Mathematical Foundations of Computer Science

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1 Broken Chessboard and Jumping With Coins

1.1 Tiling a Damaged Checkerboard

Exercise 1.1. *Re-write the proof in your own way, using simple English sentences.*

Proof. Your proof ... □

Exercise 1.2. *Another exercise ...*

Proof. Your proof ... □

2 Exclusion-Inclusion

2.1 Sets

Exercise 2.1.

1. *Proof.* As is shown in the Venn diagram below, $|A| + |B|$ add the common part $|A \cap B|$ twice. So it should be subtracted once if we want to count $|A \cup B|$. □

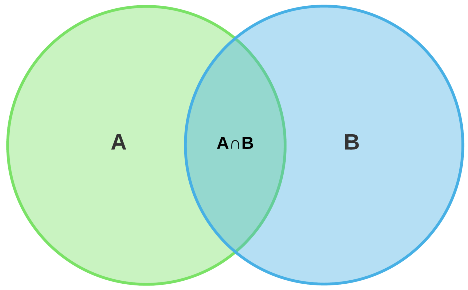

Figure 1: Venn Diagram

2. *Solution.* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ ■

3. *Solution.* $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$ ■

Exercise 2.2.

Solution. $|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j:1 \le i < j \le n} |A_i \cap A_j| + \sum_{i,j,k:1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$ ■

Exercise 2.3.

Proof. □

1. proof using induction on n

   First, let $A_{n,k} = \sum_{1 \le i_1 < i_2 < \ldots \le n} |A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{A_1, A_2, \ldots, A_n\}$.

   Then the Inclusion-exclusion principle which we want to prove is as follows:
   $|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} A_{n,k}$

The theorem holds for $n = 1$, obviously.

The theorem holds for $n = 2$, as is showed in the 2.1.1

For the induction step, we want to show if it holds for $n - 1$, then it holds for n.

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} (A_i \cap A_n)\right|
\end{aligned}
\tag{1}
$$

Let $B_i = (A_i \cap A_n)$.

Similarly, let $B_{n-1,k} = \sum_{1 \le i_1 < i_2 < \ldots \le n-1} |B_{i_1} \cap B_{i_2} \ldots \cap B_{i_k}|$, which denotes the sum of all the possible k-wise intersections in $\{B_1, B_2, \ldots, B_n - 1\}$.

(1) now becomes

$$
\sum_{k=1}^{n-1} (-1)^{k+1} A_{n-1,k} + |A_n| - \left|(\cup_{i=1}^{i=n-1} B_i\right| \tag{2}
$$

Similarly, it holds:

$$
|B_1 \cup \ldots \cup B_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k+1} B_{n-1,k} \tag{3}
$$

(2) now becomes

$$\sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} + |A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} \tag{4}$$

In addition,

$$|A_n| = (-1)^{1+1}\,|A_n| \tag{5}$$

Thus,

$$|A_n| + \sum_{k=1}^{n-1}(-1)^{k}B_{n-1,k} = \sum_{k=1}^{n}(-1)^{k+1}A_{n,k} - \sum_{k=1}^{n-1}(-1)^{k+1}A_{n-1,k} \tag{6}$$

Then equation(4) finally becomes: $\sum\limits_{k=1}^{n}(-1)^{k+1}A_{n,k}$

2. proof not using induction on n

   First, let $A = |A_1 \cup \ldots \cup A_{n-1}|$.
   Function $P_S(x)$ defined as if set S includes element x, then $P_S(x) = 1$, else $P_S(x) = 0$.

   (1) If $P_A(x) = 1$, there must exist an i that $P_{A_i}(x) = 1$. In this way:
   $(P_A(x) - P_{A_1}(x))(P_A(x) - P_{A_2}(x)) \ldots (P_A(x) - P_{A_n}(x)) = 0$

   (2) According to the properities of set, $P_{A_i}(x)P_{A_j}(x) = P_{A_i \cap A_j}(x)$.

   (3) Let $P_{n,k}$ denotes $P_{A_{i_1} \cap A_{i_2} \ldots \cap A_{i_k}}(1 \leq i_1 < i_2 < \ldots \leq n)$.
   Then decompose the first equation, we can have:
   $P_A(x) = \sum\limits_{k=1}^{n} P_{n,k}$
   which can be demonstrated as:
   $|A_1 \cup \ldots \cup A_n| = \sum\limits_{k=1}^{n}(-1)^{k+1}A_{n,k}$

3 Feasible Intersection Patterns

3.1

Exercise 3.1.

Find sets A1; A2; A3; A4 such that all pairwise intersections have size 3 and all three-wise intersections have size 1.
Formally, 1.$|A_i \bigcap A_j| = 3 for all i, j \in \binom{[4]}{2}$), 2.$|A_i \bigcap A_j \bigcap A_k| = 1$ for all $\{i, j, k\} \in \binom{[4]}{3}$).

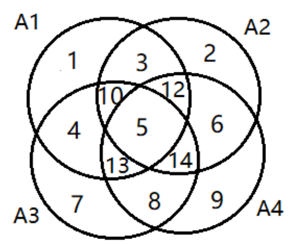

Figure 2: Area Diagram

Proof. As is shown in figure 2. From $|A_i \bigcap A_j| = 3$ for all $i, j \in \binom{[4]}{2}$).
We can infer that Domain$\{3, 10, 12, 5\}$, $\{4, 5, 10, 13\}$,$\{5, 6, 12, 14\}$,$\{5, 8, 13, 14\}$ has each 3 elements.
From$|A_i \bigcap A_j \bigcap A_k| = 1$ for all $\{i, j, k\} \in \binom{[4]}{3}$)

We can infer that Domain$\{5, 10\}$, $\{5, 12\}$, $\{5, 13\}$, $\{5, 14\}$ has each 1 elements. $\{5, 8, 13, 14\}$have 3 elements, and $\{5, 13\}$, $\{5, 14\}$ has each 1 element.
For one thing,Domain$\{8\}\{13\}\{14\}$ is empty and Domain 5 has one element.
Then it is obvious that there is 1 in$\{5\}$, 0 in$\{10\}\{12\}\{13\}\{14\}$, 2 in $\{3\}\{4\}\{6\}\{8\}$, and arbitrary number in $\{1\}\{2\}\{7\}\{8\}$.
As it is shown in the left figure in Figure 3.

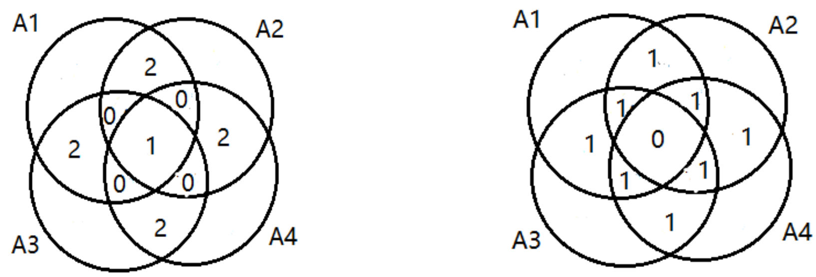

Figure 3: Distribution

For another thing, Domain{5}has no element.
And then Domain{10}{12}{13}{14}{3}{4}{6}{8}has each 1 element.

It is shown in the right one in Figure 3. □

Exercise 3.2.

In the spirit of the previous questions, let us call a sequence $(a_1, a_2, ..., a_n) \in \mathbb{N}_0$ feasible if there are sets $A_1, ..., A_n$ such that all k-wise intersections have size a_k. That is, $|Ai| = a_1$ for all i, $|A_i \bigcap A_j| = a_2$ for all $i \neq j$ and so on. The previous exercise would thus state that $(5, 3, 1, 0)$ is not feasible, but $(6, 3, 1, 0)$ is, as one solution of Exercise 3.1 shows.

Proof. Because $|A_1 \bigcap A_2 \bigcap A_3 \bigcap A_4| = 0$.
It is the same as the second situation.
From picture 3 in 3.1 we can know that there are at least 6 elements in A_i.
So $(5, 3, 1, 0)$ is not feasible. □

Exercise 3.3.

Suppose I give you a sequence $(a_1, ..., a_n)$. Find a way to determine whether such a sequence is feasible or not.

Proof. Given $A_1, ..., A_n I \subseteq 1, ..., n$ define $A_i = \bigcap\limits_{i \in I} A_i$.
Given $B_1, ..., B_n I \subseteq 1, ..., n$ define $B_i = \bigcap\limits_{i \in I} A_i \bigcup\limits_{j \notin A_i}$.
Use A_1, A_2, A_3 as an example,draw the picture below.

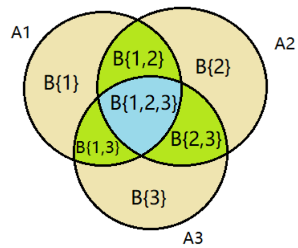

Figure 4: Example

Obviously, iff for every i,$B_i \geq 0$,the B-table is feasible.
For given $(a_1, ..., a_n), A1 = a_1, Ai, j(i \neq j)a_2, ..., Ai, 2, ..., n = a_n$.
Besides,for all $i, i, j(i \neq j), i, j...k$(i,j,...k are different) Ahas the same value.
So we can infer that $A_i = A\sum_{j=1}^{i-1} A_j$. $A_1, ..., A_n = a_1, ..., a_n$.
So iff $A_i = A\sum_{j=1}^{i-1} A_j$,the sequence is feasible. $\square$

4 third part

Exercise 4.1.

Given $A_1, A_2, \ldots, A_n$ and $I \subseteq [n]$, I is not empty. define $B_I = (\cap_{i\in I} A_i)$ $(\cup_{j\notin I} A_j)$. That is the elements that are in every $A_i, in \in I$ but in no other $A_j, j \in [n]$
I

1.Solve 3.3". Given a B-table, how to determine whether it is feasible.

2.Given a feasible B-table,how to compute A-table.

2.Given an A-table,find a way to compute the B-table. and then apply 1.