Mathematical Foundations of Computer Science

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Group: NOIDEA

7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Solution.

1. The score of a graph is the increasing sequence of degrees of all vertexes.

2. Give a increasing sequence of degrees $D = (d_1, d_2, d_3 \ldots d_n)$ with d_i denoting its element. Consider another sequnces $D^{'}$ of size $n-1$ whose element denoted as $d_i^{'}$ with:

$$d_i^{'} = \begin{cases} d_i & i < n - d_n \\ d_i - 1 & i \geq n - d_n \end{cases}$$

D is a graph score if and only if $D^{'}$ is a graph score.

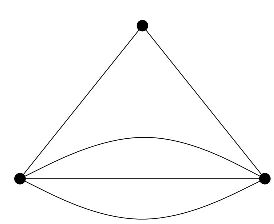
The idea of graph score algorithm is recursively using theorem to simplify the degree sequnce until it's easy to judge whether the sequnce is a graph score.

The difficult part of the proof is the reverse implication i.e. if D is a graph score then $D^{'}$ is a graph score. Since we can't just reverse the constrution process how we get the graph of D from the graph of $D^{'}$. ■

7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs. That is, something like

Theorem 7.3 (Multigraph Score Theorem). *Let* $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$. *There is a multigraph with this score if and only if* `<fill in some simple criterion here>`.

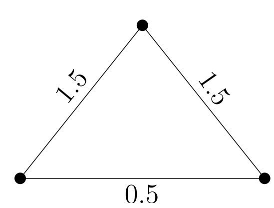
Remark. This is actually simpler than for graphs.

Solution. Let $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$. There is a multigraph with this score if and only if 1) $\sum_{i=1}^{n} a_i$ is even and 2) $\sum_{i=1}^{n-1} a_i \geq a_n$. ■

Exercise 7.4. Prove your theorem.

Proof. First, if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$ is multigraph score then it obviously satisfy the criterion according to Handshake Lemma.
To prove the reverse implication, we use induction theorem.
Hypothesis: Any sequcence $\mathbf{a} = (a'_1, \ldots, a'_{n-1})$ is a multigraph score satisfying both criterions, then it has corresponding multigraph with score $\mathbf{a}$.
Base case: $\mathbf{a}' = (a'_1, a'_2)$ meeting the conditons obviously has corresponding multigraph score $\mathbf{a} = (a_1, a_2, a_3)$.
Induction proof: For any nodecreasing order sequnce $\mathbf{a} = (a_1, a_2, \ldots a_n)$ satisfying both criterions, remove the node with degree a_1 in $\mathbf{a}$ to get $\mathbf{a}' = (a_2, a_3, \ldots a'_n)$ where $a'_n = a_n - a_1$.
If a'_n is still the largest one, then $\mathbf{a}'$ is obviously a multigraph score. Therefore $\mathbf{a}$ is also a multigraph score according to induction theorem.
If a'_n is not the largest one, which means a_{n-1} is the largest one. Since $a_n > a_{n-1}$ and $a'_n + a_2$ is at least $a_n + a_2 - a_1 > a_n \geq a_{n-1}$, which implies $\mathbf{a}'$ is multigraph score.
Then, we can conclude that the theorem is correct. □

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\mathrm{wdeg}(u) = \sum_{\{u,v\} \in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. *There is a weighted graph with this score if and only if* `<fill in some simple criterion here>`.

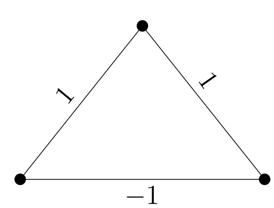
Remark. This is actually even simpler.

Solution. Let $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. There is a weighted graph with this score if and only if $\sum_{i=1}^{n-1} a_i \geq a_n$. ■

Exercise 7.7. Prove your theorem.

Proof. First, for any weighted graph, we have $\sum_{i=1}^{n-1} a_i \geq a_n$ since its each edge connected to a_n also connecting a node in $(a_1, \ldots a_{n-1})$.
Besides, for any sequnce $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$ with $\sum_{i=1}^{n-1} a_i \geq a_n$, we use induction theorem to prove the theorem.
Hypothesis: Any sequnce $\mathbf{a}\prime = (a_1^{'}, \ldots, a_{n-1}^{'}) \in \mathbb{R}_0^n$ which is weighted graph score has a corresponding sequnce $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$ meeting the criterion is a weighted graph score.
Base case: $\mathbf{a} = a_1, a_2$ with $a_1 \geq a_2$ is a weighted graph score.
Induction step: For any nodecreasing order sequnce $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$ with $\sum_{i=1}^{n-1} a_i \geq a_n$, remove vertex with degree a_1 and connected edges to get $\mathbf{a}\prime = (a_2, \ldots, a_n^{'})$ where $a_n^{'} = a_n - a_1$.
If $a_n^{'}$ is still the largest element in $\mathbf{a}^{'}$, it's obviously that $\sum_{i=2}^{n-1} a_i \geq a_n - a_1$, which means $\mathbf{a}^{'}$ is a graph weighted score.
If $a_n^{'}$ is not the largest element, which means a_{n-1} is the largest one. We have $\sum_{i=2}^{n-2} a_i + a_n - a_1 \geq a_{n-1}$, so $\mathbf{a}^{'}$ is a graph weighted score.
Accordingly, we have the theorem proved. □

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2,0,0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let* $(a_1,\ldots,a_n)\in\mathbb{R}^n$*. There is a graph with real edge weights with this score if and only if* `<fill in some simple criterion here>`*.*

(Score Theorem For Graphs With Real Edge Weights).Let$(a_1,\cdots,a_n)\in\mathbb{R}^n$.There is a graph with real edge weights with this score if and only if

$$\begin{cases} a_n = 0 & n = 1,\\ a_{n-1} = a_n & n = 2,\\ \textit{always holds} & n \geq 3. \end{cases}$$

Exercise 7.10. Prove your theorem.

Proof. **Induction Hypothesis:** For any given score $\mathbf{a} = (a_1,\cdots,a_{n-1},a_n)(n \geq 3)$, there's always a corresponding weightedgraph whose score is $\mathbf{a}$.
Base Case:
If n=1, the score must be 0 because there is no edge in the graph.
If n=2, only one edge is connecting these two vertices, and thus, their degrees should be the same..
If $n = 3$, denote by W(u,v) the weight of the edge connecting vertex u and v. Then we have

$$\begin{cases} W(a_1,a_2) = (a_1 + a_2 - a_3)/2,\\ W(a_1,a_3) = (a_1 - a_2 + a_3)/2,\\ W(a_2,a_3) = (a_2 + a_3 - a_1)/2. \end{cases}$$

Since $a_1,a_2,a_3 \in \mathbf{R}$, there is always a solution to this equation array, which corresponds to a graph with real weights.
Induction Step:
If $n > 3$,we can reduce the size of the problem to a new score **a'** $= (a'_1,\cdots,a'_{n-1})$ by the following operations:

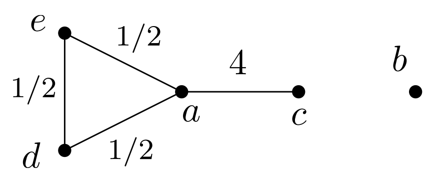
$$a'_i = \begin{cases} a_i & 1 \leq i < n-1,\\ a_i - a_{i+1} & i = n-1. \end{cases}$$

Then, we could apply the induction to this newly formed score, i.e., we could construct a weightedgraph with **a'**.
And we just connect n^{th} vertex to the $(n-1)^{th}$ with an edge of weight a_n. And that's your weightedgraph.
That concludes this proof. $\square$

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

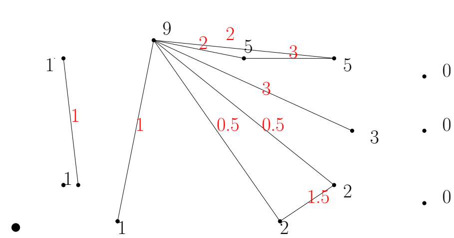
Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

1. 515021910302

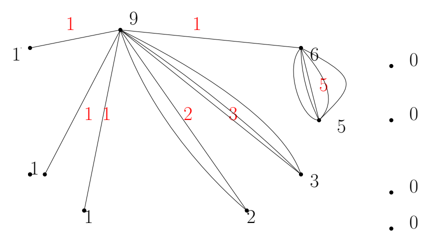
   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.
   - a multigraph score: no, the sum of the degrees can not be an odd number.

   The figure above settles for (3) and (4).

2. 516021910003

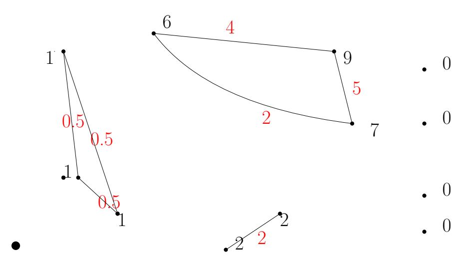
   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.

     The figure above settles for (2), (3) and (4). *P.S.for a multigraph, simply ignoring the weight value, and for a weighted graph, simply ignoring the multi-edges between 2 vertices and see them as only 1 edge.*

3. 516021910725

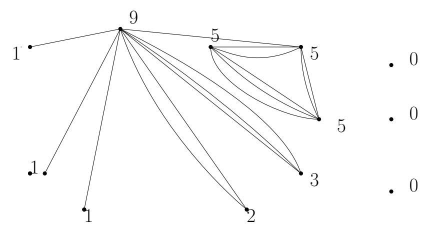
   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.
   - a multigraph score: no, the sum of the degrees can not be an odd number.

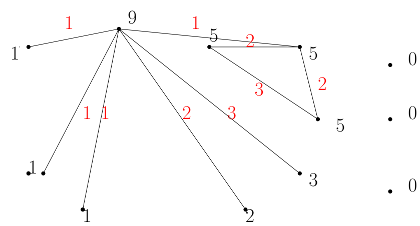
     

     The figure above settles for (3) and (4).

4. 515021910053

   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.

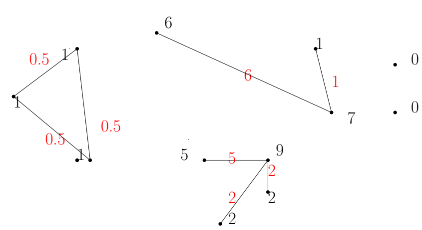




The figures above settles for (2), (3) and (4).

5. 516021910127

   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.
   - a multigraph score: no, the sum of the degrees can not be an odd number.



The figure above settles for (3) and (4).