Mathematical Foundations
of
Computer Science

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1 Broken Chessboard and Jumping With Coins

1.1 Tiling a Damaged Checkerboard

Exercise 1.1. *Re-write the proof in your own way, using simple English sentences.*

Proof. Your proof ... $\qquad\square$

Exercise 1.2. *Another exercise ...*

Proof. Your proof ... $\qquad\square$

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1.

1. *Proof.* As is shown in the Venn diagram below, $|A| + |B|$ add the common part $|A \cap B|$ twice. So it should be subtracted once if we want to count $|A \cup B|$. $\qquad\square$
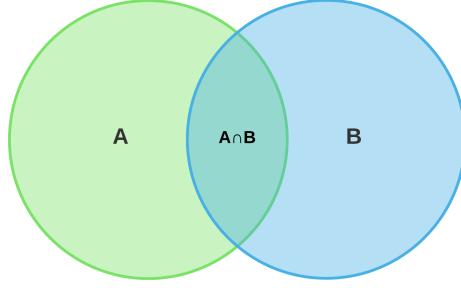
Figure 1: Venn Diagram

2. *Solution.* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ ∎

3. *Solution.* $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$ ∎

Exercise 2.2.

Solution. $|A_1 \cup \ldots \cup A_n| = \sum\limits_{i=1}^{n} |A_i| - \sum\limits_{i,j:1 \le i < j \le n} |A_i \cap A_j| +$

$\sum\limits_{i,j,k:1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$ ∎

Exercise 2.3.

Proof. □

1. proof using induction on n

   First, let $A_{n,k} = \sum\limits_{1 \le i_1 < i_2 < \ldots \le n} |A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{A_1, A_2, \ldots, A_n\}$.

   Then the Inclusion-exclusion principle which we want to prove is as follows:
   $|A_1 \cup \ldots \cup A_n| = \sum\limits_{k=1}^{n} (-1)^{k+1} A_{n,k}$

2

The theorem holds for $n = 1$, obviously.

The theorem holds for $n = 2$, as is showed in the 2.1.1

For the induction step, we want to show if it holds for $n - 1$, then it holds for n.

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left| (\cup_{i=1}^{i=n-1}(A_i \cap A_n) \right|
\end{aligned}
$$

(1)

Let $B_i = (A_i \cap A_n)$.

Similarly, let $B_{n-1,k} = \sum_{1 \leq i_1 < i_2 < \ldots \leq n-1} |B_{i_1} \cap B_{i_2} \ldots \cap B_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{B_1, B_2, \ldots, B_n-1\}$.

(1) now becomes

$$
\sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left| (\cup_{i=1}^{i=n-1} B_i \right|
$$

(2)

Similarly, it holds:

$$
|B_1 \cup \ldots \cup B_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k+1} B_{n-1,k}
$$

(3)

(2) now becomes

$$\sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} + |A_n| + (-1)^k\sum_{k=1}^{n-1}B_{n-1,k} \tag{4}$$

2. proof not using induction on n