

# Mathematical Foundations of Computer Science

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- Homework assignment published on Monday, 2018-03-05.
- Work on it and submit a first solution or questions by Sunday, 2018-03-11, 12:00 by email to me and the TAs.
- You will receive feedback by Wednesday, 2018-03-14.
- Submit your final solution by Sunday, 2018-03-18 to me and the TAs.

## 2 Partial Orderings

### 2.1 Equivalence Relations as a Partial Ordering

An equivalence relation  $R \subseteq V \times V$  is basically the same as a partition of  $V$ . A *partition* of  $V$  is a set  $\{V_1, \dots, V_k\}$  where (1)  $V_1 \cup \dots \cup V_k = V$  and (2) the  $V_i$  are pairwise disjoint, i.e.,  $V_i \cap V_j = \emptyset$  for  $1 \leq i < j \leq k$ . For example,  $\{\{1\}, \{2, 3\}, \{4\}\}$  is a partition of  $\{1, 2, 3, 4\}$  but  $\{\{1\}, \{2, 3\}, \{1, 4\}\}$  is not.

**Exercise 2.1.** Let  $E_4$  be the set of all equivalence relations on  $\{1, 2, 3, 4\}$ . Note that  $E_4$  is ordered by set inclusion, i.e.,

$$(E_4, \{(R_1, R_2) \in E_4 \times E_4 \mid R_1 \subseteq R_2\})$$

is a partial ordering.

1. Draw the Hasse diagram of this partial ordering in a nice way.
2. What is the size of the largest chain?
3. What is the size of the largest antichain?

## 2.2 Chains and Antichains

Define the partially ordered set  $(\mathbb{N}_0^n, \leq)$  as follows:  $x \leq y$  if  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . For example,  $(2, 5, 4) \leq (2, 6, 6)$  but  $(2, 5, 4) \not\leq (3, 1, 1)$ .

**Exercise 2.2.** Consider the infinite partially ordered set  $(\mathbb{N}_0^n, \leq)$ .

1. Which elements are minimal? Which are maximal?
2. Is there a minimum? A maximum?
3. Does it have an infinite chain?
4. Does it have arbitrarily large antichains? That is, can you find an antichain  $A$  of size  $|A| = k$  for every  $k \in \mathbb{N}$ ?

**\*Exercise 2.3.** Does every infinite subset  $S \subseteq \mathbb{N}_0^n$  contain an infinite chain?

**Exercise 2.4.** Show that  $(\mathbb{N}_0^n, \leq)$  has no infinite antichain. **Hint.** Use the previous exercise.

Consider the induced ordering on  $\{0, 1\}^n$ . That is, for  $x, y \in \{0, 1\}^n$  we have  $x \leq y$  if  $x_i \leq y_i$  for every coordinate  $i \in [n]$ .

**Exercise 2.5.** Draw the Hasse diagrams of  $(\{0, 1\}^n, \leq)$  for  $n = 2, 3$ .

**Exercise 2.6.** Determine the maximum, minimum, maximal, and minimal elements of  $\{0, 1\}^n$ .

**Exercise 2.7.** What is the longest chain of  $\{0, 1\}^n$ ?

**\*\*Exercise 2.8.** What is the largest antichain of  $\{0, 1\}^n$ ?

## 2.3 Infinite Sets

In the lecture (and the lecture notes) we have showed that  $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$ , i.e., there is a bijection  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . From this, and by induction, it follows quite easily that  $\mathbb{N}^k \cong \mathbb{N}$  for every  $k$ .

**Exercise 2.9.** Consider  $\mathbb{N}^*$ , the set of all finite sequences of natural numbers, that is,  $\mathbb{N}^* = \{\epsilon\} \cup \mathbb{N} \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \dots$ . Here,  $\epsilon$  is the empty sequence. Show that  $\mathbb{N} \cong \mathbb{N}^*$  by defining a bijection  $\mathbb{N} \rightarrow \mathbb{N}^*$ .

*Proof.* As we have learn before,  $\mathbb{N} \cong \{0, 1\}^*$ , so we need proof  $\mathbb{N}^* \cong \{0, 1\}^*$ .

We could define a function  $f$ :

$f : \mathbb{N}^* \rightarrow \{0, 1\}^*$ ,  $(a_1, a_2, \dots) \mapsto$  sequence of 1s with  $(a_k + 1)$ th bit replaced with 0.

In other words, for each element  $\mathbf{a} \in \mathbb{N}^*$ , its corresponding bit sequence is combination of  $a_k$  1s appended with a 0 where  $k \in [1, |\mathbf{a}|]$ .

Since  $\{0, 1\}^*$  is also a natural number sequence,  $|\{0, 1\}^*| \leq \mathbb{N}^*$ . Also, according to function  $f$ , each image of  $\mathbf{a} \in \mathbb{N}^*$  in  $\{0, 1\}^*$  has different construction, which means  $\mathbb{N}^* \leq |\{0, 1\}^*|$ . In this way, function  $f$  is bijective.  $\square$

**Exercise 2.10.** Show that  $R \cong R \times R$ . **Hint:** Use the fact that  $R \cong \{0, 1\}^{\mathbb{N}}$  and thus show that  $\{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ .

*Proof.* We can define  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$$

$$((a_{11}, a_{12}, a_{13}, \dots), (a_{21}, a_{22}, a_{23}, \dots)) \mapsto (a_{11}, a_{21}, a_{12}, a_{22}, \dots)$$

For any element  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ ,  $\mathbf{a}$  can be representated by a combination of unique ordered pair  $(\mathbf{a}_1, \mathbf{a}_2)$  where  $\mathbf{a}_1, \mathbf{a}_2 \in \{0, 1\}^{\mathbb{N}}$ . In this way,  $f$  is bijective. Then we can get  $R \cong \{0, 1\}^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}} \cong R \times R$ .  $\square$

**Exercise 2.11.** Consider  $\mathbb{R}^{\mathbb{N}}$ , the set of all infinite sequences  $(r_1, r_2, r_3, \dots)$  of real numbers. Show that  $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$ . **Hint:** Again, use the fact that  $\mathbb{R} \cong \{0, 1\}^{\mathbb{N}}$ .

*Proof.* Since  $R \cong \{0, 1\}^{\mathbb{N}}$ , we need to prove  $(\{0, 1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0, 1\}^{\mathbb{N}}$ .

The element  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$  can be represented as  $(\mathbf{a}^1, \mathbf{a}^2, \dots)$  where  $\mathbf{a}^i \in \{0, 1\}^{\mathbb{N}}$

Also,  $\mathbf{e}$  can be represented as a matrix:

$$\mathbf{e} = \begin{pmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \dots \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & \dots \\ a_1^2 & a_2^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Define a function  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}, (a_1, a_2, a_3, \dots) \mapsto \begin{pmatrix} a_1 & a_3 & a_6 & \dots \\ a_2 & a_5 & \dots & \dots \\ a_4 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

For  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$ , we place each bit of  $\mathbf{a}$  on matrix following the diagonals in turn, then we get a unique  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ .

Also, for each  $\mathbf{e} \in (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ , there is exactly one element  $\mathbf{a} \in \{0, 1\}^{\mathbb{N}}$  such that  $f(\mathbf{a}) = \mathbf{e}$ .

In this sense,  $\{0, 1\}^{\mathbb{N}} \cong (\{0, 1\}^{\mathbb{N}})^{\mathbb{N}}$ , which can be expressed as  $\mathbb{R} \cong \mathbb{R}^{\mathbb{N}}$ .  $\square$

Next, let us view  $\{0, 1\}^{\mathbb{N}}$  as a partial ordering: given two elements  $\mathbf{a}, \mathbf{b} \in \{0, 1\}^{\mathbb{N}}$ , that is, sequences  $\mathbf{a} = (a_1, a_2, \dots)$  and  $\mathbf{b} = (b_1, b_2, \dots)$ , we define  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i \in \mathbb{N}$ . Clearly,  $(0, 0, \dots)$  is the minimum element in this ordering and  $(1, 1, \dots)$  the maximum.

**Exercise 2.12.** Give a countably infinite chain in  $\{0, 1\}^{\mathbb{N}}$ . Remember that a set  $A$  is countably infinite if  $A \cong \mathbb{N}$ .

*Solution.* For set  $A \subseteq \{0, 1\}^{\mathbb{N}}$ , let  $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid \text{if } k < i, a_k = 1, \text{ else, } a_k = 0\}$ , any two elements in  $A$  are comparable

We can define a function  $f$ :

$$f : \mathbb{N} \rightarrow A, x \mapsto \mathbf{a}_x$$

which is obviously bijective. In this way,  $A$  is a countably infinite chain.  $\blacksquare$

**Exercise 2.13.** Find a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ .

*Solution.* For set  $A \subseteq \{0, 1\}^{\mathbb{N}}$ , let  $A = \{\mathbf{a}_i \in \{0, 1\}^{\mathbb{N}} \mid a_i = 1 \text{ and other bits is } 0\}$ . Any two elements in  $A$  is uncomparable since each has one bit larger than the other.

Also, we can define a function  $f$ :

$$f : N \rightarrow A, x \mapsto \mathbf{a}_x$$

which is bijective.

In this sense,  $A$  is a countably infinite antichain in  $\{0, 1\}^{\mathbb{N}}$ . ■

**Exercise 2.14.** Find an uncountable antichain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .

*Solution.* Define a function  $f$ :

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow A, (a_1, a_2, a_3 \dots) \mapsto (a_1, 1 - a_1, a_2, 1 - a_2, \dots)$$

$f$  is a bijective function, and  $\{0, 1\}^{\mathbb{N}} \cong A$

For  $\mathbf{a}, \mathbf{b} \in A$ , if  $\mathbf{a} \neq \mathbf{b}$  and  $\mathbf{a}, \mathbf{b}$  are comparable, let  $\mathbf{a} < \mathbf{b}$ .

There must exist  $a_i < b_i$ , but  $1 - a_i > 1 - b_i$ , so  $\mathbf{a}$  and  $\mathbf{b}$  are uncomparable. Then we get a contradiction.

In this way,  $A$  is a uncountable antichain. ■

**\*\*Exercise 2.15.** Find an uncountable chain in  $\{0, 1\}^{\mathbb{N}}$ . That is, an antichain  $A$  with  $A \cong \mathbb{R}$ .