{ij} + \Gamma^\lambda_{j\alpha}\Gamma^\alpha_{i\lambda} - \Gamma^\lambda_{\lambda\alpha}\Gamma^\alpha_{ij} \end{aligned}$$

1. $-\partial_t\Gamma^0_{ij} = -\partial_t(a\dot{a})\delta_{ij} = -(\dot{a}^2 + a\ddot{a})\delta_{ij}$
2. $\Gamma^\lambda_{j\alpha}\Gamma^\alpha_{i\lambda} = \Gamma^0_{jk}\Gamma^k_{i0} + \Gamma^k_{j0}\Gamma^0_{ik} = a\dot{a}\delta_{jk}\frac{\dot{a}}{a}\delta^k_i + \frac{\dot{a}}{a}\delta^k_j a\dot{a}\delta_{ik} = 2\dot{a}^2\delta_{ij}$
3. $-\Gamma^\lambda_{\lambda\alpha}\Gamma^\alpha_{ij} = -\Gamma^\lambda_{\lambda 0}\Gamma^0_{ij} = -\frac{3\dot{a}}{a}a\dot{a}\delta_{ij} = -3\dot{a}^2\delta_{ij}$

$$R_{ij} = -(\dot{a}^2 + a\ddot{a})\delta_{ij} + 2\dot{a}^2\delta_{ij} - 3\dot{a}^2\delta_{ij} = -(2\dot{a}^2 + a\ddot{a})\delta_{ij}$$

1 分

3. (共 5 分)

Ans: II.3

General solution for species with equation of state $p = \omega\rho$:

$$\rho \propto a^{-3(1+\omega)}$$

Substitute into the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \propto a^{-3(1+\omega)} \Rightarrow \begin{cases} a \propto e^{H_0 t}, & \omega = -1 \\ a \propto t^{\frac{2}{3(1+\omega)}}, & \omega \neq -1 \end{cases}$$

The age of the Universe when $a = a_0$:

$$t_0 = \int_0^{a_0} \frac{da}{aH(a)} = \frac{2}{3(1+\omega)H_0}(\omega \neq -1)$$

The particle horizon distance:

$$D_h = a(t)\int_0^t \frac{cdt'}{a(t')} = \frac{3(\omega+1)}{3\omega+1}ct$$

(1) Energy conservation for non-relativistic matter:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \Rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{-3}$$

Substitute into the Friedmann equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3} \Rightarrow a = a_0\left(\frac{3}{2}H_0\,t\right)^{\frac{2}{3}}\,[a(t=0)=0]$$

The age of the Universe when $a = a_0$:

$$t_0 = \frac{2}{3H_0}$$

The particle horizon distance:

$$D_h^m = a(t)\int_0^t \frac{cdt'}{a(t')} = 3ct$$

2 分

(2) Energy conservation for radiation:

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \Rightarrow \rho = \rho_0\left(\frac{a}{a_0}\right)^{-4}$$

Substitute into the Friedmann equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-4} \Rightarrow a = a_0\left(2H_0\,t\right)^{\frac{1}{2}}\,[a(t=0)=0]$$

The age of the Universe when $a = a_0$:

$$t_0 = \frac{1}{2H_0}$$

The particle horizon distance:

$$D_h^r = a(t)\int_0^t \frac{cdt'}{a(t')} = 2ct$$

2 分

(3) Energy conservation for cosmological constant:

$$\dot{\rho} = 0 \Rightarrow \rho = \rho_0$$

Substitute into the Friedmann equation:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0 \Rightarrow a = a_0 e^{H_0(t-t_0)}[a(t=t_0)=a_0]$$

1 分

4. (共 6 分)

Ans: II.4

(1) When $\rho_{\mathrm{m}} = \rho_{\mathrm{rad}}$:

$$\rho_{m,0}(1+z)^3 = \rho_{r,0}(1+z)^4 \Rightarrow z = \frac{\rho_{m,0}}{\rho_{r,0}} - 1 = 3123$$

0.5 分

The age of the Universe at that period:

$$t = \int_z^{+\infty} \frac{dz'}{(1+z')H(z')} \approx 6.25 \times 10^4 \mathrm{yr}$$

0.5 分

(2) When $\rho_{\mathrm{m}} = \rho_{\Lambda}$:

$$\rho_{m,0}(1+z)^3 = \rho_{\Lambda,0} \Rightarrow z = \left(\frac{\rho_{\Lambda,0}}{\rho_{m,0}}\right)^{\frac{1}{3}} - 1 = 0.44$$

0.5 分

The age of the Universe at that period:

$$t = \int_z^{+\infty} \frac{dz'}{(1+z')H(z')} \approx 9.5 \text{ Gyr}$$

0.5 分

(3.1) When z = 30 and z = 20 (first stars form):

$$t(z=30) = 0.11 \text{ Gyr}; t(z=20) = 0.19 \text{ Gyr}$$

1 分

(3.2) When z = 6 (time about reionization):

$$t(z=6) = 1 \text{ Gyr}$$

0.5 分

(3.3) When z = 2 (“cosmic noon”):

$$t(z=2) = 3.5 \text{ Gyr}$$

0.5 分

(3.4) When z = 1:

$$t(z = 1) = 6.2 \text{ Gyr}$$

The age of our sun is about 4.6 Gyr now, an the birth of the sun is about $13.8 - 4.6 = 9.2$ Gyr $> t(z = 1) = 6.2$ Gyr. So the sun is born after the cosmic noon. 1 分

(3.5) When z = 0.1:

$$t(z = 0.1) = 12.9 \text{ Gyr}$$

Dinosaurs originated in the early Triassic Period of the Mesozoic Era, around 0.23 Gyr ago, which is much later than $t(z = 0.1) = 12.9$ Gyr. So at that time, the dinosaurs haven't appeared yet. 1 分