2MBA60 Analysis 2, Group 4-4

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19.7.1

Consider the function: $f : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^2$ given by

$$f((x_1, x_2)) = \exp((x_1)^2 - (x_2)^3)$$

Show that f is twice differentiable on $\mathbb{R}^2$ by going through the following steps:

1. Show that the partial derivative functions of f, namely

$$\frac{\partial f}{\partial x_1} : \Omega \to \mathbb{R} \quad \text{and} \quad \frac{\partial f}{\partial x_2} : \Omega \to \mathbb{R}$$

   exist and compute them

2. Show that the second order partial derivative functions

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} : \Omega \to \mathbb{R} \qquad \frac{\partial^2 f}{\partial x_2 \partial x_1} : \Omega \to \mathbb{R}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} : \Omega \to \mathbb{R} \qquad \frac{\partial^2 f}{\partial x_2 \partial x_2} : \Omega \to \mathbb{R}$$

   exist and compute them.

3. Now show that the second order partial derivatives are continuous and conclude, by quoting the right theorem, that f is twice differentiable.

1. First partial derivatives

Notice that

$$\exp((x_1)^2 - (x_2)^3) = \frac{\exp((x_1)^2)}{\exp((x_2)^3)}$$

which is, viewed from x_1, a constant times the exponential of a polynomial, which are by Proposition 16.2.2 and Proposition 16.3.1 differentiable and thus by the Chain Rule their composition is differentiable.
Thus

$$\frac{\partial f}{\partial x_1}(x) = \frac{2x_1}{\exp((x_2)^3)} \exp((x_1)^2).$$

The same reasoning as above applies to when we fix x_2. In that case we have a rational function with the composition of an exponential and a polynomial function in the denominator. These are differentiable by Proposition 16.2.2 and their composition is differentiable by the Chain Rule. The quotient of a constant $\exp((x_1)^2)$ and the composition is is differentiable by the Quotient Rule.
Thus

$$\frac{\partial f}{\partial x_2}(x) = \frac{-3(x_2)^2 \exp((x_1)^2)}{\exp((x_2)^3)}.$$

2. Second partial derivatives

As in part 1. we have a quotient of an exponential function composed with a polynomial and a polynomial and times a constant factor. Hence the first derivatives are by Proposition 16.2.2, 16.2.3 and 16.3.1 differentiable. Thus we get

$$\begin{aligned}
\frac{\partial f}{\partial x_1 \partial x_1}(x) &= \frac{2}{\exp(x_2^3)} \cdot (\exp(x_1^2) + 2x_1^2 \exp(x_1^2)) \\
\frac{\partial f}{\partial x_1 \partial x_2}(x) &= \frac{-3x_2^2}{\exp(x_2^3)} \cdot 2x_1 \exp(x_1^2) \\
\frac{\partial f}{\partial x_2 \partial x_1}(x) &= 2x_1 \exp(x_1^2) \cdot \frac{-3x_2^2}{\exp(x_2^3)} \\
\frac{\partial f}{\partial x_2 \partial x_2}(x) &= -3 \exp(x_1^2) \cdot \frac{2x_2 - 3x_2^4}{\exp(x_2^3)}
\end{aligned}$$

3. Continuity and second derivative

Notice how all second derivatives are quotients of polynomials composed with exponential functions of polynomials. The denominator of each quotient only contains exponential functions which never attain 0, hence the domain of each second partial derivative includes 0. Since polynomials of positive exponents and exponential functions are continuous in $\mathbb{R}$, we have that their composition and by extension their quotient with an exponent in the denominator being continuous on all $x_1, x_2 \in \mathbb{R}$. Hence we conclude that the second partial derivatives are continuous on $\mathbb{R}^2$.

By Theorem 19.4.1: "Let $f : \Omega \to W$where Ω is an open subset of $\mathbb{R}^d$. If all partial derivatives of f of order less than or equal to n exist, and if all derivatives of order n are continuous on Ω, then f is n times differentiable on Ω.", we get that f is twice differentiable.

19.7.2

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) = (x_1)^5 (x_2)^8$$

a. For arbitrary $u \in \mathbb{R}^2$, give the function

$$D_u f : \mathbb{R}^2 \to \mathbb{R}$$

b. For arbitrary $v \in \mathbb{R}^2$, give the function

$$(D_v(D_u f)) : \mathbb{R}^2 \to \mathbb{R}$$

c. Define the vectors $u := (1, 3)$ and $v := (7, 2)$. Let $a = (1, 1) \in \mathbb{R}^2$.
Give

$$(D^2 f)_a(u, v)$$

d. For the same choice of u, v and a also give

$$(D^2 f)_a(u, v)$$

First notice that f is the product of two polynomials and hence by the Product Rule and Proposition 16.2.2 differentiable on $\mathbb{R}^2$.

a. $D_u f$

By Proposition 17.2.2 if f is differentiable in a point a, then for arbitrary $u \in \mathbb{R}^2$ we have

$$(D_u f)_a = (Df)_a(u)$$

By Proposition 17.6.2, for a vector we can be calculate the derivative in a point applied to a vector by the inner product of said vector with the gradient of the function, hence

$$(D_u f)_a = (\nabla f(a), u) = \left(\begin{pmatrix} 5a_1^4 a_2^8 \\ 8a_1^5 a_2^7 \end{pmatrix}, u \right) = 5a_1^4 a_2^8 u_1 + 8a_1^5 a_2^7 u_2$$

Hence

$$D_u f = a \mapsto 5a_1^4 a_2^8 u_1 + 8a_1^5 a_2^7 u_2$$

b. $D_v(D_u f))_a$

We apply the same logic as above, denoting $(D_u f)$ as g. Thus we find that for all $a \in \mathbb{R}^2$ we get

$$\begin{aligned}
(D_v(D_u f)_a &= (D_v g)_a \\
&= (Dg)_a(v) \\
&= (\nabla g(a), v) \\
&= \left(\begin{pmatrix} 20a_1^3 a_2^8 u_1 + 40a_1^4 a_2^7 u_2 \\ 40a_1^4 a_2^7 u_1 + 56a_1^5 a_2^6 u_2 \end{pmatrix}, v \right) \\
&= 20a_1^3 a_2^8 u_1 v_1 + 40a_1^4 a_2^7 u_2 v_1 + 40a_1^4 a_2^7 u_1 v_2 + 56a_1^5 a_2^6 u_2 v_2
\end{aligned}$$

Hence

$$D_v(D_u f) = a \mapsto 20a_1^3 a_2^8 u_1 v_1 + 40a_1^4 a_2^7 u_2 v_1 + 40a_1^4 a_2^7 u_1 v_2 + 56a_1^5 a_2^6 u_2 v_2$$

c. $(D^2 f)_a(u, v)$

By Proposition 19.3.1 we have $(D_v(D_u f))_a = (D^2 f)_a(u, v)$. Thus we know $(D^2 f)_a(u, v) = 20a_1^3 a_2^8 u_1 v_1 + 40a_1^4 a_2^7 u_2 v_1 + 40a_1^4 a_2^7 u_1 v_2 + 56a_1^5 a_2^6 u_2 v_2$.
Which after putting in the given vectors leaves us with

$$(D^2 f)_a(u, v) = 20 \cdot 1 \cdot 7 + 40 \cdot 3 \cdot 7 + 40 \cdot 1 \cdot 2 + 56 \cdot 3 \cdot 2$$

Hence

$$(D^2 f)_a(u, v) = 1396$$

d. $(D^2 f)_a(v, u)$

By the symmetry of the second derivative we know

$$(D^2 f)_a(u, v) = (D^2 f)_a(v, u) = 1396$$

19.7.3

About a certain function $f : \mathbb{R}^5 \to \mathbb{R}^2$ the following is known in a point $a \in \mathbb{R}^5$. The function is 4 times differentiable in a and

$$\begin{aligned}
(D^4 f)_a(e_2, e_3, e_3, e_5) &= (5, 0) \qquad (1) \\
(D^4 f)_a(e_2, e_3, e_5, e_5) &= (2, 3) \qquad (2) \\
\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_3 \partial x_3}(a) &= (0, 1) \qquad (3) \\
\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_5 \partial x_3}(a) &= (1, 2). \qquad (4)
\end{aligned}$$

Give

$$(D^4 f)_a(e_3 - 2e_2, 6e_5, e_3 + e_5, e_3).$$

It holds that

$$\begin{aligned}
(D^4 f)_a(e_3 - 2e_2, 6e_5, e_3 + e_5, e_3) &= (D^4 f)_a(e_3, 6e_5, e_3 + e_5, e_3) - 2(D^4 f)_a(e_2, 6e_5, e_3 + e_5, e_3) \\
&= 6(D^4 f)_a(e_3, e_5, e_3 + e_5, e_3) - 12(D^4 f)_a(e_2, e_5, e_3 + e_5, e_3) \\
&= 6(D^4 f)_a(e_3, e_5, e_3, e_3) + 6(D^4 f)_a(e_3, e_5, e_5, e_3) \\
&\quad - 12(D^4 f)_a(e_2, e_5, e_3, e_3) - 12(D^4 f)_a(e_2, e_5, e_5, e_3) \\
\text{(symmetry of higher order derivatives)} &= 6(D^4 f)_a(e_5, e_3, e_3, e_3) + 6(D^4 f)_a(e_5, e_3, e_5, e_3) \\
&\quad - 12(D^4 f)_a(e_2, e_3, e_3, e_5) - 12(D^4 f)_a(e_2, e_3, e_5, e_5) \\
(1), (2) &= 6(D^4 f)_a(e_5, e_3, e_3, e_3) + 6(D^4 f)_a(e_5, e_3, e_5, e_3) \\
&\quad - 12(5, 0) - 12(2, 3) \\
&= 6\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_3 \partial x_3}(a) + 6\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_5 \partial x_3}(a) - 12(7, 3) \\
(3), (4) &= 6(0, 1) + 6(1, 2) - 12(7, 3) \\
&= (0, 6) + (6, 12) - (84, 36) = (-78, -18)
\end{aligned}$$