2IT80 Discrete Structures

2023-24 Q2

Lecture 9: Trees (and 2-connected graphs)

EINDHOVEN UNIVERSITY OF TECHNOLOGY

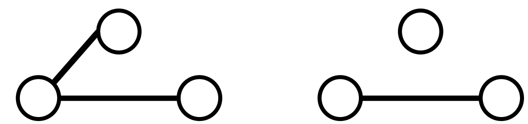
Vertex connectivity

Graph operations

Let $G = (V, E)$ be a graph

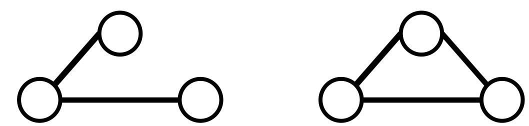
Edge deletion:

$G - e = (V, E \backslash \{e\})$, where $e \in \mathrm{E}$.



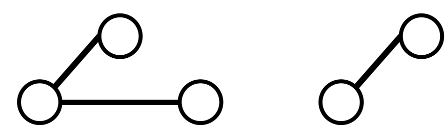
Edge insertion:

$G + e = (V, E \cup \{e\})$, where $e \in \binom{V}{2} \backslash \mathrm{E}$.



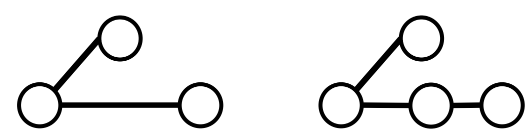
Vertex deletion:

$G - v = (V \backslash \{v\}, \{e \in \mathrm{E}: v \notin \mathrm{e}\})$, where $v \in \mathrm{V}$.



Edge subdivision:

$G\%e = (V \cup \{z\}, (E \setminus \{\{x, y\}\}) \cup \{\{x, z\}, \{z, y\}\})$,
where $e = \{x, y\} \in \mathrm{E}$ and $z \notin \mathrm{V}$.

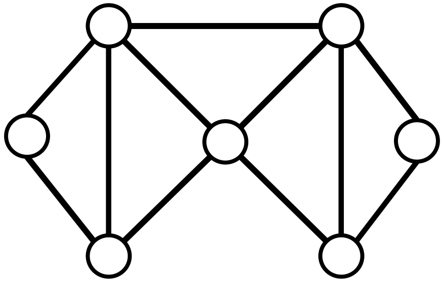


k-vertex-connectivity

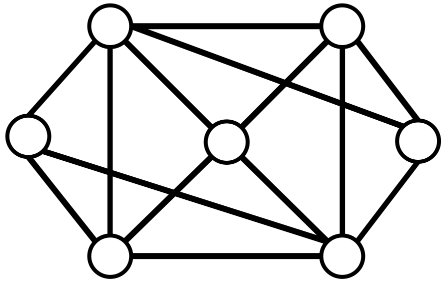
A graph G is called k-vertex-connected if it has at least $k + 1$ vertices and by deleting any $k - 1$ vertices we obtain a connected graph. Often this is abbreviated to k-connected.

For example a graph is 2-connected if it has at least 3 vertices and deleting any 1 vertex does not create a disconnected graph.

Example:



2-connected

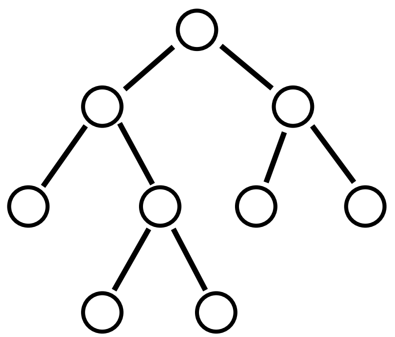


3-connected

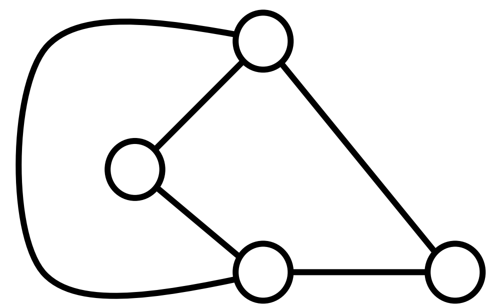
Example

For which k is the graph below k-connected.

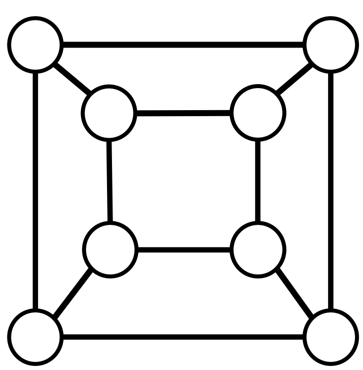
1)



2)



3)



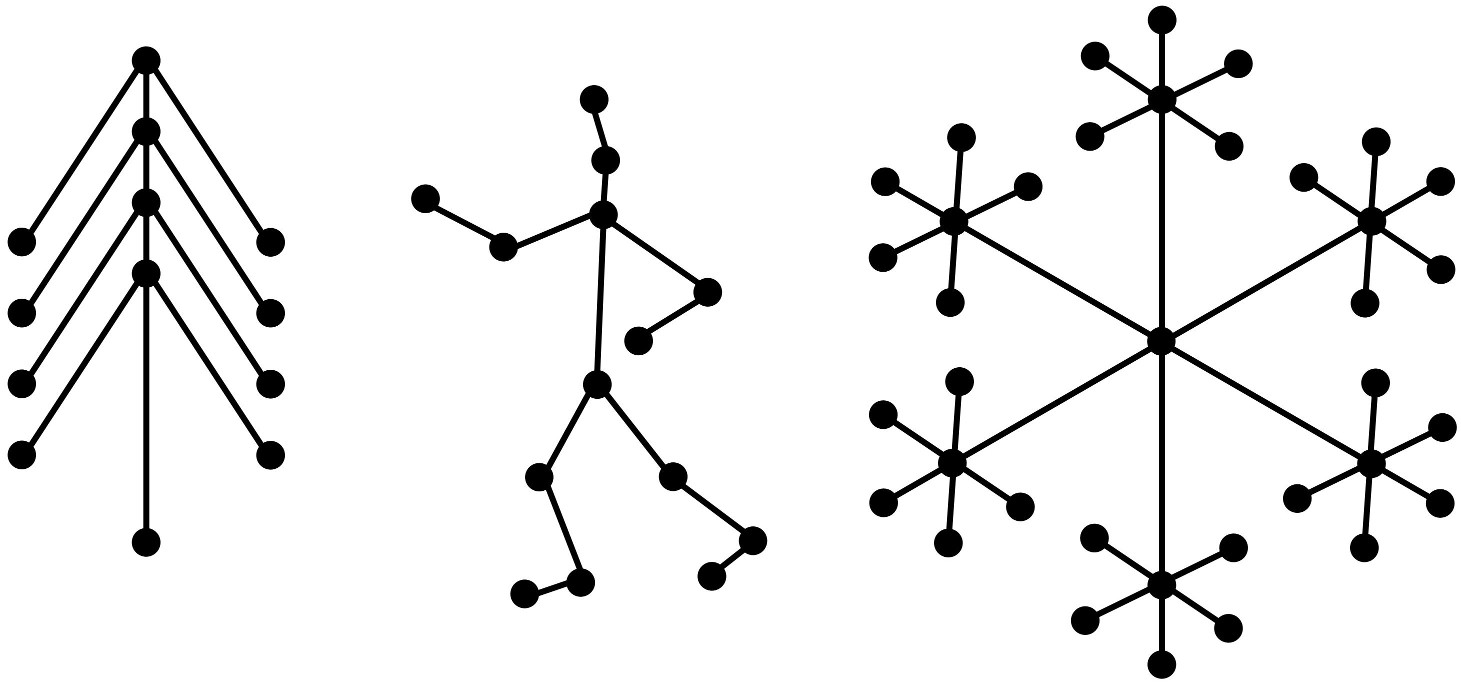
4) The complete graph K_5

5) The complete bipartite graph $K_{3,5}$

Trees

Definition and Characterizations

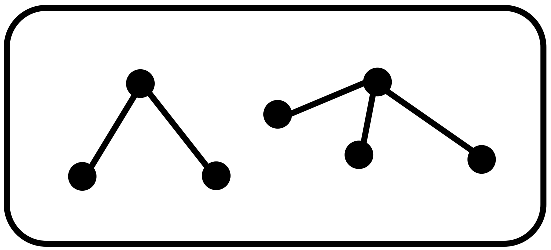
A tree is...



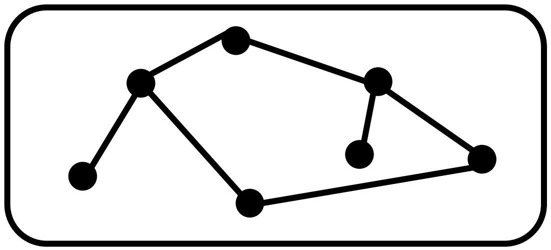
What could a precise definition look like?

Tree

A tree is a connected graph that does not contain a cycle.



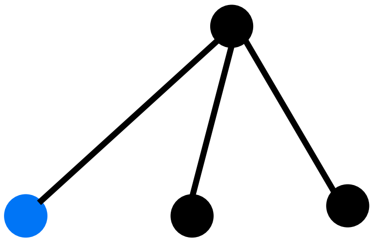
not connected



contains a cycle

Leaf

A vertex of degree 1.



Characterization of Trees

Theorem: For a non-empty graph $G = (V, E)$ the following conditions are equivalent:

i. The graph G is a tree.

ii. (unique paths)
For any two distinct vertices $x, y \in V$ there is exactly one path from x to y.

iii. (minimal connected graph)
The graph G is connected, and for any edge $e \in E$ the graph $G - e$ obtained by removing e is not connected.

iv. (maximal acyclic graph)
The graph G does not contain a cycle, and for any edge $e \in \binom{V}{2} \setminus E$ the graph $G + e$ obtained by adding e has a cycle.

v. (Euler's formula)
G is connected and $|V| = |E| + 1$.

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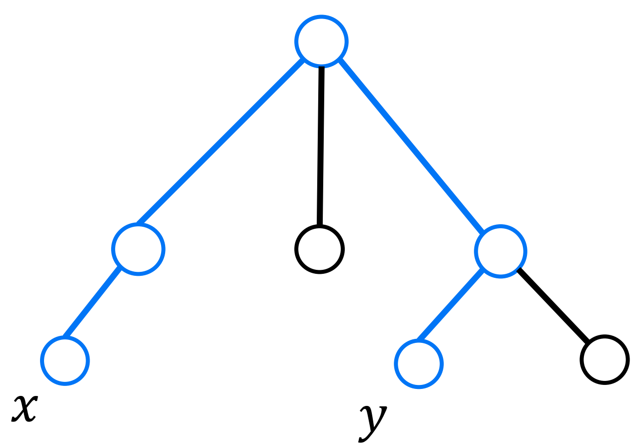

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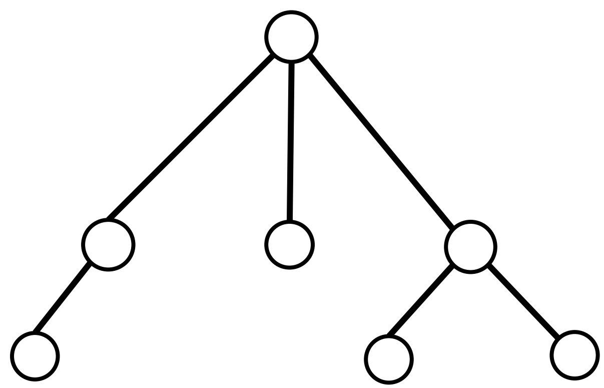

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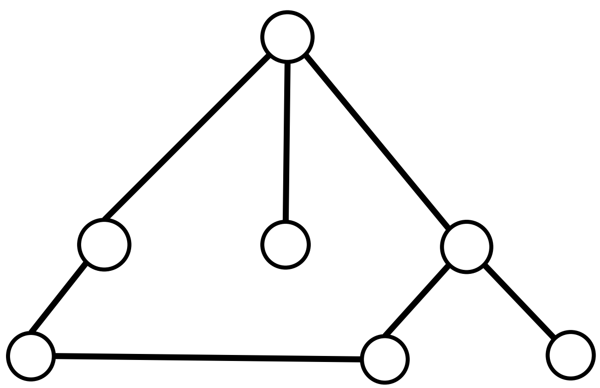

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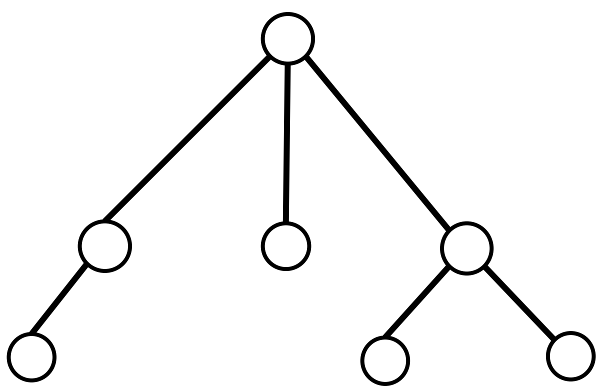

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Properties of Trees

Theorem: For a non-empty graph $= (V, E)$ the following conditions are equivalent:

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v. (Euler's formula)
is connected and $|V| = |E| + 1$.

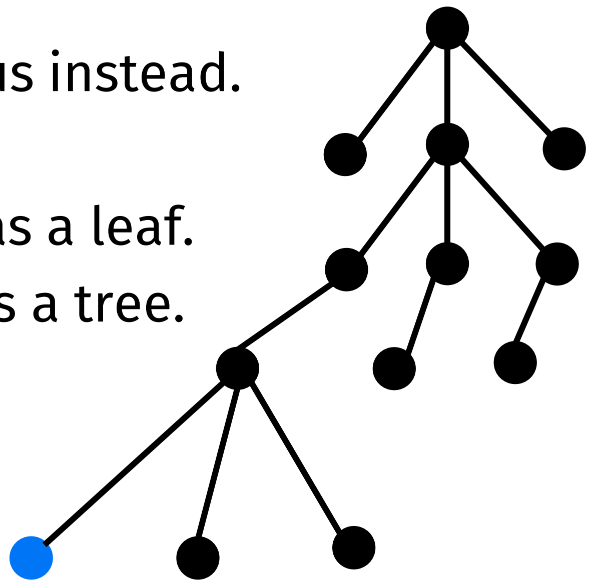
Characterization of Trees

So let's prove all the different characterizations!

Fine, then not.

Let's look at something really obvious instead.

- ☐ Every tree of at least 2 vertices has a leaf.
- ☐ Removing a leaf from a tree yields a tree.



Why do we care??

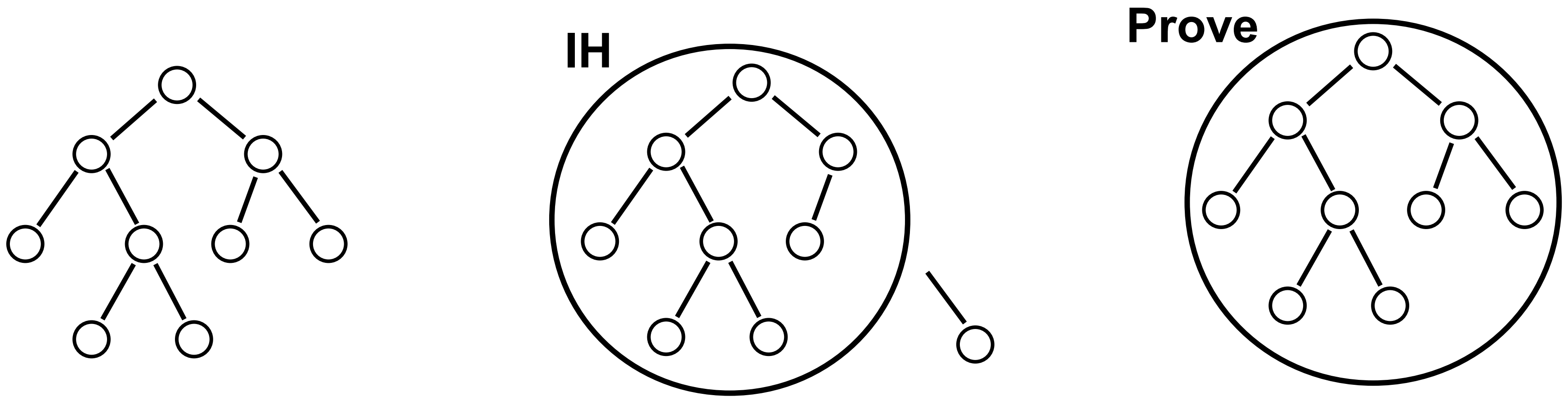
Induction on Trees

Induction on trees

We can do induction on the number of vertices.

Remove a leaf, apply inductive hypothesis to the rest.

Note the direction!



Induction on trees

Lemma: (end-vertex) Every tree with at least two vertices has at least two leaves.

Lemma: (tree-growing) Let G be a graph and v a leaf in G. Then the following statements are equivalent:

i. G is a tree

ii. $G - v$ is a tree.

Proofs in the book!

Example

Theorem: The graph G is a tree $\Rightarrow$ G is connected and $|V| = |E| + 1$.

Proof sketch: By induction on size of the tree

Step: To prove for all trees with $k + 1$ vertices. (for some $k \geq ?$)

Consider arbitrary tree $T = (V, E)$ with $k + 1$ vertices

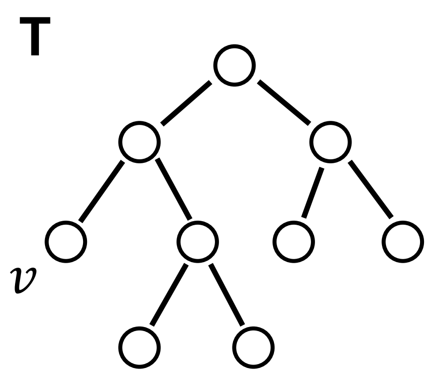
Since T is a tree, T has a leaf, say v (end-vertex lemma)

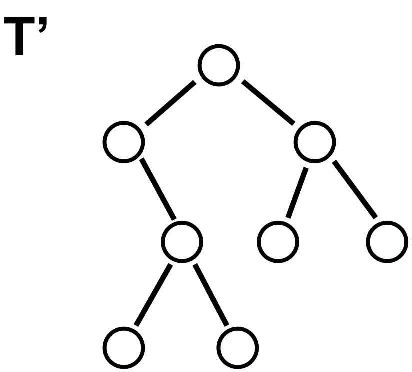
Let $T' = T - v$, then T' is a tree with k vertices (tree-growing lemma)

By IH T' has $|V'| = |E'| + 1$.

Since v is a leaf T contains 1 more vertex and one more edge than T',
so $|V| = |V'| + 1$ and $|E| = |E'| + 1$,

then $|V| = |V'| + 1 = |E'| + 2 = |E| + 1$





Rooted trees

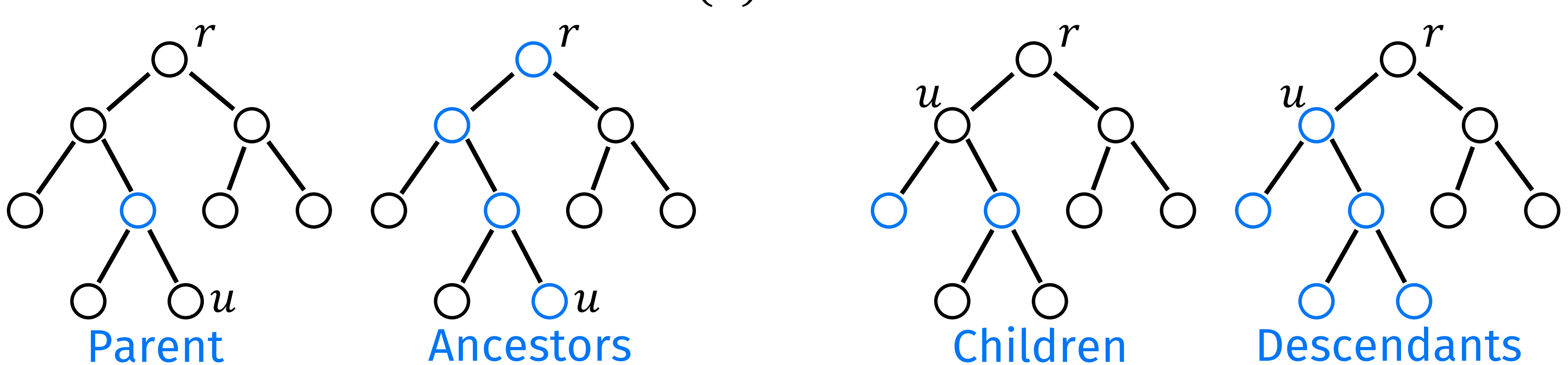
When down is up

Rooted tree

A rooted tree is a pair (T, r) where T is a tree and $r \in V(T)$ is a distinguished vertex of T called the root.

A node u in a rooted tree T may have a...

> parent (book: father): the unique vertex $v \in \mathrm{V}(T)$ such that $\{u, v\} \in E(T)$ and v lies on the unique path from u to the root.
>
> ancestor: a vertex $v \in \mathrm{V}(T)$ such that v lies on the unique path from u to the root. (This definition includes u itself...)
>
> child: a vertex $v \in \mathrm{V}(T)$ where u is the parent of v.
>
> descendant: a vertex $v \in \mathrm{V}(T)$ where u is an ancestor of v.



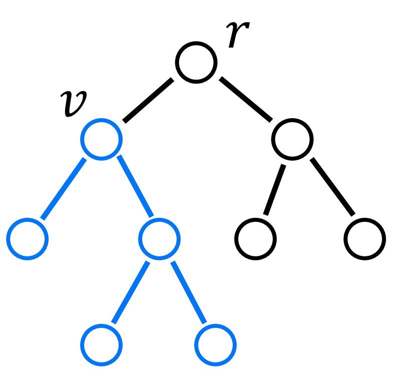
Subtree

The subtree rooted at $v \in V(T)$ in a rooted tree is the induced subgraph defined by all vertices that are descendants of v (by definition then also including v), rooted at v.*

Example:

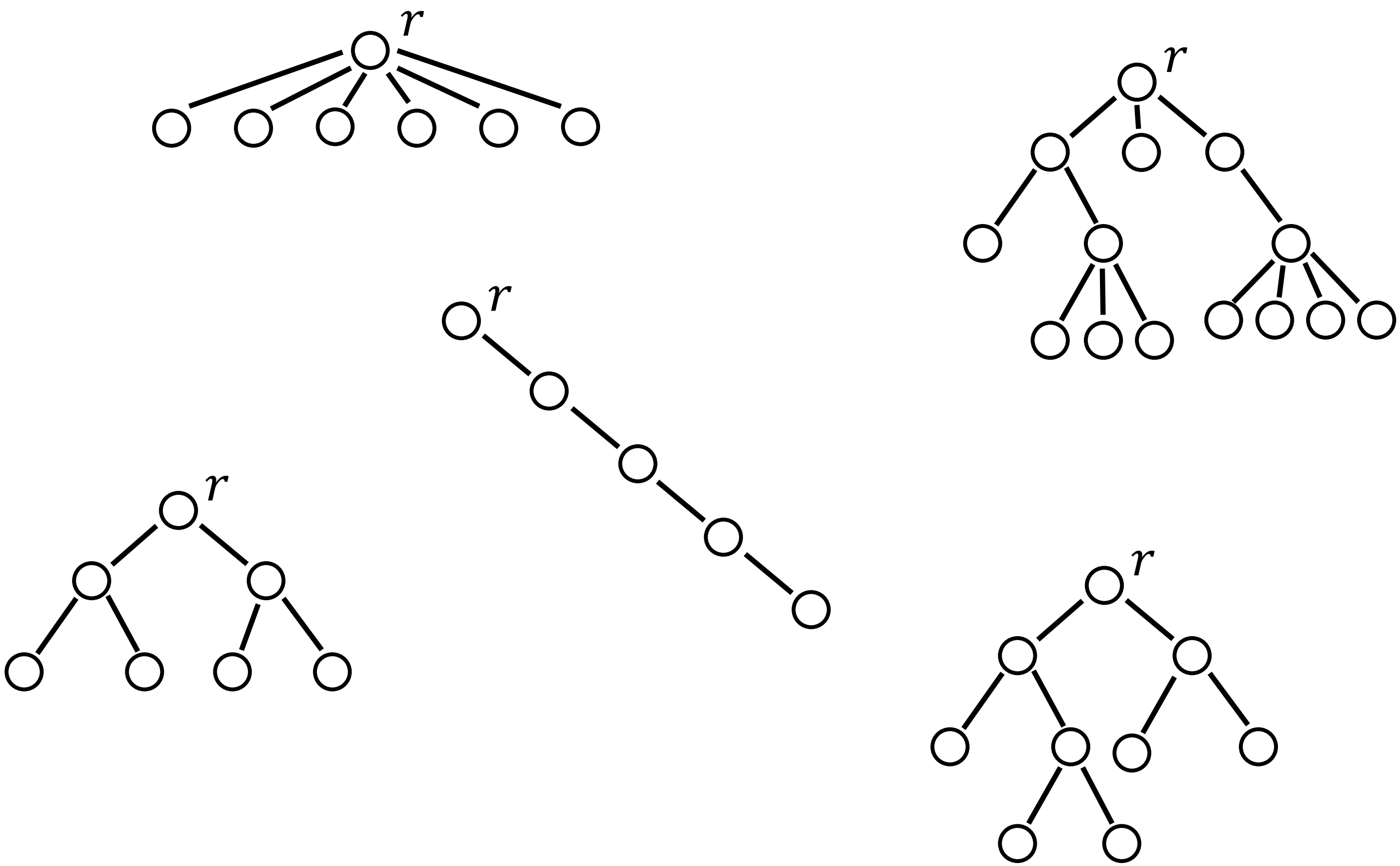
A rooted tree with root r,

The subtree rooted at v in blue.



*We will not prove that a subtree is indeed a rooted tree, but you may assume it is.

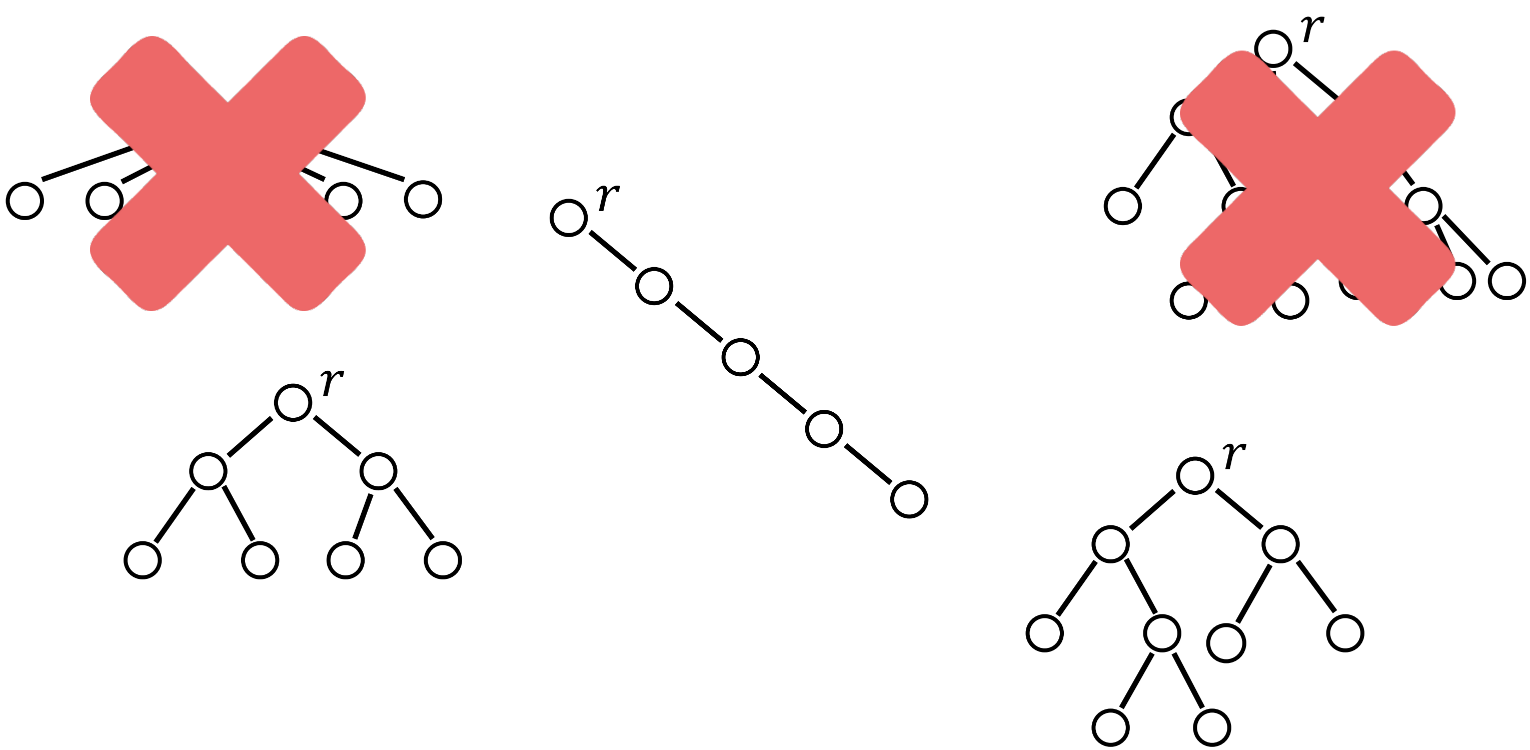
Examples rooted trees



Binary trees

We can place more restrictions on trees.

Binary tree: every vertex has at most two children.



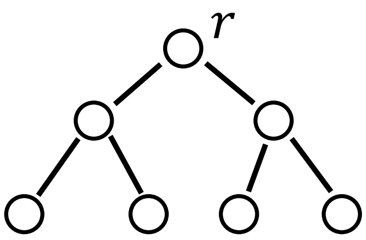
Binary trees

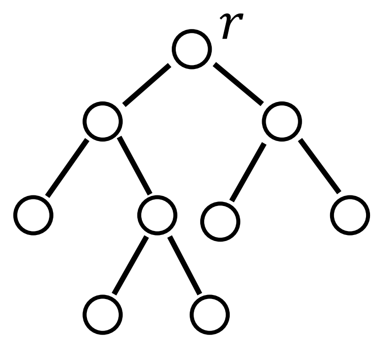
We can place more restrictions on trees.

Binary tree: every vertex has at most two children.
Strict binary tree: every vertex has zero or two children.

Binary trees

Can we bound the number of internal vertices in a strict binary tree?



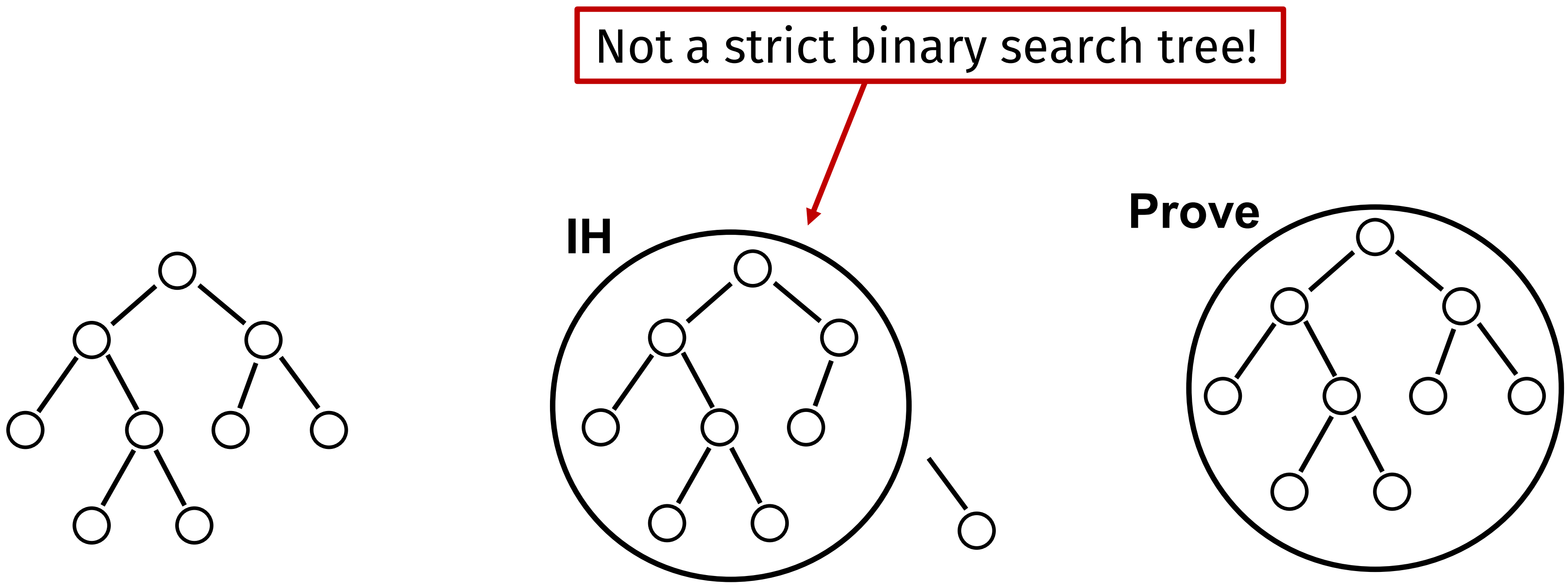


Yes, we can.

Internal vertices

Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.

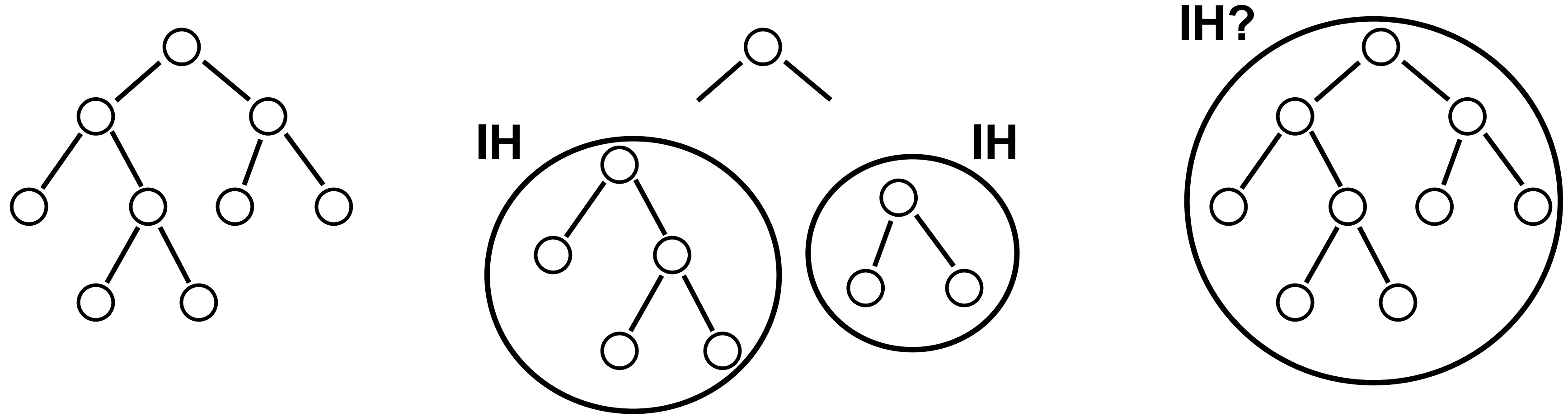
Proof: By induction on size of tree



More induction on trees

Look at the subtrees formed by children of the root.
Apply inductive hypothesis to the sub-trees.

Note the direction!



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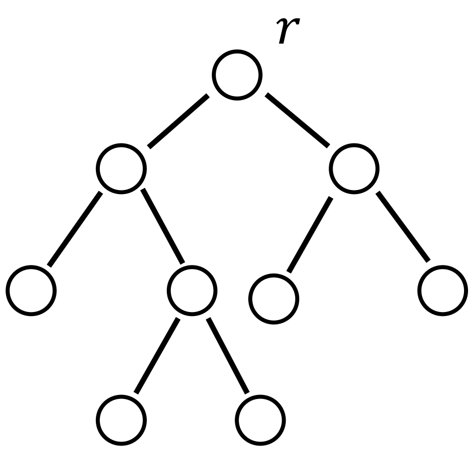
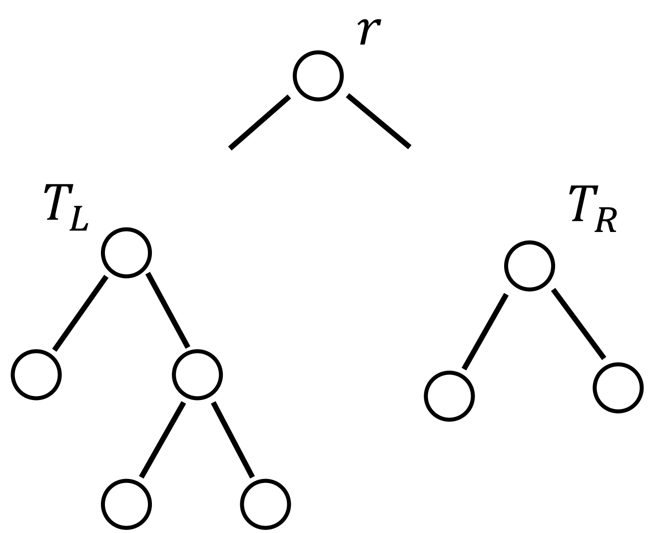
Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.





Let $l = |T_L| \geq 1$ and $r = |T_R| \geq 1$

Then $n = l + r + 1$

By IH T_L has $\frac{l-1}{2}$ internal vertices and T_R has $\frac{r-1}{2}$ internal vertices.

Internal vertices

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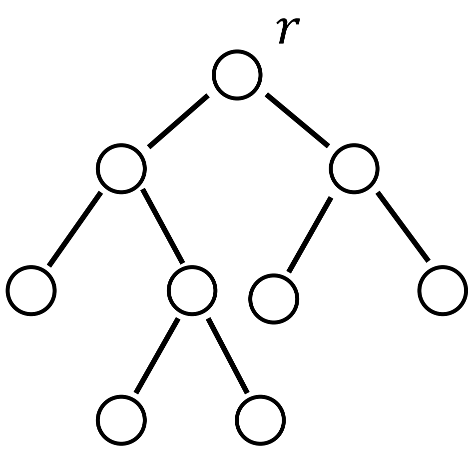

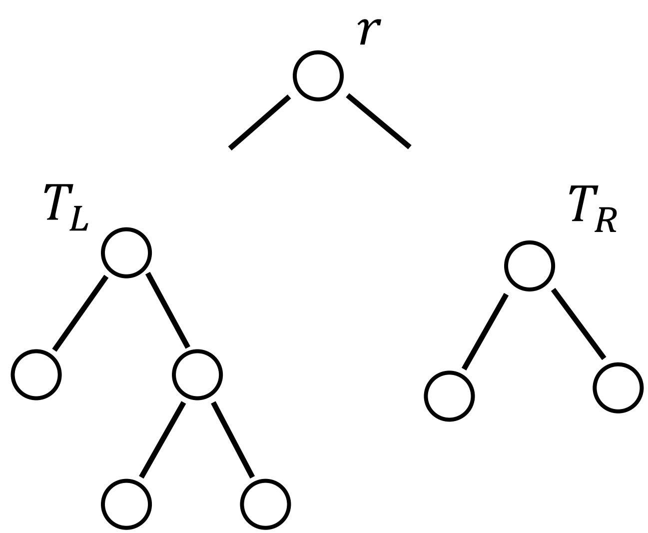

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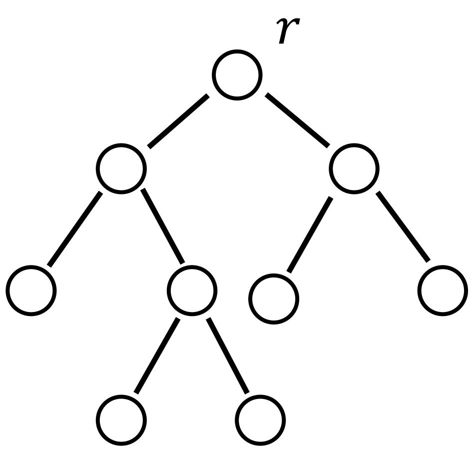
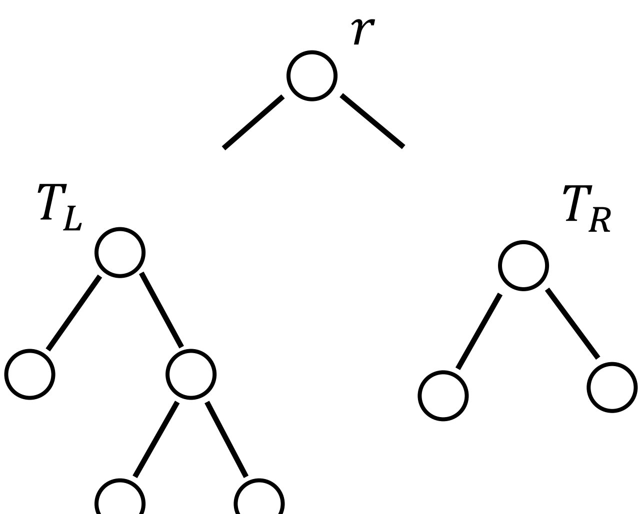
Internal vertices in T is $\frac{l-1}{2} + \frac{r-1}{2} + 1$

Internal vertices

Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.





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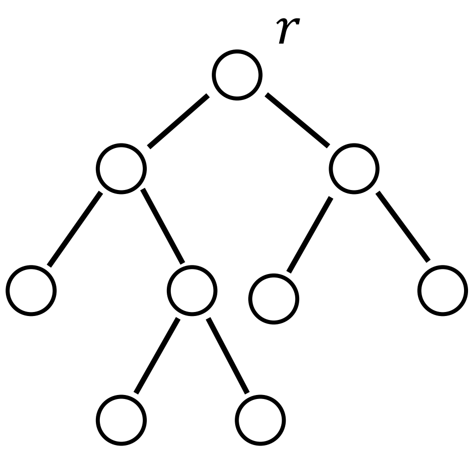
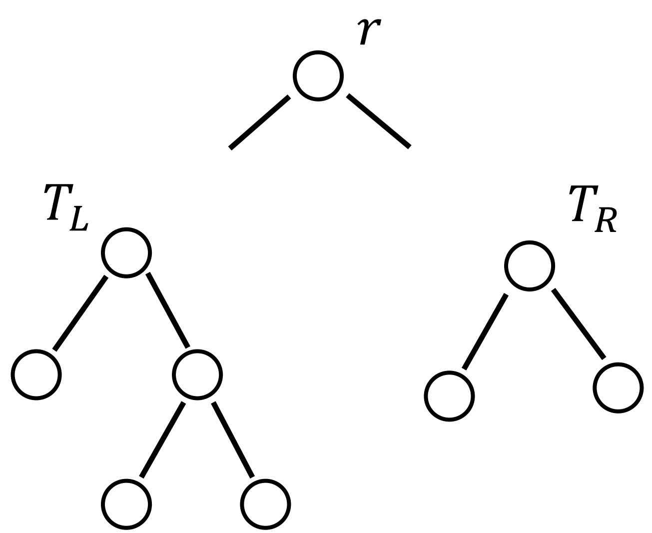
Internal vertices in T is $\frac{l-1}{2} + \frac{r-1}{2} + 1 = \frac{l+r-2+2}{2} = \frac{l+r}{2}$

Internal vertices

Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.





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Internal vertices in T is $\frac{l-1}{2} + \frac{r-1}{2} + 1 = \frac{l+r-2+2}{2} = \frac{l+r}{2} = \frac{n-1}{2}$

Internal vertices

Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.

Proof: We will prove the statement by induction.

Base $(n = 1)$:

A strict binary tree consisting of one vertex, must have only a degree zero vertex. Thus it has 0 internal nodes and 1 leaf.

Indeed the number of internal vertices is then $\frac{1-1}{2} = 0$.

Step: Let $k \geq 1$.

Induction hypothesis: for all $1 \leq k' \leq k$ a strict binary tree with k' vertices has $\frac{k'-1}{2}$ vertices.

Internal vertices

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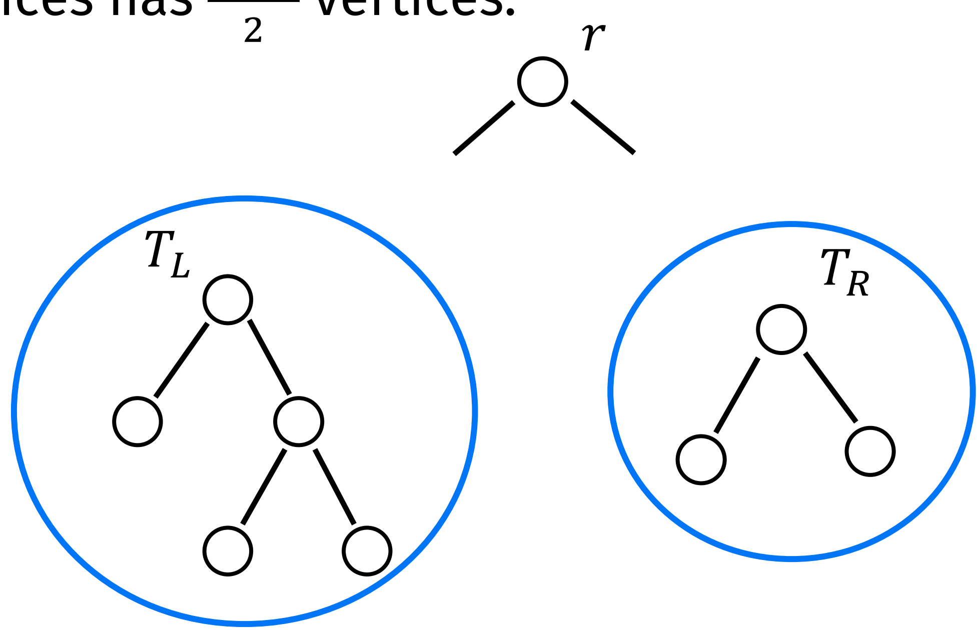

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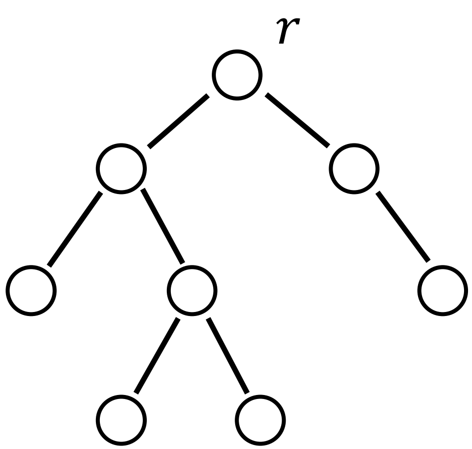
Induction hypothesis: for all $1 \leq k' \leq k$ a strict binary tree with k' vertices has $\frac{k'-1}{2}$ vertices.

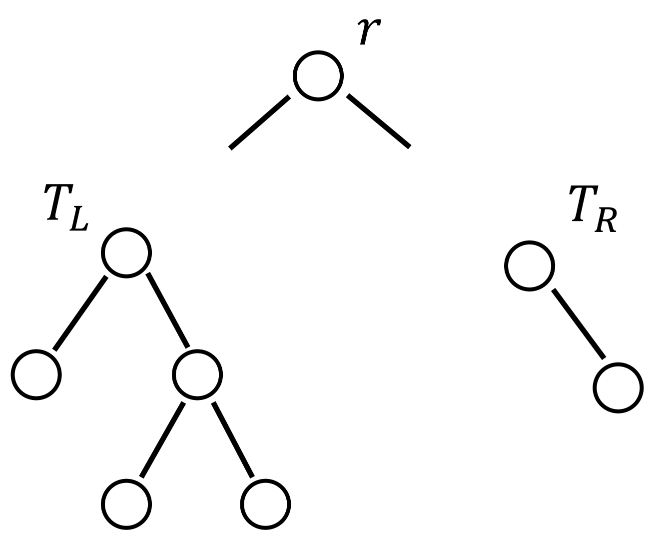
Let (T, r) be a strict binary tree with $k+1$ vertices. Let T_L and T_R be the sub-trees rooted at the left and right child of r, with l and m vertices respectively. We know $l + m + 1 = k + 1$ and $l, m \geq 1$ so $l, m \leq \mathrm{k}$.

By IH T_L has $\frac{l-1}{2}$ internal vertices and T_R has $\frac{r-1}{2}$ internal vertices. These internal vertices are also internal vertices for T. Furthermore r is also an internal vertex for T. But then T has $\frac{l-1}{2} + \frac{r-1}{2} + 1 = \frac{l+r}{2} = \frac{n-1}{2}$ internal vertices.

Internal vertices

Lemma: A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.





So why does our proof not work for (regular) binary trees?

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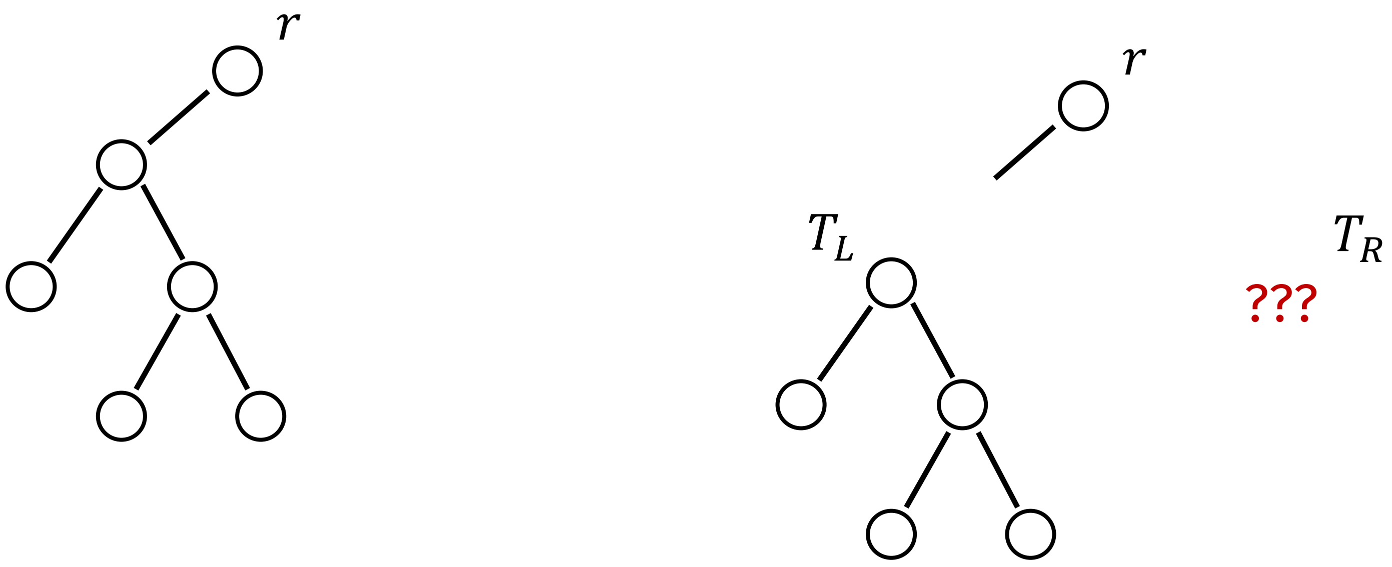

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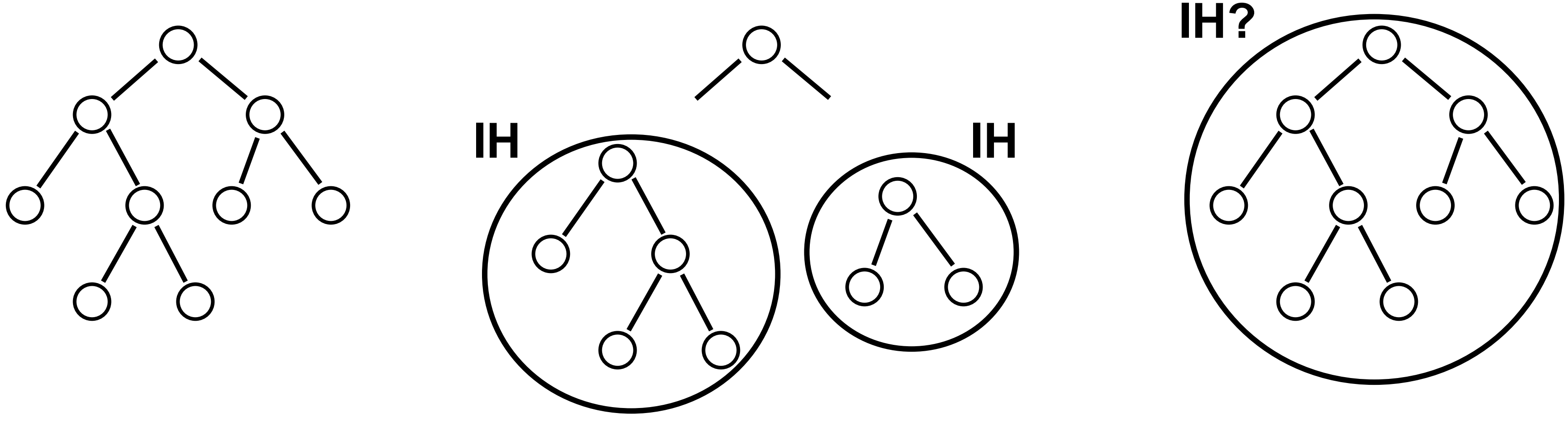
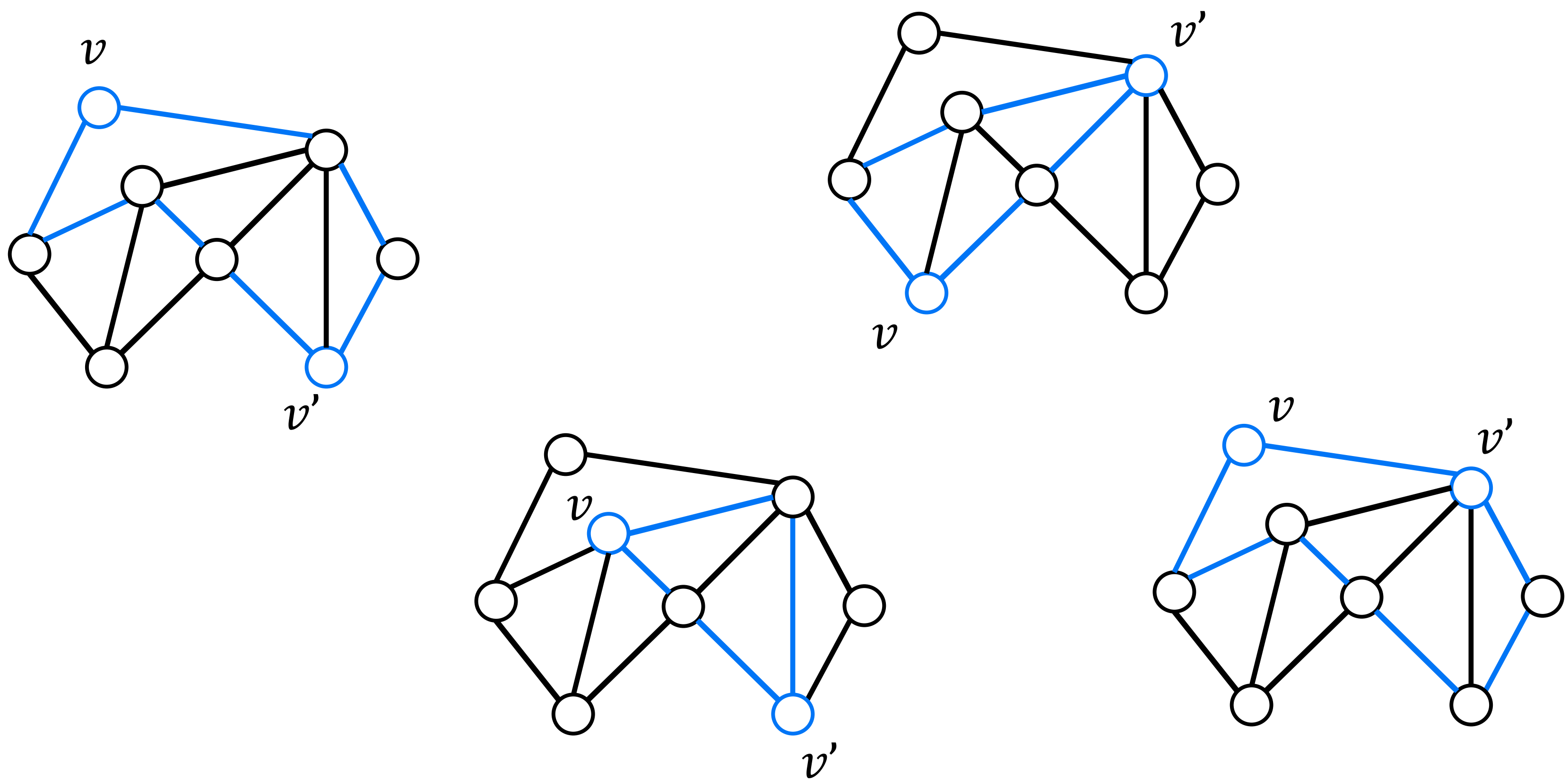


2-connected graphs

2-connected

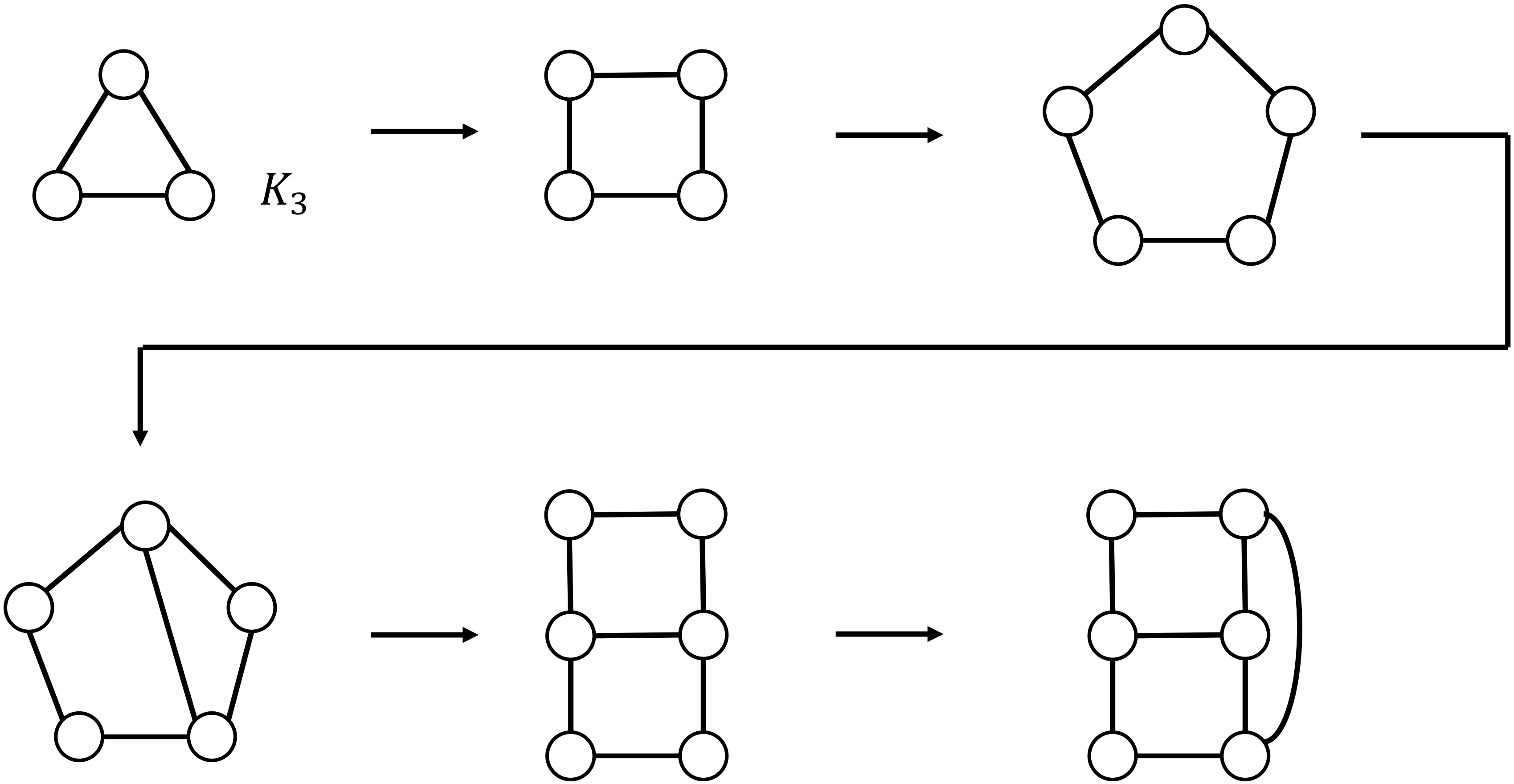
Theorem: A graph $G = (V, E)$ is 2-connected if and only if there exists, for any two vertices $v, v' \in V$, a cycle in G containing v and v'.



Ear decompositions

Nurse, a knife please...

2-connected



2-connected

Lemma: Let $G = (V, E)$ be a 2-connected graph, then

1) $G\%e$ is a 2-connected graph, where $e \in \ E$
2) $G + e$ is a 2-connected graph, where $e \notin E$

Proof sketch:

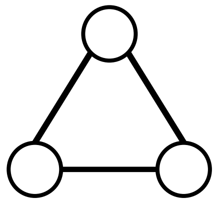
2) adding an edge is never reduces connectivity.

For 1) the proof is a little more involved, see book

2-connected

Lemma: Any graph $G = (V, E)$ created from K_3 by a sequence of edge subdivisions and edge additions is 2-connected.

Proof sketch:



K_3

K_3 is 2-connected.

By previous lemma any subdivision or edge addition maintains 2-connectedness.

So the Lemma holds.

(Can make a formal proof using induction on length of the sequence)

Are there also 2-connected graphs that cannot be created by a sequence of edge subdivisions and edge additions from K_3?

2-connected

Lemma: Any graph $G = (V, E)$ created from K_3 by a sequence of edge subdivisions and edge additions is 2-connected.

Proof sketch:

K_3 is 2-connected.

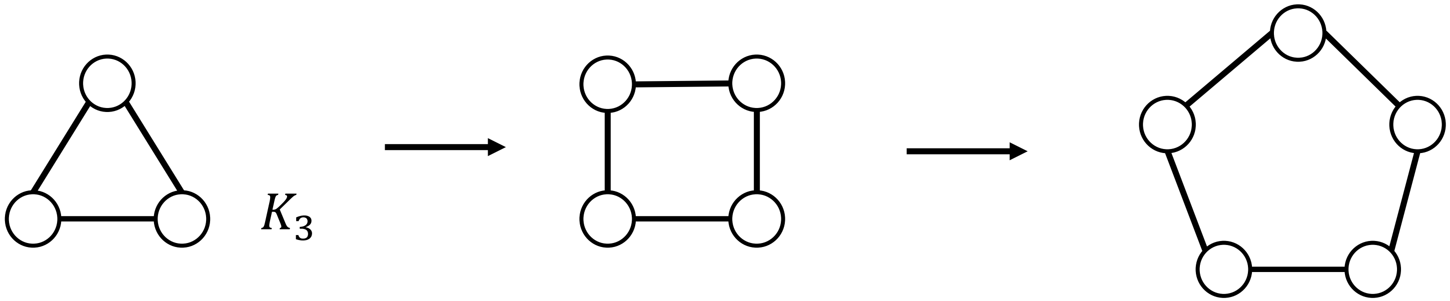
By previous lemma any subdivision or edge addition maintains 2-connectedness.

So the Lemma holds.

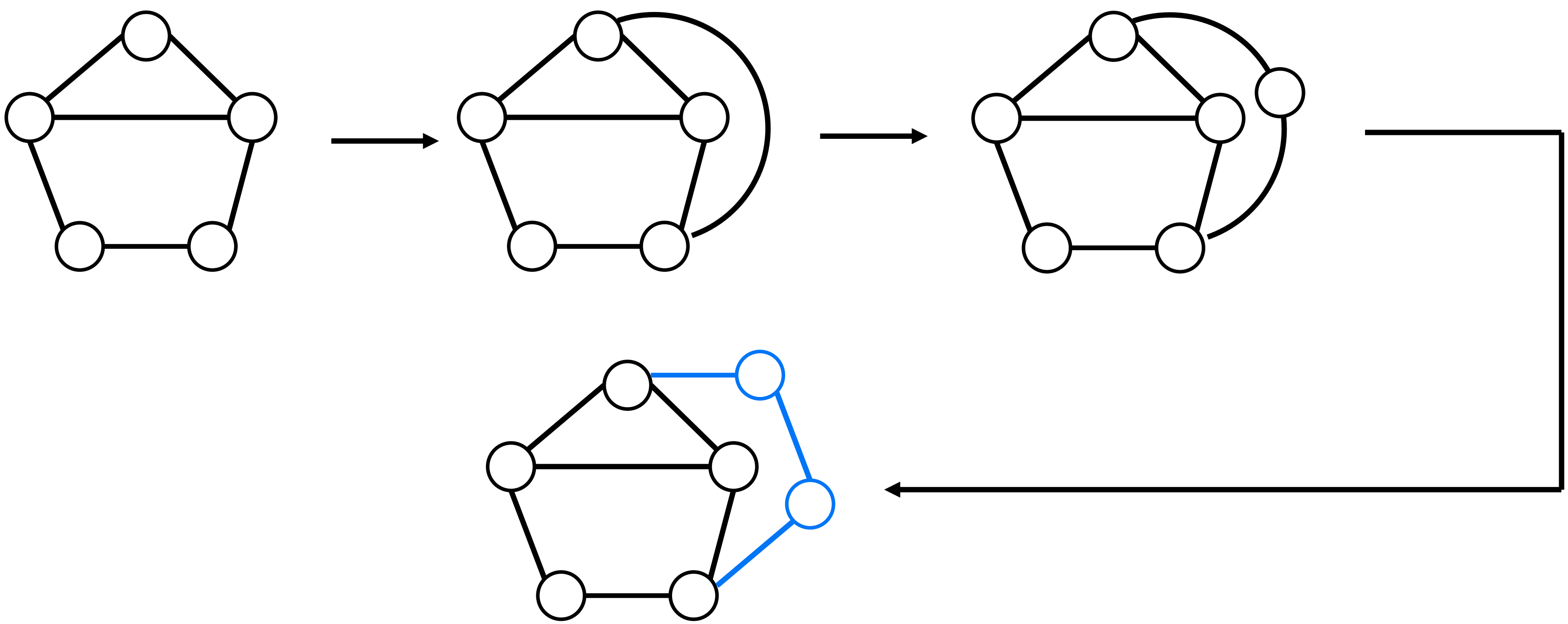
(Can make a formal proof using induction on length of the sequence)

Proposition: Any 2-connected graph $G = (V, E)$ can be created from K_3 by a sequence of edge subdivisions and edge additions.

A level up



We can easily make C_k starting from K_3.

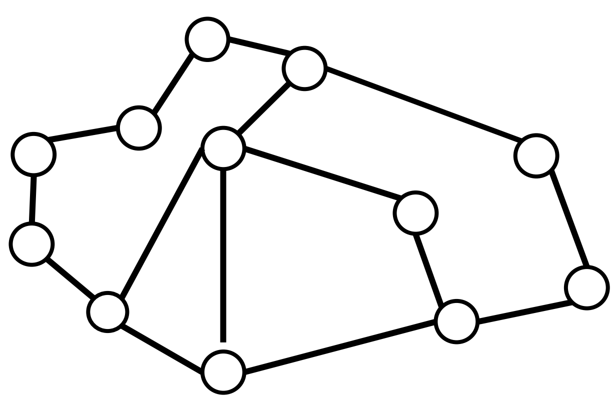


2-connected

We'll consider the problem at a higher level.

- We know we can make any cycle from K_3.
- We also know we can consider adding paths instead of edges.

We find a partition of G:



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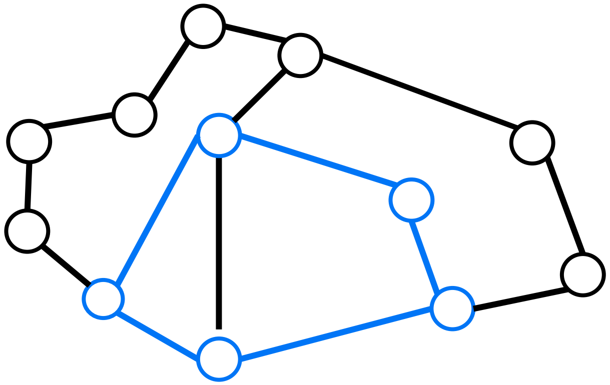

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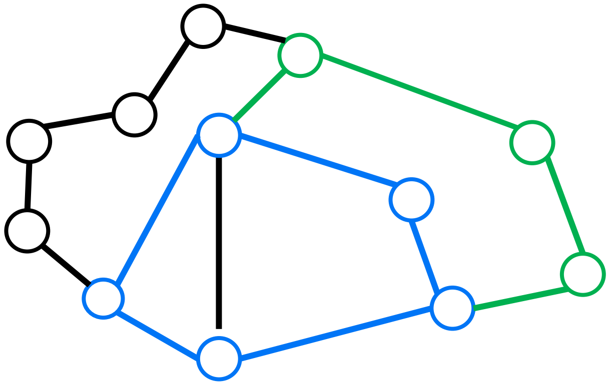

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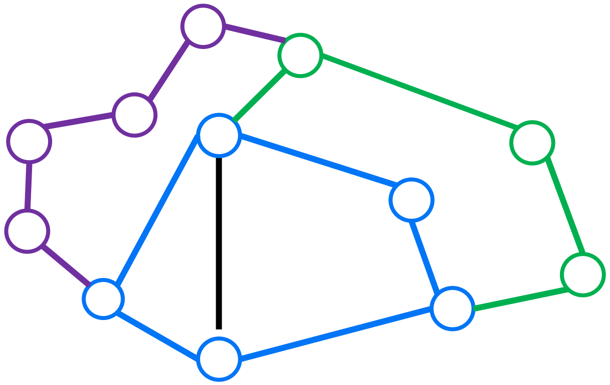

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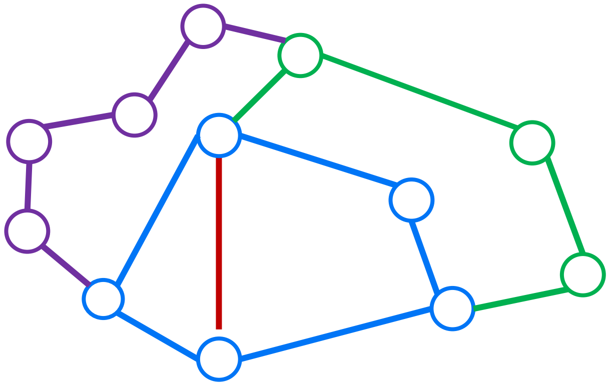


Ear Decomposition

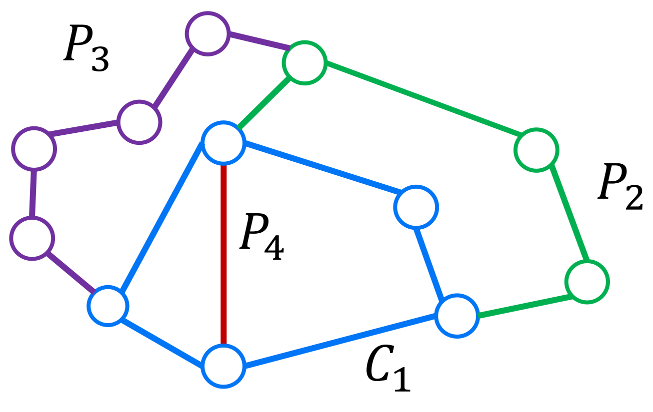
Partition of the edges of a graph into a sequence of ears $C_1, P_2, \ldots, P_k$.

The first ear C_1 is a cycle.

All remaining ears are paths $P_2, \ldots, P_k$ such that:

- For any path P_i, only the endpoints of P_i are part of a previous ear.

Example:



Notational warning:

In C_1 the 1 indicates it is the first ear, not the length of the cycle.
In P_i, the i indicates which ear it is, not the length of the path.

2-connected

Lemma: Any 2-connected graph $G = (V, E)$ has an ear decomposition.

Why is this useful?

Induction!

Every 2-connected graph has an ear-decomposition with k ears.

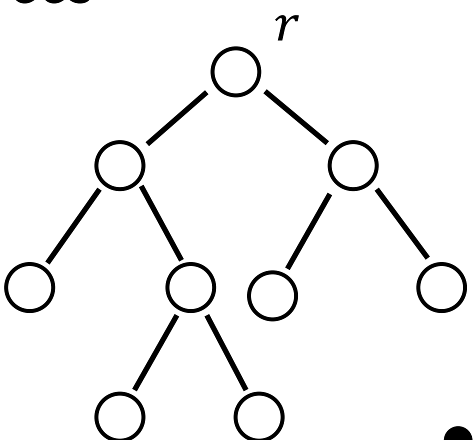
Can reduce graph by taking away an ear and the result is still 2-connected and "smaller" (fewer ears)

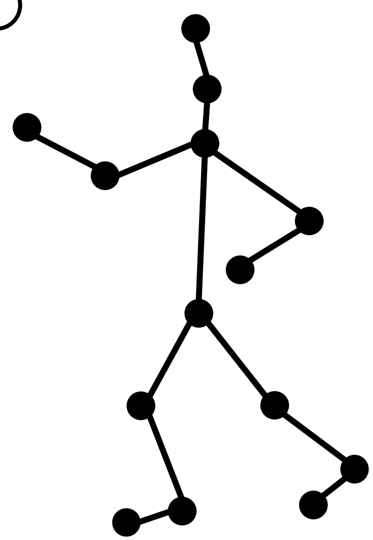
Similar to "a subtree of a rooted tree is a rooted tree".

We sometimes refer to this as structural induction

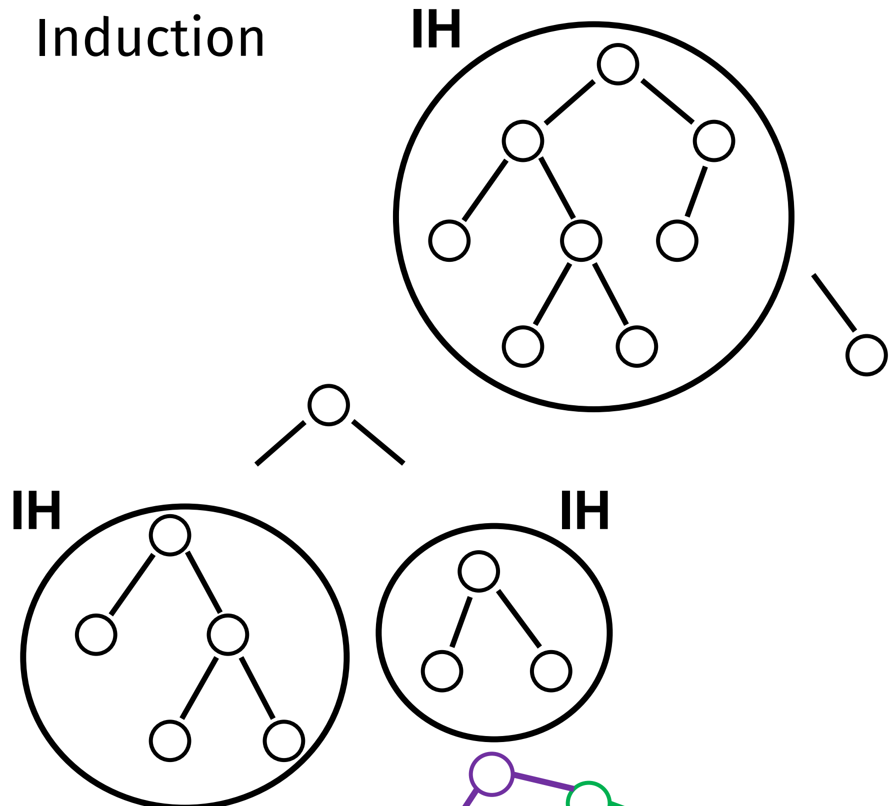
Summary

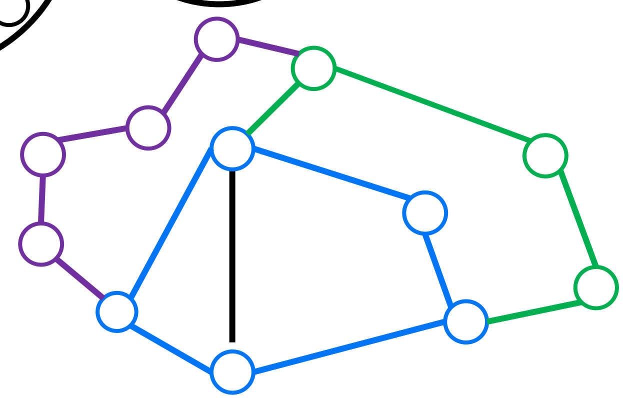
Trees





Induction





Organizational

No Lecture Thursday

A3 test next week!