prob 3.1

1. Assuming the universe is matter dominated, calculate the deviation of total matter density from the critical density in terms of $|\Omega_{tot} - 1|$ at $1\,s$ after the Big Bang (where $\Omega_{tot} = \rho_{tot}/\rho_{cr}$), if we require $|\Omega_{tot} - 1| \leqslant 1$ at the present day as is indicated by observations. Why an accelerating universe can help with solving the *flatness problem*?

(1) Friedmann eq can be written as: $|\Omega_t - 1| = \dfrac{|K|}{a^2 H^2}$

in a matter-dominated universe:

$a \propto t^{2/3} \Rightarrow H = \dfrac{\dot{a}}{a} = \dfrac{2}{3t} \Rightarrow a^2 H^2 \propto t^{-2/3}$

$\Rightarrow |\Omega_t - 1| \propto t^{2/3}$

at the present day: $|\Omega_t(t_0) - 1| \leq 1$, $t_0 \approx 1.38 \times 10^{10}\,yr \approx 4.35 \times 10^{17}\,s$

$\Rightarrow |\Omega_t(1s) - 1| = |\Omega_t(t_0 - 1)| \left(\dfrac{1s}{t_0}\right)^{2/3} \leq 1.74 \times 10^{-12}$

(2) in an accelerating universe: $\ddot{a} > 0$

Friedmann equation: $\dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}(\rho + 3p) > 0 \Rightarrow \rho + 3p < 0 \Rightarrow w < -\dfrac{1}{3}$

$$\left.\begin{array}{l} a \propto t^{\frac{2}{3(1+w)}} \\[2em] H = \dfrac{2}{3(1+w)}\dfrac{1}{t} \end{array}\right\} \Rightarrow aH \propto t^{-\frac{1+3w}{3(1+w)}} \Rightarrow |\Omega_t - 1| \propto t^{\frac{2(1+3w)}{3(1+w)}}$$

since $w < -\dfrac{1}{3}$ and $w > -1$, $|\Omega_t - 1|$ is shrinking and inflation drives $\Omega_t \to 1$

2. Assuming the universe is matter dominated, calculate the ratio between the comoving horizon today and that at $z = 1100$, where the cosmic microwave background (CMB) radiation came out. We observe that the CMB sky is smooth, homogeneous and isotropic to a level of 10^{-5} over the entire sky. Why does this indicate a *horizon problem*? Why an accelerating universe can help with solving this problem?

(1) in a matter - dominated universe:

co-moving horizon $D_h(t) = \dfrac{2c}{H} = \dfrac{2c}{(1+z)H_0} = D_h(z)$

$\Rightarrow \dfrac{D_h(z=0)}{D_h(z=1100)} = \sqrt{1+1100} \approx 33.18$

(2) Regions of the sky separated by more than the co-moving horizon at $z = 1000$ could not have been in causal contact in a standard matter-

dominated universe. But the tempertaure of the CMB is uniform across these regions.

(3) During inflation the causally connected region is shrinking. It was very large before inflation; much larger than the present horizon. Thus the present observable universe has evolved from a small patch of a much larger causally connected region.

prob 3.3

(1) $k_{start} = H_{start} \dfrac{a_{start}}{a_0} \Rightarrow a_{start} = \dfrac{k_{start}}{H_{start}} a_0$

$$N(t_{start}) = \ln \frac{a_{end}}{a_{start}} = \ln \frac{a_{end}}{a_0} \frac{H_{start}}{k_{start}} = \ln \frac{a_{end}}{a_0} \frac{H_{start}}{H_0} \frac{H_0}{k_{start}}$$

$$= \ln \frac{a_{end}}{a_0} + \ln \frac{H_{start}}{H_0} + \ln \frac{H_0}{k_{start}}$$

(2) during slow-roll inflation: $H_{start}^2 = \dfrac{8\pi G}{3} V_{SR}$

Friedmann eq. today: $H_0^2 = \dfrac{8\pi G}{3} \rho_{c0}$

$\Rightarrow \dfrac{H_{start}}{H_0} = \sqrt{\dfrac{V_{SR}}{\rho_0}} = \dfrac{V_{SR}^{1/4}}{10^{16} \, GeV} \dfrac{10^{16} \, GeV \cdot V_{SR}^{1/4}}{\sqrt{\rho_{c0}}}$

(3)

$$\left. \begin{array}{l} \rho_r(a_{eq}) = \rho_{ro} \dfrac{a_o^4}{a_{eq}^4} \approx \rho(a_{eq}) \\[4mm] \rho_{reh}\, a_{reh}^4 = \rho(a_{eq})\, a_{eq}^4 \end{array} \right\} \Rightarrow \rho_{reh} = \rho_{ro} \dfrac{a_o^4}{a_{reh}^4}$$

$$\rho_{end}\, a_{end}^3 = \rho_{reh}\, a_{reh}^3$$

$$\Rightarrow \rho_{end} = \rho_{reh} \frac{a_{reh}^3}{a_{end}^3} = \rho_{ro} \frac{a_o^4}{a_{reh}^4} \frac{a_{reh}^3}{a_{end}^3} = \rho_{ro} \frac{a_o^4}{a_{reh}\, a_{end}^3}$$

$$a_{reh}/a_o = (\rho_{ro}/\rho_{reh})^{1/4}$$

$$\Rightarrow \frac{a_{end}}{a_o} = \left(\frac{\rho_{ro}}{\rho_{end}} \frac{a_o}{a_{reh}} \right)^{1/3} = \left(\frac{\rho_{ro}}{\rho_{end}} \frac{\rho_{reh}^{1/4}}{\rho_{ro}^{1/4}} \right)^{1/3}$$

$$\Rightarrow N(t_{start}) = \frac{1}{3} \ln\left(\frac{\rho_{ro}}{\rho_{end}} \frac{\rho_{reh}^{\frac{1}{4}}}{\rho_{ro}^{1/4}} \right) + \ln\left(\frac{V_{SR}^{\frac{1}{4}}}{10^{16}\,GeV} \frac{10^{16}\,GeV \cdot V_{SR}^{\frac{1}{4}}}{\rho_{co}^{\frac{1}{2}}} \right) + \ln \frac{H_o}{k_{start}}$$

$$= \ln \frac{H_0}{k_{start}} + \ln \frac{10^{16} \, GeV \cdot \rho_{ro}^{1/4}}{\rho_{co}^{1/2}} + \frac{1}{12} \ln \frac{\rho_{reh}}{\rho_{end}} + \frac{1}{4} \ln \frac{V_{SR}}{\rho_{end}} + \ln \frac{V_{SR}^{1/4}}{10^{16} \, GeV}$$

(4) $k_{start} \leq H_0$

$$\left. \begin{array}{l} \rho_{co} \approx (3 \times 10^{-12} \, GeV)^4 \, h^2 = (3 \times 10^{-12} \, GeV)^4 \left(\frac{H_0}{100 \, km/s/Mpc} \right)^2 \\ \\ \rho_{ro} \approx (2.5 \times 10^{-13} \, GeV)^4 \end{array} \right\}$$

$$\Rightarrow N(t_{start}) \approx \ln \frac{H_0}{k_{start}} + \ln \frac{10^{16} \, GeV \cdot \rho_{ro}^{1/4}}{\rho_{co}^{1/2}} \geq \ln \frac{10^{16} \, GeV \times 2.5 \times 10^{-13} \, GeV}{(3 \times 10^{-12} \, GeV)^2 \frac{H_0}{100 \, km/s/Mpc}}$$

$$\geq 61$$

prob 3.4

(1) $V(\varphi) = \frac{1}{2} m^2 \varphi^2 \Rightarrow V'(\varphi) = m^2 \varphi \Rightarrow \begin{cases} 3H\dot{\varphi} = -m^2 \varphi \\ H^2 = \frac{1}{6} \frac{m^2}{M_{pl}^2} \varphi^2 \end{cases}$

$\Rightarrow \varepsilon = \frac{1}{2} M_{pl}^2 \left[\frac{V'(\varphi)}{V(\varphi)} \right]^2 = \frac{1}{2} M_{pl}^2 \left(\frac{m^2 \varphi}{\frac{1}{2} m^2 \varphi^2} \right)^2 = \frac{2 M_{pl}^2}{\varphi^2} = \frac{m^2}{3H^2}$

$\left. \begin{array}{l} \frac{1}{2} \dot{\varphi}^2 \ll \frac{1}{2} m^2 \varphi^2 \Rightarrow \dot{\varphi}^2 \ll m^2 \varphi^2 \\ 3H\dot{\varphi} = -m^2 \varphi \end{array} \right\} \Rightarrow \frac{\dot{\varphi}^2}{\varphi^2} = \left(-\frac{m^2}{3H} \right)^2 = \frac{m^4}{9H^2} \ll m^2$

$\therefore \varepsilon = \frac{m^2}{3H^2} \ll 1$

$H^2 = \frac{1}{6} \frac{m^2}{M_{pl}^2} \varphi^2 \Rightarrow 2H\dot{H} = \frac{1}{3} \frac{m^2}{M_{pl}^2} \varphi \dot{\varphi} = \frac{1}{3} \frac{6H^2}{\varphi^2} \varphi \frac{-m^2 \varphi}{3H} = -\frac{2}{3} m^2 H$

$\therefore 3\dot{H} = -m^2 \Rightarrow \varepsilon = \frac{m^2}{3H^2} = -\frac{\dot{H}}{H^2}$

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}^2}{a^2} \left.\begin{array}{c}\\\\\\\\\end{array}\right\} \Rightarrow \dot{H} + H^2 = \frac{\ddot{a}a}{a^2} - \frac{\dot{a}^2}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} \gg 0$$

$$\varepsilon = -\frac{\dot{H}}{H^2} \ll 1 \Rightarrow \dot{H} + H^2 \gg 0$$

(2)
$$\varepsilon = \frac{1}{2}M_{pl}^2\left[\frac{V'(\varphi)}{V(\varphi)}\right]^2 = \frac{1}{2}M_{pl}^2\left(\frac{m^2\varphi}{\frac{1}{2}m^2\varphi^2}\right)^2 = \frac{2M_{pl}^2}{\varphi^2} = 1$$

$$\therefore \varphi_{end} = \sqrt{2}\, M_{pl}$$

(3)
$$\begin{cases} 3H\dot{\varphi} = -V'(\varphi) \\ H^2 = \frac{V(\varphi)}{3M_{pl}^2} \end{cases} \Rightarrow \frac{H}{3\dot{\varphi}} = -\frac{1}{3M_{pl}^2}\frac{V(\varphi)}{V'(\varphi)}$$

$$\Rightarrow \int_{\varphi(t)}^{\varphi_{end}} \frac{H}{\dot{\varphi}}\, d\varphi = \frac{1}{M_{pl}^2}\int_{\varphi_{end}}^{\varphi(t)} \frac{V(\varphi)}{V'(\varphi)}\, d\varphi$$

$$\int_{\varphi(t)}^{\varphi_{end}} \frac{H}{\dot{\varphi}} d\varphi = \int_{t}^{t_{end}} H \, dt = \int_{a(t)}^{a(t_{end})} \frac{da}{a} = \ln \frac{a(t_{end})}{a(t)} = N(t)$$

(4) $V(\varphi) = \frac{1}{2} m^2 \varphi^2 \Rightarrow V'(\varphi) = m^2 \varphi$

$$N(t) = \frac{1}{M_{pl}^2} \int_{\varphi_{end}}^{\varphi(t)} \frac{V(\varphi)}{V'(\varphi)} d\varphi = \frac{1}{M_{pl}^2} \int_{\varphi_{end}}^{\varphi(t)} \frac{\varphi}{2} d\varphi = \frac{1}{4 M_{pl}^2} \left[\varphi^2(t) - \varphi_{end}^2 \right]$$

$$N(t_{start}) = \frac{1}{4 M_{pl}^2} \left[\varphi^2(t_{start}) - \varphi_{end}^2 \right] \geqslant 60$$

$$\varphi_{end} = \sqrt{2} \, M_{pl}$$

$$\therefore \varphi(t_{start}) \geqslant \sqrt{\varphi_{end}^2 + 240 M_{pl}^2} = \sqrt{242} \, M_{pl} \approx 15.56 \, M_{pl}$$

$$\therefore \varepsilon = \frac{2 M_{pl}^2}{\varphi^2(t_{start})} = \frac{1}{121} \ll 1$$