Mathematical Foundations
of
Computer Science

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6 Graph Theory Basics

Exercise 6.1. For each of the graphs below, compute the number of automorphisms it has.

Justify your answer!

Solution.

1. The number of automorphisms is 2. We denote the vertices v_1, v_2, v_3, v_4 from the left side to the right side, which itself is an automorphisms. The other automorphism is to denote the vertices from the right to the left.

2. The number of automorphisms is $2 \times 6 = 12$. For each vertex, we denote it v_1 and number the remaining vertex clockwise or anticlockwise in order, all of which are automorphisms.

3. The number of automorphisms is $8 \times 6 = 48$. For each vertex, we denote it v_0, and its adjacent three vertices v_1, v_2, v_3, which has $3! = 6$ choices. Each selection can form an automorphism, thus the total number is 48.

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Exercise 6.2. Show that H n has exactly 2 n n! automorphisms. Be careful: it is easy to construct 2 n n! dierent automorphisms. It is more dicult to show that there are no automorphisms other than those.

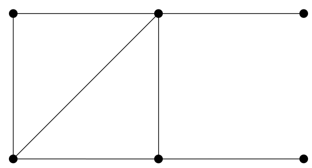
Proof.

- There are at least $2^n \cdot n!$ automorphisms. For each vertex we denote it v_0 , and denote its adjacent vertices $v_1, v_2, \ldots, v_n$, which has $n!$ choice. All $2^n \cdot n!$ will at least form one automorphism.
- There are no more than $2^n \cdot n!$ automorphisms. If there are more than $2^n \cdot n!$ automorphisms, then after we choose $v_0, v_1, v_2, \ldots, v_n$, we can still find more than one automorphism when numbering $v_{n+1}, v_{n+2}, \ldots, v_{2^n-1}$.

  Consider the n-dimensional Hamming cube as a n-dimensional linear space with each vertex having the coordinates $(x_1, x_2, \ldots, x_n)$ ($x_i = 1$ or 0) However, after choosing $v_0, v_1, \ldots, v_n$, equivalently, we have defined the origin and basis of the space, all vertices' coordinates are fixed, there is no alternative automorphism.

□

Exercise 6.3. Give an example of an asymmetric graph on six vertices.



Solution. ■

Exercise 6.4. Find an asymmetric tree.

Solution. The is no asymmetric tree on six vertices.

Proof. A tree containing only 2 leaves is a path, which has 2 automorphisms. Thus the tree must containing at least 3 leaves v_1, v_2, v_3 and the leaves have different adjacent vertices v_1', v_2', v_3'. Otherwise switching the number of 2 leaves connecting the same adjacent vertex will form a new automorphism.

There already exist 6 vertices. But if there are only 6 vertice to form a tree, v'_1, v'_2, v'_3 must be connected. But this will form automorphisms by switch the name of vertices groups $(v_1, v'_1), (v_2, v'_2), (v_3, v'_3)$.

Therefore, an asymmetric tree cannot be found on less than 7 vertices. ■

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Exercise 6.5. Show that there is no self-complementary graph on 999 vertices.

Proof. A complete graph on 999 vertices has $999 \times 998/2 = 498501$ edges, which cannot be divide by 2, that is, the edges cannot be divide into two parts with the same number of edges. □

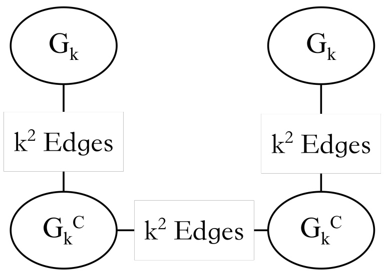
Exercise 6.6. Exercise 6.6. Characterize the natural numbers n for which there is a selfcomplementary graph G on n vertices.

Theorem. *There is a self-complementary graph on n vertices if and only if* $n \mod 4 = 0$ *or* $n \mod 4 = 1$.

Proof.

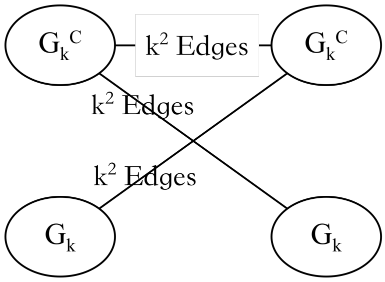
- If $n \mod 4 \neq 0$ and $n \mod 4 \neq 1$, the number of edges of the complete graph $n(n-1)/2/2$ is not an integer, namely the edges of the complete graph cannot be divide into two parts with the same number of edges. Hence there isn't any self-complementary graph on n vertices.

- If $n \mod 4 = 0$, that is $n = 4k$, where k is a positive integer. we denote an graph G_k containing k vertices whose complement is G_k^C. Now we form the following graph G on n vertices



The graph G contains four induced subgraphs G_k, G_k, G_k^C, G_k^C. Between some subgraphs there are edges between each vertex of two subgraphs, whose total number is k^2.

The complementary graph G^C is as follows. Obviously, G^C and G are isomorphic.



- If $n \mod 4 = 1$, that is $n = 4k + 1$, where k is a positive integer.

$\square$