Physical Cosmology
- CMB basics I

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1 Gaussian perturbation

If a real field $g(\mathbf{x})$ with zero mean (such as over-density $\delta \equiv \rho/\bar{\rho} - 1$ or temperature fluctuation $\delta T \equiv T/\overline{T} - 1$) is generated by an *isotropic* Gaussian random process (GRP, e.g., inflation), then this field can be simply summarized by merely two statistical properties, i.e., $\langle g(\mathbf{x}) \rangle = 0$ and $\langle g(\mathbf{x})^2 \rangle = S^2$, where S is a number. $\langle\rangle$ is taking average over all realizations of such a GRP.

We can expand $g(\mathbf{x})$ with Fourier series as $g(\mathbf{x}) = \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, with $g_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ being the Fourier coefficient for wave vector $\mathbf{k} = (k_x,\, k_y,\, k_z)$, in each dimension $k_n = 2\pi n/L$, $n = 0,\, \pm1,\, \pm2, ..$, with L being the size of the field. Each realization of the GRP will generate a specific field of $g(\mathbf{x})$ with its corresponding set of Fourier coefficient $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. Being a product of an isotropic GRP means that statistically the probability density $p(g_{\mathbf{k}})$ of $\{g_{\mathbf{k}}\}$ follows a Gaussian distribution:

$$\begin{aligned} p(g_{\mathbf{k}}) &= \frac{1}{2\pi S_{\mathbf{k}}^2} \exp\left(- \frac{|g_{\mathbf{k}}^2|}{2S_{\mathbf{k}}^2} \right) \\ &= \frac{1}{\sqrt{2\pi} S_{\mathbf{k}}} \exp\left(- \frac{a_{\mathbf{k}}^2}{2S_{\mathbf{k}}^2} \right) \frac{1}{\sqrt{2\pi} S_{\mathbf{k}}} \exp\left(- \frac{b_{\mathbf{k}}^2}{2S_{\mathbf{k}}^2} \right), \end{aligned} \tag{1}$$

where under the isotropic condition $S_{\mathbf{k}}^2 = S_k^2$ is the variance of the population, where $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$.

One can show that over the many realizations we must have the following:

- $\langle g_{\mathbf{k}} \rangle = \int p(g_{\mathbf{k}})\, g_{\mathbf{k}}\, \mathrm{d}a_{\mathbf{k}} \mathrm{d}b_{\mathbf{k}} = 0$
- $\langle |g_{\mathbf{k}}|^2 \rangle = \int p(g_{\mathbf{k}})\, |g_{\mathbf{k}}|^2\, \mathrm{d}a_{\mathbf{k}} \mathrm{d}b_{\mathbf{k}} = 2S_k^2$
- $\langle g_{\mathbf{k}} g_{\mathbf{k}'}^* \rangle = 2\delta_{\mathbf{k}\mathbf{k}'} S_k^2$ (where $g_{\mathbf{k}}^*$ is the complex conjugate of $g_{\mathbf{k}}$).

One can also show that over the many realizations we must also have:

- $\langle g(\mathbf{x}) \rangle = \langle \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \rangle = \Sigma_{\mathbf{k}} \langle g_{\mathbf{k}} \rangle e^{i\mathbf{k}\cdot\mathbf{x}} = 0$
- $\langle g(\mathbf{x})^2 \rangle = 2\Sigma_{\mathbf{k}} S_k^2$ (using the reality condition $g_{-\mathbf{k}} = g_{\mathbf{k}}^*$)
- $\langle g(\mathbf{x}) g(\mathbf{x}+\mathbf{r}) \rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}}$.

2 Theoretical power spectrum and two-point correlation function

In the case where we can expand the field from a finite size of L^3 to $\pm\infty$, we can then write the Fourier series in the form of Fourier integral with $\Sigma_{\mathbf{k}} (\frac{2\pi}{L})^3 = \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{k}$ when $L \to \infty$, we then obtain:

$$g(\mathbf{x}) = \Sigma_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \approx \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} g(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{k}, \tag{2}$$

with $g(\mathbf{k}) = \int_{-\infty}^{\infty} g(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3\mathbf{x} = L^3 g_{\mathbf{k}}$ connecting the continuous and discrete Fourier coefficients.

Let us be more specific without loss generality, we take $g(\mathbf{x})$ to be the over-density field $\delta(\mathbf{x})$. By design, $\langle \delta(\mathbf{x}) \rangle = 0$. We define power spectrum $P(\mathbf{k})$, $\mathcal{P}(k)$ and two-point correlation function $\xi(\mathbf{r})$ in the following way:

- Power spectrum $P(\mathbf{k})$ has unit of volume L^3, it is defined as:

$$P(\mathbf{k}) \equiv L^3 \langle |\delta_{\mathbf{k}}|^2 \rangle = 2L^3 S_k^2 \tag{3}$$

$$P(\mathbf{k}) = \int_{-\infty}^{\infty} \langle \delta(\mathbf{x}) \delta(\mathbf{x}+\mathbf{r}) \rangle e^{-i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{r} \tag{4}$$

- Define dimensionless power spectrum $\mathcal{P}(k)$ per $\ln k$ interval:

$$\mathcal{P}(k) \equiv k^3 P(k)/(2\pi^2) \tag{5}$$

- Correlation function $\xi(\mathbf{r})$ and power spectrum $P(\mathbf{k})$ are connected through the Fourier transformation:

$$\xi(\mathbf{r}) \equiv \langle \delta(\mathbf{x}) \delta(\mathbf{x}+\mathbf{r}) \rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} P(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \, \mathrm{d}^3\mathbf{k} \tag{6}$$

- Variance of the over density field $\langle \delta(\mathbf{x})^2 \rangle$ is given by:

$$\langle \delta(\mathbf{x})^2 \rangle = \Sigma_{\mathbf{k}} P(\mathbf{k})/L^3 = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} P(\mathbf{k}) \, \mathrm{d}^3\mathbf{k} = \int_{k=0}^{k=\infty} \mathcal{P}(k) \, \mathrm{d} \ln k \tag{7}$$

3 Theoretical angular power spectrum and angular correlation function

Now instead of having a $3d$ fluctuation field $\delta(\mathbf{x})$, we have a spherical $2d$ fluctuation field, such as the CMB temperature fluctuation $\delta T(\hat{n} = \{\theta, \phi\})$ in the sky. In this case, instead of having power spectrum and correlation function of the over-density field in $3d$ through the help of plane wave functions and Fourier transform, we will have *angular* power spectrum and *angular* correlation function all defined on a $2d$ sphere with the help of Legendre transform and spherical harmonic transform.

3.1 Legendre polynomials, angular power spectrum and angular correlation function

Legendre polynomials $P_l(x)$ are a complete and orthogonal set of basis functions, defined on $x \in [-1, 1]$ through (Rodriguez formula):

$$P_l(x) = \frac{1}{2^l\, l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l}[(x^2 - 1)^l]. \tag{8}$$

Being complete and orthogonal means:

$$\int_{-1}^{1} P_n(x) P_l(x) \mathrm{d}x = \frac{2}{2l+1} \delta_{nl}. \tag{9}$$

With two unit vectors $\hat{n}$ and $\hat{n}'$ denoting two directions in the sky, where the temperature fluctuations are $\delta T(\hat{n})$ and $\delta T(\hat{n}')$, respectively, we can define an angular correlation function $C(\bar{\theta}) \equiv \langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle$, with $\cos\bar{\theta} = \hat{n} \cdot \hat{n}'$. $C(\bar{\theta})$ can be expanded with Legendre polynomials:

$$C(\bar{\theta}) \equiv \langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle = \Sigma_l\, \mathcal{C}_l \Big(\frac{2l+1}{4\pi}\Big) P_l(\hat{n} \cdot \hat{n}'), \tag{10}$$

The coefficient $\mathcal{C}_l$ can be obtained through the inverse transformation:

$$\mathcal{C}_l = \frac{1}{4\pi} \int \mathrm{d}^2\hat{n} \int \mathrm{d}^2\hat{n}' P_l(\hat{n} \cdot \hat{n}') \langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle. \tag{11}$$

We can compare Eq. (11) to Eq. (4) and Eq. (11) to Eq. (6). Just like the power spectrum $P(\mathbf{k})$ being the Fourier transform of the two-point correlation function $\xi(\mathbf{r})$ of a $3d$ over-density field $\delta(\mathbf{x})$, here $\mathcal{C}_l$ is the angular power spectrum of the temperature fluctuation field $\delta T(\hat{n})$ on a $2d$ sphere. It is connected to the angular correlation function $C(\bar{\theta})$ through the Legendre transformation.

3.2 Spherical Harmonics

Spherical harmonics $Y_{lm}(\theta,\,\phi)$ are a complete and orthogonal set of basis functions defined for $\theta \in [0,\pi)$ and $\phi \in [0,2\pi)$. We can write out a given function, e.g., $\delta T(\hat{n} = \{\theta,\phi\})$ using a series of spherical harmonics via:

$$\delta T(\theta,\phi) = \Sigma_{lm} a_{lm} Y_{lm}(\theta,\phi). \tag{12}$$

The function $Y_{lm}(\theta,\phi)$ has the form:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}, \tag{13}$$

with $P_l^m(\cos\theta)$ being the associated Legendre polynomial, defined as:

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!}(1-x^2)^{m/2}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}[(x^2-1)^l],\ \ x = \cos\theta, \tag{14}$$

with $m = 0, \pm 1, \pm 2, ... \pm l$.

Being complete and orthogonal means:

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) = \delta_{mm'}\delta_{ll'}. \tag{15}$$

With this, the coefficient a_{lm} of the full-sky temperature fluctuation field $\delta T(\theta,\phi)$ can be obtained through the inverse transformation:

$$a_{lm} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \delta T(\theta,\phi) Y^*_{lm}(\theta,\phi). \tag{16}$$

We will see that the spherical harmonic coefficients a_{lm} are connected to the Legendre coefficients $\mathcal{C}_l$ in the next section.

- Reality condition: if $\delta T(\theta,\phi)$ is a real function, then $Y_{l\,-m} = (-1)^m\, Y^*_{lm}$, therefore $a_{l\,-m} = (-1)^m\, a^*_{lm}$.
- Closure relation: $\Sigma_m |Y_{lm}(\hat{u})|^2 = (2l+1)/(4\pi)$, independent of m.
- Addition relation: $\Sigma_m Y_{lm}(\hat{u}) Y^*_{lm}(\hat{v}) = (2l+1)/(4\pi)\, P_l(\hat{u}\cdot\hat{v})$.
- Plane wave in spherical harmonic expansion:

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\, \Sigma_{lm}\, i^l\, j_l(kx)\, Y_{lm}(\hat{e}_{\mathbf{k}}) Y^*_{lm}(\hat{e}_{\mathbf{x}}),\ \mathbf{k} = k\hat{e}_{\mathbf{k}},\ \mathbf{x} = x\hat{e}_{\mathbf{x}}, \tag{17}$$

  where $j_l(kx)$ is the spherical Bessel function given by:

$$j_l(kx) = (-1)^l (kx)^l \left[\frac{\mathrm{d}}{kx\,\mathrm{d}(kx)}\right]^l \frac{\sin(kx)}{kx} \tag{18}$$

- Plane wave in Legendre expansion:

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \Sigma_l (2l+1)\, i^l\, j_l(kx)\, P_l(\hat{e}_{\mathbf{k}}\cdot\hat{e}_{\mathbf{x}}). \tag{19}$$

3.3 A comparison

In Section 1 we have seen that a $3d$ over-density field $\delta(\mathbf{x})$ generated by an isotropic GRP has following statistical properties in the frequency domain:

- $\langle \delta_{\mathbf{k}} \rangle = 0$, where $\delta_{\mathbf{k}}$ is the discrete Fourier coefficient of $\delta(\mathbf{x})$
- $\langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle = 2\delta_{\mathbf{k}\mathbf{k}'} S_k^2$, where $\delta_{\mathbf{k}\mathbf{k}'}$ is the Kronecker delta, and $S_k^2 \equiv 1/2\langle |\delta_{\mathbf{k}}|^2 \rangle$, is the variance of the Gaussian distribution of $\{\delta_{\mathbf{k}}\}$,

and correspondingly statistical properties in real space:

- $\langle \delta(\mathbf{x}) \rangle = 0$
- $\xi(\mathbf{r}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r}) \rangle = 2\Sigma_{\mathbf{k}} S_k^2 e^{i\mathbf{k}\cdot\mathbf{r}} = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} P(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\, \mathrm{d}^3\mathbf{k}$ (Eq. 6), i.e., the two-point correlation function $\xi(\mathbf{r})$, or equivalently the power spectrum $P(\mathbf{k})$, contains all the statistical information about the Gaussian perturbation field, and the two quantities are connected through the Fourier transformation.

For a temperature fluctuation field $\delta T(\hat{n} = \{\theta, \phi\})$ in the sky which is generated through an isotropic GRP, one can also show that this field has following statistical properties in the frequency domain:

- $\langle a_{lm} \rangle = 0$, where a_{lm} is the spherical harmonic coefficient of $\delta T(\theta, \phi)$ (see Eq.16). Note that $\langle \rangle$ is again taking average over all realizations from the GRP.
- $\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'}\delta_{mm'} C_l$, where C_l under the isotropic condition (meaning independent of m) is given by:

$$C_l \equiv \langle |a_{lm}|^2 \rangle = \frac{1}{2l+1} \Sigma_m \langle |a_{lm}|^2 \rangle. \tag{20}$$

Correspondingly in real space, one can also show that this Gaussian perturbation field has the following statistical properties:

- $\langle \delta T(\theta, \phi) \rangle = 0$
- $C(\bar{\theta}) \equiv \langle \delta T(\hat{n}) \delta T(\hat{n}') \rangle = \Sigma_l C_l \left(\frac{2l+1}{4\pi}\right) P_l(\hat{n} \cdot \hat{n}')$, with $\cos\bar{\theta} = \hat{n} \cdot \hat{n}'$. Now we see that the angular power spectrum $\mathcal{C}_l$ in Eq. (11) is equal to C_l here, related to a_{lm} through Eq. (20). Again, the angular two-point correlation function $C(\bar{\theta})$, or equivalently the angular power spectrum $\mathcal{C}_l$, contains all the statistical information about the Gaussian perturbation field in the $2d$ sphere, and the two quantities are connected through the Legendre transformation.

4 Observed angular correlation function and angular power spectrum, cosmic variance

As we only have one copy of the Universe, we cannot obtain theoretical statistics (those with $\langle\rangle$ given in previous sections) averaged over many different realizations of the CMB skies that are generated from the GRP related to inflation. For the only realization of our CMB sky, we can however average over the entire 4π solid angle and define *observed* angular correlation function $\hat{C}(\bar{\theta})$ and *observed* angular power spectrum $\hat{C}_l$, i.e., using $\int \mathrm{d}\Omega/(4\pi)$ to replace $\langle\rangle$. Clearly, there are *more* sub-regions of smaller scales (larger ls) than those of larger scales (smaller ls). If we treat different sub-regions as different realizations of the GRP, we therefore also expect the full-sky averaged statistics on smaller scales shall be closer to the theoretical expectations than those on larger scales. The difference between the observed and theoretical angular power spectra is described by the cosmic variance $\langle(\hat{C}_l - C_l)^2\rangle$.

- Observed angular correlation function $\hat{C}(\bar{\theta})$:

$$\hat{C}(\bar{\theta}) = \frac{1}{4\pi}\int \mathrm{d}\Omega \left[\delta T(\hat{n})\delta T(\hat{n}')\right] \tag{21}$$

- Observed angular power spectrum $\hat{C}_l$:

$$\hat{C}_l = \frac{1}{2l+1}\Sigma_m |a_{lm}|^2 = \frac{1}{4\pi}\int \mathrm{d}^2\hat{n}\int \mathrm{d}^2\hat{n}' P_l(\hat{n}\cdot\hat{n}')\delta T(\hat{n})\delta T(\hat{n}'). \tag{22}$$

- Cosmic variance:

$$\left\langle \frac{(\hat{C}_l - C_l)^2}{C_l^2} \right\rangle = \frac{2}{2l+1}; \tag{23}$$

$$\left\langle \left(\frac{(\hat{C}_l - C_l)}{C_l}\right)\left(\frac{(\hat{C}_{l'} - C_{l'})}{C_{l'}}\right) \right\rangle = 0. \tag{24}$$