prob 2.1

1. Assuming the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ with convention $\eta_{00} = -1$, and a perfect fluid has an energy momentum tensor $T^{\mu\nu}$ taking the form

$$\begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

where ρ and $P = w\rho$ are the density and pressure of the fluid. For non-relativistic matter, radiation and dark energy, $w = 0,\ 1/3,\ -1$, respectively. Please calculate $T \equiv g_{\mu\nu}T^{\mu\nu}$ and $S_{00} = T_{00} - 1/2 g_{00}\, T$ for the three species.

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$T = g_{\mu\nu}T^{\mu\nu} = g_{00}T^{00} + g_{11}T^{11} + g_{22}T^{22} + g_{33}T^{33} = 3p - \rho = (3w-1)\rho$$

$$S_{00} = T_{00} - \frac{1}{2} g_{00} T = -\rho - \frac{1}{2}(-1)(3w-1)\rho = \frac{3(w-1)}{2}\rho$$

(1) for non-relativistic matter, $w=0 \Rightarrow T = -\rho,\ S_{00} = -\frac{1}{2}\rho$

(2) for radiation, $w=\frac{1}{3} \Rightarrow T=0,\ S_{00}=-\rho$

(3) for dark matter, $w=-1 \Rightarrow T=-4\rho,\ S_{00}=-3\rho$

prob 2.2

2. Assuming a flat space geometry (K=0), the FLRW metric takes the form: $\mathrm{d}\tau^2=\mathrm{d}t^2-a^2(t)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)$. Please calculate all the components of the Christoffel connection $\Gamma^\alpha_{\mu\nu}$, show that: $\Gamma^0_{ij}=a\dot{a}\delta_{ij}$, $\Gamma^i_{0j}=\dot{a}/a\delta_{ij}$, $\Gamma^i_{jl}=0$, and $R_{00}=3\ddot{a}/a$, $R_{ij}=-(a\ddot{a}+2\dot{a}^2)\delta_{ij}$.

$$g_{\mu\nu}=\begin{bmatrix} -1 & & & \\ & a^2(t) & & \\ & & a^2(t) & \\ & & & a^2(t)\end{bmatrix} \Rightarrow \begin{cases} g_{00}=g^{00}=-1 \\ g_{11}=g_{22}=g_{33}=a^2(t) \\ g^{11}=g^{22}=g^{33}=\dfrac{1}{a^2(t)}\end{cases}$$

only compute non-zero components:

$$\Gamma^\alpha_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\nu\beta}}{\partial x^\mu}+\frac{\partial g_{\mu\beta}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\beta}\right)=\frac{1}{2}g^{\alpha\alpha}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\mu}+\frac{\partial g_{\mu\alpha}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)$$

$$\Rightarrow \begin{cases} \Gamma^{\alpha}_{\mu\alpha} = \Gamma^{\alpha}_{\alpha\mu} = \frac{1}{2} g^{\alpha\alpha} \frac{\partial g_{\alpha\alpha}}{\partial x^{\mu}} \\ \Gamma^{\alpha}_{\mu\mu} = -\frac{1}{2} g^{\alpha\alpha} \frac{\partial g_{\mu\mu}}{\partial x^{\alpha}} \end{cases}$$

$$\therefore \begin{cases} \Gamma^{0}_{ii} = -\frac{1}{2} g^{00} \frac{\partial g_{ii}}{\partial x^{0}} = \frac{1}{2} \frac{\partial a^2}{\partial t} = a\dot{a} \\ \Gamma^{i}_{0i} = \Gamma^{i}_{i0} = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x^{0}} = \frac{1}{2a^2} \frac{\partial a^2}{\partial t} = \frac{\dot{a}}{a} \end{cases}$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

$$\therefore R_{00} = \frac{\partial \Gamma^{i}_{00}}{\partial x^{i}} - \frac{\partial \Gamma^{i}_{0i}}{\partial x^{0}} + \Gamma^{0}_{i0}\Gamma^{i}_{00} - \Gamma^{i}_{i0}\Gamma^{i}_{0i}$$

$$= -3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\,\frac{\ddot{a}a - \dot{a}^2}{a^2} - 3\,\frac{\dot{a}^2}{a^2} = -3\,\frac{\ddot{a}}{a}$$

$$R_{ii} = \frac{\partial \Gamma^0_{ii}}{\partial x^0} + \Gamma^\alpha_{0\alpha}\Gamma^0_{ii} - \Gamma^0_{ii}\Gamma^i_{i0} - \Gamma^i_{0i}\Gamma^0_{ii} = \frac{\partial}{\partial t}(a\dot{a}) + 3\dot{a}^2 - 2\dot{a}^2 = a\ddot{a} + 2\dot{a}^2$$

The different signs for R_{00} and R_{ii} are due to different metric convention.

prob 2.3

3. The Friedmann equation derived from the GR field equation ultimately determines the evolution of the scale factor $a(t)$. Now let us assume the universe is dominated by a single component i below in a flat space geometry (K=0). Through the Friedmann equation $(\dot{a}/a)^2 = 8\pi G/3\rho_i(a)$ and local energy conservation $\dot{\rho}_i + 3\dot{a}/a(1+w_i)\rho_i = 0$, work out the time dependence of a(t), the particle horizon distance $D_{\rm h}(t)$ at $a(t)$, and the age of the Universe today t_0, show that:
(1) Non-relativistic matter: $a(t) \propto t^{2/3}$, $D_{\rm h}^{\rm m}(t) = 2c/H = 3ct$, $t_0 = 2/(3H_0)$;
(2) Radiation: $a(t) \propto t^{1/2}$, $D_{\rm h}^{\rm rad}(t) = c/H = 2ct$, $t_0 = 1/(2H_0)$;
(3) Dark energy (cosmological constant): $a(t) \propto \exp(H_0 t)$;
where c is the speed of light, H_0 is the Hubble constant today.

local energy conservation: $\dot{\rho}_i + 3\frac{\dot{a}}{a}(1+w_i)\rho_i = 0$

$$\Rightarrow \frac{d\rho_i}{dt}\frac{1}{\rho_i} = -\frac{3}{a}\frac{da}{dt}(1+w_i) \Rightarrow \frac{d\rho_i}{\rho_i} = -3(1+w_i)\frac{da}{a}$$

$$\Rightarrow \ln\rho_i = -3(1+w_i)\ln a + C_1 \Rightarrow \rho_i(t) = C_2\, a(t)^{-3(1+w_i)}$$

Friedmann eq $\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_i(a) = C_3\, a^{-3(1+w_i)} \Rightarrow \dot{a}^2 = C_3\, a^{-1-3w_i}$

$$\Rightarrow \dot{a} = \frac{da}{dt} = C_3\, a^{-\frac{1}{2}-\frac{3}{2}w_i} \Rightarrow a^{\frac{1}{2}+\frac{3}{2}w_i}\, da = C_3\, dt \Rightarrow \frac{a^{\frac{3}{2}(1+w_i)}}{\frac{3}{2}(1+w_i)} = C_3 t + C_4$$

$$\Rightarrow a(t) = C_5\, t^{\frac{2}{3(1+w_i)}} + C_6 = a_0\, t^{\frac{2}{3(1+w_i)}} \propto t^{\frac{2}{3(1+w_i)}}$$

Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{2}{3(1+w_i)}\frac{1}{t}$

age of the Universe today $t_0 = \frac{2}{3(1+w_i)}\frac{1}{H_0}$

particle horizon distance

$$d_h(t) = a(t)\, c \int_0^t \frac{d\tau}{a(\tau)} = c\, t^{\frac{2}{3(1+w_i)}} \int_0^t \tau^{-\frac{2}{3(1+w_i)}} d\tau = c\, t^{\frac{2}{3(1+w_i)}} \frac{3(1+w_i)}{1+3w_i}\, t^{\frac{1+3w_i}{3(1+w_i)}}$$

$$= \frac{3(1+w_i)}{1+3w_i}\, ct = \frac{2c}{(1+3w_i)H}$$

(1) for non-relativistic matter, $w_i = 0$:

$$\Rightarrow a(t) \propto t^{2/3},\quad d_h(t) = \frac{2c}{H} = 3ct,\quad t_0 = \frac{2}{3H_0}$$

(2) for radiation, $w_i = \frac{1}{3}$:

$$\Rightarrow a(t) \propto t^{1/2},\quad d_h(t) = \frac{c}{H} = 2ct,\quad t_0 = \frac{1}{2H_0}$$

(3) for dark energy, $w_i = -1 \Rightarrow \dot{\rho}_i = 0 \Rightarrow \rho_i = C$

$$\Rightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_i = C = H_0^2 \Rightarrow \frac{da}{a} = H_0\, dt \Rightarrow a(t) \propto e^{H_0 t}$$

prob 2.4

4. Assuming $\Omega_{\mathrm{rad},0} = 8 \times 10^{-5}$, $\Omega_{\mathrm{m},0} = 0.25 - \Omega_{\mathrm{rad},0}$, $\Omega_{\Lambda,0} = 1 - \Omega_{\mathrm{m},0}$, $H_0 = 70\,\mathrm{km/s/Mpc}$, using numerical integration to calculate:

(1) Redshift and age of the universe when $\rho_{\mathrm{matter}} = \rho_{\mathrm{rad}}$.

(2) Redshift and age of the universe when $\rho_{\mathrm{matter}} = \rho_{\Lambda}$.

(3.1) Age of the universe when $z = 30$ and $z = 20$ (first stars form);

(3.2) Age of the universe when $z = 6$ (time about reionization);

(3.3) Age of the universe when $z = 2$ ("cosmic noon");

(3.4) Age of the universe when $z = 1$, was the Sun born by then?

(3.5) Age of the universe when $z = 0.1$, had dinosaurs come to exist?

$$\Omega_{m0} = \frac{\rho_{m0}}{\rho_c} = 0.25 - 8\times10^{-5}, \quad \Omega_{r0} = \frac{\rho_{r0}}{\rho_c} = 8\times10^{-5}, \quad \Omega_{\Lambda0} = \frac{\rho_{\Lambda0}}{\rho_c} = 0.75 + 8\times10^{-5}$$

$$\rho_r(z) = \rho_{r0}(1+z)^4 = \rho_c\Omega_{r0}(1+z)^4, \quad \rho_m(z) = \rho_{m0}(1+z)^3 = \rho_c\Omega_{m0}(1+z)^3, \quad \rho_\Lambda = \rho_{\Lambda0} = \rho_c\Omega_{\Lambda0}$$

Hubble parameter at redshift z: $H(z) = H_0\sqrt{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{\Lambda0}}$

age of the universe at redshift z:

$$t(z)=\int_z^{+\infty}\frac{dz'}{(1+z')H(z')}=\int_0^{\frac{1}{1+z}}\frac{1}{H_0}\frac{dx}{x\sqrt{\frac{\Omega_{r0}}{x^4}+\frac{\Omega_{m0}}{x^3}+\Omega_{\Lambda 0}}}=\frac{1}{H_0}\int_0^{\frac{1}{1+z}}\frac{dx}{\sqrt{\frac{\Omega_{r0}}{x^2}+\frac{\Omega_{m0}}{x}+\Omega_{\Lambda 0}x^2}}$$

(1) when $\rho_r(z)=\rho_m(z) \Rightarrow \rho_c\Omega_{r0}(1+z)^4=\rho_c\Omega_{m0}(1+z)^3$

$$\Rightarrow z=\frac{\Omega_{m0}}{\Omega_{r0}}-1\approx 3.123\times10^3 \Rightarrow t(z)\approx 6.249\times10^{-5}\ \text{Gyr}$$

(2) when $\rho_m(z)=\rho_\Lambda \Rightarrow \rho_c\Omega_{m0}(1+z)^3=\rho_c\Omega_{\Lambda 0}$

$$\Rightarrow z=\sqrt[3]{\frac{\Omega_{\Lambda 0}}{\Omega_{m0}}}-1\approx 4.425\times10^{-1} \Rightarrow t(z)\approx 9.471\ \text{Gyr}$$

(3) $t(30)\approx 1.065\times10^{-1}$ Gyr, $t(20)\approx 1.918\times10^{-1}$ Gyr

$t(6)\approx 1.001$ Gyr, $t(2)\approx 3.517$ Gyr

$t(1)\approx 6.225$ Gyr, $t(0.1)\approx 1.285\times10$ Gyr

The numerical integrations are computed by a Python script, which is also attached.