

Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V . A *partition* of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \dots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$. For example, $\{\{1\}, \{2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$ but $\{\{1\}, \{2, 3\}, \{1, 4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1, 2, 3, 4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

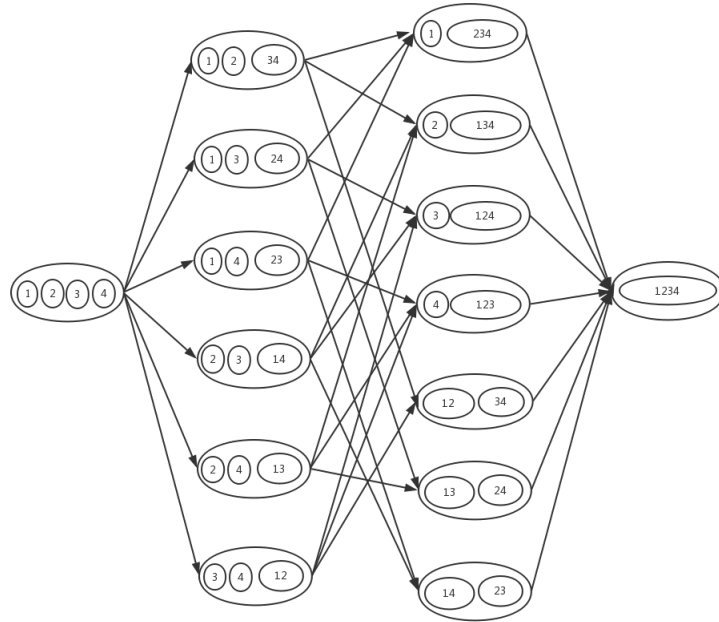


Figure 1: the Hasse diagram of the partial ordering

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?

solution:the size of the largest chain is 4.

3. What is the size of the largest antichain?

solution:the size of the largest antichain is 7.

2.2 Chains and Antichains

Define the partially ordered set (\mathbb{N}_0^n, \leq) as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2, 5, 4) \leq (2, 6, 6)$ but $(2, 5, 4) \not\leq (3, 1, 1)$.

Exercise 2.2. Consider the infinite partially ordered set (\mathbb{N}_0^n, \leq) .

1. Which elements are minimal? Which are maximal?

solution: $(0,0,\dots,0)$ is the minimal. There is no maximal element.

2. Is there a minimum? A maximum?

solution: $(0,0,\dots,0)$ is the minimum. There is no maximum element.

3. Does it have an infinite chain?

solution: Yes. If it does not have an infinite chain, it will have the maximum element which we assume it (a_1, a_2, \dots, a_n) . Then we have the element $(a_1 + 1, a_2 + 1, \dots, a_n + 1)$, which is obviously greater than (a_1, a_2, \dots, a_n) . Thus, it has an infinite chain.

4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

solution: Yes, $\{ (x_1, x_2, \dots, x_n) \in \mathbb{N}_0^n \mid x_1 + x_2 + \dots + x_n = k-1, k \in \mathbb{N} \}$ is an antichain of size k . And since k can be arbitrarily large, it has an arbitrarily large antichain.

***Exercise 2.3.** Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?

solution: For a infinite subset $S \subseteq \mathbb{N}_0^n$ define $S_I := \{\mathbf{a}_I \mid \mathbf{a} \in S\}$. Because S is infinite, there must be a division causing S_I is infinite and $S_{[n] \setminus I}$ is finite ($S_{[n] \setminus I}$ might be \emptyset).

Hence, there are infinite elements of S having the same part in $S_{[n] \setminus I}$ due to the finity.

Consider the infinite part of these elements. For any element S_i in S , we can find another element S_j in the infinite subset S , whose finite part is the same as S_i and the infinite part is all greater than S_i . Therefore, we can find an infinite chain.

Exercise 2.4. Show that (\mathbb{N}_0^n, \leq) has no infinite antichain. **Hint.** Use the previous exercise.

Solution. Consider an infinite antichain $A \subseteq \mathbb{N}_0^n$, since A is an infinite subset of \mathbb{N}_0^n , A must contain an infinite chain according to **2.3**. Then we get a contradiction.

In this way, (\mathbb{N}_0^n, \leq) has no infinite antichain. ■

Consider the induced ordering on $\{0, 1\}^n$. That is, for $x, y \in \{0, 1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0, 1\}^n, \leq)$ for $n = 2, 3$.

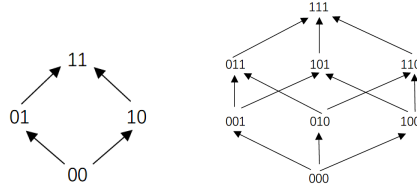


Figure 2: the Hasse diagrams of $(\{0, 1\}^n, \leq)$ for $n = 2, 3$

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0, 1\}^n$.

solution: The maximum and maximal element are both $11\dots 1$ (it has n many 1's). The minimum and minimal element are both $00\dots 0$ (it has n many 0's).

Exercise 2.7. What is the longest chain of $\{0, 1\}^n$?

solution: The longest chain of $\{0, 1\}^n$ is $00\dots 0, 00\dots 1, 00\dots 11, \dots, 11\dots 1$. The length of the chain is $n + 1$.

****Exercise 2.8.** What is the largest antichain of $\{0, 1\}^n$?

Solution. The largest antichain of $\{0, 1\}^n$ is the set of all bit sequence composed by $\lfloor n/2 \rfloor$ 1s and $(n - \lfloor n/2 \rfloor)$ 0s.

1. We partition all elements in $\{0, 1\}^n$ by the number of 1s. A_i denotes the set of all bit sequence containing i 1s, then $\{0, 1\}^n$ consists of A_1, A_2, \dots, A_n . Each partition is an antichain in itself. According to character of enumerative combination, $A_{\lfloor n/2 \rfloor}$ has the most elements.

2. Consider an antichain $C = (c_1, c_2, \dots, c_m)$ of $\{0, 1\}^n$ whose elements lie in different partitions.
 Take any subset $C^* \subseteq C$ whose elements belong to A_i where $i \neq \lfloor n/2 \rfloor$, and A_j next to A_i with $|A_i| < |A_j|$.
 Take a subset $C^{**} \subseteq A_j$ containing all elements in A_j which are comparable to elements in C^* . Obviously $|C^{**}| \geq |C^*|$ and elements C^{**} are uncomparable to elements of $C \setminus C^*$. In this way, $C \setminus C^* + C^{**}$ is also a antichain.
3. We can recursively replace C^* with C^{**} to get an antichain whose size is not smaller than the previous one until all elements lie in $A_{\lfloor n/2 \rfloor}$.

To sum up, $A_{\lfloor n/2 \rfloor}$ is the largest antichain of $\{0, 1\}^n$ with size $\binom{n}{\lfloor n/2 \rfloor}$. ■

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k .

Exercise 2.9. Consider \mathbb{N}^* , the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$. Here, ϵ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \rightarrow \mathbb{N}^*$.

Proof. As we have learn before, $\mathbb{N} \cong \{0, 1\}^*$, so we need proof $\mathbb{N}^* \cong \{0, 1\}^*$. Let b_i denotes a bit sequence which starts with i 1s and ends with a 0. Then define a function f :

$$\begin{aligned} f : \mathbb{N}^* &\rightarrow \{0, 1\}^*, \\ (a_1, a_2, \dots) &\mapsto (b_{a_1}, b_{a_2}, \dots) \end{aligned}$$

Since $\{0, 1\}^*$ is also a natural number sequence, $|\{0, 1\}^*| \leq |\mathbb{N}^*|$.

According to function f , for two different elements in \mathbb{N}^* , let natural number a_i be the first different byte, its corresponding b_i will also be different, so the their images in $\{0, 1\}^*$ are different. We can get $|\mathbb{N}^*| \leq |\{0, 1\}^*|$.

In this way, function f is bijective. □

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0, 1\}^{\mathbb{N}}$ and thus show that $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$.

Proof. We can define f :

$$f : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$$

$$((a_{11}, a_{12}, a_{13}, \dots), (a_{21}, a_{22}, a_{23}, \dots)) \mapsto (a_{11}, a_{21}, a_{12}, a_{22}, \dots)$$

For any element $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$, \mathbf{a} can be represented by a combination of unique ordered pair $(\mathbf{a}_1, \mathbf{a}_2)$ where $\mathbf{a}_1, \mathbf{a}_2 \in \{0, 1\}^{\mathbb{N}}$. In this way, f is bijective. Then we can get $R \cong \{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \cong R \times R$. \square

Exercise 2.11. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences (r_1, r_2, r_3, \dots) of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0, 1\}^{\mathbb{N}}$.

Proof. Since $R \cong \{0, 1\}^{\mathbb{N}}$, we need to prove $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$. The element $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ can be represented as $(\mathbf{a}^1, \mathbf{a}^2, \dots)$ where $\mathbf{a}^i \in \{0, 1\}^{\mathbb{N}}$.

Also, \mathbf{e} can be represented as a matrix:

$$\mathbf{e} = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \dots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \dots \\ a_1^2 & a_2^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Define a function f :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}, (a_1, a_2, a_3, \dots) \mapsto \begin{pmatrix} a_1 & a_3 & a_6 & \dots \\ a_2 & a_5 & \dots & \dots \\ a_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

For $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$, we place each bit of \mathbf{a} on matrix following the diagonals in turn, then we get a unique $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$.

Also, for each $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$, there is exactly one element $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ such that $f(\mathbf{a}) = \mathbf{e}$.

In this sense, $\{0, 1\} \cong (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$, which can be expressed as $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. \square

Next, let us view $\{0, 1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0, 1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Solution. For set $A \subseteq \{0, 1\}^{\mathbb{N}}$, let $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid \text{if } k < i, a_k = 1, \text{ else, } a_k = 0\}$, any two elements in A are comparable
We can define a function f :

$$f : \mathbb{N} \rightarrow A, x \mapsto \mathbf{a}_x$$

which is obviously bijective. In this way, A is a countably infinite chain. ■

Exercise 2.13. Find a countably infinite antichain in $\{0, 1\}^{\mathbb{N}}$.

Solution. For set $A \subseteq \{0, 1\}^{\mathbb{N}}$, let $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid a_i = 1 \text{ and other bits is } 0\}$. Any two elements in A is uncomparable since each has one bit larger than the other.

Also, we can define a function f :

$$f : \mathbb{N} \rightarrow A, x \mapsto \mathbf{a}_x$$

which is bijective.

In this scense, A is a countably infinite antichain in $\{0, 1\}^{\mathbb{N}}$. ■

Exercise 2.14. Find an uncountable antichain in $\{0, 1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Solution. Define a function f :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow A, (a_1, a_2, a_3 \dots) \mapsto (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

f is a bijective function, and $\{0, 1\}^{\mathbb{N}} \cong A$

For $\mathbf{a}, \mathbf{b} \in A$, if $\mathbf{a} \neq \mathbf{b}$ and \mathbf{a}, \mathbf{b} are comparable, let $\mathbf{a} < \mathbf{b}$.

There must exist $a_i < b_i$, but $1 - a_i > 1 - b_i$, so \mathbf{a} and \mathbf{b} are uncomparable. Then we get a contradiction.

In this way, A is a uncountable antichain. ■

****Exercise 2.15.** Find an uncountable chain in $\{0, 1\}^{\mathbb{N}}$. That is, an chain A with $A \cong \mathbb{R}$.

Solution. For $b_i \in \{0, 1\}^{\mathbb{N}}$ where $i \in \mathbb{N}$, we define matrix B_k as:

$$B_k = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \dots \\ \mathbf{b}_k \\ 1 \\ \dots \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots \\ b_{21} & b_{22} & \dots \\ \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots \\ 1 & 1 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Let B denotes the set of all B_k , then $B \cong \{0, 1\}^{\mathbb{N}}$.

Take bit sequence a_k from B_k following the diagonals, which means $a_k = (b_{11}, b_{21}, b_{12}, b_{31}, b_{22}, \dots)$. And set A is the set of all a_k . We can get $A \cong B \cong \{0, 1\}^{\mathbb{N}}$

According to the way we construct B_k , the 0s in a_{k+1} will not be less than these in a_k and exist at the same bits. In this way, any two elements in A are comparable.

So A is an uncountable chain. ■