Mathematical Foundations
of
Computer Science

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- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

2 Partial Orderings

2.1 Equivalence Relations as a Partial Ordering

An equivalence relation $R \subseteq V \times V$ is basically the same as a partition of V. A *partition* of V is a set $\{V_1, \dots, V_k\}$ where (1) $V_1 \cup \dots \cup V_k = V$ and (2) the V_i are pairwise disjoint, i.e., $V_i \cap V_j = \emptyset$ for $1 \le i < j \le k$. For example, $\{\{1\}, \{2,3\}, \{4\}\}$ is a partition of $\{1,2,3,4\}$ but $\{\{1\}, \{2,3\}, \{1,4\}\}$ is not.

Exercise 2.1. Let E_4 be the set of all equivalence relations on $\{1,2,3,4\}$. Note that E_4 is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

2.2 Chains and Antichains

Define the partially ordered set $(\mathbb{N}_0^n, \leq)$ as follows: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For example, $(2,5,4) \leq (2,6,6)$ but $(2,5,4) \not\leq (3,1,1)$.

Exercise 2.2. Consider the infinite partially ordered set $(\mathbb{N}_0^n, \leq)$.

1. Which elements are minimal? Which are maximal?
2. Is there a minimum? A maximum?
3. Does it have an infinite chain?
4. Does it have arbitrarily large antichains? That is, can you find an antichain A of size $|A| = k$ for every $k \in \mathbb{N}$?

***Exercise 2.3.** *Does every infinite subset $S \subseteq \mathbb{N}_0^n$ contain an infinite chain?*

Exercise 2.4. Show that $(\mathbb{N}_0^n, \leq)$ has no infinite antichain. **Hint.** Use the previous exercise.

Consider the induced ordering on $\{0,1\}^n$. That is, for $x, y \in \{0,1\}^n$ we have $x \leq y$ if $x_i \leq y_i$ for every coordinate $i \in [n]$.

Exercise 2.5. Draw the Hasse diagrams of $(\{0,1\}^n, \leq)$ for $n = 2, 3$.

Exercise 2.6. Determine the maximum, minimum, maximal, and minimal elements of $\{0,1\}^n$.

Exercise 2.7. What is the longest chain of $\{0,1\}^n$?

****Exercise 2.8.** *What is the largest antichain of $\{0,1\}^n$?*

Solution. The largest antichain of $\{0,1\}^n$ is the set of all bit sequnce composed by $[n/2]$ 1s and $(n - [n/2])$ 0s.

1. We partition all elements in $\{0,1\}^n$ by the number of 1s. A_i denotes the set of all bit sequence containing i 1s, then $\{0,1\}^n$ consists of $A_1, A_2, \ldots, A_n$.
   Each partition is an antichanin itself. According to character of enumerative combination, $A_{[n/2]}$ has the most elements.

2. Consider an antichain $C = (c_1, c_2, \ldots c_m)$ of $\{0,1\}^n$ whose elements lie in different partitions.
   Take any subset $C^* \subseteq C$ whose elements belong to A_i where $i \neq [n/2]$, and A_j next to A_i with $|A_i| < |A_j|$.
   Take a subset $C^{**} \subseteq A_j$ containing all elements in A_j which are comparable to elements in C^*. Obviously $|C^{**}| \geq |C^*|$ and elements C^{**} are uncomparable to elements of $C \backslash C^*$. In this way, $C \backslash C^* + C^{**}$ is also a antichain.

3. We can recursively replace C^* with C^{**} to get an antichain whose size is not smaller than the previous one until all elements lie in $A_{[n/2]}$.

To sum up, $A_{[n/2]}$ is the largest antichain of $\{0,1\}^n$ with size $\binom{n}{n/2}$. ■

2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, i.e., there is a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. From this, and by induction, it follows quite easily that $\mathbb{N}^k \cong \mathbb{N}$ for every k.

Exercise 2.9. Consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers, that is, $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \ldots$. Here, ϵ is the empty sequence. Show that $\mathbb{N} \cong \mathbb{N}^*$ by defining a bijection $\mathbb{N} \to \mathbb{N}^*$.

Proof. As we have learn before, $\mathbb{N} \cong \{0,1\}^*$, so we need proof $\mathbb{N}^* \cong \{0,1\}^*$.
Let b_i denotes a bit sequence which starts with i 1s and ends with a 0.
Then define a function f:

$$f : \mathbb{N}^* \to \{0,1\}^*,$$
$$(a_1, a_2, \ldots) \mapsto (b_{a_1}, b_{a_2}, \ldots)$$

Since $\{0,1\}^*$ is also a natural number sequence, $|\{0,1\}^*| \leq |\mathbb{N}^*|$.
According to function f, for two different elements in $\mathbb{N}^*$, let natural number a_i be the first different byte, its conresponding b_i will also be different, so the their images in $\{0,1\}^8$ are different. We can get $|\mathbb{N}^*| \leq |\{0,1\}^*|$.
In this way, function f is bijective. □

Exercise 2.10. Show that $R \cong R \times R$. **Hint:** Use the fact that $R \cong \{0,1\}^{\mathbb{N}}$ and thus show that $\{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$.

Proof. We can define f:

$$\begin{gathered} f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \\ ((a_{11}, a_{12}, a_{13}, \dots), (a_{21}, a_{22}, a_{23}, \dots)) \mapsto (a_{11}, a_{21}, a_{12}, a_{22}, \dots) \end{gathered}$$

For any element $\mathbf{a} \in \{0,1\}^{\mathbb{N}}$, $\mathbf{a}$ can be representated by a combination of unique ordered pair $(\mathbf{a_1}, \mathbf{a_2})$ where $\mathbf{a_1}, \mathbf{a_2} \in \{0,1\}^{\mathbb{N}}$. In this way, f is bijective.
Then we can get $R \cong \{0,1\}^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \cong R \times R$.

□

Exercise 2.11. Consider $\mathbb{R}^{\mathbb{N}}$, the set of all infinite sequences $(r_1, r_2, r_3, \dots)$ of real numbers. Show that $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. **Hint:** Again, use the fact that $\mathbb{R} \cong \{0,1\}^{\mathbb{N}}$.

Proof. Since $R \cong \{0,1\}^{\mathbb{N}}$, we need to prove $(\{0,1\})^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$.
The element $\mathbf{e} \in (\{0,1\})^{\mathbb{N}})^{\mathbb{N}}$ can be represented as $(\mathbf{a^1}, \mathbf{a^2}, \dots)$ where $\mathbf{a^i} \in \{0,1\}^{\mathbb{N}}$
Also, $\mathbf{e}$ can be represented as a matrix:

$$\mathbf{e} = \begin{pmatrix} \mathbf{a^1} \\ \mathbf{a^2} \\ \dots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \dots \\ a_1^2 & a_2^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Define a function f:

$$f : \{0,1\} \to (\{0,1\})^{\mathbb{N}})^{\mathbb{N}}, (a_1, a_2, a_3, \dots) \mapsto \begin{pmatrix} a_1 & a_3 & a_6 & \dots \\ a_2 & a_5 & \dots & \dots \\ a_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

For $\mathbf{a} \in \{0,1\}^{\mathbb{N}}$, we place each bit of $\mathbf{a}$ on matrix following the diagonals in turn, then we get a unique $\mathbf{e} \in (\{0,1\})^{\mathbb{N}})^{\mathbb{N}}$.

Also, for each $\mathbf{e} \in (\{0,1\})^{\mathbb{N}})^{\mathbb{N}}$, there is exactly one element $\mathbf{a} \in \{0,1\}^{\mathbb{N}}$ such that $f(\mathbf{a}) = \mathbf{e}$.
In this sense, $\{0,1\} \cong (\{0,1\})^{\mathbb{N}})^{\mathbb{N}}$, which can be expressed as$\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$. $\square$

Next, let us view $\{0,1\}^{\mathbb{N}}$ as a partial ordering: given two elements $\mathbf{a}, \mathbf{b} \in \{0,1\}^{\mathbb{N}}$, that is, sequences $\mathbf{a} = (a_1, a_2, \dots)$ and $\mathbf{b} = (b_1, b_2, \dots)$, we define $\mathbf{a} \leq \mathbf{b}$ if $a_i \leq b_i$ for all $i \in \mathbb{N}$. Clearly, $(0, 0, \dots)$ is the minimum element in this ordering and $(1, 1, \dots)$ the maximum.

Exercise 2.12. Give a countably infinite chain in $\{0,1\}^{\mathbb{N}}$. Remember that a set A is countably infinite if $A \cong \mathbb{N}$.

Solution. For set $A \subseteq \{0,1\}^{\mathbb{N}}$, let $A = \{\mathbf{a_i} \in \{0,1\}^{\mathbb{N}} |$ if $k < i$,$a_k = 1, else, a_k = 0\}$, any two elements in A are comparable
We can define a function f:

$$f : N \to A, x \mapsto \mathbf{a_x}$$

which is obviously bijective. In this way, A is a countably infinite chain. ■

Exercise 2.13. Find a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$.

Solution. For set $A \subseteq \{0,1\}^{\mathbb{N}}$, let $A = \{\mathbf{a_i} \in \{0,1\}^{\mathbb{N}} | a_i = 1$ and other bits is $0\}$. Any two elements in A is uncomparable since each has one bit larger than the other.
Also, we can define a function f:

$$f : N \to A, x \mapsto \mathbf{a_x}$$

which is bijective.
In this scense, A is a countably infinite antichain in $\{0,1\}^{\mathbb{N}}$. ■

Exercise 2.14. Find an uncountable antichain in $\{0,1\}^{\mathbb{N}}$. That is, an antichain A with $A \cong \mathbb{R}$.

Solution. Define a function f:

$$f : \{0,1\}^{\mathbb{N}} \to A, (a_1, a_2, a_3 \dots) \mapsto (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

f is a bijective function, and $\{0,1\}^{\mathbb{N}} \cong A$
For $\mathbf{a}, \mathbf{b} \in A$, if $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{a}, \mathbf{b}$ are comparable, let $\mathbf{a} < \mathbf{b}$.
There must exist $a_i < b_i$, but $1 - a_i > 1 - b_i$, so $\mathbf{a}$ and $\mathbf{b}$ are uncomparable.
Then we get a contradiction.
In this way, A is a uncountable antichain. ■

****Exercise 2.15.** *Find an uncountable chain in $\{0,1\}^{\mathbb{N}}$. That is, an chain A with $A \cong \mathbb{R}$.*

Solution. For $b_i \in \{0,1\}^{\mathbb{N}}$ where $i \in \mathbb{N}$,we define matrix B_k as:

$$\mathbf{B_k} = \begin{pmatrix} \mathbf{b_1} \\ \mathbf{b_2} \\ \dots \\ \mathbf{b_k} \\ 1 \\ \dots \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots \\ b_{21} & b_{22} & \dots \\ \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots \\ 1 & 1 & \dots \\ \dots & & \end{pmatrix}$$

Let B denotes the set of all B_k, then $B \cong \{0,1\}^{\mathbb{N}}$.
Take bit sequnce a_k from B_k following the diagonals, which means $a_k = (b_{11}, b_{21}, b_{12}, b_{31}, b_{22}, \dots)$. And set A is the set of all a_k. We can get $A \cong B \cong \{0,1\}^{\mathbb{N}}$
According to the way we construct B_k, the 0s in a_{k+1} will not be less than these in a_k and exist at the same bits. In this way, any two elements in A are comparable.
So A is an uncountable chain.

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