Relative Monads

Noah Loutchmia

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Abstract

The following document is intended as an outline for the formalisation of *Relative Monads* using the proof assistant *Lean 4*. Part of this formalisation consists in trying to establish relations between relative monads, monads and Kleisli triples.

Chapter 1

Relative Monads

1.1 Basic Definitions

Definition 1 (Relative Monads)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$ and a functor $J : \mathcal{C} \to \mathcal{D}$ *(root)*, a relative monad on $\mathcal{D}$ over $\mathcal{C}$ consists in :

- A function $T : \mathcal{C} \to \mathcal{D}$;
- A natural transformation $\eta : J \to T$ *(unit)* ;
- For each morphism in $\mathcal{D}$ of the form $f : Ja \to Tb$ (where a and b are objects of $\mathcal{C}$), a morphism $f^\dagger : Ta \to Tb$ *(extension /lifting operation)* ;

satisfying the following conditions :

- $\forall a \in obj_{\mathcal{C}}, (f : Ja \to Ta), f = f^\dagger \circ \eta_a$ *(left unitality)*
- $\forall a \in obj_{\mathcal{C}}, (\eta_a)^\dagger = \mathbb{1}_{Ta}$ *(right unitality)*
- $\forall a, b, c \in obj_{\mathcal{C}}$,
  $(f : Ja \to Ta), (g : Jb \to Tc)$,
  $g^\dagger \circ f^\dagger = g^\dagger \circ f^\dagger$

Proposition 2. *It follows from the conditions in the definition of a relative monad that :*

- T *defines a* functor *;*
- η *and* $\cdot^\dagger$ *are* natural transformations.

Definition 3 (Relative Monads morphisms)**.**

Proposition 4 (Category of Relative monads)**.**

Proof. □

1.2 Examples of Relative Monads

$T = J$

Chapter 2

External presentation of monads - Kleisli Triples

Definition 5 (Kleisli Triples)**.**

Definition 6 (Morphisms of Kleisli triples)**.**

Proposition 7. *"Given a category $\mathcal{C}$, the Kleisli triples on $\mathcal{C}$ and the morphisms between them form a category."*

Proof. $\square$

Proposition 8. *"Given a category $\mathcal{C}$, each Kleisli triple on $\mathcal{C}$ corresponds to a relative monad on $\mathcal{C}$ over itself."*

Proof. $\square$

Chapter 3

Kleisli triples are equivalent to Monads

Proposition 9. *"Given a category $\mathcal{C}$, Kleisli triples on $\mathcal{C}$ are equivalent to monads on $\mathcal{C}$."*

Proof. □

Proposition 10. *"Given a category $\mathcal{C}$, each monad on $\mathcal{C}$ corresponds to a relative monad on $\mathcal{C}$ over itself."*

Proof. □