Magnetohydrodynamics (MHD)

等离子体中的激波

本章内容

引言

- 激波的定义
  一个以大于等离子体特征速度传播的物理量（$\rho, p, T\ldots$）的突变面
- 处理激波问题时的简化
  1. 将激波看作间断面
  2. 将激波看作两个状态均匀区域的过渡区
- 激波的耗散机制
  1. 流体力学激波：碰撞耗散（激波厚度为流体质点平均自由程）
  2. 磁流体力学激波：碰撞耗散 + 焦耳耗散（可以存在小于平均自由程厚度的无碰撞激波）
- 宇宙等离子体中的激波
  1. 太阳高层大气：日冕/行星际激波（射电 II 型暴）
  2. 磁层顶：弓激波
  3. ……
- 激波的形成
  线性波 $\rightarrow$ 有限振幅波 $\rightarrow$ 激波

流体力学激波

- 相容性条件

$$\begin{cases} \rho_1 v_1 = \rho_2 v_2 \\ p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \\ \dfrac{p_1}{\rho_1} + \varepsilon_1 + \dfrac{1}{2} v_1^2 = \dfrac{p_2}{\rho_2} + \varepsilon_2 + \dfrac{1}{2} v_2^2 \\ \left(\dfrac{\gamma p_1}{(\gamma-1)\rho_1} + \dfrac{1}{2} v_1^2 = \dfrac{\gamma p_2}{(\gamma-1)\rho_2} + \dfrac{1}{2} v_2^2 \right) \end{cases}$$

- Rankine-Hugoniot (R-H) 关系

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma-1)+(\gamma+1)p_2/p_1}{(\gamma+1)+(\gamma-1)p_2/p_1} \text{ or } \frac{p_2}{p_1} = \frac{(\gamma+1)\rho_2/\rho_1-(\gamma-1)}{(\gamma+1)-(\gamma-1)\rho_2/\rho_1}$$

由 R-H 关系得 $v_1^2 = \dfrac{p_1}{2\rho_1}[(\gamma-1)+(\gamma+1)p_2/p_1]$，$v_2 = v_1\rho_1/\rho_2$

1. $p_2 = p_1$，$v_1 = v' = \sqrt{\gamma p_1/\rho_1} = c_{s1}$，声波
2. $p_2 > p_1$，$v_1 = v' > c_{s1}$，激波速度必须大于声速
3. $p_2/p_1 \to \infty$，$\rho_2/\rho_1 \to (\gamma+1)/(\gamma-1)$, $T_2/T_1 \to (\gamma-1)p_2/(\gamma+1)p_1$，强激波，密度有限增加，温度剧烈增加

- 定义上游马赫数 $M_1 = v'/c_{s1} = v_1/c_{s1}$，有

$$\frac{p_2}{p_1} = \frac{2\gamma {M_1}^2-(\gamma-1)}{\gamma+1}, \quad \frac{\rho_2}{\rho_1} = \frac{(\gamma+1){M_1}^2}{2+(\gamma-1){M_1}^2}, \quad \frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \cdots$$

MHD 激波–相容性条件

1. 磁场法向分量连续

$$\{B_n\} = 0 \to \{B_x\} = 0$$

2. 电场切向分量连续

$$\{v_n \boldsymbol{B}_t - B_n \boldsymbol{v}_t\} = 0 \to \begin{cases} \{v_x B_y - v_y B_x\} = 0 \\ \{v_x B_z - v_z B_x\} = 0 \end{cases}$$

3. 质量守恒

$$\{\rho v_n\} = 0 \to \{\rho v_x\} = 0$$

4. 动量守恒

$$\{\rho v_n \boldsymbol{v} + (p + \frac{B^2}{2\mu_0})\boldsymbol{n} - \frac{B_n}{\mu_0}\boldsymbol{B}\} = 0 \to \begin{cases} \{\rho v_x^2 + p + \dfrac{B_y^2 + B_z^2}{2\mu_0}\} = 0 \\ \{\rho v_x v_y - \dfrac{B_x B_y}{\mu_0}\} = 0 \\ \{\rho v_x v_z - \dfrac{B_x B_z}{\mu_0}\} = 0 \end{cases}$$

5. 能量守恒

$$\{(p + \rho\varepsilon + \frac{1}{2}\rho v^2) v_n + \frac{1}{\mu_0}[B^2 v_n - (\boldsymbol{B} \cdot \boldsymbol{v}) B_n]\} = 0 \to$$

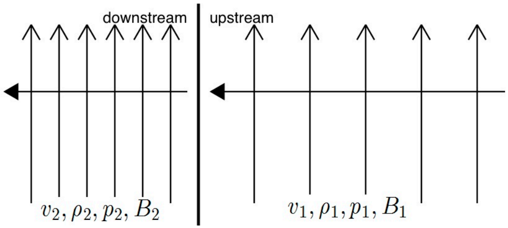
$$\{(p + \rho\varepsilon + \frac{1}{2}\rho v^2) v_x + \frac{v_x(B_y^2 + B_z^2)}{\mu_0} - B_x \frac{B_y v_y + B_z v_z}{\mu_0}\} = 0$$

- HD 间断面
  1. HD 激波 $v_n \neq 0$
  2. 切向间断 $v_n = 0$

$$\{\rho\} \neq 0, \{\boldsymbol{v}_t\} \neq 0$$

- MHD 间断
  1. 切向间断 $v_n = 0, B_n = 0$
  $$\{\boldsymbol{B}_t\} \neq 0, \{\boldsymbol{v}_t\} \neq 0, \{\rho\} \neq 0$$
  2. 接触间断 $v_n = 0, B_n \neq 0$
  $$\{\boldsymbol{v}_t\} = 0, \{\boldsymbol{B}_t\} = 0, \{p\} = 0, \{\rho\} \neq 0$$
  3. 旋转间断 $v_n \neq 0, \{\rho\} = 0$
  利用相容性条件可得 $\boldsymbol{v}_{t_2} - \frac{\boldsymbol{B}_{t_2}}{\sqrt{\mu_0\rho}} = \boldsymbol{v}_{t_1} - \frac{\boldsymbol{B}_{t_1}}{\sqrt{\mu_0\rho}} = \boldsymbol{u}$，选择以速度 $\boldsymbol{u}$ 运动的坐标系，在该坐标系中 $\boldsymbol{B}' = \boldsymbol{B}, {v_n}' = v_n, {\boldsymbol{v}_t}' = \frac{\boldsymbol{B}_t}{\sqrt{\mu_0\rho}}$，故
  $$\{B'\} = 0, \{{v_n}'\} = 0, \boldsymbol{v}' = \frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}}$$
  4. MHD 激波 $v_n \neq 0, \{\rho\} \neq 0$
     - 正激波（垂直激波）$\{\boldsymbol{v}_t\} = 0, B_n = 0$
     - 斜激波 $\{\boldsymbol{v}_t\} \neq 0, B_n \neq 0$

MHD 激波–垂直激波 1



- 上游区： $p_1, \rho_1, \boldsymbol{B} = \{0, B_1, 0\}, \boldsymbol{v} = \{v_1, 0, 0\}$
  下游区： $p_2, \rho_2, \boldsymbol{B} = \{0, B_2, B_z\}, \boldsymbol{v} = \{v_2, 0, 0\}$
  由电场切向分量连续相容性条件得： $B_z = 0$
- 代入其他相容性条件，得

$$\begin{cases} \rho_2 v_2 = \rho_1 v_1 \\ p_2 + B_2^2/(2\mu_0) + \rho_2 v_2^2 = p_1 + B_1^2/(2\mu_0) + \rho_1 v_1^2 \\ \varepsilon_2 + \dfrac{p_2}{\rho_2} + \dfrac{B_2^2}{\mu_0 \rho_2} + \dfrac{1}{2} v_2^2 = \varepsilon_1 + \dfrac{p_1}{\rho_1} + \dfrac{B_1^2}{\mu_0 \rho_1} + \dfrac{1}{2} v_1^2 \quad (*) \\ B_2 v_2 = B_1 v_1 \end{cases}$$

MHD 激波–垂直激波 2

- 定义无量纲量 $X=\rho_2/\rho_1, M_1=v_1/c_{s1}, \beta_1=2\mu_0 p_1/B_1^2=\dfrac{2c_{s1}^2}{\gamma v_{a1}^2}$，有

$$\begin{aligned} v_2/v_1 &= X^{-1} \\ B_2/B_1 &= X \\ p_2/p_1 &= \gamma M_1^2(1-X^{-1})+\beta_1^{-1}(1-X^2)+1 \end{aligned}$$

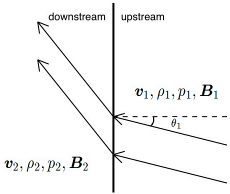
代入 (*) 可得 $(X-1)f(X)=0$，其中
$f(X)=2(2-\gamma)X^2+[2(\beta_1+1)+(\gamma-1)\beta_1 M_1^2]\gamma X-\gamma(\gamma+1)\beta_1 M_1^2$

1. $X=1$，无间断情形
2. $f(X)=0$，激波解要求有一解 $X\geq 1$，则 $f(1)\leq 0$，故有

$$M_1^2\geq 1+2/(\gamma\beta_1)=1+v_{a1}^2/c_{s1}^2 \to v_1\geq\sqrt{c_{s1}^2+v_{a1}^2}=v_{fm1}$$

$$X=\frac{(\gamma+1)M_1^2}{2+(\gamma-1)M_1^2}-\frac{2}{\beta_1\gamma}\frac{(2-\gamma)X^2+\gamma X}{2+(\gamma-1)M_1^2}$$

- 对于相同的 M_1，有 $X_{\mathrm{MHD}}<X_{\mathrm{HD}}$
- $\beta_1\to\infty$，退化到流体力学激波，$X\to X_{\mathrm{HD}}$
- $M_1\to\infty$，强激波情形，$X\to(\gamma+1)/(\gamma-1)$，故 $1<\dfrac{B_2}{B_1}<\dfrac{\gamma+1}{\gamma-1}$



相容性条件

$$
\begin{cases}
\rho_2 v_{2x} = \rho_1 v_{1x} \\
p_2 + \dfrac{B_2^2}{2\mu_0} - \dfrac{B_{2x}^2}{\mu_0} + \rho_2 v_{2x}^2 = p_1 + \dfrac{B_1^2}{2\mu_0} - \dfrac{B_{1x}^2}{\mu_0} + \rho_1 v_{1x}^2 \\
\rho_2 v_{2x} v_{2y} - \dfrac{B_{2x}B_{2y}}{\mu_0} = \rho_1 v_{1x} v_{1y} - \dfrac{B_{1x}B_{1y}}{\mu_0} \\
(p_2 + \dfrac{B_2^2}{2\mu_0}) v_{2x} - \dfrac{B_{2x}(\boldsymbol{B}_2 \cdot \boldsymbol{v}_2)}{\mu_0} + (\rho_2 \varepsilon_2 + \dfrac{1}{2}\rho_2 v_2^2 + \dfrac{B_2^2}{2\mu_0}) v_{2x} = \\
(p_1 + \dfrac{B_1^2}{2\mu_0}) v_{1x} - \dfrac{B_{1x}(\boldsymbol{B}_1 \cdot \boldsymbol{v}_1)}{\mu_0} + (\rho_1 \varepsilon_1 + \dfrac{1}{2}\rho_1 v_1^2 + \dfrac{B_1^2}{2\mu_0}) v_{1x} \\
B_{2x} = B_{1x} \\
v_{2x} B_{2y} - v_{2y} B_{2x} = v_{1x} B_{1y} - v_{1y} B_{1x}
\end{cases}
$$

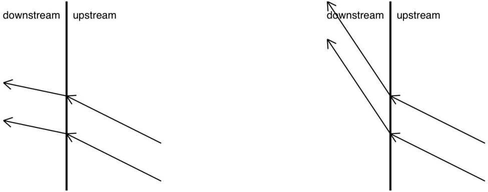
- 选择一运动坐标系（de Hoffmann & Teller 坐标系），该坐标系以 $v_{1x}B_{1y}/B_{1x}$ 沿 $-y$ 轴运动，则在这样的坐标系中有 $v_{1y}=v_{1x}B_{1y}/B_{1x}$，故
$$v_{2x}B_{2y}-v_{2y}B_{2x}=v_{1x}B_{1y}-v_{1y}B_{1x}=0$$
即上下游均有 $\boldsymbol{v}\parallel\boldsymbol{B}$，电场切向分量相容性条件消失
进一步，能量守恒相容性条件退化为流体力学形式
$$\frac{\gamma p_2}{(\gamma-1)\rho_2}+\frac{1}{2}v_2^2=\frac{\gamma p_1}{(\gamma-1)\rho_1}+\frac{1}{2}v_1^2$$
- 定义 $X=\dfrac{\rho_2}{\rho_1}, c_{s1}=\sqrt{\gamma\dfrac{p_1}{\rho_1}}, v_{a1}=\dfrac{B_1}{\sqrt{\mu_0\rho_1}}$，代入相容性条件，有
$$\frac{v_{2x}}{v_{1x}}=X^{-1}$$
$$\frac{v_{2y}}{v_{1y}}=\frac{v_1^2-v_{a1}^2}{v_1^2-Xv_{a1}^2}$$
$$\frac{B_{2x}}{B_{1x}}=1$$
$$\frac{B_{2y}}{B_{1y}}=\frac{(v_1^2-v_{a1}^2)X}{v_1^2-Xv_{a1}^2}$$
$$\frac{p_2}{p_1}=X+\frac{(\gamma-1)Xv_1^2}{2c_{s1}^2}\left(1-\frac{v_2^2}{v_1^2}\right)$$

MHD 激波–斜激波 3

- X 满足

$$(v_1^2 - Xv_{a1}^2)^2\{Xc_{s1}^2 + \frac{1}{2}v_1^2\cos^2\theta[X(\gamma-1)-(\gamma+1)]\}+$$
$$\frac{1}{2}v_{a1}^2 v_1^2\sin^2\theta X\{[\gamma+X(2-\gamma)]v_1^2 - Xv_{a1}^2[(\gamma+1)-X(\gamma-1)]\} = 0 \quad (*$$

1. 线性波：$X \to 1$，$(v_1^2 - v_{a1}^2)[v_{1x}^4 - (c_{s1}^2 + v_{a1}^2)v_{1x}^2 + c_{s1}^2 v_{a1}^2\cos^2\theta] = 0$
   三种模式：Alfvén 波 + 快慢磁声波
2. 激波：$X > 1$，有 $p_2 > p_1$，同时要求 $B_{2y}/B_{1y} > 0$
   1. $v_1^2 \le v_{a1}^2 (< Xv_{a1}^2)$，有 $B_2 < B_1$：慢激波（左）
   2. $v_1^2 \ge Xv_{a1}^2 (> v_{a1}^2)$，有 $B_2 > B_1$：快激波（右）



$\theta \to \pi/2$，快激波 $\to$ 垂直激波，慢激波 $\to$ 切向间断

MHD 激波–斜激波 4

- 对于慢激波，当 $v_1 = v_{a1}$ 且 $X \neq 1$，有 $B_{2y} = 0$（只要 $B_{1y} \neq 0$）该慢激波称消去（switch-off）激波，此时 (*)→
$(2c_{s1}^2/v_{a1}^2 + \gamma - 1)X^2 - [2c_{s1}^2/v_{a1}^2 + \gamma(1 + \cos^2\theta)]X + (\gamma + 1)\cos^2\theta = 0$
方程必有一根 $X > 1$
- 当 $\theta = 0$，(*)→ $(v_1^2 - Xv_{a1}^2)^2\{Xc_{s1}^2 + \frac{1}{2}v_1^2[X(\gamma - 1) - (\gamma + 1)]\} = 0$
    1. $\{\ \} = 0 \rightarrow X = \dfrac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2}$，此时（慢）激波退化为 HD 激波
    2. $v_1^2 = Xv_{a1}^2$，$X > 1$ 要求 $v_1 > v_{a1}$，此时快激波称诱生（switch-on）激波，有 $B_{2y}^2/B_{2x}^2 = (X - 1)[(\gamma + 1) - (\gamma - 1)X - 2c_{s1}^2/v_{a1}^2]$，则
    $$1 < X \leq \frac{\gamma + 1 - 2c_{s1}^2/v_{a1}^2}{\gamma - 1}$$
    上式要求 $v_{a1} > c_{s1}$，即诱生激波只存在于上游 Alfvén 速度大于声速的等离子体中！