# Problem 1.1

作业1

① 度规张量 $ds^2=dr^2+r^2d\theta^2$ 描述了____的几何性质?

a) 2D sphere

b) 2D flat

- 请写出: $g_{rr}, g_{r\theta}, g_{\theta\theta}, g^{rr}, g^{r\theta}, g^{\theta\theta}$ 分别的表达(以$r,\theta$)
- 请写出 $\Gamma^a_{bc}$ 的全部非零分量, 用$r,\theta$表达. 计算求证 $\nabla_a g_{mn}=0$

(1) 2D flat

(2) obviously $g_{rr}=1$, $g_{r\theta}=0$, $g_{\theta\theta}=r^2 \Rightarrow$ metric tensor $\begin{bmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 1 & \\ & r^2 \end{bmatrix}$

invert the metric tensor, we get $g^{rr}=1$, $g^{r\theta}=0$, $g^{\theta\theta}=\frac{1}{r^2}$

(3) $\Gamma^a_{bc}=\frac{1}{2}g^{ad}\left(\frac{\partial g_{cd}}{\partial x^b}+\frac{\partial g_{bd}}{\partial x^c}-\frac{\partial g_{bc}}{\partial x^d}\right)$

for the given metric, $\Gamma^{a}_{bc} \neq 0$ only if $a=d$, such that

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{aa} \left( \frac{\partial g_{ac}}{\partial x^b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^a} \right)$$

if $a=b$, then $\Gamma^{a}_{ac} = \frac{1}{2} g^{aa} \frac{\partial g_{aa}}{\partial x^c} \Rightarrow \Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{2r^2} 2r = \frac{1}{r}$

if $b=c$, then $\Gamma^{a}_{bb} = \frac{1}{2} g^{aa} \left( - \frac{\partial g_{bb}}{\partial x^a} \right) \Rightarrow \Gamma^{r}_{\theta\theta} = -\frac{1}{2} 2r = -r$

(4) $g_{mn;a} = \frac{\partial g_{mn}}{\partial x^a} - \Gamma^{b}_{am} g_{bn} - \Gamma^{b}_{an} g_{bm}$

only if $m=n$ such that $g_{mn} \neq 0 \Rightarrow g_{mm;a} = \frac{\partial g_{mm}}{\partial x^a} - 2\Gamma^{b}_{am} g_{bm} = \frac{\partial g_{mm}}{\partial x^a} - 2\Gamma^{m}_{am} g_{mm}$

$$\Rightarrow \begin{cases} g_{rr;a} = -2\Gamma^{r}_{ar} g_{rr} = 0 \\ g_{\theta\theta;a} = \frac{\partial g_{\theta\theta}}{\partial x^a} - 2\Gamma^{\theta}_{a\theta} g_{\theta\theta} \Rightarrow g_{\theta\theta;r} = \frac{\partial g_{\theta\theta}}{\partial r} - 2\Gamma^{\theta}_{r\theta} g_{\theta\theta} = 2r - 2\frac{1}{r} r^2 = 0 \end{cases}$$

$\therefore g_{mn;a} = 0$

Problem 1.2 ② 在低速弱引力场条件下，时空近似平直。我们可以假设度规是Minkowski时空度规上加一个微小的扰动项：$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$

$\eta_{\mu\nu}\Rightarrow\begin{pmatrix}-1 & & & 0\\ & 1 & & \\ & & 1 & \\ 0 & & & 1\end{pmatrix}$，$h_{\mu\nu}$ 为小量。 $\begin{pmatrix}ds^2=g_{\mu\nu}dx^\mu dx^\nu\\ d\tau=\sqrt{-ds^2}\end{pmatrix}$

将测地线方程：$\frac{d^2x^\mu}{\partial\tau^2}+\Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}=0$ 在上述条件下近似：

通过与牛顿引力对比，证明：

(1) $h_{00}=-2\phi$，这里 $\phi$ 是牛顿引力势 ($\nabla^2\phi=4\pi G\rho$.)

(2) 我们对引力场中静钟计时间隔 $dt\simeq(1-\phi)d\tau$，其中 $d\tau$ 为固有时间间隔.

(1) Under the assumption of low velocities ($\beta=\frac{v}{c}\ll1$) and weak fields, we can make the following approximations:

① $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, neglect high-order terms in $h_{\mu\nu}$ and its derivatives;

② spatial components of velocity $\frac{dx^i}{dt}$ are much smaller than $c\,(=1)$;

③ use coordinate time $t$ instead of proper time $\tau$

With these assumptions, the geodesic eq simplifies to: $\frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{00} = 0$

here $\Gamma^\mu_{\sigma\nu} = \frac{1}{2} g^{\mu\rho} \left( \frac{\partial g_{\nu\rho}}{\partial x^\sigma} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\rho} \right) \approx \frac{1}{2} \eta^{\mu\rho} \left( \frac{\partial h_{\nu\rho}}{\partial x^\sigma} + \frac{\partial h_{\sigma\rho}}{\partial x^\nu} - \frac{\partial h_{\sigma\nu}}{\partial x^\rho} \right)$

$$\Rightarrow \Gamma^\mu_{00} = \frac{1}{2} \eta^{\mu\rho} \left( 2 \frac{\partial h_{\rho 0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\rho} \right) = \eta^{\mu\rho} \left( \frac{\partial h_{\rho 0}}{\partial x^0} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^\rho} \right)$$

note that $\frac{\partial h_{\rho 0}}{\partial x^0} = \frac{\partial h_{\rho 0}}{\partial t} \frac{1}{c} \ll \frac{\partial h_{00}}{\partial t} \frac{dt}{dx^\rho} = \frac{\partial h_{00}}{\partial x^\rho}$, so:

$$\Gamma^\mu_{00} \approx -\frac{1}{2} \eta^{\mu\rho} \frac{\partial h_{00}}{\partial x^\rho} \Rightarrow \begin{cases} \Gamma^0_{00} = -\frac{1}{2} \eta^{0\rho} \frac{\partial h_{00}}{\partial x^\rho} = \frac{1}{2} \frac{\partial h_{00}}{\partial x^0} \\ \Gamma^i_{00} = -\frac{1}{2} \eta^{i\rho} \frac{\partial h_{00}}{\partial x^\rho} = -\frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \end{cases}$$

$\therefore$ geodesic eq $\begin{cases} \dfrac{d^2x^0}{dt^2} + \dfrac{1}{2}\dfrac{\partial h_{00}}{\partial x^0} = 0 \text{ (identical eq)} \\ \dfrac{d^2x^i}{dt^2} = \dfrac{1}{2}\dfrac{\partial h_{00}}{\partial x^i} \end{cases}$

(2) In Newton grav, the eq of motion is: $\dfrac{d^2x^i}{dt} = -\dfrac{\partial \varphi}{\partial x^i}$

Compare this with geodesic eq, we get: $\dfrac{1}{2}\dfrac{\partial h_{00}}{\partial x^i} = -\dfrac{\partial \varphi}{\partial x^i} \Rightarrow h_{00} = -2\varphi + C$

Since $h_{00}$ should vanish at inf where $\varphi \to 0$, the constant $C$ is zero. So:

$h_{00} = -2\varphi$

(3) for a clock at rest in the grav field, $dx^i = 0$:

$ds^2 = g_{00}\, dx^0 dx^0 = (\eta_{00} + h_{00})\, dt^2 = -(1+2\varphi)\, dt^2$

$\Rightarrow d\tau = \sqrt{-ds^2} = \sqrt{(1+2\varphi)\, dt^2} \approx (1+\varphi)\, dt \Leftrightarrow dt = (1-\varphi)\, d\tau$

Problem 1.3

③ 利用①中极坐标表达考虑时空度规 $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2$，$d\tau = \sqrt{-ds^2}$，写出测地线方程所有非零项对应的表达。

另：自由粒子平直空间中 Lagrangian：$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$，在极坐标下利用 Euler-Lagrange Equation，写出自由粒子 Equation of motion。考察和理解二者等价。

(1) geodesic eq: $$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

non-zero Christoffel connection (from prob 1.1): $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$

$$\Rightarrow \begin{cases} \dfrac{d^2 r}{d\tau^2} - r\left(\dfrac{d\theta}{d\tau}\right)^2 = 0 \\[2ex] \dfrac{d^2\theta}{d\tau^2} + \dfrac{2}{r}\dfrac{dr}{d\tau}\dfrac{d\theta}{d\tau} = 0 \Rightarrow \dfrac{d}{d\tau}\left(r^2 \dfrac{d\theta}{d\tau}\right) = 0 \end{cases}$$

(2) Lagrange eq: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ and Lagrange func $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$$\Rightarrow \begin{cases} q = r, \quad 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 \Rightarrow \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 \\ q = \theta, \quad 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(mr^2\dot{\theta}) \Rightarrow \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0 \end{cases}$$

Since there is no grav here, $dt = d\tau$, these eqs of motion are equivalent to the geodesic eqs.