Magnetohydrodynamics · Homework 3

1. (1) $\begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} = B_0 \begin{bmatrix} \frac{l}{k} J_1(kr) e^{-lz} \\ \sqrt{1-\frac{l^2}{k^2}} J_1(kr) e^{-lz} \\ J_0(kr) e^{-lz} \end{bmatrix}$ 为柱坐标系下一轴对称磁场(与θ无关) $= B_0 e^{-lz} \begin{bmatrix} \frac{l}{k} J_1(kr) \\ \sqrt{1-\frac{l^2}{k^2}} J_1(kr) \\ J_0(kr) \end{bmatrix}$

$$\nabla\times\vec{B} = \left(\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial \theta}\right]\hat{z}$$

$$= -\frac{\partial B_\theta}{\partial z}\hat{r} + \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z}$$

$$= B_0 \begin{bmatrix} -\sqrt{1-\frac{l^2}{k^2}} J_1(kr)(-l)e^{-lz} \\ \frac{l}{k} J_1(kr)(-l)e^{-lz} - e^{-lz}\frac{d}{dr}J_0(kr) \\ \frac{1}{r}\sqrt{1-\frac{l^2}{k^2}} e^{-lz} \frac{d}{dr}[rJ_1(kr)] \end{bmatrix}$$

注意到：$\frac{d}{dx}J_0(x) = -J_1(x)$ 和 $\frac{d}{dx}[xJ_1(x)] = xJ_0(x)$，于是：

$$\nabla\times\vec{B} = B_0 e^{-lz} \begin{bmatrix} l\sqrt{1-\frac{l^2}{k^2}} J_1(kr) \\ -\frac{l^2}{k}J_1(kr) + kJ_1(kr) \\ k\sqrt{1-\frac{l^2}{k^2}} J_0(kr) \end{bmatrix} = \sqrt{k^2-l^2}\, B_0 e^{-lz} \begin{bmatrix} \frac{l}{k} J_1(kr) \\ \sqrt{1-\frac{l^2}{k^2}} J_1(kr) \\ J_0(kr) \end{bmatrix} = \sqrt{k^2-l^2}\,\vec{B}$$

因k, l为常数，故此磁场为线性无力场

(2) $\vec{B} = \begin{bmatrix} B_r \\ B_\theta \\ B_z \end{bmatrix} = B_0 \begin{bmatrix} 0 \\ \frac{br}{1+b^2r^2} \\ \frac{1}{1+b^2r^2} \end{bmatrix}$ 也为一柱坐标下的轴对称磁场(与θ无关) $= \frac{B_0}{1+b^2r^2}\begin{bmatrix} 0 \\ br \\ 1 \end{bmatrix}$

且$B_r = 0$

且与z也无关

$$\nabla\times\vec{B} = -\frac{\partial B_z}{\partial r}\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z}$$

$$= B_0\left[\frac{2b^2r}{(1+b^2r^2)^2}\hat{\theta} + \frac{1}{r}\frac{d}{dr}\left(\frac{br^2}{1+b^2r^2}\right)\hat{z}\right] = B_0\left[\frac{2b^2r}{(1+b^2r^2)^2}\hat{\theta} + \frac{2b}{(1+b^2r^2)^2}\hat{z}\right]$$

$$= B_0\frac{2b}{(1+b^2r^2)^2}\begin{bmatrix} 0 \\ br \\ 1 \end{bmatrix} = \frac{2b}{1+b^2r^2}\vec{B}$$ 为一非线性无力场

2. (1) 无作用力场基本方程 $\begin{cases} \nabla\times\vec{B} = \alpha\vec{B} & ① \\ \nabla\cdot\vec{B} = 0 & ② \end{cases}$ 对①式两边取旋度得：

对①式两边取旋度得：$\nabla\times(\nabla\times\vec{B}) = \nabla\times(\alpha\vec{B})$

两边均用矢量恒等式展开得：

$$\nabla\cdot(\nabla\cdot\vec{B})-\nabla^2\vec{B}=\alpha\nabla\times\vec{B}+\nabla\alpha\times\vec{B}$$

再将①②式代入可得:

$$\vec{B}\times\nabla\alpha=\nabla^2\vec{B}+\alpha(\alpha\vec{B})=(\nabla^2+\alpha^2)\vec{B}$$

(2) 已知 $\begin{cases} (\nabla^2+\alpha^2)\vec{B} = \vec{B}\times\nabla\alpha & ③ \\ \nabla\cdot\vec{B}=0 & ② \end{cases}$

由矢量恒等式得:

$$\nabla^2\vec{B} = \nabla(\nabla\cdot\vec{B}) - \nabla\times\nabla\times\vec{B} = -\nabla\times(\nabla\times\vec{B})$$

代入③式得:

$$\alpha^2\vec{B} - \nabla\times(\nabla\times\vec{B}) = \vec{B}\times\nabla\alpha \quad ④$$

对④式两边取散度得: $\longrightarrow \nabla\times(\nabla\times\vec{B}-\alpha\vec{B})+\alpha(\nabla\times\vec{B}-\alpha\vec{B})=0$

$$\nabla\cdot(\alpha^2\vec{B})-\nabla\cdot\nabla\times(\nabla\times\vec{B})=\nabla\cdot(\vec{B}\times\nabla\alpha)$$

其中对一矢量场 $\nabla\times\vec{B}$ 总有: $\nabla\cdot\nabla\times(\nabla\times\vec{B})=0$, 于是:

$$\nabla\cdot(\alpha^2\vec{B})=\nabla\cdot(\vec{B}\times\nabla\alpha)$$

其中:

$$\nabla\cdot(\alpha^2\vec{B})=\alpha^2\nabla\cdot\vec{B}+\vec{B}\cdot\nabla\alpha^2=2\alpha\vec{B}\cdot\nabla\alpha$$

$$\nabla\cdot(\vec{B}\times\nabla\alpha)=\nabla\alpha\cdot(\nabla\times\vec{B})-\vec{B}\cdot\nabla\times\nabla\alpha=\nabla\alpha\cdot(\nabla\times\vec{B})$$

已使用对一标量场 α 总有 $\nabla\times\nabla\alpha=0$, 于是:

$$2\alpha\vec{B}\cdot\nabla\alpha=(\nabla\times\vec{B})\cdot\nabla\alpha$$

$$\Rightarrow(\nabla\times\vec{B}-2\alpha\vec{B})\cdot\nabla\alpha=0$$

$\Rightarrow \nabla\times\vec{B}=2\alpha\vec{B}$ 或 $(\nabla\times\vec{B}-2\alpha\vec{B})\perp\nabla\alpha$

即所对应磁场不一定是无力场, 满足 $(\nabla\times\vec{B}-2\alpha\vec{B})\perp\nabla\alpha$ 即可

3. 磁流体静力学平衡方程组为:

$$
\begin{cases}
\nabla p = \vec{j} \times \vec{B} \\
\vec{j} = \frac{1}{\mu_0} \nabla \times \vec{B} \\
\nabla \cdot \vec{B} = 0
\end{cases}
$$

(1) $\nabla\cdot(\vec{B}\times\nabla p)=\nabla p\cdot(\nabla\times\vec{B})-\vec{B}\cdot\nabla\times(\nabla p)$

注意到对任意标量场p都有 $\nabla\times\nabla p=0$，于是

$$\nabla\cdot(\vec{B}\times\nabla p)=\nabla p\cdot(\nabla\times\vec{B})=(\vec{j}\times\vec{B})\cdot\mu_0\vec{J}=(\mu_0\vec{J}\times\vec{J})\cdot\vec{B}=0$$

(2) 注意到:

$$\nabla\times(\vec{j}\times\vec{B})=\vec{j}(\nabla\cdot\vec{B})-\vec{B}(\nabla\cdot\vec{j})+(\vec{B}\cdot\nabla)\vec{j}-(\vec{j}\cdot\nabla)\vec{B}$$

$$=-\vec{B}(\nabla\cdot\vec{j})+(\vec{B}\cdot\nabla)\vec{j}-(\vec{j}\cdot\nabla)\vec{B}$$

其中:

$$\nabla\cdot\vec{j}=\nabla\cdot\left(\frac{1}{\mu_0}\nabla\times\vec{B}\right)=\frac{1}{\mu_0}\nabla\cdot(\nabla\times\vec{B})=0\Rightarrow\vec{B}(\nabla\cdot\vec{j})=0$$

$$\nabla p = -\frac{1}{2\mu_0}\nabla \vec{B}^2 + \frac{1}{\mu_0}(\vec{B}\cdot\nabla)\vec{B}$$

$$\nabla \times (\vec{j} \times \vec{B}) = \nabla \times \nabla p = 0$$

故：$(\vec{B}\cdot\nabla)\vec{j} = (\vec{j}\cdot\nabla)\vec{B}$