Physical Cosmology
- Cosmological distances and cosmic times

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1 Cosmological age and look-back time

Recall that the Hubble parameter $H(a)$ is given by:

$$H^2(a) = H_0^2\left(\Omega_{\mathrm{m},0}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{\mathrm{rad},0}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0}\left(\frac{a}{a_0}\right)^{-2}\right). \tag{1}$$

Now let us define $x \equiv a/a_0 = 1/(1+z)$. Therefore $H \equiv \dot{a}/a = \mathrm{d}a/(a\mathrm{d}t) = \mathrm{d}x/(x\mathrm{d}t) = -\mathrm{d}z/((1+z)\mathrm{d}t)$. We then have:

$$\mathrm{d}t = \frac{\mathrm{d}x}{xH(x)} = \frac{\mathrm{d}x}{H_0 xE(x)} \tag{2}$$

$$= -\frac{\mathrm{d}z}{(1+z)H(z)} = -\frac{\mathrm{d}z}{H_0(1+z)E(x)}, \tag{3}$$

where $E(x) \equiv \sqrt{\Omega_{\mathrm{m},0}x^{-3} + \Omega_{\mathrm{rad},0}x^{-4} + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0}x^{-2}}$, and $E(z) \equiv \sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{\Lambda,0} + \Omega_{\mathrm{K},0}(1+z)^2}$.

With this, the age of the Universe for an event at redshift z is given by:

$$t(z) = \int_z^\infty \frac{\mathrm{d}z}{(1+z)H(z)} = \int_0^x \frac{\mathrm{d}x}{xH(x)}. \tag{4}$$

While the *look-back time* of this event is given by:

$$t_{\mathrm{LB}}(z) = \int_0^z \frac{\mathrm{d}z}{(1+z)H(z)} = \int_x^1 \frac{\mathrm{d}x}{xH(x)}. \tag{5}$$

2 Cosmological distances

For light, $\mathrm{d}\tau^2 = 0$. A *radially* moving light ray thus satisfies $c^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\chi^2 = 0$ or $c^2\mathrm{d}t^2 - a^2(t)(1-Kr^2)^{-1}\mathrm{d}r^2 = 0$, where $r = S_K(\chi)$. In particular, for a light ray which travels towards us, $-\mathrm{d}\chi = -\mathrm{d}r/\sqrt{1-Kr^2} = c/a(t)\mathrm{d}t$. Therefore a comoving coordinate distance χ_e (and r_e) between us and a light beam which emits at time t_e and arrives at us at time t_0 is then give by:

$$\chi_e = \int_0^{\chi_e} \mathrm{d}\chi' = \int_0^{r_e} \frac{\mathrm{d}r'}{\sqrt{1-Kr'^2}} = \int_{t_e}^{t_0} \frac{c\,\mathrm{d}t}{a(t)}. \tag{6}$$

Several important cosmological distances are defined based on the comoving coordinate distances χ_e and $r_e = S_K(\chi_e)$.

2.1 Line-of-sight comoving distance

When the comoving *coordinate* distance $\chi(z)$ is rescaled by today's scale factor a_0, this gives the *line-of-sight comoving* distance $D_{\mathrm{com}}^{\mathrm{los}}(z)$:

$$D_{\mathrm{com}}^{\mathrm{los}}(z) \equiv a_0\chi(z) = a_0 \int_{t(z)}^{t_0} \frac{c\,\mathrm{d}t'}{a(t')} = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')} = \frac{c}{H_0}\int_x^1 \frac{\mathrm{d}x'}{x'^2E(x')}. \tag{7}$$

2.2 Transverse comoving distance

When the comoving *coordinate* distance $r(z) = S_K(\chi(z))$ is rescaled by a_0, this gives the *transverse comoving* distance $D_{\mathrm{com}}^{\mathrm{trans}}(z)$:

$$D_{\mathrm{com}}^{\mathrm{trans}}(z) \equiv a_0 r(z) = a_0 S_K\left(\frac{c}{a_0H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right). \tag{8}$$

Using definition of $\Omega_{\mathrm{K},0} = -Kc^2/(a_0H_0)^2$ to re-write a_0, Eq. (8) is given by:

$$= \frac{c}{H_0}\begin{cases} \sqrt{\Omega_{K,0}}^{-1}\sinh\left[\sqrt{\Omega_{K,0}}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right] & \text{if } K=-1 \\ \int_0^z \frac{\mathrm{d}z'}{E(z')} & \text{if } K=0 \\ \sqrt{|\Omega_{K,0}|}^{-1}\sin\left[\sqrt{|\Omega_{K,0}|}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right] & \text{if } K=1 \end{cases}$$

One can further use the fact that $\sinh(ix) = i\sin(x)$ to write:

$$D_{\mathrm{com}}^{\mathrm{trans}}(z) = \frac{c}{H_0\sqrt{\Omega_{K,0}}}\sinh\left[\sqrt{\Omega_{K,0}}\int_0^z \frac{\mathrm{d}z'}{E(z')}\right]. \tag{9}$$

2.3 Proper distance

An object that emits a light ray at redshift z towards us and arrives at us today (i.e., the world line of this object intercepts our today's past light cone at z) has a proper distance $D_{\rm prop}(z)$, which is the distance measured at the past time $t(z)$ corresponding to z, and given by:

$$D_{\rm prop}(z) \equiv a(z)\chi(z) = a(t)\int_{t(z)}^{t_0} \frac{c\,{\rm d}t'}{a(t')} \tag{10}$$

$$= \frac{c}{H_0(1+z)}\int_0^z \frac{{\rm d}z'}{E(z')} \tag{11}$$

$$\left(= \frac{c\,x}{H_0}\int_x^1 \frac{{\rm d}x'}{x'^2 E(x')} \right). \tag{12}$$

We shall note that the proper distance is the comoving *coordinate* distance $\chi(z)$ rescaled by $a(z)$ at time of light emission instead of today's scale factor a_0. In particular, in the case of a flat space geometry (K=0), we have:

$$D_{\rm prop}(z) = \frac{D_{\rm com}^{\rm los}(z)}{1+z} = \frac{D_{\rm com}^{\rm trans}(z)}{1+z} = \frac{c}{H_0(1+z)}\int_0^z \frac{{\rm d}z'}{E(z')}. \tag{13}$$

2.4 Angular diameter distance

An object at redshift z has an angular diameter $D_{\rm AD}(z)$, which is a distance defined using angular size argument ($\theta = \Delta X/X$) assuming an Euclidean geometry (i.e., a local Universe approximation). $D_{\rm AD}(z)$ is essentially the comoving *coordinate* distance $r(z)$ rescaled by $a(z)$ at time of light emission (instead of today's scale factor a_0), and given by:

$$D_{\rm AD}(z) \equiv a(z)r(z) = \frac{a_0 r(z)}{1+z} = \frac{D_{\rm com}^{\rm trans}(z)}{1+z}, \tag{14}$$

which in a flat universe (K=0) equals to proper distance $D_{\rm prop}(z)$ (Eq. 13). Note (1) this distance is used when measurements involve angular sizes, such as gravitational lensing; (2) both $D_{\rm DA}$ and $D_{\rm prop}$ can be non-increasing (or even decreasing) with z increasing under certain cosmological parameters, essentially reflecting the fact that the universe was smaller at earlier times.

2.5 Luminosity distance

An object at redshift z has a luminosity distance $D_{\rm lum}(z)$, which is a distance defined using surface brightness argument ($s = L/(4\pi D^2)$) assuming an Euclidean geometry (i.e., a local Universe approximation). It is given by:

$$D_{\rm lum}(z) \equiv (1+z)a_0 r(z) = (1+z)D_{\rm com}^{\rm trans}. \tag{15}$$

Note (1) this distance is used when measurements essentially involve luminosities, such as SNIa distances; (2) it is important to see that:

$$D_{\rm AD}(z) < D_{\rm com}^{\rm trans}(z) < D_{\rm lum}(z). \tag{16}$$

2.6 Particle horizon

At any given time t, a particle/event/influence can maximally travel to a distance given by:

$$a(t)\int_0^t \frac{c\,{\rm d}t'}{a(t')} = \frac{c}{H_0(1+z)}\int_z^\infty \frac{{\rm d}z'}{E(z')} = \frac{c\,x}{H_0}\int_0^x \frac{{\rm d}x'}{x'^2E(x')}. \tag{17}$$

This distance is the *particle horizon* at time t. The particle horizon today is $\sim 14\,{\rm Gyr}$, which is the maximal distance a particle can travel since $t=0$.

2.7 Event horizon

The *event horizon* at any given time t is a proper distance $D_{\rm EH}(t)$ (to us) of an object , the photons which emit then will arrive at us at time $t \to \infty$ (i.e., this object is on our light cone joining us at $t \to \infty$). This means that objects at and beyond $D_{\rm EH}(t)$ (at given time t) will never be seen by us. This distance is given by $D_{\rm EH}(t) = a(t)\int_t^\infty \frac{c\,{\rm d}t'}{a(t')}$. One can show that if a universe is dominated by a cosmological constant with energy density $\rho_\Lambda > 0$, then the event horizon has a finite value proportional to $1/\sqrt{\rho_\Lambda}$. The smaller ρ_Λ is, the bigger the event horizon is.

2.8 Hubble sphere

At any given time t, there is a proper distance $D_{\rm HS}(t)$ (to us) where galaxies recede from us (at that time) with the speed of light, i.e., $c = H(t)D_{\rm HS}(t)$. These distances (at all times) compose the *Hubble sphere*. Note that all light cones that join us in the past, present or future satisfy $\dot{D}(t) = H(t)D(t) - c$, where $D(t) = a(t)\chi(t)$ is the proper distance of photons which are at comoving coordinate distance $\chi(t)$ from us at time t. Any one of our light cones always first increases with time (i.e., $\dot{D}(t) > 0$, meaning the Hubble flow is moving faster than the speed of light), until reaching a "turn-around" point where $\dot{D}(t) = 0$ (i.e., $H(t)D(t) = c$), and then starts decreasing (i.e.,

$\dot{D}(t) < 0$, meaning the Hubble flow is moving slower than the speed of light), and eventually reaching us at $t > 0$. We can see that the Hubble sphere is essentially composed of all light-cone distances $D(t)$ at time of "turn-around". In a universe for which the cosmological constant $\Lambda > 0$, the Hubble sphere converges to the event horizon at $t \to \infty$. Question: *can we see galaxies that recede from us with the Hubble flow at speed of light?*