2IT80 Discrete Structures

2023-24 Q2

Lecture 4: Orderings

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Relation

A relation between two sets X and Y is a subset $R \subseteq X \times Y$ of the Cartesian product.

Important special case: $X = Y$

We then speak of a relation on X.

This is an arbitrary subset $R \subseteq X \times X$.

For $(x, y) \in R$:

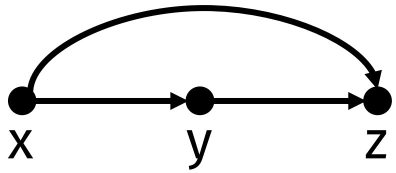
say: x and y are related by R

write: xRy

Properties of relations

A relation R on a set X is

- Reflexive, if xRx for all $x \in X$
- Irreflexive, if there is no $x \in X$ with xRx
- Symmetric, if xRy implies yRx for all $x, y \in X$
- Antisymmetric if, xRy and yRx implies $x = y$
- Transitive, if xRy and yRz implies xRz for all $x, y, z \in X$



Quiz

Are the following relations reflexive, irreflexive, symmetric, anti-symmetric and / or transitive?

On set of students

sRt iff student s graduates before student t

On set of integers

xRy iff $x + y = 10$

Ordering Relations

Motivation

Usual ordering $\leq$ on $\mathbb{N}$ is a relation.

We would like to order other sets as well ...

Example: apartments have a size s and a monthly rental cost r.
Can we order apartments $A_1 = (s_1, r_1)$ and $A_2 = (s_2, r_2)$?
A_1 is better or equal to A_2 if

- it has at least the same size and
- at most the same prize.

What are the crucial properties that make $\leq$ useful?

Ordering relation

An ordering relation on a set X is a reflexive, antisymmetric, transitive relation on X.

A partially ordered set (poset) is a pair (X, R) where X is a set and R is an ordering relation on X.

We often use notation $\leq$ and $\preccurlyeq$.

A relation R is a linear or total ordering if for every two elements x, y we have xRy or yRx.

Reflexive, if xRx for all $x \in X$

Antisymmetric if, xRy and yRx implies $x = y$

Transitive, if xRy and yRz implies xRz for all $x, y, z \in X$

Orderings

Linear orderings (or total orderings):

Smaller or equal on natural numbers: $(\mathbb{N}, \leq)$

Lexicographic order of words

Partial orderings: (to emphasize they are not necessarily linear)

$\Delta = \{\, (x, x) \mid x \in X\}$; Δ is the usual "equal" relation

Divisors: $(\mathbb{N}, |)$, where $a \mid b$ if and only if a is a divisor of b.

Inclusion of subsets of X: $(2^X, \subseteq)$, where $\subseteq$ is the usual "subset-of" relation

Domination orderings: $(\mathbb{R}^3, \preccurlyeq)$ where $(x_1, x_2, x_3) \preccurlyeq (y_1, y_2, y_3)$ if and only if $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \leq y_3$.
The "worse in every respect" relation for multiple criteria

Exercise

Come up with a two orderings (one linear ordering and one that is not linear) on the gives set.

Example: on natural numbers we have seen
“is smaller or equal too” and “divides”

1 – Real numbers

2 – Courses (at the university)

3 – People (all people currently on the planet)

Orderings

Given an ordering $\preccurlyeq$, we can derive the following relations

Strict inequality: $\prec$ as $a \prec b$ if and only if $a \preccurlyeq b$ and $a \neq b$

Reverse ordering: $\succcurlyeq$ as $a \succcurlyeq b$ if and only if $b \preccurlyeq a$

Examples:

$(\mathbb{N}, \leq)$, $(\mathbb{R}, \leq)$ where $\leq$ is the usual ordering.

The corresponding strict ordering is the usual ordering $<$.

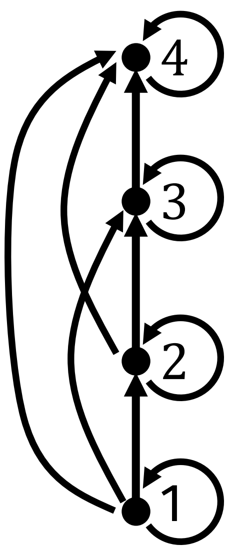
The reverse ordering is $\geq$.

Drawing as a relation

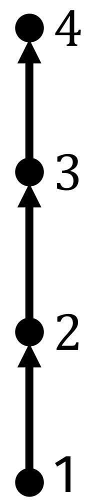
Elements are drawn as points

Arrows indicate relation a → b iff aRb

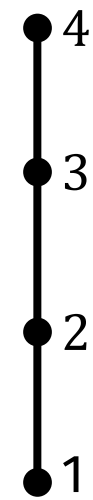
Example ≤ on {1,2,3,4}



Transitive and reflexive arrows are implied



Convention: arrows point up



A lot of excessive arrows if we know it is an ordering ....

Representing orderings

Let $(X, \preccurlyeq)$ be a poset
An element $x \in X$ is an immediate predecessor of element $y \in X$ iff

- $x \prec y$ and
- there is no $t \in X$ with $x \prec t \prec y$.

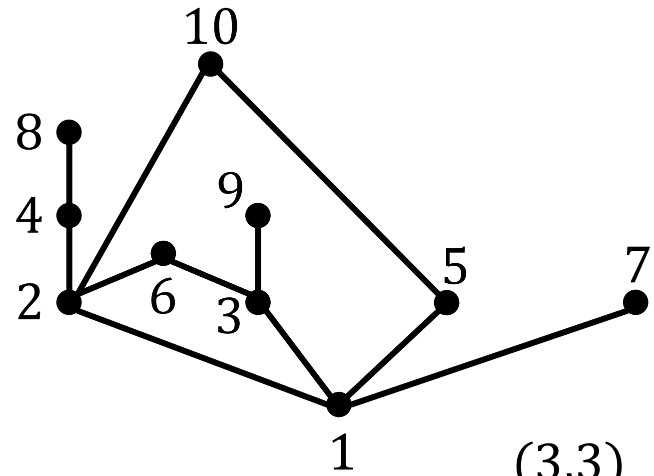
Write $x \lhd y$ if x is the immediate predecessor of y

Theorem: Let $(X, \preccurlyeq)$ be a finite poset and let $\lhd$ be the corresponding "immediate predecessor" relation. Then for any $x, y \in X$, $x \prec y$ holds if and only if there is a chain of elements $x_1, x_2, \ldots, x_k \in X$ such that $x \lhd x_1 \lhd \cdots \lhd x_k \lhd y$ (possibly $k = 0$, i.e., $x \lhd y$).

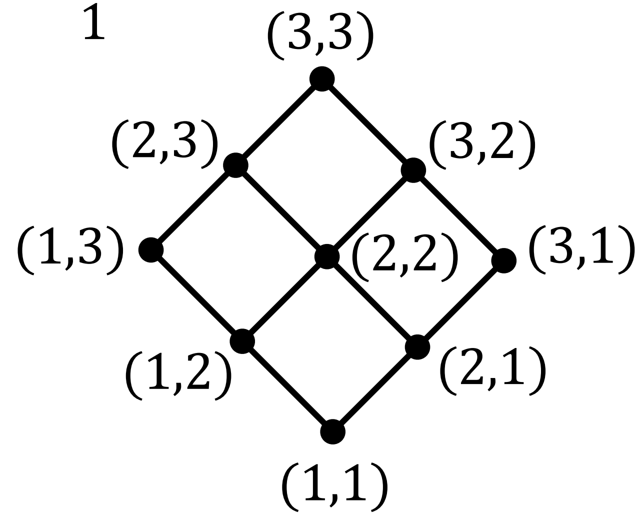
Proof: In book!

Hasse diagrams

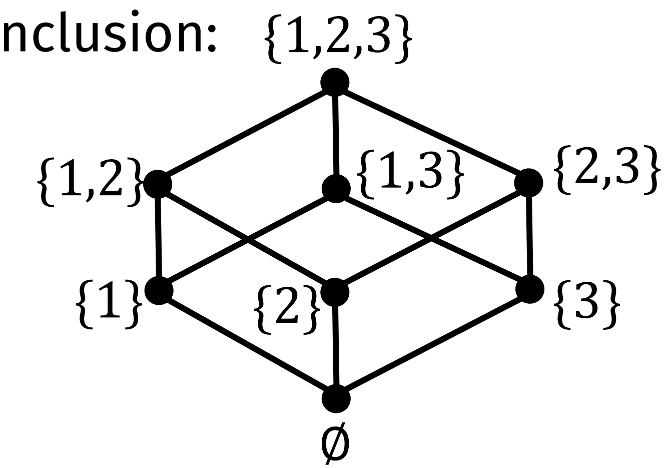
Divisibility relation: $(\{1,2,\ldots,10\},|)$:



Set $\{1,2,3\} \times \{1,2,3\}$ where $(a_1, b_1) \preccurlyeq (a_2, b_2)$
if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$:



Subsets of $\{1,2,3\}$ ordered by inclusion:



Orderings and Linear Orderings

Linear ordering

Every linear ordering is also a (partial) ordering.

The converse statement is false:

divisibility in $\mathbb{N}$

Theorem: Let $(X, \preccurlyeq)$ be a finite partially ordered set. Then there exists a linear ordering $\leq$ of X such that $x \preccurlyeq y$ implies $x \leq y$.

This means: each partial ordering can be extended to a linear ordering: linear extension

Need some more concepts for the proof …

Minimal element

Let $(X, \preccurlyeq)$ be an ordered set. An element $a \in X$ is called a minimal element of $(X, \preccurlyeq)$ if there is no $x \in X$ such that $x \prec a$.

A maximal element a is defined analogously (there is no $x \succ a$).

Theorem: Every finite partially ordered set $(X, \preccurlyeq)$, with $|X| \geq 1$, has at least one minimal element.

Does not hold for infinite sets!

Example: $(\mathbb{Z}, \leq)$, the integers with the usual ordering:
no minimal element.

Smallest element

Let $(X, \preccurlyeq)$ be an ordered set. An element $a \in X$ is called a smallest element of $(X, \preccurlyeq)$ if for every $x \in X$ we have $a \preccurlyeq x$.

A largest element is defined analogously.

Smallest/largest elements are sometimes called minimum/maximum element.

Beware: minimal $\neq$ smallest, maximal $\neq$ largest!

Existence of minimal elements

Theorem: Every finite partially ordered set $(X, \preccurlyeq)$, with $|X| \geq 1$, has at least one minimal element.

Intuition: Choose $x_0 \in X$ arbitrarily.

If x_0 is minimal, we are done.

Otherwise, there exists some $x_1 \prec x_0$.

If x_1 is minimal, we are done.

Otherwise, there exists some $x_2 \prec x_1$

...

...

...

After finitely many steps we arrive at a minimal element.

Otherwise X would have infinitely many elements $x_0, x_1, x_2, \dots$.

Alternative proof

Theorem: Every finite partially ordered set $(\mathrm{X}, \preccurlyeq)$, with $|X| \geq 1$, has at least one minimal element.

Proof:

Choose $x \in X$ such that $L_x = \{\, y \colon y \preccurlyeq x\}$ is the smallest over all elements in X.

We prove that $|L_x| \leq 1$ by contradiction.

Assume $|L_x| > 1$.

Then $y \in L_x$ with $x \neq y$ exists and $\left|L_y\right| < |L_x|$ since $L_y \subset L_x$

This contradicts the choice of x, so $|L_x| \leq 1$

Also $|L_x| \geq 1$, since $x \in |L_x|$

So x is a minimal element.

Linear extensions

Theorem: Let $(X, \preccurlyeq)$ be a finite partially ordered set. Then there exists a linear ordering $\leq$ of X such that $x \preccurlyeq y$ implies $x \leq y$.

Proof: Proof by induction on $n = |X|$.

Base case: $n = 0$: The empty ordering is linear.

Linear extensions

Theorem: Let $(X, \preccurlyeq)$ be a finite partially ordered set. Then there exists a linear ordering $\leq$ of X such that $x \preccurlyeq y$ implies $x \leq y$.

Proof: Inductive step: Let $k \geq 0$

IH: For any set $(X', \preccurlyeq')$ with $0 \leq |X'| \leq k$ there exists a linear ordering $(X', \leq')$ such that $x \preccurlyeq' y$ implies $x \leq' y$ for all $x, y \in X'$.

Consider an partially ordered set $(X, \preccurlyeq)$ with $|X| = k + 1$.

Let $x_0 \in X$ be a minimal element in $(X, \preccurlyeq)$

Set $X' = X \setminus \{x_0\}$, and let $\preccurlyeq'$ be the relation $\preccurlyeq$ restricted to the set X'.

$(X', \preccurlyeq')$ is an ordered set with $|X'| = k$

By IH linear ordering $(X', \leq')$ exists such that $x \preccurlyeq' y$ implies $x \leq' y$

We define relation $\leq$ on X as follows:

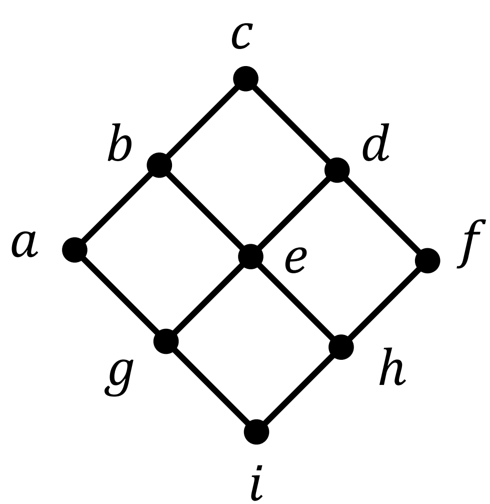
$$
\begin{array}{lcl}
x_0 \leq y & \text{for each} & y \in X; \\
x \leq y & \text{whenever} & x \leq' y.
\end{array}
$$

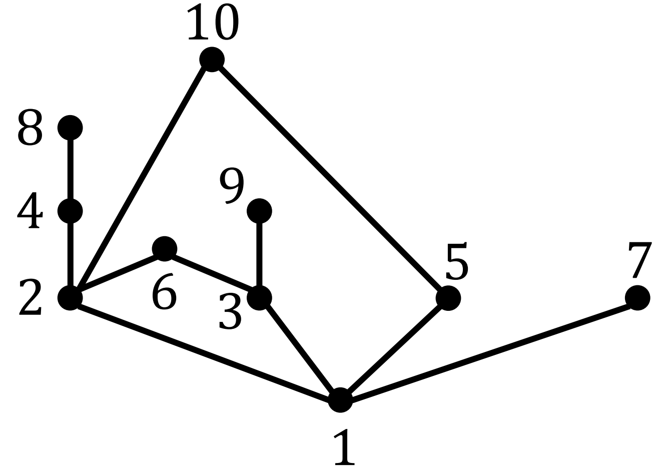
To prove: $x \preccurlyeq y$ implies $x \leq y$.

To prove: $\leq$ is a linear ordering. ■

Lets make some linear extensions

Given the ordering defined by the Hasse diagram below. Give a linear extension of the ordering.

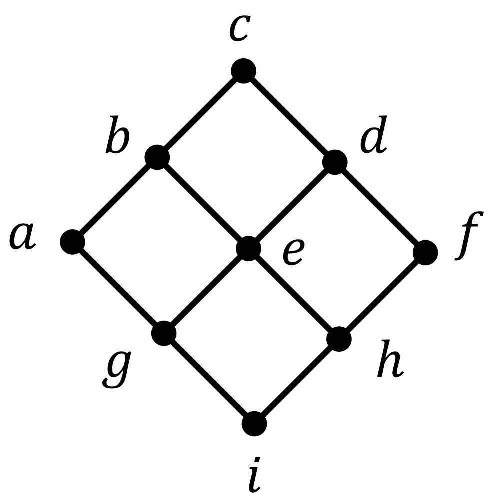


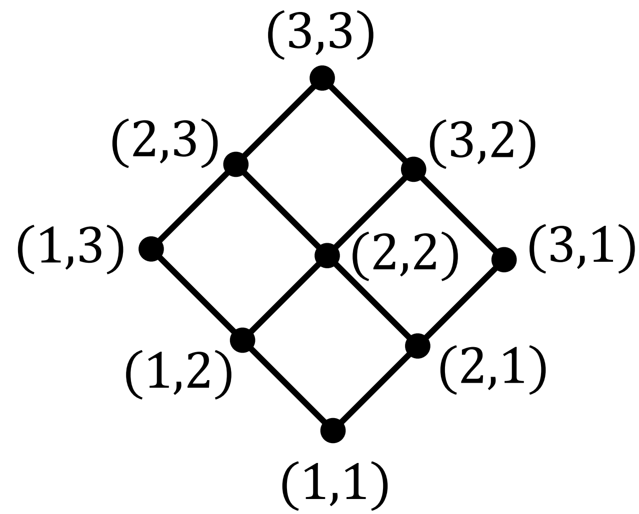


Ordering by Inclusion

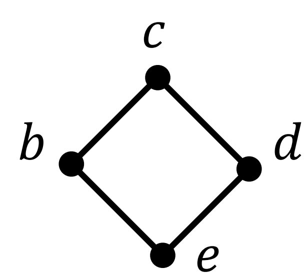
Relating orderings

Some orderings appear to be similar.



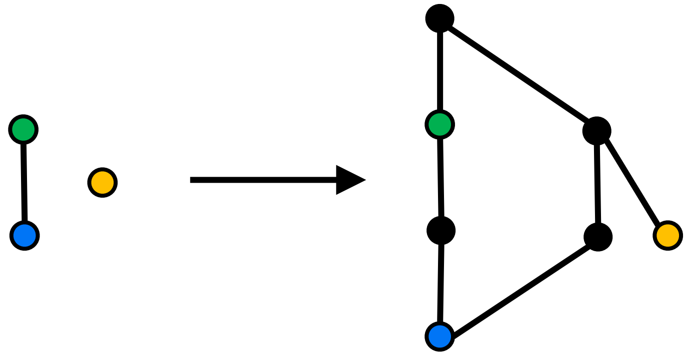


Or one is an extended version of another.



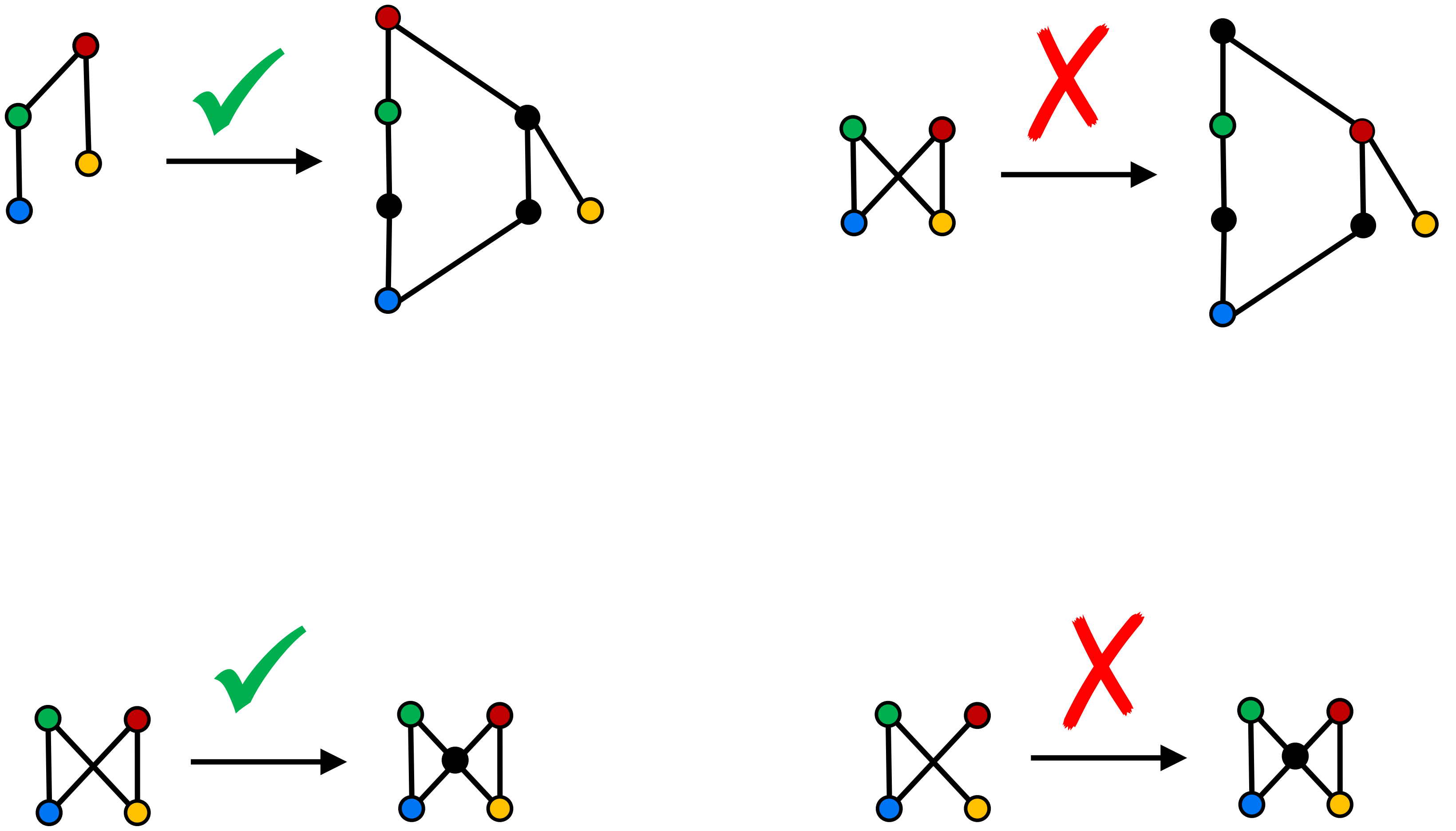
Embedding

Let $(X, \preccurlyeq)$ and $(X', \preccurlyeq')$ be ordered sets. A mapping $f : X \to X'$ is called an embedding of $(X, \preccurlyeq)$ into $(X', \preccurlyeq')$ if the following conditions hold:
(i) f is injective;
(ii) $f(x) \preccurlyeq' f(y)$ if and only if $x \preccurlyeq y$.



If an embedding is surjective, then it is an isomorphism.

Embeddings



This is how far we got in the lecture. I have left the remaining slides for reference, but the book has the same information as the last two topics, so I suggest reading that.

Ordering by Inclusion

Theorem: For every ordered set $(X, \preccurlyeq)$ there exists an embedding into the ordered set $(2^X, \subseteq)$.

Proof (sketch): Define $f : X \to 2^X$ by $f(x) = \{a \in X : a \preccurlyeq x\}$.

Remains to check: f is an embedding.

1. f is injective: Assume $f(x) = f(y)$.

   $x \in f(x) = f(y) \Rightarrow x \preccurlyeq y$

   $y \in f(y) = f(x) \Rightarrow y \preccurlyeq x$

   $\Rightarrow x = y$ (anti-symmetry)

2. Show: if $x \preccurlyeq y$, then $f(x) \subseteq f(y)$.

   Let $x \preccurlyeq y$.

   If $z \in f(x)$, then $z \preccurlyeq x$. By transitivity $z \preccurlyeq y$.
   But then $z \in f(y)$.

3. Show: if $f(x) \subseteq f(y)$, then $x \preccurlyeq y$.

   If $f(x) \subseteq f(y)$, then $x \in f(y)$. Hence $x \preccurlyeq y$. ∎

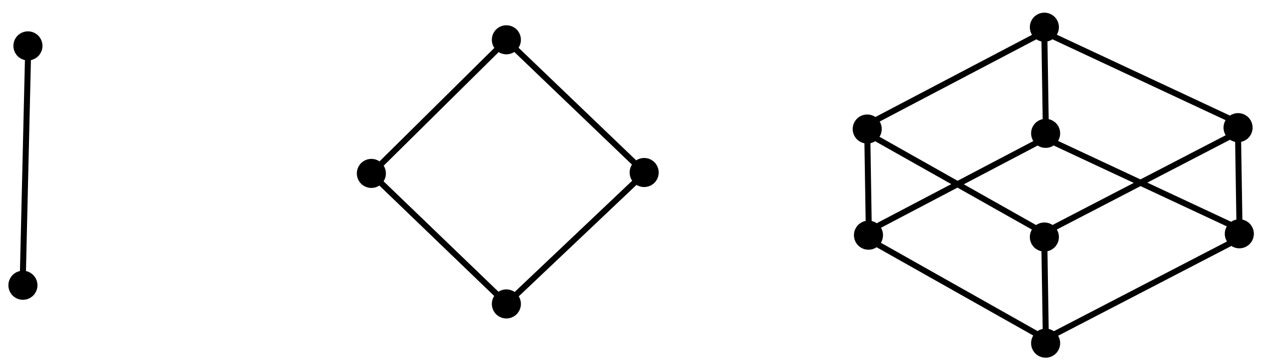
Remarks

"Order by inclusion" holds for infinite sets as well.

The ordered sets $(2^X, \subseteq)$ are universal:

they contain a copy of every ordered set

For $X = \{1,2,\ldots,n\}$ the set $(2^X, \subseteq)$ is often denoted by $\mathcal{B}_n$.



$\mathcal{B}_1$ $\mathcal{B}_2$ $\mathcal{B}_3$

Boolean lattice, n-dimensional cube ...

Large Implies Tall or Wide

The Erdős-Szekeres theorem

Theorem: An arbitrary sequence $(x_1, \ldots, x_{n^2+1})$ of real numbers contains a monotone subsequence of length $n+1$.

Subsequence is determined by indices $i_1, i_2, \ldots, i_m$, $i_1 < i_2 < \cdots < i_m$. It has the form $(x_{i_1}, x_{i_2}, \ldots, x_{i_m})$.

A subsequence is monotone if either

$$x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_m}, \text{ or}$$
$$x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_m}.$$

(3,5,6,2,8,1,4,7)

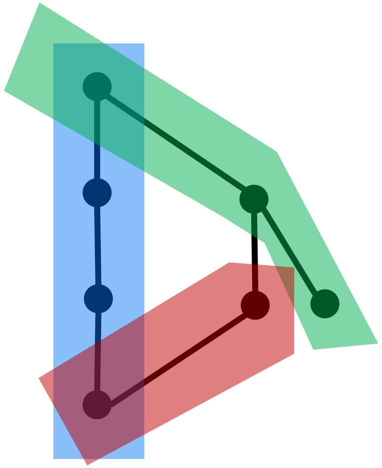
(with $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 5$)

Comparability and chains

Let $P = (X, \preccurlyeq)$ be a poset.

Elements x, y are called
comparable if either $x \preccurlyeq y$ or $y \preccurlyeq x$
Incomparable if neither $x \preccurlyeq y$ nor $y \preccurlyeq x$

A set $A \subseteq X$ is called a chain in P if every two of its elements are comparable (in P).

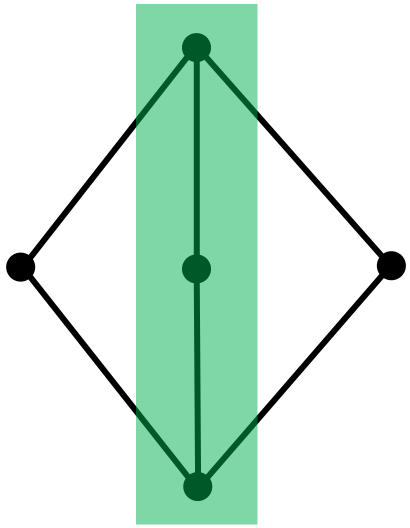


Maximum chains

We use

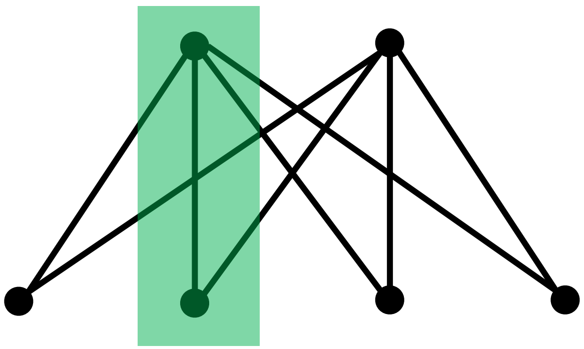
$$\omega(P) = \max\{|A| : A \text{ chain in } P\}$$

to denote the maximum size of a chain in P.



P_1

$\omega(P_1) = 3$



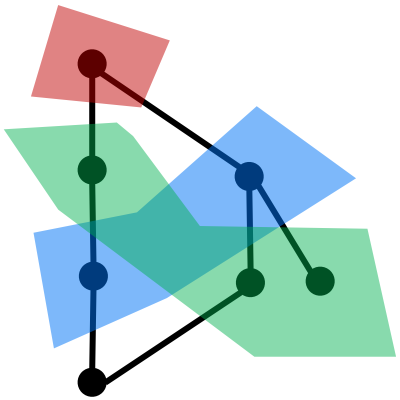
P_2

$\omega(P_2) = 2$

Independence

A set $A \subseteq X$ is called independent in P if any two of its elements are incomparable.

Independent sets are also called antichains



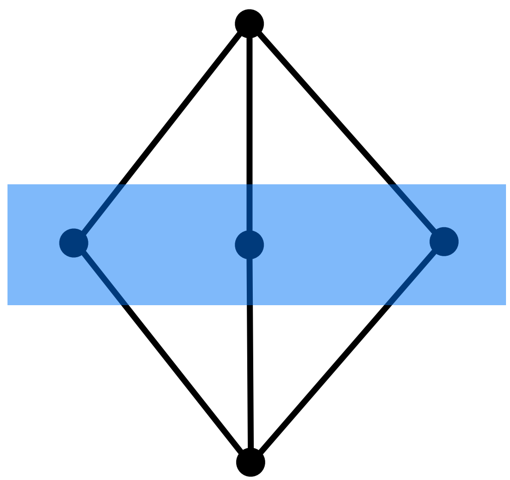
Observation: The set of all minimal elements in P is independent.

Maximum independent sets

We use

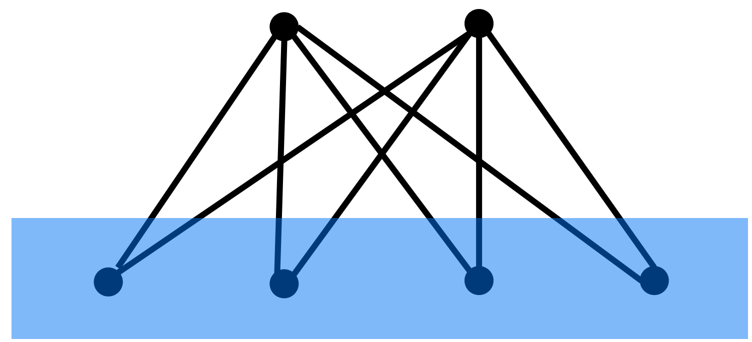
$$\alpha(P) = \max\{|A| : A \text{ independent in } P\}$$

to denote the maximum size of an independent set in P



P_1

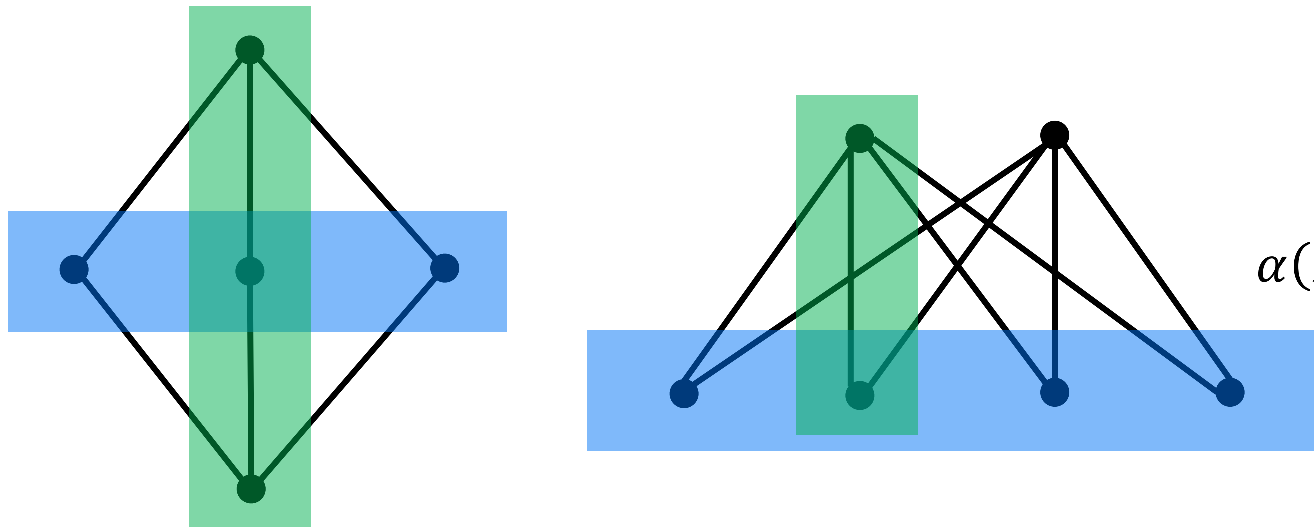
$\alpha(P_1) = 3$



P_2

$\alpha(P_2) = 4$

Maximum chains and antichains



$\alpha(P_1) = 3$ $\alpha(P_2) = 4$

P_1 P_2

$\omega(P_1) = 3$ $\omega(P_2) = 2$

Intuition: $\omega(P)$ measures "height"
$\alpha(P)$ measures "width"

Claim: A large poset cannot have low height and low width, i.e.,
$\alpha(P) \cdot \omega(P) \geq |X|$.

Either tall or wide …

Theorem: For every finite ordered set $P = (X, \preccurlyeq)$ we have

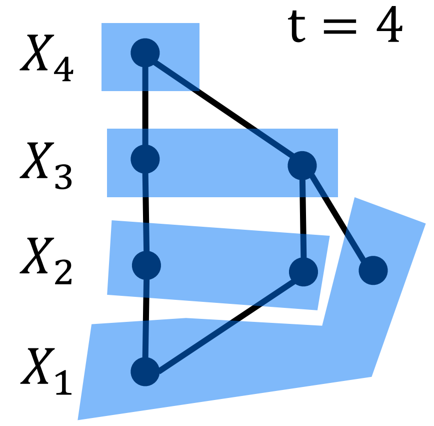
$$\alpha(P) \cdot \omega(P) \geq |X|.$$

Proof: Define sets $X_1, X_2, \ldots, X_t$ inductively:

X_1: all minimal elements of P

Inductive step: $X_1, \ldots, X_\ell$ already defined:

- $X'_\ell = X \setminus \bigcup_{i=1}^{\ell} X_i$
- if $X'_\ell = \emptyset$, then put $t = \ell$, construction finished
- Otherwise: let $\preccurlyeq'$ be $\preccurlyeq$ restricted to X'_ℓ
- $X_{\ell+1}$: all minimal elements of $(X'_\ell, \preccurlyeq')$



$X_{\ell+1} \neq \emptyset$ since X_ℓ, is finite

Claims: (1) $X_1, \ldots, X_t$ form a partition of X

(2) Each X_i is an independent set in P

(3) $\omega(P) \geq t$

Either tall or wide ...

Claims: (1) $X_1, \ldots, X_t$ form a partition of X

(2) Each X_i is an independent set in P

(3) $\omega(P) \geq t$

Together these claims finish the proof:

$$(1) \;\Rightarrow\; |X| = \sum_{i=1}^{t} |X_i| \overset{(2)}{\leq} \sum_{i=1}^{t} \alpha(P) = t \cdot \alpha(P) \overset{(3)}{\leq} \omega(P) \cdot \alpha(P)$$

(1), (2) follow by construction of $X_1, X_2, \ldots, X_t$ and the observation that minimal elements are independent

Remains to prove claim (3).

Either tall or wide ...

Claim: $\omega(P) \geq t$

Idea: Inductively construct a chain of length t to prove the claim

Choose $x_t \in X_t$ arbitrarily
$x_t \notin X_{t-1} \Rightarrow$ there exists $x_{t-1} \in X_{t-1}$ so that $x_{t-1} \prec x_t$.
Repeat this argument:

- Have constructed $x_t \in X_t, x_{t-1} \in X_{t-1}, \ldots, x_{k+1} \in X_{k+1}$
- Then $x_{k+1} \notin X_k \Rightarrow$ there exists $x_k \in X_k$ with $x_k \prec x_{k+1}$

The set $\{x_1, \ldots, x_t\}$ constructed this way is a chain.
Therefore $\omega(P) \geq t$. ■

The Erdős-Szekeres theorem

Theorem: An arbitrary sequence $(x_1, \dots, x_{n^2+1})$ of real numbers contains a monotone subsequence of length $n+1$.

Subsequence is determined by indices $i_1, i_2, \dots, i_m$, $i_1 < i_2 < \cdots < i_m$. It has the form $(x_{i_1}, x_{i_2}, \dots, x_{i_m})$.

A subsequence is monotone if either

$$x_{i_1} \le x_{i_2} \le \cdots \le x_{i_m}, \text{ or}$$
$$x_{i_1} \ge x_{i_2} \ge \cdots \ge x_{i_m}.$$

(3,5,6,2,8,1,4,7)

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(with $i_1 = 1, i_2 = 2, i_3 = 3, i_4 = 5$)

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$$(3,5,6,2,8,1,4,7)$$
$$(\text{with } i_1 = 3, i_2 = 4, i_3 = 6)$$

The Erdős-Szekeres theorem

Theorem: An arbitrary sequence $(x_1, \dots, x_{n^2+1})$ of real numbers contains a monotone subsequence of length $n+1$.

Proof: Let a sequence $(x_1, \dots, x_{n^2+1})$ be given.

Let $X = \{1, 2, \dots, n^2+1\}$. Define a relation $\preccurlyeq$ on X by
$i \preccurlyeq j$ if and only if both $i \leq j$ and $x_i \leq x_j$.

$\preccurlyeq$ is a partial ordering of X

It is $\alpha(X, \preccurlyeq) \cdot \omega(X, \preccurlyeq) \geq n^2 + 1$; therefore $\alpha(X, \preccurlyeq) > n$ or $\omega(X, \preccurlyeq) > n$.

Chain $i_1 \prec i_2 \prec \cdots \prec i_m$ in $\preccurlyeq$ corresponds to non-decreasing subsequence $x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_m}$ (note $i_1 < i_2 < \cdots < i_m$)

Independent set $\{i_1, i_2, \dots, i_m\}$ corresponds to decreasing subsequence: Choose numbering so that $i_1 < i_2 < \cdots < i_m$, then $x_{i_1} > x_{i_2} > \cdots > x_{i_m}$.

by contradiction: $x_{i_1} \leq x_{i_2}$and $i_1 < i_2$ would mean $i_1 \prec i_2$. ■

Organizational

- Practice set:
  - Exercises 2,3 for Discussion group (you can decide differently with your group)

- In-class test A1
  - Do the SEB test
  - Be on time
  - Any questions can go in slack