

Mathematical Foundations of Computer Science

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10 Network Flow

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- Submit questions and first solution by Sunday, 2018-05-13, 12:00
- Submit final solution by Sunday, 2018-05-20.

Exercise 10.1. [From the video lecture] Recall the definition of the value of a flow: $\text{val}(f) = \sum_{v \in V} f(s, v)$. Let $S \subseteq V$ be a set of vertices that contains s but not t . Show that

$$\text{val}(f) = \sum_{u \in S, v \in V \setminus S} f(u, v) .$$

That is, the total amount of flow leaving s equals the total amount of flow going from S to $V \setminus S$. **Remark.** It sounds obvious. However, find a formal proof that works with the axiomatic definition of flows.

Proof. By flow conservation, $\sum_w f(v, w) = 0$, for all $v \in V \setminus \{s, t\}$. Also by definition of the value of a flow, we have:

$$\begin{aligned}
\text{val}(f) &= \sum_{w \in V} f(s, w) \\
&= \sum_{u \in S, w \in V} f(u, w) \\
&= \sum_{u \in S} \left(\sum_{w_1 \in S} f(u, w_1) + \sum_{w_2 \in V \setminus S} f(u, w_2) \right) \\
&= \sum_{u \in S, w_1 \in S} f(u, w_1) + \sum_{u \in S, w_2 \in V \setminus S} f(u, w_2) \\
&= \sum_{u \in S, v \in V \setminus S} f(u, v) + \sum_{u \in S, w_1 \in S} f(u, w_1)
\end{aligned}$$

Recall the skew-symmetry that $f(u, v) = -f(v, u)$, we have:

$$\sum_{u \in S, w_1 \in S} f(u, w_1) = \sum_{w_1 \in S, u \in S} -f(w_1, u) = - \sum_{u \in S, w_1 \in S} f(u, w_1) = 0 \quad (1)$$

In conclusion, $\text{val}(f) = \sum_{u \in S, v \in V \setminus S} f(u, v)$. \square

Exercise 10.2. Let $G = (V, E, c)$ be a flow network. Prove that flow is “transitive” in the following sense: If there is a flow from s to r of value k , and a flow from r to t of value k , then there is a flow from s to t of value k . **Hint.** The solution is extremely short. If you are trying something that needs more than 3 lines to write, you are on the wrong track.

Proof. Suppose there is no flow from s to t of value k . Then there is a s - t cut $s \in S, t \in V \setminus S$ such that $c(S, V \setminus S) < k$. If $r \in S$, then there is a r - t cut $(S, V \setminus S)$ such that the capacity is less than k , so there’s no flow of value k from r to t . Else if $r \in V \setminus S$, then there is a s - r cut $(S, V \setminus S)$, whose capacity is less than k and no flow value of k exists. In either of these cases, we reach a contradiction, so flow must be transitive. \square

10.1 An Algorithm for Maximum Flow

Recall the algorithm for Maximum Flow presented in the video. It is usually called the Ford-Fulkerson method.

We proved in the lecture that f is a maximum flow and S is a minimum cut, by showing that upon termination of the while-loop, $\text{val}(f) = \text{cap}(S)$.

Algorithm 1 Ford-Fulkerson Method

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1: procedure FF( $G = (V, E), s, t, c$ )
2:   Initialize  $f$  to be the all-0-flow.
3:   while there is a path  $p$  from  $s$  to  $t$  in the residual network  $G_f$  do
4:      $c_{\min} := \min\{c_f(e) \mid e \in p\}$ 
5:     let  $f_p$  be the flow in  $G_f$  that routes  $c_{\min}$  flow along  $p$ 
6:      $f := f + f_p$ 
7:   end while
8:   // now  $f$  is a maximum flow
9:    $S := \{v \in V \mid G_f \text{ contains a path from } s \text{ to } v\}$ 
10:  //  $S$  is a minimum cut
11:  return ( $f, S$ )
12: end procedure
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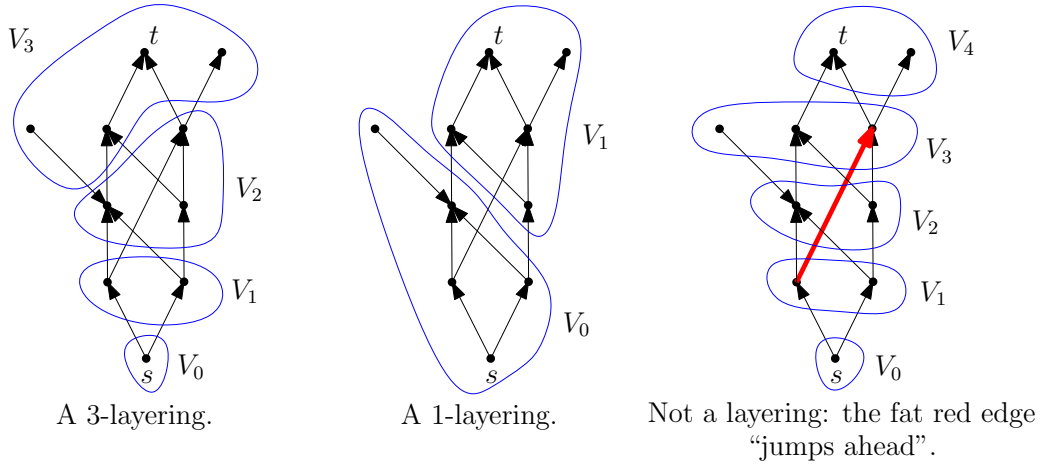
The problem is that the while-loop might not terminate. In fact, there is an example with capacities in \mathbb{R} for which the while loop does not terminate, and the value of f does not even converge to the value of a maximum flow. As indicated in the video, a little twist fixes this:

Edmonds-Karp Algorithm: Execute the above Ford-Fulkerson Method, but in every iteration choose p to be a shortest s - t -path in G_f . Here, “shortest” means minimum number of edges.

In a series of exercises, you will now show that this algorithm always terminates after at most $n \cdot m$ iterations of the while loop (here $n = |V|$ and $m = |E|$).

Definition 10.3. Let (G, s, t, c) be a flow network and $k \in \mathbb{N}_0$. A k -layering is a partition of $V = V_0 \cup \dots \cup V_k$ such that (1) $s \in V_0$, (2) $t \in V_k$, (3) for every edge $(u, v) \in E$ the following holds: suppose $u \in V_i$ and $v \in V_j$. Then $j \leq i + 1$. In words, point (3) states that every edge moves at most one level forward.

The figure below illustrates this concept: for one network we show two possible layerings and something that looks like a layering but is not:



Exercise 10.4. Suppose the network (G, s, t, c) has a k -layering. Show that $\text{dist}(s, t) \geq k$. That is, every s - t -path in G has at least k edges.

Proof. Prove by contradiction:

Suppose that there is a s - t -path, say $s-v_1 \dots -v_n-t$, in G that has less than k edges ($n+1 < k$). By definition, $s \in V_0$ and $t \in V_k$ and every edge moves at most one level forward. Hence, for the s - t -path, $s-v_1 \dots -v_n-t$, we have:

$$\begin{aligned}
 s &\in V_0 \\
 v_1 &\in V_{a_1}, a_1 \leq 1 \\
 v_2 &\in V_{a_2}, a_2 \leq a_1 + 1 \leq 2 \\
 \dots v_n &\in V_{a_n}, a_n \leq a_{n-1} + 1 \leq n \\
 t &\in V_{a_{n+1}}, a_{n+1} \leq a_n + 1 \leq n + 1 < k
 \end{aligned}$$

However, $t \in V_k$, a contradiction. □

Exercise 10.5. Conversely, suppose $\text{dist}(s, t) = k$. Show that (G, s, t, c) has a k -layering.

Proof. Prove by construction:

$\text{dist}(s, t) = k$, suppose the shortest path between s and t are $s-v_1 \dots -v_{k-1}-t$. Construction steps are as follows:

- First, add s to V_0 , v_1 to V_1 , \dots , v_{k-1} to V_{k-1} , t to V_k .
- Consider the vertices that are directly connected with vertices that are already in the partitions, iteratively add them to any partitions as long as the adding satisfies *POINT 3* above.

- Recursively do step 2 above until no vertices left (since a flow network is a connected component as a whole).

□

Let (G, s, t, c) be a flow network and V_0, \dots, V_k a k -layering. We call this layering *optimal* if $\text{dist}_G(s, t) = k$. Here, $\text{dist}_G(u, v)$ is the shortest-path distance from s to t (measured by number of edges). If there is no path from s to t , we set $\text{dist}_G(s, t) = \infty$. In this case, no layering is optimal. For example, the 3-layering in the above figure is optimal, but the 1-layering in the middle of the above figure is not. Let us explore how layerings and the Ford-Fulkerson Method interact.

Exercise 10.6. Let (G, s, t, c) be a flow network and V_0, V_1, \dots, V_k be an optimal layering (that is, $k = \text{dist}_G(s, t)$). Let p be a path from s to t of length k . Suppose we route some flow f along p (of some value $c_{\min} > 0$) and let (G_f, s, t, c_f) be the residual network. Show that V_0, V_1, \dots, V_k is a layering of (G_f, s, t, c_f) , too. Obviously, condition (1) and (2) in the definition of k -layerings still hold, so you only have to check condition (3).

Proof. Consider the route operation, changes made on original edges in G are only limited on the capacity value and the added new edges each has a corresponding reversely directed edge in G , which is to say, that if any edge in G satisfied *POINT 3*, then any edge in G_f will not violate *POINT 3*. □

Exercise 10.7. Show that every network (G, s, t, c) has an optimal layering, provided there is a path from s to t .

Proof. Since there is a path from s to t , there is a shortest path, say the $\text{dist}(s, t) = k$. According to **Exercise 10.5**, G has a k -layering, which is the optimal layering by definition. □

Exercise 10.8. Imagine we are in some iteration of the while-loop of the Ford-Fulkerson method. Let V_0, \dots, V_k be an optimal layering of (G, s, t, c) . Show that after at most m iterations of the while-loop, V_0, \dots, V_k ceases to be an optimal layering. **Remark.** Note that it is the *network* that changes from iteration to iteration of the while-loop, not the partition V_0, \dots, V_k . We consider the partition V_0, \dots, V_k to be fixed in this exercise.

Exercise 10.9. Show that the Edmonds-Karp algorithm terminates after $n \cdot m$ iterations of the while-loop. **Hint.** Initially, compute an optimal k -layering (which?). Then keep this layering as long as it's optimal. Once it ceases to be optimal, compute a new optimal layering. Note that the Edmonds-Karp algorithm does not actually need to compute any layering. It's us who compute it to show that $n \cdot m$ bound on the number of iterations.

Exercise 10.10. Show that every network has a maximum flow f . That is, a flow f such that $\text{val}(f) \geq \text{val}(f')$ for every flow f' . **Remark.** This sounds obvious but it is not. In fact, there might be an infinite sequence of flows f_1, f_2, f_3, \dots of increasing value that does not reach any maximum. Use the previous exercises!