1. (1) b)

(2) $ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ and $g^{\mu\nu} = (g_{\mu\nu})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$ $\mu, \nu = 1, 2\ (r, \theta)$

then $g_{rr} = 1$, $g_{r\theta} = 0$, $g_{\theta\theta} = r^2$, $g^{rr} = 1$, $g^{r\theta} = 0$, $g^{\theta\theta} = \frac{1}{r^2}$

(3) since the space is locally flat, so we have $g_{\mu\nu;\lambda} = 0$, $g_{\nu\lambda;\mu} = 0$ and $g_{\lambda\mu;\nu} = 0$

$$\Rightarrow \begin{cases} g_{\mu\nu,\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu} - \Gamma^\alpha_{\nu\lambda} g_{\mu\alpha} = 0 & ① \\ g_{\nu\lambda,\mu} - \Gamma^\alpha_{\nu\mu} g_{\alpha\lambda} - \Gamma^\alpha_{\lambda\mu} g_{\nu\alpha} = 0 & ② \\ g_{\lambda\mu,\nu} - \Gamma^\alpha_{\lambda\nu} g_{\alpha\mu} - \Gamma^\alpha_{\mu\nu} g_{\lambda\alpha} = 0 & ③ \end{cases}$$

$$\xrightarrow{\frac{1}{2}(②+③-①)} \Gamma_{\lambda\mu\nu} = g_{\lambda\alpha}\Gamma^\alpha_{\mu\nu} = \frac{1}{2}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})$$

$$\Rightarrow \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})$$

so $\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} \cdot (g_{\theta r,\theta} + g_{\theta r,\theta} - g_{\theta\theta,r}) = \frac{1}{2} \cdot 1 \cdot (-\frac{\partial r^2}{\partial r}) = -r$

$\Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta}(g_{r\theta,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}) = \frac{1}{2}\frac{1}{r^2} \cdot 2r = \frac{1}{r}$

$\Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta}(g_{\theta\theta,r} + g_{r\theta,\theta} - g_{\theta r,\theta}) = \frac{1}{2}\frac{1}{r^2} \cdot 2r = \frac{1}{r}$

2. (1) based on the conditions given, $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\rho}(h_{\rho\alpha,\beta} + h_{\rho\beta,\alpha} - h_{\alpha\beta,\rho})$ ①

then rewrite the geodesic equation into 2 equations:

$$\begin{cases} \frac{d^2x^0}{d\tau^2} = 0 & ② \\ \frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}(\frac{dx^0}{d\tau})^2 = 0 & ③ \end{cases}$$

② $\Rightarrow x^0 = \alpha\tau +$ const., where α is a constant ④

③ ④ $\Rightarrow \frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0$ ⑤

also because of ①, $\Gamma^i_{00} = \frac{1}{2}\eta^{ii}(h_{i0,0} + h_{i i,0} - h_{00,i}) = \frac{1}{2} \cdot 1 \cdot (-h_{00,i}) = -\frac{1}{2}h_{00,i}$

compare ⑤ to the Newton equation : $\frac{d^2x^i}{dt^2} + \phi_{,i} = 0$

$\Rightarrow \Gamma^i_{00} = \phi_{,i}$

$\Rightarrow -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} = \frac{\partial \phi}{\partial x^i}$

$\Rightarrow h_{00} = -2\phi$

(2) $g_{00} = \eta_{00} + h_{00} = -(1+2\phi)$

$d\tau = \sqrt{-ds^2} = \sqrt{-g_{00}dx^0dx^0} = \sqrt{-g_{00}}\,dt = \sqrt{1+2\phi}\,dt \Rightarrow dt = \frac{d\tau}{\sqrt{1+2\phi}}$

use the Talor expansion : $(1+2\phi)^{-\frac{1}{2}} = 1 - \phi + \frac{3}{2}\phi^2 + \cdots \approx 1-\phi$

so $dt \approx (1-\phi)d\tau$

3. (1) $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 \Rightarrow g_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & r^2 \end{pmatrix}, \; g^{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \frac{1}{r^2} \end{pmatrix}$ $\mu, \nu = 0, 1, 2$

rewrite the geodesic equation :

$$\begin{cases} \frac{d^2x^0}{d\tau^2} = 0 \\ \frac{d^2x^1}{d\tau^2} + \Gamma^1_{22}\left(\frac{dx^2}{d\tau}\right)^2 = 0 \\ \frac{d^2x^2}{d\tau^2} + \Gamma^2_{21}\frac{dx^2}{d\tau}\frac{dx^1}{d\tau} + \Gamma^2_{12}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2t}{d\tau^2} = 0 & ① \\ \frac{d^2r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2 = 0 & ② \\ \frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} = 0 & ③ \end{cases}$$

(2) ① $\Rightarrow t = \alpha\tau + \text{const.}$, where α is a constant ④

use ④, ②③ $\Rightarrow$
$$\begin{cases} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 & ⑤ \\ \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{d\theta}{dt}\frac{dr}{dt} = 0 & ⑥ \end{cases}$$

Lagrangian of a prime in polar coordinates is : $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$

Euler-Lagrangian equation :
$$\begin{cases} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\dot{r} = r\dot{\theta}^2 \\ \frac{d}{dt}(r^2\dot{\theta}) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 \\ r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{dr}{dt} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 & \text{, i.e. ⑤} \\ \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{d\theta}{dt}\frac{dr}{dt} = 0 & \text{, i.e. ⑥} \end{cases}$$
which means they are equivalent.