

Mathematical Foundations of Computer Science

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Group: NOIDEA

7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Solution.

1. The score of a graph is the increasing sequence of degrees of all vertexes.

2. Give a increasing sequence of degrees $D = (d_1, d_2, d_3 \dots d_n)$ with d_i denoting its element. Consider another sequences D' of size $n - 1$ whose element denoted as d'_i with:

$$d'_i = \begin{cases} d_i & i < n - d_n \\ d_i - 1 & i \geq n - d_n \end{cases}$$

D is a graph score if and only if D' is a graph score.

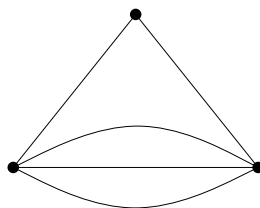
The idea of graph score algorithm is recursively using theorem to simplify the degree sequence until it's easy to judge whether the sequence is a graph score.

The difficult part of the proof is the reverse implication i.e. if D is a graph score then D' is a graph score. Since we can't just reverse the construction process how we get the graph of D from the graph of D' . ■

7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs. That is, something like

Theorem 7.3 (Multigraph Score Theorem). *Let $(a_1, \dots, a_n) \in \mathbb{N}_0^n$. There is a multigraph with this score if and only if <fill in some simple criterion here>.*

Remark. This is actually simpler than for graphs.

Solution. Let $(a_1, \dots, a_n) \in \mathbb{N}_0^n$. There is a multigraph with this score if and only if 1) $\sum_{i=1}^n a_i$ is even and 2) $\sum_{i=1}^{n-1} a_i \geq a_n$. ■

Exercise 7.4. Prove your theorem.

Proof. First, if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}_0^n$ is multigraph score then it obviously satisfy the criterion according to Handshake Lemma.

To prove the reverse implication, we use induction theorem.

Hypothesis: Any sequence $\mathbf{a}' = (a'_1, \dots, a'_{n-1})$ is a multigraph score satisfying both criterions, then it has corresponding multigraph with score \mathbf{a}' .

Base case: $\mathbf{a}' = (a'_1, a'_2)$ meeting the conditons obviously has corresponding multigraph score $\mathbf{a} = (a_1, a_2, a_3)$.

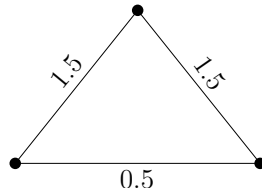
Induction proof: For any nodecreasing order sequence $\mathbf{a} = (a_1, a_2, \dots, a_n)$ satisfying both criterions, remove the node with degree a_1 in \mathbf{a} to get $\mathbf{a}' = (a_2, a_3, \dots, a'_n)$ where $a'_n = a_n - a_1$.

If a'_n is still the largest one, then \mathbf{a}' is obviously a multigraph score. Therefore \mathbf{a} is also a multigraph score according to induction theorem.

If a'_n is not the largest one, which means a_{n-1} is the largest one. Since $a_n > a_{n-1}$ and $a'_n + a_2$ is at least $a_n + a_2 - a_1 > a_n \geq a_{n-1}$, which implies \mathbf{a}' is multigraph score.

Then, we can conclude that the theorem is correct. □

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e . In such a graph the *weighted degree* of a vertex u is $\text{wdeg}(u) = \sum_{\{u,v\} \in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem). *Let $(a_1, \dots, a_n) \in \mathbb{R}_0^n$. There is a weighted graph with this score if and only if <fill in some simple criterion here>.*

Remark. This is actually even simpler.

Solution. Let $(a_1, \dots, a_n) \in \mathbb{R}_0^n$. There is a weighted graph with this score if and only if $\sum_{i=1}^{n-1} a_i \geq a_n$. ■

Exercise 7.7. Prove your theorem.

Proof. First, for any weighted graph, we have $\sum_{i=1}^{n-1} a_i \geq a_n$ since its each edge connected to a_n also connecting a node in (a_1, \dots, a_{n-1}) . Besides, for any sequence $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_0^n$ with $\sum_{i=1}^{n-1} a_i \geq a_n$, we use induction theorem to prove the theorem.

Hypothesis: Any sequence $\mathbf{a}' = (a'_1, \dots, a'_{n-1}) \in \mathbb{R}_0^n$ which is weighted graph score has a corresponding sequence $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_0^n$ meeting the criterion is a weighted graph score.

Base case: $\mathbf{a} = a_1, a_2$ with $a_1 \geq a_2$ is a weighted graph score.

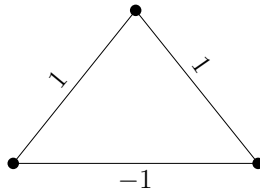
Induction step: For any nodecreasing order sequence $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_0^n$ with $\sum_{i=1}^{n-1} a_i \geq a_n$, remove vertex with degree a_1 and connected edges to get $\mathbf{a}' = (a_2, \dots, a'_n)$ where $a'_n = a_n - a_1$.

If a'_n is still the largest element in \mathbf{a}' , it's obviously that $\sum_{i=2}^{n-1} a_i \geq a_n - a_1$, which means \mathbf{a}' is a graph weighted score.

If a'_n is not the largest element, which means a_{n-1} is the largest one. We have $\sum_{i=2}^{n-2} a_i + a_n - a_1 \geq a_{n-1}$, so \mathbf{a}' is a graph weighted score.

Accordingly, we have the theorem proved. □

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This “graph with real edge weights” has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights). *Let $(a_1, \dots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if* **<fill in some simple criterion here>**.

(Score Theorem For Graphs With Real Edge Weights). Let $(a_1, \dots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if

$$\begin{cases} a_n = 0 & n = 1, \\ a_{n-1} = a_n & n = 2, \\ \text{always holds} & n \geq 3. \end{cases}$$

Exercise 7.10. Prove your theorem.

Proof. **Induction Hypothesis:** For any given score $\mathbf{a} = (a_1, \dots, a_{n-1}, a_n)$ ($n \geq 3$), there's always a corresponding weighted graph whose score is \mathbf{a} .

Base Case:

If $n=1$, the score must be 0 because there is no edge in the graph.

If $n=2$, only one edge is connecting these two vertices, and thus, their degrees should be the same..

If $n = 3$, denote by $W(u,v)$ the weight of the edge connecting vertex u and v . Then we have

$$\begin{cases} W(a_1, a_2) = (a_1 + a_2 - a_3)/2, \\ W(a_1, a_3) = (a_1 - a_2 + a_3)/2, \\ W(a_2, a_3) = (a_2 + a_3 - a_1)/2. \end{cases}$$

Since $a_1, a_2, a_3 \in \mathbf{R}$, there is always a solution to this equation array, which corresponds to a graph with real weights.

Induction Step:

If $n > 3$, we can reduce the size of the problem to a new score $\mathbf{a}' = (a'_1, \dots, a'_{n-1})$ by the following operations:

$$a'_i = \begin{cases} a_i & 1 \leq i < n-1, \\ a_i - a_{i+1} & i = n-1. \end{cases}$$

Then, we could apply the induction to this newly formed score, i.e., we could construct a weighted graph with \mathbf{a}' .

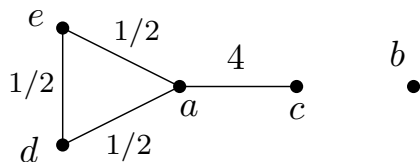
And we just connect n^{th} vertex to the $(n - 1)^{th}$ with an edge of weight a_n . And that's your weighted graph.

That concludes this proof. \square

Exercise 7.11. For each student ID (a_1, \dots, a_n) in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

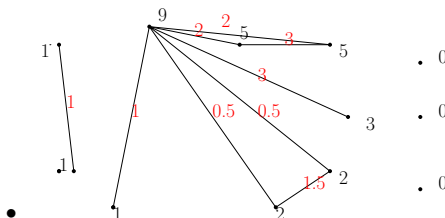
Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

1. 515021910302

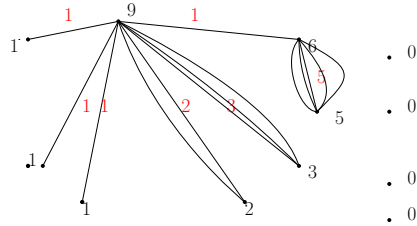
- a graph score: There are only eight vertices but there is one 9, so it's impossible for graph score.
- a multigraph score: no, the sum of the degrees can not be an odd number.



The figure above settles for (3) and (4).

2. 516021910003

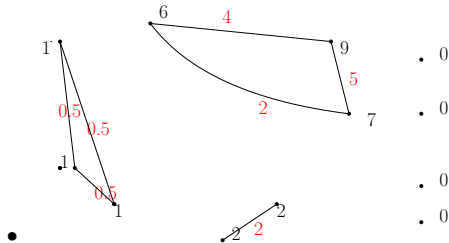
- a graph score: There are only seven vertices but there is one 9, so it's impossible for graph score.



The figure above settles for (2), (3) and (4). *P.S. for a multigraph, simply ignoring the weight value, and for a weighted graph, simply ignoring the multi-edges between 2 vertices and see them as only 1 edge.*

3. 516021910725

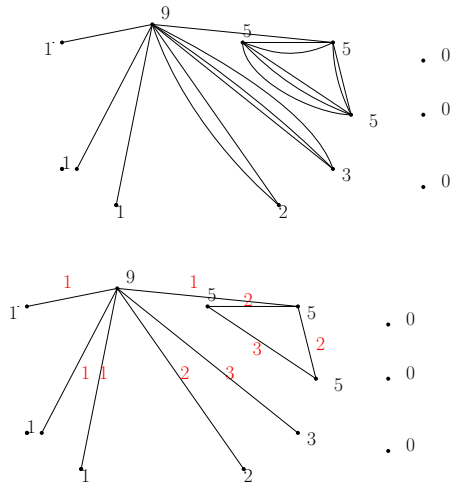
- a graph score: There are only seven vertices but there is one 9, so it's impossible for graph score.
- a multigraph score: no, the sum of the degrees can not be an odd number.



The figure above settles for (3) and (4).

4. 515021910053

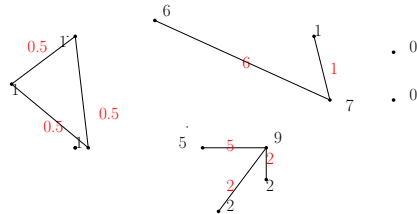
- a graph score: There are only eight vertices but there is one 9, so it's impossible for graph score.



The figures above settles for (2), (3) and (4).

5. 516021910127

- a graph score: According to the graph score theorem: (001111225679) \rightarrow (00000011456), since in the second step it is obvious that the score cannot denote a graph. Hence, it's impossible for graph score.
- a multigraph score: no, the sum of the degrees can not be an odd number.



The figure above settles for (3) and (4).