Discrete Structures

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1 Counting

2 Graphs

2.1 Graphs

Definition 2.1.1 – **Isomorhpism** Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic* if there exists a bijection $f : V \to V'$ such that $\forall x, y \in V, x \neq y[\{x, y\} \in E \iff \{f(x), f(y)\} \in E']$ Such a bijection f is called an *isomorphism* from G to G'.

Theorem 2.1.2 – **Counting graphs** Let $V = \{1, 2, \ldots, n\}$. There are $2^{\binom{n}{2}}$ possible graphs.

2.2 Subgraphs

Definition 2.2.1 – **Subgraph** A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 2.2.2 – **Induced subgraph** A graph $G' = (V', E')$ is an *induced subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{x, y\} \in E \mid x, y \in V'\}$.

2.3 Common graphs

Complete graphs

Definition 2.3.1 – **Complete graph** A graph $G = (V, E)$ is a *complete graph* if $\forall x, y \in V, x \neq y[\{x, y\} \in E]$.

Proposition 2.3.2 – Let $V = \{1, 2, \ldots, n\}$ and K_n be the complete graph on V. Then K_n has $\binom{n}{2}$ edges.

Star graphs

Definition 2.3.3 – **Star graph** A graph $G = (V, E)$ is a *star graph* if $V = \{u\} \cup \{v_1, \ldots, v_n\}$ and $E = \{\{u, v_j\} \mid j = 1, 2, \ldots, n\}$.

Proposition 2.3.4 – Let $V = \{1, 2, \ldots, n\}$ and S_n be the star graph on V. Then S_n has $n - 1$ edges.

Complete bipartite graphs

Definition 2.3.5 – **Complete bipartite graph** A graph $G = (V, E)$ is a *complete bipartite graph* if $V = V_1 \cup V_2$ and $E = \{\{x, y\} \mid x \in V_1, y \in V_2\}$.

Proposition 2.3.6 – Let $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{n+1, n+2, \ldots, n+m\}$ and $K_{n,m}$ be the complete bipartite graph on $V_1 \cup V_2$. Then $K_{n,m}$ has $n \cdot m$ edges.

Paths

Definition 2.3.7 – **Path** A graph $G = (V, E)$ is a *path* if $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \ldots, n-1\}$.

Proposition 2.3.8 – Let $V = \{1, 2, \ldots, n\}$ and P_n be the path on V. Then P_n has $n-1$ edges.

Cycles

Definition 2.3.9 – **Cycle** A graph $G = (V, E)$ is a *cycle* if $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \ldots, n-1\} \cup \{\{v_n, v_1\}\}$.

Proposition 2.3.10 – Let $V = \{1, 2, \ldots, n\}$ and C_n be the cycle on V. Then C_n has n edges.

2.4 Walks

Definition 2.4.1 – **Walk** A *walk* in a graph $G = (V, E)$ is a sequence of vertices $v_1, v_2, \ldots, v_n$ such that $\forall i \in \{1, 2, \ldots, n-1\}[\{v_i, v_{i+1}\} \in E]$.
Another definition is that a walk is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $\forall i \in \{1, 2, \ldots, n-1\}[e_i = \{v_i, v_{i+1}\}]$.

2.5 Connected and Components

Definition 2.5.1 – **Connected** A graph $G = (V, E)$ is *connected* if $\forall x, y \in V$ [$\exists$ a walk from x to y].

Definition 2.5.2 – **Component** The components of a graph G are the equivalence classes defined by the relation on the set $V(G)$, where $x \;\; y \iff \exists$a walk from x to y in G

Theorem 2.5.3 – Any graph $G = (V, E)$ where each vertex $v \in V$ has $\deg_G(v) \geq \frac{n-1}{2}$ is connected, where $n = |V|$.

2.6 Graph distance

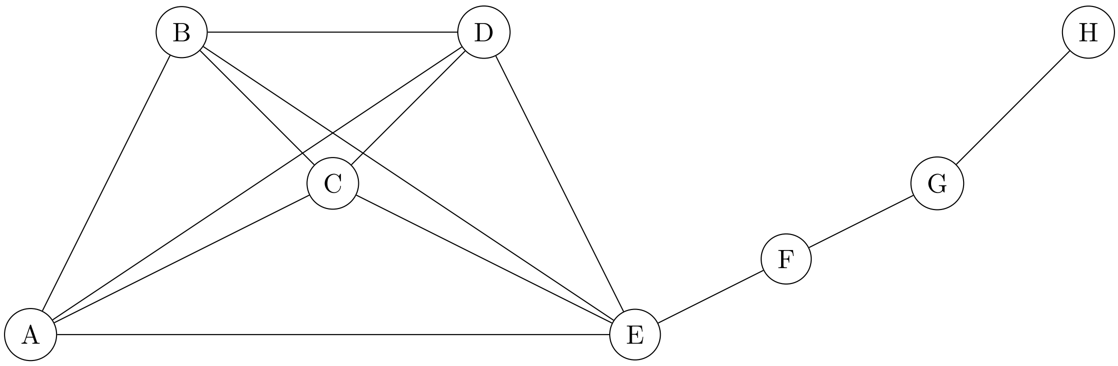
Definition 2.6.1 – **Distance** The *distance* between two vertices x and y in a graph G is the length of the shortest walk from x to y in G.

Theorem 2.6.2 – Let $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a^k_{i,j}$ denote the element of A^k at position (i, j). Then $a^k_{i,j}$ is the number of walks of length k from v_i to v_j.

2.7 Degree sequence

Definition 2.7.1 – **Degree sequence** The *degree sequence* of a graph $G = (V, E)$ is the sequence of degrees of the vertices of G.
The order doesn't matter. Generally, we sort the sequence in non-decreasing order.

Example 2.7.2 The degree sequence of the graph G is $(1, 2, 2, 4, 4, 4, 4, 4)$.



Example 2.7.3 A path of length n has degree sequence $(2, 1, 1, \ldots, 1, 2)$.
Example 2.7.4 A cycle of length n has degree sequence $(2, 2, \ldots, 2)$.
Example 2.7.5 A complete graph of n vertices has degree sequence $(n-1, n-1, \ldots, n-1)$.
Example 2.7.6 A complete bipartite graph $K_{n,m}$ has degree sequence $(m, m, \ldots, m, n, n, \ldots, n)$.
Remark 2.7.7. A sequence of non-negative integers is a degree sequence of some graph if and only if the sum of the integers is even.

Lemma 2.7.8 – Hankshake lemma For any graph $G = (V, E)$ the number of vertices of odd degree is even.

Theorem 2.7.9 – Let $D = (d_1, \ldots, d_n)$ be a sequence of natural numbers, for $n > 1$, where $d_1 \leq d_2 \leq \cdots \leq d_n \leq n_1$ and let D' denote the sequence $(d'_1, \ldots, d'_{n-1})$ where

$$d'_i = \begin{cases} d_i & \text{for } i < n - d_n \\ d_i - 1 & \text{for } i \geq n - d_n \end{cases}$$

Then D is a degree sequence if and only if D' is a degree sequence.

2.8 Eulerian graphs

Definition 2.8.1 – **Closed Eulerian tour** A closed walk containing all the vertices and edges, and each edge exactly once, is called a *closed Eulerian tour*.

Definition 2.8.2 – **Eulerian graph** A graph $G = (V, E)$ is *Eulerian* if it has a closed Eulerian tour.

Theorem 2.8.3 – A graph $G = (V, E)$ is Eulerian if and only if G is connected and each vertex has even degree.

2.9 Hamiltonian cycle

Definition 2.9.1 – **Hamiltonian cycle** A *Hamiltonian cycle* is a cycle that contains all the vertices of a graph.

2.10 Graph operations

Definition 2.10.1 – **Graph operations** Let $G = (V, E)$ be a graph

1. Edge deletion: $G - e = (V, E \setminus \{e\})$, where $e \in E$.
2. Edge insertion: $G + e = (V, E \cup \{e\})$, where $e \in \binom{V}{2} \setminus E$
3. Vertex deletion: $G - v = (V - \{v\}, \{e \in E \mid v \in e\})$, where $v \in V$.

4. Edge subdivision: $G\%e = (V \cup \{z\}, (E \setminus \{\{x,y\}\}) \cup \{\{x,z\},\{z,y\}\})$ where $e = \{x,y\} \in E$ and $z \notin V$.

2.11 K-vertex-connectivity

Definition 2.11.1 – A graph G is called *k-vertex-connected* if $|V(G)| \geq k+1$ and $G - v$ is connected for every $v \in V(G)$. Often we say G is k-connected.

Example 2.11.2 K_n is $(n-1)$-connected.

Theorem 2.11.3 – A graph $G = (V, E)$ is 2-connected if and only if for any two vertices $v, w \in V$, there exists a cycle containing v and w.

3 Trees

3.1 Definition

Definition 3.1.1 – **Tree** A *tree* is a connected graph with no cycles.

Theorem 3.1.2 – For a non-empty graph $G = (V, E)$, the following are equivalent:

1. The graph G is a tree.
2. For any two distinct vertices $u, v \in V$, there is a unique path from u to v. (unique paths)
3. The graph G is connected and $\forall e \in E, G - e$ is disconnected. (minimal connected graph)
4. The graph G is acyclic and $\forall e \in \binom{V}{2} \setminus E, G + e$ contains a cycle. (maximal acyclic graph)
5. G is connected and $|V| = |E| + 1$. (Euler's formula)

3.2 Induction on trees

Lemma 3.2.1 – **end-vertex** Every tree with at least two vertices has at least two leaves.

Lemma 3.2.2 – **tree-growing** Let G be a graph and v be a leaf in G. Then $G - v$ is a tree.

3.3 Rooted trees

Definition 3.3.1 – **Rooted tree** A *rooted tree* is a pair (T, r) where T is a tree and $r \in V(T)$ is a distinguished vertex of T called *the root*.
A node u in a rooted tree T may have a:

1. parent: the unique vertex $v \in V(T)$ such that $\{u, v\} \in E(T)$ and v lies on the unique path from u to r,
2. ancestor: a vertex $v \in V(T)$ such that v lies on the unique path from u to r,
3. child: a vertex $v \in V(T)$ where u is the parent of v,
4. descendant: a vertex $v \in V(T)$ where u is an ancestor of v,

3.4 Subtree

Definition 3.4.1 – **Subtree** The *subtree rooted at* $v \in V(T)$ in a rooted tree is the induced subgraph defined by all vertices that are descendants of v, rooted at v.

3.5 Binary trees

Definition 3.5.1 – **Binary tree** A *binary tree* is a rooted tree where each node has at most two children.

Definition 3.5.2 – **Strict binary tree** A *strict binary tree* is a rooted tree where each node has exactly zero or two children.

Lemma 3.5.3 – A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.

3.6 Ear decomposition

Lemma 3.6.1 – Let $G = (V, E)$ be a 2-connected graph, then

1. $G\%e$ is 2-connected graph, where $e \in E$
2. $G + e$ is a 2 connected graph, where $e \in \binom{V}{2} \setminus E$

Proposition 3.6.2 – Any 2-connected graph $G = (V, E)$ can be connected from K_3 by a sequence of edges subdivisions and edge additions.

Definition 3.6.3 – **Ear decomposition** An *ear decomposition* of a graph $G = (V, E)$ is a sequence of subgraphs $G_0, G_1, \ldots, G_k$ of G such that

1. G_0 is a cycle,
2. $G_k = G$,
3. $G_i = G_{i-1}\%e_i$ or $G_i = G_{i-1} + e_i$ for $i = 1, 2, \ldots, k$.

Theorem 3.6.4 – Any 2-connected graph G has an ear decomposition.

4 Directed Graphs

4.1 Definition

Definition 4.1.1 – **Directeed graph** A directed graph G is an order pari (V, E), where V is some set of elements and $E \subseteq V \times V$.
A *directed edge* $e = (u, v)$, called an edge from u to v, has *head* v and *tail* u.
The *indegree* $\deg_G^+(v)$ of a vertex v is the number of edges having v as head. The *outdegree* $\deg_G^-(v)$ is the number of edges having v as tail.

4.2 Connectedness

Definition 4.2.1 – **Symmetrization** The *symmetrization* of a directed graph $G = (V, E)$ is the undirected graph $\mathrm{Sym}(G) = (V, \overline{E})$ where $\overline{E} = \{\{u, v\} \mid (u, v) \in E \vee (v, u) \in E\}$

Definition 4.2.2 – **Weakly connected** A directed graph G is called *weakly connected* if its symmetrization $\mathrm{Sym}(G)$ is connected.

Definition 4.2.3 – **Strongly connected** A directed graph G is called *strongly connected* if for every two vertices $u, v \in V$ there is a directed path from u to v and a directed path from v to u.

Definition 4.2.4 – **Weakly conneccted components** *Weakly connected components* of a directed graph G are the equivalence classes defined by the relation on the set $V(G)$, where $x \ y \iff \exists$a walk from x to y in $\mathrm{Sym}(G)$

Definition 4.2.5 – **Strongly connected components** *Strongly connected components* of a directed graph G are the equivalence classes defined by the relation on the set $V(G)$, where $x \ y \iff \exists$a directed walk from x to y and from y to x in G

4.3 Eulerian directed graphs

Definition 4.3.1 – **Eulerian directed tour** A closed directed walk containing all the vertices and edges, and each edge exactly once is an *Eulerian directed tour.*

Definition 4.3.2 – **Eulerian directed graph** A directed graph G is *Eulerian* if it has an Eulerian directed tour.

Theorem 4.3.3 – A directed graph is Eulerian if and only if its symmetrization is connected and $\deg^+(v) = \deg^-(v)$ for all $v \in V$.

4.4 De Bruijn Graphs

Lemma 4.4.1 – Every vertex v in a De Bruijn graph has $\deg^+(v) = \deg^-(v)$.

Lemma 4.4.2 – For any De Bruijn graph G, $\mathrm{Sym}(G)$ is connected.

4.5 Directed acyclic graphs

Definition 4.5.1 – **Directed acyclic graph** A *directed acyclic graph* is a directed graph with no directed cycles.

Definition 4.5.2 – **Source** A *source* in a directed graph G is a vertex v such that $\deg^+(v) = 0$.

Definition 4.5.3 – **Sink** A *sink* in a directed graph G is a vertex v such that $\deg^-(v) = 0$.

Theorem 4.5.4 – Every (finite) DAG $G = (V, E)$ has at least one sink.

5 Planar Graphs

5.1 Definitions

Definition 5.1.1 – **Planar graph** A *planar graph* is a graph that can be drawn in the plane without any edges crossing.

Definition 5.1.2 – **Arc** An *arc* is an injective continuous function $\gamma : [0, 1] \to \mathbb{R}^2$.

Definition 5.1.3 – **Drawing** A *drawing* of a graph $G = (V, E)$ is an assignment:

- to every vertex $v \in V$ a point $b(v)$ of the plane
- to every edge $e = \{u, v\} \in E$, assign an arc $a(e)$ in the plane with endpoints $b(u)$ and $b(v)$

such that

- the mapping b is injective
- no point $b(v)$ lies on any of the arcs $a(e)$ unless it is an endpoint of that arc

Definition 5.1.4 – **Topological graph** A *topological graph* is a graph together with a drawing.

5.2 Faces

Definition 5.2.1 – **Face** A *face* of a drawing of a graph G is a maximal connected subset of the plane whose boundary consists of arcs $a(e)$ for edges $e \in E$.

Definition 5.2.2 – **Jordan curve** A *Jordan curve* is an arc whose endpoints coincide.

Theorem 5.2.3 – Any Jordan curve k divides the plane into exactly two connected parts, the "interior" and the "exterior" of k, and k is the boundary of both the interior and exterior.

5.3 Planar graphs

Proposition 5.3.1 – K_1, K_2, K_3, K_4 are planar.
K_5 is not planar.

Proposition 5.3.2 – $K_{3,3}$ is not planar

Theorem 5.3.3 – **Kuratowski's theorem** A graph G is planar if and only if it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or to a subdivision of K_5

5.4 Properties of planar graphs

Theorem 5.4.1 – **Euler's formula** Let $G = (V, E)$ be a connected planar graph and let f be the number of faces of some planar drawing of G. Then we have

$$|V| - |E| + f = 2$$

Theorem 5.4.2 – Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then

$$|E| \leq 3|V| - 6.$$

Corollary 5.4.3 – Every planar graph contains a vertex of degree at most 5.

5.5 Coloring maps

Definition 5.5.1 – A mapping $c : V \to \{1, 2, \dots, k\}$ is called a *coloring* of a graph $G = (V, E)$ if $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

Definition 5.5.2 – **Chromatic number** The *chromatic number*, denoted by $\chi(G)$, of a graph G is the smallest k such that G has a coloring $c : V \to \{1, 2, \dots, k\}$.

Example 5.5.3 $\chi(K_n) = n$.
Example 5.5.4 $\chi(K_{n,m}) = 2$.
Example 5.5.5 $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}.$
Example 5.5.6 $\chi(P_n) = 2$.
Example 5.5.7 $\chi(T_n) = 2$.

Theorem 5.5.8 – Any planar graph satisfies $\chi(G) \leq 4$.

6 Double Counting

6.1 Double Counting

Theorem 6.1.1 – If $G = (V, E)$ is a triangle-free graph with n vertices, then G has at most $\frac{n^2}{4}$ edges.

Theorem 6.1.2 – If $G = (V, E)$ is a n-vertex graph without a $K_{2,2}$ subgraph, then G has at most $\frac{1}{2}(n^{\frac{3}{2}} + n)$ edges