

# Mathematical Foundations of Computer Science

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- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

## 2 Partial Orderings

### 2.1 Equivalence Relations as a Partial Ordering

An equivalence relation  $R \subseteq V \times V$  is basically the same as a partition of  $V$ . A *partition* of  $V$  is a set  $\{V_1, \dots, V_k\}$  where (1)  $V_1 \cup \dots \cup V_k = V$  and (2) the  $V_i$  are pairwise disjoint, i.e.,  $V_i \cap V_j = \emptyset$  for  $1 \leq i < j \leq k$ . For example,  $\{\{1\}, \{2, 3\}, \{4\}\}$  is a partition of  $\{1, 2, 3, 4\}$  but  $\{\{1\}, \{2, 3\}, \{1, 4\}\}$  is not.

**Exercise 2.1.** Let  $E_4$  be the set of all equivalence relations on  $\{1, 2, 3, 4\}$ . Note that  $E_4$  is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

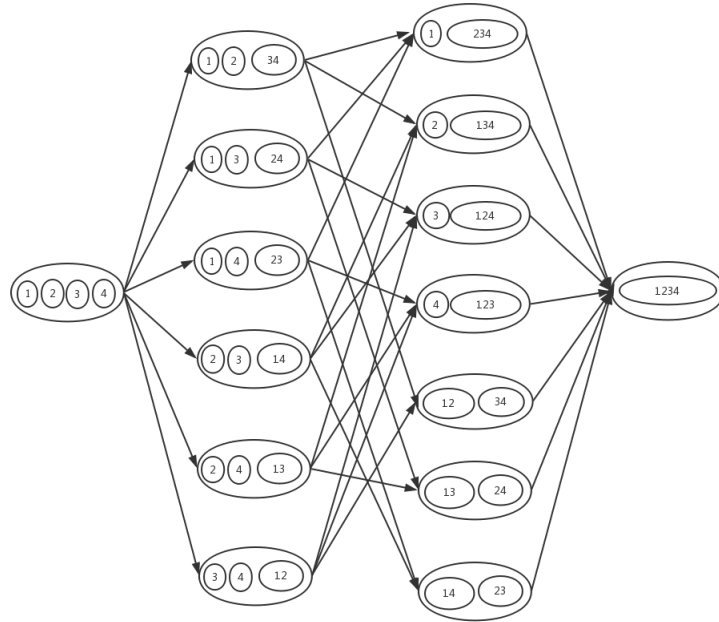


Figure 1: the Hasse diagram of the partial ordering

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?

**solution:**the size of the largest chain is 4.

3. What is the size of the largest antichain?

**solution:**the size of the largest antichain is 6.

## 2.2 Chains and Antichains

Define the partially ordered set  $(\mathbb{N}_0^n, \leq)$  as follows:  $x \leq y$  if  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . For example,  $(2, 5, 4) \leq (2, 6, 6)$  but  $(2, 5, 4) \not\leq (3, 1, 1)$ .

**Exercise 2.2.** Consider the infinite partially ordered set  $(\mathbb{N}_0^n, \leq)$ .

1. Which elements are minimal? Which are maximal?

**solution:**  $(0,0,\dots,0)$  is the minimal. There is no maximal element.

2. Is there a minimum? A maximum?

**solution:**  $(0,0,\dots,0)$  is the minimum. There is no maximum element.

3. Does it have an infinite chain?

**solution:** Yes. If it does not have an infinite chain, it will have the maximum element which we assume it  $(a_1, a_2, \dots, a_n)$ . Then we have the element  $(a_1 + 1, a_2 + 1, \dots, a_n + 1)$ , which is obviously greater than  $(a_1, a_2, \dots, a_n)$ . Thus, it has an infinite chain.

4. Does it have arbitrarily large antichains? That is, can you find an antichain  $A$  of size  $|A| = k$  for every  $k \in \mathbb{N}$ ?

**solution:** Yes. For every  $k \in \mathbb{N}$ , there is an antichain like  $(1, 0, \dots, 0)$ ,  $(0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$ , which is an antichain.

**\*Exercise 2.3.** Does every infinite subset  $S \subseteq \mathbb{N}_0^n$  contain an infinite chain?

**solution:** For a subset  $S \subseteq \mathbb{N}_0^n$  define  $S_I := \{\mathbf{a}_I \mid \mathbf{a} \in S\}$ . Because  $S$  is infinite, there must be a division causing  $S_I$  is infinite and  $S_{[n] \setminus I}$  is finite ( $S_{[n] \setminus I}$  might be  $\emptyset$ ).

Hence, there are infinite elements of  $S$  having the same part in  $S_{[n] \setminus I}$  due to the finity.

Consider the infinite part of these elements. For any element  $S_i$  in  $S$ , we can find another element  $S_j$  in the infinite subset  $S$ , whose finite part is the same as  $S_i$  and the infinite part is all greater than  $S_i$ . Therefore, we can find an infinite chain.

**Exercise 2.4.** Show that  $(\mathbb{N}_0^n, \leq)$  has no infinite antichain. **Hint.** Use the previous exercise.

*Solution.* For any two elements  $\mathbf{e}^1, \mathbf{e}^2 \in (\mathbb{N}_0^n, \leq)$  are incomparable, there must be at least two corresponding bytes of them have opposite relations such as  $e_i^1 < e_i^2$  and  $e_k^1 > e_k^2$ .

Consider a element  $\mathbf{e}$  in an antichain, for a byte  $e_m$  of  $\mathbf{e}$ , the numbers in the interval  $[1, e_m]$  is finite. Also, the number of bytes in  $\mathbf{e}$  is finite.

In this way, the number of elements which are incomparable to  $\mathbf{e}$  is also finite. So  $(\mathbb{N}_0^n, \leq)$  has no infinite antichain. ■

Consider the induced ordering on  $\{0, 1\}^n$ . That is, for  $x, y \in \{0, 1\}^n$  we have  $x \leq y$  if  $x_i \leq y_i$  for every coordinate  $i \in [n]$ .

**Exercise 2.5.** Draw the Hasse diagrams of  $(\{0, 1\}^n, \leq)$  for  $n = 2, 3$ .

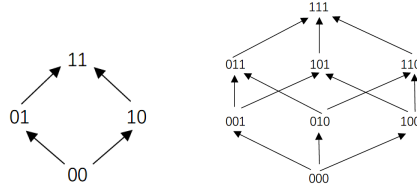


Figure 2: the Hasse diagrams of  $(\{0, 1\}^n, \leq)$  for  $n = 2, 3$

**Exercise 2.6.** Determine the maximum, minimum, maximal, and minimal elements of  $\{0, 1\}^n$ .

**solution:** The maximum and maximal element are both  $11\dots 1$  (it has  $n$  many 1's). The minimum and minimal element are both  $00\dots 0$  (it has  $n$  many 0's).

**Exercise 2.7.** What is the longest chain of  $\{0, 1\}^n$ ?

**solution:** The longest chain of  $\{0, 1\}^n$  is  $00\dots 0, 00\dots 1, 00\dots 11, \dots, 11\dots 1$ . The length of the chain is  $n + 1$ .

**\*\*Exercise 2.8.** What is the largest antichain of  $\{0, 1\}^n$ ?

*Solution.* The largest antichain of  $\{0, 1\}^n$  is the set of all bit sequence composed by  $\lfloor n/2 \rfloor$  1s and  $(n - \lfloor n/2 \rfloor)$  0s.

1. We partition all elements in  $\{0, 1\}^n$  by the number of 1s.  $A_i$  denotes the set of all bit sequence containing  $i$  1s, then  $\{0, 1\}^n$  consists of  $A_1, A_2, \dots, A_n$ . Each partition is an antichain in itself. According to character of enumerative combination,  $A_{\lfloor n/2 \rfloor}$  has the most elements.

2. Consider an antichain  $C = (c_1, c_2, \dots, c_m)$  of  $\{0, 1\}^n$  whose elements lie in different partitions.  
 Take any subset  $C^* \subseteq C$  whose elements belong to  $A_i$  where  $i \neq \lfloor n/2 \rfloor$ , and  $A_j$  next to  $A_i$  with  $|A_i| < |A_j|$ .  
 Take a subset  $C^{**} \subseteq A_j$  containing all elements in  $A_j$  which are comparable to elements in  $C^*$ . Obviously  $|C^{**}| \geq |C^*|$  and elements  $C^{**}$  are uncomparable to elements of  $C \setminus C^*$ . In this way,  $C \setminus C^* + C^{**}$  is also a antichain.
3. We can recursively replace  $C^*$  with  $C^{**}$  to get an antichain whose size is not smaller than the previous one until all elements lie in  $A_{\lfloor n/2 \rfloor}$ .

To sum up,  $A_{\lfloor n/2 \rfloor}$  is the largest antichain of  $\{0, 1\}^n$  with size  $\binom{n}{\lfloor n/2 \rfloor}$ . ■

## 2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that  $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ , i.e., there is a bijection  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . From this, and by induction, it follows quite easily that  $\mathbb{N}^k \cong \mathbb{N}$  for every  $k$ .

**Exercise 2.9.** Consider  $\mathbb{N}^*$ , the set of all finite sequences of natural numbers, that is,  $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$ . Here,  $\epsilon$  is the empty sequence. Show that  $\mathbb{N} \cong \mathbb{N}^*$  by defining a bijection  $\mathbb{N} \rightarrow \mathbb{N}^*$ .

*Proof.* As we have learn before,  $\mathbb{N} \cong \{0, 1\}^*$ , so we need proof  $\mathbb{N}^* \cong \{0, 1\}^*$ . Let  $b_i$  denotes a bit sequence which starts with  $i$  1s and ends with a 0. Then define a function  $f$ :

$$\begin{aligned} f : \mathbb{N}^* &\rightarrow \{0, 1\}^*, \\ (a_1, a_2, \dots) &\mapsto (b_{a_1}, b_{a_2}, \dots) \end{aligned}$$

Since  $\{0, 1\}^*$  is also a natural number sequence,  $|\{0, 1\}^*| \leq |\mathbb{N}^*|$ .

According to function  $f$ , for two different elements in  $\mathbb{N}^*$ , let natural number  $a_i$  be the first different byte, its corresponding  $b_i$  will also be different, so the their images in  $\{0, 1\}^*$  are different. We can get  $|\mathbb{N}^*| \leq |\{0, 1\}^*|$ .

In this way, function  $f$  is bijective. □

**Exercise 2.10.** Show that  $R \cong R \times R$ . **Hint:** Use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$  and thus show that  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ .

*Proof.* We can define  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$$

$$((a_{11}, a_{12}, a_{13}, \dots), (a_{21}, a_{22}, a_{23}, \dots)) \mapsto (a_{11}, a_{21}, a_{12}, a_{22}, \dots)$$

For any element  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ ,  $\mathbf{a}$  can be represented by a combination of unique ordered pair  $(\mathbf{a}_1, \mathbf{a}_2)$  where  $\mathbf{a}_1, \mathbf{a}_2 \in \{0, 1\}^{\mathbb{N}}$ . In this way,  $f$  is bijective. Then we can get  $R \cong \{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \cong R \times R$ .  $\square$

**Exercise 2.11.** Consider  $\mathbb{R}^{\mathbb{N}}$ , the set of all infinite sequences  $(r_1, r_2, r_3, \dots)$  of real numbers. Show that  $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$ . **Hint:** Again, use the fact that  $\mathbb{R} \cong \{0, 1\}^{\mathbb{N}}$ .

*Proof.* Since  $R \cong \{0, 1\}^{\mathbb{N}}$ , we need to prove  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$ . The element  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$  can be represented as  $(\mathbf{a}^1, \mathbf{a}^2, \dots)$  where  $\mathbf{a}^i \in \{0, 1\}^{\mathbb{N}}$ .

Also,  $\mathbf{e}$  can be represented as a matrix:

$$\mathbf{e} = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \dots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \dots \\ a_1^2 & a_2^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Define a function  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}, (a_1, a_2, a_3, \dots) \mapsto \begin{pmatrix} a_1 & a_3 & a_6 & \dots \\ a_2 & a_5 & \dots & \dots \\ a_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

For  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ , we place each bit of  $\mathbf{a}$  on matrix following the diagonals in turn, then we get a unique  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ .

Also, for each  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ , there is exactly one element  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$  such that  $f(\mathbf{a}) = \mathbf{e}$ .

In this sense,  $\{0, 1\} \cong (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ , which can be expressed as  $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$ .  $\square$

Next, let us view  $\{0, 1\}^{\mathbb{N}}$  as a partial ordering: given two elements  $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbb{N}}$ , that is, sequences  $\mathbf{a} = (a_1, a_2, \dots)$  and  $\mathbf{b} = (b_1, b_2, \dots)$ , we define  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ . Clearly,  $(0, 0, \dots)$  is the minimum element in this ordering and  $(1, 1, \dots)$  the maximum.

**Exercise 2.12.** Give a countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$ . Remember that a set  $A$  is countably infinite if  $A \cong \mathbb{N}$ .

*Solution.* For set  $A \subseteq \{0, 1\}^{\mathbb{N}}$ , let  $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid \text{if } k < i, a_k = 1, \text{ else, } a_k = 0\}$ , any two elements in  $A$  are comparable  
We can define a function  $f$ :

$$f : \mathbb{N} \rightarrow A, x \mapsto \mathbf{a}_x$$

which is obviously bijective. In this way,  $A$  is a countably infinite chain. ■

**Exercise 2.13.** Find a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ .

*Solution.* For set  $A \subseteq \{0, 1\}^{\mathbb{N}}$ , let  $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid a_i = 1 \text{ and other bits is } 0\}$ . Any two elements in  $A$  is uncomparable since each has one bit larger than the other.

Also, we can define a function  $f$ :

$$f : \mathbb{N} \rightarrow A, x \mapsto \mathbf{a}_x$$

which is bijective.

In this scense,  $A$  is a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ . ■

**Exercise 2.14.** Find an uncountable antichain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .

*Solution.* Define a function  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow A, (a_1, a_2, a_3 \dots) \mapsto (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

$f$  is a bijective function, and  $\{0, 1\}^{\mathbb{N}} \cong A$

For  $\mathbf{a}, \mathbf{b} \in A$ , if  $\mathbf{a} \neq \mathbf{b}$  and  $\mathbf{a}, \mathbf{b}$  are comparable, let  $\mathbf{a} < \mathbf{b}$ .

There must exist  $a_i < b_i$ , but  $1 - a_i > 1 - b_i$ , so  $\mathbf{a}$  and  $\mathbf{b}$  are uncomparable. Then we get a contradiction.

In this way,  $A$  is a uncountable antichain. ■

**\*\*Exercise 2.15.** Find an uncountable chain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an chain  $A$  with  $A \cong \mathbb{R}$ .

*Solution.* For  $b_i \in \{0, 1\}^{\mathbb{N}}$  where  $i \in \mathbb{N}$ , we define matrix  $B_k$  as:

$$B_k = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \dots \\ \mathbf{b}_k \\ 1 \\ \dots \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots \\ b_{21} & b_{22} & \dots \\ \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots \\ 1 & 1 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Let  $B$  denotes the set of all  $B_k$ , then  $B \cong \{0, 1\}^{\mathbb{N}}$ .

Take bit sequence  $a_k$  from  $B_k$  following the diagonals, which means  $a_k = (b_{11}, b_{21}, b_{12}, b_{31}, b_{22}, \dots)$ . And set  $A$  is the set of all  $a_k$ . We can get  $A \cong B \cong \{0, 1\}^{\mathbb{N}}$

According to the way we construct  $B_k$ , the 0s in  $a_{k+1}$  will not be less than these in  $a_k$  and exist at the same bits. In this way, any two elements in  $A$  are comparable.

So  $A$  is an uncountable chain. ■