Mathematical Foundations
of
Computer Science

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1 Broken Chessboard and Jumping With Coins

1.1 Tiling a Damaged Checkerboard

Exercise 1.1. *Re-write the proof in your own way, using simple English sentences.*

Proof. Your proof ... □

Exercise 1.2. *Another exercise ...*

Proof. Your proof ... □

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1.

1. *Proof.* As is shown in the Venn diagram below, $|A| + |B|$ add the common part $|A \cap B|$ twice. So it should be subtracted once if we want to count $|A \cup B|$. □

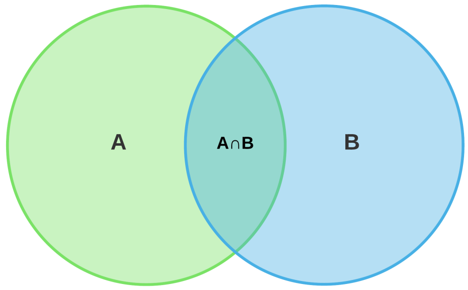

Figure 1: Venn Diagram

2. *Solution.* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ ■

3. *Solution.* $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$ ■

Exercise 2.2.

Solution. $|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j:1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{i,j,k:1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$ ■

Exercise 2.3.

Proof. □

1. proof using induction on n

   First, let $A_{n,k} = \sum_{1 \leq i_1 < i_2 < \ldots \leq n} |A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{A_1, A_2, \ldots, A_n\}$.

   Then the Inclusion-exclusion principle which we want to prove is as follows:
   $|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} A_{n,k}$

The theorem holds for $n = 1$, obviously.

The theorem holds for $n = 2$, as is showed in the 2.1.1

For the induction step, we want to show if it holds for $n - 1$, then it holds for n.

$$\begin{aligned} |A_1 \cup \ldots \cup A_n| &= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\ &= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\ &= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} (A_i \cap A_n)\right| \end{aligned} \tag{1}$$

Let $B_i = (A_i \cap A_n)$.

Similarly, let $B_{n-1,k} = \sum_{1 \le i_1 < i_2 < \ldots \le n-1} |B_{i_1} \cap B_{i_2} \ldots \cap B_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{B_1, B_2, \ldots, B_n - 1\}$.

(1) now becomes

$$\sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} B_i\right| \tag{2}$$

Similarly, it holds:

$$|B_1 \cup \ldots \cup B_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k+1} B_{n-1,k} \tag{3}$$

(2) now becomes

$$\sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} + |A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} \tag{4}$$

In addition,

$$|A_n| = (-1)^{1+1}\,|A_n| \tag{5}$$

Thus,

$$|A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} = \sum_{k=1}^{n}(-1)^{k+1}A_{n,k} - \sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} \tag{6}$$

Then equation(4) finally becomes: $\sum_{k=1}^{n}(-1)^{k+1}A_{n,k}$

2. proof not using induction on n

First, let $A = |A_1 \cup \ldots \cup A_{n-1}|$.
Function $P_S(x)$ defined as if set S includes element x, then $P_S(x) = 1$, else $P_S(x) = 0$.

(1) If $P_A(x) = 1$, there must exist an i that $P_{A_i}(x) = 1$. In this way:
$(P_A(x) - P_{A_1}(x))(P_A(x) - P_{A_2}(x)) \ldots (P_A(x) - P_{A_n}(x)) = 0$

(2) According to the properities of set, $P_{A_i}(x)P_{A_j}(x) = P_{A_i \cap A_j}(x)$.

(3) Let $P_{n,k}$ denotes $P_{A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}} (1 \le i_1 < i_2 < \ldots \le n)$.
Then decompose the first equation, we can have:
$P_A(x) = \sum_{k=1}^{n} P_{n,k}$
which can be demonstrated as:
$|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n}(-1)^{k+1}A_{n,k}$

3 Feasible Intersection Patterns

3.1

Exercise 3.1.

Find sets A_1, A_2, A_3, A_4 such that all pairwise intersections have size 3 and all three-wise intersections have size 1.
Formally, $1.|A_i \bigcap A_j| = 3 for all i, j \in \binom{[4]}{2}$, $2.|A_i \bigcap A_j \bigcap A_k| = 1$ for all $\{i, j, k\} \in \binom{[4]}{3}$.
$A_1 = \{1, 2, 3, 5, 6, 9\}$;
$A_2 = \{1, 2, 3, 4, 7, 8\}$;
$A_3 = \{1, 5, 7, 8, 9, 10\}$;
$A_4 = \{2, 4, 5, 6, 8, 10\}$;

Exercise 3.2.

In the spirit of the previous questions, let us call a sequence $(a_1, a_2, ..., a_n) \in \mathbb{N}_0$ feasible if there are sets $A_1, ..., A_n$ such that all k-wise intersections have size a_k. That is, $|Ai| = a_1$ for all i, $|A_i \bigcap A_j| = a_2$ for all $i \neq j$ and so on. The previous exercise would thus state that $(5, 3, 1, 0)$ is not feasible, but $(6, 3, 1, 0)$ is, as one solution of Exercise 3.1 shows.

Proof. □

Assume that there exist such sets A,B,C,D to which $(5, 3, 1, 0)$ is feasible.
$|A| = |B| = |C| = |D| = 5$;
Since $|A \bigcap B| = 3$,$|A \bigcap B \bigcap C| = 1$,
A and B have 3 same elements. A, B and C have 1.
There are 2 elements in $A \bigcap B$ that are not in C;
Since $|A \bigcap c| = 3$, there are 2 elements in A and C but not in B.
There are 2 elements in B and C but not in A.
So, now we have 1 elements in A, B, C, 2 in A,C but not B, 2 in B,C but not A.
Then, set D has the same requirement with C.
Similarly,We have 1 elements in A, B, D, 2 in A,D but not B, 2 in B,D but not A.
A has only 5 elements. So dose B.
Then there is a contradiction. D will have at least 5 same elements with C.
$|C \bigcap D| = 5 \neq 3$.

Exercise 3.3.

Suppose I give you a sequence $(a_1, ..., a_n)$. Find a way to determine whether such a sequence is feasible or not.

Proof. $\square$

Definition:
Given $A_1, A_2, A_3, ..., A_n$,
define

$$A_{\{I\}} = \bigcap_{i \in I} A_i.$$

In other words,

$$A_{\{i,j,k,...\}} = A_i \bigcap A_j \bigcap A_k \bigcap ...$$

$$a_{|I|} = |A_{\{I\}}|.$$

$$B_{\{I\}} = \bigcap_{i \in I} A_i \cap \bigcap_{j \notin I} \bar{A}_j.$$

In other words,

$$B_{\{I\}} \text{ is the number of elements in } \bigcap_{i \in I} A_i \text{ but not in } \bigcap_{j \notin I} A_j.$$

$$b_{|I|} = |B_{\{I\}}|.$$

Because every set B_i is not divided by existing boundaries, we can put any number of elements in set B_i.

Thus, $(b_1, b_2, ..., b_n)$ is feasible for all situations as long as b_i are nonnegative integers.

Obviously, $(a_1, a_2, ..., a_n)$ has something to do with $(b_1, b_2, ..., b_n)$.We have

$$A_{\{I\}} = \bigcup_{J \geq I} B_{\{J\}};$$

$$\text{Thus, } a_{|I|} = \sum_{J \geq I} b_{|J|} = \sum_{j=i}^{j=n} \binom{n-i}{j-i} b_{|J|}$$

$$\text{Then,} b_i = \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} a_i + k$$

For $(a_1, a_2, ..., a_n)$, replace the a_i into the formula.

If all the results are nonnegative integers.The sequence is feasible.