0.1 数学准备 (Math Preparation)

设f,g为**标量**(scalar)；$\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D}$为**向量**(vector)，则有如下恒等式：

1. $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=(\boldsymbol{B}\times\boldsymbol{C})\cdot\boldsymbol{A}=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{C}\times\boldsymbol{A})\cdot\boldsymbol{B}$
2. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{C}\times\boldsymbol{B})\times\boldsymbol{A}=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}$
3. $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})+\boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A})+\boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B})=0$
4. $(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C})$
5. $(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})=((\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{D})\boldsymbol{C}-((\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C})\boldsymbol{D}$
6. $\nabla(fg)=f\nabla g+g\nabla f$
7. $\nabla\cdot(f\boldsymbol{A})=f\nabla\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\nabla f$
8. $\nabla\times(f\boldsymbol{A})=f\nabla\times\boldsymbol{A}+\nabla f\times\boldsymbol{A}$
9. $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=\boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$
10. $\nabla\times(\boldsymbol{A}\times\boldsymbol{B})=(\nabla\cdot\boldsymbol{B})\boldsymbol{A}-(\nabla\cdot\boldsymbol{A})\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$
11. $\boldsymbol{A}\times(\nabla\times\boldsymbol{B})=\nabla(\boldsymbol{B}\cdot\boldsymbol{A})-(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$
12. $\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}$
13. $\nabla^2 f=\nabla\cdot(\nabla f)$
14. $\nabla^2\boldsymbol{A}=\nabla(\nabla\cdot\boldsymbol{A})-\nabla\times(\nabla\times\boldsymbol{A})$
15. $\nabla\times(\nabla f)=0$
16. $\nabla\cdot(\nabla\times\boldsymbol{A})=0$

设 $\overset{\to\to}{T}$ 为**张量**(tensor)； $\overset{\to\to}{I}$ 为单位张量，则有如下恒等式：

1. $\nabla\cdot(\boldsymbol{A}\boldsymbol{B})=(\nabla\cdot\boldsymbol{A})\boldsymbol{B}+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$
2. $\nabla\cdot(f\overset{\to\to}{T})=\nabla f\cdot\overset{\to\to}{T}+f\nabla\cdot\overset{\to\to}{T}$

设$\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$是直角坐标系中从原点指向$(x,y,z)$处的位置向量，则有如下恒等式：

1. $\nabla\cdot\boldsymbol{r}=3$
2. $\nabla\times\boldsymbol{r}=0$
3. $\nabla r=\dfrac{\boldsymbol{r}}{r}$
4. $\nabla(\dfrac{1}{r})=-\dfrac{\boldsymbol{r}}{r^3}$
5. $\nabla\cdot(\dfrac{\boldsymbol{r}}{r^3})=4\pi\delta(\boldsymbol{r})$
6. $\nabla\boldsymbol{r}=\overset{\to\to}{I}$

在直角坐标系(x,y,z)中，有：

1. $\nabla f=\dfrac{\partial f}{\partial x}\boldsymbol{i}+\dfrac{\partial f}{\partial y}\boldsymbol{j}+\dfrac{\partial f}{\partial z}\boldsymbol{k}$

2. $\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x}\boldsymbol{i} + \frac{\partial A_y}{\partial y}\boldsymbol{j} + \frac{\partial A_z}{\partial z}\boldsymbol{k}$
3. $\nabla \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{k}$
4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2}\boldsymbol{i} + \frac{\partial^2 f}{\partial y^2}\boldsymbol{j} + \frac{\partial^2 f}{\partial z^2}\boldsymbol{k}$

在柱坐标系(ρ, φ, z)中，有：

1. $\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$
2. $\nabla \cdot \boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
3. $\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\varphi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{z}$
4. $\nabla^2 f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$

在球坐标系(r, θ, φ)中，有：

1. $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{\varphi}$
2. $\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$
3. $\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial \varphi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\hat{\varphi}$
4. $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$