Assignment 10

Group 1-1

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1 Exercise 13.11.3

> **Problem** Let $(X, \mathrm{dist}_X) := (\mathbb{R}, \mathrm{dist}_\mathbb{R})$ and set $D := \mathbb{N} \subseteq \mathbb{R}$. Let (Y, dist_Y) be a metric space and let $a : \mathbb{N} \to Y$ be a function. Show that $a : \mathbb{N} \to Y$ is continuous (when viewed as a function defined on $D := \mathbb{N}$ as a subset of the metric space (X, dist_X) mapping to the metric space (Y, dist_Y)).

Proof. Need to show that for all $n \in D := \mathbb{N}$, a is continuous at n. I.e. for all $\varepsilon > 0$, there exists $\delta > 0$, for all $x \in D = \mathbb{N}$, if $0 < \mathrm{dist}_X(x, a) < \delta$, then $\mathrm{dist}_Y(a(x), a(n)) < \varepsilon$.
Let $\varepsilon > 0$.
Choose $\delta = 1/2$.
Take $x \in \mathbb{N}$.
Need to show that $0 < |x - a| < \delta \implies \mathrm{dist}_Y(a(x), a(n)) < \varepsilon$.
Since $x, a \in N$ and, we have $|x - a| >= 1$ or $|x - a| = 0$
$0 < |x - a| < \delta \implies \mathrm{dist}_Y(a(x), a(n)) < \varepsilon$ is true.
We conclude a is continuous on $\mathbb{N}$. □

2 Exercise 13.11.5

> **Problem** Let (X, dist_X) and (Y, dist_Y) be metric spaces, let $D \subseteq X$ and let $f : D \to Y$. Assume that $f : D \to Y$ is *Lipschitz continuous*, which means that there exists a constant $M > 0$ such that for all $x, z \in D$,
>
> $$\mathrm{dist}_Y(f(x), f(z)) \leq M \mathrm{dist}_X(x, z).$$
>
> Show that $f : D \to Y$ is uniformly continuous on D.

Proof. Need to show that

for all $\varepsilon > 0$,
  there exists $\delta > 0$,
    for all $p, q \in D$,
      $0 < \mathrm{dist}_X(p, q) < \delta \implies \mathrm{dist}_Y(f(p), f(q)) < \varepsilon$

Let $\varepsilon > 0$.
Since f is Lipschitz continuous, there exists a constant $M > 0$ such that for all $x, z \in D$, $\mathrm{dist}_Y(f(x), f(z)) \leq M \mathrm{dist}_X(x, z)$.

Obtain such M.
Choose $\delta = \frac{\varepsilon}{M}$,
Let $p, q \in D$.
Then it holds that $\text{dist}_Y(f(p), f(q)) \leq M\text{dist}_X(p, q)$
Need to show that $0 < \text{dist}_X(p, q) < \delta \implies \text{dist}_Y(f(p), f(q)) < \varepsilon$.
Assume $0 < \text{dist}_X(p, q) < \delta$, then it holds that

$$\text{dist}_Y(f(p), f(q)) \leq M\text{dist}_X(p, q) < M\delta = \varepsilon$$

We conclude that f is uniformly continuous on D. $\square$

3 Exercise 14.12.2

Problem Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{\exp\left(x_1^2 - 3x_2\right)}{x_1^2 + x_2^2}.$$

Prove that $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is continuous considered as a function mapping from the domain $\mathbb{R}^2 \setminus \{0\}$ in the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$.

Proof. Note:

1. $p(x_1, x_2) = x_1^2 - 3x_2$ is a polynomial function, thus continuous.
2. $\exp : \mathbb{R} \to \mathbb{R}$ is continuous. (Standard function)
3. $\exp(p(x_1, x_2)) = \exp\left(x_1^2 - 3x_2\right)$ is continuous. (Composition of continuous functions)
4. $q(x_1, x_2) = x_1^2 + x_2^2$ is a polynomial function, thus continuous, and $q(x_1, x_2) \neq 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0, 0\}$.

Then it holds that $f(x) = \frac{\exp(p(x))}{q(x)} = \frac{\exp\left(x_1^2 - 3x_2\right)}{x_1^2 + x_2^2}$ is continuous. $\square$

4 Exercise 14.12.4

Problem Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x_1^4 + 2x_2^4}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0, 0), \\ 0 & \text{if } (x_1, x_2) = (0, 0), \end{cases}$$

is continuous as a function from the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ to the normed vector space $(\mathbb{R}, |\cdot|)$.

Proof. Note:

1. $p(x_1, x_2) = x_1^4 + 2x_2^4$ is a polynomial function, thus continuous.

2. $q(x_1, x_2) = x_1^2 + x_2^2$ is a polynomial function, thus continuous, and $q(x_1, x_2) \neq 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0, 0\}$.

Then it holds that $f(x) = \frac{x_1^4 + 2x_2^4}{x_1^2 + x_2^2}$ is continuous on $\mathbb{R}^2 \setminus \{0, 0\}$.

We need to show that f is continuous at $(0, 0)$.
By the $\varepsilon - \delta$ characterization of continuity, it suffices to show that

$$\begin{aligned}
&\text{for all } \varepsilon > 0 \\
&\quad \text{there exists } \delta > 0 \\
&\quad\quad \text{for all } x \in \mathbb{R}^2 \\
&\quad\quad\quad 0 < \|x - (0,0)\| < \delta \implies |f(x) - f(0,0)| < \varepsilon
\end{aligned}$$

Let $\varepsilon > 0$.
Choose $\delta = \sqrt[4]{\frac{\varepsilon}{2}}$.
Take $x \in \mathbb{R}^2$.
Assume $0 < \|x\| < \delta$, then it holds that $x \neq (0, 0)$ and $x_1^2 + x_2^2 < \delta^2$
Need to show that $|f(x) - f(0, 0)| < \varepsilon$.
It holds that

$$\begin{aligned}
|f(x) - f(0,0)| &= \left| \frac{x_1^4 + 2x_2^4}{x_1^2 + x_2^2} - 0 \right| \\
&= \frac{|x_1^4 + 2x_2^4|}{|x_1^2 + x_2^2|} \\
&< |x_1^4 + 2x_2^4| \\
&\leq |x_1^4| + |2x_2^4| \\
&= x_1^4 + 2x_2^4 \\
&< 2(x_1^4 + x_2^4) \\
&< 2(x_1^2 + x_2^2)^2 \\
&< 2\delta^4 \\
&= 2 \cdot \sqrt[4]{\frac{\varepsilon}{2}}^4 \\
&= \varepsilon
\end{aligned}$$

We conclude that f is continuous on $\mathbb{R}^2$. $\square$