Magnetohydrodynamics Problems

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1. 动量守恒方程写为:(不考虑外力)

$$\rho \frac{d\vec{v}}{dt} = \nabla \cdot \overleftrightarrow{P} + \vec{j} \times \vec{B}$$

其中Lorentz力$\vec{f}$可写为:

$$\vec{f} = \vec{j} \times \vec{B} = \nabla \cdot \left(\frac{\vec{B}\vec{B}}{\mu_0} - \frac{B^2}{2\mu_0} \overleftrightarrow{I} \right)$$

利用矢量恒等式有:

$$\nabla\cdot(\rho\vec{v}\vec{v})=[\nabla\cdot(\rho\vec{v})]\vec{v}+(\rho\vec{v}\cdot\nabla)\vec{v}$$

注意到:

$$\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}$$

$$\Rightarrow\rho\frac{d\vec{v}}{dt}=\rho\frac{\partial\vec{v}}{\partial t}+\rho(\vec{v}\cdot\nabla)\vec{v}$$

$$=\rho\frac{\partial\vec{v}}{\partial t}+\nabla\cdot(\rho\vec{v}\vec{v})-[\nabla\cdot(\rho\vec{v})]\vec{v}$$

利用连续性方程得:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

$$\Rightarrow\rho\frac{d\vec{v}}{dt}=\rho\frac{\partial\vec{v}}{\partial t}+\nabla\cdot(\rho\vec{v}\vec{v})+\frac{\partial\rho}{\partial t}\vec{v}$$

$$=\frac{\partial(\rho\vec{v})}{\partial t}+\nabla\cdot(\rho\vec{v}\vec{v})$$

最后代入动量守恒方程:

$$\frac{\partial(\rho\vec{v})}{\partial t}=\rho\frac{d\vec{v}}{dt}-\nabla\cdot(\rho\vec{v}\vec{v})=\nabla\cdot\overleftrightarrow{P}+\vec{j}\times\vec{B}-\nabla\cdot(\rho\vec{v}\vec{v})$$

$$=\nabla\cdot(\overleftrightarrow{P}+\frac{\vec{B}\vec{B}}{\mu_0}-\frac{B^2}{2\mu_0}\overleftrightarrow{I}-\rho\vec{v}\vec{v})$$

2. 当$\vec{q}=0$, $\vec{P}=-p\vec{I}$时，能量方程写为:

$$\rho\frac{d}{dt}(\varepsilon+\frac{v^2}{2})=-\nabla\cdot(p\vec{v})+\vec{E}\cdot\vec{j}$$

机械能方程为(相应地)

$$\rho\frac{d}{dt}(\frac{v^2}{2})=-\vec{v}\cdot\nabla p+(\vec{j}\times\vec{B})\cdot\vec{v}$$

又不计Ohm耗散，即:

$$\frac{j^2}{\sigma}=\vec{E}\cdot\vec{j}-(\vec{j}\times\vec{B})\cdot\vec{v}=0$$

$$\Rightarrow\rho\frac{d\varepsilon}{dt}=-\nabla\cdot(p\vec{v})+\vec{v}\cdot\nabla p=-p\nabla\cdot\vec{v}$$

由热力学关系$\varepsilon=\frac{p}{(\gamma-1)\rho}$和连续性方程$\frac{d\rho}{dt}+\rho\nabla\cdot\vec{v}=0$代入得:

$$\frac{\rho}{\gamma-1}(\frac{1}{\rho}\frac{dp}{dt}-\frac{p}{\rho^2}\frac{d\rho}{dt})-\frac{p}{\rho}\frac{d\rho}{dt}=0$$

$$\Rightarrow\frac{dp}{dt}-\gamma\frac{p}{\rho}\frac{d\rho}{dt}=0$$

$$\Rightarrow\frac{d}{dt}(\frac{p}{\rho^\gamma})=0$$

3. 用Green函数解得:

$$\begin{cases}B(x,t)=\int_{-\infty}^{+\infty}B(y,0)\,G(x,y,t)\,dy\\ G(x,y,t)=\frac{1}{\sqrt{4\pi\eta_m t}}e^{-(x-y)^2/4\eta_m t}\end{cases}$$

将初始条件$B(x,0)$代入得:

$$B(x,t) = -B_0\int_{-\infty}^{0} G(x,y,t)\,dy + B_0\int_{0}^{+\infty} G(x,y,t)\,dy$$

$$= \frac{B_0}{2\sqrt{\pi\eta_m t}}\int_0^{+\infty}\left[e^{-(x-y)^2/4\eta_m t} - e^{-(x+y)^2/4\eta_m t}\right]dy$$

$$= \frac{B_0}{\sqrt{\pi}}\int_0^{+\infty}\left[e^{-(\beta-\frac{x}{2\sqrt{\eta_m t}})^2} - e^{-(\beta+\frac{x}{2\sqrt{\eta_m t}})^2}\right]d\beta$$

$$= \frac{B_0}{\sqrt{\pi}}\left(\int_{-\frac{x}{2\sqrt{\eta_m t}}}^{+\infty} e^{-\alpha^2}\,d\alpha - \int_{\frac{x}{2\sqrt{\eta_m t}}}^{+\infty} e^{-\alpha^2}\,d\alpha\right) = \frac{B_0}{\sqrt{\pi}}\left(\int_{-\frac{x}{2\sqrt{\eta_m t}}}^{0} e^{-\alpha^2}\,d\alpha + \int_0^{\frac{x}{2\sqrt{\eta_m t}}} e^{-\alpha^2}\,d\alpha\right)$$

$$= \frac{2B_0}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{\eta_m t}}} e^{-\alpha^2}\,d\alpha$$