Analysis 1

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Contents

1 Sets, Spaces and Function

1.1 Metric Space

Definition 1.1.1 – distance Let X be a set. A function $d : X \times X \to X$ is called a *distance* on X if it satisfies the following properties:

(i) Positivity: For all $a, b \in X$, it holds that $d(a, b) \geq 0$.

(ii) Non-degeneracy: For all $a, b \in X$, if $d(a, b) = 0$, then $a = b$.

(iii) Symmetry: For all $a, b \in X$, it holds that $d(a, b) = d(b, a)$.

(iv) Triangle inequality: For all $a, b, c \in X$, it holds that $d(a, c) \leq d(a, b) + d(b, c)$.

(v) Reflexivity: For all $a \in X$, it holds that $d(a, a) = 0$.

Usually conditions (ii) and (v) are combined into one condition: For all $a, b \in X, d(a, b) = 0$ if and only if $a = b$.

Definition 1.1.2 – metric space A metric space is a pair $(X, dist)$, where X is a set and $dist$ is a distance function $dist : X \times X \to \mathbb{R}$ on X.

Example 1.1.3 Let $X = \{$Die Hard, Barbie, Oppenheimer$\}$

d	Die Hard	Barbie	Oppenheimer
Die Hard	0	5	2
Barbie	5	0	3
Oppenheimer	2	3	0

Then d is a distance function on X

Definition 1.1.4 – ball in a metric space Let (X, d) be a metric space. Let $c \in X$ and $r \in \mathbb{R}$. The ball of radius r centered at c is the set

$$B(c, r) = \{x \in X | d(c, x) < r\}$$

Example 1.1.5 If $(X, d) = (\mathbb{R}, d_{\mathbb{R}})$, then $B(1, 3) = (-2, 4) = \{x \in \mathbb{R} \mid |x - 1| < 3\}$

Example 1.1.6 Let $X := \{$Die Hard, Barbie, Oppenheimer$\}$, with distance defined before. Then $B($Barbie, 4$) = \{$ Barbie, Oppenheimer $\} = \{x \in X \mid d(x, Barbie) < 3\}$.

1.2 Normed Vector Spaces

Definition 1.2.1 – norm Let V be a vector space over $\mathbb{R}$. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

- Positivity: for all $u, v \in V$ we have $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$.
- Non-degeneracy: for all $u \in V$ if $\|u\| = 0$ then $u = 0$.
- Absolute Homogeneity: for all $u \in V$ and for all $\lambda \in \mathbb{R}$ we have $\|\lambda u\| = |\lambda|\|u\|$.
- Triangle inequality: for all $u, v \in V$ we have $\|u + v\| \leq \|u\| + \|v\|$.

Example 1.2.2 Let $V = \mathbb{R}^n$. Then $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ is a norm on $\mathbb{R}^n$.

Proposition 1.2.3 – Let $(V, \|\cdot\|)$ be a normed vector space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(u, v) = \|u - v\|$ is a distance on V. And (V, d) is a metric space.

Remark 1.2.4 (Notation for Euclidean distance on $\mathbb{R}^d$ and $\mathbb{R}$). We will usually write $\text{dist}_{\mathbb{R}^d}$ instead of $\text{dist}_{\|\cdot\|_2}$ for the standard (Euclidean) distance on $\mathbb{R}^d$. In particular, if $d \geq 2$, we have

$$\text{dist}_{\mathbb{R}^d}(v, w) = \|v - w\|_2 = \sqrt{\sum_{i=1}^{d} (v_i - w_i)^2}$$

and if $d = 1$ we just have

$$\text{dist}_{\mathbb{R}} = |v - w|$$

And if there is no room for confusion, we will just leave out the subscript altogether and write dist instead of $\text{dist}_{\mathbb{R}^d}$.

1.3 The reverse triangle inequality

Lemma 1.3.1 – **Reverse triangle inequality** Let $(V, \|\cdot\|)$ be a normed vector space. Then for all $u, v \in V$ we have,

$$|\|v\| - \|w\|| \leq \|v - w\|$$

2 Real Numbers

2.1 What are the real numbers?

Definition 2.1.1 – Real numbers The real numbers are a complete totally ordered field.

2.2 The completeness axiom

Definition 2.2.1 – Upper and Lower bound We say a number $M \in \mathbb{R}$ is an *upper bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \leq M].$$

We say a number $m \in \mathbb{R}$ is a *lower bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \geq M].$$

Given the definition of upper and lower bounds, we define what it means for a set to be bounded from above, bounded from below and just bounded.

Definition 2.2.2 – bounded from above, bounded from below, bounded A set $A \subseteq \mathbb{R}$ is *bounded from above* if there exists an upper bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded from below* if there exists a lower bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded* if it is bounded from above and bounded from below.

Definition 2.2.3 – Least upper bound (supremum) Precisely, M is a *least upper bound* of a subset A if both

1. M is an upper bound of A.
2. For every upper bound $L \in \mathbb{R}$ of A, it holds that $M \leq L$.

Proposition 2.2.4 – Suppose both M and W are a least upper bound of a subset $A \subseteq \mathbb{R}$. Then $M = W$.

Axiom 2.2.5 – Completeness axiom We say that a totally ordered field $\mathbf{R}$ satisfies the *completeness axiom* if every nonempty subset of $\mathbf{R}$ that is bounded from above has a least upper bound.

Lemma 2.2.6 – Every non-empty subset of the real line that is bounded from below has a *largest lower bound*.

Definition 2.2.7 – infimum We usually call the largest lower bound of a non-empty set $A \subseteq \mathbb{R}$ that is bounded from below the *infimum* of A, and we denote it by $\inf A$.

2.3 Alternative characterizations of suprema and infima

Proposition 2.3.1 – alternative characterizationa of supremum Let $A \subseteq \mathbb{R}$ be non-empty and bounded from above. Let $M \in \mathbb{R}$. Then M is the supremum of A if and only if

1. M is an upper bound for A,
2. and

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } a \in A,\\ &\qquad a > M - \epsilon.\end{aligned}$$

Proposition 2.3.2 – **alternative characterizationa of infimum** Let $A \subseteq \mathbb{R}$ be non-empty and bounded from below. Let $m \in \mathbb{R}$. Then m is the infimum of A if and only if

1. m is a lower bound for A,
2. and

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } a \in A,\\ &\qquad a < m + \epsilon.\end{aligned}$$

These alternative characterizations of the supremum and infimum really provide a standard way to determining the supremum and infimum of subsets of the real line.

2.4 Maxima and minima

Definition 2.4.1 – **maximum and minimum** Let $A \subseteq \mathbb{R}$ be a subset of the real numbers. We say that $y \in A$ is the *maximum* of A, and write $y = \max A$, if

$$\begin{aligned}&\text{for all } a \in A,\\ &\quad a \leq y.\end{aligned}$$

We say that $x \in A$ is the *minimum* of A, and write $x = \min A$, if

$$\begin{aligned}&\text{for all } a \in A,\\ &\quad a \geq x.\end{aligned}$$

Remark 2.4.2. Even if a set $A \subseteq \mathbb{R}$ is non-empty and bounded, it may not have a maximum or minimum. For example, the set $(0, 1)$ has no maximum or minimum.

Proposition 2.4.3 – Let A be a subset of $\mathbb{R}$. If A has a maximum, then A is non-empty and bounded from above, and $\sup A = \max A$. If A has a minimum, then A is non-empty and bounded from below, and $\inf A = \min A$.

Proposition 2.4.4 – Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from above. If $\sup A \in A$ then A has a maximum and $\max A = \sup A$.

Proposition 2.4.5 – Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from below. If $\inf A \in A$ then A has a minimum and $\min A = \inf A$.

2.5 The Archimedean property

Proposition 2.5.1 – **Archimedeean property** For every real number $x \in \mathbb{R}$ there exists a natural number $n \in \mathbb{N}$ such that $x < n$.

Given this proposition, we can define the ceiling function.

Definition 2.5.2 – **ceiling function** The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is defined as follows. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer $z \in \mathbb{Z}$ such that $x \leq z$.

Proposition 2.5.3 – For every two real numbers $a, b \in \mathbb{R}$ with $a < b$ there exists a $q \in \mathbb{Q}$ with $a < q < b$.

2.6 Computation rules for suprema

In the proposition below, we use the defintions

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for subsets $A, B \subseteq \mathbb{R}$ and a scalar $\lambda \in \mathbb{R}$.

Proposition 2.6.1 – Let A, B, C, D be non-empty subsets of $\mathbb{R}$. Assume that A and B are bounded from above and C and D are bounded from below. Then

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(C + D) = \inf C + \inf D$.
3. For all $\lambda \geq 0$, $\sup(\lambda A) = \lambda \sup A$.
4. For all $\lambda \leq 0$, $\sup(\lambda A) = \lambda \inf A$.
5. $\sup(-C) = -\inf C$.
6. $\inf(-C) = -\sup C$.

2.7 Bernoulli's inequality

Proposition 2.7.1 – **Bernoulli's inequality** Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

1. If $x \geq -1$, then $(1 + x)^n \geq 1 + nx$.
2. If $x \geq 0$ and $n \geq 2$, then $(1 + x)^n \geq 1 + nx$.

3 Sequences

3.1 Sequence

Definition 3.1.1 – **Sequence** A sequence is a function for which the domain is $\mathbb{N}$.

$$a : \mathbb{N} \to Y$$

Y can be any set.

Example 3.1.2 Here are some functions that are sequences:

1. $a : \mathbb{N} \to \mathbb{Q}$
2. $b : \mathbb{N} \to (\mathbb{N} \to Y)$
3. $c : \mathbb{N} \to \mathbb{N}$

And some functions that are not sequences:

1. $d : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$
2. $e : \mathbb{Q} \to \mathbb{N}$

3.2 Terminology around sequences

3.2.1 Bounded sequences

Definition 3.2.2 – **bouneded sequence** Let (X, dist) be a metric space. We say a sequence $a : \mathbb{N} \to X$ is bounded if

$$\begin{aligned}
&\text{there exists } q \in X, \\
&\quad \text{there exists } M > 0, \\
&\quad\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad\quad \text{dist}(a_n, q) \leq M.
\end{aligned}$$

In a normed linear space, we can use a simpler criterion to check whether a sequence is bounded. That is the content of the following proposition.

Proposition 3.2.3 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. The sequence a is bounded if and only if

$$\begin{aligned}
&\text{there exists } M > 0, \\
&\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad \|a_n\| \leq M.
\end{aligned}$$

3.3 Convergence of sequences

Definition 3.3.1 – **Convergence of sequences** Let (X, dist) be a metric space. We say that a sequence $a : \mathbb{N} \to X$ converges to a point $p \in X$ if

$$\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists } N \in \mathbb{N}, \\
&\quad\quad \text{for all } n \geq N, \\
&\quad\quad\quad \text{dist}(a_n, p) < \epsilon.
\end{aligned}$$

We sometimes write

$$\lim_{n\to\infty} a_n = p$$

to express that the sequence (a_n) converges to p.

Definition 3.3.2 – Divergence of sequences Let (X, dist) be a metric space. A sequence $a : \mathbb{N} \to X$ is called *divergent* is it is not convergent.

3.4 Examples and limits of simple sequences

Proposition 3.4.1 – The constant sequence Let (X, dist) be a metric space. Let $p \in X$ and assume that the sequence (a_n) is given by $a_n = p$ for every $n \in \mathbb{N}$. We also say that (a_n) is a constant sequence. Then $\lim_{n\to\infty} = p$.

Example 3.4.2 A standard limit Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence such that $a_n = 1/n$ for $n \geq 1$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to 0.

Proof. Let $\epsilon > 0$. Choose $N = \lceil 1/\epsilon \rceil + 1$. Take $n \geq N$. Then

$$\mathrm{dist}_{\mathbb{R}}(a_n, 0) = |a_n - 0| = |1/n| = 1/n \leq 1/N < \epsilon.$$

□

3.5 Uniqueness of limits

Proposition 3.5.1 – Uniqueness of limits Let (X, dist) be a metric space and let $a : \mathbb{N} \to \mathbb{R}$ be a sequence in X. Assume that $p, q \in X$ and assume that

$$\lim_{n\to\infty} = p \text{ and } \lim_{n\to\infty} a_n = q$$

Then $p = q$.

3.6 More properties of convergent sequences

Proposition 3.6.1 – Let (X, dist) be a metric space and suppose that $a : \mathbb{N} \to X$ is a sequence. Let $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the real-valued sequence

$$n \mapsto \mathrm{dist}(a_n, p)$$

converges to 0 in $\mathbb{R}$.

Proposition 3.6.2 – Convergent sequences are bounded Let (X, dist) be a metric space. Let $a : \mathbb{N} \to X$ be a sequence in X converging to $p \in X$. Then the sequence $a : \mathbb{N} \to X$ is bounded.

Proposition 3.6.3 – Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences. Let $p \in X$ and suppose that $\lim_{n\to\infty} a_n = p$. Then $\lim_{n\to\infty} b_n = p$ if and only if

$$\lim_{n\to\infty} \mathrm{dist}(a_n, b_n) = 0$$

Corollary 3.6.4 – Eventually equal sequences have the same limit Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences such that there exists an

$N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n = b_n$$

Then the sequence $a : \mathbb{N} \to X$ converges if and only if the sequence $b : \mathbb{N} \to X$ converges. If the sequences converge, they have the same limit.

3.7 Limit theorems for sequences taking values in a normed vector space

Theorem 3.7.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Assume that the $\lim_{n\to\infty} a_n$ exists and is equal to $p \in V$ and that the $\lim_{n\to\infty} b_n$ exists and is equal to $q \in V$. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $\mu \in \mathbb{R}$. Assume that $\lim_{n\to\infty} \lambda_n = \mu$. Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $p + q$.
2. The $\lim_{n\to\infty}(\lambda_n a_n)$ exists and is equal to μp.

3.8 Index shift

Proposition 3.8.1 – **Index shift** Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence. Let $k \in \mathbb{N}$ and $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the sequence $(a_{n+k})_n$ (i.e. the sequence $n \mapsto a_{n+k}$) converges to p.

4 Real-valued sequences

4.1 Terminology

Definition 4.1.1 – **increasing, decreasing and monotone sequences** We say a sequence (a_n) is

1. *increasing* if for every $n \in \mathbb{N}, a_{n+1} \geq a_n$
2. *strictly increasing* if for every $n \in \mathbb{N}, a_{n+1} > a_n$
3. *decreasing* if for every $n \in \mathbb{N}, a_{n+1} \leq a_n$
4. *strictly decreasing* if for every $n \in \mathbb{N}, a_{n+1} < a_n$
5. *monotone* if it is either increasing or decreasing
6. *strictly monotone* if it is either strictly increasing or strictly decreasing

Definition 4.1.2 – **upper bound and lower bound for a sequence** We say that a number $M \in \mathbb{R}$ is an *upper bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \leq M$$

We say that a number $m \in \mathbb{R}$ is a *lower bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \geq m$$

Definition 4.1.3 – **bounded sequence** We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that M is an upper bound for a.
We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded below* if there exists an $m \in \mathbb{R}$ such that m is a lower bound for a.

Proposition 4.1.4 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below.

4.2 Monotone, bounded sequences and convergent

Theorem 4.2.1 – Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) convergent and

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad (= \sup\{a_n \mid n \in \mathbb{N}\})$$

Theorem 4.2.2 – Let (a_n) be a decreasing sequence that is bounded from below. Then (a_n) is convergent and

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n \quad (= \inf\{a_n \mid n \in \mathbb{N}\})$$

4.3 Limit theorems

Theorem 4.3.1 – **Limit theorems for real-valued sequences** Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be two converging sequences, and let $c, d \in \mathbb{R}$ be real numbers such that

$$\lim_{n\to\infty} a_n = c \text{ and } \lim_{n\to\infty} b_n = d.$$

Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $c + d$.
2. The $\lim_{n\to\infty}(a_n b_n)$ exists and is equal to $c \cdot d$.
3. If $d \neq 0$, then $\lim_{n\to\infty}(\frac{a_n}{b_n}$ exists and is equal to $\frac{c}{d}$.
4. For every non-negative integer $m \in \mathbb{N}$, the limit $\lim_{n\to\infty}(a_n)^m$ exists and is equal to c^m.
5. If for every $n \in \mathbb{N}$, the number a_n is non-negative, then for every positive integer $k \in \mathbb{N} \setminus \{0\}$, the limit $\lim_{n\to\infty}(a_n)^{\frac{1}{k}}$ exists and is equal to $c^{\frac{1}{k}}$.

4.4 The squeeze theorem

Theorem 4.4.1 – **The squeeze theorem** Let $a, b, c : \mathbb{N} \to \mathbb{R}$ be three sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$a_n \leq b_n \leq c_n$$

and assume $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n - L$ for some $L \in \mathbb{R}$. Then $\lim_{n\to\infty} b_n$ exists and is equal to L.

4.5 Divergence to ∞ and $-\infty$

Definition 4.5.1 – We say a sequence $a : \mathbb{N} \to \mathbb{R}$ *diverges to* ∞ and write

$$\lim_{n\to\infty} = \infty$$

if

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad a_n > M.\end{aligned}$$

Similarly, we say a sequence (a_n) *diverges* to $-\infty$ and write

$$\lim_{n\to\infty} a_n = -\infty$$

if

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad a_n < M.\end{aligned}$$

Proposition 4.5.2 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} a_n = \infty.$$

Then the sequence (a_n) is bounded from below.
Similarly, let $b : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} b_n = -\infty.$$

Then the sequence (b_n) is bounded from above.

4.6 Limit theorems for improper limits

Theorem 4.6.1 – **Limit theorems for improper limits** Let $a, b, c, d : \mathbb{N} \to \mathbb{R}$ be four sequences such that

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} c_n = -\infty$$

the sequence (b_n) is bounded from below and the sequence (d_n) is bounded from above. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence bounded below by some $\mu > 0$. Then

i. $\lim_{n\to\infty}(a_n + b_n) = \infty$

ii. $\lim_{n\to\infty}(c_n + d_n) = -\infty$

iii. $\lim_{n\to\infty}(\lambda_n a_n) = \infty$

iv. $\lim_{n\to\infty}(\lambda_n c_n) = -\infty$

Proposition 4.6.2 – Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be two sequences. Then

1. $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}(-a_n) = -\infty$.

2. $\lim_{n\to\infty} b_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

4.7 Standard sequences

4.7.1 Geometric sequence

Proposition 4.7.2 – **Standard limit of of geometric sequence** Let $q \in \mathbb{R}$. The sequence (a_n) defined by $a_n := q^n$ for $n \in \mathbb{N}$

- converges to 0 if $q \in (-1, 1)$
- converges to 1 if $q = 1$
- diverges to ∞ if $q > 1$
- diverges, but not to ∞ or $-\infty$ if $q \leq -1$

4.7.3 The n^{th} root of n

Proposition 4.7.4 – **Standard limit of the n^{th} root of n** The sequence (a_n) defined by $a_n := \sqrt[n]{n}$ for $n \in \mathbb{N}$ converges to 1.

Corollary 4.7.5 – Let $a > 0$. Then the sequence (b_n) defined by $b_n := \sqrt[n]{a}$ converges to 1.

4.7.6 The number e

First let's define the sequence (a_n) by

$$a_n := \left(1 + \frac{1}{n}\right)^n .$$

We show that (a_n) is increasing and bounded from above by 3. Hence (a_n) converges to some $e \in \mathbb{R}$ by the monotone convergence theorem.

Lemma 4.7.7 – The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is increasing.

Lemma 4.7.8 – The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is bounded from above by 3.

By these two lemmas, the sequence

$$n \mapsto \left(1 + \frac{1}{n}\right)^n$$

converges.

Definition 4.7.9 – **(Standard limit of e)** We define the number e by

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

4.7.10 Exponentials beat powers

Proposition 4.7.11 – Let $a \in (1, \infty)$ and let $p \in (0, \infty)$. Then

$$\lim_{n \to \infty} \frac{n^p}{a^n} = 0.$$

4.8 Sequences with values in $\mathbb{R}^d$

Proposition 4.8.1 – Consider the metric space $(\mathbb{R}^d, \|\cdot\|_2)$. Let $z \in \mathbb{R}^d$ and let $x : \mathbb{N} \to \mathbb{R}^d$ be a sequence. Denote by y_i the ith componenet of a vector $y \in \mathbb{R}^d$. Then the seequence $(x^{(n)})$ converges to z if and only if for all $i \in \{1, \ldots, d\}$, teh sequence $(x_i^{(n)})$ converges to z_i.

5 Series

5.1 Definition

Definition 5.1.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Let $K \in \mathbb{N}$. We say that a series

$$\sum_{n=K}^{\infty} a_n$$

is *convergent* if the associatedd *sequence of partial sums* $S_k : \mathbb{N} \to V$, i.e. the sequece $(S_K^n)_{n\in\mathbb{N}}$ converges. The term S_K^n is, for $n \in \mathbb{N}$, defined as

$$S_K^n := \sum_{k=K}^{n} a_k$$

If $K = 0$, we ussually jusst write S^n or even S_n instead of S_0^n.
If the series $\sum_{n=K}^{\infty} a_n$ is convergent, the *value* of the series is by defintion equal to the limit of the sequence of partial sums, i.e.

$$\sum_{k=K}^{\infty} a_k := \lim_{n\to\infty} S_k^n = \lim_{n\to\infty} \sum_{k=K}^{\infty} a_k$$

5.2 Geometric series

Proposition 5.2.1 – Let $a \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Proof. We consider

$$\begin{aligned}
(1-a)\sum_{k=0}^{n} a^k &= \sum_{k=0}^{n} a^k - a\sum_{k=0}^{n} a^k \\
&= \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} \\
&= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k \\
&= 1 - a^{n+1}
\end{aligned}$$

□

Proposition 5.2.2 – **Geometric series** Let $a \in (-1, 1)$. Then the series

$$\sum_{k=0}^{\infty} a^k$$

is convergent and has the value
$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

5.3 The harmonic series

Proposition 5.3.1 – **Harmonic series** The series
$$\sum_{k=1}^{\infty} \frac{1}{k}$$
diverges.

5.4 The hyperharmonic series

Proposition 5.4.1 – **Hyperharmonic series** Let $p > 1$. Then the series
$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$
converges.

Example 5.4.2 Here is an example of a series taking values in the normed vector space $(\mathbb{R}^2, \|\cdot\|)$:
$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2}, \left(\frac{1}{2}\right)^k \right)$$

5.5 Only the tail matters for convergence

Lemma 5.5.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence taking values in V. Let $K, L \in \mathbb{N}$. The series
$$\sum_{n=K}^{\infty} a_n$$
is conovergent is and only if the series
$$\sum_{n=L}^{\infty} a_n$$
is convergent. Moreover, if either the series converges, and $K < L$, then
$$\sum_{n=K}^{\infty} a_n = \sum_{n=K}^{L-1} + \sum_{n=L}^{\infty} a_n.$$

Proposition 5.5.2 – Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and assume that the series
$$\sum_{k=M}^{\infty} a_k$$

is convergent. Then

$$\lim_{m\to\infty}\sum_{k=m}^{\infty} a_k = 0.$$

Proposition 5.5.3– **Index shift for series** Let $a:\mathbb{N}\to V$ be a sequence, let $M\in\mathbb{N}$ and let $\ell\in\mathbb{N}$. Then the series

$$\sum_{k=M}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=M}^{\infty} a_{k+\ell}$$

converges. Moreoever, if either series converges, then

$$\sum_{k=M}^{\infty} a_{k+\ell} = \sum_{k=M+\ell}^{\infty} a_k.$$

5.6 Divergence test

Proposition 5.6.1– Let $(V,\|\cdot\|)$ be a normed vector space, and let $a:\mathbb{N}\to V$ be a sequence in V. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent to $L\in V$. Then

$$a_n = S_n - S_{n-1}$$

where S_n denote the partial sum $\sum_{k=0}^{n} a_k$. Because S_n and S_{n-1} are both convergent to L, the sequence (a_n) is convergent as well and converges to $L-L=0$. □

Theorem 5.6.2– **Divergence test** Let $(V,\|\cdot\|)$ be a normed vector space and let $a:\mathbb{N}\to V$ be a sequence in V. Suppose the limit $\lim_{n\to\infty} a_n$ does not exist or is not equal to 0. Then the series

$$\sum_{n=0}^{\infty} a_n$$

is divergent.

5.7 Limit laws for series

Theorem 5.7.1– **Limit laws for series** Let $(V,\|\cdot\|)$ be a normed vector space and let

$a, b : \mathbb{N} \to V$ be sequences in V. Suppose the series

$$\sum_{n=0}^{\infty} a_n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n$$

are convergent. Suppose $\lambda \in \mathbb{R}$. Then

1. The series
$$\sum_{n=0}^{\infty} (a_n + b_n)$$
is converget and converges to
$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

2. The series
$$\sum_{n=0}^{\infty} \lambda a_n$$
is convergent and converges to
$$\lambda \sum_{n=0}^{\infty} a_n.$$

6 Series with positive terms

6.1 Comparison test

Theorem 6.1.1 – **Comparison test** Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences. Assume that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \leq b_n$. Then

1. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Then the series $\sum_{n=1}^{\infty} a_n$ converges as well.
2. Suppose the series $\sum_{n=1}^{\infty} a_n$ diverges. Then the series $\sum_{n=1}^{\infty} b_n$ diverges as well.

Example 6.1.2 Consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}.$$

We first observe that for all $k \geq 2$ we have

$$\frac{k}{k^2 - 1} \geq \frac{k}{k^2} = \frac{1}{k}.$$

Because the series

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

diverges, the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}$$

diverges as well byt the comparison test.

6.2 Limit comparison test

Theorem 6.2.1 – **Limit comparison test** Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences.

1. Assume the series $\sum_{k=1}^{\infty} b_k$ converges and assume the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists. Then the series $\sum_{k=1}^{\infty} a_k$ converges as well.

2. Assume the series $\sum_{k=1}^{\infty} b_k$ diverges and assume the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is strictly larger than zero, or that the limit is infinity. Then the series $\sum_{k=1}^{\infty} a_k$ diverges as well.

Example 6.2.2 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}.$$

We use sequences $a, b : \mathbb{N} \to [0, \infty)$ defined for $k \geq 2$ by

$$a_k = \frac{k}{k^2 + 1}$$

and

$$b_k = \frac{1}{k}.$$

Then

$$\frac{a_k}{b_k} = \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \frac{1}{1 + \frac{1}{k^2}}.$$

By limit laws, we find that the limit of the denominator is 1, i.e.

$$\lim_{k\to\infty} \left(1 + \frac{1}{k^2}\right) = \lim_{k\to\infty} 1 + \lim_{k\to\infty} \frac{1}{k^2} = 1 + 0 = 1.$$

Therefore, we may apply the limit law for the quoteient and conclude that

$$lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{\lim_{k\to\infty}(1 + \frac{1}{k^2})} = \frac{1}{1} = 1.$$

The series $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges and therefore it follows from the Limit Comparison Test that the series

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} \frac{k}{k^2 + 1}$$

diverges as well.

6.3 Ratio test

Theorem 6.3.1 – **Ratio Test** Let $a : \mathbb{N} \to [0, \infty)$ be a sequence.

1. if there exists an $N \in \mathbb{N}$ and a $q \in (0, 1)$ such that for all $n \geq N$, it holds that

$$\frac{a_n + 1}{a_n} \leq q$$

, then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. if there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it hodls that

$$\frac{a_n + 1}{a_n} \geq 1,$$

then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.4 Limit ratio test

Theorem 6.4.1 – **Limit Ratio Test** Let $a : \mathbb{N} \to (0, \infty)$ be a sequence.

1. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q \in [0, 1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.4.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

6.5 Root test

Theorem 6.5.1 – **Root test** Let (a_n) be a sequence of non-negative real numbers.

1. If there exists an $N \in \mathbb{N}$ and a $q \in (0, 1)$ such that for all $n \geq N$, it holds that

$$\sqrt[n]{a_n} \leq q,$$

   then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that

$$\sqrt[n]{a_n} \geq 1,$$

   then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.6 Limit root test

Theorem 6.6.1 – **Limit Root Test** Let (a_n) be a sequence of non-negative real numbers.

1. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q \in [0, 1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.6.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1.$$

7 Series with general terms

7.1 Series with real terms: the Leibniz test

Theorem 7.1.1 – Leibniz test, a.k.a Alternating series test Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences such that for all $k \in \mathbb{N}$, $b_k = (-1)^k a_k$. Assume that there exists a $K \in \mathbb{N}$ such that

1. $a_k \geq 0$ for every $k \geq K$,
2. $a_k \geq a_{k+1}$ for every $k \geq K$,
3. $\lim_{k\to\infty} a_k = 0$.

Then, the series

$$\sum_{k=K}^{\infty} b_k = \sum_{k=K}^{\infty} (-1)^k a_k$$

is convergent. In addition, the following esitmate holds for every $N \geq K$,

$$\left| S_N - \sum_{k=K}^{\infty} b_k \right| \leq a_{N+1}.$$

where for all $n \in \mathbb{N}$, $S_n := \sum_{k=K}^{\infty} b_k$.

Example 7.1.2 We claim that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

We would like to apply the Alternating series test. To do so, we need toe check its conditions. We define the sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_k := \frac{1}{k}$$

for $k \geq 1$ (and $a_0 = a_1 = 1$).

We now check the conditions for the Alternating Series Test.

1. We need to show that $a_k \geq 0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq 0.$$
2. We need to show that $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq \frac{1}{k+1} = a_{k+1}.$$
3. We need to show that
$$\lim_{k\to\infty} a_k = 0$$
. This follow as this is a standard limit.

It follows from the Alternating Series Test that the series

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$$

converges.

7.2 Series charactersization of completeness in normed vector space

Definition 7.2.1 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence of vectors in V. We say the series

$$\sum_{k=0}^{\infty} a_k$$

converges *absolutely* if

$$\sum_{k=0}^{\infty} \|a_k\|$$

converges.

Definition 7.2.2 – **Series characterization of completeness** We say a normed vector space $(V, \|\cdot\|)$ satisfies the *series characterization of completeness* if every series in V that is absolutely convergent is also convergent.

Proposition 7.2.3 – Every finite-dimensional normed vector space satisfies the series characterization of completeness.

Example 7.2.4 Consider the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}.$$

Since this is not an alternataing series, we cannot apply the Leibniz test.

However, for every k
$in\mathbb{N} \setminus \{0\}$, we have

$$\left|\frac{\sin(k)}{k^2}\right| \leq \frac{1}{k^2}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a standard hyperharmonic seris, of which we know that it converges. By the Cmomparison Test, we conclude that the series

$$\sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^2}\right|$$

converges as well.

Therefore, the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges absolutely. Since $(\mathbb{R}, |\cdot|)$ is complete, we find that

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges.

Definition 7.2.5 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. We say that a series

$$\sum_{k=0}^{\infty} a_k$$

converges *conditionally* if it converges but does not converge absolutely.

7.3 The Cauchy product

Theorem 7.3.1 – **Cauchy product** Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences. Assume that the series

$$\sum_{k=0}^{\infty} a_k$$

and

$$\sum_{k=0}^{\infty} b_k$$

converge absolutely. Then, the series

$$\sum_{k=0}^{\infty} c_k$$

converges absolutely as well, where

$$c_k := \sum_{\ell=0}^{k} a_\ell b_{k-\ell},$$

and

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

8 Subsequences, $\limsup$ and $\liminf$

8.1 Index sequences and subsequences

Definition 8.1.1 – **Index sequence** We say a sequence $n : \mathbb{N} \to \mathbb{N}$ is an *index sequence* if it is strictly increasing.

Example 8.1.2 The sequence $n : \mathbb{N} \to \mathbb{N}$ defined by

$$n_k := 2k$$

is a strictly increasing sequence of natural numbers. In other words, it is an index sequence.

Definition 8.1.3 – **Subsequence** Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. A sequence $b : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of a if there exists an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that $b = a \circ n$

Just as we often write $(a_n)_{n\in\mathbb{N}}$ for a sequence called a, we often write $(a_{n_k})_{k\in\mathbb{N}}$ for the subsequence $a \circ n$.

8.2 (Sequential) accumulation points

Definition 8.2.1 – **(Sequential accumulation points)** Let (X, dist) be a metric space. A point $p \in X$ is called an *accumulation point* of a sequence $a : \mathbb{N} \to X$ if there is a subsequence $a \circ n$ of a such that $a \circ n$ converges to p.

8.3 Subsequences of a converging sequence

Proposition 8.3.1 – Let (X, dist) be a metric space. Let (a_n) be a sequence in X converging to $p \in X$. Then every subsequence of (a_n) is convergent to p.

8.4 $\limsup$

Consider a real-valued sequence (a_n) that is bounded from above and does not diverge to $-\infty$. We can then define a new sequence

$$k \mapsto \sup_{n\geq k} a_n.$$

Note that this sequence is decreasing, beacuse for larger k the supremum is taken over a smaller set.

Lemma 8.4.1 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence that is bounded from above and does not diverge to $-\infty$. Then, the sequence $k \mapsto \sup_{n\geq k} a_n$ is bounded from below.

Since the sequence $k \mapsto \sup_{n\geq k} a_n$ is decreasing and bounded from below, it has a limit, and the limit is in fact equal to the infumum of the sequence. This limit is called the $\limsup$

$$\begin{aligned}\limsup_{n\to\infty} a_n &:= \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n \\ &= \lim_{k\to\infty} \left(\sup_{n\geq k} a_n \right)\end{aligned}$$

Proposition 8.4.2 – **Alternative characterization of** $\limsup$ Let (a_n) be a real-valued sequence. Let $M \in \mathbb{R}$. Then, $M = \limsup_{n\to\infty} a_n$ if and only if

i. For every $\epsilon > 0$,
there exists $N \in \mathbb{N}$,
for all $\ell \geq N$,
$a_\ell < M + \epsilon$

ii. For every $\epsilon > 0$,
for all $k \in \mathbb{N}$,
there exists $m \geq k$,
$a_m > M - \epsilon$

Theorem 8.4.3 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded from above and does not diverge to $-\infty$. Then $\limsup_{\ell\to\infty} a_\ell$ is a (sequential) accumulation point of a, i.e. there exists a subsequences of a that converges to $\limsup_{\ell\to\infty} a_\ell$.

Corollary 8.4.4 – **Bolzano-Weierstrass** Every bounded, real-valued sequence has a subsequence that converges in $(\mathbb{R}, \mathrm{dist}_\mathbb{R})$.

Theorem 8.4.5 – Suppose a sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$. Then

$$\limsup_{\ell\to\infty} a_\ell$$

is the maximum of the set of sequential accumulation points.

8.5 lim inf

Similarly to the lim sup, we can define the lim inf. In some sense,

$$\liminf_{\ell\to\infty} a_\ell = -\limsup_{\ell\to\infty}(-a_\ell)$$

More precisely,

$$\begin{aligned}\liminf_{\ell\to\infty} a_\ell &:= \sup_{\ell\in\mathbb{N}} \inf_{k\geq\ell} a_k \\ &= \lim_{\ell\to\infty}\left(\inf_{k\geq\ell} a_k\right)\end{aligned}$$

Proposition 8.5.1 – **Alternative characterization of** lim inf Let $a : \mathbb{N} \to \mathbb{R}$ and $M \in \mathbb{R}$. Then M equals $\liminf_{\ell\to\infty} a_\ell$ if and only if

1. For every $\epsilon > 0$,
there exists $N \in \mathbb{N}$,
for all $\ell \geq N$,
$a_\ell > M - \epsilon$

2. For every $\epsilon > 0$,
for all $K \in \mathbb{N}$,
there exists $m \geq K$,
$a_m < M + \epsilon$

Theorem 8.5.2 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is a sequential accumulation point of the sequence a, i.e. there is a subsequence of a that converges to $\liminf_{\ell\to\infty} a_\ell$.

Theorem 8.5.3 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is the minimum of the set of sequential accumulation points.

8.6 Relations between $\lim$, $\limsup$ and $\liminf$

Proposition 8.6.1 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} = L$$

Proposition 8.6.2 – Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences, such that there exists an $N \in \mathbb{N}$ such that for all $\ell \geq N$, $a_\ell \leq b_\ell$. Then

$$\limsup_{\ell\to\infty} a_\ell \leq \limsup_{\ell\to\infty} b_\ell$$

and

$$\liminf_{\ell\to\infty} a_\ell \leq \liminf_{\ell\to\infty} b_\ell.$$

9 Point-set topology of metric spaces

Here we introduce three properties for subsets of a metric space: *closedness*, *completeness*, and *compactness*. For those three properties we known that every compact set is complete, and every complete set is closed. However, not every closed set is complete, and not every complete set is compact.

9.1 Open sets

Definition 9.1.1 – Open set Let (X, dist) be a metric space. We say that a subset $O \subseteq X$ is *open* if every $x \in O$ is an interior point of O.

Now we need to say what it means to be an interior point.

Definition 9.1.2 – Interior point Let (X, dist) be a metric space and let A be subset of X. A point $a \in A$ is called an *interior point* of A if

$$\text{there exists } r > 0$$
$$B(a, r) \subseteq A$$

where $B(a, r)$ is an (open) ball around point a with radius r (definition 1.1.4).

Proposition 9.1.3 – Let (X, dist) be a metric space. The ball

$$B(p, r) := \{x \in X | \mathrm{dist}(x, p) < r\}$$

is indeed open.

Proposition 9.1.4 – 'Open' intervals are open Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals $(a, b), (-\infty, b), (a, \infty)$ are all open subsets of $\mathbb{R}$.

Proposition 9.1.5 – Let (X, dist) be a metric space. Then both the empty set $\emptyset$ and the set X itself (both of these are subsets of X) are open.

Proof. We first show that the empty set is open. We argue by contradiction. Suppose there exists a point $x \in \emptyset$ such that x is not an interior point of X. Then we have a contradiction, because the empty set has no elements.

We will now show that X is open. Let $x \in X$. We will show that x is an interior point, i.e. we will show that there exists an $r > 0$ such that $B(x, r) \subseteq X$.
Choose $r := 1$. Then $B(x, r) = B(x, 1) \subseteq X$. $\square$

The set of all interior points of a subset $A \subseteq X$ is called the *interior* of the set A.

Definition 9.1.6 – The interior of a set Let (X, dist) be a metric space and let $A \subseteq X$ be a subset of X. Then the *interior* of the set A, denoted by int A is the set of all interior points of A, i.e int A is defined as

$$\mathrm{int}\ A := \{x \in A \mid x \text{ is an interior point of } A\}.$$

Example 9.1.7 The interior of the interval $[2, 5)$ (viewed as subset of $(\mathbb{R}, |\cdot|)$) is the interval $(2, 5)$.

The interior of a set is always open.

Proposition 9.1.8 – Let (X, dist) be a metric space and let $A \subseteq X$. Then $\operatorname{int} A$ is open.

The union of open sets is always open

Unions of open sets are always open. You may recall that if $\mathcal{I}$ is some set and if for every $\alpha \in \mathcal{I}$ we have a subset $A_\alpha \subseteq X$, then the union

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha$$

is defined as

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha := \{x \in X \mid \text{ there exists } \alpha \in \mathcal{I} \text{ such that } x \in A_\alpha\}$$

9.2 Closed sets

9.3 Cauchy sequences

9.4 Completeness

9.5 Series characterization of completeness in normed vector spaces

10 Compactness

10.1 Boundedness and total boundedness

10.2 Alternative characterization of compactness

11 Limits and continuity

11.1 Accumulation points

11.2 Limit in an accumulation point

11.3 Uniqueness of limits

11.4 Sequential characterization of limits

11.5 Limit laws

11.6 Continuity

11.7 Sequential characterization of continuity

11.8 Rules for continuous functions

11.9 Images of compact sets under continuous functions are compact

11.10 Uniform continuity

12 Real-valued functions

12.1 More limit laws

12.2 Building of standard functions

12.3 Continuity of standard functions

12.4 Limits from the left and from the right

12.5 The extended real line

12.6 Limits to ∞ or $-\infty$

12.7 Limits at ∞ and $-\infty$

12.8 The Intermediate Value Theorem

12.9 The Extreme Value Theorem

12.10 Equivalence of norms

12.11 Bounded linear maps and operator norms

13 Differentiability

13.1 The derivative as a function

13.2 Constant and linear maps are differentiable

13.3 Bases and coordinates

13.4 The matrix representation

13.5 The chain rule

13.6 Sum, product and quotient rules

13.7 Differentiability of components

13.8 Differentiability implies continuity

13.9 Derivative vanishes in local maxima and minima

13.10 The Mean Value Theorem

14 Differentiability of standard functions

14.1 Global context

14.2 Polynomials and rational functions are differentiable

14.3 Differentiability of the standard functions

15 Directional and partial derivatives

15.1 A recurring and very important construction

15.2 Directional derivatives

15.3 Partial derivatives

15.4 The Jacobian of a map

15.5 Linearization and tangent planes

15.6 The gradient of a function

16 The Mean-Value Inequality

16.1 The mean-value inequality for functions defined on an interval

16.2 The mean-value inequality for functions on general domains

16.3 Continuous partial derivatives imply differentiability

17 Higher order derivatives

17.1 Multilinear maps

17.2 Relation to n-fold directional derivatives

17.3 A criterion for higher differentiability

17.4 Symmetry of second order derivatives

17.5 Symmetry of higher-order derivatives

18 Polynomials and approximation by polynomials

18.1 Homogeneous polynomials

18.2 Taylor's theorem

18.3 Taylor approximations of standard functions

19 Banach fixed point theorem

20 Implicit function theorem

20.1 The objective

20.2 Notation

20.3 The implicit function theorem

20.4 The inverse function theorem

21 Function sequences

21.1 Point-wise convergence

21.2 Uniform convergence

21.3 Preservation of continuity under uniform convergence

21.4 Differentiability theorem

21.5 The normed vector space of bounded functions

22 Function series

22.1 The Weierstrass M-test

22.2 Conditions for differentiation of function series

23 Power series

23.1 Convergence of power series

23.2 Standard functions defined as power series

23.3 Operations with power series

23.4 Differentiation of power series

23.5 Taylor series

24 Riemann integration in one dimension

24.1 Riemann integrable functions and the Riemann integral

24.2 Sums, products of Riemann integrable functions

24.3 Continuous functions are Riemann integrable

24.4 The fundamental theorem of calculus

25 Riemann integration in multiple dimensions

25.1 Partitions in multiple dimensions

25.2 Riemann integral on rectangles in $\mathbb{R}^n$

25.3 Properties of the multidimensional Riemann integral

25.4 Continuous functions are Riemann integrable

25.5 Fubini's theorem

25.6 The (topological) boundary of a set

25.7 Jordan content

25.8 Integration over general domains

25.9 The volume of bounded sets

26 Change-of-variables Theorem

26.1 Polar coordinates

26.2 Cylindrical coordinates

26.3 Spherical coordinates