Solutions to Problems 3

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3.1

Suppose that in the optical and near-UV band, the extinction efficiency is

$$Q_\mathrm{e}(a,\lambda) \approx \begin{cases} 2\left(\dfrac{\pi a}{2\lambda}\right)^\beta, & \dfrac{\pi a}{\lambda} < 2 \\ \\ 2, & \dfrac{\pi a}{\lambda} \geqslant 2 \end{cases} \tag{3.1.1}$$

Assume that the dust density is proportional to n_H, with a simple power-law size distribution

$$\frac{1}{n_\mathrm{H}}\frac{\mathrm{d}n}{\mathrm{d}a} = \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p},\ 0 < a \leqslant a_\mathrm{max} \tag{3.1.2}$$

where $a_0 = 0.1\ \mu$m is a fiducial length, A_0 is dimensionless, and $p < 4$. Let $\sigma_\mathrm{e}(\lambda)$ be the extinction cross-section per H atom at wavelength λ.

3.1.1

Assume that $a_\mathrm{max} < 0.28\ \mu$m. Obtain an expression for $\dfrac{\sigma_\mathrm{e}(\lambda)}{A_0\pi a_0^2}$ that would be valid for $\lambda = \lambda_V$ or λ_B. Evaluate this ratio for $\beta = 1.5$, $p = 3.5$, $a_\mathrm{max} = 0.25\,\mu$m, and $\lambda = \lambda_V$.

Solution: Using Equation 3.1.2, the extinction cross section per H atom at wavelength λ is defined as

$$\begin{aligned} \sigma_\mathrm{e}(\lambda) &= \int_0^{a_\mathrm{max}} \pi a^2 Q_\mathrm{e}(a,\lambda)\frac{1}{n_\mathrm{H}}\frac{\mathrm{d}n}{\mathrm{d}a}\mathrm{d}a \\ &= \int_0^{a_\mathrm{max}} \pi a^2 Q_\mathrm{e}(a,\lambda)\cdot\frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p}\mathrm{d}a \end{aligned} \tag{3.1.3}$$

Under the condition $a_\mathrm{max} < 0.28\ \mu$m and given that $\lambda = \lambda_B \approx 445$ nm or $\lambda_V \approx 551$ nm (from Wikipedia/Photometric system), we have

$$\begin{aligned} &\frac{\pi a}{\lambda} < \frac{\pi \times 0.28\ \mu\mathrm{m}}{\lambda_B} \approx 1.98 < 2 \\ &\implies Q_\mathrm{e}(a,\lambda) \approx 2\left(\frac{\pi a}{2\lambda}\right)^\beta \end{aligned} \tag{3.1.4}$$

Thus

$$
\begin{aligned}
\sigma_{\mathrm{e}}(\lambda) &= \int_0^{a_{\max}} \pi a^2 \cdot 2\left(\frac{\pi a}{2\lambda}\right)^{\beta} \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
\implies \frac{\sigma_{\mathrm{e}}(\lambda)}{A_0 \pi a_0^2} &= 2\int_0^{a_{\max}} \frac{a^2}{a_0^2}\left(\frac{\pi a}{2\lambda}\right)^{\beta} \frac{1}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
&= 2\left(\frac{\pi a_0}{2\lambda}\right)^{\beta} \int_0^{a_{\max}/a_0} x^{2+\beta-p}\mathrm{d}x \\
&= \frac{2}{\beta-p+3}\left(\frac{\pi a_0}{2\lambda}\right)^{\beta}\left(\frac{a_{\max}}{a_0}\right)^{\beta-p+3} \qquad (3.1.5)
\end{aligned}
$$

Given $\beta = 1.5, p = 3.5, a_{\max} = 0.25\ \mu\mathrm{m}$ and $\lambda = \lambda_V \approx 551$ nm, substitute into the expression

$$
\begin{aligned}
\frac{\sigma_{\mathrm{e}}(\lambda_V)}{A_0 \pi a_0^2} &= \frac{2}{\beta-p+3}\left(\frac{\pi a_0}{2\lambda_V}\right)^{\beta}\left(\frac{a_{\max}}{a_0}\right)^{\beta-p+3} \\
&= \frac{2}{1.5-3.5+3}\left(\frac{\pi \times 0.1 \times 10^{-6}}{2 \times 551 \times 10^{-9}}\right)^{1.5}\left(\frac{0.25 \times 10^{-6}}{0.1 \times 10^{-6}}\right)^{1.5-3.5+3} \\
&\approx 0.761 \qquad (3.1.6)
\end{aligned}
$$

3.1.2

For $a_{\max} < 0.28\ \mu\mathrm{m}$, using your result from **3.1.1**, obtain an expression for the ratio $\dfrac{\sigma_{\mathrm{e}}(\lambda_B)}{\sigma_{\mathrm{e}}(\lambda_V)}$, and evaluate this ratio for $\beta = 1.5$.

Solution: Using Equation 3.1.5, the ratio of extinction cross sections at λ_B and λ_V is:

$$
\begin{aligned}
\frac{\sigma_{\mathrm{e}}(\lambda_B)}{\sigma_{\mathrm{e}}(\lambda_V)} &= \frac{\dfrac{\sigma_{\mathrm{e}}(\lambda_B)}{A_0\pi a_0^2}}{\dfrac{\sigma_{\mathrm{e}}(\lambda_V)}{A_0\pi a_0^2}} = \frac{\dfrac{2}{\beta-p+3}\left(\dfrac{\pi a_0}{2\lambda_B}\right)^{\beta}\left(\dfrac{a_{\max}}{a_0}\right)^{\beta-p+3}}{\dfrac{2}{\beta-p+3}\left(\dfrac{\pi a_0}{2\lambda_V}\right)^{\beta}\left(\dfrac{a_{\max}}{a_0}\right)^{\beta-p+3}} = \left(\frac{\lambda_V}{\lambda_B}\right)^{\beta} \\
&= \left(\frac{551 \times 10^{-9}}{445 \times 10^{-9}}\right)^{1.5} \approx 1.38 \qquad (3.1.7)
\end{aligned}
$$

3.1.3

For $a_{\max} < 0.28\ \mu\mathrm{m}$, obtain an expression for R_V and evaluate this for $\beta = 1.5$.

Solution: The parameter R_V is defined as the ratio of total to selective extinction, using Equation 3.1.7

$$
\begin{aligned}
R_V &= \frac{A_V}{A_B - A_V} = \frac{\sigma_{\mathrm{e}}(\lambda_V)}{\sigma_{\mathrm{e}}(\lambda_B) - \sigma_{\mathrm{e}}(\lambda_V)} = \frac{1}{\dfrac{\sigma_{\mathrm{e}}(\lambda_B)}{\sigma_{\mathrm{e}}(\lambda_V)} - 1} = \frac{1}{\left(\dfrac{\lambda_V}{\lambda_B}\right)^{\beta} - 1} \\
&= \frac{1}{\left(\dfrac{551 \times 10^{-9}}{445 \times 10^{-9}}\right)^{1.5} - 1} \approx 2.65 \qquad (3.1.8)
\end{aligned}
$$

3.1.4

Suppose that $a_{\max} > \dfrac{2\lambda}{\pi}$, obtain an expression for $\dfrac{\sigma_{\mathrm{e}}(\lambda)}{A_0 \pi a_0^2}$.

Solution: For $a > 2\lambda/\pi$, recall Equation 3.1.3

$$\begin{aligned}
\sigma_{\mathrm{e}}(\lambda) &= \int_0^{a_{\max}} \pi a^2 Q_{\mathrm{e}}(a,\lambda) \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
&= \int_0^{2\lambda/\pi} \pi a^2 Q_{\mathrm{e}}(a,\lambda) \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a + \int_{2\lambda/\pi}^{a_{\max}} \pi a^2 Q_{\mathrm{e}}(a,\lambda) \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
&= \int_0^{2\lambda/\pi} \pi a^2 \cdot 2\left(\frac{\pi a}{2\lambda}\right)^{\beta} \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a + \int_{2\lambda/\pi}^{a_{\max}} \pi a^2 \cdot 2 \cdot \frac{A_0}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
\Longrightarrow \quad \frac{\sigma_{\mathrm{e}}(\lambda)}{A_0 \pi a_0^2} &= \frac{2}{\beta - p + 3}\left(\frac{\pi a_0}{2\lambda}\right)^{\beta}\left(\frac{2\lambda}{\pi a_0}\right)^{\beta - p + 3} + 2\int_{2\lambda/\pi}^{a_{\max}} \frac{a^2}{a_0^2} \cdot \frac{1}{a_0}\left(\frac{a}{a_0}\right)^{-p} \mathrm{d}a \\
&= \frac{2}{\beta - p + 3}\left(\frac{2\lambda}{\pi a_0}\right)^{-p+3} + 2\int_{2\lambda/(\pi a_0)}^{a_{\max}/a_0} x^{2-p}\mathrm{d}x \\
&= \frac{2}{\beta - p + 3}\left(\frac{2\lambda}{\pi a_0}\right)^{3-p} + \frac{2}{3-p}\left[\left(\frac{a_{\max}}{a_0}\right)^{3-p} - \left(\frac{2\lambda}{\pi a_0}\right)^{3-p}\right] \\
&= \frac{2}{3-p}\left(\frac{a_{\max}}{a_0}\right)^{3-p} - \frac{2\beta}{(\beta - p + 3)(3-p)}\left(\frac{2\lambda}{\pi a_0}\right)^{3-p}
\end{aligned} \tag{3.1.9}$$

3.1.5

If $a_{\max} = 0.35\ \mu\mathrm{m}$, $p = 3.5$ and $\beta = 2$, evaluate $\dfrac{\sigma_{\mathrm{e}}(\lambda_V)}{A_0 \pi a_0^2}$, $\dfrac{\sigma_{\mathrm{e}}(\lambda_B)}{A_0 \pi a_0^2}$ and R_V.

Solution: We use the expression from Equation 3.1.9

$$\begin{aligned}
\frac{\sigma_{\mathrm{e}}(\lambda_V)}{A_0 \pi a_0^2} &= \frac{2}{3-p}\left(\frac{a_{\max}}{a_0}\right)^{3-p} - \frac{2\beta}{(\beta - p + 3)(3-p)}\left(\frac{2\lambda_V}{\pi a_0}\right)^{3-p} \\
&= \frac{2}{3-3.5}\left(\frac{0.35 \times 10^{-6}}{0.1 \times 10^{-6}}\right)^{3-3.5} - \frac{2 \times 2}{(2 - 3.5 + 3)(3 - 3.5)}\left(\frac{2 \times 551 \times 10^{-9}}{\pi \times 0.1 \times 10^{-6}}\right)^{3-3.5} \\
&\approx 0.7095
\end{aligned} \tag{3.1.10}$$

$$\begin{aligned}
\frac{\sigma_{\mathrm{e}}(\lambda_B)}{A_0 \pi a_0^2} &= \frac{2}{3-p}\left(\frac{a_{\max}}{a_0}\right)^{3-p} - \frac{2\beta}{(\beta - p + 3)(3-p)}\left(\frac{2\lambda_B}{\pi a_0}\right)^{3-p} \\
&= \frac{2}{3-3.5}\left(\frac{0.35 \times 10^{-6}}{0.1 \times 10^{-6}}\right)^{3-3.5} - \frac{2 \times 2}{(2 - 3.5 + 3)(3 - 3.5)}\left(\frac{2 \times 445 \times 10^{-9}}{\pi \times 0.1 \times 10^{-6}}\right)^{3-3.5} \\
&\approx 1.0306
\end{aligned} \tag{3.1.11}$$

$$R_V = \frac{1}{\dfrac{\sigma_{\mathrm{e}}(\lambda_B)}{\sigma_{\mathrm{e}}(\lambda_V)} - 1} \approx \frac{1}{\dfrac{0.7095}{1.0306} - 1} \approx 2.210 \tag{3.1.12}$$

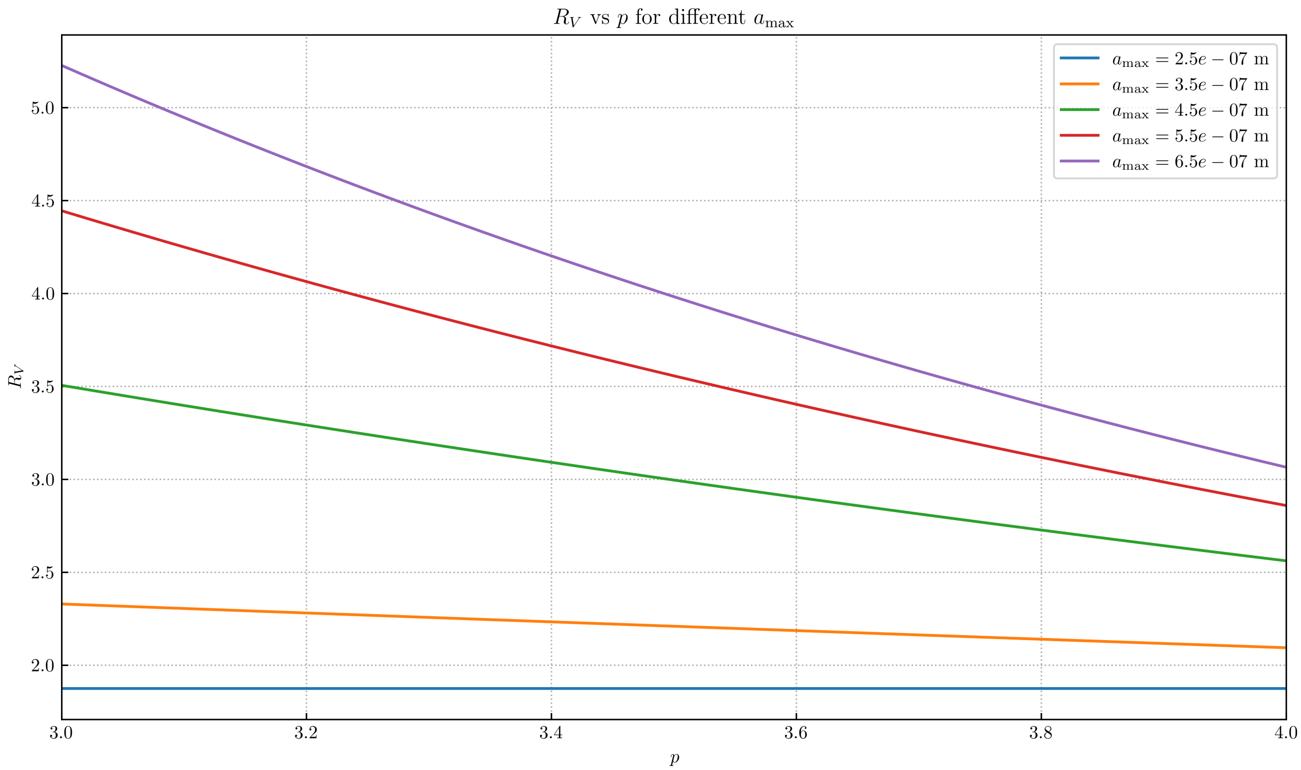

Figure 1: R_V vs p for different $a_{\max}$.

3.1.6

Plot R_V as a function of $p \in [3, 4]$ for different $a_{\max}$ (e.g., 0.25, 0.35, 0.45, 0.55, 0.65 μm).

Solution: See Figure 1.

3.2

Consider a hot plasma with density n_{H} in an elliptical galaxy. Suppose that planetary nebulae and other stellar outflows are injecting dust into the plasma with a rate per unit grain radius

$$\frac{\mathrm{d}\dot{N}_{\mathrm{d}}}{\mathrm{d}a} = \frac{A_0}{a_{\max}} \left(\frac{a}{a_{\max}} \right)^{-p} \tag{3.2.1}$$

3.2.1

Obtain an expression for the total rate $\dfrac{\mathrm{d}M_{\mathrm{d}}}{\mathrm{d}t}$ at which dust mass is being injected into the plasma, in terms of A_0, $a_{\max}$, p and the density ρ of the grain material.

Solution: Using Equation 3.2.1, the total dust mass injection rate $\dfrac{\mathrm{d}M_{\mathrm{d}}}{\mathrm{d}t}$ is obtained by integrating the mass contribution from grains of all sizes:

$$
\begin{aligned}
\frac{\mathrm{d}M_\mathrm{d}}{\mathrm{d}t} &= \int_0^{a_\mathrm{max}} \frac{4}{3}\pi a^3 \rho \frac{\mathrm{d}\dot{N}_\mathrm{d}}{\mathrm{d}a}\mathrm{d}a \\
&= \int_0^{a_\mathrm{max}} \frac{4}{3}\pi a^3 \rho \frac{A_0}{a_\mathrm{max}} \left(\frac{a}{a_\mathrm{max}}\right)^{-p} \mathrm{d}a \\
&= \frac{4}{3}\pi \rho A_0 a_\mathrm{max}^{p-1} \int_0^{a_\mathrm{max}} a^{3-p} \mathrm{d}a \\
&= \frac{4}{3}\pi \rho A_0 \frac{a_\mathrm{max}^3}{4-p} \\
&= \frac{4\pi\rho A_0 a_\mathrm{max}^3}{3(4-p)}
\end{aligned} \tag{3.2.2}
$$

3.2.2

Upon injection into the plasma, the grains are subject to sputtering at a rate $\frac{\mathrm{d}a}{\mathrm{d}t} = -\beta n_\mathrm{H}$, where β is a constant. Find the steady-state solution for $\frac{\mathrm{d}N_\mathrm{d}}{\mathrm{d}a}$, where $N_\mathrm{d}(a)$ is the number of dust grains present with radii $\leqslant a$.

Solution: In steady state, the number of grains with radii between a and $a + \mathrm{d}a$ is constant over time. The rate at which grains are injected into this size bin must balance the rate at which grains leave the bin due to sputtering:

$$
\begin{aligned}
0 = \frac{\mathrm{d}\dot{N}_\mathrm{d}}{\mathrm{d}a} &= \left(\frac{\mathrm{d}\dot{N}_\mathrm{d}}{\mathrm{d}a}\right)_\mathrm{inject} + \left(\frac{\mathrm{d}\dot{N}_\mathrm{d}}{\mathrm{d}a}\right)_\mathrm{sputter} \\
&= \frac{A_0}{a_\mathrm{max}} \left(\frac{a}{a_\mathrm{max}}\right)^{-p} - \frac{\mathrm{d}}{\mathrm{d}a}\left[\frac{\mathrm{d}N_\mathrm{d}}{\mathrm{d}a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_\mathrm{sputter}\right] \\
&= \frac{A_0}{a_\mathrm{max}} \left(\frac{a}{a_\mathrm{max}}\right)^{-p} + \beta n_\mathrm{H} \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\mathrm{d}N_\mathrm{d}}{\mathrm{d}a}\right) \\
\implies \frac{\mathrm{d}N_\mathrm{d}}{\mathrm{d}a} &= -\frac{1}{\beta n_\mathrm{H}} \frac{A_0}{a_\mathrm{max}} \int_0^a \left(\frac{a'}{a_\mathrm{max}}\right)^{-p} \mathrm{d}a' \\
&= -\frac{1}{\beta n_\mathrm{H}} \frac{A_0}{a_\mathrm{max}^{1-p}} \frac{a^{1-p}}{1-p} \\
&= \frac{A_0}{(p-1)\beta n_\mathrm{H}} \left(\frac{a}{a_\mathrm{max}}\right)^{1-p}
\end{aligned} \tag{3.2.3}
$$

3.2.3

Obtain an expression for the steady-state dust mass M_d and the characteristic survival time $\tau_\mathrm{s} = \frac{M_\mathrm{d}}{\mathrm{d}M_\mathrm{d}/\mathrm{d}t}$ in terms of a_max, p and $\frac{\mathrm{d}a}{\mathrm{d}t}$.

Solution: Using Equation 3.2.3, the steady-state dust mass M_d is given by

$$
\begin{aligned}
M_\mathrm{d} &= \int_0^{a_\mathrm{max}} \frac{4}{3}\pi a^3 \rho \frac{\mathrm{d}N_\mathrm{d}}{\mathrm{d}a}\mathrm{d}a \\
&= \int_0^{a_\mathrm{max}} \frac{4}{3}\pi a^3 \rho \frac{A_0}{(p-1)\beta n_\mathrm{H}} \left(\frac{a}{a_\mathrm{max}}\right)^{1-p} \mathrm{d}a \\
&= \frac{4\pi\rho A_0}{3(p-1)\beta n_\mathrm{H} a_\mathrm{max}^{1-p}} \int_0^{a_\mathrm{max}} a^{4-p}\mathrm{d}a \\
&= \frac{4\pi\rho A_0}{3(p-1)\beta n_\mathrm{H} a_\mathrm{max}^{1-p}} \frac{a_\mathrm{max}^{5-p}}{5-p} \\
&= \frac{4\pi\rho A_0 a_\mathrm{max}^4}{3(p-1)(5-p)\beta n_\mathrm{H}}
\end{aligned}
\tag{3.2.4}
$$

Using Equation 3.2.2, the characteristic survival time τ_s is given by

$$
\tau_\mathrm{s} = \frac{M_\mathrm{d}}{\frac{\mathrm{d}M_\mathrm{d}}{\mathrm{d}t}} = \frac{\frac{4\pi\rho A_0 a_\mathrm{max}^4}{3(p-1)(5-p)\beta n_\mathrm{H}}}{\frac{4\pi\rho A_0 a_\mathrm{max}^3}{3(4-p)}} = \frac{(4-p)a_\mathrm{max}}{(p-1)(5-p)\beta n_\mathrm{H}}
\tag{3.2.5}
$$

3.2.4

Consider a passive elliptical galaxy NGC 4564 containing hot plasma $k_\mathrm{B}T \approx 0.5$ keV and a core density $n_\mathrm{H} \approx 0.01$ cm^{-3}. Assuming $\beta = 10^{-6}$ μm cm^3 yr^{-1}, $p = 3.5$ and $a_\mathrm{max} = 0.3$ μm, estimate the survival time τ_s. If the dust injection rate from evolved stars in the central kpc is 1.3×10^{-4} $M_\odot$ yr^{-1}, estimate the steady-state dust mass M_d. The observed upper limit of the dust mass is $M_\mathrm{d} < 8700$ $M_\odot$. Is your estimate in agreement with observations?

Solution: Using Equation 3.2.5, the characteristic survival time τ_s is calculated by

$$
\begin{aligned}
\tau_\mathrm{s} &= \frac{(4-p)a_\mathrm{max}}{(p-1)(5-p)\beta n_\mathrm{H}} \\
&= \frac{(4-3.5)\times 0.3\ \mu\mathrm{m}}{(3.5-1)\times(5-3.5)\times 10^{-6}\times 0.01\ \mu\mathrm{m\ yr}^{-1}} \\
&= 4\times 10^6\ \mathrm{yr}
\end{aligned}
\tag{3.2.6}
$$

The steady-state dust mass M_d is estimated by

$$
\begin{aligned}
M_\mathrm{d} &\approx \tau_s \frac{\mathrm{d}M_\mathrm{d}}{\mathrm{d}t} = 4\times 10^6 \times 1.3\times 10^{-4}\ M_\odot \\
&= 520\ M_\odot < 8700\ M_\odot
\end{aligned}
\tag{3.2.7}
$$

Therefore, the estimate is in agreement with observations.

3.3

Consider a diffuse molecular cloud with $n_\mathrm{H} = 100$ cm^{-3}. The hydrogen is predominantly molecular, with $n(\mathrm{H_2}) = 50$ cm^{-3}. The oxygen is primarily atomic, with $n(\mathrm{O}) \approx 4\times 10^{-4} n_\mathrm{H}$. Assume that cosmic ray ionization maintains an abundance $n(\mathrm{H_3^+}) \approx 5\times 10^{-8} n_\mathrm{H}$ and cosmic ray ionization plus starlight photoionization of metals maintains $n_e \approx 10^{-4} n_\mathrm{H}$. Consider the reaction network for OH formation at $T_2 = 100$ K.

3.3.1

What is the steady-state density $n(\mathrm{OH}^+)$?

Solution: We analyze the primary formation and destruction pathways of OH in Table 1).

Reactions	**Rate coefficients**	**Rates**
$\mathrm{O} + \mathrm{H}_3^+ \to \mathrm{OH}^+ + \mathrm{H}_2$	$k_1 = 8.40 \times 10^{-10}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$	$k_1 n(\mathrm{O}) n(\mathrm{H}_3^+)$
$\mathrm{OH}^+ + \mathrm{H}_2 \to \mathrm{H}_2\mathrm{O}^+ + \mathrm{H}$	$k_2 = 1.01 \times 10^{-9}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$	$k_2 n(\mathrm{OH}^+) n(\mathrm{H}_2)$
$\mathrm{H}_2\mathrm{O}^+ + e^- \to \begin{cases} \mathrm{OH} + \mathrm{H} & (20\%) \\ \mathrm{O} + \mathrm{H}_2 & (9\%) \\ \mathrm{O} + \mathrm{H} + \mathrm{H} & (71\%) \end{cases}$	$k_3 = 4.30 \times 10^{-7}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$	$k_3 n(\mathrm{H}_2\mathrm{O}^+) n_e$
$\mathrm{H}_2\mathrm{O}^+ + \mathrm{H}_2 \to \mathrm{H}_3\mathrm{O}^+ + \mathrm{H}$	$k_4 = 6.40 \times 10^{-10}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$	$k_4 n(\mathrm{H}_2\mathrm{O}^+) n(\mathrm{H}_2)$
$\mathrm{H}_3\mathrm{O}^+ + e^- \to \begin{cases} \mathrm{O} + \mathrm{H}_2 + \mathrm{H} & (1\%) \\ \mathrm{OH} + \mathrm{H}_2 & (14\%) \\ \mathrm{OH} + \mathrm{H} + \mathrm{H} & (60\%) \\ \mathrm{H}_2\mathrm{O} + \mathrm{H} & (25\%) \end{cases}$	$k_5 = 7.48 \times 10^{-7} T_2^{-0.5}\ \mathrm{cm}^3\ \mathrm{s}^{-1}$	$k_5 n(\mathrm{H}_3\mathrm{O}^+) n_e$
$\mathrm{OH} + h\nu \to \mathrm{O} + \mathrm{H}$	$k_6 = 3.50 \times 10^{-10}\ \mathrm{s}^{-1}$	$k_6 n(\mathrm{OH})$

Table 1: Reactions, their rate coefficients and their rates

At steady state, the formation rate of OH^+ equals the destruction rate:

$$k_1 n(\mathrm{O}) n(\mathrm{H}_3^+) = k_2 n(\mathrm{OH}^+) n(\mathrm{H}_2) \tag{3.3.1}$$

$$\implies n(\mathrm{OH}^+) = \frac{k_1 n(\mathrm{O}) n(\mathrm{H}_3^+)}{k_2 n(\mathrm{H}_2)} \approx 3.327 \times 10^{-3}\ \mathrm{m}^{-3} \tag{3.3.2}$$

3.3.2

What is the steady-state density $n(\mathrm{H}_2\mathrm{O}^+)$?

Solution: Using Table 1 and Equation 3.3.1, the formation rate of $\mathrm{H}_2\mathrm{O}^+$ equals the destruction rate at steady state:

$$k_2 n(\mathrm{OH}^+) n(\mathrm{H}_2) = k_3 n(\mathrm{H}_2\mathrm{O}^+) n_e + k_4 n(\mathrm{H}_2\mathrm{O}^+) n(\mathrm{H}_2) \tag{3.3.3}$$

$$\implies n(\mathrm{H}_2\mathrm{O}^+) = \frac{k_2 n(\mathrm{OH}^+) n(\mathrm{H}_2)}{k_3 n_e + k_4 n(\mathrm{H}_2)} = \frac{k_1 n(\mathrm{O}) n(\mathrm{H}_3^+)}{k_3 n_e + k_4 n(\mathrm{H}_2)} \approx 4.628 \times 10^{-3}\ \mathrm{m}^{-3} \tag{3.3.4}$$

3.3.3

What is the steady-state OH abundance relative to hydrogen, $\dfrac{n(\mathrm{OH})}{n_\mathrm{H}}$?

Solution: Using Table 1 and Equation 3.3.4, the formation rate of $\mathrm{H}_3\mathrm{O}^+$ and OH equals the destruction rate at steady state:

$$k_4 n(\mathrm{H}_2\mathrm{O}^+) n(\mathrm{H}_2) = k_5 n(\mathrm{H}_3\mathrm{O}^+) n_e \tag{3.3.5}$$

$$20\%k_3n(\mathrm{H_2O^+})n_e + (14\% + 60\%)k_5n(\mathrm{H_3O^+})n_e = k_6n(\mathrm{OH}) \tag{3.3.6}$$

$$\begin{aligned}
\implies n(\mathrm{OH}) &= 20\%\frac{1}{k_6}k_3n(\mathrm{H_2O^+})n_e + 74\%\frac{1}{k_6}k_5n(\mathrm{H_3O^+})n_e \\
&= 0.2\frac{1}{k_6}k_3n(\mathrm{H_2O^+})n_e + 0.74\frac{1}{k_6}k_4n(\mathrm{H_2O^+})n(\mathrm{H_2}) \\
&= \left(0.2\frac{k_3}{k_6}n_e + 0.74\frac{k_4}{k_6}n(\mathrm{H_2})\right)\frac{k_1n(\mathrm{O})n(\mathrm{H_3^+})}{k_3n_e + k_4n(\mathrm{H_2})}
\end{aligned} \tag{3.3.7}$$

$$\implies \frac{n(\mathrm{OH})}{n_\mathrm{H}} = \left(0.2\frac{k_3}{k_6}n_e + 0.74\frac{k_4}{k_6}n(\mathrm{H_2})\right)\frac{k_1n(\mathrm{O})n(\mathrm{H_3^+})}{k_3n_e + k_4n(\mathrm{H_2})}\frac{1}{n_\mathrm{H}} \approx 3.245\times10^{-3} \tag{3.3.8}$$

3.3.4

There is more than 1 reaction that can reproduce OH. Which is the most important for the given conditions?

Solution: From Table 1, there are 3 reactions that can reproduce OH (see Table 2).

Reactions	**Rates**
$\mathrm{H_2O^+} + e^- \to \mathrm{OH} + \mathrm{H}$	$20\%k_3n(\mathrm{H_2O^+})n_e$
$\mathrm{H_3O^+} + e^- \to \mathrm{OH} + \mathrm{H_2}$	$14\%k_5n(\mathrm{H_3O^+})n_e$
$\mathrm{H_3O^+} + e^- \to \mathrm{OH} + \mathrm{H} + \mathrm{H}$	$60\%k_5n(\mathrm{H_3O^+})n_e$

Table 2: Reactions that can reproduce OH

Using Equation 3.3.5, compare their reaction rates:

$$\begin{aligned}
&\frac{20\%k_3n(\mathrm{H_2O^+})n_e}{60\%k_5n(\mathrm{H_3O^+})n_e} = \frac{0.2k_3n(\mathrm{H_2O^+})n_e}{0.6k_4n(\mathrm{H_2O^+})n(\mathrm{H_2})} = \frac{k_3n_e}{3k_4n(\mathrm{H_2})} \approx 0.0448 \\
&14\%k_5n(\mathrm{H_3O^+})n_e < 60\%k_5n(\mathrm{H_3O^+})n_e
\end{aligned} \tag{3.3.9}$$

So the most important reaction for reproducing OH under the given conditions is:

$$\mathrm{H_3O^+} + e^- \to \mathrm{OH} + \mathrm{H} + \mathrm{H} \tag{3.3.10}$$

3.4

Consider the collapse of a uniform density (ρ_0) spherical cloud with no gas pressure to counteract gravity, the so-called free-fall condition. Derive the free-fall time τ_{ff}, which is the time it takes for a given gas shell starting at radius r_0 to collapse to the centre of the cloud. How does τ_{ff} depend on the starting radius r_0? What does this dependence mean?

Solution: We consider the motion of a thin spherical shell of gas under the influence of gravity. The equation of motion for the shell at radius r (starting at radius r_0) is:

$$
\begin{aligned}
\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} &= -\frac{G \cdot \frac{4}{3}\pi r_0^3 \rho_0}{r^2} = -\frac{4\pi G r_0^3 \rho_0}{3r^2} \\
\implies \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\frac{\mathrm{d}r}{\mathrm{d}t} &= -\frac{4\pi G \rho_0 r_0^3}{3r^2} \\
\implies v\mathrm{d}v &= -\frac{4\pi G \rho_0 r_0^3}{3r^2}\mathrm{d}r \\
\implies \int_0^v v\mathrm{d}v &= -\frac{4\pi G \rho_0 r_0^3}{3}\int_{r_0}^{r} \frac{1}{r^2}\mathrm{d}r \\
\implies \frac{1}{2}v^2 &= \frac{4\pi G \rho_0 r_0^3}{3}\left(\frac{1}{r} - \frac{1}{r_0}\right) \\
\implies \frac{\mathrm{d}r}{\mathrm{d}t} &= -\sqrt{\frac{8\pi G \rho_0 r_0^3}{3}\left(\frac{1}{r} - \frac{1}{r_0}\right)} \\
\implies \int_0^{\tau_{\mathrm{ff}}} \mathrm{d}t &= -\int_{r_0}^{0} \frac{1}{\sqrt{\frac{8\pi G \rho_0 r_0^3}{3}\left(\frac{1}{r} - \frac{1}{r_0}\right)}}\mathrm{d}r
\end{aligned}
\tag{3.4.1}
$$

So the the free-fall time τ_{ff} is:

$$
\begin{aligned}
\tau_{\mathrm{ff}} &= \sqrt{\frac{3}{8\pi G \rho_0 r_0^3}}\int_0^{r_0} \frac{1}{\sqrt{\frac{1}{r} - \frac{1}{r_0}}}\mathrm{d}r \\
&= \sqrt{\frac{3}{8\pi G \rho_0}}\int_0^1 \frac{1}{\sqrt{\frac{1}{x} - 1}}\mathrm{d}x = \sqrt{\frac{3}{8\pi G \rho_0}}\int_0^1 \sqrt{\frac{x}{1-x}}\mathrm{d}x \\
&= \sqrt{\frac{3}{8\pi G \rho_0}}\mathrm{B}\left(\frac{3}{2}, \frac{1}{2}\right) = \sqrt{\frac{3}{8\pi G \rho_0}}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \\
&= \sqrt{\frac{3}{8\pi G \rho_0}}\frac{\sqrt{\pi}}{2}\sqrt{\pi} = \sqrt{\frac{3\pi}{32 G \rho_0}}
\end{aligned}
\tag{3.4.2}
$$

The free-fall time τ_{ff} depends solely on the initial uniform density ρ_0 of the cloud and is **independent** of the starting radius r_0. This implies that all layers of the cloud, regardless of their initial positions, collapse towards the centre simultaneously.

3.5

A pulsar is observed at 1610 and 1660 MHz. The plane of polarization at these 2 frequencies differs by 57.5°.

3.5.1

What is the minimum possible magnitude of the rotation measure |RM| toward this source? Why is it a minimum? What would be the next largest possible value of |RM|?

Solution: The rotation measure (RM) quantifies the amount of Faraday rotation experienced by the polarized electromagnetic waves as they propagate through a magnetized plasma. The change in the polarization angle between 2 frequencies ($\Delta\theta$) can be expressed as:

$$\Delta\theta = \mathrm{RM}\left(\lambda_1^2 - \lambda_2^2\right) + n\pi$$
$$\implies \mathrm{RM} = \frac{\Delta\theta - n\pi}{\lambda_1^2 - \lambda_2^2} = \frac{\Delta\theta - n\pi}{\dfrac{c^2}{\nu_1^2} - \dfrac{c^2}{\nu_2^2}} = \frac{(\Delta\theta - n\pi)\nu_1^2\nu_2^2}{c^2(\nu_2^2 - \nu_1^2)}$$
$$\implies |\mathrm{RM}| = \frac{|\Delta\theta - n\pi|\nu_1^2\nu_2^2}{c^2(\nu_2^2 - \nu_1^2)} \tag{3.5.1}$$

where $\nu_1 = 1610$ MHz and $\nu_2 = 1660$ MHz; n is an integer. The minimum possible $|\mathrm{RM}|$ occurs when n is chosen such that $\Delta\theta$ is within $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, choose $n = 0$:

$$|\mathrm{RM}|_{\mathrm{min}} = \frac{\Delta\theta\nu_1^2\nu_2^2}{c^2(\nu_2^2 - \nu_1^2)} \approx 487.81\ \mathrm{m}^{-2} \tag{3.5.2}$$

The next largest possible $|RM|$ occurs when $n = 1$:

$$|\mathrm{RM}|_{\mathrm{next}} = \frac{|\Delta\theta - \pi|\nu_1^2\nu_2^2}{c^2(\nu_2^2 - \nu_1^2)} \approx 1039.26\ \mathrm{m}^{-2} \tag{3.5.3}$$

3.5.2

If the source has a dispersion measure DM = 200 pc cm^{-3}. Using the minimum $|\mathrm{RM}|$ derived in **3.5.1**, what is the electron-density-weighted component of magnetic field along the line-of-sight?

Solution: The rotation measure (RM) and dispersion measure (DM) are related to the electron density (n_e) and the magnetic field component along the line of sight ($B_{||}$) by the following equations:

$$\mathrm{RM} = \frac{e^3}{2\pi m_e^2 c^4}\int_0^L n_e B_{||}\mathrm{d}l \tag{3.5.4}$$

$$\mathrm{DM} = \int_0^L n_e \mathrm{d}l \tag{3.5.5}$$

where L is the path length. The electron-density-weighted component of the magnetic field along the line of sight is:

$$\overline{B}_{||} = \frac{\displaystyle\int_0^L n_e B_{||}\mathrm{d}l}{\displaystyle\int_0^L n_e \mathrm{d}l} = \frac{2\pi m_e^2 c^4}{e^3}\frac{\mathrm{RM}}{\mathrm{DM}} \tag{3.5.6}$$

Using Equation 3.5.2, calculate $\overline{B}_{||}$:

$$\overline{B}_{||} = \frac{2\pi m_e^2 c^4}{e^3}\frac{|\mathrm{RM}|_{\mathrm{min}}}{\mathrm{DM}} \approx 8.094 \times 10^8\ \mathrm{T} \tag{3.5.7}$$