Mathematical Foundations
of
Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

Group: NOIDEA

7 The Graph Score Theorem

- Homework assignment published on Monday, 2018-04-09.
- Submit questions and first solution by Sunday, 2018-04-15, 12:00 by email to dominik.scheder@gmail.com and the TAs.
- You will receive feedback by Wednesday, 2018-04-18.
- Submit your final solution by Sunday, 2018-04-22 to me and the two TAs.

Exercise 7.1. Describe, in simple sentences with a minimum of mathematical formalism, (1) the score of a graph, (2) what the graph score theorem is, (3) the idea of the graph score algorithm, (4) where the difficult part of its proof is. Imagine you have a friend who does not take this class, and think about how to answer the above questions to them.

Solution.

1. The score of a graph is the increasing sequence of degrees of all vertexes.

2. Give a increasing sequence of degrees $D = (d_1, d_2, d_3 \dots d_n)$ with d_i denoting its element. Consider another sequences $D^{'}$ of size $n-1$ whose element denoted as $d_i^{'}$ with:

$$d_i^{'} = \begin{cases} d_i & i < n - d_n \\ d_i - 1 & i \geq n - d_n \end{cases}$$

D is a graph score if and only if $D^{'}$ is a graph score.

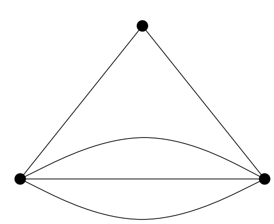
The idea of graph score algorithm is recursively using theorem to simplify the degree sequence until it's easy to judge whether the sequence is a graph score.

The difficult part of the proof is the reverse implication i.e. if D is a graph score then $D^{'}$ is a graph score. Since we can't just reverse the constrution process how we get the graph of D from the graph of $D^{'}$. ■

7.1 Alternative Graphs

Now we will look at different notions of graphs. As defined in class and in the video lectures, a graph is a pair $G = (V, E)$ where V is a (usually finite) set, called the *vertices*, and $E \subseteq \binom{V}{2}$, called the set of *edges*.

Multigraphs. A *multigraph* is like a graph, but you can have several parallel edges between two vertices. You cannot, however, have self-loops. That is, there cannot be an edge from u to u itself. This is an example of a multigraph:



We can define degree and score for multigraphs, too. For example, this multigraph has score $(4, 4, 2)$. Obviously no graph can have this score.

Exercise 7.2. State a score theorem for multigraphs. That is, something like

Theorem 7.3 (Multigraph Score Theorem). *Let* $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$. *There is a multigraph with this score if and only if* `<fill in some simple criterion here>`.

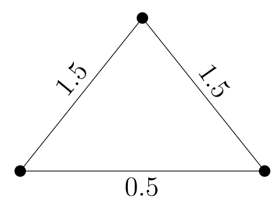
Remark. This is actually simpler than for graphs.

Solution. Let $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$. There is a multigraph with this score if and only if regarding its reordered sequence in ascending order–$(b_1, \ldots, b_n) \in \mathbb{N}_0^n$ with $b_1 \leq b_2 \leq \ldots b_n$ which satisfies 1) $\sum_{i=1}^{n} b_i$ is even and 2) $\sum_{i=1}^{n-1} b_i \geq b_n$, when $n \geq 2$; and when $n = 1$, $a_1 = 0$. ∎

Exercise 7.4. Prove your theorem.

Proof. First, if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$ is multigraph score then it obviously satisfy the criterion according to Handshake Lemma.
To prove the reverse implication, we use induction theorem.
Hypothesis: If any sequence $\mathbf{a}^{'} = (a_1^{'}, \ldots, a_{n-1}^{'})$ satisfying both criterion is a multigraph score . Then for any sequences $\mathbf{a}$ that satisfies both criterion, a corresponding multigraph can be constructed using a multi-graph of size $(n-1)$ of a score sequence $\mathbf{a}'$.
Base case: When $n = 2$, $\mathbf{a}' = (a_1', a_2')$ meeting the conditions obviously has $a_1' = a_2'$, then it is a multigraph score.
When $n = 3$, let $\mathbf{a} = (a_1, a_2, a_3)$ meeting the conditions be in non-decreasing order. If $a_1 + a_2 = a_3$, we can construct a multigraph with two nodes have a_1 and a_2 edges connected to third node. Else, we recursively reduce a_1 and a_2 by 1 until $a_1 + a_2 = a_3$, we can construct the multigraph according above graph by adding edges between vertices v_1 and v_2.
Induction proof: For any sequence $\mathbf{a} = (a_1, a_2, \ldots a_n)$ satisfying both criterion, we sort it to get an ascending order sequence $\mathbf{b} = (b_1, b_2, \ldots b_n)$ with v_i denoting the node with degree b_i.
Let v_1 has all edges connected to v_n, remove v_1 and we get new sequence $\mathbf{b}^{'} = (b_2, b_3, \ldots, b_{n-1}, b_n - b_1)$
If $b_n - b_1$ is the largest one in $\mathbf{b}^{'}$, then $\mathbf{b}^{'}$ is obviously meeting the criterion therefore a multigraph score.
If $b_n - b_1$ is not the largest one, which means b_{n-1} is the largest one. Since $b_n > b_{n-1}$ and $b_n + b_2 - b_1 \geq b_n \geq b_{n-1}$, $\mathbf{b}^{'}$ still meet the criterion therefore being a multigraph score.
In all cases we can construct a multigraph with score $\mathbf{a}$ meeting the theorem. □

Weighted graphs. A weighted graph is a graph in which every edge e has a non-negative weight w_e. In such a graph the *weighted degree* of a vertex u is $\mathrm{wdeg}(u) = \sum_{\{u,v\}\in E} w_{\{u,v\}}$.



This is an example of a weighted graph, which has score $(3, 2, 2)$. Obviously no graph and no multigraph can have this score.

Exercise 7.5. State a score theorem for weighted graphs. That is, state something like

Theorem 7.6 (Weighted Graph Score Theorem)**.** *Let* $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. *There is a weighted graph with this score if and only if* `<fill in some simple criterion here>`.

Remark. This is actually even simpler.

Solution. Let $(a_1, \ldots, a_n) \in \mathbb{R}_0^n$. There is a weighted graph with this score if and only if corresponding sorted non-decreasing sequence $(b_1, \ldots, b_n)$ meets $\sum_{i=1}^{n-1} b_i \geq b_n$. ■

Exercise 7.7. Prove your theorem.

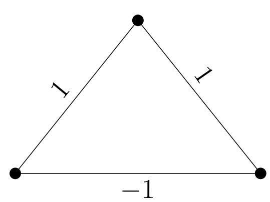
Proof. First, for any weighted graph, we have $\sum_{i=1}^{n-1} b_i \geq b_n$ since its each edge connected to b_n also connecting a node in $(b_1, \ldots b_{n-1})$.
Besides, for any non-decreasing sequence $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$ with $\sum_{i=1}^{n-1} a_i \geq a_n$, we use induction theorem to prove the theorem.
Without loss of generosity, we assume that all the score sequences mentioned below are in non-decreasing order.
Hypothesis: If any sequence $\mathbf{a}' = (a_1^{'}, \ldots, a_{n-1}^{'}) \in \mathbb{R}_0^{n-1}$ which satisfies the criterion above is a weighted graph score. Then for any score sequence of size n $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$ a corresponding weighted graph can be constructed from using a weighted graph of size $n-1$ denoted by the score sequence $\mathbf{a}'$.
Base case: When $n = 2$, obviously we have $a_1 = a_2$, then (a_1, a_2) must be

a weighted graph score.
When $n = 3$, let w_i denote the weight of each edge. We have,

$$\begin{array}{rcl} w_1 + w_2 & = & a_2 \\ w_2 + w_3 & = & a_3 \\ w_3 + w_1 & = & a_1 \end{array}$$

Solving the equations we can find $w_1, w_2, w_3 \geq 0$, then we have $\mathbf{a}$ meeting the criterion is weighted graph score.
Induction step: For any sequence $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_0^n$, in its corresponding sequence $\mathbf{b}$, remove vertex with degree b_1 assuming it has only an edge connected to node with degree b_n. Then we have new sequence $\mathbf{b}' = (b_1, \ldots, b_{n-1}, b_n - b_1)$.
If $b_n - b_1$ is still the largest element in $\mathbf{b}'$, it's obviously that $\sum_{i=2}^{n-1} b_i \geq b_n - b_1$, which means $\mathbf{b}'$ is a graph weighted score.
If $b_n - b_1$ is not the largest element, which means b_{n-1} is the largest one. We have $\sum_{i=2}^{n-2} a_i + b_n - b_1 \geq b_n \geq b_{n-1}$, so $\mathbf{b}'$ is a graph weighted score.
Accordingly, we can always construct a weighted graph with score $\mathbf{a}$ meeting the theorem. $\square$

Allowing negative edge weights. Suppose now we allow negative edge weights, like here:



This "graph with real edge weights" has score $(2, 0, 0)$. This score is impossible for graphs, multigraphs, and weighted graphs with non-negative edge weights.

Exercise 7.8. State a score theorem for weighted graphs when we allow negative edge weights. That is, state a theorem like

Theorem 7.9 (Score Theorem for Graphs with Real Edge Weights)**.** *Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$. There is a graph with real edge weights with this score if and only if* <fill in some simple criterion here>.

(Score Theorem For Graphs With Real Edge Weights).Let$(a_1, \cdots, a_n) \in \mathbb{R}^n$.There is a graph with real edge weights with this score if and only if

$$\begin{cases} a_n = 0 & n = 1, \\ a_{n-1} = a_n & n = 2, \\ always\ holds & n \geq 3. \end{cases}$$

Exercise 7.10. Prove your theorem.

Proof. **Induction Hypothesis:** For any given score $\mathbf{a} = (a_1, \cdots, a_{n-1}, a_n)(n \geq 3)$, there's always a corresponding weightedgraph whose score is $\mathbf{a}$.
Base Case:
If n=1, the score must be 0 because there is no edge in the graph.
If n=2, only one edge is connecting these two vertices, and thus, their degrees should be the same..
If $n = 3$, denote by W(u,v) the weight of the edge connecting vertex u and v. Then we have

$$\begin{cases} W(a_1, a_2) = (a_1 + a_2 - a_3)/2, \\ W(a_1, a_3) = (a_1 - a_2 + a_3)/2, \\ W(a_2, a_3) = (a_2 + a_3 - a_1)/2. \end{cases}$$

Since $a_1, a_2, a_3 \in \mathbf{R}$, there is always a solution to this equation array, which corresponds to a graph with real weights.
Induction Step:
If $n > 3$,we can reduce the size of the problem to a new score **a'** $= (a'_1, \cdots, a'_{n-1})$ by the following operations:

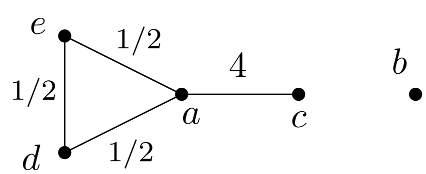
$$a'_i = \begin{cases} a_i & 1 \leq i < n-1, \\ a_i - a_{i+1} & i = n-1. \end{cases}$$

Then, we could apply the induction to this newly formed score, i.e., we could construct a weightedgraph with **a'**.
And we just connect n^{th} vertex to the $(n-1)^{th}$ with an edge of weight a_n.
And that's your weightedgraph.
That concludes this proof. □

Exercise 7.11. For each student ID $(a_1, \ldots, a_n)$ in your group, check whether this is (1) a graph score, (2) a multigraph score, (3) a weighted graph score, or (4) the score of a graph with real edge weights.

Whenever the answer is *yes*, show the graph, when it is *no*, give a short argument why.

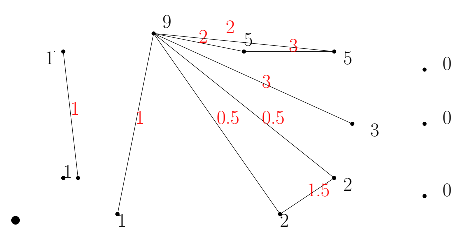
Example Solution. My work ID is 50411. This is a weighted graph score, as shown by this picture:



This settles (3). It is not a multigraph score, because BLABLABLA. I won't give more details, as it might give too many hints about Exercise 7.2. Alright, this settles (2). Note that I do *not* need to answer (4), as this is already answered by (3). Neither do I need to answer (1), as a "no" for (2) implies a "no" for (1).

1. 515021910302

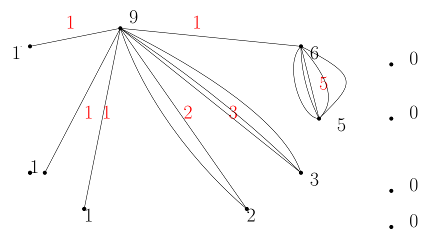
    - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.
    - a multigraph score: no, the sum of the degrees can not be an odd number.

    The figure above settles for (3) and (4).

2. 516021910003

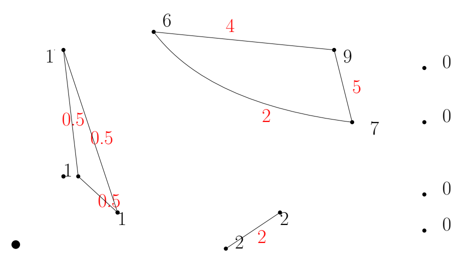
    - a graph score:There are only seven vertices but there is one 9, so it's impossible for graph score.



The figure above settles for (2), (3) and (4). *P.S.for a multigraph, simply ignoring the weight value, and for a weighted graph, simply ignoring the multi-edges between 2 vertices and see them as only 1 edge.*

3. 516021910725

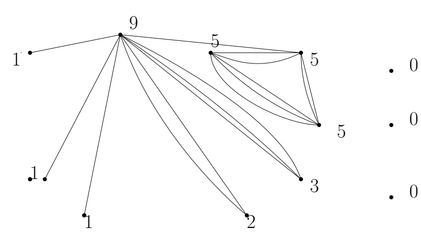
   - a graph score:There are only seven vertices but there is one 9, so it's impossible for graph score.
   - a multigraph score: no, the sum of the degrees can not be an odd number.

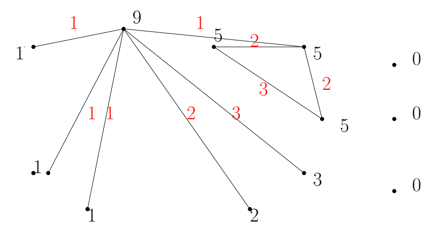


The figure above settles for (3) and (4).

4. 515021910053

   - a graph score:There are only eight vertices but there is one 9, so it's impossible for graph score.

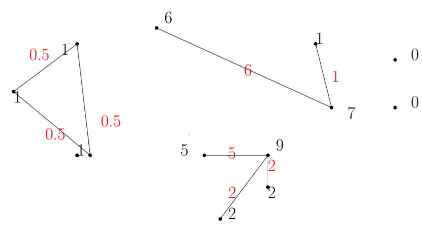




The figures above settles for (2), (3) and (4).

5. 516021910127

   - a graph score:According to the graph score theorem: (001111225679) –>(00000011456), since in the second step it is obvious that the score cannot denote a graph. Hence, it's impossible for graph score.
   - a multigraph score: no, the sum of the degrees can not be an odd number.

   The figure above settles for (3) and (4).