Physics Cosmology Homework - Inflation

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1. Assuming the universe is matter dominated, calculate the deviation of total matter density from the critical density in terms of $|\Omega_{\rm tot} - 1|$ at $1\,s$ after the Big Bang (where $\Omega_{\rm tot} = \rho_{\rm tot}/\rho_{\rm cr}$), if we require $|\Omega_{\rm tot} - 1| \leqslant 1$ at the present day as is indicated by observations. Why an accelerating universe can help with solving the *flatness problem*?

2. Assuming the universe is matter dominated, calculate the ratio between the comoving horizon today and that at $z = 1100$, where the cosmic microwave background (CMB) radiation came out. We observe that the CMB sky is smooth, homogeneous and isotropic to a level of 10^{-5} over the entire sky. Why does this indicate a *horizon problem*? Why an accelerating universe can help with solving this problem?

3. Estimate the number of e-folds $N(t) \equiv \ln[a(t_{\rm end})/a(t_{\rm start})]$ of the universe from a starting time $t_{\rm start}$ to the end of inflation at $t_{\rm end}$, if we require a cosmic inflation at an energy level of $(10^{16}\,{\rm GeV})^4$ to solve the horizon problem. We note that 10^{16}GeV is the highest inflation energy scale allowed by observations. The real energy scale can be lower than this but bound by a few MeV, at which the Big Bang nucleosynthesis (BBN) happens.

The solution requires that the observed largest comoving scale in the sky with isotropy and homogeneity to have once been within the horizon distance and *exited* the horizon during inflation (and *re-entered* in later times) [combining Question 2 understand why so]. This scale is determined by observing the CMB sky, the radiation of which came out at a very early time in the Universe. The comoving size of this scale is very close to the comoving horizon today, given by H_0^{-1}. We note that we refer to the inflation after the CMB-scale mode exiting horizon as the *observable* inflation. This is because inflation could already start before this point, but no later than this, if it would solve the horizon problem.

Back to inflation, we can Fourier expand the inflaton field and consider the development of each individual mode of perturbation, which has a comoving momentum k_c corepsonding to a comoving wavelength of $\lambda_c = 2\pi/k_c$. During inflation, the modes exit the horizon from the largest wavelengths (smaller k_c) to the smallest wavelengths (bigger k_c), as the comoving horizon size shrinks during inflation. This essentially allows us to "label" different epochs during inflation with different modes k_c that exit (or later reenter after inflation) the horizon at different times, such that the comoving horizon distance $H_c^{-1}(t) = a_0/[H(t)a(t)]$ at time t is (by definition) equal to λ_c of the mode that just exits the horizon at time t, or equivalently $k_c = H_c(t) = H(t)a(t)/a_0$. In this case, the requirement as stated above to solve the horizon problem is essentially asking $H_c^{-1}(t_{\rm start}) = \lambda_c(t_{\rm start}) \geq H_0^{-1}$, or equivalently $k_c(t_{\rm start}) \leq H_0$ (dropping pre-factors).

Now we can estimate the constraint on $N(t_{\rm start})$ for the observable inflation. Using subscripts *start* and *end* to indicate quantities defined at the start and end of inflation, respectively, we have $k_{\rm start} = H_{\rm start} a_{\rm start}/a_0$.

(3.1) Show that:

$$N(t_{\rm start}) = \ln\left(\frac{a_{\rm end}}{a_0}\right) + \ln\left(\frac{H_{\rm start}}{H_0}\right) + \ln\left(\frac{H_0}{k_{\rm start}}\right). \tag{1}$$

(3.2) Show that the argument inside the second logarithmic term:

$$\left(\frac{H_{\rm start}}{H_0}\right) = \left(\frac{{V_{\rm SR}}^{1/4}}{10^{16}{\rm GeV}}\right)\left(\frac{10^{16}{\rm GeV}\, {V_{\rm SR}}^{1/4}}{{\rho_{\rm cr,0}}^{1/2}}\right), \tag{2}$$

where $V_{\rm SR}$ is the inflaton potential energy during slow-roll inflation, normalized by inflation energy scale at 10^{16}GeV; $\rho_{\rm cr,0} \approx 2\times 10^{-29} h^2\,{\rm g/cm^3} \approx (3\times 10^{-12}{\rm GeV})^4\, h^2$ is the critical density today with $h = H_0/[100\,{\rm km/s/Mpc}]$.

(3.3) We will work out the argument inside the first logarithmic term with the help of $a_{\rm reh}$ defined at the epoch of reheating, and $a_{\rm eq}$ defined at the epoch of radiation-matter equality. To simplify the calculation, we make two approximations: (a) before $a_{\rm reh}$ the energy density follows $\rho(a) \propto a^{-3}$ (for damping oscillation); and (b) after $a_{\rm reh}$ and till $a_{\rm eq}$ the universe is very quickly dominated by radiation (neglecting energy density variation of the radiation species due to all kinds of annihilation processes), during this epoch the energy density follows $\rho(a) \propto a^{-4}$. We note that from $a_{\rm eq}$ till today, the radiation density follows $\rho_{\rm r}(a) = \rho_{\rm r,0}(a_0/a)^4$, where $\rho_{r,0} \approx 8\times 10^{-34}\,{\rm g/cm^3} \approx (2.5\times 10^{-13}{\rm GeV})^4$ is the radiation density today (including all relativistic species). Let us also assume that $\rho(a_{\rm eq}) \approx \rho_{\rm r}(a_{\rm eq})$. With all preparations above, show that:

$$
\begin{aligned}
N(t_{\rm start}) = \ln\left(\frac{H_0}{k_{\rm start}}\right) + \ln\left(\frac{10^{16}{\rm GeV}\,{\rho_{\rm r,0}}^{1/4}}{{\rho_{\rm cr,0}}^{1/2}}\right) \\
+\frac{1}{3}\ln\left(\frac{\rho_{\rm reh}}{\rho_{\rm end}}\right)^{1/4} + \ln\left(\frac{V_{\rm SR}}{\rho_{\rm end}}\right)^{1/4} + \ln\left(\frac{{V_{\rm SR}}^{1/4}}{10^{16}{\rm GeV}}\right).
\end{aligned}
\tag{3}
$$

(3.4) Note that under the slow-roll inflation with $\rho_{\rm end} \approx V_{\rm SR} \approx (10^{16}{\rm GeV})^4$, the sum of the last two terms will essentially vanish. The largest uncertainty comes from models of the reheating era, which however has to end, also, by several MeV when the BBN happens, i.e., $\rho_{\rm reh} \geq (1\,{\rm MeV})^4$, resulting in $1/3\ \ln\left(\rho_{\rm reh}/\rho_{\rm end}\right)^{1/4} \in (-3.65,\,0)$. For this reason, let us for now ignore the last three terms and combine the present-day energy densities $\rho_{\rm r,0}$ and $\rho_{\rm cr,0}$ to work out the number of e-folds $N(t_{\rm start})$ during a slow-roll inflation at $(10^{16}\,{\rm GeV})^4$ that is required to solve the horizon problem.

4. Using the observational constraint of a minimal of 60 e-folds from the start to the end of inflation, constrain some basic properties of a large-field inflation model that has a potential of $V(\phi) = m^2\phi^2/2$.

(4.1) In a FLRW universe, the equation of motion for the inflaton field and the Friedmann's equation are given by:

$$
\begin{cases}
\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \\
H^2 = \left(\dot{\phi}^2/2 + V(\phi)\right)/\,3M_{\rm pl}^2,
\end{cases}
\tag{4}
$$

where $M_{\rm pl} = \sqrt{c\hbar/(8\pi G)} \approx 2.4\times 10^{18}{\rm GeV}$ is the reduced Planck mass. Under the slow-roll condition:

$$
\begin{cases}
\dot{\phi}^2/2 \ll V(\phi), \\
|\ddot{\phi}| \ll |3H\dot{\phi}|,
\end{cases}
\tag{5}
$$

the Eqs. (4) become:

$$
\begin{cases}
3H\dot{\phi} = -V'(\phi), \\
H^2 = V(\phi)\,/\,3M_{\rm pl}^2.
\end{cases}
\tag{6}
$$

We define the shape parameter $\epsilon \equiv M_{\rm pl}^2/2\,(V'/V)^2$. Show that:
(1) under slow-roll condition given by Eqs. (5) we have $\epsilon \ll 1$;
(2) essentially $\epsilon = -\dot{H}/H^2$;
(3) $\epsilon \ll 1$ suggests $\ddot{a}/a \gg 0$, i.e., an accelerating universe.

(4.2) To end slow-roll inflation, we set $\epsilon = 1$, under the limit of slow-roll condition, this corresponds to $\ddot{a}/a = 0$. Show that for $V(\phi) = m^2\phi^2/2$, $\phi_{\rm end} = \sqrt{2}M_{\rm pl}$ at $\epsilon = 1$.

(4.3) Using $\mathrm{d}(\ln a) = \mathrm{d}a/a = H\mathrm{d}t$ and $\mathrm{d}t = \mathrm{d}\phi/\dot{\phi}$, show that under the slow-roll condition, the number of e-folds N(t) from time t to the end of inflation $t_{\rm end}$ is given by:

$$N(t) \equiv \ln \frac{a(t_{\rm end})}{a(t)} = \int_{\phi(t)}^{\phi_{\rm end}} \frac{H\,\mathrm{d}\phi}{\dot{\phi}} = \frac{1}{M_{\rm pl}^2} \int_{\phi_{\rm end}}^{\phi(t)} \frac{V(\phi)}{V'(\phi)}\,\mathrm{d}\phi, \tag{7}$$

where the subscript *end* denotes the end of inflation.

(4.4) For a potential model $V(\phi) = m^2\phi^2/2$, using Eq. (7), write down the expression of $N(t)$ in terms of $\phi(t)$ and $\phi_{\rm end}$. If from the start to the end of inflation, we request $N(t_{\rm start}) \geq 60$, work out:
(1) the value of $\phi(t_{\rm start})$ in unit of $M_{\rm pl}$;
(2) the shape parameter ϵ corresponding to $\phi(t_{\rm start})$. *Does this satisfy* $\epsilon \ll 1$ *for slow-roll inflation?*.