Mathematical Foundations of Computer Science

CS 499, Shanghai Jiaotong University, Dominik Scheder

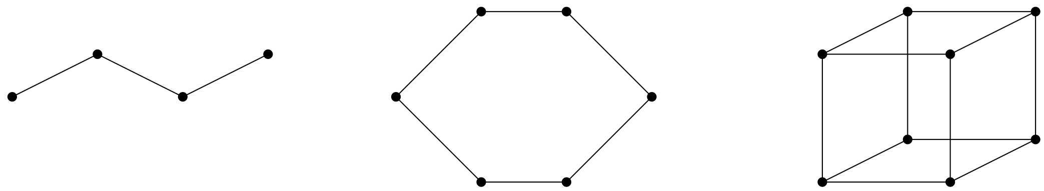
6 Graph Theory Basics

- Homework assignment published on Monday, 2018-04-02.
- Submit first solutions and questions by Sunday, 2018-04-08, 12:00, by email to dominik.scheder@gmail.com and to the TAs.
- You will receive feedback by Wednesday, 2018-04-11.
- Submit final solution by Sunday, 2018-04-15 to me and the TAs.

Let $G = (V, E)$ and $H = (V', E')$ be two graphs. A *graph isomorphism* from G to H is a bijective function $f : V \to V'$ such that for all $u, v \in V$ it holds that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. If such a function exists, we write $G \cong H$ and say that G and H are *isomorphic*. In other words, G and H being isomorphic means that they are identical up to the names of its vertices.

Obviously, every graph G is isomorphic to itself, because the identity function $f(u) = u$ is an isomorphism. However, there might be several isomorphisms f from G to G itself. We call such an isomorphism from G to itself an *automorphism* of G.

Exercise 6.1. For each of the graphs below, compute the number of automorphisms it has.



Justify your answer!

Solution.

1. The number of automorphisms is 2. We denote the vertices v_1, v_2, v_3, v_4 from the left side to the right side, which itself is an automorphisms. The other automorphism is to denote the vertices from the right to the left.

2. The number of automorphisms is $2 \times 6 = 12$. For each vertex, we denote it v_1 and number the remaining vertex clockwise or anticlockwise in order, all of which are automorphisms.

3. The number of automorphisms is $8 \times 6 = 48$. For each vertex, we denote it v_0, and its adjacent three vertices v_1, v_2, v_3, which has $3! = 6$ choices. Each selection can form an automorphism, thus the total number is 48.

■

Consider the n-dimensional Hamming cube H_n. This is the graph with vertex set $\{0,1\}^n$, and two vertices $x, y \in \{0,1\}^n$ are connected by an edge if they differ in exactly one edge. For example, the right-most graph in the figure above is H_3.

Exercise 6.2. Show that H_n has exactly $2^n \cdot n!$ automorphisms. Be careful: it is easy to construct $2^n \cdot n!$ different automorphisms. It is more difficult to show that there are no automorphisms other than those.

Proof.

- There are at least $2^n \cdot n!$ automorphisms. For each vertex we denote it v_0 , and denote its adjacent vertices $v_1, v_2, \ldots, v_n$, which has $n!$ choice. All $2^n \cdot n!$ will at least form one automorphism.

- There are no more than $2^n \cdot n!$ automorphisms. If there are more than $2^n \cdot n!$ automorphisms, then after we choose $v_0, v_1, v_2, \ldots, v_n$, we can still find more than one automorphism when numbering $v_{n+1}, v_{n+2}, \ldots, v_{2^n-1}$.

  Consider the n-dimensional Hamming cube as a n-dimensional Euclidean space, each vertex equals to a point in the Euclidean space with the coordinates $(x_1, x_2, \ldots, x_n)$ ($x_i = 1$ or 0).

  First we choose v_0, we denote it the point $(0, 0, 0, \ldots, 0)$, namely, the origin of the Euclidean space.

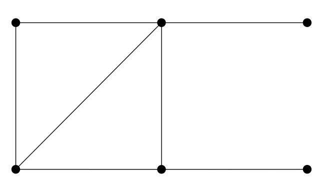
  After choosing the adjacent vertices $v_1, \ldots, v_n$, we have chosen the vertices corresponding to $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$ $\cdots$ $(0, 0, 0, \ldots, 1)$, each of that form a unit vector with v_0: (v_0, v_1), (v_0, v_2) $\cdots$ (v_0, v_n). All the n vectors form a basis of the Euclidean space.

  Once we choose a origin and a basis of the Euclidean space, all 2^n points' coordinates are fixed, which means that the way to number all the vertices is also fixed. Hence there exists no more automorphisms.

$\square$

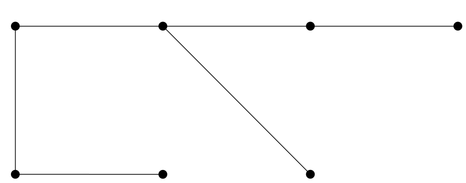
A graph G is called *asymmetric* if the identity function $f(u) = u$ is the only automorphism of G. That is, if G has exactly one automorphism.

Exercise 6.3. Give an example of an asymmetric graph on six vertices.



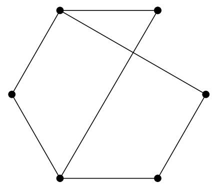
Solution. $\blacksquare$

Exercise 6.4. Find an asymmetric tree.

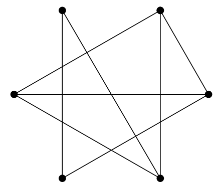


Solution. $\blacksquare$

For a graph $G = (V, E)$, let $\bar{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote its *complement graph.*

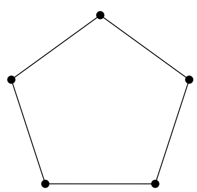


A graph H on six vertices

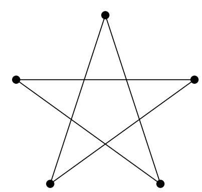


Its complement $\bar{H}$.

We call a graph *self-complementary* if $G \cong \bar{G}$. The above graph is not self-complementary. Here is an example of a self-complementary graph:



The pentagon G.



$\bar{G}$, the pentagram.

Exercise 6.5. Show that there is no self-complementary graph on 999 vertices.

Proof. A complete graph on 999 vertices has $999 \times 998/2 = 498501$ edges, which cannot be divide by 2, that is, the edges cannot be divide into two parts with the same number of edges. $\square$

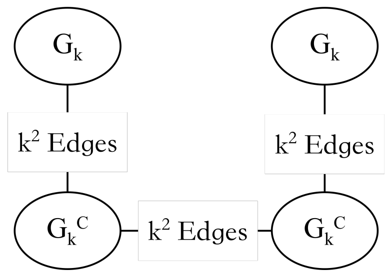
Exercise 6.6. Characterize the natural numbers n for which there is a self-complementary graph G on n vertices. That is, state and prove a theorem of the form "There is a self-complementary graph on n vertices if and only if n `<put some simple criterion here>`."

Theorem. *There is a self-complementary graph on n vertices if and only if $n \mod 4 = 0$ or $n \mod 4 = 1$.*

Proof.

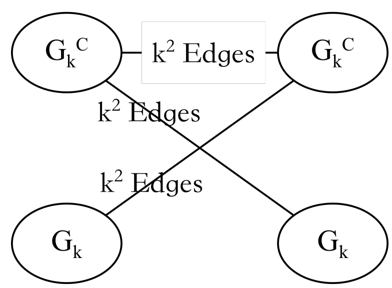
- If $n \mod 4 \neq 0$ and $n \mod 4 \neq 1$, the number of edges of the complete graph $n(n-1)/2/2$ is not an integer, namely the edges of the complete graph cannot be divide into two parts with the same number of edges. Hence there isn't any self-complementary graph on n vertices.

- If $n \mod 4 = 0$, that is $n = 4k$, where k is a positive integer. we denote an graph G_k containing k vertices whose complement is G_k^C. Now we form the following graph G on n vertices



The graph G contains four induced subgraphs G_k, G_k, G_k^C, G_k^C. Between some subgraphs there are edges between each vertex of two subgraphs, whose total number is k^2.

The complementary graph G^C is as follows. Obviously, G^C and G are isomorphic.

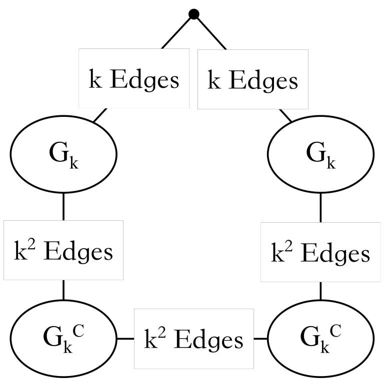


- If $n \mod 4 = 1$, that is $n = 4k + 1$, where k is a positive integer.

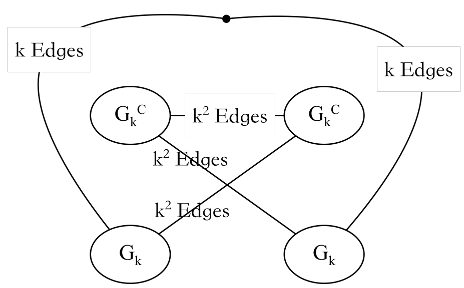
  Based on the graph we formed when $n \mod 4 = 1$, we add one vertex to form a new graph G' on $4k + 1$ vertices and its complement $G^{C'}$.

  Similarly, G' and $G^{C'}$ are isomorphic.

$\square$



k Edges
k Edges
G_k
G_k
k^2 Edges
k^2 Edges
$G_k{}^C$
k^2 Edges
$G_k{}^C$



k Edges
k Edges
$G_k{}^C$
k^2 Edges
$G_k{}^C$
k^2 Edges
k^2 Edges
G_k
G_k