Lineaire Algebra 1 (2MBA20)

Worked examples on vector geometry (Ch. 1)

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Contents

1 Introduction

This document contains a number of completely solved problems on the material in Chapter 1 of the lecture notes Linear Algebra 1. I should emphasize however that exercises can often be solved in more than one way, and only one solution is provided. The solutions also show how to write down these solutions in typical mathematical style.

2 Vectors, vector descriptions of lines, planes

1. *Suppose $\underline{a}$, $\underline{b}$ and $\underline{c}$ are three vectors in 3-space which are not in one plane through the origin. Provide a vector parametric description of the plane V through (the 'endpoints' of) the three vectors.*

   Solution. (Make a picture to visualize the situation.) We need a position (or support) vector and two direction vectors. We take as position vector the vector $\underline{a}$. As direction vectors we take $\underline{b}-\underline{a}$ and $\underline{c}-\underline{a}$. This way we arrive at the vector parametric description

   $$V : \underline{x} = \underline{a} + \lambda(\underline{b}-\underline{a}) + \mu(\underline{c}-\underline{a}).$$

   We check that all three vectors are on it: for $\underline{a}$ take $\lambda = \mu = 0$. For $\underline{b}$ take $\lambda = 1$, $\mu = 0$. For $\underline{c}$ take $\lambda = 0$ and $\mu = 1$.
   We also need to check that $\underline{b}-\underline{a}$ and $\underline{c}-\underline{a}$ are not multiples of one another. So suppose first that $\underline{b}-\underline{a} = t(\underline{c}-\underline{a})$ for some t. Then $\underline{b} = (1-t)\underline{a} + t\underline{c}$, so $\underline{b}$ is in a plane through the origin and containing $\underline{a}$ and $\underline{c}$ contradicting the assumption on $\underline{a}$, $\underline{b}$, $\underline{c}$. So $\underline{b}-\underline{a}$ is not a multiple of $\underline{c}-\underline{a}$. In a similar way you show that $\underline{c}-\underline{a}$ is not a multiple of $\underline{b}-\underline{a}$.
   Remark. Of course, $\underline{b}$ and $\underline{c}$ are also good and easy to find choices for the position vector. Similarly, the vector $\underline{c}-\underline{b}$ can be used as a direction vector (instead of one of the other two). Note that the direction vectors should be linearly independent, i.e., not multiples of one another. Linear independence is discussed in generality in Chapter 4.

2. *Determine a vector parametric equation of the plane with equation $x + y - 2z = 3$.*
   Solution. We solve this problem by solving the equation and rewriting the solutions in vector language. To solve the equation, note that if we assign any value to y and any value to z then there is exactly one corresponding value for x. Every solution is obtained this way: if (u, v, w) is a solution, then take y to be v and z to be w; then x must be u. So assign arbitrary values λ to z and μ to y, then $x = 3 - \mu + 2\lambda$. So the solutions of the equation are given by the vectors (written horizontally)

   $$\underline{x} = (3 - \mu + 2\lambda, \mu, \lambda).$$

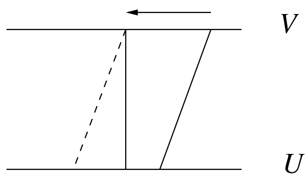
   Rewriting this as $\underline{x} = (3, 0, 0) + \lambda(2, 0, 1) + \mu(-1, 1, 0)$ we have our vector parametric description with position vector $(3, 0, 0)$ and direction vectors $(2, 0, 1)$ and $(-1, 1, 0)$.

Remark 1. There are more ways to solve this problem. For instance, take three vectors (but be careful, not just any three; why?) on the plane and construct a position vector and two direction vectors from them. Or find two independent vectors perpendicular to $(1, 1, -2)$ which serve as direction vectors.
Remark 2. More on solving linear equations in Chapter 3 on systems of linear equations.

3 Distances, angles, and the inner product

1. *If $\underline{a}$, $\underline{b}$, $\underline{c}$ are mutually perpendicular vectors in 3-space (so $\underline{a} \perp \underline{b}$, $\underline{a} \perp \underline{c}$ and $\underline{b} \perp \underline{c}$), then*
$$\| \underline{a} + \underline{b} + \underline{c} \|^2 = \| \underline{a} \|^2 + \| \underline{b} \|^2 + \| \underline{c} \|^2$$
Prove this.
Solution. We use Pythagoras and apply it first to the vectors $\underline{a} + \underline{b}$ and $\underline{c}$. To apply Pythagoras we need to know that $\underline{a} + \underline{b}$ and $\underline{c}$ are perpendicular. So we compute their inner product
$$(\underline{a} + \underline{b}, \underline{c}) = (\underline{a}, \underline{c}) + (\underline{b}, \underline{c}) = 0 + 0 = 0.$$
Since the vectors are perpendicular, Pythagoras then implies
$$\| (\underline{a} + \underline{b}) + \underline{c} \|^2 = \| \underline{a} + \underline{b} \|^2 + \| \underline{c} \|^2 .$$
Next we apply Pythagoras to the perpendicular vectors $\underline{a}$ and $\underline{b}$ and find:
$$\| \underline{a} + \underline{b} \|^2 = \| \underline{a} \|^2 + \| \underline{b} \|^2 .$$
Together:
$$\| \underline{a} + \underline{b} + \underline{c} \|^2 = \| \underline{a} \|^2 + \| \underline{b} \|^2 + \| \underline{c} \|^2 .$$

2. *Determine the distance between the planes $U : x + y + z = 3$ and $V : x + y + z = 6$, i.e., the minimal distance between a vector of U and a vector of V.*
Solution. (Making a good sketch is really helpful!) Note that the planes are parallel. A normal vector for both is $(1, 1, 1)$. Our strategy is to find vectors $\underline{p}$ and $\underline{q}$, with $\underline{p}$ in U and $\underline{q}$ in V, so that the vector $\underline{q} - \underline{p}$ is perpendicular to the planes. The distance is then equal to $\| \underline{q} - \underline{p} \|$. To carry out this strategy we use a line through a vector on U and perpendicular to both planes.
Choose $\underline{p} = (3, 0, 0)$ in U and consider the line $\ell : \underline{x} = (3, 0, 0) + \lambda(1, 1, 1)$ perpendicular to both planes. Now intersect with V. This leads to the equation (substituting $3 + \lambda$, λ, λ for x, y, z respectively in the equation of V): $(3 + \lambda) + \lambda + \lambda = 6$ for λ. So $\lambda = 1$. We then obtain $\underline{q} = (3, 0, 0) + 1 \cdot (1, 1, 1) = (4, 1, 1)$ on V. The distance $\| \underline{q} - \underline{p} \|$ is
$$\| \underline{q} - \underline{p} \| = \| (1, 1, 1) \| = \sqrt{3}.$$
Remark. If you want to prove that this is really the minimum distance between a vector from U and one from V, then you could reason as follows.



Let $\underline{s}$ be an arbitrary vector in V and $\underline{r}$ one in U. The distance between these two vectors doesn't change if we add the same vector to both (why?). Now we add $\underline{q}-\underline{s}$ to both so that the first new vector becomes $\underline{q}$. This will simplify matters. Since $\underline{q}-\underline{s}$ is a direction vector of V and of U, the two new vectors, $\underline{q}$ and $\underline{r}+\underline{q}-\underline{s}$ are in V and U, respectively. Let's put $\underline{t}=\underline{r}+\underline{q}-\underline{s}$. So $\underline{t}$ is in U.

Then $\underline{p}-\underline{t}$ is the difference of two vectors in U and so perpendicular to $(1,1,1)$. So $\underline{p}-\underline{t}\perp\underline{q}-\underline{p}$. Their sum is $\underline{q}-\underline{t}$. By Pythagoras we then get

$$\| \underline{q}-\underline{t} \|^2 = \| \underline{q}-\underline{p} \|^2 + \| \underline{p}-\underline{t} \|^2 .$$

So $\| \underline{q}-\underline{t} \| \geq \| \underline{q}-\underline{p} \|$. Taking it all together:

$$\| \underline{s}-\underline{r} \| = \| \underline{q}-\underline{t} \| \geq \| \underline{q}-\underline{p} \| .$$

4 Geometry with vectors

1. *Prove that if the vectors $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ are perpendicular, then the vectors $\underline{a}$ and $\underline{b}$ have the same length.*
   Solution. We need to prove that $\| \underline{a} \| = \| \underline{b} \|$. We use the inner product and its properties to exploit the perpendicularity condition. Now

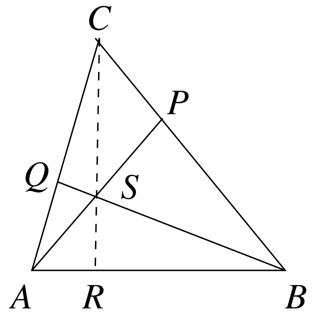
$$\underline{a}+\underline{b}\perp\underline{a}-\underline{b} \Leftrightarrow (\underline{a}+\underline{b},\underline{a}-\underline{b})=0.$$

   Since (for the 2nd =-sign use the properties of the inner product)

$$0=(\underline{a}+\underline{b},\underline{a}-\underline{b})=(\underline{a},\underline{a})-(\underline{b},\underline{b})-(\underline{a},\underline{b})+(\underline{b},\underline{a})=\| \underline{a} \|^2-\| \underline{b} \|^2$$

   we find that $\| \underline{a} \|^2 = \| \underline{b} \|^2$ and hence $\| \underline{a} \| = \| \underline{b} \|$ (here we use that the two lengths are nonnegative).
   Remark. In the last line we infer from $a^2=b^2$ that $a=b$. This is a valid conclusion since a and b are lengths and so nonnegative numbers.

2. *In $\triangle ABC$ in the plane, point P is on segment BC such that $|BP|:|PC|=2:1$, and point Q is on segment AC such that $|CQ|:|QA|=2:1$. The lines AP and BQ intersect in a point S. The line CS intersects AB in point R. Determine the ratio $|AR|:|RB|$.*
   Remark. We have used the notation $|BP|$ for the distance between B and P. It is common to simply use BP if it is clear from the context that the length of the segment is meant. So note that, depending on the context, BP may refer to the line through B and P, the segment between the two points (including B and P), or the length of the segment.



Solution. We use vectors and choose the origin in A. We denote by $\underline{b}$ te vector corresponding to B, etc. The strategy is to use vector parametric equations for the various lines. First we write the various vectors in terms of $\underline{b}$ and $\underline{c}$. Now $\underline{p} = \underline{b} + \frac{2}{3}(\underline{c} - \underline{b}) = \frac{1}{3}\underline{b} + \frac{2}{3}\underline{c}$. Likewise $\underline{q} = \frac{1}{3}\underline{c}$ (note that $\underline{a} = \underline{0}$). Then vector parametric descriptions of AP and BQ are:

$$AP : \underline{x} = \lambda(\frac{1}{3}\underline{b} + \frac{2}{3}\underline{c}), \quad BQ : \underline{x} = \underline{b} + \mu(\frac{1}{3}\underline{c} - \underline{b}).$$

Now we intersect the two lines. To do this we first look for λ and μ such that

$$\lambda(\frac{1}{3}\underline{b} + \frac{2}{3}\underline{c}) = \underline{b} + \mu(\frac{1}{3}\underline{c} - \underline{b}).$$

This leads to $(\lambda - 3 + 3\mu)\underline{b} = (\mu - 2\lambda)\underline{c}$. Since $\underline{b}$ and $\underline{c}$ are not multiples of one another (since we assume of course that the triangle is 'non-degenerate'), we conclude that $\lambda - 3 + 3\mu = 0$ and $\mu = 2\lambda$. This leads to the solution $\lambda = 3/7$ (and $\mu = 6/7$). So the point S corresponds to $\underline{s} = \frac{1}{7}\underline{b} + \frac{2}{7}\underline{c}$.
The line CS has vector parametric description

$$CS : \underline{x} = \underline{c} + \rho(\frac{1}{7}\underline{b} + \frac{2}{7}\underline{c} - \underline{c})$$

or $CS : \underline{x} = \underline{c} + \rho(\frac{1}{7}\underline{b} - \frac{5}{7}\underline{c})$. To intersect with AB we look for a value of ρ so that $\underline{c} - \frac{5}{7}\rho\underline{c} = 0$. We conclude that $\rho = \frac{7}{5}$ and that the vector $\underline{r}$ corresponding to R equals

$$\underline{r} = \frac{1}{5}\underline{b}.$$

So $\underline{b} - \underline{r} = \frac{4}{5}\underline{b}$ and therefore $|AR| : |RB| = 1 : 4$.
Remark. Taking the origin in A simplifies the computations. Without such a choice you would find that S corresponds to

$$\frac{4}{7}\underline{a} + \frac{1}{7}\underline{b} + \frac{2}{7}\underline{c}$$

and R corresponds to

$$\underline{r} = \frac{4}{5}\underline{a} + \frac{1}{5}\underline{b}.$$

5 Geometry with complex numbers

By the way, in the chapters on vectors we will encounter more ways of dealing with geometry problems. Sometimes, vectors are more suited for a certain problem.

1. *Use the complex numbers to prove that the two diagonals of a square $ABCD$ are perpendicular.*

   Solution. Put the origin in A. The idea is to find complex numbers representing the diagonals and to show that one of the diagonals can be obtained from the other by a multiplicaton by i (or $-i$).
   Step 1. Introducing complex numbers.
   Let b be the complex number corresponding to B, and let c the complex number corresponding to C, and let d the complex number corresponding to D. Since AB and AD are perpendicular and of equal length, we have $d = ib$ or $d = -ib$. We continue the proof in the case $d = ib$ (the other case is similar, or just relabel the vertices). Then $c = b + ib$.
   Step 2: describing the diagonals.
   The diagonal AC is represented by the complex number $b + ib - 0 = b + ib$, and the diagonal BD is represented by the complex number $ib - b$. Now $i(b + ib) = ib - b$, so indeed the diagonal AC is transformed into BD by a rotation over $90°$.
   Remark. Note that we use that if F and G are two (distinct) points in the plane, and if f and g are the corresponding complex numbers, then $|g - f|$ is the length of the segment FG and $\arg(g - f) \pmod{2\pi}$ is the angle FG makes with the positive real axis.

2. *The line ℓ given by $\mathrm{Re}(z) = \mathrm{Im}(z)$ makes an angle of $\pi/4$ radians with the positive real axis. Use rotations over $\pm\pi/4$ radians and the fact that reflecting in the real axis corresponds to complex comjugation, to show that the reflection in the line ℓ transforms each complex number z into $i \cdot \overline{z}$.*

   Solution. The reflection can also be seen as the following composition of transformations: First rotate around 0 over an angle of $-\pi/4$ radians, then reflect in the real axis, and finally rotate around 0 over $\pi/4$ radians.
   First step: z transforms into $z \cdot e^{-i\pi/4}$.
   Second step: $z \cdot e^{-i\pi/4}$ transforms into $\overline{z \cdot e^{-i\pi/4}} = \overline{z} \cdot e^{i\pi/4}$.
   Finally, $\overline{z} \cdot e^{i\pi/4}$ is transformed into $\overline{z} \cdot e^{i\pi/4} \cdot e^{i\pi/4} = i\overline{z}$.

3. *If we rotate z around u over an angle of t radians in the positive direction (counter clockwise), then z is transformed into $e^{it}(z - u) + u$. Prove this.*

   Solution. We use the fact that rotating over an angle of t radians around the origin corresponds to the transformation $z \mapsto e^{i}t \cdot z$.

   To rotate z around u, first translate z back: $z - u$. Then rotate $z - u$ around the origin to get $e^{it}(z - u)$. Finally, translate over u. The final result is $e^{it}(z - u) + u$.