prob 3.1

1. Assuming the universe is matter dominated, calculate the deviation of total matter density from the critical density in terms of $|\Omega_{\text{tot}} - 1|$ at 1 s after the Big Bang (where $\Omega_{\text{tot}} = \rho_{\text{tot}}/\rho_{\text{cr}}$), if we require $|\Omega_{\text{tot}} - 1| \leqslant 1$ at the present day as is indicated by observations. Why an accelerating universe can help with solving the *flatness problem*?

(1) Friedmann eq can be written as: $|\Omega_t - 1| = \dfrac{|K|}{a^2 H^2}$

in a matter-dominated universe:

$a \propto t^{2/3} \Rightarrow H = \dfrac{\dot{a}}{a} = \dfrac{2}{3t} \Rightarrow a^2 H^2 \propto t^{-2/3}$

$\Rightarrow |\Omega_t - 1| \propto t^{2/3}$

at the present day: $|\Omega_t(t_0) - 1| \leq 1$, $t_0 \approx 1.38 \times 10^{10}$ yr

$\Rightarrow |\Omega_t(1s) - 1| = |\Omega_t(t_0 - 1)| \left(\dfrac{1s}{t_0}\right)^{2/3} \leq$

(2) in an accelerating universe: $\ddot{a} > 0$

Friedmann equation: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0 \Rightarrow \rho + 3p < 0 \Rightarrow w < -\frac{1}{3}$

$$\left.\begin{aligned} a &\propto t^{\frac{2}{3(1+w)}} \\ H &= \frac{2}{3(1+w)} \frac{1}{t} \end{aligned}\right\} \Rightarrow aH \propto t^{-\frac{1+3w}{3(1+w)}} \Rightarrow |\Omega_t - 1| \propto t^{\frac{2(1+3w)}{3(1+w)}}$$

since $w < -\frac{1}{3}$ and $w > -1$, $|\Omega_t - 1|$ is shrinking and inflation drives $\Omega_t \to 1$

prob 3.2

2. Assuming the universe is matter dominated, calculate the ratio between the comoving horizon today and that at $z = 1100$, where the cosmic microwave background (CMB) radiation came out. We observe that the CMB sky is smooth, homogeneous and isotropic to a level of 10^{-5} over the entire sky. Why does this indicate a *horizon problem*? Why an accelerating universe can help with solving this problem?

(1) in a matter-dominated universe:

co-moving horizon $D_h(t) = \frac{2c}{H} = \frac{2c}{(1+z)H_0} = D_h(z)$

$$\Rightarrow \frac{D_h(z=0)}{D_h(z=1100)} = \sqrt{1+1100} \approx 33.18$$

(2) Regions of the sky separated by more than the co-moving horizon at $z=1000$ could not have been in causal contact in a standard matter-

dominated universe. But the tempertaure of the CMB is uniform across these regions.

(3) During inflation the causally connected region is shrinking. It was very large before inflation; much larger than the present horizon. Thus the present observable universe has evolved from a small patch of a much larger causally connected region.

prob 3.3

(1) $k_{start} = H_{start} \dfrac{a_{start}}{a_0} \Rightarrow a_{start} = \dfrac{k_{start}}{H_{start}} a_0$

$$N(t_{start}) = \ln \frac{a_{end}}{a_{start}} = \ln \frac{a_{end}}{a_0} \frac{H_{start}}{k_{start}} = \ln \frac{a_{end}}{a_0} \frac{H_{start}}{H_0} \frac{H_0}{k_{start}}$$

$$= \ln \frac{a_{end}}{a_0} + \ln \frac{H_{start}}{H_0} + \ln \frac{H_0}{k_{start}}$$

(2) during slow-roll inflation: $H_{start}^2 = \dfrac{8\pi G}{3} V_{SR}$

Friedmann eq. today: $H_0^2 = \dfrac{8\pi G}{3} \rho_{c0}$

$$\Rightarrow \frac{H_{start}}{H_0} = \sqrt{\frac{V_{SR}}{\rho_0}} = \frac{V_{SR}^{1/4}}{10^{16}\,\mathrm{GeV}} \frac{10^{16}\,\mathrm{GeV} \cdot V_{SR}^{1/4}}{\sqrt{\rho_{c0}}}$$

(3)

$$\left.\begin{aligned} &\rho_r(a_{eq}) = \rho_{r0}\frac{a_0^4}{a_{eq}^4} \approx \rho(a_{eq}) \\ &\rho_{reh}\, a_{reh}^4 = \rho(a_{eq})\, a_{eq}^4 \end{aligned}\right\} \Rightarrow \rho_{reh} = \rho_{r0}\frac{a_0^4}{a_{reh}^4}$$

$$\rho_{end}\, a_{end}^3 = \rho_{reh}\, a_{reh}^3$$

$$\Rightarrow \rho_{end} = \rho_{reh}\frac{a_{reh}^3}{a_{end}^3} = \rho_{r0}\frac{a_0^4}{a_{reh}^4}\frac{a_{reh}^3}{a_{end}^3} = \rho_{r0}\frac{a_0^4}{a_{reh}a_{end}^3}$$

$$a_{reh}/a_0 = (\rho_{r0}/\rho_{reh})^{1/4}$$

$$\Rightarrow \frac{a_{end}}{a_0} = \left(\frac{\rho_{r0}}{\rho_{end}}\frac{a_0}{a_{reh}}\right)^{1/3} = \left(\frac{\rho_{r0}}{\rho_{end}}\frac{\rho_{reh}^{1/4}}{\rho_{r0}^{1/4}}\right)^{1/3}$$

$$\Rightarrow N(t_{start}) = \frac{1}{3}\ln\left(\frac{\rho_{r0}}{\rho_{end}}\frac{\rho_{reh}^{\frac{1}{4}}}{\rho_{r0}^{1/4}}\right) + \ln\left(\frac{V_{SR}^{\frac{1}{4}}}{10^{16}\,\mathrm{GeV}}\frac{10^{16}\,\mathrm{GeV}\cdot V_{SR}^{\frac{1}{4}}}{\rho_{c0}^{\frac{1}{2}}}\right) + \ln\frac{H_0}{k_{start}}$$

$$= \ln \frac{H_0}{k_{start}} + \ln \frac{10^{16}\,\mathrm{GeV} \cdot \rho_{r0}^{1/4}}{\rho_{c0}^{1/2}} + \frac{1}{12} \ln \frac{\rho_{reh}}{\rho_{end}} + \frac{1}{4} \ln \frac{V_{SR}}{\rho_{end}} + \ln \frac{V_{SR}^{1/4}}{10^{16}\,\mathrm{GeV}}$$

(4) $$\left.\begin{aligned} & k_{start} \leq H_0 \\ & \rho_{c0} \approx (3\times 10^{-12}\,\mathrm{GeV})^4 h^2 = (3\times 10^{-12}\,\mathrm{GeV})^4 \left(\frac{H_0}{100\,\mathrm{km/s/Mpc}}\right)^2 \\ & \rho_{r0} \approx (2.5\times 10^{-13}\,\mathrm{GeV})^4 \end{aligned}\right\}$$

$$\Rightarrow N(t_{start}) \approx \ln \frac{H_0}{k_{start}} + \ln \frac{10^{16}\,\mathrm{GeV} \cdot \rho_{r0}^{1/4}}{\rho_{c0}^{1/2}} \geq \ln \frac{10^{16}\,\mathrm{GeV} \times 2.5\times 10^{-13}\,\mathrm{GeV}}{(3\times 10^{-12}\,\mathrm{GeV})^2 \frac{H_0}{100\,\mathrm{km/s/Mpc}}}$$

$$\geq 61$$

prob 3.4

(1) $V(\varphi) = \frac{1}{2}m^2\varphi^2 \Rightarrow V'(\varphi) = m^2\varphi \Rightarrow \begin{cases} 3H\dot{\varphi} = -m^2\varphi \\ H^2 = \frac{1}{6}\frac{m^2}{M_{pl}^2}\varphi^2 \end{cases}$

$$\left.\begin{array}{l} \Rightarrow \varepsilon = \frac{1}{2}M_{pl}^2\left[\frac{V'(\varphi)}{V(\varphi)}\right]^2 = \frac{1}{2}M_{pl}^2\left(\frac{m^2\varphi}{\frac{1}{2}m^2\varphi^2}\right)^2 = \frac{2M_{pl}^2}{\varphi^2} = \frac{m^2}{3H^2} \\ \left.\begin{array}{l} \frac{1}{2}\dot{\varphi}^2 \ll \frac{1}{2}m^2\varphi^2 \Rightarrow \dot{\varphi}^2 \ll m^2\varphi^2 \\ 3H\dot{\varphi} = -m^2\varphi \end{array}\right\} \Rightarrow \frac{\dot{\varphi}^2}{\varphi^2} = \left(-\frac{m^2}{3H}\right)^2 = \frac{m^4}{9H^2} \ll m^2 \end{array}\right\}$$

$$\therefore \varepsilon = \frac{m^2}{3H^2} \ll 1$$

$$3H\dot{\varphi} = -m^2\varphi \Rightarrow 3H\ddot{\varphi} + 3\dot{H}\dot{\varphi} = -m^2\dot{\varphi} \approx 3\dot{H}\dot{\varphi} \Rightarrow m^2 = -3\dot{H}$$

$$\therefore \varepsilon = \frac{m^2}{3H^2} = -\frac{\dot{H}}{H^2}$$

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}a - \dot{a}^2}{a^2}$$

$$\Rightarrow \varepsilon = -\frac{\dot{H}}{H^2} = -\frac{\ddot{a}a - \dot{a}^2}{a^2}\frac{a^2}{\dot{a}^2} = \frac{\dot{a}^2 - a\ddot{a}}{\dot{a}^2} = 1 - \frac{a\ddot{a}}{\dot{a}^2} \ll 1$$

$$\therefore \frac{\ddot{a}a}{\dot{a}^2} = \left(\frac{a}{\dot{a}}\right)^2\frac{\ddot{a}}{a} = \frac{1}{H^2}\frac{\ddot{a}}{a} \gg 0 \Rightarrow \frac{\ddot{a}}{a} \gg 0$$

(2)

$$\varepsilon = \frac{1}{2}M_{pl}^2\left[\frac{V'(\varphi)}{V(\varphi)}\right]^2 = \frac{1}{2}M_{pl}^2\left(\frac{m^2\varphi}{\frac{1}{2}m^2\varphi^2}\right)^2 = \frac{2M_{pl}^2}{\varphi^2} = 1$$

$$\therefore \varphi_{end} = \sqrt{2}\,M_{pl}$$

(3)

$$\begin{cases} 3H\dot{\varphi} = -V'(\varphi) \\ H^2 = \dfrac{V(\varphi)}{3M_{pl}^2} \end{cases} \Rightarrow \frac{H}{3\dot{\varphi}} = -\frac{1}{3M_{pl}^2}\frac{V(\varphi)}{V'(\varphi)}$$

$$\Rightarrow \int_{\varphi(t)}^{\varphi_{end}} \frac{H}{\dot{\varphi}} d\varphi = \frac{1}{M_{pl}^2} \int_{\varphi_{end}}^{\varphi(t)} \frac{V(\varphi)}{V'(\varphi)} d\varphi$$

$$\int_{\varphi(t)}^{\varphi_{end}} \frac{H}{\dot{\varphi}} d\varphi = \int_{t}^{t_{end}} H\,dt = \int_{a(t)}^{a(t_{end})} \frac{da}{a} = \ln \frac{a(t_{end})}{a(t)} = N(t)$$

(4) $V(\varphi) = \frac{1}{2} m^2 \varphi^2 \Rightarrow V'(\varphi) = m^2 \varphi$

$$N(t) = \frac{1}{M_{pl}^2} \int_{\varphi_{end}}^{\varphi(t)} \frac{V(\varphi)}{V'(\varphi)} d\varphi = \frac{1}{M_{pl}^2} \int_{\varphi_{end}}^{\varphi(t)} \frac{\varphi}{2} d\varphi = \frac{1}{4M_{pl}^2} \left[\varphi^2(t) - \varphi_{end}^2\right]$$

$$N(t_{start}) = \frac{1}{4M_{pl}^2} \left[\varphi^2(t_{start}) - \varphi_{end}^2\right] \geqslant 60$$

$$\varphi_{end} = \sqrt{2} M_{pl}$$

$$\therefore \varphi(t_{start}) \geqslant \sqrt{\varphi_{end}^2 + 240 M_{pl}^2} = \sqrt{242} M_{pl} \approx 15.56 M_{pl}$$

$$\therefore \varepsilon = \frac{2M_{pl}^2}{\varphi^2(t_{start})} = \frac{1}{121} \ll 1$$