Mathematical Foundations
of
Computer Science

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1 Broken Chessboard and Jumping With Coins

1.1 Tiling a Damaged Checkerboard

Exercise 1.1. *Re-write the proof in your own way, using simple English sentences.*

Proof. Your proof ... □

Exercise 1.2. *Another exercise ...*

Proof. Your proof ... □

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1.

1. *Proof.* As is shown in the Venn diagram below, $|A| + |B|$ add the common part $|A \cap B|$ twice. So it should be subtracted once if we want to count $|A \cup B|$. □

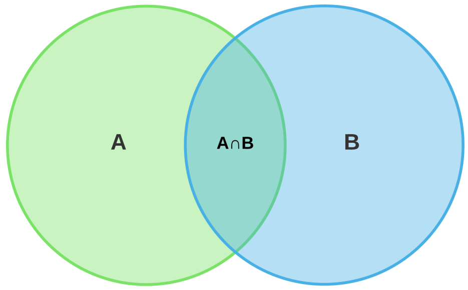

Figure 1: Venn Diagram

2. *Solution.* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ ∎

3. *Solution.* $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$ ∎

Exercise 2.2.

Solution. $|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j:1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{i,j,k:1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$ ∎

Exercise 2.3.

Proof. □

1. proof using induction on n

   First, let $A_{n,k} = \sum_{1 \leq i_1 < i_2 < \ldots \leq n} |A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{A_1, A_2, \ldots, A_n\}$.

   Then the Inclusion-exclusion principle which we want to prove is as follows:
   $|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} A_{n,k}$

The theorem holds for $n = 1$, obviously.

The theorem holds for $n = 2$, as is showed in the 2.1.1

For the induction step, we want to show if it holds for $n - 1$, then it holds for n.

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} (A_i \cap A_n)\right|
\end{aligned}
\tag{1}
$$

Let $B_i = (A_i \cap A_n)$.

Similarly, let $B_{n-1,k} = \sum_{1 \le i_1 < i_2 < \ldots \le n-1} |B_{i_1} \cap B_{i_2} \ldots \cap B_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{B_1, B_2, \ldots, B_n - 1\}$.

(1) now becomes

$$
\sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} B_i\right| \tag{2}
$$

Similarly, it holds:

$$
|B_1 \cup \ldots \cup B_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k+1} B_{n-1,k} \tag{3}
$$

(2) now becomes

$$\sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} + |A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} \tag{4}$$

In addition,

$$|A_n| = (-1)^{1+1}|A_n| \tag{5}$$

Thus,

$$|A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} = \sum_{k=1}^{n}(-1)^{k+1}A_{n,k} - \sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} \tag{6}$$

Then equation(4) finally becomes: $\sum\limits_{k=1}^{n}(-1)^{k+1}A_{n,k}$

2. proof not using induction on n

First, let $A = |A_1 \cup \ldots \cup A_{n-1}|$.
Function $P_S(x)$ defined as if set S includes element x, then $P_S(x) = 1$, else $P_S(x) = 0$.

(1) If $P_A(x) = 1$, there must exist an i that $P_{A_i}(x) = 1$. In this way:
$(P_A(x) - P_{A_1}(x))(P_A(x) - P_{A_2}(x)) \ldots (P_A(x) - P_{A_n}(x)) = 0$

(2) According to the properities of set, $P_{A_i}(x)P_{A_j}(x) = P_{A_i \cap A_j}(x)$.

(3) Let $P_{n,k}$ denotes $P_{A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}}(1 \leq i_1 < i_2 < \ldots \leq n)$.
Then decompose the first equation, we can have:
$P_A(x) = \sum\limits_{k=1}^{n} P_{n,k}$
which can be demonstrated as:
$|A_1 \cup \ldots \cup A_n| = \sum\limits_{k=1}^{n}(-1)^{k+1}A_{n,k}$