Magnetohydrodynamics (MHD)

波动理论

本章内容

- 基本方程组

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \dfrac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \dfrac{1}{\rho}\nabla p \end{cases}$$

- 平衡时：不可压缩均质流体 $\rho = \mathrm{const}$，均匀磁场 $\boldsymbol{B}_0 = B_0 \mathbf{e}_z$。引入小扰动:$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}', \boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}' = \boldsymbol{v}', p = p_0 + p'$，并设扰动沿 z 方向传播（$\partial/\partial x = 0, \partial/\partial y = 0$ ）。将小扰动方程线性化，得:

$$\begin{cases} \nabla \cdot \boldsymbol{v}' = 0 \\ \dfrac{\partial \boldsymbol{B}'}{\partial t} = \nabla \times (\boldsymbol{v}' \times \boldsymbol{B}_0) \\ \dfrac{\partial \boldsymbol{v}'}{\partial t} = \dfrac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}') \times \boldsymbol{B}_0 - \dfrac{1}{\rho}\nabla p' \end{cases} \rightarrow \begin{cases} \dfrac{\partial {v_z}'}{\partial z} = 0 \rightarrow {v_z}' = 0 \\ \dfrac{\partial {B_x}'}{\partial t} = B_0 \dfrac{\partial {v_x}'}{\partial z} \ \left(\dfrac{\partial {B_y}'}{\partial t} = B_0 \dfrac{\partial {v_y}'}{\partial z}\right) \\ \dfrac{\partial {B_z}'}{\partial t} = 0 \ \dfrac{\partial {B_z}'}{\partial z} = 0 \rightarrow {B_z}' = 0 \\ \dfrac{\partial {v_x}'}{\partial t} = \dfrac{B_0}{\mu_0 \rho} \dfrac{\partial {B_x}'}{\partial z} \ \left(\dfrac{\partial {v_y}'}{\partial t} = \dfrac{B_0}{\mu_0 \rho} \dfrac{\partial {B_y}'}{\partial z}\right) \\ \dfrac{\partial {v_z}'}{\partial t} = -\dfrac{1}{\rho} \dfrac{\partial p'}{\partial z} = 0 \rightarrow p' = 0 \end{cases}$$

完全导电理想流体中的 Alfvén 波 2

- 不失一般性，仅考虑 y 方向的扰动

$$\begin{cases} \dfrac{\partial B_y{}'}{\partial t} = B_0 \dfrac{\partial v_y{}'}{\partial z} \\ \dfrac{\partial v_y{}'}{\partial t} = \dfrac{B_0}{\mu_0 \rho} \dfrac{\partial B_y{}'}{\partial z} \end{cases} \rightarrow \begin{cases} \dfrac{\partial^2 B_y{}'}{\partial t^2} - \dfrac{B_0{}^2}{\mu_0 \rho} \dfrac{\partial^2 B_y{}'}{\partial z^2} = 0 \\ \dfrac{\partial^2 v_y{}'}{\partial t^2} - \dfrac{B_0{}^2}{\mu_0 \rho} \dfrac{\partial^2 v_y{}'}{\partial z^2} = 0 \end{cases}$$

关于 $B_y{}'$ 和 $v_y{}'$ 的波动方程，其解为以 $t \pm z/v_A$ 为参量的任意函数。其传播速度

$$v_A = \pm \frac{B_0}{\sqrt{\mu_0 \rho}}$$

称为 Alfvén 速度，而这种磁场和流体的波动就是 Alfvén 波。

- 导电流体与磁场的“冻结”，取 $B_y{}' = A \sin \omega (t - z/v_A)$

磁力线方程 $\dfrac{dy}{B_y{}'} = \dfrac{dz}{B_0} \rightarrow y = y_0 + \dfrac{A}{\omega} \dfrac{v_A}{B_0} \cos \omega (t - z/v_A)$

扰动速度 $\dfrac{\partial v_y{}'}{\partial z} = \dfrac{1}{B_0} \dfrac{\partial B_y{}'}{\partial t} \rightarrow v_y{}' = -\dfrac{A}{\sqrt{\mu_0 \rho}} \sin \omega (t - z/v_A)$

比较可得

$$v_y{}' = \frac{dy}{dt}$$

阻尼 Alfvén 波 1

- 考虑有限电导率、粘性不可压缩均质流体

$$\begin{cases} \nabla \cdot \boldsymbol{v} = 0 \\ \rho \dfrac{\partial \boldsymbol{v}}{\partial t} + (\rho \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \dfrac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \eta \nabla^2 \boldsymbol{v} \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta_m \nabla^2 \boldsymbol{B} \\ \nabla \cdot \boldsymbol{B} = 0 \end{cases}$$

- 平衡位型与前例相同，仍然引入沿 z 方向传播的小扰动 $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}', \boldsymbol{v} = \boldsymbol{v}', p = p_0 + p'$。线性化方程，得

$$\begin{cases} \rho \dfrac{\partial \boldsymbol{v}'}{\partial t} = \dfrac{B_0}{\mu_0} \dfrac{\partial \boldsymbol{B}'}{\partial z} - \nabla \left(p' + \dfrac{B_0 {B_z}'}{\mu_0} \right) + \eta \nabla^2 \boldsymbol{v}' \to (\dfrac{\partial}{\partial t} - \nu \nabla^2)\boldsymbol{v}' = \dfrac{B_0}{\mu_0 \rho} \dfrac{\partial \boldsymbol{B}'}{\partial z} \\ \dfrac{\partial \boldsymbol{B}'}{\partial t} = B_0 \dfrac{\partial \boldsymbol{v}'}{\partial z} + \eta_m \nabla^2 \boldsymbol{B}' \to (\dfrac{\partial}{\partial t} - \eta_m \nabla^2)\boldsymbol{B}' = B_0 \dfrac{\partial \boldsymbol{v}'}{\partial z} \\ \nabla \cdot \boldsymbol{B}' = 0 \to {B_z}' = 0 \\ \nabla \cdot \boldsymbol{v}' = 0 \to {v_z}' = 0 \end{cases}$$

阻尼 Alfvén 波 2

- 消去 $\boldsymbol{B}'$ 得

$$(\partial/\partial t - \eta_m \nabla^2)(\partial/\partial t - \nu \nabla^2)\boldsymbol{v}' = \frac{{B_0}^2}{\mu_0 \rho}\frac{\partial^2 \boldsymbol{v}'}{\partial z^2}$$

- 考虑到 $\partial/\partial x = \partial/\partial y = 0, v_z' = 0$，故

$$\frac{\partial^2 \boldsymbol{v}_\perp{}'}{\partial t^2} - {v_{a0}}^2 \frac{\partial^2 \boldsymbol{v}_\perp{}'}{\partial z^2} = (\eta_m + \nu)\frac{\partial^3 \boldsymbol{v}_\perp{}'}{\partial t \partial z^2} \quad (v_{a0} = B_0/\sqrt{\mu_0 \rho})$$

$\eta_m = \nu = 0 \rightarrow$ 完全导电理想流体中 Alfvén 波传播方程

- 考虑 $\exp i(\omega t - kz)$ 形式的平面波解，带入波动方程，得色散关系

$$\omega^2 - {v_{a0}}^2 k^2 - i(\eta_m + \nu)\omega k^2 = 0 \rightarrow$$

$$\omega = \frac{1}{2}\left[i(\eta_m + \nu)k^2 \pm \sqrt{4{v_{a0}}^2 k^2 - (\eta_m + \nu)^2 k^4}\right]$$

1. 平面波解以 $e^{-(\eta_m + \nu)k^2 t/2}$ 衰减
2. 平面波解存在要求 $v_{a0} > \dfrac{\eta_m + \nu}{2}k$，即波速大于（磁）扩散速度

恒星大气中的 Alfvén 波 1

- 考虑恒星大气的不均匀性，并考虑平衡磁位型 $\boldsymbol{B} = B_0\boldsymbol{e}_z$，由
$$\frac{dp}{dz} = -\rho_0 g$$
得 $\rho_0(z) = \rho_0(0)\exp(-z/H)$，其中密度标高 $H = kT/mg$
- 基本方程
$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \\ \rho\dfrac{\partial\boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \dfrac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} + \rho\boldsymbol{g} \\ \dfrac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) \\ \nabla\cdot\boldsymbol{B} = 0 \end{cases}$$
- 引入小扰动 $\rho = \rho_0 + \rho', p = p_0 + p', \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}', \boldsymbol{v} = \boldsymbol{v}'$：
$$\begin{cases} \dfrac{\partial\rho'}{\partial t} + \nabla\cdot(\rho_0\boldsymbol{v}') = 0 \\ \rho_0\dfrac{\partial\boldsymbol{v}'}{\partial t} = \dfrac{1}{\mu_0}(\nabla\times\boldsymbol{B}')\times\boldsymbol{B}_0 + \rho'\boldsymbol{g} - {c_{s0}}^2\nabla\rho' \\ \dfrac{\partial\boldsymbol{B}'}{\partial t} = \nabla\times(\boldsymbol{v}'\times\boldsymbol{B}_0) \\ \nabla\cdot\boldsymbol{B}' = 0 \to {B_z}' = 0 \end{cases}$$

恒星大气中的 Alfvén 波 2

- 考虑到 Alfvén 波为沿 z 方向传播的横波，故

$$\left.\begin{cases}\dfrac{\partial\rho'}{\partial t}=0\rightarrow\rho'=0\\ \rho_0\dfrac{\partial\boldsymbol{v}'}{\partial t}=\dfrac{B_0}{\mu_0}\dfrac{\partial\boldsymbol{B}'}{\partial z}\\ \dfrac{\partial\boldsymbol{B}'}{\partial t}=B_0\dfrac{\partial\boldsymbol{v}'}{\partial z}\end{cases}\right\}\rightarrow\frac{\partial^2 v'}{\partial t^2}-{v_{a0}}^2\frac{\partial^2 v'}{\partial z^2}=0$$

${v_{a0}}^2=\dfrac{{B_0}^2}{\mu_0\rho_0}$ 为 z 的函数：${v_{a0}}^2={v_0}^2e^{z/H}$ $\left(v_0=\dfrac{B_0}{\sqrt{\mu_0\rho_0(0)}}\right)$

- 讨论一维谐波解，即令 $v'=v(z)e^{i\omega t}$，故

$$\omega^2 v+{v_{a0}}^2\frac{d^2v}{dz^2}=0$$

变量代换，令 $\xi=e^{-z/2H}$，故

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{dv}{d\xi}\right)+\frac{4H^2\omega^2}{{v_0}^2}v=0\rightarrow v=c_1J_0\left(\frac{2H\omega}{v_0}\xi\right)+c_2Y_0\left(\frac{2H\omega}{v_0}\xi\right)$$

- Alfvén 波能流密度

$$\boldsymbol{S}=\frac{1}{\mu_0}\boldsymbol{E}'\times\boldsymbol{B}'\quad(\boldsymbol{E}'=-\boldsymbol{v}'\times\boldsymbol{B}_0)$$

- 考虑导电流体的可压缩性 ($d\rho/dt \neq 0$)，忽略粘滞 ($\eta = 0$)，电导率无穷大 ($\sigma \to \infty$)，绝热过程 ($dS/dt = 0$)，故

$$
\begin{cases}
\dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \\
\nabla \cdot \boldsymbol{B} = 0 \\
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\
\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \dfrac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \dfrac{1}{\rho}\nabla p \\
\dfrac{\partial S}{\partial t} + (\boldsymbol{v} \cdot \nabla)S = 0 \\
p = p(\rho, S)
\end{cases}
$$

未知量 $\boldsymbol{v}, \boldsymbol{B}, \rho, p, S$，方程数（除去磁场控制方程）9，故方程组封闭。

- 引入扰动 $\boldsymbol{B}=\boldsymbol{B}_0+\boldsymbol{B}'$,$\boldsymbol{v}=\boldsymbol{v}_0+\boldsymbol{v}'$,$p=p_0+p'$, $\rho=\rho_0+\rho'$,$S=S_0+S'$。线性化方程，并用 $\boldsymbol{u}_0=\boldsymbol{B}_0/\sqrt{\mu_0\rho_0}$, $\boldsymbol{u}'=\boldsymbol{B}'/\sqrt{\mu_0\rho_0}$ 替代 $\boldsymbol{B}_0,\boldsymbol{B}'$:

$$\begin{cases}\dfrac{\partial \boldsymbol{u}'}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{u}'=(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{v}'-\boldsymbol{u}_0(\nabla\cdot\boldsymbol{v}')\\ \nabla\cdot\boldsymbol{u}'=0\\ \dfrac{\partial \rho'}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\rho'=-\rho_0(\nabla\cdot\boldsymbol{v}')\\ \dfrac{\partial \boldsymbol{v}'}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}'=-\dfrac{1}{\rho_0}\nabla(p'+\rho_0\boldsymbol{u}'\cdot\boldsymbol{u}_0)+(\boldsymbol{u}_0\cdot\nabla)\boldsymbol{u}'\\ \dfrac{\partial S'}{\partial t}+(\boldsymbol{v}_0\cdot\nabla)S'=0\\ p'=\left(\dfrac{\partial p}{\partial \rho}\right)_S\rho'+\left(\dfrac{\partial p}{\partial S}\right)_\rho S'={c_s}^2\rho'+bS'\end{cases}$$

- 均匀磁场 $\boldsymbol{B}_0$ 位于 xoz 平面中，与 x 轴夹角为 θ，将所有扰动量表示为沿 x 轴传播的平面谐波：

$$Q'(x,t) = Q' e^{i(\omega t - kx)}$$

考虑到 $\partial/\partial y = \partial/\partial z = 0, \partial/\partial x = -ik, \partial/\partial t = i\omega$，有

$$\begin{cases} (\omega - kv_{0x})u_x{}' = 0 \\ (\omega - kv_{0x})u_y{}' + ku_{0x}v_y{}' = 0 \\ (\omega - kv_{0x})u_z{}' + ku_{0x}v_z{}' - ku_{0z}v_x{}' = 0 \\ ku_x{}' = 0 \\ (\omega - kv_{0x})v_x{}' - kp'/\rho_0 - ku_{0z}u_z{}' = 0 \\ (\omega - kv_{0x})v_y{}' + ku_{0x}u_y{}' = 0 \\ (\omega - kv_{0x})v_z{}' + ku_{0x}u_z{}' = 0 \\ p' - c_s{}^2\rho' - bS' = 0 \\ (\omega - kv_{0x})\rho' - k\rho_0 v_x{}' = 0 \\ (\omega - kv_{0x})S' = 0 \end{cases}$$

- 令 $\omega_0 = \omega - kv_{0x}$，即 $v = \dfrac{\omega_0}{k} = \dfrac{\omega}{k} - v_{0x}$。方程组有非零解，要求系数行列式 $=0$，即

$$
\begin{array}{c|ccccccccc|}
 & {u_x}' & {u_y}' & {u_z}' & {v_x}' & {v_y}' & {v_z}' & p' & \rho' & S' \\
\hline
1 & \omega_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \omega_0 & 0 & 0 & ku_{0x} & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & \omega_0 & -ku_{0z} & 0 & ku_{0x} & 0 & 0 & 0 \\
4 & 0 & 0 & -ku_{0z} & \omega_0 & 0 & 0 & -k/\rho_0 & 0 & 0 \\
5 & 0 & ku_{0x} & 0 & 0 & \omega_0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & ku_{0x} & 0 & 0 & \omega_0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -{c_s}^2 & -b \\
8 & 0 & 0 & 0 & -k\rho_0 & 0 & 0 & 0 & \omega_0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_0
\end{array} = 0
$$

考虑到 $u_{0x} = u_0 \cos\theta, u_{0z} = u_0 \sin\theta$，可得色散关系

$$
{\omega_0}^2[{\omega_0}^2 - (ku_0\cos\theta)^2][{\omega_0}^4 - k^2({c_s}^2 + {u_0}^2){\omega_0}^2 + k^2{c_s}^2(ku_0\cos\theta)^2] = 0
$$

1. $\omega_0 = 0$

$$u_x{}' = u_y{}' = u_z{}' = 0$$
$$v_x{}' = v_y{}' = v_z{}' = 0$$
$$p' = 0,\ S' \neq 0,\ \rho' = -\frac{b}{{c_s}^2} S'$$

熵波：仅有密度和熵的扰动，且扰动不传播

2. ${\omega_0}^2 - (ku_0 \cos\theta)^2 = 0 \to v_a = \frac{\omega_0}{k} = \pm u_0 \cos\theta = \pm \frac{B_0}{\sqrt{\mu_0 \rho_0}} \cos\theta$

斜 Alfvén 波：传播方向与磁力线有一夹角

$$u_x{}' = u_z{}' = 0,\ u_y{}' \neq 0$$
$$v_x{}' = v_z{}' = 0,\ v_y{}' = \pm u_y{}'$$
$$p' = 0,\ S' = 0,\ \rho' = 0$$

3. ${\omega_0}^4 - k^2({c_s}^2 + {u_0}^2){\omega_0}^2 + k^4 {c_s}^2 (u_0 \cos\theta)^2 = 0$

$$({v_m}^2)_\pm = \frac{({\omega_0}^2)_\pm}{k^2} = \frac{1}{2}\{({c_s}^2 + {u_0}^2) \pm \sqrt{({c_s}^2 + {u_0}^2)^2 - 4{c_s}^2 {u_0}^2 \cos^2\theta}\}$$

快磁声波 v_{m+}+ 慢磁声波 v_{m-}

- $\theta = 0 \rightarrow ({v_m}^2)_\pm = \frac{1}{2}\{({c_s}^2 + {u_0}^2) \pm |{c_s}^2 - {u_0}^2|\}$，则

$$
\begin{aligned}
v_{m+} &= \max(c_s, u_0) \\
v_{m-} &= \min(c_s, u_0)
\end{aligned}
$$

沿磁场方向传播的快慢磁声波就是 Alfvén 波和声波

当 $v_m = c_s$ 时，有

$$
\begin{gathered}
{u_x}' = {u_y}' = {u_z}' = 0 \\
{v_x}' = \frac{\rho'}{\rho_0} c_s,\ \ {v_y}' = {v_z}' = 0 \\
S' = 0,\ \ p' = {c_s}^2 \rho'
\end{gathered}
$$

声波模式

当 $v_m = u_0$ 时，有

$$
\begin{gathered}
{u_x}' = 0,\ \ {u_y}' \neq 0,\ \ {u_z}' \neq 0 \\
{v_x}' = 0,\ \ {v_y}' = \pm {u_y}',\ \ {v_z}' = \pm {u_z}' \\
S' = 0,\ \ p' = 0,\ \ \rho' = 0
\end{gathered}
$$

Alfvén 波模式

可压缩磁流体中的波动–垂直磁场方向传播的磁声波

- $\theta = \pi/2 \to ({v_m}^2)_\pm = \frac{1}{2}\{({c_s}^2 + {u_0}^2) \pm ({c_s}^2 + {u_0}^2)\}$，则

$$
\begin{aligned}
v_{m+} &= \sqrt{{c_s}^2 + {u_0}^2} = \sqrt{{c_s}^2 + {v_a}^2} \\
v_{m-} &= 0
\end{aligned}
$$

v_{m-} 对应熵波模式

当 $v_{m+} = \sqrt{{c_s}^2 + {u_0}^2}$ 时，有

$$
\begin{aligned}
{u_x}' = {u_y}' = 0,\ {u_z}' &= \frac{u_0}{\sqrt{{c_s}^2 + {u_0}^2}} {v_x}' \\
{v_x}' = \frac{\rho'}{\rho_0}\sqrt{{c_s}^2 + {u_0}^2},\ {v_y}' &= {v_z}' = 0 \\
S' = 0,\ p' &= {c_s}^2 \rho'
\end{aligned}
$$

1. 热压力和磁压力的“弹性”在 $\boldsymbol{k} \perp \boldsymbol{B}_0$ 情形下产生最佳叠加。
2. 速度扰动在波传播方向，磁场扰动垂直波传播方向。故波为一种纵波和横波的混杂波。

- $0<\theta<\pi/2$，一般情形，有

$$\max(c_s, v_a) \le (v_m)_+ \le \sqrt{{c_s}^2 + {v_a}^2}$$
$$0 \le (v_m)_- \le \min(c_s, v_a)$$

对任一给定方向，有

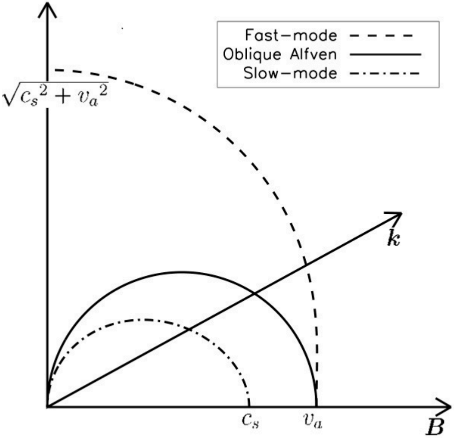
$$(v_m)_- \le v_a \le (v_m)_+$$
$$(v_m)_- \le c_s \le (v_m)_+$$

当 $v=v_m$ 时，有

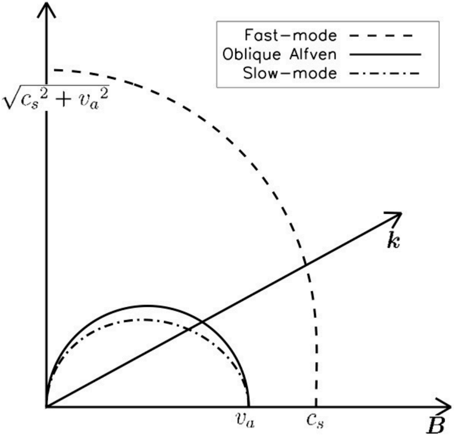
$$u_x{}' = u_y{}' = 0,\ \ u_z{}' = \frac{1}{\rho_0 u_0 \sin\theta}({v_m}^2 - {c_s}^2)\rho'$$
$$v_x{}' = \frac{\rho'}{\rho_0} v_m,\ \ v_y{}' = 0,\ \ v_z{}' = -\frac{\rho'}{\rho_0 v_m} \cot\theta({v_m}^2 - {c_s}^2)$$
$$S' = 0,\ \ p' = {c_s}^2\rho'$$

沿扰动传播方向，只有速度扰动；垂直扰动传播方向，速度和磁场扰动均存在，故快慢磁声波都是混杂波。

可压缩磁流体中的波动–波法图



$(v_a > c_s)$



$(c_s > v_a)$

简单波–HD 情形

- 基本方程组，理想流体

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\
\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\dfrac{1}{\rho}\nabla p \\
\dfrac{dS}{dt} = 0
\end{cases}
\rightarrow
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho v)}{\partial x} = 0 \\
\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0
\end{cases}
$$

假定 p, ρ, v 互为单值函数

$$
\begin{array}{l}
\dfrac{\partial \rho}{\partial t} + \dfrac{d(\rho v)}{d\rho}\dfrac{\partial \rho}{\partial x} = 0 \\
\dfrac{\partial v}{\partial t} + (v + \dfrac{1}{\rho}\dfrac{dp}{dv})\dfrac{\partial v}{\partial x} = 0
\end{array}
\rightarrow
\begin{array}{l}
\left(\dfrac{\partial x}{\partial t}\right)_\rho = v + \rho\dfrac{dv}{d\rho} \\
\left(\dfrac{\partial x}{\partial t}\right)_v = v + \dfrac{1}{\rho}\dfrac{dp}{dv}
\end{array}
\rightarrow \rho\dfrac{dv}{d\rho} = \dfrac{1}{\rho}\dfrac{dp}{dv}
$$

故 $\dfrac{dv}{d\rho} = \pm\dfrac{c_s}{\rho}$, $dv = \pm\dfrac{c_s}{\rho}d\rho = \pm\dfrac{dp}{\rho c_s}$

一维非定常流动一般解

$$
x = (v \pm c_s)t + f(v), \quad \text{or} \quad v = F[x - (v \pm c_s)t]
$$

简单波–MHD 情形

- 基本方程组，完全导电理想流体

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \\ \dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\dfrac{1}{\rho}\nabla p + \dfrac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \end{cases} \rightarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\mu_0 \rho}\left(B_y \frac{\partial B_y}{\partial x} + B_z \frac{\partial B_z}{\partial x}\right)$$

$$B_x \frac{\partial B_y}{\partial x} = 0, \quad B_x \frac{\partial B_z}{\partial x} = 0$$

$$\frac{\partial B_x}{\partial t} = 0, \quad \frac{\partial B_y}{\partial t} = -\frac{\partial}{\partial x}(vB_y), \quad \frac{\partial B_z}{\partial t} = -\frac{\partial}{\partial x}(vB_z)$$

1. $B_x \neq 0$，运动方程退化为 $\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$，退化到 HD 情形
2. $B_x = 0$，有 $B = \alpha\rho$（磁冻结），引入 $p_m = p + B^2/(2\mu_0)$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = -\rho\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p_m}{\partial x}$$

与 HD 形式上一致，$dv = \pm\dfrac{c_m}{\rho}d\rho = \pm\dfrac{dp_m}{c_m \rho}$，其中 $c_m = \sqrt{c_s^2 + v_A^2}$，

而流动一般解为 $v = F[x - (v \pm c_m)t]$ $(x = (v \pm c_m)t + f(v))$

简单波–激波

- 简单波与线性波的区别
  - 线性波：波轮廓上各点速度 $c_s(c_m)$ 相同，波形不发生变化
  - 简单波：波轮廓上各点速度 $v \pm c_s(v \pm c_m)$ 不同，扰动大（小）的点传播速度快（慢），波形不断变化
- 简单波传播速度
  考虑绝热情况 $p\rho^{-\gamma} = \mathrm{const} = A$

$$c_s = \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}$$

$$v = \pm \int \frac{c_s}{\rho} d\rho = \pm \frac{2\sqrt{A\gamma}}{\gamma - 1}\left(\rho^{\frac{\gamma-1}{2}} - \rho_0^{\frac{\gamma-1}{2}}\right) = \pm \frac{2}{\gamma - 1}(c_s - c_{s0}) \text{ or}$$

$$c_s = c_{s0} \pm \frac{\gamma - 1}{2} v$$

- 简单波 $\rightarrow$ 激波
  波峰上 $U_p = c_{s0} + \dfrac{\gamma+1}{2} v_m$；波谷上 $U_v = c_{s0} - \dfrac{\gamma+1}{2} v_m$ 形成激波的条件（t^*, X^*）

$$\int_0^{t^*} (U_p - U_v)\, dt = \int_0^{t^*} (\gamma + 1) v_m dt = \frac{\lambda}{2}, \quad \int_0^{X^*} \frac{\gamma+1}{c_{s0}} v_m dx = \frac{\lambda}{2}$$

  或者，从数学上考虑，当激波形成时

$$\left(\frac{\partial v}{\partial x}\right)_t \rightarrow \infty, \text{or} \left(\frac{\partial x}{\partial v}\right)_t \rightarrow 0, \text{and} \left(\frac{\partial^2 x}{\partial v^2}\right)_t = 0$$