

# Solutions to Problems 2

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## 2

### 2.1

Assume that you have both emission and absorption spectra for the 21 cm line in a particular direction, and can therefore determine  $\tau_\nu$  across the line profile.

#### 2.1.1

Show that the measured excitation temperature  $T_e$  in any line of sight is actually the harmonic mean of all components with various excitation temperatures in that direction. That is, if there are  $M$  separate clouds along the line of sight, each with column density  $N_i$  and excitation temperature  $T_i$  and total column density  $N_{\text{tot}} = \sum_{i=1}^M N_i$ , then

$$\frac{1}{T_e} = \frac{1}{N_{\text{tot}}} \sum_{i=1}^M \frac{N_i}{T_i} \quad (2.1.1)$$

**Solution:** For each cloud  $i$ , if the excitation temperature  $T_i$  is constant along the line of sight, the optical depth at frequency  $\nu$  is given by:

$$\tau_{\nu,i} = \frac{g_u}{g_l} \sigma_{ul} (1 - e^{-h\nu_{ul}/k_B T_i}) N_i \quad (2.1.2)$$

where  $\sigma_{ul}$  is the cross section for stimulated emission. When looking at the 21 cm line, we are in the regime where  $k_B T_i$  is much greater than  $h\nu_{ul} = 5.87 \mu\text{eV}$ . With  $k_B T_i \gg h\nu_{ul}$ ,

$$\tau_{\nu,i} \approx \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_i} N_i \quad (2.1.3)$$

Similarly, the total optical depth along the line of sight at frequency  $\nu$  is given by:

$$\tau_\nu \approx \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_e} N_{\text{tot}} \quad (2.1.4)$$

The total optical depth is also the sum of the optical depths of individual clouds:

$$\tau_\nu = \sum_{i=1}^M \tau_{\nu,i} \approx \sum_{i=1}^M \frac{g_u}{g_l} \sigma_{ul} \frac{h\nu_{ul}}{k_B T_i} N_i = \frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \sum_{i=1}^M \frac{N_i}{T_i} \quad (2.1.5)$$

Compare Equation 2.1.4 and Equation 2.1.5, we get:

$$\begin{aligned} \frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \frac{N_{\text{tot}}}{T_e} &= \frac{g_u \sigma_{ul} h\nu_{ul}}{g_l k_B} \sum_{i=1}^M \frac{N_i}{T_i} \\ \implies \frac{1}{T_e} &= \frac{1}{N_{\text{tot}}} \sum_{i=1}^M \frac{N_i}{T_i} \end{aligned} \quad (2.1.6)$$

### 2.1.2

Typical measured excitation temperatures for lines of sight near the galactic plane are  $T_e \approx 150$  K. Assume that all emission is optically thin, and all the H I is either in cold neutral clouds or in a warm neutral phase (this is the 2-phase model of the ISM). What is the fraction of the H I in each phase, given  $T_e = 150$  K?

**Solution:** Since all the H I gas is either in cold neutral clouds or in a warm neutral phase, utilize Equation 2.1.1:

$$\begin{cases} N_{\text{tot}} = N_c + N_w \\ \frac{1}{T_e} = \frac{1}{N_{\text{tot}}} \left( \frac{N_c}{T_c} + \frac{N_w}{T_w} \right) \end{cases} \quad (2.1.7)$$

where subscript  $_c$  refers to the variables of H I in the cold phase; subscript  $_w$  refers to the variables of H I in the warm phase. Typical temperature in CNM is  $T_c \approx 80$  K and the one in WNM is  $T_w \approx 8000$  K. Now, substitute these values into Equation 2.1.7 and solve this linear system:

$$\begin{cases} \frac{N_c}{N_{\text{tot}}} = \frac{T_c(T_w - T_e)}{T_e(T_w - T_c)} \approx 52.86\% \\ \frac{N_w}{N_{\text{tot}}} = \frac{T_w(T_c - T_e)}{T_e(T_c - T_w)} \approx 47.14\% \end{cases} \quad (2.1.8)$$

## 2.2

Consider a region containing only partially-ionized hydrogen. Let  $\zeta$  be the ionization rate per H atom and  $\alpha$  be the recombination coefficient.

### 2.2.1

Determine the steady-state ionization fraction  $x_{\text{ss}}$  in terms of  $n_{\text{H}} = n(\text{HI}) + n(\text{HII})$ ,  $\zeta$  and  $\alpha$ .

**Solution:** The ionization fraction  $x(t)$  is defined as:

$$x(t) = \frac{n(\text{HII})}{n_{\text{H}}} \quad (2.2.1)$$

Then the neutral fraction is:

$$1 - x(t) = \frac{n(\text{HI})}{n_{\text{H}}} \quad (2.2.2)$$

The ionization rate per unit volume is  $\zeta n(\text{HI}) = \zeta n_{\text{H}}(1 - x)$  and the recombination rate per unit volume is  $\alpha n_e n(\text{HII}) = \alpha n^2(\text{HII}) = \alpha n_{\text{H}}^2 x^2$ . Notice that we have used  $n_e = n(\text{HII})$  since each ionized hydrogen atom contributes 1 electron. The rate of change of the ionized hydrogen density is given by the balance between ionization and recombination:

$$\frac{dn(\text{HII})}{dt} = \zeta n_{\text{H}}(1 - x) - \alpha n_{\text{H}}^2 x^2 = n_{\text{H}} \frac{dx}{dt} \quad (2.2.3)$$

where  $n_{\text{H}}$  is constant. At steady state:

$$\begin{aligned} \frac{dx}{dt} &= 0 \\ \implies 0 &= \zeta(1 - x_{\text{ss}}) - \alpha n_{\text{H}} x_{\text{ss}}^2 \end{aligned} \quad (2.2.4)$$

This is a quadratic equation. Use the quadratic formula:

$$x_{\text{ss}} = \frac{-\zeta \pm \sqrt{\zeta^2 - 4\alpha n_{\text{H}}(-\zeta)}}{2\alpha n_{\text{H}}} = \frac{\zeta}{2\alpha n_{\text{H}}} \left( -1 \pm \sqrt{1 + \frac{4\alpha n_{\text{H}}}{\zeta}} \right) \quad (2.2.5)$$

Since ionization fractions must be between 0 and 1, we discard the negative root. Thus, the steady-state ionization fraction is:

$$x_{\text{ss}} = \frac{\zeta}{2\alpha n_{\text{H}}} \left( \sqrt{1 + \frac{4\alpha n_{\text{H}}}{\zeta}} - 1 \right) \quad (2.2.6)$$

### 2.2.2

Suppose that the fractional ionization at time  $t = 0$  is given by  $x(0) = x_{\text{ss}} + \delta(0)$ . If  $|\delta(0)| \ll x_{\text{ss}}$ , determine the solution  $x(t)$  by assuming  $n_{\text{H}}$ ,  $\zeta$ , and  $\alpha$  to be constant.

**Solution:** Start from the rate equation, Equation 2.2.3:

$$\frac{dx}{dt} = \zeta(1 - x) - \alpha n_{\text{H}} x^2 \quad (2.2.7)$$

We can write  $x(t)$  as a perturbation around the steady state:

$$x(t) = x_{\text{ss}} + \delta(t) \quad (2.2.8)$$

Given that  $|\delta| \approx |\delta(0)| \ll x_{\text{ss}}$ , we can linearize the rate equation by expanding to 1st order in  $\delta$ . Substitute  $x(t)$  into Equation 2.2.7:

$$\begin{aligned} \frac{dx}{dt} &= \frac{d\delta}{dt} = \zeta(1 - x_{\text{ss}} - \delta) - \alpha n_{\text{H}}(x_{\text{ss}} + \delta)^2 \\ &\approx \zeta(1 - x_{\text{ss}}) - \zeta\delta - \alpha n_{\text{H}}(x_{\text{ss}}^2 + 2x_{\text{ss}}\delta) \\ &= \zeta(1 - x_{\text{ss}}) - \alpha n_{\text{H}}x_{\text{ss}}^2 - (\zeta + 2\alpha n_{\text{H}}x_{\text{ss}})\delta \\ &= -(\zeta + 2\alpha n_{\text{H}}x_{\text{ss}})\delta \end{aligned} \quad (2.2.9)$$

Notice that  $\zeta(1 - x_{\text{ss}}) - \alpha n_{\text{H}}x_{\text{ss}}^2 = 0$  due to the steady-state condition (Equation 2.2.4). This is a 1st-order linear differential equation with solution:

$$\begin{aligned} \delta(t) &= \delta(0)e^{-(\zeta + 2\alpha n_{\text{H}}x_{\text{ss}})t} \\ &= \delta(0)e^{-\sqrt{\zeta^2 + 4\alpha n_{\text{H}}\zeta}t} \end{aligned} \quad (2.2.10)$$

Substitute back into Equation 2.2.8:

$$x(t) = x_{\text{ss}} + \delta(0)e^{-\sqrt{\zeta^2 + 4\alpha n_{\text{H}}\zeta}t} \quad (2.2.11)$$

## 2.3

We have worked out the methodology for the determination of the electron temperature in a photoionized medium using the 4363 Å and 5007 Å lines of [O III], because the electron density dependence is cancelled out in this ratio.

### 2.3.1

If we add the 4959 Å in the diagnostics, derive the expression for the  $j(4959\text{Å} + 5007\text{Å})/j(4363\text{Å})$  by also including the term for collisional de-excitation of the  $^1\text{D}_2$  level. Solve for the relation between the observed emissivity ratio and the temperature. [Hint: in this solution, the temperature-dependent is again through the collisional rate, but there will be an additional term that includes the electron density.]

**Solution:** Consider the relevant energy levels of the [O III] ion:

- ground state (level 0, 1, 2):  $^3\text{P}_J$  ( $J = 0, 1, 2$ )
- 1st excited state (level 3):  $^1\text{D}_2$
- 2nd excited state (level 4):  $^1\text{S}_0$

The relevant transitions and their associated wavelengths are:

- auroral line (level 4  $\rightarrow$  3):  $^1\text{S}_0 \rightarrow ^1\text{D}_2$  (4363Å)
- nebular line (level 3  $\rightarrow$  1):  $^1\text{D}_2 \rightarrow ^3\text{P}_1$  (4959Å)
- nebular line (level 3  $\rightarrow$  2):  $^1\text{D}_2 \rightarrow ^3\text{P}_2$  (5007Å)

We assume that the radiative de-excitation will be primarily spontaneous rather than stimulated emission. In excitation equilibrium, the rate of collisional excitation from the ground state to level 4 will be balanced by radiative de-excitation from level 4.

$$n_0 n_e C_{04} = n_4 (A_{41} + A_{43}) \quad (2.3.1)$$

where  $n_i$  is the population of level  $i$ ;  $n_e$  is the electron number density;  $C_{lu}$  is the collisional excitation rate coefficient from lower level  $l$  to upper level  $u$ ;  $A_{ul}$  is the spontaneous emission coefficient from upper level  $u$  to lower level  $l$ . The emissivity of the 4  $\rightarrow$  3 transition is:

$$j(4 \rightarrow 3) = \frac{1}{4\pi} n_4 A_{43} h\nu_{43} = \frac{1}{4\pi} n_0 n_e C_{04} \frac{A_{43}}{A_{41} + A_{43}} h\nu_{43} \quad (2.3.2)$$

where  $h\nu_{ul}$  is the energy difference between upper level  $u$  and lower level  $l$ . Level 3 can be populated in 2 ways: by collisional excitation directly from the ground state, or by collisional excitation from the ground state to level 4, followed by radiative de-excitation to level 3. We also include the term for collisional de-excitation of level 3.

$$n_0 n_e C_{03} + n_0 n_e C_{04} \frac{A_{43}}{A_{41} + A_{43}} = n_3 (A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}) \quad (2.3.3)$$

The emissivity of the 3  $\rightarrow$  2 transition is

$$\begin{aligned} j(3 \rightarrow 2) &= \frac{1}{4\pi} n_3 A_{32} h\nu_{32} \\ &= \frac{1}{4\pi} n_0 n_e \left( C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}} h\nu_{32} \end{aligned} \quad (2.3.4)$$

Similarly, the emissivity of the 3  $\rightarrow$  1 transition is

$$\begin{aligned} j(3 \rightarrow 1) &= \frac{1}{4\pi} n_3 A_{31} h\nu_{31} \\ &= \frac{1}{4\pi} n_0 n_e \left( C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{31}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}} h\nu_{31} \end{aligned} \quad (2.3.5)$$

The observed emissivity ratio is given by:

$$\begin{aligned}
\frac{j(4959\text{\AA}) + j(5007\text{\AA})}{j(4363\text{\AA})} &= \frac{j(3 \rightarrow 1) + j(3 \rightarrow 2)}{j(4 \rightarrow 3)} \\
&= \frac{\frac{1}{4\pi} n_0 n_e \left( C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{31} h \nu_{31} + A_{32} h \nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}}{\frac{1}{4\pi} n_0 n_e C_{04} \frac{A_{43}}{A_{41} + A_{43}} h \nu_{43}} \\
&= \frac{\left( C_{03} + C_{04} \frac{A_{43}}{A_{41} + A_{43}} \right) \frac{A_{31} \nu_{31} + A_{32} \nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}}{C_{04} \frac{A_{43} \nu_{43}}{A_{41} + A_{43}}} \\
&= \left( \frac{C_{03}}{C_{04}} \frac{A_{41} + A_{43}}{A_{43}} + 1 \right) \frac{A_{31} \nu_{31} + A_{32} \nu_{32}}{A_{31} + A_{32} + n_e C_{30} + n_e C_{31} + n_e C_{32}}
\end{aligned} \tag{2.3.6}$$

The collisional excitation and de-excitation rate coefficients are given by:

$$C_{lu} = \sqrt{\frac{2\pi \hbar^4}{k_B m_e^3}} \frac{\Omega_{ul}}{g_l} \frac{e^{-h\nu_{ul}/k_B T}}{\sqrt{T}} \tag{2.3.7}$$

$$C_{ul} = \sqrt{\frac{2\pi \hbar^4}{k_B m_e^3}} \frac{\Omega_{ul}}{g_u} \frac{1}{\sqrt{T}} \tag{2.3.8}$$

where  $\Omega_{ul}$  is the dimensionless collision strength for the transition from lower level  $l$  to upper level  $u$ ;  $g_i$  is the degeneracy (statistical weight) of level  $i$ . The ratio of the collisional excitation rates is then:

$$\begin{aligned}
\frac{C_{03}}{C_{04}} &= \frac{\Omega_{30}}{\Omega_{40}} e^{-h(\nu_{30}-\nu_{40})/k_B T} \\
&= \frac{\Omega_{30}}{\Omega_{40}} e^{-h\nu_{43}/k_B T}
\end{aligned} \tag{2.3.9}$$

Substitute Equation 2.3.9 into Equation 2.3.7, we get the relation:

$$\frac{j(4959\text{\AA}) + j(5007\text{\AA})}{j(4363\text{\AA})} = \left( \frac{\Omega_{30}}{\Omega_{40}} \frac{A_{41} + A_{43}}{A_{43}} e^{-h\nu_{43}/k_B T} + 1 \right) \frac{A_{31} \nu_{31} + A_{32} \nu_{32}}{A_{31} + A_{32} + n_e \sqrt{\frac{2\pi \hbar^4}{k_B m_e^3}} \frac{\Omega_{30} + \Omega_{31} + \Omega_{32}}{g_3} \frac{1}{\sqrt{T}}} \tag{2.3.10}$$

### 2.3.2

Make a plot of the predicted observed ratio as a function of temperature for electron density of  $10^3$ ,  $10^4$  and  $10^5 \text{ cm}^{-3}$ .

**Solution:** We use following numerical values for the parameters:

$$\left\{ \begin{array}{l} \nu_{31} = c/4959\text{\AA} \\ \nu_{32} = c/5007\text{\AA} \\ \nu_{43} = c/4363\text{\AA} \\ A_{31} = 6.951 \times 10^{-3} \\ A_{32} = 2.029 \times 10^{-2} \\ A_{43} = 1.685 \\ \Omega_{30} = 0.243 \\ \Omega_{31} = 0.243 \times 3 \\ \Omega_{32} = 0.243 \times 5 \\ \Omega_{40} = 0.0321 \end{array} \right. \quad (2.3.11)$$

The plot is shown in Figure 1.

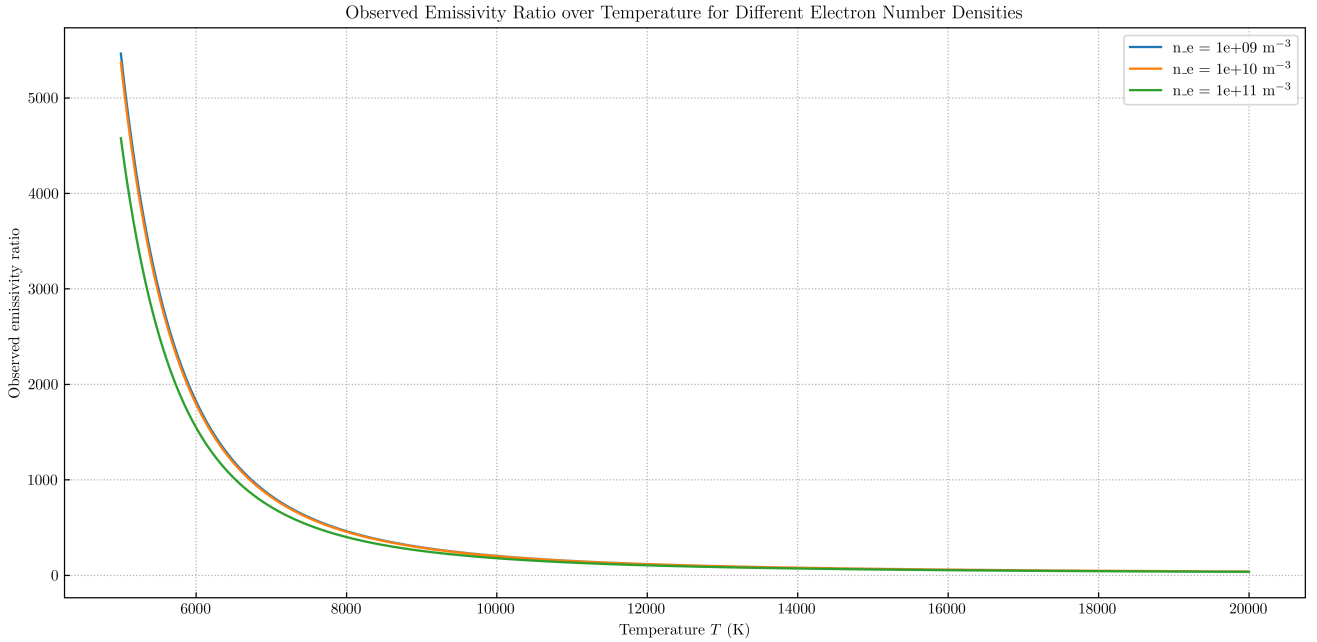


Figure 1: Observed Emissivity Ratio over Temperature for Different Electron Number Densities

## 2.4

Derive the Rankine-Hugoniot jump condition for the density, pressure and temperature ratios across the shock front.

**Solution:** Consider a steady, 1D shock wave moving through a stationary medium. The subscripts <sub>1</sub> and <sub>2</sub> denote the upstream (pre-shock) and downstream (post-shock) conditions, respectively. Mass, momentum and energy conservation give:

$$\rho_1 v_1 = \rho_2 v_2 \quad (2.4.1)$$

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \quad (2.4.2)$$

$$\frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} \quad (2.4.3)$$

where  $\rho$  is the density,  $v$  is the fluid velocity,  $p$  is the pressure and  $\gamma$  is the adiabatic index. The Mach number is defined as:

$$\mathcal{M}_1 = \sqrt{\frac{\rho_1 v_1^2}{\gamma p_1}} \quad (2.4.4)$$

Rewrite momentum conservation equation (Equation 2.4.2) using mass conservation equation (Equation 2.4.1):

$$p_1 - p_2 = \rho_2 v_2^2 - \rho_1 v_1^2 = \rho_2 \left( \frac{\rho_1}{\rho_2} v_1 \right)^2 - \rho_1 v_1^2 = \rho_1 v_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) \quad (2.4.5)$$

Similarly, rewrite energy conservation equation (Equation 2.4.3) using Equation 2.4.1:

$$\begin{aligned} \left( \frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} \right) \rho_1 \rho_2 &= \left( \frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} \right) \rho_1 \rho_2 \\ \Rightarrow \frac{1}{2} \rho_1 \rho_2 v_1^2 + \frac{\gamma}{\gamma - 1} p_1 \rho_2 &= \frac{1}{2} \rho_1 \rho_2 \left( \frac{\rho_1}{\rho_2} v_1 \right)^2 + \frac{\gamma}{\gamma - 1} p_2 \rho_1 \\ \Rightarrow \frac{\gamma}{\gamma - 1} (p_1 \rho_2 - p_2 \rho_1) &= \frac{1}{2} \frac{\rho_1^3}{\rho_2} v_1^2 - \frac{1}{2} \rho_1 \rho_2 v_1^2 \\ \Rightarrow \frac{\gamma}{\gamma - 1} (p_1 \rho_2 - p_2 \rho_1) &= \frac{1}{2} \rho_1 \rho_2 v_1^2 \left( \frac{\rho_1^2}{\rho_2^2} - 1 \right) \end{aligned} \quad (2.4.6)$$

Rewrite the left-hand side using Equation 2.4.4 and Equation 2.4.5:

$$\begin{aligned} p_1 \rho_2 - p_2 \rho_1 &= p_1 \rho_2 - \left[ p_1 - \rho_1 v_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) \right] \rho_1 \\ &= (\rho_2 - \rho_1) p_1 + \rho_1^2 v_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) \\ &= (\rho_2 - \rho_1) \frac{\rho_1 v_1^2}{\gamma \mathcal{M}_1^2} + \rho_1^2 v_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) \end{aligned} \quad (2.4.7)$$

Substitute this into Equation 2.4.6:

$$\begin{aligned} \frac{\gamma}{\gamma - 1} \left[ (\rho_2 - \rho_1) \frac{\rho_1 v_1^2}{\gamma \mathcal{M}_1^2} + \rho_1^2 v_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) \right] &= \frac{1}{2} \rho_1 \rho_2 v_1^2 \left( \frac{\rho_1^2}{\rho_2^2} - 1 \right) \\ \Rightarrow (\rho_2 - \rho_1) \frac{\rho_1}{\mathcal{M}_1^2} + \gamma \rho_1^2 \left( \frac{\rho_1}{\rho_2} - 1 \right) &= \frac{1}{2} (\gamma - 1) \rho_1 \rho_2 \left( \frac{\rho_1^2}{\rho_2^2} - 1 \right) \\ \Rightarrow \left( \frac{\rho_2}{\rho_1} - 1 \right) \frac{1}{\mathcal{M}_1^2} + \gamma \left( \frac{\rho_1}{\rho_2} - 1 \right) &= \frac{1}{2} (\gamma - 1) \frac{\rho_2}{\rho_1} \left( \frac{\rho_1^2}{\rho_2^2} - 1 \right) \\ \Rightarrow \left( \frac{\rho_2}{\rho_1} - 1 \right) \frac{1}{\mathcal{M}_1^2} + \frac{1}{2} (\gamma - 1) \frac{\rho_2}{\rho_1} &= \frac{1}{2} (\gamma - 1) \frac{\rho_1}{\rho_2} - \gamma \left( \frac{\rho_1}{\rho_2} - 1 \right) \\ \Rightarrow \left( \frac{1}{\mathcal{M}_1^2} + \frac{\gamma - 1}{2} \right) \frac{\rho_2}{\rho_1} - \frac{1}{\mathcal{M}_1^2} &= -\frac{\gamma + 1}{2} \frac{\rho_1}{\rho_2} + \gamma \\ \Rightarrow \left( \frac{1}{\mathcal{M}_1^2} + \frac{\gamma - 1}{2} \right) \frac{\rho_2^2}{\rho_1^2} - \left( \frac{1}{\mathcal{M}_1^2} + \gamma \right) \frac{\rho_2}{\rho_1} + \frac{\gamma + 1}{2} &= 0 \end{aligned} \quad (2.4.8)$$

This is a quadratic equation with 1 unknown. Use the corresponding formula:

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{\left(\frac{1}{\mathcal{M}_1^2} + \gamma\right) \pm \sqrt{\left(\frac{1}{\mathcal{M}_1^2} + \gamma\right)^2 - 4\left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{2}\right)\frac{\gamma+1}{2}}}{2\left(\frac{1}{\mathcal{M}_1^2} + \frac{\gamma-1}{2}\right)} \\
&= \frac{1 + \gamma\mathcal{M}_1^2 \pm \mathcal{M}_1^2 \sqrt{\frac{1}{\mathcal{M}_1^4} + \frac{2\gamma}{\mathcal{M}_1^2} + \gamma^2 - \frac{2(\gamma+1)}{\mathcal{M}_1^2} - (\gamma-1)(\gamma+1)}}{2 + (\gamma-1)\mathcal{M}_1^2} \\
&= \frac{1 + \gamma\mathcal{M}_1^2 \pm \mathcal{M}_1^2 \sqrt{\frac{1}{\mathcal{M}_1^4} - \frac{2}{\mathcal{M}_1^2} + 1}}{(\gamma-1)\mathcal{M}_1^2 + 2} \\
&= \frac{1 + \gamma\mathcal{M}_1^2 \pm (1 - \mathcal{M}_1^2)}{(\gamma-1)\mathcal{M}_1^2 + 2} \\
&= 1 \text{ (discarded) or } \frac{(\gamma+1)\mathcal{M}_1^2}{(\gamma-1)\mathcal{M}_1^2 + 2} \tag{2.4.9}
\end{aligned}$$

This is the density ratio across the shock front. From Equation 2.4.4, Equation 2.4.5 and the density ratio (Equation 2.4.9):

$$\begin{aligned}
\frac{p_2}{p_1} &= \frac{p_1 - \rho_1 v_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right)}{p_1} \\
&= 1 - \gamma\mathcal{M}_1^2 \left[ \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} - 1 \right] \\
&= 1 - \gamma\mathcal{M}_1^2 \frac{(\gamma-1)\mathcal{M}_1^2 + 2 - (\gamma+1)\mathcal{M}_1^2}{(\gamma+1)\mathcal{M}_1^2} \\
&= 1 - \frac{\gamma}{\gamma+1} 2(1 - \mathcal{M}_1^2) \\
&= \frac{\gamma+1 - 2\gamma(1 - \mathcal{M}_1^2)}{\gamma+1} \\
&= \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1} \tag{2.4.10}
\end{aligned}$$

Using the ideal gas law  $p\mu = \rho RT$ , the temperature ratio is:

$$\begin{aligned}
\frac{T_2}{T_1} &= \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} \\
&= \frac{2\gamma\mathcal{M}_1^2 - (\gamma-1)}{\gamma+1} \frac{(\gamma-1)\mathcal{M}_1^2 + 2}{(\gamma+1)\mathcal{M}_1^2} \\
&= \frac{[2\gamma\mathcal{M}_1^2 - (\gamma-1)][(\gamma-1)\mathcal{M}_1^2 + 2]}{(\gamma+1)^2\mathcal{M}_1^2} \tag{2.4.11}
\end{aligned}$$

## 2.5

The evolution of supernova remnant in the Sedov-Taylor phase can be derived by simple dimensional analysis:

$$R_s = AE^{1/5} \rho_0^{-1/5} t^{2/5} \tag{2.5.1}$$



### 2.5.1

Obtain an estimate of the dimensionless factor  $A$  by assuming that 50% of the total energy will be in ordered kinetic energy  $Mv_s^2/2$ , where  $M$  is the swept-up mass.

**Solution:** Mathematically, the assumption can be expressed as:

$$\frac{1}{2}E = \frac{1}{2}Mv_s^2 \quad (2.5.2)$$

We first determine the speed of the spherical shock front  $v_s$  by differentiating the radius of the spherical shock front  $R_s$  with respect to time  $t$ :

$$v_s = \frac{dR_s}{dt} = \frac{2}{5}AE^{1/5}\rho_0^{-1/5}t^{-3/5} = \frac{2}{5}\frac{R_s}{t} \quad (2.5.3)$$

Next, the swept-up mass  $M$  is related to the ambient density  $\rho_0$  and  $R_s$  by:

$$M = \frac{4}{3}\pi R_s^3 \rho_0 \quad (2.5.4)$$

Substituting  $M$  (Equation 2.5.4),  $v_s$  (Equation 2.5.3) and the Sedov-Taylor solution (Equation 2.5.1) into the energy equation (Equation 2.5.2):

$$\begin{aligned} \frac{1}{2}E &= \frac{1}{2} \left( \frac{4}{3}\pi R_s^3 \rho_0 \right) \left( \frac{2}{5}\frac{R_s}{t} \right)^2 \\ &= \frac{8\pi}{75} R_s^5 \rho_0 t^{-2} \\ &= \frac{8\pi}{75} (AE^{1/5}\rho_0^{-1/5}t^{2/5})^5 \rho_0 t^{-2} \\ &= \frac{8\pi}{75} A^5 E \\ \implies A &= \left( \frac{75}{16\pi} \right)^{1/5} \approx 1.083 \end{aligned} \quad (2.5.5)$$

### 2.5.2

Above we considered the case of uniform ambient density  $\rho$  and constant total energy  $E$ . Suppose that we instead assume that the ambient density decreases as

$$\rho(r) = \rho_0 \left( \frac{r}{r_0} \right)^\delta \quad (2.5.6)$$

and energy is increasing with time as a power law:

$$E(t) = E_0 \left( \frac{t}{t_0} \right)^\varepsilon \quad (2.5.7)$$

where  $\delta > -3$  and  $\varepsilon \geq 0$ . Find  $\gamma$  in the time evolution of the radius of the blast wave  $R_s \propto t^\gamma$  as a function of  $\delta$  and  $\varepsilon$ .

**Solution:** Blast wave radius  $R_s$  satisfy:

$$R_s(t) \propto t^\gamma \quad (2.5.8)$$

The velocity of the spherical shock front  $v_s$  is the time derivative of the blast wave radius:

$$v_s = \frac{dR_s}{dt} \propto t^{\gamma-1} \quad (2.5.9)$$

The mass  $M$  swept up by the blast wave up to time  $t$  is given by:

$$M = \int_0^{R_s(t)} 4\pi r^2 \rho(r) dr \propto \int_0^{R_s(t)} r^{\delta+2} dr \propto R_s(t)^{\delta+3} \propto t^{\gamma(\delta+3)} \quad (2.5.10)$$

The total energy  $E(t)$  is assumed to be proportional to the kinetic energy of the swept-up mass:

$$E(t) = E_0 \left( \frac{t}{t_0} \right)^\varepsilon \propto \frac{1}{2} M v_s^2 \propto t^{\gamma(\delta+3)} t^{2(\gamma-1)} = t^{(\delta+5)\gamma-2} \quad (2.5.11)$$

The exponents of  $t$  on both sides must be equal. Therefore, we equate the powers of  $t$ :

$$\begin{aligned} \varepsilon &= (\delta + 5)\gamma - 2 \\ \implies \gamma &= \frac{\varepsilon + 2}{\delta + 5} \end{aligned} \quad (2.5.12)$$

### 2.5.3

If  $R_s \propto t^\gamma$ , how does the shock temperature  $T_s$  vary with time?

**Solution:** The shock temperature  $T_s$  is proportional to the squared velocity of the spherical shock front:

$$T_s \propto v_s^2 \propto t^{2(\gamma-1)} = t^{2\gamma-2} \quad (2.5.13)$$

### 2.5.4

Suppose that the density profile in the ambient medium is  $\rho \propto r^{-2}$ , as would apply to a constant-velocity steady stellar wind pre-process the gas before supernova explosion. Suppose that there is a sudden explosion depositing an energy  $E_0 = \text{Constant}$ . What will be  $\gamma$  for this case?

**Solution:** Since  $\delta = -2$  and  $\varepsilon = 0$ , we can calculate  $\gamma$  using Equation 2.5.12:

$$\gamma = \frac{\varepsilon + 2}{\delta + 5} = \frac{2}{3} \quad (2.5.14)$$