1. $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$

① non-relativistic matter $w=0 \Rightarrow p = w\rho = 0$

$T = g_{\mu\nu}T^{\mu\nu} = -\rho$, $S_{00} = T_{00} - \frac{1}{2}g_{00}T = g_{00}T^{00} - \frac{1}{2}g_{00}T = -\rho - \frac{1}{2}\rho = -\frac{3}{2}\rho$

② radiation $w = \frac{1}{3} \Rightarrow p = w\rho = \frac{1}{3}\rho$

$T = g_{\mu\nu}T^{\mu\nu} = -\rho + \frac{1}{3}\rho \times 3 = 0$, $S_{00} = T_{00} - \frac{1}{2}g_{00}T = -\rho$

③ dark energy $w=-1 \Rightarrow p = w\rho = -\rho$

$T = g_{\mu\nu}T^{\mu\nu} = -\rho - \rho\times 3 = -4\rho$, $S_{00} = T_{00} - \frac{1}{2}g_{00}T = -\rho + \frac{1}{2}(-4\rho) = -3\rho$

2. $d\tau^2 = dt^2 - a^2(t)(dx^2+dy^2+dz^2) \Rightarrow g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -a^2(t) & & \\ & & -a^2(t) & \\ & & & -a^2(t) \end{pmatrix}$

(1) $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})$

$\Rightarrow \Gamma^0_{ij} = \frac{1}{2}g^{00}(g_{i0,j} + g_{j0,i} - g_{ij,0}) = \frac{1}{2}\cdot 1\cdot(0+0+\delta_{ij}\,2a\dot{a}) = a\dot{a}\delta_{ij}$

$\Gamma^i_{0j} = \frac{1}{2}g^{ii}(g_{0i,j} + g_{ji,0} - g_{0j,0}) = \frac{1}{2}(-\frac{1}{a^2})(-2a\dot{a}) = \frac{\dot{a}}{a}\delta_{ij}$

$\Gamma^i_{jl} = \frac{1}{2}g^{ii}(g_{ji,l} + g_{li,j} - g_{jl,i}) = \frac{1}{2}(-\frac{1}{a^2})\cdot 0 = 0$

(2) $R_{\mu\nu} = \Gamma^\alpha_{\alpha\mu,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\mu\alpha} - \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{\mu\nu}$

$\Rightarrow R_{00} = \Gamma^\alpha_{\alpha 0,0} - \Gamma^\alpha_{00,\alpha} + \Gamma^\alpha_{0\lambda}\Gamma^\lambda_{0\alpha} - \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{00}$

$= 3\frac{d(\dot{a}/a)}{dt} + 3(\frac{\dot{a}}{a})^2 = 3\frac{\ddot{a}}{a} + 3\dot{a}(-\frac{1}{a^2})\cdot\dot{a} + 3(\frac{\dot{a}}{a})^2 = 3\frac{\ddot{a}}{a}$

$R_{ij} = \Gamma^\alpha_{\alpha i,j} - \Gamma^\alpha_{ij,\alpha} + \Gamma^\alpha_{j\lambda}\Gamma^\lambda_{i\alpha} - \Gamma^\alpha_{\lambda\alpha}\Gamma^\lambda_{ij}$

$= -\Gamma^0_{ij,0} + (\Gamma^0_{jj}\Gamma^j_{i0} + \Gamma^i_{j0}\Gamma^0_{ii}) - \Gamma^\alpha_{0\alpha}\Gamma^0_{ij}$

$= -(\dot{a}^2 + a\ddot{a})\delta_{ij} + 2a\dot{a}\frac{\dot{a}}{a} - 3\frac{\dot{a}}{a}a\dot{a}\delta_{ij} = -(a\ddot{a} + 2\dot{a}^2)\delta_{ij}$

3. Energy conservation: $\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \frac{d\ln\rho}{dt} = -3(1+w)\frac{d\ln a}{dt}$

$\Rightarrow d\ln\rho = -3(1+w)d\ln a \Rightarrow \ln\rho = -3(1+w)\ln a + C_1$

$\Rightarrow \rho = C_2 a^{-3(1+w)} \Rightarrow \rho \propto a^{-3(1+w)}$ ①

Friedmann equation: $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho(a) = H^2(a)$

$\xRightarrow{①} (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)}$

since $K=0$, so $\frac{8\pi G}{3}\rho_0 = \frac{8\pi G}{3}\rho_{0,cr} = H_0^2$

$\Rightarrow \dot{a}^2 = H_0^2 a^{-(1+3w)}$

$\Rightarrow da/dt = H_0 a^{-\frac{1}{2}(1+3w)}$

$\Rightarrow a^{\frac{1}{2}(1+3w)}da = H_0\,dt$ ②

(1) non-relativistic matter, $w=0$, ② $\Rightarrow a^{\frac{3}{2}} = H_0 t \Rightarrow a(t) \propto t^{\frac{2}{3}}$

$H = \frac{\dot{a}}{a} = \left(\frac{2}{3}t^{-\frac{1}{3}}\right)/t^{\frac{2}{3}} = \frac{2}{3t} \Rightarrow t = \frac{2}{3H}$, i.e. $t_0 = \frac{2}{3H_0}$

$D_h^m = a(t)\int_0^t \frac{c}{a(t')}dt' = ct^{\frac{2}{3}}\int_0^t t'^{-\frac{2}{3}}dt' = ct^{\frac{2}{3}}\cdot 3t^{\frac{1}{3}} = 3ct = \frac{2c}{H}$

(2) radiation, $w = \frac{1}{3}$, ② $\Rightarrow a^2 = H_0 t \Rightarrow a(t) \propto t^{\frac{1}{2}}$

$H = \frac{\dot{a}}{a} = \left(\frac{1}{2}t^{-\frac{1}{2}}\right)/t^{\frac{1}{2}} = \frac{1}{2t} \Rightarrow t = \frac{1}{2H}$, i.e. $t_0 = \frac{1}{2H_0}$

$D_h^r = a(t)\int_0^t \frac{c}{a(t')}dt' = ct^{\frac{1}{2}}\int_0^t t'^{-\frac{1}{2}}dt' = ct^{\frac{1}{2}}\cdot 2t^{\frac{1}{2}} = 2ct = \frac{c}{H}$

(3) dark energy, $w=-1$ ② $\Rightarrow \ln a = H_0 t + C_1 \Rightarrow a(t) \propto e^{H_0 t}$

4. $\Omega_{r,0} = 8\times10^{-5}$, $\Omega_{m,0} = 0.25 - \Omega_{r,0}$, $\Omega_{\Lambda,0} = 1-\Omega_{m,0}$, $H_0 = 70$ km/s/Mpc

(1) $\rho_m = \rho_r \Rightarrow \Omega_{m,0}(1+z)^3 = \Omega_{r,0}(1+z)^4$

$1+z = \Omega_{m,0}/\Omega_{r,0}$

$1+z = 0.25/8\times10^{-5} - 1$

$z = 3125 - 2 = 3123$

(2) $\rho_m = \rho_\Lambda \Rightarrow \Omega_{m,0}(1+z)^3 = \Omega_{\Lambda,0}$

$(1+z)^3 = (1-\Omega_{m,0})/\Omega_{m,0} \approx \frac{1}{0.25} - 1 = 3$

$z \approx 0.442$

(3) $H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{\Lambda,0}}$

$a = \frac{1}{1+z} \Rightarrow da = -\frac{1}{(1+z)^2}dz$

$t = \int_0^t dt' = \int_0^a da'\frac{dt'}{da'} = \int_0^a da'\frac{1}{\dot{a}'} = \int_0^a \frac{1}{a'H}da' = \int_z^\infty \frac{1+z'}{H}\left(-\frac{1}{(1+z')^2}dz'\right) = \int_z^\infty \frac{dz'}{(1+z')H(z')}$

Use this equatien to do numerical integration, we get:

```
At redshift z = 30, the universe was 0.10659 Gyr old.
At redshift z = 20, the universe was 0.19194 Gyr old.
At redshift z = 6, the universe was 1.00187 Gyr old.
At redshift z = 2, the universe was 3.51920 Gyr old.
At redshift z = 1, the universe was 6.22980 Gyr old.
At redshift z = 0.1, the universe was 12.85869 Gyr old.
```

sun × (4.6 Gyr ago)

dinosaurs × (0.23Gyr ago)