Statistics & Numerical Method, Problem Set 3 (due Nov. 12, 2024)

1. Keplerian motion (17 pts)

Consider the motion of a particle around a central object due to a central $1/r^2$ gravity force. Of course, its trajectory can be solved analytically, and in general it is given by a closed ellipse if the total energy (kinetic plus potential) of the particle is negative. We will treat the problem as an initial value problem, and measure the accuracy of various integration algorithms by comparison to the analytical solution.

(1). Derive the equations of motion for the particle, using a Cartesian $(x - y)$ coordinate system. For simplicity, you can assume $GM = 1$. You can choose coordinates such that the orbit lies in the $x - y$ plane, so that the problem is two-dimensional. Write the equations as a set of coupled first-order ODEs. Construct an initial condition that should give pure circular motion at an orbital radius $r_0 = 1$. Show the analytic expressions of the orbit, and give the value of total energy (kinetic and gravitational) and angular momentum (2 pts).

For (2) - (4) below, you are asked to write your own numerical integration schemes from the initial condition that you just derived, and attach your codes.

(2). Write your own forward-Euler method (1 pt), and

- Show the particle trajectory for two orbital periods using a timestep $h = 0.01$, and compare with the analytical trajectory. (1 pt)

- Show the position error $|\mathbf{r}_{\text{numerical}} - \mathbf{r}_{\text{analytical}}|$ at the end of the integration, for integration timestep of $h = (0.5, 1, 2, 4) \times 10^{-3}$. Discuss the order of accuracy of the numerical solution based on your results. (2 pts)

- Repeat the above, but for the error in total energy and angular momentum. (1 pt)

(3). Repeat (2), but using the classic fourth-order Runge-Kutta method. Show the trajectory with $h = 0.5$, and calculate errors for $h = 0.05, 0.1, 0.2, 0.4$. What happens when using $h = 1.0$? (5 pts)

(4). Write your own leapfrog integrator and integrate the same equations as in (2) and (3) but for 10 orbits (1 pt). Show the particle trajectory with $h = 0.3$, as well as *the time evolution* of the position error, and error in energy and angular momentum. (2 pts) To what extent does the method conserve total energy and angular momentum? (1 pt) Where is the source of position error? (1 pt)

2. Aerodynamics of interstellar dust (15 pts)

The interstellar medium consists of gas and dust, who interact by mutual aerodynamic drag. For a dust of given size, the drag law is expressed as

$$\frac{dv_d}{dt} = -\frac{v_d - v_g(t)}{t_s} \, , \tag{1}$$

where v_d, v_g are the velocities of gas and dust, and t_s is the stopping time, determined by dust size and gas density. For simplicity, we only consider one-dimensional problem here and hence v_d, v_g are both scalars, and t_s is constant. Also note that v_g can be a function of time due to external forcing (e.g., turbulence).

(1). We need to write an ODE solver to integrate Equation (1) from time 0 to T. Discuss under what circumstances the equation can become stiff, and what would be the general strategy to design your integrator (you can mention what method to use, but no need to provide detailed algorithm here). (2 pts)

In reality, interstellar dust is characterized by a size distribution, and hence dust of different sizes have different t_s. Moreover, due to momentum conservation, gas should feel the backreaction from dust, whose dynamics is also affected by dust motion (therefore, we no longer consider $v_g(t)$ as known). If we discretize the dust size distribution into m bins, with the ith bin characterized by a dust species with stopping time $t_{s,i}$, and a dust-to-gas mass ratio ϵ_i, the composite equations of motion in a uniform medium read

$$
\begin{aligned}
\frac{dv_{d,i}}{dt} &= -\frac{v_{d,i} - v_g}{t_{s,i}} , \quad (i = 1, \cdots, m) \\
\frac{dv_g}{dt} &= \sum_{i=1}^{m} \epsilon_i \frac{v_{d,i} - v_g}{t_{s,i}} ,
\end{aligned}
\tag{2}
$$

where now $v_{d,i}$ represents the velocity of each dust species. For our purpose, t_i and ϵ_i are considered to be constants.

(2). Generalize your discussions in (1) to handle Equation (2) on when the equations become stiff, and how you would design your integrator. (2 pts)

(3). Suppose that accuracy is NOT our primary concern. Give your integration algorithm so that you can integrate both (1) and (2) in all regimes (non-stiff and stiff) as an initial value problem. Write and attach your code. (3 pts) Comment on whether your algorithm conserves total momentum at the level of machine precision. (1 pt) Note: you are not allowed to use any built-in ODE solvers.

4). Using your integrator for Equation (1), and solve the initial value problems with $v_d(t = 0) = 0$, for the following three cases. It is up to you to decide the integration timestep.

4a). $v_g = 1$, $t_s = 1$. Integrate the equation to $t = 10$. Show the time evolution of v_d from your numerical solution. Compare with analytical solution and discuss the accuracy of your algorithm. (2 pts)

4b). $v_g = \sin(t)$, $t_s = 1$. Integrate the equation to $t = 10$. Show the time evolution of v_d from your numerical solution. (1 pt)

4c). $v_g = \sin(t)$, $t_s = 10^{-5}$. Integrate the equation to $t = 10$. Show the time evolution of v_d from your numerical solution. (1 pt)

5). Consider 10 dust species, with stopping time being $t_{s,i} = 10^{-6+i}$ ($i = 1, \cdots, 10$) and $\epsilon_i = 0.1$ for each species (exaggerated compared to reality). For initial condition, let $v_{d,i} = 0$ for all i, and $v_g = 1$. Solve the ODE to time $t = 10^5$, with a constant dt of your own choice. (3 pts)

To show your solution, make the following two plots.

5a). Show the time evolution of gas velocity and the velocities of the first five dust species, for the first ten steps.

5b). Show the time evolution of gas velocity and the velocities of the remaining five dust species, with time shown as $\log(t)$ (so that the $t = 0$ initial condition is ignored in the plot).