0.1 数学准备 (Math Preparation)

设f,g为**标量**(scalar);A,B,C,D为**向量**(vector),则有如下恒等式:

1.
$$m{A}\cdot(m{B} imes m{C})=(m{A} imes m{B})\cdot m{C}=m{B}\cdot(m{C} imes m{A})=(m{B} imes m{C})\cdot m{A}=m{C}\cdot(m{A} imes m{B})=(m{C} imes m{A})\cdot m{B}$$

2.
$$m{A} imes (m{B} imes m{C}) = (m{C} imes m{B}) imes m{A} = (m{A} \cdot m{C}) m{B} - (m{A} \cdot m{B}) m{C}$$

3.
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

4.
$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$$

5.
$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{D})\mathbf{C} - ((\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C})\mathbf{D}$$

6.
$$\nabla (fq) = f \nabla q + q \nabla f$$

7.
$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

8.
$$abla imes (foldsymbol{A}) = f
abla imes oldsymbol{A} +
abla f imes oldsymbol{A}$$

9.
$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$$

10.
$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$

11.
$$m{A} imes (
abla imes m{B}) =
abla (m{B} \cdot m{A}) - (m{A} \cdot
abla) m{B}$$

12.
$$abla(m{A}\cdotm{B}) = m{A} imes(
abla imes m{B}) + m{B} imes(
abla imes m{A}) + (m{A}\cdot
abla)m{B} + (m{B}\cdot
abla)m{A}$$

13.
$$abla^2 f =
abla \cdot (
abla f)$$

14.
$$abla^2 m{A} =
abla (
abla \cdot m{A}) -
abla imes (
abla imes m{A})$$

15.
$$\nabla imes (\nabla f) = 0$$

16.
$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$

 \overrightarrow{T} 为**张量**(tensor); \overrightarrow{I} 为单位张量,则有如下恒等式:

1.
$$abla \cdot (m{A}m{B}) = (
abla \cdot m{A}) m{B} + (m{A} \cdot
abla) m{B}$$

2.
$$\nabla \cdot (f\overset{
ightarrow}{T}) = \nabla f \cdot \overset{
ightarrow}{T} + f \nabla \cdot \overset{
ightarrow}{T}$$

设 $m{r}=xm{i}+ym{j}+zm{k}$ 是直角坐标系中从原点指向(x,y,z)处的位置向量,则有如下恒等式:

1.
$$abla \cdot oldsymbol{r} = 3$$

2.
$$abla imes oldsymbol{r} = 0$$

3.
$$abla r = rac{m{r}}{r}$$

$$4. \nabla \left(\frac{1}{r}\right) = -\frac{r}{r^3}$$

5.
$$\nabla \cdot (\frac{m{r}}{r^3}) = 4\pi \delta(m{r})$$

6.
$$abla oldsymbol{r} = \stackrel{\prime}{I} \stackrel{
ightarrow}{I}$$

在直角坐标系(x, y, z)中,有:

1.
$$abla f = rac{\partial f}{\partial x}m{i} + rac{\partial f}{\partial y}m{j} + rac{\partial f}{\partial z}m{k}$$

2.
$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} \boldsymbol{i} + \frac{\partial A_y}{\partial y} \boldsymbol{j} + \frac{\partial A_z}{\partial z} \boldsymbol{k}$$
3. $\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \boldsymbol{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \boldsymbol{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \boldsymbol{k}$
4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} \boldsymbol{i} + \frac{\partial^2 f}{\partial y^2} \boldsymbol{j} + \frac{\partial^2 f}{\partial z^2} \boldsymbol{k}$

在柱坐标系 (ρ, φ, z) 中,有:

1.
$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \hat{\varphi} + \frac{\partial f}{\partial z} \hat{z}$$
2.
$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_{\rho}) + \frac{1}{\rho} \frac{\partial A_{\varphi}}{\partial \varphi} + \frac{\partial A_{z}}{\partial z}$$
3.
$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_{z}}{\partial \varphi} - \frac{\partial A_{\varphi}}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_{z}}{\partial \rho} \right) \hat{\varphi} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_{\varphi}) - \frac{\partial A_{\rho}}{\partial \varphi} \right) \hat{z}$$
4.
$$\nabla^{2} f = \frac{\partial^{2} f}{\partial \rho^{2}} + \frac{1}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^{2}} \frac{\partial^{2} f}{\partial \varphi^{2}} + \frac{\partial^{2} f}{\partial z^{2}}$$

在球坐标系 (r, θ, φ) 中,有:

1.
$$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \hat{\varphi}$$
2.
$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$$
3.
$$\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\sin \theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\varphi}$$
4.
$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}$$