Assignment 3

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1 5.9.1

Problem 1.1 Let (X, dist) be a metric space. Let $p \in X$ and assume that the sequence (a_n) is given by $a_n = p$ for every $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} a_n = p$.

Proof. To show that the sequence (a_n) converges to p, we need to show that

$$\begin{gathered}\text{for all } \epsilon > 0,\\ \text{there exists } N > 0,\\ \text{for all } n > N,\\ \text{dist}(a_n, p) < \epsilon.\end{gathered}$$

Let $\epsilon > 0$.
Choose $N = 1$.
Take $n > N$,
It holds that $\text{dist}(a_n, p) = \text{dist}(p, p) = 0 < \epsilon$.
We conclude that $\lim_{n\to\infty} a_n = p$. □

2 5.9.2

Problem 2.1 Let (x, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence in X. Let $k \in \mathbb{N}$ and $p \in X$. Then the sequence (a_n) converges to p if and only if the sequence (a_{n+k}) converges to p.

Proof. To show that the sequence (a_n) converges to p if and only if the sequence (a_{n+k}) converges to p, we need to show both directions.
1. Forward direction:

Assume that the sequence (a_n) converges to p.
It holds that (for all $\epsilon > 0$, there exists $N > 0$ for which $n > N$ implies $\text{dist}(a_n, p) < \epsilon$).
We need to show that (for all $\epsilon > 0$, there exists $N > 0$ for which $n > N$ implies $\text{dist}(a_{n+k}, p) < \epsilon$).
□

3 5.9.3

Problem 3.1 Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Assume that the limit $\lim_{n\to\infty} a_n$ exists and is equal to $p \in V$ and that the limit $\lim_{n\to\infty} b_n$ exists and is equal to $q \in V$. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $\mu \in \mathbb{R}$. Assume that $\lim_{n\to\infty} \lambda_n = \mu$. Prove that the limit $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $p + q$.

Proof. □

4 5.9.4

Problem 4.1 Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a bounded sequence in X. Let $p \in X$. Define also the sequence $s : \mathbb{N} \to \mathbb{R}$ by

$$s_k = \sup\{\mathrm{dist}(a_l, p) \mid l \in \mathbb{N}, l \geq k\}.$$

Show that $\lim n \to \infty a_n = p$ if and only if

$$\inf_{k\in\mathbb{N}} s_k = 0$$

Proof. □