Assignment 5 Exercise 7.8.1

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1 7.8.1

Problem 1.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Define the sequence $b : \mathbb{N} \to \mathbb{R}$ by

$$b_n := a_{n+1} - a_n, \text{ for } n \in \mathbb{N}$$

1. Show that the series
$$\sum_{n=0}^{\infty} b_n$$
converges if and only if a converges.

2. Show that if the sequence $a : \mathbb{N} \to \mathbb{R}$ converges, then
$$\lim_{n\to\infty} a_n = a_0 + \sum_{n=0}^{\infty} b_n$$

Proof. 1. We need to show both directions of the biimplication

$$a \text{ converges} \iff \sum_{n=0}^{\infty} b_n \text{ converges}$$

1. First we show the foward direction.
Suppose a converges to $p \in \mathbb{R}$ We need to show that $\lim_{n\to\infty} \sum_{k=0}^{n} b_k$ converges.
Consider the partial sum

$$S_n := \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_{k+1} - a_k) = a_{n+1} - a_0$$

Then we have

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (a_{n+1} - a_0) = \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_0 = p - a_0$$

2. Now we show the backward direction.
Suppose $\sum_{n=0}^{\infty} b_n$ converges to $q \in \mathbb{R}$.
We need to show that a converges.

Since the series $\sum_{n=0}^{\infty} b_n$ converges, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=0}^{n} b_k &= q \\
\lim_{n\to\infty} \sum_{k=0}^{n} (a_{k+1} - a_k) &= q \\
\lim_{n\to\infty} (a_{n+1} - a_0) &= q \\
\lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_0 &= q \\
\lim_{n\to\infty} a_{n+1} - a_0 &= q \\
\lim_{n\to\infty} a_{n+1} &= q + a_0 \text{ by index shift we get} \\
\lim_{n\to\infty} a_n &= q + a_0
\end{aligned}$$

Thus we have shown both directions of the biimplication, and we conclude that

$$a \text{ converges} \iff \sum_{n=0}^{\infty} b_n \text{ converges}$$

Further more if a converges, then

$$\lim_{n\to\infty} a_n = a_0 + q$$

$$\lim_{n\to\infty} a_n = a_0 + \sum_{n=0}^{\infty} b_n$$

□

2 Exercise 7.8.4

2.1 e)

Problem 2.1

$$\sum_{k=0}^{\infty} \frac{2k+3}{(k+1)^2(k+2)^2}$$

2.2 f)

Problem 2.2

$$\sum_{k=1}^{\infty} \sqrt[k]{2^{-k} + 3}$$

Claim: the series diverges.

Proof. Consider the sequence $a_k = \sqrt[k]{2^{-k}}$. Note that $\sqrt[k]{2^{-k}} < \sqrt[k]{2^{-k}+3}$ for all $k \in \mathbb{N}$. Note that

$$\sum_{k=1}^{\infty} \sqrt[k]{2^{-k}} = \sum_{k=1}^{\infty} \frac{1}{2}$$

diverges. By the comparison test, we conclude that

$$\sum_{k=1}^{\infty} \sqrt[k]{2^{-k}+3}$$

diverges. $\square$