Complex Numbers

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2023

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1 Complex numbers

The equation

$$x^2 = -1$$

does not have a solution in the real numbers. Already in the sixteenth century it became clear that having a solution x to this equation would make various computations easier and lead to the determination of solutions of not only quadratic equations like $x^2 = -1$ or $x^2 + 2x + 10 = 0$, but also cubic ones like $x^3 + x + 1 = 0$. The most famous ones are the formulas of Cardano, providing solutions to arbitrary cubic equations. See Section 3.8.

So, we introduce a new number (which is **not** real) with the name i satisfying

$$i^2 = -1.$$

At first, this may look weird. But introducing new numbers is something we have seen before. Indeed, think of the introduction of zero or negative integers. We started with counting objects and learned about $1, 2, 3, \ldots$, the positive integers. The numbers $0, -1, -2, \ldots$ were introduced later. They have been introduced as solutions to equations of the form

$$x + a = b$$

where a and b are positive integers with $a \geqslant b$. Then we learned about fractions to solve equations like

$$ax + b = c,$$

with a, b, c arbitrary integers, and (square) roots to solve equations like

$$ax^2 + bx + c = 0.$$

Finally irrational numbers like π or e have been introduced. All these numbers are real numbers. But now we enter the new world of complex numbers. These extensions correspond to the following chain of sets of numbers:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Here $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers (i.e. fractions), $\mathbb{R}$ the set of real numbers and finally $\mathbb{C}$ denotes the set of complex numbers.

1.1 Complex numbers

A *complex number* is an expression of the form $a+bi$ where a and b are real numbers. For i holds that $i \cdot i = i^2 = -1$. So the symbol i is now reserved for this special complex number.

We denote the set of all complex numbers with $\mathbb{C}$.

If b equals 1, then write $a+i$ instead of $a+1i$. If $b=-1$, then write $a-i$ instead of $a-1i$. If $a=0$, then just write bi, and if $b=0$, then just write a. In particular, write i instead of $0+1i$. And instead of $0+0i$ we write 0.

Instead of $a+bi$ we also write $a+ib$, especially if there is a risk of confusion. For example, one prefers to write $i\sqrt{2}$ than $\sqrt{2}i$, because in the last formula perhaps it is not so clear that the square root only applies to 2.

We may also write $bi+a$ instead of $a+bi$.

The complex numbers $a+0i$ with a real are the ordinary real numbers. So every real number a is also a complex number.

Complex numbers are often represented by a single letter. The letter z is then favorite, followed by the letter w, for example: *define* $z = 1 + i\sqrt{3}$. The use of these letters is not an obligation, but more of a habit among scientists.

If we are introducing a complex number as $a+bi$, then our convection is that we consider a and b to be real numbers.

1.1.1 Examples.

$2 = 2 + 0 \cdot i$.

$1+i$ is short for the complex number $1 + 1 \cdot i$.

$i-1$ is the same as the complex number $-1+i$.

$i\sqrt{3}$ is short for the complex number $0 + \sqrt{3} \cdot i$.

1.2 Real and imaginary part

Two complex numbers $a+bi$ and $c+di$ are equal only if $a=c$ and $b=d$. Every pair of real numbers (a,b) therefore is associated to exactly *one* complex number, namely $a+bi$. The number a is called the *real part* of $a+bi$, and the number b is called the *imaginary part* of $a+bi$. So both the real and imaginary parts of a complex number are themselves real.

A complex number whose real part equals 0 is called *purely imaginary*. So it is of the form bi with b real.

Special notations have been introduced for real part and imaginary part:

$$\mathrm{Re}(a+bi)=a$$

and

$$\mathrm{Im}(a+bi)=b.$$

1.2.1 Examples.

$\mathrm{Re}(i\sqrt{3}+2)=2$.

$\mathrm{Im}(7-6i)=-6$. Pay attention: the imaginary part is itself a real number.

The numbers $3+4i$ and $3+5i$ are different because their imaginary parts are different: $\mathrm{Im}(3+4i)=4\neq 5=\mathrm{Im}(3+5i)$.

The complex number $i\sqrt{3}$ is purely imaginary: $\mathrm{Re}(i\sqrt{3})=0$, but $1+i\sqrt{3}$ is not purely imaginary: $\mathrm{Re}(1+i\sqrt{3})=1\neq 0$.

1.3 Addition and subtraction

1.3.1 Definition. The *sum* of two complex numbers $a+bi$ and $c+di$ is defined as:

$$(a+bi)+(c+di)=(a+c)+(b+d)i.$$

The real part of the sum is the sum of the real parts of the two original complex numbers. The imaginary part of the sum is the sum of the imaginary parts of the two complex numbers. In a similar way we can define the *difference* of two complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

These operations generalise the addition and subtraction of reals: if a and c are real numbers, and add them as complex numbers $a+0\cdot i$ and $c+0\cdot i$, we obtain:

$$(a+0\cdot i)+(c+0\cdot i)=(a+c)+(0+0)i=(a+c)+0\cdot i.$$

The complex number $(a+c)+0\cdot i$ is of course the real number $a+c$.

1.3.2 Examples. We can add two complex numbers. For example:

$$(3+i)+(2-4i)=(3+2)+(1-4)i=5-3i.$$

And we can also subtract two complex numbers:

$$(23-7i)-(8+32i)=(23-8)+(-7-32)i=15-39i.$$

1.3.3 Example. If bi and di are purely imaginary, then so is their sum:

$$bi+di=(0+bi)+(0+di)=(0+0)+(b+d)i=(b+d)i.$$

Similarly we have for subtraction:

$$bi-di=(0+bi)-(0+di)=(0-0)+(b-d)i=(b-d)i.$$

1.4 Multiplication

We will now define multiplication of complex numbers.

We will base the definition of multiplication on the usual rules of multiplication of real numbers, and the extra rule that

$$i^2=-1.$$

So, for the product $(a+bi)(c+di)$ of the two complexe numbers $a+bi$ and $c+di$ we find

$$(a+bi)(c+di)=ac+adi+bic+bidi=ac+bdi^2+(ad+bc)i.$$

Now using that $i^2=-1$ we find:

$$(a+bi)(c+di)=ac+bdi^2+(ad+bc)i=ac+bd(-1)+(ad+bc)i=(ac-bd)+(ad+bc)i.$$

This leads to the following definition of the product.

1.4.1 Definition. The *product* of the complex numbers $a+bi$ and $c+di$ is defined as:

$$(a+bi)\cdot(c+di):=ac-bd+(ad+bc)i.$$

(Notice that we often do not write the multiplication symbol $\cdot$ and write $(a+bi)(c+di)$ for $(a+bi)\cdot(c+di)$.)

It is now straightforward to check the following rules for addition and multiplication. Let $a+bi$, $c+di$ and $e+fi$ be three complex numbers. Then we have

- $(a+bi)+(c+di) = (c+di)+(a+bi)$, addition is commutative.
- $(a+bi)(c+di) = (c+di)(a+bi)$, multiplication is commutative.
- $((a+bi)+(c+di))+(e+fi) = (a+bi)+((c+di)+(e+fi))$, addition is associative.
- $((a+bi)\cdot(c+di))\cdot(e+fi) = (a+bi)\cdot((c+di)\cdot(e+fi))$, multiplication is associative.
- $(a+bi)\cdot((c+di)+(e+fi)) = (a+bi)\cdot(c+di)+(a+bi)\cdot(e+fi)$, multiplication distributes over addition.

Multiplication of complex numbers generalises multiplication of real numbers. Indeed, starting with two real numbers a and c, the product of the complex numbers $a+0\cdot i$ and $c+0\cdot i$ is

$$(a+0\cdot i)(c+0\cdot i) = (ac-0\cdot 0)+(a\cdot 0+0\cdot c)i = ac+0\cdot i.$$

So, the real number ac is the product of the two complex numbers a and c.

1.4.2 Example. According to the definition of the product we find $(2+3i)(1-2i)$ to be equal to

$$(2+3i)(1-2i) = 2\cdot 1-3\cdot(-2)+(2\cdot(-2)+3\cdot 1)i = 2+6+(-4+3)i = 8+(-1)i = 8-i.$$

We can also use the common rules for multiplication together with $i^2=-1$:

$$(2+3i)(1-2i) = 2\cdot 1+3i\cdot 1-2\cdot(2i)-(3i)\cdot(2i) = 2+3i-4i-6i^2 = 2-i+6 = 8-i.$$

1.4.3 Example. The product of two purely imaginary numbers is real:

$$ai\cdot bi = ab\cdot i^2 = ab\cdot -1 = -ab.$$

1.4.4 Example. As multiplication of complex numbers satisfies the common rules, we also have

$$(a+bi)(a-bi) = a^2-(bi)^2 = a^2-(-b^2) = a^2+b^2.$$

This can also be deduced using the definition of the product:

$$(a+bi)(a-bi) = a^2-(b\cdot -b)+(a\cdot -b+b\cdot a)i = a^2+b^2+0\cdot i = a^2+b^2.$$

1.5 Fractions and division

Division in $\mathbb{C}$ is also possible:

1.5.1 Definition. For every complex number $z = a + bi \neq 0$ we define

$$\frac{1}{z} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

The number $\frac{1}{z}$ is also denoted by z^{-1} or $1/z$ and is called the *inverse* of z.

For all complex numbers w and $z \neq 0$ we define

$$\frac{w}{z} = w \cdot \frac{1}{z}.$$

This definition is inspired by the following:

$$(a + bi)(a - bi) = a^2 + b^2.$$

So, we find for all $z = a + bi \neq 0$:

$$\begin{array}{rcl} z \cdot \frac{1}{z} & = & (a + bi)\left(\frac{a-bi}{a^2+b^2}\right) \\ & = & \frac{a^2+b^2}{a^2+b^2} \\ & = & 1, \end{array}$$

which is what we expect from the definition of $1/z$.

Several calculation rules apply to fractions and division, which we also know for real numbers. We name a few.

- If $zw = 1$, then $w = 1/z$ and $z = 1/w$.
- For all w, $z \neq 0$ and $u \neq 0$ we have
$$\frac{w}{z} = \frac{wu}{zu}.$$
- For all w, $z \neq 0$ and u we have
$$\frac{w}{z} + \frac{u}{z} = \frac{w + u}{z}.$$
- For all $w \neq 0$ and $z \neq 0$ we have
$$\frac{1}{w/z} = \frac{z}{w}.$$

1.5.2 Example. Calculate $\dfrac{1+i}{2+i}$.

We can use the definition for w/z, but it can be done differently. If we come across a fraction with denominator $a+bi$, multiply numerator and denominator by $a-bi$. According to the second calculation rule, the value of the fraction does not change. The trick is that $(a+bi)(a-bi)$ is a real number: $(a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2$. We illustrate this method by means of an example.

$$\begin{aligned}\frac{1+i}{2+i} &= \frac{(1+i)(2-i)}{(2+i)(2-i)}\\ &= \frac{2-i+2i-i^2}{2^2+1^2}\\ &= \frac{3+i}{5}\\ &= \frac{3}{5}+\frac{1}{5}i.\end{aligned}$$

1.6 Exponentiation

1.6.1 Definition. For every complex number z and for every positive integer n the n-th power of z is defined as

$$z^n = \overbrace{z\cdot z\cdot z\cdots z}^{n\text{ times}}.$$

For exponents less than or equal to 0, we make the following convention:

$$z^0 = 1.$$
$$z^n = \frac{1}{z^{-n}} \quad \text{if } n<0.$$

The usual exponentiation properties are also valid for complex numbers. We mention a few:

- $z^m \cdot z^n = z^{m+n}$,
- $(z^m)^n = z^{mn}$,
- $(zw)^n = z^n w^n$

We may think that once we start defining negative exponents, we will immediately continue with fractions as exponents, but that is disappointing.

Just consider $(3-4i)^{1/2}$ or $\sqrt{3-4i}$. If we try to define this number we are looking for a complex number z with $z^2 = 3-4i$. There are two solutions: $z = 2-i$ and $z = i-2$. Which one should we use?

This is similar to the square root of a positive real number. There are two solutions of the equation $x^2 = 4$, namely $x = 2$ and $x = -2$. We then choose the *positive* solution as the definition of $\sqrt{4}$. But none of the solutions of $z^2 = 3-4i$ is positive. There is also no other way to come up with a sensible ordering of the complex numbers so that we can unambiguously define $\sqrt{3-4i}$. Therefore, we define powers of complex numbers only for *integer* exponents.

1.6.2 Example. The complex number z^{-1} is equal to $1/z$. The notation z^{-1} is therefore often used as an alternative notation for $1/z$.

1.6.3 Example. Define $z = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, then

$$z^2 = \frac{1}{4} - 2. \cdot \frac{1}{2} \cdot \frac{1}{2}\sqrt{3} - \frac{1}{4} \cdot 3 = -\frac{1}{2} - \frac{1}{2}i\sqrt{3},$$

and

$$z^3 = z{\cdot}z^2 = \left(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)\left(-\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right) = \left(-\frac{1}{2}\right)^2 - \left(\frac{1}{2}i\sqrt{3}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1.$$

So there is a complex number, unequal to 1, whose cube is equal to 1.

1.6.4 Example. Take z from the previous example. Define $z_1 = z^2$, then the cube of z_1 is also equal to 1:

$$z_1^3 = \left(z^2\right)^3 = z^6 = \left(z^3\right)^2 = 1^2 = 1.$$

We now have three complex numbers whose cube is equal to 1, namely $z = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, $z_1 = -\frac{1}{2} - \frac{1}{2}i\sqrt{3}$ and 1. These are also the only ones, as we will see later in Section 3.6.

1.7 Complex conjugation

1.7.1 Definition. For each complex number $z = a + bi$ we define the *complex conjugate* $\overline{z}$ of z by

$$\overline{z} = a - bi.$$

We often say *conjugate* instead of complex conjugate.

Properties

- A complex number z is real if and only if $\overline{z} = z$.
- $\mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$.
- $\mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$.
- If $z = a + bi$, then $z \cdot \overline{z} = a^2 + b^2$.
- $\overline{\overline{z}} = z$.
- $\overline{z + w} = \overline{z} + \overline{w}$.
- $\overline{z - w} = \overline{z} - \overline{w}$.
- $\overline{zw} = \overline{z} \cdot \overline{w}$.
- $\overline{z/w} = \overline{z}/\overline{w}$.
- $\overline{z^n} = (\overline{z})^n$ for all integer exponents n.

The expression $z\overline{z}$ is useful in many calculations. An example of this can be found when writing $1/z$ in standard form. Indeed, for $z = a + bi$ we have:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

1.7.2 Example. The conjugate of $2 + 3i$ is $2 - 3i$. The conjugate of $1 - 2i$ is $1 + 2i$. The conjugate of i is $-i$.

1.7.3 Example. The conjugate of $\sqrt{2}$ is $\sqrt{2}$, because $\sqrt{2}$ is a real number.

1.7.4 Example. Calculate $\dfrac{1+i}{2+i}$.

$$\begin{aligned}
\frac{1+i}{2+i} &= \frac{(1+i)(\overline{2+i})}{(2+i)(\overline{2+i})} \\
&= \frac{(1+i)(2-i)}{2^2 + 1^2} \\
&= \frac{3+i}{5} \\
&= \frac{3}{5} + \frac{1}{5} i.
\end{aligned}$$

1.7.5 Example. Calculate $(1+i)^4(\overline{1+i})^4$.

We can of course calculate the fourth power of $1+i$ and $1-i$, but it can be done faster:

$$(1+i)^4(\overline{1+i})^4 = \big((1+i)(\overline{1+i})\big)^4 = (1^2+1^2)^4 = 2^4 = 16.$$

1.8 Exercises

Write the following complex numbers in the form $a+bi$, where $a, b \in \mathbb{R}$

1.8.1 Exercise. Write the following sums and differences of complex numbers in the form $a+bi$, where $a, b \in \mathbb{R}$.

a. $(1+i)+(3-i)$

b. $(1+2i)+(2+i)$

c. $(2-i)-(2+i)$

d. $(3+2i)+(1+2i)$

1.8.2 Exercise. Write the following products of complex numbers in the form $a+bi$, where $a, b \in \mathbb{R}$.

a. $(1+i)^2$

b. $(1+2i)(2+i)$

c. $(2-i)(2+i)$

d. $(3+2i)(1+2i)$

1.8.3 Exercise. Write the following fractions of complex numbers in the form $a+bi$, where $a, b \in \mathbb{R}$.

a. $\frac{1}{(1+i)^2}$

b. $\frac{1+2i}{2+i}$

c. $\frac{2-i}{2-i}$

d. $\frac{3+2i}{1+2i}$

2 The complex plane

2.1 The complex plane

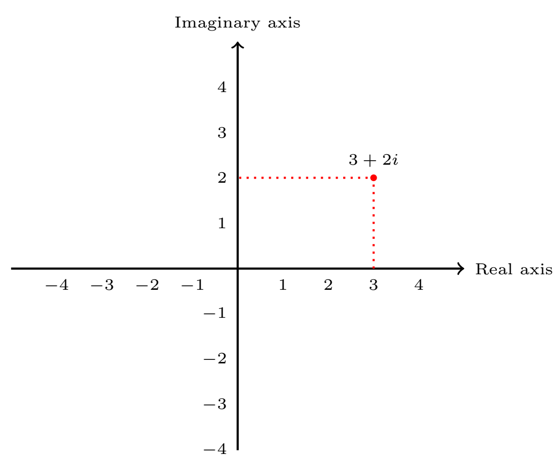
A complex number can be seen as a point in the plane. The complex number $a + bi$ then corresponds to the point with coordinates (a, b). This way we can think of the set of complex numbers as the set of points in $\mathbb{R}^2$.

So, while complex numbers are simply points in the plane, several concepts get other names referring to the complex numbers. The plane itself is now called the *complex plane*. The x-axis is called *real axis*, and the y-axis is called *imaginary axis*. The positive real axis is the portion of the x-axis with points with a positive x-coordinate, and the negative real axis is the portion of the x-axis with points with a negative x-coordinate.

From now the point $(0, 0)$ is called 0, and the point $(1, 0)$ is called 1 and the point $(0, 1)$ is called i.

The x-coordinate of a complex number z is the real part of z, and the y-coordinate is the imaginary part.

2.1.1 Example. We draw the complex number $z = 3 + 2i$ in the plane:



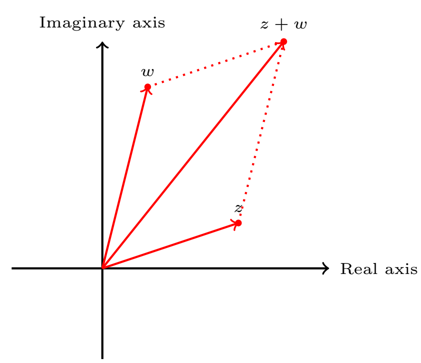
2.1.2 Examples. Consider a complex number $z = a + bi$.

If $b = 0$ then $z = a + 0 \cdot i = a + 0 = a$, in other words: z is a real number. On the other hand, the complex number z corresponds to the point $(a, 0)$, and therefore lies at the real axis. So the real axis (the x-axis) corresponds to the real line.

If $a = 0$ holds $z = 0 + b \cdot i = bi$, in other words: z is purely imaginary. On the other hand, the complex number z corresponds to the point $(0, b)$, and therefore lies at the imaginary axis. The imaginary axis (the y-axis) thus contains the purely imaginary complex numbers.

2.2 Addition in the plane

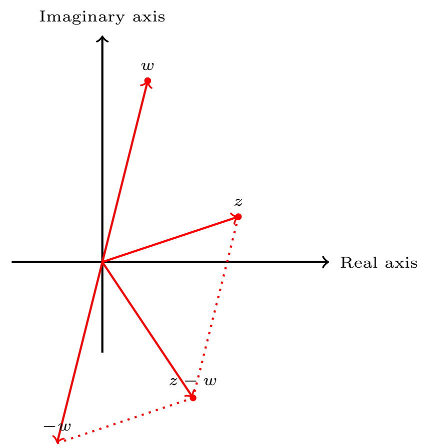
The addition of two complex numbers z and w can be geometrically constructed with the *parallelogram construction*. This means that the points 0, z, w and $z + w$ form the vertices of a parallelogram. Here 0 and $z + w$ are one diagonal, and z and w are on the other diagonal.



Subtraction can also be described using a parallelogram construction. We can interpret the difference $z - w$ as the addition

$$z - w = z + (-w).$$

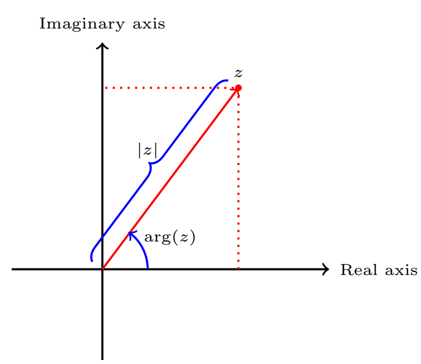
The complex number $-w$ is obtained by mirroring w by 0. So we can also get $z - w$ as a vertex of a parallelogram with vertices 0, z, $-w$ and $z - w$.



2.3 The absolute value and the argument

2.3.1 Definition. The *absolute value* or *modulus* of a complex number $z = a + bi$ is defined as the distance in the plane from 0 to z, and is written as $|z|$.

The *argument* of $z \neq 0$ is defined as the angle between the segment from 0 to z and the positive real axis. It is written as $\arg(z)$.



From the Pythagorean theorem it follows that $|z| = \sqrt{a^2 + b^2}$. Note that $|z|^2 = a^2 + b^2 = z\overline{z}$.

The argument is measured in radians unless explicitly stated otherwise. The argument of 0 is not defined.

The measurement of the argument is made from the positive real axis to z. If we turn it to the left (that is, counterclockwise), the argument is positive. And if we rotate clockwise (i.e. clockwise), the argument is negative.

When determining the angle, it is allowed to make one or more extra turns. This makes no difference to the final direction in which z lies. So the argument is determined *up to an integer multiple of* 2π.

We can always find one (but no more than one) value for the argument in the interval $(-\pi, \pi]$. This is called the *principal value* of the argument.

2.3.2 Example. The absolute value of i is 1: $|i| = \sqrt{0^2 + 1^2} = 1$. The argument is the angle that the imaginary axis makes with the real axis, where we start at the real axis. If we rotate counterclockwise, the angle is $90° = \frac{\pi}{2}$ radians. If we turn clockwise, the angle is $270° = \frac{3\pi}{2}$ radians. So we can say both $\arg(i) = \frac{\pi}{2}$ and $\arg(i) = -\frac{3\pi}{2}$. But $\arg(i) = \frac{5\pi}{2}$ is also allowed.

There are actually infinitely many possible values for $\arg(i)$: for every integer k holds $\arg(i) = \frac{\pi}{2} + 2k\pi$. There is exactly one principal value of $\arg(i)$, namely $\frac{\pi}{2}$.

2.3.3 Example. For a complex number x on the positive real axis holds: $|x| = x$ and $\arg(x) = 0$.

2.3.4 Example. The principal value of the argument of z is the smallest angle that determines the direction of z. Only if a complex number x is on the negative real axis are these angles equal. In that case $\arg(x) = \pi$ applies, but $\arg(x) = -\pi$ is also good. By definition, the principal value is equal to $+\pi$. This choice is arbitrary and has no further meaning.

2.4 Polar coordinates and multiplication

2.4.1 Definition. A complex number z, different from 0, is determined by its modulus $|z|$ together with its argument $\arg(z)$. Indeed we have

$$z = |z| \cdot (\cos(\arg(z)) + \sin(\arg(z))).$$

Together $(|z|, \arg(z))$ form the *polar coordinates* of z.

Although for $z = 0$ the argument is not defined, we can still assign $(0, 0)$ to be its polar coordinates.

Suppose two complex numbers z and w are represented in polar coordinates (r, ϕ) and (s, θ) respectively, where $r = |z|$, $s = |w|$, $\phi = \arg(z)$ and $\theta = \arg(w)$. So

$$z = r(\cos(\phi) + i\sin(\phi))$$

and

$$w = s(\cos(\theta) + i\sin(\theta)).$$

Then for their product we find, using trigonometric formulas for $\sin(\phi+\theta)$ and $\cos(\phi+\theta)$:

$$\begin{aligned} zw &= r(\cos(\phi) + i\sin(\phi)) \cdot s(\cos(\theta) + i\sin(\theta)) \\ &= rs \cdot (\cos(\phi)\cos\theta - \sin(\phi)\sin(\theta) + i \cdot (\cos(\phi)\sin(\theta) + \sin(\phi)\cos(\theta))) \\ &= rs(\cos(\phi+\theta) + i \cdot \sin(\phi+\theta)). \end{aligned}$$

The representation in polar coordinates of zw is equal to

$$zw = (rs)(\cos(\phi+\theta) + i\sin(\phi+\theta)).$$

In other words:

$$|zw| = |z| \cdot |w|$$

and

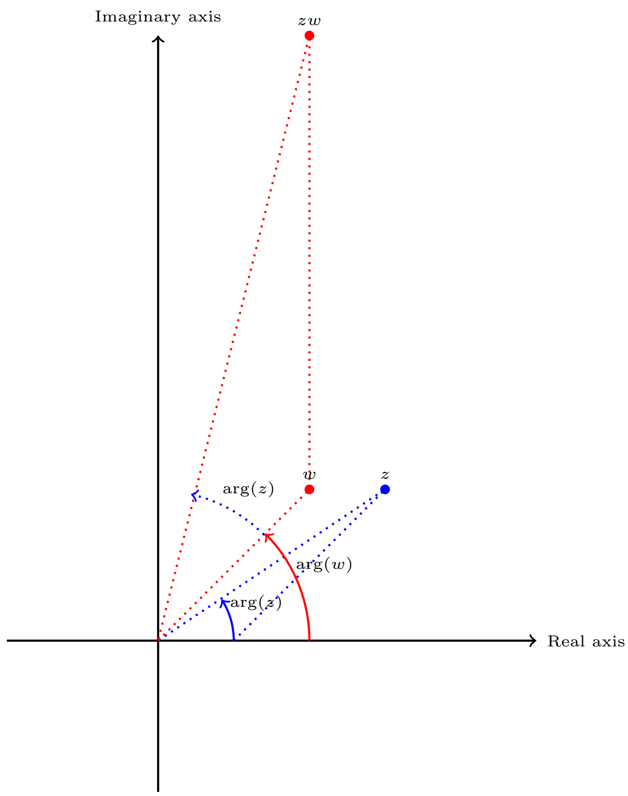
$$\arg(zw) = \arg(z) + \arg(w).$$

A warning is in order with the last equality: because the argument of a complex number is determined up to an integer multiple of 2π it can happen that the equality holds up to multiples of 2π. So we say

$$\arg(zw) = \arg(z) + \arg(w) + 2k\pi$$

for some integer k.

The product of z and w can be constructed geometrically. Namely, it holds that the triangle with vertices 0 , z and zw is similar to the triangle with vertices 0, 1 and w. Moreover the triangle with vertices 0 , w and zw is similar to the triangle with vertices 0, 1 and z.



2.4.2 Example. From z and w it is given that

$$\begin{aligned} |z| &= 2 \\ \arg(z) &= \frac{\pi}{3} \end{aligned}$$

and

$$\begin{aligned} |w| &= 3 \\ \arg(z) &= \frac{\pi}{6} \end{aligned}$$

Calculate zw.

For the product zw applies

$$\begin{aligned} |zw| &= |z| \cdot |w| = 2 \cdot 3 = 6, \\ \arg(zw) &= \arg(z) + \arg(w) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}. \end{aligned}$$

So

$$zw = 6\left(\cos(\pi/2) + i \sin(\pi/2)\right) = 6(0 + 1 \cdot i) = 6i.$$

2.5 Polar coordinates and division

Suppose two complex numbers z and w are represented by polar coordinates (r, ϕ) and (s, θ) respectively, so

$$z = r(\cos(\phi) + i\sin(\phi))$$

and

$$w = s(\cos(\theta) + i\sin(\theta)),$$

where $r = |z|$, $s = |w|$, $\phi = \arg(z)$ and $\theta = \arg(w)$. Then the representation in polar coordinates of z/w is equal to

$$\frac{z}{w} = \frac{r}{s}(\cos(\phi - \theta) + i\sin(\phi - \theta)).$$

In other words:

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

and

$$\arg(z/w) = \arg(z) - \arg(w) + 2k\pi$$

for some integer k.

A special case is $1/z$. In that case applies

$$\frac{1}{z} = \frac{1}{r}(\cos(0 - \phi) + i\sin(0 - \phi)) = \frac{1}{r}(\cos(\phi) - i\sin(\phi)).$$

If z is on the unity circle, then $|z| = 1$, and so

$$z^{-1} = \frac{1}{z} = \cos(\phi) - i\sin(\phi) = (\cos(\phi) - i\sin(\phi) = \overline{z}.$$

For points on the unit circle, inverting is the same as conjugating. That's no surprise, after all $z^{-1} = 1/z = \frac{\overline{z}}{z\overline{z}}$.

The quotient of z and w can be constructed geometrically. Namely, it holds that the triangle has vertices 0 , z/w and z is similar to the triangle with vertices 0, 1 and w. The other side is the triangle with vertices 0 , w and z is similar to the triangle with vertices 0, 1 and z/w.

2.5.1 Example. From z and w it is given that

$$\begin{aligned} |z| &= 2 \\ \arg(z) &= \frac{\pi}{3} \end{aligned}$$

and

$$\begin{aligned} |w| &= 3 \\ \arg(z) &= \frac{\pi}{6} \end{aligned}$$

Calculate z/w.

For the quotient z/w holds

$$\begin{aligned} |z/w| &= \frac{|z|}{|w|} = \frac{2}{3}, \\ \arg(z/w) &= \arg(z) - \arg(w) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \end{aligned}$$

So

$$\frac{z}{w} = \frac{2}{3}\left(\cos(\pi/6) + i\sin(\pi/6)\right) = \frac{2}{3}\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right) = \frac{1}{3}\sqrt{3} + \frac{1}{3}i.$$

2.6 Polar coordinates and exponents

Because exponentiation is repeated multiplication, we can use the polar coordinates when exponentiating as well. Let

$$z = r(\cos(\phi) + i\sin(\phi)),$$

and let n be an integer. Then the representation in polar coordinates of z^n is equal to

$$z^n = r^n(\cos(n\phi) + i\sin(n\phi)).$$

In other words:

$$|z^n| = |z|^n$$

and

$$\arg(z^n) = n \cdot \arg(z) + 2k\pi$$

for some integer k.

2.6.1 Example. The advantage of using polar coordinates only comes into its own with large exponents. Suppose we want to calculate $(1+i)^{20}$. If we were to do this the usual way, we would have to eliminate parentheses in the expression

$$\underbrace{(1+i)(1+i)\cdots(1+i)}_{20 \text{ times}}.$$

It is clear that this is hopeless work. But with the help of polar coordinates it is becomes straightforward.

First we write $1+i$ using polar coordinates:

$$1+i=\sqrt{2}(\cos(\pi/4)+i\sin(\pi/4)).$$

Then

$$\begin{aligned}(1+i)^{20} &= \left(\sqrt{2}\right)^{20}(\cos(20\cdot\frac{\pi}{4})+i\sin(20\cdot\frac{\pi}{4}))\\ &= \left(2^{1/2}\right)^{20}(\cos(5\pi)+i\sin(5\pi))\\ &= 2^{10}(-1+0\cdot i)=-2^{10}=-1024.\end{aligned}$$

2.7 Polar coordinates and conjugates

Let

$$z=r(\cos(\phi)+i\sin(\phi)),$$

Then the representation in polar coordinates of the conjugate $\overline{z}$ is equal to

$$\overline{z}=r(\cos(\phi)-i\sin(\phi)).$$

Note that $\cos(-\phi)=\cos(\phi)$, and that $\sin(-\phi)=-\sin(\phi)$. So we can also write:

$$\overline{z}=r(\cos(-\phi)+i\sin(-\phi)).$$

In other words:

$$|\overline{z}|=|z|$$

and

$$\arg(\overline{z})=-\arg(z).$$

Geometrically, to conjugate means to mirror with respect to the real axis. We can achieve the same effect by changing the angle we need to turn.

2.7.1 Example. Suppose $z=1+i$, then $\arg(z)=\pi/4$. For the conjugate of z holds: $\arg(\overline{z})=-\arg(z)=-\pi/4$.

2.8 The complex exponential function

2.8.1 **Definition.** Let $z = a + bi$ be a complex number. Then e^z is defined to be the complex number

$$e^a \cdot (\cos(b) + i\sin(b)).$$

The function $\mathrm{Exp} : \mathbb{C} \to \mathbb{C}$ defined by

$$\mathrm{Exp}(z) = e^z$$

is called the complex exponential function.

Notice that when we restrict the function Exp to the real axis $\mathbb{R}$, then it is just the well known real exponential function.

2.8.2 **Examples.** We have the following identities:
$e^{\frac{\pi}{3}i} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$,
$e^{\frac{\pi}{2}i} = i$,
$e^{\pi i} = -1$.
This latter formula is often written as

$$e^{i\pi} + 1 = 0$$

known as Euler's formula. The famous physicist Richard Feynman called it *the most remarkable formula in mathematics.*

Notice that for $z = a + bi$ and $w = c + di$ we have

- $|e^z| = e^a$;
- $\arg(e^z) = b$; The polar coordinates of e^z are (e^a, b). In particular, we find that for a purely imaginary z we have that $|e^z| = 1$.
- $e^{z+w} = e^z \cdot e^w$.
- $\overline{e^z} = e^{\overline{z}}$
- $e^{-z} = \frac{1}{e^z}$.

Furthermore we find

- $\cos(b) = \frac{e^{ib}+e^{-ib}}{2}$
- $\sin(b) = \frac{e^{ib}-e^{-ib}}{2i}$

We can also extend the latter formulas to extend the functions cos and sin to complex functions where for all $z \in \mathbb{C}$ we have

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2} \text{ and } \sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$$

2.9 Exercises

2.9.1 Exercise. Draw the numbers z, $\overline{z}$ and $\frac{1}{z}$ in the complex plane, where z equals:

a. $1+i$

b. $2+i2\sqrt{3}$

c. $\sqrt{3}+i$

d. $\frac{1}{1+i\sqrt{3}}$

2.9.2 Exercise. Determine the modulus and the argument of the following complex numbers.

a. $1+i$

b. $1+i\sqrt{3}$

c. $\frac{1}{1+i\sqrt{3}}$

d. $4-4i$

e. $(1+i)^5$

f. $(\sqrt{3}+i)^7$

g. $\frac{1}{(1-i)^4}$

2.9.3 Exercise. Write the following complex numbers in the form $a+bi$:

a. e^{1-i}

b. $e^{2-i\pi}$

c. $e^{\ln(2)+i\frac{\pi}{3}}$

d. e^{i}

e. $\cos(i)$

f. $\sin(\pi+i)$

g. $\cos(i\pi)$

2.9.4 Exercise. Write the following complex numbers in the form e^{a+bi}:

a. $3i$

b. $4+4i$

c. $1+i\sqrt{3}$

d. $1-i$

2.9.5 Exercise. Determine the modus and argument of yje following complex numbers, and draw them in the plane.

a. $\sqrt{3}-2i$

b. $-\sqrt{3}-2i$

c. $(\sqrt{3}-2i)+\overline{\sqrt{3}-2i}$

d. $(\sqrt{3}-2i)\cdot(\overline{\sqrt{3}-2i})$

2.9.6 Exercise. Let $\phi \in \mathbb{R}$. Express $\cos(3\phi)$ as well as $\sin(3\phi)$ in terms of $\cos(\phi)$ and $\sin(\phi)$.

3 Complex equations

3.1 Soving equations

Complex numbers come in handy when solving equations. The real numbers sometimes fall short in that respect. The equation $z^2 = -1$ has no real solutions, but does allow for complex ones.

Complex equations can be considered to be *systems* of real equations, because a complex number is determined by two quantities, namely the real part and the imaginary part. An equation of the form $f(z) = g(z)$, with $f(z)$ and $g(z)$ complex expressions in z can we then split it into a real and an imaginary part:

$$\begin{aligned} \mathrm{Re}(f(z)) &= \mathrm{Re}(g(z)), \\ \mathrm{Im}(f(z)) &= \mathrm{Im}(g(z)). \end{aligned}$$

This yields a system of two equations with two unknowns $Re(z)$ and $Im(z)$. However, it is not always necessary to split the equation into the real and imaginary parts, as we can see in the first example.

There are endless types of equations, each with its own solution method. On this site we will review the most important ones. On this page we give some examples, so that we can get a taste for it.

3.1.1 Example. Solve $2z = iz + 5$.

First we isolate z. Bring the term iz to the left-hand side:

$$2z - iz = 5.$$

Remove z from parentheses:

$$(2 - i)z = 5.$$

Divide left and right sides by $2 - i$:

$$z = \frac{5}{2 - i}.$$

This solves the equation.

We are not done yet, the solution is not yet in standard form.

$$z = \frac{5}{2 - i} = \frac{5(2 + i)}{(2 - i)(2 + i)} = \frac{5(2 + i)}{5} = 2 + i.$$

Finally, we check if $2 + i$ is indeed a solution:

$$iz + 5 = i(2 + i) + 5 = 2i - 1 + 5 = 2i + 4 = 2(i + 2) = 2z.$$

3.1.2 Example. Solve $z = i\overline{z}$.

Write $z = a + bi$, then

$$\begin{aligned} z &= i\overline{z} \\ a+bi &= i(\overline{a+bi}) \\ a+bi &= i(a-bi) \\ a+bi &= ai+b \\ a-b+(b-a)i &= 0 \end{aligned}$$

This means that $a = b$, or $z = a + bi = a + ai = a(1 + i)$. Each value of a yields a solution. Suppose $z = a(1 + i)$, then

$$i\overline{z} = i\overline{a(1+i)} = ai(1-i) = ai + a = a(i+1) = z.$$

So there are infinitely many solutions. The solutions form a line, namely the line through 0 and $1 + i$.

3.1.3 Example. Solve $|z - i| = |z - 1|$.

We can solve this equation by interpreting absolute values as a way of measuring distances: $|v - w|$ is the distance between the points u and v. The solution of $|z - i| = |z - 1|$ consists of all points z whose distance to 1 is equal is on the distance to i. So this is the perpendicular bisector of i and 1, in other words: these are the complex numbers $a(1 + i)$ with a any real number.

The equation can also be solved algebraically. Since there are absolute values in the equation, square roots are likely to occur in the calculation to arise. To avoid this, we square the left and right sides of the equation first.

$$\begin{aligned} |z-i|^2 &= |z-1|^2 \\ (z-i)\overline{(z-i)} &= (z-1)\overline{(z-1)} \\ (z-i)(\overline{z}+i) &= (z-1)(\overline{z}-1) \\ z\overline{z}+iz-i\overline{z}+1 &= z\overline{z}-z-\overline{z}+1 \\ iz-i\overline{z} &= -z-\overline{z} \\ iz+z &= i\overline{z}-\overline{z} \\ (i+1)z &= (i-1)\overline{z} \\ z &= \frac{i-1}{i+1}\overline{z} \end{aligned}$$

Since $\frac{i-1}{i+1} = i$ the equation $|z - i| = |z - 1|$ can be reduced to $z = i\overline{z}$, the comparison from the previous example.

3.1.4 Example. Solve $z^2 - 2z + 2 = 0$.

This is a quadratic equation. We can solve quadratic equations using *completion of the square*:

$$\begin{aligned} z^2 - 2z + 2 &= 0 \\ z^2 - 2z + 1 + 1 &= 0 \\ (z-1)^2 + 1 &= 0 \\ (z-1)^2 &= -1 \end{aligned}$$

Substitute $w = z - 1$, the equation becomes: $w^2 = -1$. One solution is obvious: $w = i$. But $w = -i$ is also a solution, so

$$w = z - 1 = \pm i,$$

in other words

$$z = 1 \pm i.$$

3.2 The Power Equation

The equation $z^n = w$ (with unknown z and given complex number $w \neq 0$ and positive integer n) solve using polar coordinates. Suppose the polar notation of w is $w = s(\cos(\theta) + i\sin(\theta))$, with $s = |w|$ and $\theta = \arg(w)$. We also write z in polar form: $z = r(\cos(\phi) + i\sin(\phi))$. Note that $r = |z|$ and $\phi = \arg(z)$ are unknown. We determine r and ϕ separately.

First we take the absolute value of the left and right sides: $|z|^n = |w|$, that is

$$r^n = s.$$

Because $s = |w| > 0$, r can be calculated using the nth root: $r = \sqrt[n]{s}$.

We determine the argument ϕ of z by considering that except for an integer multiple of 2π $\arg(z^n)$ is equal to $n \cdot \arg(z)$. So

$$n\phi = n\arg(z) = \arg(z^n) + 2k\pi = \arg(w) + 2k\pi = \theta + 2k\pi$$

for an integer k. So

$$\phi = \frac{\theta}{n} + \frac{2k\pi}{n}.$$

The solutions have been determined:

$$z_k = \sqrt[n]{|w|}(\cos(\phi_k) + i\sin(\phi_k)),$$

with $\phi_k = \frac{\arg(w)}{n} + \frac{2k\pi}{n}$. It suffices to choose the numbers k in $0, 1, 2 \ldots, n-1$. So we always get exactly n different solutions z_0, z_1, $z_2 \ldots, z_{n-1}$.

3.2.1 Example. Solve $z^4 = -4$.

So given is $n = 4$ and $w = -4$. It holds that $s = |w| = 4$ and $\theta = \arg(w) = \pi$. So $|z| = \sqrt[4]{4} = 4^{1/4} = (2^2)^{1/4} = 2^{1/2} = \sqrt{2}$.

For the z argument, use the formula for ϕ_k:

$$\phi_k = \frac{\arg(w)}{n} + \frac{2k\pi}{n} = \frac{\pi}{4} + \frac{2k\pi}{4} = \frac{\pi}{4} + k\frac{\pi}{2}$$

with $k = 0, 1, 2, 3$. This calculates four values:

$$\begin{aligned} \phi_0 &= \frac{\pi}{4}, \\ \phi_1 &= \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}, \\ \phi_2 &= \frac{\pi}{4} + 2\frac{\pi}{2} = \frac{5\pi}{4}, \\ \phi_3 &= \frac{\pi}{4} + 3\frac{\pi}{2} = \frac{7\pi}{4}. \end{aligned}$$

This provides four solutions:

$$\begin{aligned} z_0 &= \sqrt{2}(\cos(\phi_0) + i\sin(\phi_0)) \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) \\ &= 1 + i. \end{aligned}$$

With similar calculations we get the other solutions $z_1 = -1+i$, $z_2 = -1-i$ and $z_3 = 1-i$.

3.2.2 Example. The equation $z^2 = w$ has one solution if $w = 0$ (namely $z = 0$), and two solutions as $w \neq 0$. In that case the solutions are (with $\theta = \arg(w)$)

$$z_0 = \sqrt{|w|}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$

and

$$z_1 = \sqrt{|w|}\left(\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \ pi\right)\right).$$

Note that $\cos(a + \pi) = -\cos(a)$ and $\sin(a + \pi) = -\sin(a)$, so $z_1 = -z_0$. That is why the solutions of $z^2 = w$ are often described as

$$z_{0,1} = \pm\sqrt{|w|}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$

3.3 Equations involving the exponential funtion

Below you find some examples in which e^z plays a role.

3.3.1 Example. Solve
$$e^z = e^{\overline{z}}.$$
Suppose
$$e^z = e^{\overline{z}}.$$
Then
$$e^{z-\overline{z}} = 1$$
from which it follows that
$$e^{2i\cdot \mathrm{Im}(z)} = e^0.$$
Hence $2\mathrm{Im}(z) = 2k\pi$ and $\mathrm{Im}(z) = k\pi$ with $k \in \mathbb{Z}$.

3.3.2 Example. Find all $z \in \mathbb{C}$ with
$$e^z = i \cdot e^{iz}.$$
As $i = e^{i\pi/2}$ we find
$$e^z = e^{i\pi/2} \cdot e^{iz} = e^{i\pi/2+iz},$$
from which it follows that
$$e^{z-i\pi/2-iz} = 1.$$
But then
$$(1-i)z - i\pi/2 = 2k\pi \cdot i$$
and
$$z = \frac{i\pi}{1-i} \cdot (\frac{1}{2} + 2k) = \frac{(i-1)\pi}{2} \cdot (\frac{1}{2} + 2k)$$
with $k \in \mathbb{Z}$.

3.4 Polynomial equations

3.4.1 Definition. A *polynomial of degree n in z* is an expression of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers and where $a_n \neq 0$. The numbers $a_0, a_1, \ldots, a_n$ are called the *coefficients* of the polynomial.

3.4.2 Definition. A *polynomial equation of degree n* is an equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers and where $a_n \neq 0$. The number n is called the *degree* of the polynomial.

3.4.3 Definition. A complex number u is a *solution* or *root* of the equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0,$$

if

$$a_n u^n + a_{n-1} u^{n-1} + \cdots + a_2 u^2 + a_1 u + a_0 = 0.$$

A solution u is also called a *zero* of the polynomial $a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$.

If $n = 2$ one also speaks of an *quadratic equation*. We can solve quadratic equations with the *abc*-formula, even if they have complex coefficients. The *abc*-formula expresses the solution in a formula in which the square root plays a role.

There is also a formula for cubic polynomial equations, the so-called *Formula of Cardano*. In this formula we will find square and cube roots.

Even for polynomial equations of degree 4 there do exist formulae for the roots, just like the *abc*-formula and Cardano's formula. However, the famous Abel-Ruffini theorem says that for n-th degree polynomial equations with $n \geqslant 5$ it is, in general, not possible to give a formula for the roots. That's not to say there aren't solutions. In fact, the *Fundamental Theorem of Algebra* 3.6.1 just says that *every* polynomial equation has solutions.

3.4.4 Example. The equation $z^2 - (2+2i)z + 1 + 2i = 0$ is a quadratic equation with complex coefficients. This equation has two roots. The number 1 is a root:

$$1^2 - (2+2i) \cdot 1 + 1 + 2i = 1 - 2 - 2i + 1 + 2i = 0.$$

The other square root is $1 + 2i$:

$$(1+2i)^2 - (2+2i)(1+2i) + 1 + 2i = ((1+2i) - (2+2i) + 1)(1+2i) = 0(1+2i) = 0.$$

3.4.5 Example. The complex number $\zeta = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$ is solution of the fifth degree equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$. This simply follows from equality

$$(z-1)(z^5 + z^4 + z^3 + z^2 + z + 1) = z^6 - 1.$$

So

$$z^5 + z^4 + z^3 + z^2 + z + 1 = \frac{z^6 - 1}{z - 1}.$$

The complex number ζ is the point on the unit circle with argument $\pi/6$, so $\zeta^6 = 1$, and with that we calculate easily

$$\zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = \frac{\zeta^6 - 1}{\zeta - 1} = \frac{1 - 1}{-\frac{1}{2} + \frac{1}{2}i\sqrt{3}} = 0.$$

3.5 Quadratic equations

A quadratic equation is an equation of the form $az^2 + bz + c = 0$ with unknown z and given complex numbers $a \neq 0$, b and c. The *abc*-formula is usually not useful, because it contains the square root. The square root function is not defined for complex numbers.

We use a method to solve the quadratic equation by a method which is called *completing the square*. This method takes advantage of the fact that:

$$(z + \tfrac{1}{2}\alpha)^2 = z^2 + 2 \cdot \tfrac{1}{2}\alpha z + \tfrac{1}{4}\alpha^2 = z^2 + \alpha z + \tfrac{1}{4}\alpha^2,$$

So

$$z^2 + \alpha z = (z + \tfrac{1}{2}\alpha)^2 - \tfrac{1}{4}\alpha^2.$$

This explains the name *completing squares*: we wrote $z^2 + \alpha z$ as a sum of a square (namely $(z + \frac{1}{2}\alpha)^2$) and a constant ($-\frac{1}{4}\alpha^2$).

Before we can complete the square, we must first divide the quadratic equation by a.

$$\begin{array}{lll} az^2 + bz + c & = 0 & \Leftrightarrow \\ z^2 + \frac{b}{a}z + \frac{c}{a} & = 0 & \Leftrightarrow \\ z^2 + \frac{b}{a}z & = -\frac{c}{a} & \Leftrightarrow \\ \left(z + \frac{1}{2}\frac{b}{a}\right)^2 - \frac{1}{4}\left(\frac{b}{a}\right)^2 & = -\frac{c}{a} & \Leftrightarrow \\ \left(z + \frac{b}{2a}\right)^2 & = \frac{1}{4}\left(\frac{b}{a}\right)^2 - \frac{c}{a} & \\ & = \frac{b^2-4ac}{4a^2} & \end{array}$$

The number $b^2 - 4ac$ is called the *discriminant* of the equation. We denote the discriminant with D. Multiply left and right by $4a^2 = (2a)^2$:

$$(2az + b)^2 = D.$$

If we define $u = 2az + b$ the equation is reduced to

$$u^2 = D.$$

In general, D is a complex number. There are now two options:

$D = 0$ The only solution is $u = 0$, so $z = -b/2a$.

$D \neq 0$ Solve the equation as power equation: write D polar: $D = s(\cos(\theta) + i\sin(\theta))$, with $s = |D|$ and $\theta = \arg(D)$. The solutions of $u^2 = D$ are

$$u = \pm\rho\sqrt{|D|},$$

where $\rho = \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)$. Then holds for z

$$z = \frac{-b \pm \rho\sqrt{|D|}}{2a}$$

This is the *abc* formula for complex quadratic equations.

Note that if $D = 0$ the *abc* formula also yields the solution $-b/2a$.

3.5.1 Example. If a, b and c are real numbers, the discriminant D is also real. For $D \geqslant 0$ $s = |D| = D$ and $\theta = \arg(D) = 0$, so $\rho = 1$. The *abc* formula then becomes

$$z = \frac{-b \pm \sqrt{D}}{2a},$$

The old familiar *abc* formula as we learned it for real coefficients.

For $D < 0$ the real *abc* formula gives no solutions, but the complex one does. In that case $s = |D| = -D$ and $\theta = \arg(D) = \pi$, so $\rho = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = 0 + 1 \cdot i = i$. We can then reduce the *abc* formula to

$$z = \frac{-b \pm i\sqrt{-D}}{2a},$$

Note that one solution is the conjugate of the other.

For example, take the equation $x^2 + x + 1 = 0$. The discriminant is $D = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$. The solutions are

$$z = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}.$$

These solutions lie on the unit circle and have argument $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$.

3.5.2 Example. Solve $z^2 - (2+2i)z + 1 + 2i = 0$.

The discriminant is $D = (2+2i)^2 - 4(1+2i) = 4 + 8i - 4 - 4 - 8i = -4$. The argument of D is $\theta = \pi$, so $\rho = \cos(\pi) + i\sin(pi) = i$, and so

$$z = \frac{-b \pm \rho\sqrt{|D|}}{2a} = \frac{-(-(2+2i)) \pm i\sqrt{4}}{2} = \frac{2 + 2i \pm 2i}{2} = 1{+}i{\pm}i = 1 \text{ of } 1{+}2i.$$

3.6 The fundamental theorem of algebra

The fundamental theorem of algebra says that every polynomial equation of degree $n \geqslant 1$ has solutions. Precisely stated:

3.6.1 Theorem. (Fundamental Theorem of Algebra) *Suppose*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$$

is a polynomial of degree n, then there are complex numbers $z_1, z_2, \ldots, z_n$ such that

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = a_n(z - z_1)(z - z_2)\cdots(z - z_n).$$

It is clear that $z_1, z_2, \ldots, z_n$ are zeros of $p(z)$. These are also immediately *all* zeros of $p(z)$.

The product $a_n(z - z_1)(z - z_2)\cdots(z - z_n)$ is called an *decomposition into linear factors*. The *linear factors* are the expressions $z - z_1$, $z - z_2$ and so on. The decomposition is unique, except for the order of the linear factors.

The roots $z_1, z_2, \ldots, z_n$ don't all have to be different. However, the number of factors is equal to the degree of the polynomial. The polynomial z^2+2z+1 can be factored as $(z+1)^2 = (z+1)(z+1)$.

The main theorem just says there are roots, but the theorem doesn't tell us how to find them. For degree n polynomial equations with $n \geqslant 5$ there isn't even a general square root formula. This is a theorem: the Abel-Ruffini theorem.

Sometimes we can make a global statement. For example

> If all coefficients are real, and the degree is odd, then there is at least one real zero.

Or

> If all coefficients are real, and z is a square root, then $\overline{z}$ is also a square root.

3.6.1 Example. The third-degree polynomial $z^3 - 2z^2 + 1$ has real coefficients, so it has at least one real zero. The plot of the function $f(x) = x^3 - 2x^2 + 1$ shows that there are even three real zeros. The zeros are 1, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. The decomposition into linear factors is

$$z^3 - 2z^2 + 1 = (z-1)\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right).$$

3.6.2 Example. The third degree polynomial $z^3 - z + 1$ has real coefficients, so it has at least one real zero. The plot of the function $f(x) = x^3 - x + 1$ shows that there is one real zero point. The real zero point is

$$\sqrt[3]{-\frac{1}{2} + \frac{1}{18}\sqrt{69}} + \sqrt[3]{-\frac{1}{2} - \frac{1}{18}\sqrt{69}} \approx -1.3247.$$

There are two more zeros, but they are not real.

3.6.3 Example. The fifth-degree polynomial $z^5 - z + 1$ has real coefficients, so it has at least one real zero. The plot of the function $f(x) = x^5 - x + 1$ shows that this zero point is between -2 and -1. That zero point is the only real zero point: the other zero points are all imaginary. A numerical approximation for this zero point is -1.16730. However, there is no nice square root formula for this solution.

3.6.4 Example. Note: it is not the case that fifth- or higher-degree polynomials do not have zeros, or that these zeros cannot be written using square root functions. The Abel-Ruffini theorem says that there is no *general* root formula. The fifth-degree polynomial $z^5 - 15z^4 + 85z^3 - 225z^2 + 274z - 120$ has five real zeros, namely the numbers 1, 2, 3, 4 and 5.

3.7 Division with remainder

Just like with integers, we can also divide polynomials by each other. The division produces a quotient and a remainder. The method is called *long division*, and the method is basically the same as the long division on whole numbers. We explain long division with polynomials using an example.

We want to divide $2z^3 - 3z^2 + 2$ by $z^2 - 1$. We start like this:

$$z^2 - 1 \;/\; 2z^3 - 3z^2 + 2 \;\backslash$$

Now we look at the highest degree terms (boxed):

$$\boxed{z^2} - 1 \;/\; \boxed{2z^3} - 3z^2 + 2 \;\backslash$$

The quotient of the boxed terms is $2z^3/z^2 = 2z$, put this to the right of the right slanted steep:

$$z^2 - 1/\; 2z^3 - 3z^2 + 2 \;\backslash\; 2z$$

The expression $2z$ will turn out to be one of the terms of the quotient. We multiply this by the polynomial $z^2 - 1$:

$$2z \cdot (z^2 - 1) = 2z^3 - 2z$$

We write this under the middle polynomial. We have written the polynomials so that terms with equal degree are below each other. There is no term with z, so we write an artificial term $0 \cdot z$:

$$\begin{array}{lllll} z^2 - 1 & / & 2z^3 - 3z^2 + 0z + 2 & \backslash & 2z \\ & & 2z^3 \qquad\quad - 2z & & \end{array}$$

Subtract the polynomials:

$$\begin{array}{lllll} z^2 - 1 & / & 2z^3 - 3z^2 + 0z + 2 & \backslash & 2z \\ & & \underline{2z^3 \qquad\quad - 2z \qquad} & & \\ & & \quad\; - 3z^2 + 2z + 2 & & \end{array}$$

We repeat the steps just made. This creates a "tail" of polynomials, which explains the name.

We again frame the highest degree terms. Divide the boxed terms: $-3z^2/z^2 = -3$, and write the result on the right: Multiply z^2-1 by -3, and write this at the bottom of the tail:

$$\begin{array}{lll} z^2-1 & / \; 2z^3 - 3z^2 + 0z + 2 & \backslash \; 2z-3 \\ & \underline{2z^3 \qquad\quad - 2z \quad\;\;} & \\ & \quad\; - 3z^2 + 2z + 2 & \\ & \quad\; - 3z^2 + \qquad 3 & \end{array}$$

and subtract:

$$\begin{array}{lll} z^2-1 & / \; 2z^3 - 3z^2 + 0z + 2 & \backslash \; 2z-3 \\ & \underline{2z^3 \qquad\quad - 2z \quad\;\;} & \\ & \quad\; - 3z^2 + 2z + 2 & \\ & \quad\; \underline{-3z^2 + \qquad 3} & \\ & \qquad\qquad\quad 2z - 1 & \end{array}$$

Now we can't go any further:

$$\begin{array}{lll} \boxed{z^2}-1 & / \; 2z^3 - 3z^2 + 0z + 2 & \backslash \; 2z-3 \\ & \underline{2z^3 \qquad\quad - 2z \quad\;\;} & \\ & \quad\; - 3z^2 + 2z + 2 & \\ & \quad\; \underline{-3z^2 + \qquad 3} & \\ & \qquad\qquad\quad \boxed{2z} - 1 & \end{array}$$

The quotient of $2z/z^2$ no longer yields the power of z that we used with the quotient on the right can add to the long division. In general, we stop when the degree of the polynomial at the bottom of the tail is 0, or when the degree *smaller* is than the degree of the polynomial we are dividing by.

The polynomial to the right of the right bar is the *quotient*, and the polynomial at the bottom of the tail is the *remainder*.

If we divide a polynomial $p(z)$ by $f(z)$ with long division, and the result is a quotient $q(z)$ and a remainder $r(z)$, then

$$p(z) = f(z)q(z) + r(z).$$

Actually we have the following result:

3.7.1 Theorem. *Given a polynomial $p(z)$ and a polynomial $f(z)$ of degree n, then there exist unique polynomials $q(z)$ and $r(z)$, with the degree of $r(z)$ smaller than n such that*

$$p(z) = f(z)q(z) + r(z).$$

The polynomial $q(z)$ and $r(z)$ can be obtained as quotient and remainder, respectively, of long division of $p(z)$ by $f(z)$.

3.7.1 Example. Divide $z^2 - 1$ by $z - 1$:

$$\begin{array}{rl} z-1 \; / & z^2 \qquad -1 \; \backslash \; z+1 \\ & \underline{z^2 - z \qquad} \\ & \qquad z - 1 \\ & \qquad \underline{z - 1} \\ & \qquad\quad 0 \end{array}$$

In this case the remainder is 0. That in itself is not strange, after all $z^2 - 1 = (z+1)(z-1)$ holds.

3.7.2 Example. If we divide a polynomial $p(z)$ by a polynomial whose degree is greater than the degree of $p(z)$ the algorithm stops immediately. We then agree that the quotient is 0, and the remainder is $p(z)$.

Obviously if we can write a polynomial $p(z)$ as a product in the form

$$p(z) = (z - \alpha)q(z)$$

with $q(z)$ another polynomial, that α is a zero of $p(z)$. Just enter α in $p(z)$:

$$p(\alpha) = (\alpha - \alpha)q(\alpha) = 0 \cdot q(\alpha) = 0.$$

However, the reverse is also true:

> If α is a zero of the polynomial $p(z)$, then there is a polynomial $q(z)$ such that
> $$p(z) = (z - \alpha)q(z).$$

The principle of division is actually based on long division. If we do a long division where we divide $p(z)$ by $z - \alpha$ we get a quotient $q(z)$ and a remainder $r(z)$. This remainder $r(z)$ is ór 0, ór a polynomial with degree smaller than the degree of $x - \alpha$. Since the degree of $z - \alpha$ is 1, $r(z)$ is a constant polynomial, i.e.

$$p(z) = (z - \alpha)q(z) + c$$

for some complex number c. By entering α we see that c must be equal to zero.

With the help of division we can determine the factorization of polynomials. We demonstrate this using an example.

3.7.3 Example. Consider the cubic polynomial $z^3 - 2z^2 + 1$. We can easily see that 1 is a zero of this polynomial. This means that there exists a (second degree) polynomial $q(z)$ such that $z^3 - 2z^2 + 1 = (z-1)q(z)$. The polynomial $q(z)$ can be found by performing long division:

$$\begin{array}{rl} z-1 \ / & z^3 - 2z^2 \qquad + 1 \ \backslash \ z^2 - z - 1 \\ & \underline{z^3 - \ \ z^2 \qquad\quad\ \ } \\ & \qquad - z^2 \qquad + 1 \\ & \qquad \underline{-z^2 + z \qquad\ } \\ & \qquad\qquad - z + 1 \\ & \qquad\qquad \underline{-z + 1} \\ & \qquad\qquad\quad 0 \end{array}$$

So $q(z) = z^2 - z - 1$. This is a quadratic polynomial, whose zeros can be determined with the *abc* formula:

$$z = \frac{1 \pm \sqrt{5}}{2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}.$$

So the linear factorization of $q(z)$ is

$$q(z) = (z - \frac{1}{2} + \frac{1}{2}\sqrt{5})(z - \frac{1}{2} - \frac{1}{2}\sqrt{5}),$$

and the factorization of $p(z)$ is thus

$$p(z) = z^3 - 2z^2 + 1 = (z-1)q(z) = (z-1)(z - \frac{1}{2} + \frac{1}{2}\sqrt{5})(z - \frac{1}{2} - \frac{1}{2}\sqrt{5}).$$

3.7.4 Example. Consider the equation $z^4 - 2z^3 + 2z^2 - z + 1 = 0$. After some searching we find the square root $z = 1$. Divide the factor $(z-1)$:

$$
\begin{array}{llllll}
z-1 \;/ & z^4 - 2z^3 & +\,2z^2 & -\,2z & +\,1 & \backslash\; z^3 - z^2 + z - 1 \\
 & \underline{z^4 - z^3} & & & & \\
 & - z^3 & \underline{+\,2z^2} & & & \\
 & \underline{ - z^3} & \underline{+\,z^2} & & & \\
 & & z^2 & -\,2z & & \\
 & & \underline{z^2} & \underline{-\,z} & & \\
 & & & -\,z & +\,1 & \\
 & & & \underline{-\,z} & \underline{+\,1} & \\
 & & & & 0 &
\end{array}
$$

So $z^4 - 2z^3 + 2z^2 - 2z + 1 = (z-1)(z^3 - z^2 + z - 1)$

Of the quotient $z^3 - z^2 + z - 1$, 1 is again a zero, and we divide the factor $z - 1$ again:

$$
\begin{array}{lllll}
z-1 \;/ & z^3 - z^2 & +\,z & -\,1 & \backslash\; z^2 + 1 \\
 & \underline{z^3 - z^2} & \underline{} & & \\
 & & z & -\,1 & \\
 & & \underline{z} & \underline{-\,1} & \\
 & & & 0 &
\end{array}
$$

So $z^3 - z^2 + z - 1 = (z-1)(z^2+1)$.

Finally, we factor $z^2 + 1$:

$$z^2 + 1 = (z - i)(z + i)$$

If we combine everything, the final result is the decomposition of $z^4 - 2z^3 + 2z^2 - z + 1$ into linear factors:

$$z^4 - 2z^3 + 2z^2 - z + 1 = (z-1)(z-1)(z-i)(z+i),$$

what we usually write as

$$z^4 - 2z^3 + 2z^2 - z + 1 = (z-1)^2(z-i)(z+i).$$

3.8 Cardano's formula [not for exam]

Cardano's formula gives a general solution to the cubic equation

$$az^3 + bz^2 + cz + d = 0,$$

with a, b, c and d complex numbers and $a \neq 0$. The equation is first reduced by defining the constants p and q as follows:

$$p = \frac{c}{a} - \frac{b^2}{3a^2}$$

and

$$q = \frac{d}{a} + \frac{2b^3 - 9abc}{27a^3}.$$

A solution of the equation is now given by

$$z = -\frac{b}{3a} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

This formula is due to Nicolo Tartaglia (1500-1557), but is nevertheless known as the formula of Cardano (1501-1576). Incidentally, only polynomials with real coefficients were studied in the sixteenth century.

The above formula is actually not complete. Somewhere in the derivation an equation of type $u^3 = w$ must be solved, and this means that three complex solutions should appear. In Tartaglia's time complex numbers were not yet known, so he only wrote down the (apparently) real solution $u = \sqrt[3]{w}$. Apart from that, complex numbers can also occur in Tartaglia's formula (for example if $\frac{p^3}{27} + \frac{q^2}{4}$ is negative). Cardano's formula was actually the reason for the discovery of the complex numbers.

A better formulation of Cardano's formula reads:

$$z_k = -\frac{b}{3a} + \rho^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}} + \rho^{-k} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{27} + \frac{q^2}{4}}}.$$

where $k = 0$, 1 or 2, and where $\rho = -\frac{1}{2} + \frac{1}{2}i\sqrt{3} = \cos(2\pi/3) + i\sin(2\pi/3)$, the complex number on the unit circle with argument $2\pi/3$.

3.8.1 Example. Determine the solutions of the equation $z^3 - 3z^2 - 84z - 44 = 0$.

Filling in the formulas for p and q yields $p = -87$ and $q = -130$. The number under the radical sign is $\frac{p^3}{27} + \frac{q^2}{4} = -20164 = -142^2$. This number is negative, so the square root of this is $142i$ or $-142i$. It turns out that it doesn't matter which one we choose. Cardano's formula provides the solutions:

$$z_k = 1 + \rho^k \sqrt[3]{65 + 142i} + \rho^{-k} \sqrt[3]{65 - 142i}.$$

Here we can immediately see what the problem is with Cardano's formula: we need to search for a complex number u with the property $u^3 = 65+142i$. One solution is $u = 5 + 2i$. All solutions can be written as $\rho^k(5 + 2i)$ with $k = 0$, 1 or 2. This means that all solutions are given by

$$\begin{aligned} z_k &= 1 + \rho^k(5 + 2i) + \rho^{-k}(5 - 2i) \\ &= 1 + \rho^k(5 + 2i) + \overline{\rho^k(5 + 2i)} \\ &= 1 + 2\mathrm{Re}(\rho^k(5 + 2i)). \end{aligned}$$

For $k = 0$ this gives

$$z_0 = 1 + 2 \cdot 5 = 11.$$

For $k = 1$ this yields

$$z_1 = 1 + 2\mathrm{Re}((-\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3})(5 + 2i)) = 1 + 2\mathrm{Re}(-\tfrac{5}{2} - \sqrt{3}) = -4 - 2\sqrt{3}.$$

For $k = 2$ this yields

$$z_2 = 1 + 2\mathrm{Re}((-\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3})(5 + 2i)) = 1 + 2\mathrm{Re}(-\tfrac{5}{2} + \sqrt{3}) = -4 + 2\sqrt{3}.$$

With this we have determined the factorization of the polynomial $z^3 - 3z^2 - 84z - 44 = 0$:

$$z^3 - 3z^2 - 84z - 44 = (z - 11)(z + 4 - 2\sqrt{3})(z + 4 - 2\sqrt{3}).$$

By expanding parentheses in the right-hand side, we can easily check that the equality is correct.

3.9 Exercises

3.9.1 Exercise. Solve the following linear equations for z:

a. $5z = 2iz + 3$

b. $(1 + i)z = (1 - i)z + i$

c. $z - iz = 1$

3.9.2 Exercise. Solve the following quadratic equations for z:

a. $5z^2 = 2z - 3$

b. $(1 + i)z^2 = (1 - i)z + i$

c. $z^2 - iz^2 = 1$

3.9.3 Exercise. Solve the following equations for z:

a. $z^5 + 1 = 0$

b. $z^3 + 3z^2 + 3z + 1 = 0$

c. $z^6 = 1 + i\sqrt{3}$

d. $z^7 - z^3 = 0$

3.9.4 Exercise. Find all $z \in \mathbb{C}$ with

a. $z\overline{z} = |(1-z)^2|$

b. $z\overline{z} = (1-z)^2$

3.9.5 Exercise. Describe the set of all $z \in \mathbb{C}$ satisfying:

a. $|z - 1 - i| = |z + 1 + i|$

b. $|z^2 - 1| = |z - 1|^2$

c. $|z - 1| \leqslant |z - 2i|$

3.9.6 Exercise. Determine all $z \in \mathbb{C}$ satisfying the following equation:

a. $(4 + 3i)z + (4 - 3i)\overline{z} = 10$

b. $|z - 3i| = |4 + 2i|$

c. $(1 + i)z = (2 - i)\overline{z}$

d. $|z + i| = |z - 1|$

3.9.7 Exercise. The polynomial $p(z) = z^4 - 4z^3 + 14z^2 - 4z + 13$ has the complex number i as a root. Determine all roots of the polynomial.

3.9.8 Exercise. Give a real polynomial $p(z)$ of degree 3, with the property that 1 and $1 + i$ are roots of $p(z)$.

3.9.9 Exercise. Write the following polynomials in z as products of linear factors.

a. $z^3 + 1$

b. $z^4 - 16$

c. $z^3 + -2z^2 + 4z - 8$

d. $z^3 + 3iz^2 - 2z$

3.9.10 Exercise. Suppose $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is a complex polynomial with real coefficients $a_n \neq 0, \ldots, a_0$. Assume that $r \in \mathbb{C}$ is a root of $p(z)$.

a. Prove that $\overline{r}$ is also a root of $p(z)$.

b. Prove, using the Fundamental Theorem of Algebra, that $p(z)$ can be written as a product of degree 1 and degree 2 polynomials with real coefficients.

c. Prove that if n is odd, that $p(z)$ has a real root.

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