2IT80 Discrete Structures

2023-24 Q2

Lecture 5: Counting I

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

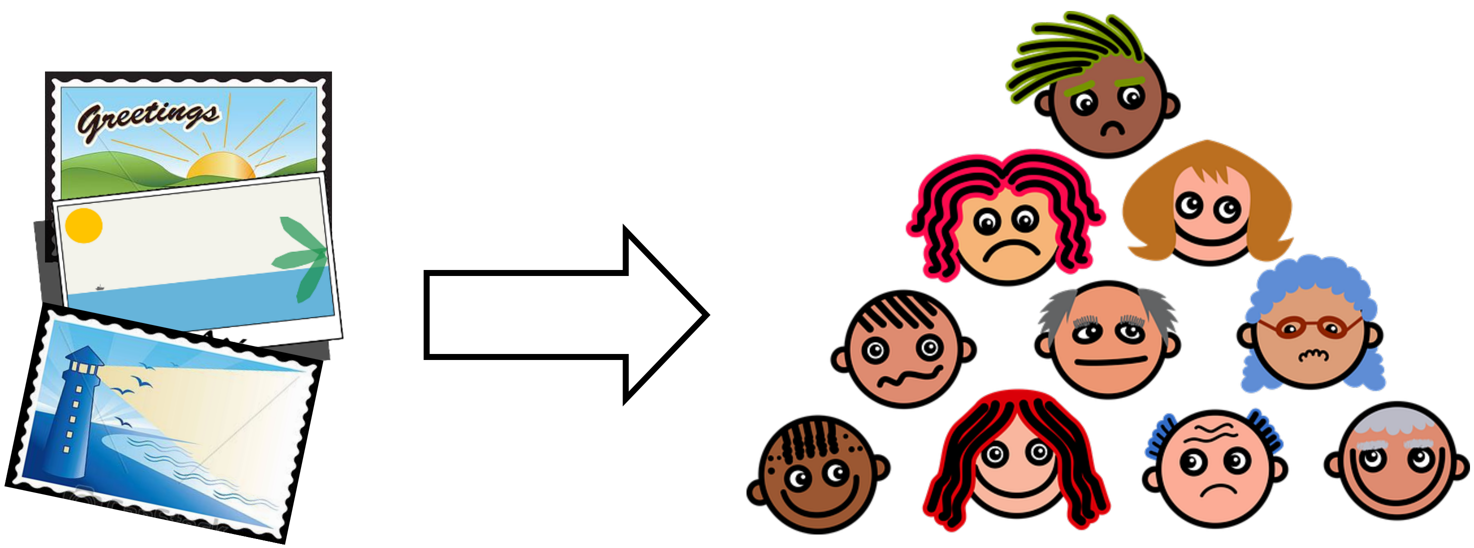
Counting

Functions and Subsets

Postcards

Problem. Professor X, having completed a successful short-term visit at the School of Mathematical Contemplation and Machine Cleverness in the city of Y., strolls around one sunny day and decides to send a postcard to each of his friends Alice, Bob, Arthur, Merlin, and HAL-9000. A street vendor nearby sells 26 kinds of postcards with great sights of Y.'s historical center. How many possibilities does Professor X have for sending postcards to his 5 friends?

26 choices per friend ➡ 26^5 choices



Postcards

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Same postcard may be sent to several friends.

But: each friend receives exactly one postcard.

Choice of postcards corresponds to function

$$\{\text{friends}\} \to \{\text{postcards}\}$$

More abstractly: What is the number of mappings from a 5-element set (friends) to a 26-element set (postcards)?

Words as mappings

Problem: How many different words with 5 letters exist?

(We use an alphabet with 26 characters and do not insist on these words having a meaning)

Again: 26^5 choices. Words = Mappings from {1,2,...,5} to {a,b,...,z}

Transformations of counting problems are important:
encode one type of objects with another type of objects.

From now on: n-set := n-element set

Theorem: Let N be an n-set (possibly empty) and let M be an m-set, $m \geq 1$. Then the number of functions $f : N \to M$ is m^n.

Number of functions

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Proof: Induction on n.

Base case $n = 0$: What are functions $f : \emptyset \to M$?

Recall: $f \subseteq N \times M$,
if $N = \emptyset$ then $N \times M = \emptyset$ and $f \subseteq \emptyset$

$f = \emptyset$ is indeed a function $f : \emptyset \to M$, i.e., there is exactly one such function for every M.

Formula says: $m^0 = 1$ for every M.

The formula holds for $n = 0$, which establishes the base case.

Number of functions

Theorem: Let N be an n-set (possibly empty) and let M be an m-set, $m \geq 1$. Then the number of functions $f : N \to M$ is m^n.

Proof: Induction on n.

Blackboard

Number of functions

Theorem: Let N be an n-set (possibly empty) and let M be an m-set, $m \geq 1$. Then the number of functions $f : N \to M$ is m^n.

Proof: Induction on n.

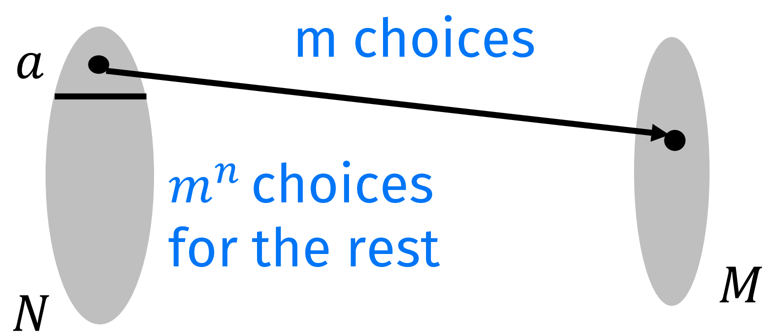
Inductive step: $k \geq 0$

IH: Assume theorem holds for all $0 \leq k' \leq k$ and all $m \geq 1$.

To prove for $k + 1$, consider $(k + 1)$-set N and m-set M.

Let $a \in N$. Giving a function $f: N \to M$ is the same as

- specifying $f(a) \in M$, and
- giving a function $f': N \setminus \{a\} \to M$.



Total: $m \cdot m^k = m^{k+1}$ functions. ∎

Counting subsets

Theorem: Every n-set has exactly 2^n subsets (for all $n \geq 0$).

First proof: Use induction on n.

see book ...

Counting subsets

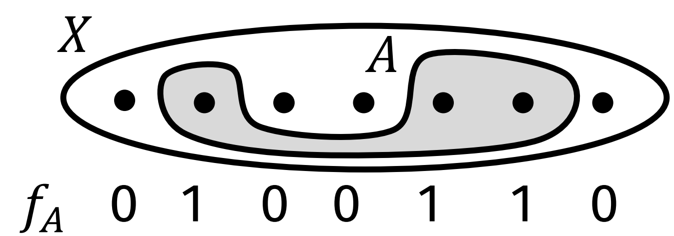
Theorem: Every n-set has exactly 2^n subsets (for all $n \geq 0$).

Second proof: Reduce to known result.

For a subset $A \subseteq X$ of an n-set X, we define a function

$$f_A: X \to \{0,1\} \text{ via } f_A(x) = \begin{cases} 1, & if\ x \in A \\ 0, & if\ x \notin A \end{cases}$$

f_A is the characteristic function of A



Different sets have different characteristic functions. (Injective)

Every function $f: X \to \{0,1\}$ determines a subset $A \subseteq X$ with $f_A = f$. (Surjective)

Number of subsets is equal to number of functions $X \to \{0,1\} = 2^n$ ■

More subset counting

Theorem: Let $n \geq 1$. Every n-set X has exactly 2^{n-1} subsets with an odd number of elements as well as exactly 2^{n-1} subsets with an even number of elements.

Proof: see book

Back to postcards

Problem. Professor X., after having spent some time contemplating the approximately 12 million possibilities of sending his postcards, returned to the street vendor and wanted to buy a selection. But the vendor had already sold all the postcards and was about to close. After some discussion, he admitted he **still had one sample of each of the 26 postcards**, and was willing to sell 5 of them to Professor X. for $5 a piece. In this situation, Professor X. has to make a new decision about which postcard is best for whom. How many possibilities does he have now?

Which mappings from friends to postcards are not allowed now?

Which mappings from friends to postcards do we need to count?

Counting injective functions

Theorem: For given numbers $n, m \geq 0$, there exist exactly

$$m(m-1) \cdot \dots \cdot (m-n+1) = \prod_{i=0}^{n-1} (m-i)$$

injective functions of a given n-set to a given m-set.

Proof: Induction on n.

Base case $n = 0$: $f = \emptyset$ is the only function, and it is injective(!),
so there is exactly 1injective function.
$\prod_{i=0}^{-1} (m-i) = 1$.

Counting injective functions

Theorem: For given numbers $n, m \geq 0$ and $m \geq n$, there exist exactly

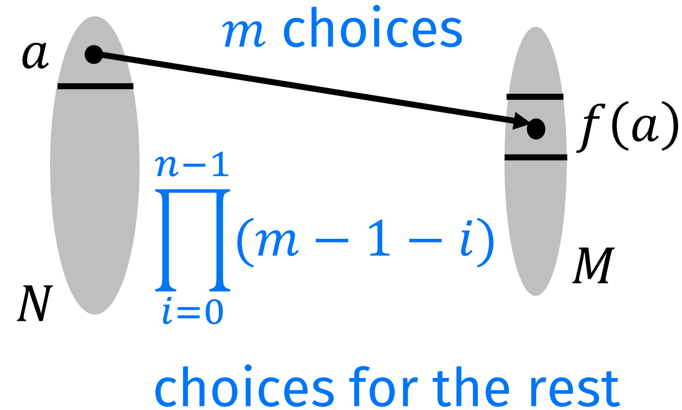
$$m(m-1) \cdot \dots \cdot (m-n+1) = \prod_{i=0}^{n-1}(m-i)$$

injective functions of a given n-set to a given m-set.

Proof sketch: Induction on n.

Inductive step: Let $k \geq 0$:

IH: Thm. holds for all $0 \leq k' \leq k$ and $m \geq k'$.



For $k+1$. Consider $(k+1)$-set N and m-set M

Fix $a \in N$. Choosing injective function $f: N \to M$ is same as choosing

- $f(a) \in \mathrm{M}$, and
- injective function $f': N \setminus \{a\} \to M \setminus \{f(a)\}$

Total: $m \cdot (m-1)(m-2) \cdot \dots \cdot (m-k) = \prod_{i=0}^{k}(m-i)$ ∎

Permutations

… mix things up

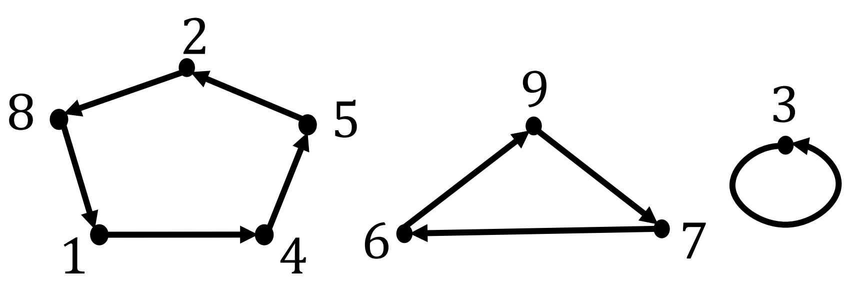
Permutation

A permutation is a bijective function of a finite set X to itself.

If the elements of X are arranged in some order, we can imagine a permutation as rearranging the elements in a different order.

Example: $p = \binom{a\ b\ c\ d}{b\ d\ c\ a}$ $\quad p(a) = b, p(b) = d, p(c) = c, p(d) = a$

Permutations can be written as cycles.



Example: $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 3 & 5 & 2 & 9 & 6 & 1 & 7 \end{pmatrix}$ has cycles $p = (1\ 4\ 5\ 2\ 8)(3)(6\ 9\ 7)$

Permutations

Theorem: The number of permutations of an n-set is

$$n! = \prod_{i=1}^{n} i\,.$$

Proof: Let X be an n-set.

$f : X \to X$ is bijective if and only if it is injective

Number of injective functions from n-set to n-set is

$$\prod_{i=0}^{n-1} (n-i) = \prod_{i=1}^{n} i = n!$$

∎

Counting functions: overview

Let A be an n-set and B be an m-set:

Number of functions $f : A \to B$? m^n

Number of injective functions $f : A \to B$? $\prod_{i=0}^{n-1}(m-i)$

Number of surjective functions?

... later: harder to compute, no nice formula!

Sinterklaas quiz

Sinterklaas comes to class with 150 different presents. He wants to hand the presents out to the 75 students in the room (sorry online people). In how many different ways can he distribute the presents?

What if Sinterklaas is more fair and wants to give every student exactly one present? (so he distributes only 75 presents)

What is he wants to distribute all presents and each student should get at least one?

What if everyone should get exactly 2 presents?

Binomial Coefficients

Some unrelated questions

Let X be an n-set and let $k \leq n$.

How many subsets of size k does X have?

X = {1,2,3,4,5}, k = 3

{1,2,3},{1,2,4},{1,2,5},{1,3,4},{1,3,5},
{1,4,5},{2,3,4},{2,3,5},{2,4,5},{3,4,5}
$\rightarrow$ 10 subsets of size 3

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

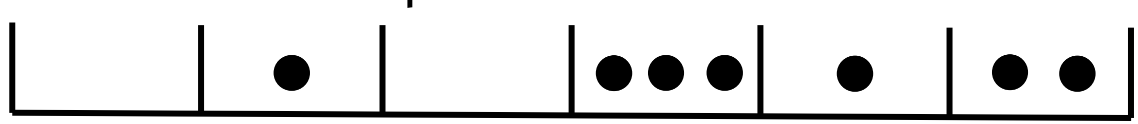
m = 3, r = 2

3 = 0 + 3
3 = 1 + 2
3 = 2 + 1
3 = 3 + 0

four possibilities

In how many ways can we distribute m indistinguishable balls into r bins?

Example: 7 balls and 6 bins



Binomial coefficient

For $n \geq k \geq 0$, the binomial coefficient $\binom{n}{k}$, read as "n choose k", is defined as

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1)}{k \cdot (k-1) \cdot \cdots \cdot 2 \cdot 1} = \frac{\prod_{i=0}^{k-1} (n-i)}{k!}$$

An equivalent formulation is

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Subsets of a set

Let X be a set and let k be a non-negative integer. We denote by

$$\binom{X}{k}$$

the set of all k-element subsets of the set X.

Example: $\binom{\{a,b,c\}}{2} = \{\{a,b\},\{a,c\},\{b,c\}\}$

Note: $\binom{x}{k}$ has different meaning depending on whether x is a number or a set!

Theorem: A finite set X has $\binom{|X|}{k}$ subsets with k elements.

As a formula: $\left|\binom{X}{k}\right| = \binom{|X|}{k}$.

Subsets of a set

Theorem: A finite set X has $\binom{|X|}{k}$ subsets with k elements.

As a formula: $\left|\binom{X}{k}\right| = \binom{|X|}{k}$

Proof: Let $n = |X|$. Use technique called double counting.

Count ordered k-tuples of X (without repetition) in two ways.

1) There are $n \cdot (n-1) \cdot \dots \cdot (n-k+1)$ ordered k-tuples (injective maps from $\{1, \dots, k\} \to \{1, \dots, n\}$).
2) There are $k! \left|\binom{X}{k}\right|$ ordered k-tuples:
   Every k-element subset can be ordered in $k!$ ways, and every k-tuple is obtained from exactly one k-element subset M this way.

Hence: $n \cdot (n-1) \cdot \dots \cdot (n-k+1) = k! \left|\binom{X}{k}\right|$ ■

Some unrelated questions

Let X be an n-set and let $k \leq n$.
How many subsets of size k does X have?

?

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

?

In how many ways can we distribute m indistinguishable balls into r bins?

?

Some unrelated questions

Let X be an n-set and let $k \leq n$.
How many subsets of size k does X have?

Answer: $\binom{n}{k}$

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

In how many ways can we distribute m indistinguishable balls into r bins?

Sums and balls into bins

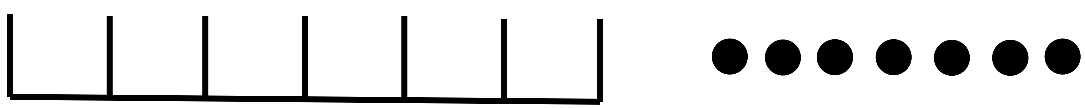
Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

In other words: What is the number of ordered r-tuples $(i_1, i_2, \ldots, i_r)$ of non-negative integers satisfying $i_1 + i_2 + \cdots + i_r = m$?

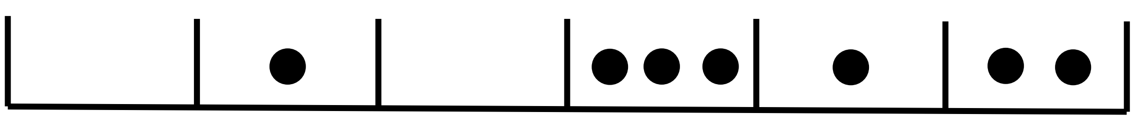
i.e., they are distinguishable

i.e., they are indistinguishable

Consider r numbered bins and m identical balls



Distribute balls into bins and let $i_1, \ldots, i_r$ denote the number of balls in the bins.



$\leftrightarrow$ 0+1+0+3+1+2 = 7

$\Rightarrow$ Number of sums = Number of distributions of m balls into r bins.

Some unrelated questions

Let X be an n-set and let $k \leq n$.
How many subsets of size k does X have? ✓

Answer: $\binom{n}{k}$

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers? ?

In how many ways can we distribute m indistinguishable balls into r bins? ?

Some unrelated questions

Let X be an n-set and let $k \leq n$.
How many subsets of size k does X have?

Answer: $\binom{n}{k}$

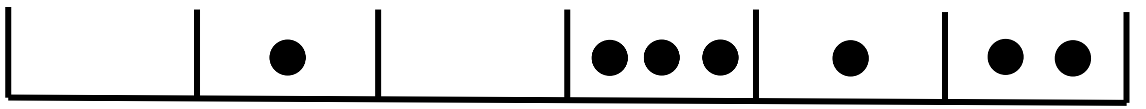
Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

Answer: same answer as next question!

In how many ways can we distribute m indistinguishable balls into r bins?

Balls into bins

In how many ways can we distribute m indistinguishable balls into r bins?



Drop bottom as well as left/right side

Keep m balls and $r-1$ separating walls

Distribution into bins is equivalent to picking positions of separators

- There are $m+r-1$ objects (balls and separation walls)
- We can choose which of them are balls and which are walls
- May pick $r-1$ positions as walls
- There are $\binom{m+r-1}{r-1}$ ways to do this.

Some unrelated questions

Let X be an n-set and let $k \leq n$.
How many subsets of size k does X have?

Answer: $\binom{n}{k}$

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

Answer: same answer as next question!

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Some unrelated questions

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How many subsets of size k does X have?

Answer: $\binom{n}{k}$

Let $m \geq 0$ be an integer. In how many ways can we express m as a sum of r non-negative integers?

Answer: same answer as next question!

In how many ways can we distribute m indistinguishable balls into r bins?

Answer: $\binom{m+r-1}{r-1}$

Basic properties of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof: Follows directly from the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. ■

Intuitively: Number of k-element subsets of an n-set is the same as the number of $(n-k)$-element subsets:

we select elements that should be in the set or

we select elements that are not in the set

More precisely: If X is an n-set, then

$$f\colon \binom{X}{k} \to \binom{X}{n-k},\ f(M) = X \setminus M \text{ is a bijection}$$

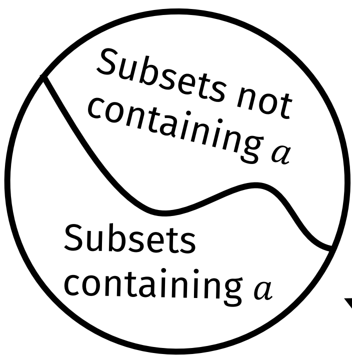
Basic properties of binomial coefficients

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof sketch: We use a combinatorial interpretation.

$\binom{n}{k}$ is the number of k-element subsets of an n-set X.

Pick $a \in X$, divide k-element subsets of X into two groups



k-element subsets of X

Bijection to k-subsets of $X \setminus \{a\}$.
There are $\binom{n-1}{k}$.

Bijection to $(k-1)$-subsets of $X \setminus \{a\}$.
There are $\binom{n-1}{k-1}$.

Binomial theorem

Theorem (Binomial theorem):
For every integer $n \geq 0$ it holds that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof idea. Use induction and $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ ■

Binomial theorem

Theorem (Binomial theorem):
For every integer $n \geq 0$ it holds that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

For $x = y = 1$ we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

counting all subsets of an n-set grouped by size ...

Binomial theorem

Theorem (Binomial theorem):
For every integer $n \geq 0$ it holds that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

For $x = -1, y = 1$ we have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Adding $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ gives

$$2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots\right] = 2^n$$

Hence an n-set has 2^{n-1} subsets of even size.

Basic properties of binomial coefficients

Theorem:

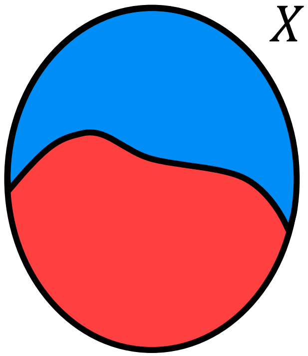
$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

Proof:

Let X be a $2n$ -set. There are $\binom{2n}{n}$ ways to pick an n-element subset.

Color n elements of X red and the remaining n elements blue.

A n-element subset of X contains i red elements and $n - i$ blue elements, where $i \in \{0, \dots, n\}$.



$\binom{n}{i}$ choices for red subset, $\binom{n}{n-i}$ choices for blue.

$\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$ choices for n-element subset of X.

$$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2$$

∎

Summary

Map objects you want to count to (combinations of) mathematical objects (tuples/functions/permutations etc.) and use known results.

Choosing, order matters:

- may choose objects more then once $\rightarrow$ functions
- may choose each object only once $\rightarrow$ injective functions

Choosing, order irrelevant:

- may choose objects only once $\rightarrow$ binomial coefficients
- may choose objects multiple times $\rightarrow$ balls into bins

Organizational

- Practice set:
  - Ex. 4,5,6
- A2 test already next week