Linear Algebra

Jiaqi Wang

October 1, 2023

1 Vectors in two and three dimensions

1.1 Vectors

Definition 1.1 – Linear combination Let $\underline{\mathrm{v}}_1$, $\underline{\mathrm{v}}_2$, $\ldots$, $\underline{\mathrm{v}}_n$ be n vectors and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be scalars. A vector of the form

$$\lambda_1 \underline{\mathrm{v}}_1 + \lambda_2 \underline{\mathrm{v}}_2 + \ldots \lambda_n \underline{\mathrm{v}}_n$$

is called *linear combination* of the vectors $\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \ldots, \underline{\mathrm{v}}_n$.

1.2 Vector descriptions of lines and planes

1.2.1 Lines

$$l : \underline{\mathrm{x}} = \underline{\mathrm{v}} + \lambda \underline{\mathrm{u}}$$

where $\underline{\mathrm{v}}$ lands on the line and is called the *position vector*, and $\underline{\mathrm{u}}$ is vector that is in the "direction" of the line, called the direction vector.

1.2.2 Plane

$$V : \underline{\mathrm{x}} = \underline{\mathrm{u}} + \lambda \underline{\mathrm{v}} + \mu \underline{\mathrm{w}}$$

where $\underline{\mathrm{u}}$ is a vector that lands on the plane, and $\underline{\mathrm{v}}$ and $\underline{\mathrm{w}}$ are two linearly independant vectors in the plane.

1.3 Bases, coordinates, and equations

1.3.1 Basis a coordinates

- The plane
  The set of two vectors $\underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2$ that are not a multiple of each other (linearly independant) is called a basis.
  Every vector $\underline{\mathrm{v}}$ can be expressed as a linear combination of the vectors $\underline{\mathrm{e}}_1$ and $\underline{\mathrm{e}}_2$.

  $$\underline{\mathrm{v}} = v_1 \underline{\mathrm{e}}_1 + v_2 \underline{\mathrm{e}}_2$$

  A basis where the two vectors are with length 1 and perpendicular is called an *orthonormal basis*.

- 3-dimensional space
  In space we choose three vectors $\underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2, \underline{\mathrm{e}}_3$ that are not coplanar. Then any vector $\underline{\mathrm{v}}$ can be written as a linear combination of these three vectors:

  $$\underline{\mathrm{v}} = v_1 \underline{\mathrm{e}}_1 + v_2 \underline{\mathrm{e}}_2 + v_3 \underline{\mathrm{e}}_3$$

1.3.2 The vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$

1.3.3 Describing lines in the plane with coordinates

$$\ell : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

1.3.4 Describing lines in space with coordinates

$$\ell : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

1.3.5 Describing planes in space with coordinates

- Parametric descriptions in 'row notation'

  $$V : (x_1, x_2, x_3) = (a_1, a_2, a_3) + \mu(u_1, u_2, u_3) + \lambda(v_1, v_2, v_3)$$

- Parametric descriptions in 'column notation'

  $$V : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- Each coordinate separately

$$x_1 = a_1 + \lambda u_1 + \mu v_1$$
$$x_2 = a_2 + \lambda u_2 + \mu v_2$$
$$x_3 = a_3 + \lambda u_3 + \mu v_3$$

1.4 Distances, Angles and the Inner Product

Definition 1.2

$$\|\underline{\mathrm{x}}\| \text{ (length of vector } \underline{\mathrm{x}})$$

$$\|\underline{\mathrm{x}} - \underline{\mathrm{y}}\| \text{ (distance between } \underline{\mathrm{x}}, \underline{\mathrm{y}})$$

! If $\underline{\mathrm{a}}$ and $\underline{\mathrm{b}}$ are perpendicular, then

$$\|\underline{\mathrm{a}} - \underline{\mathrm{b}}\| = \|a\|^2 + \|b\|^2$$

$$\|\underline{a} + \underline{b}\| = \|\underline{a}\|^2 + \|\underline{b}\|^2$$

Definition 1.3 – (Inner product) The inner product of two vectors $\underline{a} = (a_1, a_2)$ and $\underline{b} = (b_1, b_2)$ (or $\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$ in space) is defined as:

$$a_1 b_1 + a_2 b_2 \text{ (or in space } a_1 b_1 + a_2 b_2 + a_3 b_3).$$

It is denoted by $(\underline{a}, \underline{b})$.

2 Matrices and systems of linear equation

2.1 Matrices

2.1.1 What is a matrix?

Definition 2.1 – Matrix A *matrix* is a rectangular array of numbers or elements from some arithmetical structure, like

$$A = \begin{pmatrix} 1 & 0 & 4 & -2 \\ 0 & 2 & 0 & 1 \end{pmatrix}$$

2.1.2 Matrix arithmetic: addition

Add each element of matrix A to the corresponding element to matrix B.

2.1.3 Properties

1. $A + B = B + A$ (commutativity)
2. $(A + B) + C = A + (B + C)$ (associativity)

2.1.4 Matrix arithmetic: scalar multiplication

Let λ be a scalar and A be a $m \times n$-matrix. The matrix λA is obtained by multiplying *every* element of A with λ.

2.1.5 Matrix arithmetic: multiplication

We only define the product AB of two matrices A and B if the *row* of A have the same length as the columns of B

! AB and BA are not necessarily the same. Thus matrix multiplication is not commutative

2.1.6 Property

For matrices with correc dimensions, various arithmetic rules hold.

$$A(B + C) = AB + AC$$

$$(E + F)G = EG + FG$$
$$(\lambda A)B = \lambda(AB)$$
$$\lambda(\mu A) = (\lambda\mu)A$$
$$(AB)C = A(BC)$$

2.1.7 Zero matrix

$$O = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

2.1.8 The opposite matrix

The opposite matrix of A is $(-1)A$ or $-A$. And it satisfies $A + (-A) = O$.

2.1.9 The identity matrix

The $n \times n$-matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

It satisfies $IA = AI = A$ for every $n \times n$-matrix A.

2.1.10 The inverse matrix

A^{-1} is called the *inverse* of A if $A \times A^{-1} = I$ and A is called *invertible*.

2.1.11 Property

1. Let A and B be invertible $n \times n$ matrices. Then the prodcut AB is also invertible and
$$(AB)^{-1} = A^{-1}B^{-1}$$
2. Let A be an invertible n by n matrix and m be a positive integer. Then A^m is invertible
$$(A^m)^{-1} = (A^{-1})^m = A^{-m}$$
3. $A^k \cdot A^l = A^{k+l}$

2.1.12 The transpose of a matrix

If $A = (a_{ij})$ is an $m \times n$ matrix, then its *transpose* A^T is the $n \times m$ matrix whose entries in position i, j equal a_{ji}.
A matrix A is called symmetric if $A = A^T$

2.1.13 Property

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ (\lambda A)^T &= \lambda A^T \\ (AB)^T &= B^T A^T \\ (A^T)^T &= A\end{aligned}$$

2.1.14 Matrix Algebra rules

1.	$A + B = B + A$	Commutative Law of Addition
2.	$A + (B + C) = (A + B) + C$	Associative Law of Addition
3.	$\lambda(A + B)\lambda A + \lambda B$, where $\lambda \in \mathbb{R}$	Distributive Law of a Scalar over Matrices
4.	$(\lambda + \mu)A = \lambda A + \mu A$, where $\lambda, \mu \in \mathbb{R}$	Distributive Law of Scalars over a Matrix
5.	$\lambda(\mu A) = (\lambda \cdot \mu)A$, where $\lambda, \mu \in \mathbb{R}$	Associative Law of Scalar Multiplication
6.	$OA = O$, where O is the zero matrix	Zero Matrix Annihilates all Products
7.	$0A = O$, where 0 on the left is the scalar 0	Zero Scalar Annihilates all Products
8.	$A + O = A$	Zero Matrix is an identity for Addition
9.	$A + (-1)A = O$	Negation produced additive inverses
10.	$(B + C)A = BA + CA$	Right Distributive Law of Matrix Multiplication
11.	$A(B + C) = AB + AC$	Left Distributive Law of Matrix Multiplication
12.	$A(BC) = (AB)C$	Associative Law of Matrix Multiplication
13.	$IA = A$ and $AI = A$	Identity Matrix is a Multiplicative Identity
14.	If A^{-1} exists, $(A^{-1})^{-1} = A$	Involution Property of Inverses
15.	If A^{-1} and B^{-1} exist, $(AB)^{-1} = B^{-1}A^{-1}$	Inverse of Product Rule

2.2 Row reduction

2.2.1 Elementary row operations

1. Interchange the order of the rows.
2. Multiply every entry in a row by a non-zero constant.
3. Replace a row by the sum of this row and a scalar multiple of another row.

2.2.2 Reduced row echalon form

A matrix in reduced row ecchalon form has the following properties:

- Every row starts with (possibly zero) zeros.
- Its first non-zero entry (if there is any) is 1 (its leading entry). The column containing this 1 has zeros in all other entries.
- Every non-zero row starts with more zeros than the row directly above it. In particular, if there are any 'zero rows', they are all below the non-zero rows.

2.3 Systems of linear equations

Definition 2.2 – (System of linear equations) A linear equation in the variables $x_1, x_2, \dots, x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \dots, a_n, b$ are scalars from the field, coefficients of the equation.

A *system of linear equations* in the unknowns $x_1, \dots, x_2$ consists of m such equations:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

The matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

is called the *coefficient matrix* and the row $\underline{b} = (b_1, \dots, b_m)$ is called the *right-hand side.*

If $b_i = 0$ for all i then the system is called *homogeneous*, and otherwise is called *inhomogeneous.*

3 Vector spaces and linear subspaces

3.1 Vector spaces and linear subspaces

Definition 3.1 – (Vector space) Let V be a non-empty set whose elements we'll call vectors and will denote like vectors $\underline{a}$. Suppose a map $V \times V \to V$ and a map $\mathbb{K} \times V \to V$ are given. Denote the result of applying the first map to $\underline{u}$ and $\underline{v}$ by $\underline{u} \oplus \underline{v}$. Denote the result of applying the second map to the scalar λ and $u \in V$ by $\lambda * \underline{u}$. Suppose these operations satisfy the vector space axioms for all $\underline{u}, \underline{v}$ and $\underline{w} \in V$ and $\lambda, \mu \in \mathbb{K}$, then V is called a $\mathbb{K}$-vector space or a vector space over $\mathbb{K}$, the operation $\oplus$ is called *vector addition*, and the operation * is called *scalar multiplication*. In the case $\mathbb{K} = \mathbb{R}$ we also speak of a *real vector space*, in the case $\mathbb{K} = \mathbb{C}$ - of a *complex vector space*.

3.1.1 Axioms of vector space

1. $\underline{u} \oplus \underline{v} = \underline{v} \oplus \underline{u}$ (commutativity)
2. $(\underline{u} \oplus \underline{v}) \oplus \underline{w} = \underline{u} \oplus (\underline{v} \oplus \underline{w})$ (associativity)
3. There is a *zero vector* $\underline{0}$ with the property $\underline{v} \oplus \underline{0} = \underline{v}$
4. Every vector $\underline{u}$ has an opposite $-\underline{u}$ such that $\underline{u} \oplus -\underline{u} = \underline{0}$
5. $1 * \underline{u} = \underline{u}$
6. $(\lambda\mu) * \underline{u} = \lambda * (\mu * \underline{u})$
7. $(\lambda + \mu) * \underline{u} = \lambda * \underline{u} + \mu * \underline{u}$ (distributivity)
8. $\lambda * (\underline{u} \oplus \underline{v}) = \lambda * \underline{u} + \lambda * \underline{v}$ (distributivity)

3.1.2 Additional arithmetic rules

1. There is pricisely one zero vector in V
2. The opposite of a vector is unique
3. For every vector $\underline{v} \in V$ we have $0 \cdot \underline{v} = \underline{0}$
4. For every scalar λ we have $\lambda \cdot \underline{0} = \underline{0}$

3.2 Spans, linearly (in)dependant systems

Definition 3.2 – (Span of vectors) Given $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ vectors in V.
Span of $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$: $< \underline{a}_1, \underline{a}_2, \ldots, \underline{a}_3 >= \{\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \cdots + \lambda_n \underline{a}_n | \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$

Theorem 3.1 – (Spans are linear subspaces) If $a_1, \ldots, a_n$ are vectors in the vector space V, then $< a_1, \ldots, a_n >$ is a linear subspace of V .

Proof. Of course, the span is non-empty (it contains the zero vector, obtained by taking all scalars equal to 0).
Now let $\underline{p}$ and $\underline{q}$ be vectors in $< \underline{a}_1, \ldots, \underline{a}_n >$ and suppose

$$\underline{p} = p_1 \underline{a}_1 + p_2 \underline{a}_2 + \cdots + p_n \underline{a}_n \text{ and } \underline{q} = q_1 \underline{a}_1 + q_2 \underline{a}_2 + \cdots + q_n \underline{a}_n.$$

Then

$$\begin{aligned}\underline{p} + \underline{q} &= (p_1 \underline{a}_1 + p_2 \underline{a}_2 + \cdots + p_n \underline{a}_n) + (q_1 \underline{a}_1 + q_2 \underline{a}_2 + \cdots + q_n \underline{a}_n) \\ &= (p_1 + q_1)\underline{a}_1 + \cdots + (p_n + q_n)\underline{a}_n \in < \underline{a}_1, \ldots, \underline{a}_n >\end{aligned}$$

Also, for every scalar λ

$$\begin{aligned}\lambda p &= \lambda(p_1 \underline{a}_1 + p_2 \underline{a}_2 + \cdots + p_n \underline{a}_n) \\ &= \lambda p_1 \underline{a}_1 + \cdots + \lambda p_n \underline{a}_n \in < \underline{a}_1, \ldots, \underline{a}_n >\end{aligned}$$

Thus every span is a linear subspace. □

3.2.1 Span operations

1. Swap 2 vectors.
2. Multiply $\underline{\mathrm{a}}_i$ by $\lambda \neq 0$.
3. Add $\lambda\underline{\mathrm{a}}_i$ to $\underline{\mathrm{a}}_j$ $(i \neq j)$.
4. insert/append $\underline{0}$ or leave out $\underline{0}$.
5. insert/append $\lambda_1\underline{\mathrm{a}}_1 + \cdots + \lambda_n\underline{\mathrm{a}}_n$.
6. leave out $\underline{\mathrm{a}}_i$ if it can be written as a linear combination.

Theorem 3.2 – (Exchange theorem) If $V =< \underline{a}_1, < \ldots, \underline{a}_n$ and $\underline{b} = \lambda_1\underline{a}_1 + \cdots + \lambda_i\underline{a}_i + \cdots + \lambda_n\underline{a}_n$ with $\lambda_i \neq 0$ for some i, then

$$V =< \underline{a}_1, \ldots, \underline{a}_n >=< \underline{a}_1, \ldots, \underline{a}_{i-1}, \underline{b}, \underline{a}_{i+1}, \ldots, \underline{a}_n >$$

3.2.2 Linear (in)dependence

Definition 3.3 – (Linear (in)dependent set of vectors) A set or system of vectors $\underline{a}_1, \ldots, \underline{a}_n$ is called *linearly dependent* if at least one of the vectors is a linear combination of the others.

The vectors are called *linearly independent* if none of the vectors is a linear combination of the others.

Definition 3.4 – (Linear (in)dependent vectors: practical version) The system of vectors $\underline{a}_1, \ldots, \underline{a}_2$ is linearly independent if and only if the only solution of the equation

$$\lambda_1\underline{a}_1 + \lambda_2\underline{a}_2 + \cdots + \lambda_n\underline{a}_n = \underline{0}$$

in $\lambda_1, \lambda_2, \ldots, \lambda_n$ is: $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

The system is called *linearly dependent* if it is not linearly independent.

3.2.3 Example

The functions sin and cos in the space of real functions $\mathbb{R} \to \mathbb{R}$ are linearly independent. Suppose

$$a \sin + b \cos = 0 \text{ (the zero function)}$$

Then, *since this is an equality of functions*, we find that for every $t \in \mathbb{R}$ the relation $a \sin t + b \cos t = 0$ holds. Now we choose a few 'smart' value for t to deduce that deduce that a and b are 0: for $t = 0$ we get $b \cos 0 = 0$ so $b = 0$, and fot $t = \frac{\pi}{2}$ we get $a \sin \frac{\pi}{2} = 0$, so $a = 0$.

Theorem 3.3 Suppose $V =< \underline{a}_1, \ldots, \underline{a}_n >$ and suppose $\underline{b}_1, \ldots, \underline{b}_m$ is a linearly independent set of vectors in V. Then $m \leq n$.

Theorem 3.4 If the vector space V is the span of each of the systems of independent vectors $\underline{a}_1, \ldots, \underline{a}_n$ and $\underline{b}_1, \ldots, \underline{b}_m$, then $n = m$.

Definition 3.5 – (Basis and dimension) A linearly independent set spanning a vector space V is called a *basis* of V. The number of elements in the basis is called the *dimension* of V and is denoted as $\dim(V)$ (or $\dim_{\mathbb{K}}(V)$ if we want to emphasize the scalars). If no finite basis of V exists (and V does not contain only the zero vector), then we say the V is infinite-dimensional and write $\dim(V) = \infty$. We define the dimension of a vector space consisting of the zero vector only to be 0.

3.2.4 Examples

(a) Geometrically it is clear that $\dim(E^3) = 3$ and $\dim(E^2) = 2$

(b) In $\mathbb{R}^n$ the set containing the vectors:

$$\begin{aligned} \underline{e}_1 &= (1, 0, 0, \ldots, 0), \\ \underline{e}_2 &= (0, 1, 0, \ldots, 0), \\ &\vdots \\ \underline{e}_n &= (0, 0, 0, \ldots, 1) \end{aligned}$$

is a linearly independent set spanning $\mathbb{R}^n$. So $\dim(\mathbb{R}^n) = n$

(c) Let P_n be the set of all (real or complex) polynomials in x of degree at most n. Then $P_n =< 1, x, x^2, \ldots, x^n >$. $\dim(P_n) = n + 1$

(d) The space P of all polynomials can be shown to have infinite dimension.

3.2.5 Finding bases

Theorem 3.5 If the set of vectors $\{\underline{a}_1, \ldots, \underline{a}_n\}$ in the vector space V satisfies

$$a_1 \neq \underline{0}, a_2 \notin < a_1 >, a_3 \notin < a_1, a_2 >, \ldots, a_n \notin < \underline{a}_1, \ldots, \underline{a}_n >,$$

then the vectors are linearly independent.

3.3 Coordinates

Definition 3.6 – (Coordinates) Let $\underline{\mathrm{v}}_1, \underline{\mathrm{v}}_2, \ldots, \underline{\mathrm{v}}_n$ be a basis of the vector space V over $\mathbb{K}$. If

$$\underline{\mathrm{v}} = \lambda_1 \underline{\mathrm{v}}_1 + \cdots + \lambda_n \underline{\mathrm{v}}_n,$$

then the vector $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is called the *coordinate vector* and is itself a vector in $\mathbb{K}^n$.

! The coordinate vector of $\underline{\mathrm{v}}$ is unique.

3.3.1 Example

3.4 Constructing vector spaces

4 Rank and inverse of a matrix, determinants

5 Inner product spaces

6 Introduction to linear maps