Physical Cosmology Homework - Early Universe

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1. The total energy density $\rho(T)$ of all relativistic species under thermal equilibrium in early universe is given via an effective number of degree of freedom $g_*(T)$ as: $\rho(T) = \pi^2/30\, g_*(T)\, T^4$ (assuming $c = \hbar = k = 1$), where

$$g_*(T) = \Sigma_{i \in \text{bonson}}\, g_i(T) + 7/8\, \Sigma_{i \in \text{fermion}}\, g_i(T). \tag{1}$$

Please calculate $g_*(T)$ at different era in early universe (for g for each particle species, refer to Table 1):

(1.1) At $T > 175\,\text{GeV}$, all standard model particles are relativistic, calculate $g_*(T)$ at that temperature.

(1.2) Let us assume that electroweak phase transition once happened, and particles acquired mass due to the *Higgs mechanism.* Then at $T \sim 100\,\text{GeV}$, *top* quarks and anti-quarks eventually annihilated, followed by Higgs boson H^0, and electroweak gauge boson Z^0 and $W^\pm$. At $T \sim 10\,\text{GeV}$, assuming all annihilation mentioned above was completed by then, what was $g_*(T)$?

(1.3) At $T \leq 1\,\text{GeV}$, *bottom* and *charm* quarks annihilated with their anti-quarks, followed by tau leptons τ and their antiparticles. Assuming all annihilation above was done *before* the QCD phase transition at 150 MeV, which quark and lepton species remain in the relativistic sea? What was $g_*(T)$ by then?

(1.4) After the QCD phase transition, all free quarks that used to be in the relativistic sea were then confined into *baryons* (composed of 3 quarks with three different colors) and *mesons* (composed of one quark with any color and an antiquark with the matching anticolor), as well as their antiparticles. Note that this actually happened before *strange*, *up* and *down* quarks

Table 1: The particles in the standard model of particle physics
Particle Data Group, 2018

Quarks	t	$173.0 \pm 0.4\,\mathrm{GeV}$	$\bar{t}$	spin=$\frac{1}{2}$	$g = 2 \cdot 3 = 6$	
	b	4.15–$4.22\,\mathrm{GeV}$	$\bar{b}$	3 colors		
	c	$1.27 \pm 0.03\,\mathrm{GeV}$	$\bar{c}$			
	s	92–$104\,\mathrm{MeV}$	$\bar{s}$			
	d	4.4–$5.2\,\mathrm{MeV}$	$\bar{d}$			
	u	1.8–$2.7\,\mathrm{MeV}$	$\bar{u}$			
						72
Gluons	8 massless bosons			spin=1	$g = 2$	16
Leptons	τ^-	$1776.86 \pm 0.12\,\mathrm{MeV}$	τ^+	spin=$\frac{1}{2}$	$g = 2$	
	μ^-	$105.658\,\mathrm{MeV}$	μ^+			
	e^-	$510.999\,\mathrm{keV}$	e^+			
						12
	ν_τ	$< 2\,\mathrm{eV}$	$\bar{\nu}_\tau$	spin=$\frac{1}{2}$	$g = 1$	
	ν_μ	$< 2\,\mathrm{eV}$	$\bar{\nu}_\mu$			
	ν_e	$< 2\,\mathrm{eV}$	$\bar{\nu}_e$			
						6
Electroweak gauge bosons	W^+	$80.379 \pm 0.012\,\mathrm{GeV}$	W^-	spin=1	$g = 3$	
	Z^0	$91.1876 \pm 0.0021\,\mathrm{GeV}$				
	γ	$0 \quad (< 1 \times 10^{-18}\,\mathrm{eV})$			$g = 2$	
						11
Higgs boson (SM)	H^0	$125.18 \pm 0.16\,\mathrm{GeV}$		spin=0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$

and anti-quarks yet had time to annihilate. In particular, the *up* and *down* quarks were confined into protons and neutrons (so were anti-*up* quarks and anti-*down* quarks confined into anti-protons and anti-neutrons), which annihilated right after they were born (*think of why*). The Π^0 and $\Pi^\pm$ mesons were also produced (with a total degree of freedom $g_{\text{pion}} = 3$) (assuming they had *not* yet completed the annihilation), what was $g_*(T)$ by then?

(1.5) At $T < 100\,\text{MeV}$, Π^0, $\Pi^\pm$, as well as *Muon* leptons μ all finished the annihilation processes. Which species remain in the relativistic sea? What was $g_*(T)$ by then?

(1.6) Finally at $T < 1\,\text{MeV}$, two important events happened: neutrinos (ν_e, ν_μ, ν_τ) and their antiparticles decoupled from the rest of the relativistic sea; and electrons and positrons annihilated. After neutrinos decoupled and before the cosmic temperature dropped to $1\,\text{eV}$, were they in thermal equilibrium with photons, and were they relativistic? Why? After $e^\pm$ annihilation finished and before the cosmic temperature dropped to $3000\,\text{K}$, were the remaining free electrons in thermal equilibrium with photons, and were they relativistic? Why?

[Note that after $e^\pm$ annihilation, the interaction rate $\Gamma_\gamma \sim n_e \langle \sigma_\text{T} \rangle$ for Thomson scatter of a photon by remaining free electrons only dropped below the Hubble expansion rate H at a temperature of $T \sim 130\,K$. However, the cosmic recombination happened around $T \sim 3000\,K$, when free electrons were captured by protons to form neutral hydrogen (which were than re-ionized at the epoch of cosmic re-ionization around $z \sim 6 - 7$). While for free electrons it is a different story as $\Gamma_e \sim n_\gamma \langle \sigma_\text{T} \rangle \gg \Gamma_\gamma$, due to an extremely large photon-to-electron ratio $n_\gamma / n_e \sim \eta^{-1} \sim 10^9$, free electrons could always have been surrounded and bombarded by a vast sea of photons.]

2. At below 1 MeV, neutrinos and anti-neutrinos decoupled from the rest of the relativistic sea, as the weak interaction rate $\Gamma(T) \propto T^5$ drops much below the cosmic expansion rate $H(T) \propto T^2$ at this energy level. They would have remained the same temperature as photons, if electron-positron annihilation had not happened. Before the $e^\pm$ annihilation, both species share the same temperature T_0; but by the time the annihilation was finished, the energy level of the photon sea (but not of the neutrino sea) was raised up by the energy input from $e^\pm$ annihilation, resulting in photon temperature $T_{\gamma,1}$ higher than neutrino temperature $T_{\nu,1}$, with the ratio fixed and lasting until today. We will now calculate the temperature ratio $T_{\nu,1}/T_{\gamma,1}$ and the energy density ratio $\rho_{\nu,1}/\rho_{\gamma,1}$ between the two relativistic species right after the annihilation.

(2.1) Write down the effective degree of freedom $g_{*,0}$ and the entropy density $s(T_0)$ in terms of $g_{*,0}$ at temperature T_0 *before* electron-positron annihilation, assuming that photons, electrons, neutrinos and their antiparticles are all in thermal equilibrium at this stage.

(2.2) *After* electron-positron annihilation, neutrinos and photons have different temperatures $T_{\nu,1}$ and $T_{\gamma,1}$, respectively. This is the case where multiple relativistic species co-exist, each with temperature T_i that may or may not be the same as the equilibrium temperature T of the majority species. In this case, we can define an effective degree of freedom $g_*(T)$ through energy density $\rho(T)$:

$$g_*(T) \equiv \frac{30\,\rho(T)}{\pi^2\,T^4} = \Sigma_{i\in\text{bonson}}\, g_i \left(\frac{T_i}{T}\right)^4 + 7/8\,\Sigma_{i\in\text{fermion}}\, g_i \left(\frac{T_i}{T}\right)^4. \qquad (2)$$

We can also define an effective degree of freedom $g_{*s}(T)$ through entropy density $s(T)$ via:

$$g_{*s}(T) \equiv \frac{45\,s(T)}{2\pi^2\,T^3} = \Sigma_{i\in\text{bonson}}\, g_i \left(\frac{T_i}{T}\right)^3 + 7/8\,\Sigma_{i\in\text{fermion}}\, g_i \left(\frac{T_i}{T}\right)^3. \qquad (3)$$

Only if all species are in thermal equilibrium, i.e., $T_i = T$, then $g_* = g_{*s}$. Now let us follow the photon species, and choose $T_{\gamma,1}$ as the equilibrium temperature for *after* electron-positron annihilation. Write down the effective degree of freedom $g_{*s,1}$ and the entropy density $s(T_{\gamma,1})$ in terms of $g_{*s,1}$.

(2.3) The conservation of comoving-volume entropy requires $s(T_0)a_0^3 = s(T_{\gamma,1})a_1^3$, where the scale factor a_0 and a_1 for before and after the annihilation can be linked by the neutrino temperature, which simply scales as $T_\nu \propto a^{-1}$. With this, show that:

$$\frac{T_{\nu,1}}{T_{\gamma,1}} = \left(\frac{g_{*s,1}}{g_{*,0}}\right)^{1/3} = \left(\frac{4}{11}\right)^{1/3} \text{(for 3 neutrino families).} \qquad (4)$$

(2.4) Using Eq. (2) and (4), calculate the numerical values of effective $g_*(T)$ before and after electron-positron annihilation.

(2.5) Work out the numerical values of $\rho_{\nu,1}/\rho_{\gamma,1}$ between the two relativistic species *after* the annihilation. Note that this is why the CMB background temperature $T_{\rm CMB} \sim 2.75\,{\rm K}$, while the neutrino background temperature $T_{\rm CNB} \sim 1.96\,{\rm K}$. The present-day energy density of all relativistic species $\rho_{\rm R,\,today} = [1 + 7/8 \times 3 \times (4/11)^{4/3}]\,\rho_{\gamma,\,{\rm today}} \sim 8 \times 10^{-34}\,{\rm g/cm^3}$, with the CMB radiation density today $\rho_{\gamma,\,{\rm today}} \sim 4.6 \times 10^{-34}\,{\rm g/cm^3}$.

3. Consider a flat universe dominated by relativistic species with an effective degree of freedom $g_*(T)$ (given by Eq. (2)). The Hubble parameter $H(t)$ can be linked to temperature T and $g_*(T)$ through the Friedmann equation.

(3.1) Evaluate in the *radiation* era the power-law slope α in $H(T_\gamma) \propto T_\gamma^\alpha$ for the relation between the Hubble parameter H and photon temperature T_γ, and the power-law slope β in $T_\gamma(t) \propto t^\beta$ for the relation between T_γ and cosmic time t. Specifically show that during the *radiation* era, we can express time t as a function of T:

$$t(T) = \sqrt{\frac{45}{16\pi^3\,G}}\,\frac{T^{-2}}{\sqrt{g_*(T)}}. \tag{5}$$

Note that this is a very useful relation between t and T in early universe. We can re-normalize this into:

$$t(T) = \frac{2.4}{\sqrt{g_*(T)}}\left(\frac{T}{\rm MeV}\right)^{-2}{\rm sec}. \tag{6}$$

(3.2) Now let us estimate the mass fraction X_4 of primordial helium ^{4}He via the approximation of $X_4 \approx 2X_{\rm n}(t_{\rm d})$, where $t_{\rm d}$ is the time of efficient deuterium production ("deuterium bottleneck"). The mass fraction $X_{\rm n}(t_{\rm d})$ of free neutrons is essentially controlled by two crucial time stamps, i.e., the time of neutrino decouple $t_{\rm ND}$ (at energy scale of $T_{\rm ND} \sim 0.8\,{\rm MeV}$) and $t_{\rm d}$ (at energy scale of $T_{\rm d} \sim 0.07\,{\rm MeV}$), in between free neutrons decay with a half time of $\tau_{\rm n} = 887\,s$. If we take the thermal equilibrium abundance $X_{\rm n}(t_{\rm ND}) = 0.1657$ at just before neutrino decouple, now use Eq. (6) and the result of (2.4), work out $t_{\rm ND}$, $t_{\rm d}$, $X_{\rm n}(t_{\rm d})$ and finally X_4.

(3.3) In our Universe, as the baryon-to-photon ratio $\eta \equiv n_{\rm B}/n_\gamma \sim 10^{-9}$, the thermal status has always been governed by the relativistic species, while the dynamics in the *matter* era is governed by the non-relativistic matter component. Evaluate in the *matter* era the power-law slope α in $H(T_\gamma) \propto T_\gamma^\alpha$, and the power-law slope β in $T_\gamma(t) \propto t^\beta$. [Also think about how η can affect the primordial helium abundance.]