Linear Algebra 2

Understanding linear maps

$$A = S \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} S^{-1}$$

Kathrin Hövelmanns

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Faculty of Mathematics and Computer Science

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Preface

Linear Algebra 1 introduced you to vector spaces and maps that respect (or 'play nicely with') their linear structures (hence called linear maps). LinA 2 takes a deeper dive concerning linear maps: We will discuss how to describe and deal with linear maps by representing them via matrices, and how to determine the nature of a linear map by analysing its matrix representation. (For instance, how to recognize that a matrix represents a reflection.)

This involves techniques to

- switch between different vector bases; and to
- find a particularly useful basis (called eigenvector basis) by analysing characteristic properties of the matrix (called eigenvalues).

Eigenvectors allow to describe linear maps in a simpler and more economical way. We will use this to study interesting special classes of maps like orthogonal maps (maps that 'preserve' shapes, like reflections and rotations) and symmetric maps (which can be used to analyse quadratic curves and surfaces like ellipses and hyperboloids).

How does this relate to other courses, and why do we care about linear maps? In general, eigenvectors/-values characterise transformations and hence play an important role in all areas that apply linear algebra (from geology to quantum mechanics). Here are some application examples:

- ***Numerical methods.*** In many numerical methods, an important first step is to bring a matrix into a simpler form (like diagonal or triangular). This often uses the techniques developed in this course. You might encounter this in *Introduction to Numerical Analysis* or *Numerical Linear Algebra*.
- ***Analysing quadratic surfaces.*** Imagine you are handed a convoluted quadratic equation of which you are told that it describes some geometric object (like a curve or a surface), and your goal is to determine the object's geometric properties. In Section 3.2, we will use the techniques developed in Chapter 2 to end up with a nice description of the object from which the most important geometric properties can be easily extracted.
- ***Evolutions of physical systems - differential equations.*** The concept of eigenvalues was devised in the context of dealing with systems of linear differential equations. It hence comes as no surprise that *Theory and Application of Ordinary Differential Equations* draws heavily on the techniques and language developed in this course.

- ***Quantum mechanics.*** In quantum mechanics, linear maps play a mayor role: physical quantities (like velocity and momentum) are described by linear maps in vector spaces of functions. In this context, eigenvalues/vectors have a special physical interpretation.
- ***Analysis 2, tensor calculus and differential geometry.*** In Analysis 2, linear maps appear as first–order approximations of differentiable multi-variable functions: Say $f : \mathbb{R}^n \to \mathbb{R}^n$ is such a function and $\underline{p} \in \mathbb{R}^n$ is a vector. Then the matrix of this first–order approximation in $\underline{p}$ is the $n \times n$ matrix whose (i, j)-th element is $\partial f_i / \partial x_j$. This is also useful to study curved spaces.
- ***A more general view - algebra.*** Vector spaces are only one example for sets with an additional mathematical structure, and there are many more examples. (E.g., fields, rings, vector-space-like structures over rings, ...) Studying maps that play nicely with the given structure will prove very useful also in these other examples. This theme will play a huge role in your courses on Algebra.

Changelog for students who are revisiting this course: These lecture are based on the ones by Hans Sterk who taught this course before I took over, and before the Bachelor curriculum's redesign. The main changes are that I

- assume the basic knowledge about linear maps that was moved to *Linear Algebra I*;
- put more emphasis on a certain matrix simplification technique (called Jordan Normal Form);
- do not cover dual spaces (these were previously covered in the lecture notes, but to my knowledge not in the lecture itself); and
- do not cover certain matrix decompositions (LU-decomposition, Singular Value Decomposition), since these will be covered in Numerical courses.

For students that are curious about the removed/de-emphasised parts, Hans' previous lecture notes are provided on Canvas.

As I will try to improve my lecture notes with each year,

any constructive feedback is more than welcome!

Contents

Chapter 1

Setting the stage: fundamental definitions/concepts for linear maps

What happens in Chapter 1?

⚠ ***This course assumes familiarity with the knowledge on linear maps that was introduced in Linear Algebra I.***

To jog your memory (and simplify referencing), this chapter starts by summarising the most important concepts related to linear maps in Section 1.1. Something that is very important (and useful!) is the close connection between matrices and linear maps, recapped in Section 1.2.

If you're very comfortable with the material covered in *Linear Algebra 1*, you can skip sections 1.1 and 1.2 ***except for the following two parts***:

- a new example (example 1.1.1) that mixes linear maps with quotient spaces; and
- a new remark (remark 1.2.4) that translates our result on particular solutions for vector equations (Theorem 1.1.14) to the setting where we deal with systems of linear equations.

After these recaps, we begin tackling new material in Section 1.3: we dig deeper into the connection between matrices and linear maps.

Learning Goals of Chapter 1: You can

- explain and work with the concept of a linear map between vectors spaces
  - for concrete vector spaces like $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{K}^n$ as well as
  - for vector spaces that are handled as abstract objects;
- reason about the relation between linear maps and matrices;
- set up matrix representations of linear maps;
- analyse how the coordinates of a vector change when switching to another basis;

- perform basis transformations to switch between representations for different bases and compute matrices that represent such a basis transition.

Why do we care? The techniques developed in this chapter lay the groundwork for the rest of this course: In Chapter 2, our main goal will be to develop tricks to transform matrices into a 'nicer' (simpler) shape. On a high level, the main trick will be to find a certain basis for which the matrix transforms into something nice. This will use the observations we made in this section.

1.1. Recap: Linear maps (as seen in *Linear Algebra 1*), plus another nice example

In this recap section, we recall the most fundamental concept of this course, *linear maps*. Besides recapping the notion itself, this section covers:

- that linear maps can be composed, added, multiplied (with each other and with scalars), and sometimes also inverted;
- the null space $\mathcal{N}$ and the range $\mathcal{R}$ of a linear map;
- how $\mathcal{N}$ and $\mathcal{R}$ relate to injectivity and surjectivity;
- how to specify a linear map on a given basis; and
- (very important) the dimension theorem.

This section requires familiarity with the following concepts which you can look up in the appendices when needed: fields (see Definition B.1.1), vector spaces over fields (see Definition B.2.1), bases (see Definition B.2.5), inner product spaces (see Definition B.2.7), matrix multiplication (see Definition B.3.3), and basic notions concerning maps like injectivity/surjectivity (see the table in Appendix A.2).

Let V and W be $\mathbb{K}$-vector spaces for some field $\mathbb{K}$. (E.g., think of real vector spaces, meaning $\mathbb{K} = \mathbb{R}$, or complex vector spaces, meaning $\mathbb{K} = \mathbb{C}$.) A map $\mathcal{A} : V \to W$ associates to *each* vector $\underline{v}$ in V exactly one vector $\mathcal{A}(\underline{v})$ (or short: $\mathcal{A}\underline{v}$) in W.

Definition 1.1.1 (Linear map). *Let V and W be two $\mathbb{K}$-vector spaces. A map $\mathcal{A} : V \to W$ is called* **$\mathbb{K}$-linear** *(or simply* **linear***) if for all vectors $\underline{x}, \underline{y} \in V$ and all field elements λ ('scalars') the following holds:*

i) $\mathcal{A}(\underline{0}) = \underline{0}$ — *0 is mapped to 0*

ii) $\mathcal{A}(\underline{x} + \underline{y}) = \mathcal{A}\underline{x} + \mathcal{A}\underline{y}$ — *addition works linearly*

iii) $\mathcal{A}(\lambda \underline{x}) = \lambda \mathcal{A}\underline{x}$ — *scalar multiplication works linearly*

A bijective linear map is also called an **isomorphism**.

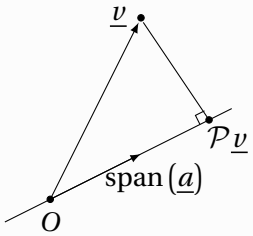
Equivalently, linearity can be defined through the following single requirement: $\mathcal{A}$ is linear if for all $\underline{x}, \underline{y} \in V$ and all scalars $\alpha, \beta \in \mathbb{K}$, one has

$$\mathcal{A}(\alpha \underline{x} + \beta \underline{y}) = \alpha \mathcal{A}\underline{x} + \beta \mathcal{A}\underline{y} \,.$$

This means that for linear maps, the image of a linear combination of two vectors is the same as the linear combination of their image vectors. In other words, ***we can switch the order of linearly combining and applying*** $\mathcal{A}$. (We will use this in this course very, very often.)

Example 1.1.2 ***(Orthogonal projections)*** Let V be a real inner product space, and let $l = \text{span}(\underline{a})$ be a line through the origin in V. The map that associates to each vector in V the orthogonal projection on l we will call $\mathcal{P}$. If $\underline{a}$ has length one, then $\mathcal{P}$ is given by the formula:

$$\mathcal{P}\underline{v} = (\underline{v}, \underline{a})\underline{a} \quad .$$



More generally, we saw that ***orthogonal projections on general linear subspaces of a real inner product space are linear***.

Example 1.1.3 ***(Trivial examples)*** For every vector space V, we have the so-called ***identity map*** (or just *identity*) $\mathcal{I} : V \to V$ given by $\mathcal{I}\underline{v} := \underline{v}$. It is very straightforward to verify that $\mathcal{I}$ is linear.
When V and W are two vector spaces that are either both real or both complex, then the so-called ***zero map*** $\mathcal{O} : V \to W$ given by $\mathcal{O}\underline{v} := \underline{0}$ is also linear.

Example 1.1.4 ***(Multiplication with a matrix)*** Let A be a real $m \times n$-matrix with entries in $\mathbb{R}$. In this example, we treat elements from $\mathbb{R}^n$ and $\mathbb{R}^m$ as columns. We defined the map $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by

$$L_A(\underline{v}) := A\underline{v} \ ,$$

and saw that L_A is linear.
Note that the argument you saw in LinA I would have worked for any other base field $\mathbb{K}$ just the same, meaning if the matrix A has entries in an arbitrary field $\mathbb{K}$ instead of $\mathbb{R}$, then the map

$$\mathbb{K}^n \xrightarrow{\underline{v} \mapsto A\underline{v}} \mathbb{K}^m \ ,$$

is also linear.

Example 1.1.5 ***(Differentiation is a linear map.)*** The derivative D is defined by $f' = Df$. We saw that D behaves in a linear manner.
⚠ Remember, however, that we always need to be specific concerning the question on which vector space D is acting.

Now that we've recalled some examples, we recall another very useful observation: we learnt that

switching the order of linearly combining and using $\mathcal{A}$ does work for any number of vectors:

Theorem 1.1.6. *$\mathcal{A} : V \to W$ is linear if and only if*

$$\mathcal{A}\left(\sum_{i=1}^{n} \alpha_i \underline{v}_i\right) = \sum_{i=1}^{n} \alpha_i \mathcal{A}\underline{v} \tag{1.1}$$

for all vectors $\underline{v}_1, \ldots, \underline{v}_n$ in V and all scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$.

We will also recall that compositions, inverses, sums and scalar multiples of linear maps are again linear maps.

Definition/theorem 1.1.7 (Composition or product). *Let $\mathcal{A} : V \to W$ and $\mathcal{B} : U \to V$ be linear maps. Take a vector $\underline{u} \in U$. First applying $\mathcal{B}$ to $\underline{u} \in U$, we get $\mathcal{B}\underline{u} \in V$. Now applying $\mathcal{A}$ to $\mathcal{B}\underline{u}$, we get $\mathcal{A}(\mathcal{B})\underline{u} \in W$. In a diagram:*

$$\begin{array}{ccccc} U & \xrightarrow{\mathcal{B}} & V & \xrightarrow{\mathcal{A}} & W \\ \underline{u} & \longrightarrow & \mathcal{B}\underline{u} & \longrightarrow & \mathcal{A}(\mathcal{B}\underline{u}) \end{array} \tag{1.2}$$

The **composition** *$\mathcal{A}\,\mathcal{B} : U \to W$ is defined by*

$$(\mathcal{A}\mathcal{B})\underline{u} := \mathcal{A}(\mathcal{B}\underline{u}) \text{ for all } \underline{u} \in U\,.$$

It is itself linear.

⚠ **Cave:** The notation $\mathcal{A}\mathcal{B}$ suggests that the composition can be understood as a 'product' of $\mathcal{A}$ and $\mathcal{B}$. Remember, however, that ***this is not a product in the sense as we know it, e.g., from fields***: Say U, the 'target space' of $\mathcal{B}$, also is the 'starting space' of $\mathcal{A}$, meaning we can indeed compose $\mathcal{A}$ and $\mathcal{B}$ into $\mathcal{A}\mathcal{B}$. This does not mean that $\mathcal{B}\mathcal{A}$ exists - we have no guarantee that the 'target space' of $\mathcal{A}$ also is the 'starting space' of $\mathcal{B}$. And even if $\mathcal{B}\mathcal{A}$ also exists, it is not necessarily equal to $\mathcal{A}\mathcal{B}$.

Definition/theorem 1.1.8 (Inverse map). *If the linear map $\mathcal{A} : V \to W$ is a bijection, then the* **inverse map**

$$\mathcal{A}^{-1} : W \xrightarrow{w \mapsto v \text{ s. th. } Av = w} V$$

is also linear.

For maps that map from some vector space into itself (endomorphisms), we can define their powers:

Definition 1.1.9 (Powers of endomorphisms). *For $\mathcal{A} : V \to V$, we define $\mathcal{A}^2 := \mathcal{A}\mathcal{A}$. More generally, for $n = 2, 3, \ldots$, we define $\mathcal{A}^n := \mathcal{A}^{n-1}\mathcal{A}$. We take for $\mathcal{A}^0$ the identity map $\mathcal{I}$. If $\mathcal{A}$ is invertible (so if the map has an inverse $\mathcal{A}^{-1}$), then we define for positive integers n the map $\mathcal{A}^{-n}$ as the composite map $\left(\mathcal{A}^{-1}\right)^n$.*

Definition/theorem 1.1.10 (Sum, scalar multiple). *Let $\mathcal{A}: V \to W$ and $\mathcal{B}: V \to W$ be two linear maps. Then the* **sum** *$\mathcal{A}+\mathcal{B}: V \to W$ is defined by*

$$(\mathcal{A}+\mathcal{B})\underline{v} := \mathcal{A}\underline{v} + \mathcal{B}\underline{v} \,.$$

If α is a scalar, then the **scalar multiple** *$\alpha\mathcal{A}: V \to W$ is defined by*

$$(\alpha\mathcal{A})\underline{v} := \alpha(\mathcal{A}\underline{v}) \,.$$

Both constructions are linear.

Since linear maps are also ordinary maps, we can talk about the image of a vector or a subset of the vector space (notation: $\mathcal{A}(D)$ if $\mathcal{A}$ is the linear map and D the subset), and about the (complete) inverse image of a subset.

We recall two very important linear subspaces that are associated to linear maps: the first is a generalization of the solution space of a homogeneous system of linear equations, the second is a generalization of the column space of a matrix.

Definition/theorem 1.1.11 (Null space and range). *Let $\mathcal{A}: V \to W$ be a linear map. We define $\mathcal{N}$, the* **null space** *(or* **kernel***) of $\mathcal{A}$, by*

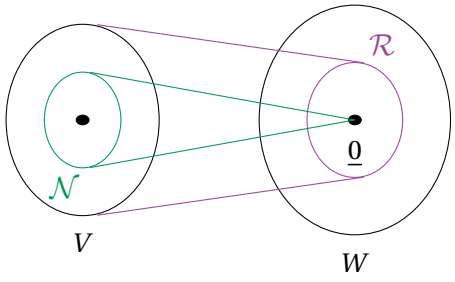
$$\mathcal{N}(\mathcal{A}) := \{\underline{v} \in V \mid \mathcal{A}\underline{v} = \underline{0}\} \quad ,$$

and $\mathcal{R}$, the **range** *of $\mathcal{A}$, by*

$$\mathcal{R}(\mathcal{A}) := \{\mathcal{A}\underline{v} \in W \mid \underline{v} \in V\} \,.$$

The range can also be denoted by $\mathcal{A}(V)$. If the context is clear, we will simply write $\mathcal{N}/\mathcal{R}$ instead of $\mathcal{N}(\mathcal{A})/\mathcal{R}(\mathcal{A})$.

$\mathcal{N}$ is a linear subspace of V and $\mathcal{R}$ is a linear subspace of W .



Notice that the null space is precisely the inverse image $\mathcal{A}^{-1}\{\underline{0}\}$ of the origin under $\mathcal{A}$.

Example 1.1.12 ***(Multiplication with a matrix)*** We first revisit example 1.1.4: The null space of the linear map L_A consists of the vectors $\underline{v}$ that satisfy $L_A(\underline{v}) = \underline{0}$, so all solutions of the homogeneous system $A\underline{v} = \underline{0}$.
The range of L_A consists of all vectors of the form $L_A(\underline{v})$. If the columns of A are $\underline{a}_1, \ldots, \underline{a}_n$,

then this is the set

$$\{x_1\underline{a}_1 + \cdots + x_n\underline{a}_n \mid x_1, \ldots, x_n \text{ arbitrary}\} \quad ,$$

which is the column space of A.
So ***null space and range generalise two notions from the matrix world***.

Example 1.1.13 ***(Orthogonal projections)*** We also revisit example 1.1.2, the orthogonal projection $\mathcal{P}$ on a line $\ell = \text{span}(\underline{a})$: Geometrically, we see that the range $\mathcal{R}(\mathcal{P})$ (so the collection of vectors that occur as an image of $\mathcal{P}$) is ℓ itself. The null space $\mathcal{N}(\mathcal{P})$ consists of all vectors that are mapped to $\underline{0}$, so all vectors in $\ell^\perp$, the orthogonal complement.

We now recall how injectivity, surjectivity and the inverse image are connected to the null space and the range:

Theorem 1.1.14. *Consider a linear map $\mathcal{A} : V \to W$.*

1. *$\mathcal{N} = \{\underline{0}\} \Leftrightarrow \mathcal{A}$ is injective.*
2. *$\mathcal{R} = W \Leftrightarrow \mathcal{A}$ is surjective.*
3. *Let $\underline{b} \in \mathcal{R}$. Then there is a vector $\underline{p}$ satisfying $\mathcal{A}\underline{p} = \underline{b}$; we say that $\underline{p}$ is a* **particular solution** *of the vector equation $\mathcal{A}\underline{x} = \underline{b}$. Together all solutions of the vector equation $\mathcal{A}\underline{x} = \underline{b}$ are given by the set*

$$\underline{p} + \mathcal{N} := \{\underline{p} + \underline{w} \mid \underline{w} \in \mathcal{N}\} \quad ,$$

the **coset**. *In particular, the equation $\mathcal{A}\underline{x} = \underline{b}$ has exactly one solution if $\mathcal{N} = \{\underline{0}\}$.*

In particular, ***Theorem 1.1.14 tells us how to we get all solutions of a vector equation $\mathcal{A}\underline{x} = \underline{b}$:*** find a single particular solution, and then add to that all solutions of the corresponding homogeneous equation $\mathcal{A}\underline{x} = \underline{0}$.

If we know how $\mathcal{A}$ acts on a basis (meaning we know the images for all basis vectors), ***it is possible to determine the image of any vector simply from the basis vector images***, due to the linearity of $\mathcal{A}$,

We also learnt that ***any basis-images combination yields a unique linear map***:

Theorem 1.1.15. *Let V and W be vector spaces, let $\{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for V, and let $\underline{w}_1, \ldots, \underline{w}_n$ be an n–tuple of vectors in W. Then there exists a unique linear map $\mathcal{A} : V \to W$ satisfying $\mathcal{A}\underline{a}_i = \underline{w}_i$ for $i = 1, \ldots, n$.*

... and that ***the range of a map hence can also be written in terms of basis vector images***, by using the linear span (which was recalled in Definition B.2.3):

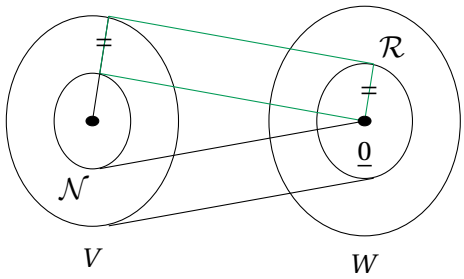
Theorem 1.1.16. *Consider a linear map $\mathcal{A} : V \to W$ with $V = \text{span}(\underline{a}_1, \ldots, \underline{a}_n)$. Then*

$$\mathcal{R} = \text{span}(\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_n) \quad .$$

While it is not easy to give a comparably simple characterisation of $\mathcal{N}$, we have at least the following important result that says something about the dimension:

Theorem 1.1.17 (Dimension Theorem). *Let $\mathcal{A} : V \to W$ be a linear map with $\dim(V) < \infty$. Then*

$$\dim(V) = \dim(\mathcal{N}) + \dim(\mathcal{R}) \, .$$



Example 1.1.18 ***(Orthogonal projections)*** Revisiting orthogonal projections $\mathcal{P}$ on a subspace W of the inner product space V (example 1.1.2) once more: We can easily convince ourselves that the range of $\mathcal{P}$ equals W, while the null space consists of all vectors perpendicular (orthogonal) to W, so $\mathcal{N} = W^\perp$. The dimension theorem above therefore implies that

$$\dim V = \dim W + \dim W^\perp \, .$$

Lastly, we recall a nice ***criterion that tells us when a linear map with finite-dimensional starting space is invertible***:

Theorem 1.1.19. *Let V be a vector space with $\dim(V) < \infty$, and let $\mathcal{A} : V \to W$ be linear map. $\mathcal{A}$ has an inverse if and only if $\dim(V) = \dim(W)$ and $\mathcal{N} = \{\underline{0}\}$.*

If we do not feel like checking the condition $\mathcal{N} = \{\underline{0}\}$ (for whatever reason), we can alternatively replace this condition with the condition that $\mathcal{R} = W$.

1.1.1 The new example for linear maps: the quotient map

In LinA 1, you learned about quotient spaces modulo a subspace U (recalled in Appendix B.2.2), where two vectors $\underline{v}$ and $\underline{w}$ lie in the same residue class $[\underline{v}]$ if $\underline{v} - \underline{w} \in U$. ***We will now revisit the following example from LinA 1 and connect it to linear maps:***

Example 1.1.20 Let $V = R^3$ and let $U = \operatorname{span}((1,0,0))$. Then any two vectors (x,a,b) and (y,a,b) are equivalent modulo U, since their difference $(x-y,0,0) \in U$. So the residue class is $[(x,a,b)] = [(y,a,b)] = [(0,a,b)]$, meaning V/U ***looks a bit like the linear subspace spanned by*** $\underline{e}_2, \underline{e}_3$. This will be made precise in *Linear Algebra 2* using linear maps.

This section will make the meaning of the emphasised sentence in the example more precise by proving a fundamental theorem about linear maps (Theorem 1.1.21 below). Originally, Theo-

rem 1.1.21 was devised by Emmy Noether, a last-century mathematician who made many important contributions to abstract algebra and mathematical physics, but wasn't allowed to officially teach for a long time due to her gender. (She did it anyways, under David Hilbert's name!)

We quickly introduce a concept needed for the theorem: Since for any linear map $\mathcal{A}: V \to W$, the null space $\mathcal{N}$ is a linear subspace of V, we can also define the respective quotient space $V/\mathcal{N}$.

> **Theorem 1.1.21 (Noether's fundamental theorem on homomorphisms, vector space edition).** *For any linear map $\mathcal{A}: V \to W$, there exists a linear bijection between its range $\mathcal{R}$ and the quotient space $V/\mathcal{N}$, where $\mathcal{N}$ is the null space of $\mathcal{A}$.*
>
> *In other words, $\mathcal{R}$ and $V/\mathcal{N}$ are isomorphic (or even shorter: $\mathcal{R} \cong V/\mathcal{N}$).*

How does this relate to example 1.1.20? To make the connection, we use as $\mathcal{A}$ the projection unto the subspace span$(\underline{e}_2, \underline{e}_3)$:

$$\mathcal{P}: \mathbb{R}^3 \xrightarrow{(x,a,b)\mapsto(0,a,b)} \mathbb{R}^3 \ .$$

As we reminded ourselves in example 1.1.18, the null space of a projection is the orthogonal complement of the subspace unto which we project. So $\mathcal{N}(\mathcal{P}) = \text{span}(\underline{e}_2, \underline{e}_3)^\perp = \text{span}((1,0,0)) = U$, and the range of $\mathcal{P}$ is $\mathcal{R}(\mathcal{P}) = \text{span}(\underline{e}_2, \underline{e}_3)$. Plugging this into Theorem 1.1.21, we get that span$(\underline{e}_2, \underline{e}_3)$ is equivalent to V/U, up to an isomorphism.

To see that Theorem 1.1.21 is true and how the isomorphism looks, we'll go through the proof:

(I interspersed the proof with our example to illustrate the main ideas.)

Proof of Theorem 1.1.21. For any linear map $\mathcal{A}$ with null space $\mathcal{N}$, we can define a map π that maps vectors to their residue classes:

$$\pi: V \xrightarrow{\underline{v}\mapsto[\underline{v}]} V/\mathcal{N} \ .$$

(Applying this to example 1.1.20: π maps any vector (x,a,b) to $[(0,a,b)] \in V/U$.)

To practice a bit with the involved concepts, your homework will include convincing yourself that the map π always is linear, surjective (for the definition, see the table in Appendix A.2.2); and that its null space is exactly the null space of $\mathcal{A}$.

We can also define a 'quotient version' of $\mathcal{A}$, by that we mean a map $\overline{\mathcal{A}}$ that maps residue classes to the image of an element of the residue class:

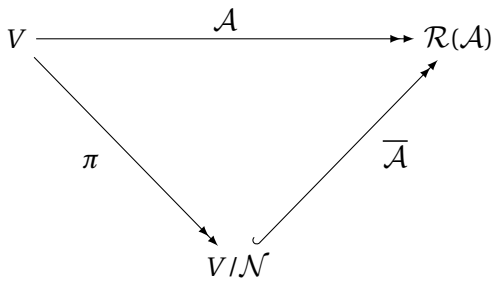
$$\overline{\mathcal{A}}: V/\mathcal{N} \xrightarrow{[\underline{v}]\mapsto\mathcal{A}\underline{v}} W \ .$$

(Applying this to example 1.1.20: $\overline{\mathcal{P}}$ maps any residue class $[(0,a,b)]$ to the projection $\mathcal{P}(0,a,b) = (0,a,b)$.)

But wait, is our mapping rule well-defined? There can be many different elements $\underline{v}_1, \underline{v}_2, \ldots$ in the same residue class $[\underline{v}]$ – which image $\mathcal{A}\underline{v}_1, \mathcal{A}\underline{v}_2, \ldots$ do we pick for $\overline{\mathcal{A}}[\underline{v}]$? Luckily, the definition still makes sense because all elements sharing the same residue class actually are mapped to the same image: If $\underline{v}$ and $\underline{w}$ are in the same residue class, then $\underline{v} - \underline{w} \in \mathcal{N}$, meaning $\mathcal{A}(\underline{v} - \underline{w}) = \underline{0}$. Due to the linearity of $\mathcal{A}$, we get $\mathcal{A}(\underline{v}) = \mathcal{A}(\underline{w})$.

So $\overline{\mathcal{A}}$ is a well-defined map from $V/\mathcal{N}$ to $\mathcal{R}(\mathcal{A})$ – we can restrict the 'target space' of $\overline{\mathcal{A}}$ to the actual range rather than using the whole space W – that is naturally surjective due to the restriction. (For example 1.1.20, this means restricting the target space to the subspace span$(\underline{e}_2, \underline{e}_3)$.)

We get the following diagram:



What's missing to verify that $\overline{\mathcal{A}}$ is an isomorphism? We still need to show that $\overline{\mathcal{A}}$ always is

- linear and
- injective (the definition of injectivity is also recalled in Appendix A.2.2) ,

which you will do in your homework.

So $\overline{\mathcal{A}}$ is a linear bijection from $V/\mathcal{N}$ to $\mathcal{R}(\mathcal{A})$, which proves Theorem 1.1.21. □

Given that $\overline{\mathcal{A}}$ is a bijection, there also exists an inverse $\overline{\mathcal{A}}^{-1} : \mathcal{R}(\mathcal{A}) \to V/\mathcal{N}$. ***But how does the inverse look, concretely?*** Let $w \in \mathcal{R}(\mathcal{A})$. Since w is in the range of $\mathcal{A}$, there always exists a pre-image $v \in \mathcal{A}^{-1}(w)$, so we can set $\overline{\mathcal{A}}^{-1}(w) := [v]$.

1.2. Recap: Connecting matrices and linear maps (as seen in *Linear Algebra 1*)

In this recap section, we recall the close connection between matrices and linear maps that are of the form $\mathbb{K}^n \to \mathbb{K}^m$, where $\mathbb{K}$ is an arbitrary field. (E.g., you can think of $\mathbb{K} = \mathbb{R}$, or of $\mathbb{K} = \mathbb{C}$, but $\mathbb{K}$ can actually be any set that satisfies the rules for fields.)

This section covers:

- how every such matrix is connected to a linear map, and vice versa;
- how to compute the respective matrix for a given map;
- the connection with systems of linear equations;
- that the connection between maps and matrices behaves natural: we can easily switch between maps and matrices even when dealing, e.g., with compositions or inverses

This section requires familiarity with matrices and matrix multiplication which you can revisit in Appendix B.3 when needed.

In example 1.1.4, ***we saw that any matrix A defines a linear map***

$$\mathcal{A} : \mathbb{K}^n \xrightarrow{\underline{v} \mapsto A\underline{v}} \mathbb{K}^m \ .$$

In a bit more detail, we also showed in LinA I by substitution that ***the columns of the matrix are the images of the standard base vectors:*** Let $\underline{e}_1, \ldots, \underline{e}_n$ be the standard basis of $\mathbb{K}^n$, and let $\underline{c}_1, \ldots, \underline{c}_n$ be the columns of the matrix A. Then

$$\mathcal{A}\underline{e}_1 = A\underline{e}_1 = \underline{c}_1, \ \mathcal{A}\underline{e}_2 = \underline{c}_2, \ \ldots, \mathcal{A}\underline{e}_n = \underline{c}_n \ .$$

We will now recall that ***vice versa, we can also associate to each linear map a matrix that 'represents' the map, meaning we found a new (and very important) use for matrices***:

> **Definition/theorem 1.2.1.** *Every linear map $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^m$ is determined by an $m \times n$-matrix A, called* **the matrix of the linear map** *$\mathcal{A}$, whose columns are $\mathcal{A}\underline{e}_1, \ldots, \mathcal{A}\underline{e}_n$. The image of the vector $\underline{v}$ under $\mathcal{A}$ can be computed as the matrix product $A\underline{v}$.*

Proof. Let's take an arbitrary linear map $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^m$, and an arbitrary vector $\underline{v} \in \mathbb{K}^n$. We can always write a $\underline{v}$ as a linear combination of the standard basis vectors: $\underline{v} = x_1\underline{e}_1 + \cdots + x_n\underline{e}_n$. Then $\mathcal{A}\underline{v} = x_1\mathcal{A}\underline{e}_1 + \cdots + x_n\mathcal{A}\underline{e}_n$ due to the linearity of $\mathcal{A}$. Collecting all image vectors $\mathcal{A}\underline{e}_1, \ldots, \mathcal{A}\underline{e}_n$ as columns in an $m \times n$–matrix A, we obtain that $\mathcal{A}\underline{v} = x_1\mathcal{A}\underline{e}_1 + \cdots + x_n\mathcal{A}\underline{e}_n$ equals the matrix product $A\underline{v}$. □

Remark 1.2.2 ***(How to compute the matrix if we only know the images on some basis?)*** Assume we know how $\mathcal{A}$ acts on a basis $\alpha = \underline{a}_1, \ldots, \underline{a}_n$ for $\mathbb{K}^n$, meaning we are given a description of $\mathcal{A}$ that only tells us what $\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_n$ are. As mentioned in Section 1.1, such a description is sufficient to uniquely determine $\mathcal{A}$, but how do we determine the corresponding matrix?

- If α is the standard basis, we can simply write down the matrix using the previous Definition/theorem 1.2.1.
- If α is not the standard basis, we learnt in LinA how we can apply row reduction techniques to find the image vectors of the standard basis, which leads us back to the first case, as in the example below.

Example 1.2.3 The linear map $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{A}(-1,0,1) = (-4,2,4), \ \mathcal{A}(1,1,0) = (1,-1,-1), \ \mathcal{A}(0,1,2) = (-5,4,4) \ .$$

We put this data in three (vector, image)-rows (for clarity, we visually separate the vector and its image with a vertical bar):

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 1 & -4 & 2 & 4 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & -5 & 4 & 4 \end{array}\right) .$$

Performing row-reduction, we obtain the (row-reduced) normal form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 3 & 1 \end{array}\right) ,$$

in which the left rows are the standard basis vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ and the right rows are their images $\mathcal{A}\underline{e}_1, \mathcal{A}\underline{e}_2, \mathcal{A}\underline{e}_3$. Using Definition/theorem 1.2.1 (and translating the rows on the right-hand side back into columns), we find that the matrix of $\mathcal{A}$ is

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 1 & -2 & 3 \\ -3 & 2 & 1 \end{pmatrix} .$$

It is easy to verify correctness of our computations by using this matrix to compute the images of the vectors $(-1,0,1)$, $(1,1,0)$ and $(0,1,2)$.

Remark 1.2.4 ***(Connection with systems of linear equations)*** Consider the following system of linear equations

$$\begin{array}{ccccc} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = & b_1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + & a_{mn}x_n = & b_m \end{array}$$

with $m \times n$–coefficient matrix A. Then A determines a linear map $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^m$, so ***systems of equations can be written as a vector equation***

$$\mathcal{A}\underline{x} = \underline{b} \quad .$$

Let's look how we can use what we learnt so far to ***find all solutions of the equation*** $\mathcal{A}\underline{x} = \underline{b}$ ***(if they exist):*** Theorem 1.1.14 tells us that we find all solutions to $\mathcal{A}\underline{x} = \underline{b}$ by finding one particular solution $\underline{p}$, and adding to it all vectors from the null space, meaning the solution set is of the form

$$\underline{p} + \mathcal{N} := \{\underline{p} + \underline{w} \mid \underline{w} \in \mathcal{N}\} \quad .$$

But does such a particular solution $\underline{p}$ exist? To decide this, we first identify the range $\mathcal{R}$ of $\mathcal{A}$, using Theorem 1.1.16:

$$\mathcal{R} = \operatorname{span}\left(\mathcal{A}\underline{e}_1, \ldots, \mathcal{A}\underline{e}_n\right) ,$$

which is exactly the column space of the matrix A. The equation $\mathcal{A}\underline{x} = \underline{b}$ has a solution (at least one) if and only if $\underline{b} \in \mathcal{R}$, so if and only if $\underline{b}$ is contained in the column space of the matrix.

Now that we've found a criterion to determine whether particular solutions exist, it only remains to quickly identify the null space $\mathcal{N}$: $\mathcal{N}$ consists of all vectors $\underline{v}$ with $\mathcal{A}\underline{v} = \underline{0}$, so all solutions of the homogeneous system $A\underline{x} = \underline{0}$.

The rest of this section briefly recalls that ***matrices of compositions/sums/inverses can be found in a very natural way***: e.g., the matrix of a sum $\mathcal{A} + \mathcal{B}$ of linear maps $\mathcal{A}$ and $\mathcal{B}$ is exactly the sum of their two representing matrices A and B. This makes it very easy to switch between matrix-POVs and map-POVs!

Theorem 1.2.5 (Matrix of sums and scalar multiples). *Let A and B be two matrices of the same size, and let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding linear maps. Then $A+B$ is the matrix of the sum $\mathcal{A}+\mathcal{B}$.*

For every scalar $\alpha \in \mathbb{K}$, the matrix of the linear map $\alpha\mathcal{A}$ is αA.

Theorem 1.2.6 (Matrix of compositions). *Let A and B be two matrices such that the composition AB exists, and let $\mathcal{A}$ and $\mathcal{B}$ be the corresponding linear maps. Then AB is the matrix of the composition map $\mathcal{AB}$.*

Theorem 1.2.7 (Matrix of the inverse). *Let $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^n$ be a linear map with matrix A. Then the map $\mathcal{A}$ has an inverse if and only if its matrix A has an inverse, in which case the matrix for the map's inverse $\mathcal{A}^{-1}$ is exactly the inverse A^{-1} of the matrix A.*

Remark 1.2.8 ***(Applying polynomials to maps/matrices)*** We have seen that we can add and compose linear maps, and that switching between maps and their matrices preserves addition and multiplication. In Definition 1.1.9, we defined powers of linear maps that map from some space V into itself. If we have such a linear map $\mathcal{A} : V \to V$ with matrix A, and if we have a polynomial $p(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ whose scalars $a_n, \cdots, a_0$ are taken from V's scalar field, we can hence also define the linear map

$$p(\mathcal{A}) \coloneqq a_n \mathcal{A}^n + a_{n-1} \mathcal{A}^{n-1} + \cdots + a_1 \mathcal{A} + a_0 \cdot \mathcal{I} \ ,$$

whose matrix is

$$p(A) \coloneqq a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 \cdot \mathcal{I} \ .$$

1.2.1 Computing a matrix inverse via row reduction

Lastly, we recall how our accumulated knowledge was used to close a gap that had remained so far:

When matrix inverses were introduced, it was mentioned without proof that $AB = I$ implies $BA = I$ (for square matrices A and B). ***We will now revisit how we used the connection between linear maps and matrices to prove that*** $AB = I$ ***implies*** $BA = I$***:***

Theorem 1.2.9. *Let A and B be $n \times n$-matrices. If $AB = I$, then $BA = I$.*

Proof. Instead of proving the claim by dealing with matrices, we'll look at it through a linear maps lens: Let $\mathcal{A}, \mathcal{B} : \mathbb{K}^n \to \mathbb{K}^n$ be the linear maps corresponding to A and B. (Like at the beginning of 1.2.) Now, Theorem 1.2.6 tells us that the matrix for the composition $\mathcal{AB}$ is the composition of the maps A and B (so the identity matrix), hence $\mathcal{AB} = \mathcal{I}$. If we could prove that also $\mathcal{BA} = \mathcal{I}$ is true, then we'd already be done, as we could translate this equation back into matrix terms and get that $BA = I$ as desired.

How do we prove that $\mathcal{BA} = \mathcal{I}$ is true? Since $\mathcal{AB} = \mathcal{I}$, we have in other words that $\mathcal{AB}\underline{v} = \underline{v}$ for all $\underline{v} \in \mathbb{K}^n$, meaning that $\mathcal{A}$ is the left-hand inverse of $\mathcal{B}$. For bijective maps, the right-hand inverse and the left-hand inverse is the same, meaning that also $\mathcal{BA} = \mathcal{I}$, and we're done. □

The connection between maps and matrices also ***explains why/how we can do row reduction to compute the inverse of square matrices (if it exists)***, since we can compute the matrix of a map's inverse by working with base vector images (and doing row reduction). We saw the following two examples:

Example 1.2.10 ***(Computing the inverse via row reduction)*** Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{pmatrix}.$$

For the corresponding linear map $\mathcal{A}$, we have $\mathcal{A}\underline{e}_1 = (1,0,4)$, $\mathcal{A}\underline{e}_2 = (0,1,-3)$ and $\mathcal{A}\underline{e}_3 = (3,2,8)$. For the inverse map $\mathcal{A}^{-1}$, it follows that $\mathcal{A}^{-1}(1,0,4) = \underline{e}_1$, $\mathcal{A}^{-1}(0,1,3) = \underline{e}_2$ and $\mathcal{A}^{-1}(3,2,8) = \underline{e}_3$. (Because, e.g., $\mathcal{A}^{-1}(1,0,4) = \mathcal{A}^{-1}\mathcal{A}\underline{e}_1 = \underline{e}_1$.) We can now determine the matrix of $\mathcal{A}^{-1}$ by computing $\mathcal{A}^{-1}\underline{e}_1$, $\mathcal{A}^{-1}\underline{e}_2$ and $\mathcal{A}^{-1}\underline{e}_3$, since we know that these vectors constitute the column vectors of $\mathcal{A}^{-1}$. We get $\mathcal{A}^{-1}\underline{e}_1$, $\mathcal{A}^{-1}\underline{e}_2$ and $\mathcal{A}^{-1}\underline{e}_3$ from $\mathcal{A}^{-1}(1,0,4),\ldots,\mathcal{A}^{-1}(3,2,8)$ exactly as in example 1.2.3:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 3 & 2 & 8 & 0 & 0 & 1 \end{array}\right).$$

Row reduction yields the normal form

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 4 & -2 \\ 0 & 1 & 0 & -\frac{9}{2} & -2 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & \frac{1}{2} \end{array}\right)$$

Translating the rows on the right-hand side back into columns gives us the inverse:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 14 & -9 & -3 \\ 8 & -4 & -2 \\ -4 & 3 & 1 \end{pmatrix}.$$

Example 1.2.11 ***(Showing that there is no inverse via row reduction)*** We consider a similar matrix that differs only in the last row:

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 6 \end{pmatrix}.$$

For the inverse $\mathcal{A}^{-1}$ of the corresponding linear map $\mathcal{A}$, (provided it exists), we must have $\mathcal{A}^{-1}(1,0,4) = \underline{e}_1$, $\mathcal{A}^{-1}(0,1,-3) = \underline{e}_2$, $\mathcal{A}^{-1}(3,2,6) = \underline{e}_3$, yielding the system

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{array}\right).$$

Partial row reduction gives

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{array}\right).$$

This shows that the columns of A are dependent, and that there hence is no inverse matrix.

1.3. Representation matrices for finite-dimensional spaces $\neq \mathbb{K}^n$, using coordinates

What will we do? So far, we saw that linear maps from $\mathbb{K}^n \to \mathbb{K}^m$ can be described using matrices. ($\mathbb{K}$ was an arbitrary field – e.g., you can think of $\mathbb{K} = \mathbb{R}$, or of $\mathbb{K} = \mathbb{C}$, but $\mathbb{K}$ can actually be any set that satisfies the rules for fields.) What helped us there was that these spaces have standard bases, which allows us to identify the base images with the column vectors of the matrix. What happens if we have an abstract $\mathbb{K}$-vector space instead of $\mathbb{K}^n$, meaning we might not have such a nice standard basis? ***The result of Section 1.3 is that we still can come up with matrices representing the map!*** Our ***main idea*** will be: Pick a basis and ***translate the whole setting back to the $\mathbb{K}^n \to \mathbb{K}^m$ case by working with coordinates***.

In more detail, we'll

- analyse how the coordinates of a vector change when switching to another basis;
- connect linear maps to matrices for arbitrary vector spaces (depending on the choice of basis);
- use the results to analyse how these matrices transform when switching to another basis.

Why do we care? As mentioned in the 'Why do we care?' paragraph at the beginning of this chapter, we will use these techniques in the later chapters ***a lot***, essentially whenever we find and use 'nicer' bases to bring matrices into a 'nicer' (simpler) shape.

1.3.1 Coordinates

We start by recalling coordinates from LinA I:

Definition 1.3.1 (Coordinates). *Let V be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$. Every vector $\underline{v} \in V$ can be expressed as a linear combination of the basis vectors in exactly one way:*

$$\underline{v} = x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n .$$

The numbers $x_1, \dots, x_n$ are the **coordinates of $\underline{v}$ with respect to the basis** *α and $(x_1, \dots, x_n)$ is the* **coordinate vector of $\underline{v}$ with respect to the basis** *α.*

⚠ Clearly, the coordinates depend on the choice of the basis α.

Example 1.3.2 ***(Coordinates for the real-polynomial vector space)*** Consider the vector space V of real polynomials of degree at most 2, and take the polynomial $p := 1+2x+3x^2$ (which is a vector in V). We pick the basis $\alpha := \{1, x, x^2\}$.
The α-coordinates of p are 1, 2, 3, so the α-coordinate vector of p is $(1,2,3)$.

Convince yourself that the map associating to each vector $\underline{v}$ the corresponding coordinate vector is a bijection. We make an even stronger observation: The following theorem tells us that ***mapping vectors to their coordinate vectors yields an invertible linear map***.

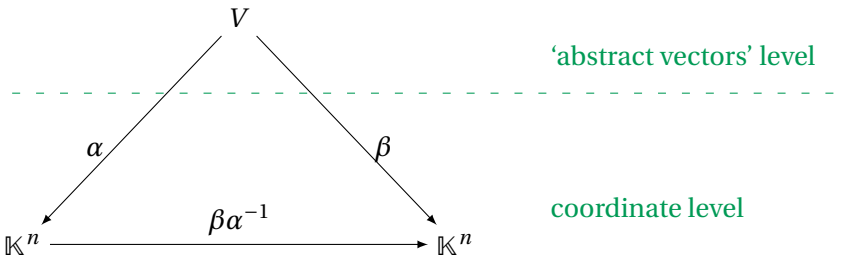
Definition/theorem 1.3.3. *Let V be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$. We will* **denote the map sending each vector $\underline{v}$ to its coordinates with respect to basis α also by α.**

Then α is an invertible linear map from V to $\mathbb{K}^n$.

With this notation, $\alpha(\underline{v})$ is the coordinate vector of the vector $\underline{v} \in V$ with respect to the basis α.

Proof of Theorem 1.3.3. As we learnt in LinAI, the coordinates of the sum $\underline{v}_1 + \underline{v}_2$ are the sum of the coordinates of $\underline{v}_1$ and of $\underline{v}_2$, and the coordinates of $\alpha\underline{v}$ are precisely α times the coordinates of $\underline{v}$. □

What happens if we switch from one basis α to another basis β? Then any vector $\underline{v} \in V$ corresponds to two different sets of coordinates: $\alpha(\underline{v})$ with respect to the basis α, and $\beta(\underline{v})$ with respect to the basis β. ***How are the α-coordinates and the β-coordinates of $\underline{v}$ related?*** Starting with the α-coordinates (viewed as a vector in $\mathbb{K}^n$), we first apply the map α^{-1} which gives us $\underline{v} \in V$, and then we apply the map β which gives us $\beta(\underline{v}) \in \mathbb{K}^n$.



Definition 1.3.4 (Coordinate transformation (map)). *Let α and β be bases of an n-dimensional vector space V. The map $\beta\alpha^{-1} : \mathbb{K}^n \to \mathbb{K}^n$ is called the* **coordinate transformation (map)** *from α to β.*

The coordinate transformation $\beta\alpha^{-1}$ is itself a linear map: We already know that α and β are linear, hence linearity of $\beta\alpha^{-1}$ follows directly from the linearity of compositions/inverses. ***Like for any linear map, we can hence associate $\beta\alpha^{-1}$ with a corresponding matrix.***

1.3.2 Coordinate transformations from a matrix POV: The basis transition matrix

Given that the coordinate transformation is a linear map from $\mathbb{K}^n$ to itself, it corresponds to an $n \times n$–matrix. ***We will use this matrix a lot in the following chapters, therefore we fix notation***:

Definition 1.3.5 (Transition matrix). *Let α and β be bases of an n-dimensional vector space V. We call the $n \times n$–matrix associated to the linear map $\beta\alpha^{-1}$ the* **transition matrix** *from basis α to basis β, and* **denote it by** ${}_\beta S_\alpha$.

The following theorem states that - as one might expect - ***multiplication with the matrix*** ${}_\beta S_\alpha$ ***translates*** α***- into*** β***-coordinates***, and gives a ***direct description of how the matrix looks, entry-wise:***

> **Theorem 1.3.6.** *Let α and β be bases of an n-dimensional vector space V, and let ${}_\beta S_\alpha$ be the basis transition matrix, i.e., the matrix of $\beta\alpha^{-1}$. Let $\underline{x} := \alpha(\underline{v})$ be the α–coordinate vector of a vector $\underline{v} \in V$. Then the β–coordinate vector of $\underline{v}$ is equal to the product ${}_\beta S_\alpha\, \underline{x}$. Furthermore, the columns of matrix ${}_\beta S_\alpha$ are the β–coordinate vectors of the α–basis vectors.*

Proof. To show that the product ${}_\beta S_\alpha\, \underline{x}$ yields the β–coordinate vector of $\underline{v}$, we can simply use that applying the map and taking the matrix product is equivalent: ${}_\beta S_\alpha\, \underline{x} = (\beta\alpha^{-1})(\underline{x}) = (\beta\alpha^{-1})(\alpha(\underline{v})) \overset{\star}{=} \beta(\alpha^{-1}(\alpha(\underline{v}))) = \beta(\underline{v})$, where $\star$ used associativity of map compositions.

To show that the columns of ${}_\beta S_\alpha$ indeed are the β–coordinate vectors of the α–basis vectors, we recall that the columns of a matrix are the images of the standard base vectors $\underline{e}_1, \ldots, \underline{e}_n$ under the corresponding linear map. The i-th column of ${}_\beta S_\alpha$ hence is equal to ${}_\beta S_\alpha \underline{e}_i = (\beta\alpha^{-1})(\underline{e}_i)$. Since α^{-1} (per definition) maps coordinate vectors back into the respective linear combination of the base vectors in α, we have that $\alpha^{-1}(\underline{e}_i) = \underline{a}_i$, where $\underline{a}_i$ is the i-th vector in the base α. Plugging this in, we get that $\beta(\alpha^{-1}(\underline{e}_i)) = \beta(\underline{a}_i)$, the β–coordinate vector of $\underline{a}_i$. □

Example 1.3.7 ***(Basis transition matrix for real-polynomial vector space)*** We again consider the vector space V of real polynomials of degree at most 2. We pick the two bases $\alpha := \{x-1, x^2-1, x^2+1\}$ and $\beta := \{1, x, x^2\}$. We can easily express the basis vectors of α in the basis vectors of β:

$$\begin{aligned} x-1 &= (-1)\cdot 1 + 1\cdot x + 0\cdot x^2 \\ x^2-1 &= (-1)\cdot 1 + 0\cdot x + 1\cdot x^2 \\ x^2+1 &= 1\cdot 1 + 0\cdot x + 1\cdot x^2 \quad . \end{aligned}$$

We therefore know the β–coordinates of the vectors of α. We can use them to give the basis transition matrix, since its columns are the β–coordinate vectors of the α–basis vectors:

$${}_\beta S_\alpha = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

It is of course also possible to switch from β–coordinates to α–coordinates, which is done with the matrix ${}_\alpha S_\beta$. When looking at the product matrix ${}_\alpha S_\beta\, {}_\beta S_\alpha$ of the two matrices, we notice that it first transforms α–coordinates to β–coordinates, and then the β–coordinates back to α–coordinates, so it doesn't do anything:

$${}_\alpha S_\beta\, {}_\beta S_\alpha = I, \text{ so } {}_\alpha S_\beta = {}_\beta S_\alpha{}^{-1} \quad .$$

Example 1.3.8 ***(Example 1.3.7, continued)*** As we just noticed, the transition matrix ${}_\alpha S_\beta$ is the inverse of matrix ${}_\beta S_\alpha$. Computing the inverse, we find

$$ {}_\alpha S_\beta = {}_\beta S_\alpha{}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. $$

The columns of this matrix should consist of the α–coordinates of the base vectors of β. Let's quickly verify this:

- The first column vector is $(0, -\frac{1}{2}, \frac{1}{2})$, translating this back via α^{-1} corresponds to vector $0\,(x-1) - \frac{1}{2}(x^2-1) + \frac{1}{2}(x^2+1) = 1$. ✓
- In the same way, we can verify that the second column consists of the α–coordinates of x and that the third column consists of the α–coordinates of x^2. ✓

Computing the α–coordinates of some vector: As an example, let's pick the vector $v := 2x^2 - 3x + 4$. The β-coordinates of this vector are $(4, -3, 2)$. We transform them to α–coordinates using the transition matrix ${}_\alpha S_\beta$:

$$ {}_\alpha S_\beta \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}. $$

We verify the result:

$$ -3(x-1) + \frac{1}{2}(x^2-1) + \frac{3}{2}(x^2+1) = 2x^2 - 3x + 4 \quad . \checkmark $$

If a third basis γ enters the picture, we could transform α– to β–coordinates, and these subsequently to γ–coordinates. The following theorem tells us that ***doing the β-detour is unnecessary and we can directly switch from α to γ***:

Theorem 1.3.9. *Let α, β and γ be bases of an n-dimensional vector space V, with respective basis transition matrices ${}_\beta S_\alpha$, ${}_\gamma S_\alpha$ and ${}_\gamma S_\beta$. Then*

$$ {}_\gamma S_\beta \, {}_\beta S_\alpha = {}_\gamma S_\alpha $$

Proof. The product of the two matrices is the matrix of their composition (remember Theorem 1.2.6). For the composition under consideration, we have $(\gamma\beta^{-1})(\beta\alpha^{-1}) = \gamma(\beta^{-1}\beta)\alpha^{-1} = \gamma\alpha^{-1}$. □

⚠ ***It is important to distinguish between calculating with vectors*** (so elements of the vector space V) ***and calculating with coordinates*** (so sequences of elements from $\mathbb{K}^n$). In example 1.3.7, the separation between the two concepts was clear: the vectors were polynomials and the coordinate vectors were sequences of real numbers.

We need to be more careful with this distinction if our vector space is $\mathbb{K}^n$ itself, as a sequence could either represent a vector from the vector space $\mathbb{K}^n$ or a sequence of coordinates. We note that this

strict separation is less crucial if we talk about coordinates w.r.t. the standard basis $\varepsilon = \{\underline{e}_1, \ldots, \underline{e}_n\}$:

$$(1,2,3) = 1\underline{e}_1 + 2\underline{e}_2 + 3\underline{e}_3 \, .$$

While having $\mathbb{K}^n$ as the vector space itself might make it more complicated to separate between vectors and their coordinate vectors, the following example shows that $\mathbb{K}^n$ ***allows for a nice short-cut when you need to compute basis transition matrices***.

Example 1.3.10 ***(Computing a basis transition matrix in*** $\mathbb{R}^3$***)*** Let's consider the following two bases:

$$\alpha = \{(1,0,2),\ (-1,1,0),\ (0,-2,1)\}\,;$$
$$\beta = \{(0,1,1),\ (1,2,-1),\ (1,0,1)\} \quad .$$

We are looking for the transition matrix that transforms α– into β–coordinates, so ${}_\beta S_\alpha$.
Method like in example 1.3.7. We'll first solve the problem directly, by finding a way to write each of the α-vectors as a linear combination of the β-vectors. This means having to solve three systems of equations in three unknowns:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 1 & -2 \\ 1 & -1 & 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{\text{row reduction}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & \frac{5}{4} & -1 & \frac{3}{4} \end{array}\right).$$

Now the last three columns are the β-coordinates of the three vectors of α, hence

$${}_\beta S_\alpha = \frac{1}{4}\begin{pmatrix} 2 & 4 & -2 \\ -1 & 0 & -3 \\ 5 & -4 & 3 \end{pmatrix}.$$

Method with shortcut. Our alternative method uses that ***in*** $\mathbb{K}^n$***, the matrix for switching to the standard basis*** ε ***is super-easy to compute***: We know the standard-basis coordinates of both bases. E.g., the first vector of α is $(1,0,2) = 1\underline{e}_1 + 0\underline{e}_2 + 2\underline{e}_3$, meaning the first column of ${}_\varepsilon S_\alpha$ must be (1,0,2), so the first base vector from α itself, and so on. So, ***without any computational work we already know the transition matrices*** ${}_\varepsilon S_\alpha$ ***and*** ${}_\varepsilon S_\beta$:

$${}_\varepsilon S_\alpha = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix},\ {}_\varepsilon S_\beta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

To be as lazy as possible on our way to finding the α–to–β transition matrix, we now use that we can do a detour via basis ε: ${}_\beta S_\alpha = {}_\beta S_\varepsilon \, {}_\varepsilon S_\alpha$ (due to Theorem 1.3.9). We'll also use that switching the direction of the base transition is the same as taking the inverse: ${}_\beta S_\varepsilon = {}_\varepsilon S_\beta{}^{-1}$. Combining both observations, we get the identity ${}_\beta S_\alpha = {}_\varepsilon S_\beta{}^{-1} \, {}_\varepsilon S_\alpha$.
So, we first determine the inverse of ${}_\varepsilon S_\beta$ and find

$${}_\beta S_\varepsilon = {}_\varepsilon S_\beta{}^{-1} = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}.$$

We can now multiply this with ${}_\varepsilon S_\alpha$ from the right and get

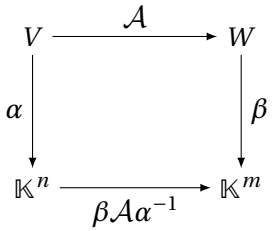
$$ {}_\beta S_\alpha = \frac{1}{4}\begin{pmatrix} -2 & 2 & 2 \\ 1 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 2 & 0 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 4 & -2 \\ -1 & 0 & -3 \\ 5 & -4 & 3 \end{pmatrix}. $$

1.3.3 Generalising the map-matrix connection for spaces that aren't $\mathbb{K}^n$

So far, we saw that linear maps from $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^m$ can be described using matrices. ***We will now generalise the connection***: we will replace $\mathbb{K}^n/\mathbb{K}^m$ with arbitrary finite-dimensional $\mathbb{K}$-vector spaces V/W. ***Our main idea will be to translate the whole thing back to the*** $\mathbb{K}^n \to \mathbb{K}^m$ ***case by picking a basis and working with coordinates.***

So, let V be a finite-dimensional vector space with basis α, let W be a finite-dimensional vector space with basis β, and let $\mathcal{A} : V \to W$ be a linear map.

Consider the following diagram:



- To every vector $\underline{v} \in V$, there corresponds a unique coordinate vector $\alpha(\underline{v})$.
- To $\mathcal{A}\underline{v}$, the image of $\underline{v}$ under map $\mathcal{A}$, there corresponds a unique coordinate vector $\beta(\mathcal{A}\underline{v})$.

This diagram might look complicated at a first glance, but in fact, I already tricked you into using the coordinate concept in the last chapter! We did this in examples 1.3.7 and 1.3.8, where the bases were polynomials instead of vectors in $\mathbb{R}^n$.

The composite linear map $\beta\mathcal{A}\alpha^{-1} : \mathbb{K}^n \to \mathbb{K}^m$ maps the coordinate vector $\alpha(\underline{v})$ to the coordinate vector $\beta(\mathcal{A}\underline{v})$. Since $\beta\mathcal{A}\alpha^{-1}$ is a linear map between $\mathbb{K}^n$ and $\mathbb{K}^m$, we already know how to represent it with a matrix (remember Definition/theorem 1.2.1). ***This gives us a representation matrix for the linear map*** $\mathcal{A}$ ***on the coordinate level:***

Definition 1.3.11 (Matrix of a linear map). *Let V, W, α, β, $\mathcal{A}$ be as above. We denote the matrix of the linear map $\beta\mathcal{A}\alpha^{-1}$ by ${}_\beta A_\alpha$ and call it the* **matrix of $\mathcal{A}$ w.r.t. the bases α and β** .

If $V = W$ and $\beta = \alpha$, then we simplify notation by denoting the corresponding matrix by A_α; we call it the **matrix of $\mathcal{A}$ w.r.t. basis α**.

In other words: The matrix A_α maps the α–coordinates of a vector $\underline{v}$ to the α–coordinates of $\mathcal{A}\underline{v}$.

We will spend a lot of time with the special case $V = W$ and $\beta = \alpha$ during this course, but let us first say a few things about the general case.

How does the matrix look? According to Definition/theorem 1.2.1, the columns of ${}_\beta A_\alpha$ are

$$(\beta \mathcal{A} \alpha^{-1})(\underline{e}_i) = \beta(\mathcal{A}\underline{a}_i)\,, \quad i = 1, \ldots, n\,,$$

meaning the i–th column consists of the β–coordinates of the image $\mathcal{A}\underline{a}_i$ of the i–th base vector in α.

This indeed gives us a description of $\mathcal{A}$ on the coordinate level: for example, to find the image of a vector $\underline{v} \in V$, we can

- determine the coordinate vector $\alpha(\underline{v})$ of $\underline{v}$;
- multiply $\alpha(\underline{v})$ with the representation matrix ${}_\beta A_\alpha$, yielding the coordinate vector of $\mathcal{A}\underline{v}$; and
- translate the coordinate vector of $\mathcal{A}\underline{v}$ back to the corresponding vector in W.

The last thing we'll do before we look at some examples is to make sure that ***our new, more general view does not clash with our definition in the old, less general case where*** $V = \mathbb{K}^n$ ***and*** $W = \mathbb{K}^m$ with their respective standard basis:

Observation 1.3.12 (Connection with *the* matrix). *If $\mathcal{A} : \mathbb{K}^n \to \mathbb{K}^m$ is a linear map, and α and β are the standard bases for these spaces, then ${}_\beta A_\alpha$, the matrix of $\mathcal{A}$ w.r.t. these bases as defined in Definition 1.3.11, is just the matrix of $\mathcal{A}$ as defined in Section 1.2. In this case, the coordinate maps α and β both are the identity maps.*

Example 1.3.13 ***(Determining a vector image from the representation matrix)*** Say V is two-dimensional with basis $\alpha = \{\underline{a}, \underline{b}\}$, and the linear map $\mathcal{A} : V \to V$ has the matrix

$$A_\alpha = \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$$

w.r.t. α. The matrix tells us that $\mathcal{A}\underline{a} = 1 \cdot \underline{a} - 2 \cdot \underline{b}$ (first column) and $\mathcal{A}\underline{b} = 4 \cdot \underline{a} + 3 \cdot \underline{b}$ (second column). To compute the image of some vector $\underline{v} = \lambda \underline{a} + \mu \underline{b}$, we multiply A_α with $(\lambda, \mu)^\top$:

$$\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda + 4\mu \\ -2\lambda + 3\mu \end{pmatrix}$$

Hence the image of $\underline{v}$ is $(\lambda + 4\mu)\underline{a} + (-2\lambda + 3\mu)\underline{b}$.

Example 1.3.14 ***(Representation matrix for differentiation.)*** Consider the vector space V of real polynomials of degree at most 2, and the linear map $\mathcal{D} : V \to V$ defined by $\mathcal{D}p = p'$. We already saw that this is a linear map (remember example 1.1.5). Take for V the basis $\alpha = \{1, x, x^2\}$. The derivatives of the basis vectors are:

$$\begin{aligned} \mathcal{D}1 &= 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2\,, \\ \mathcal{D}x &= 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2\,, \\ \mathcal{D}x^2 &= 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2\,. \end{aligned}$$

The matrix D_α is therefore

$$D_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

To illustrate how we can now differentiate using the matrix D_α, take the polynomial $p(x) = 2x^2 - 3x + 5$: The coordinate-vector of p w.r.t. α is $(5,-3,2)$ and

$$D_\alpha \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}.$$

$(-3,4,0)$ is the coordinate vector of $4x-3$ and this is indeed the derivative of p.

1.3.4 How do base changes affect matrices of linear maps?

The rules for coordinate transformations allow us to switch between bases. ***The following observation concerns how the different representation matrices are related to each other, and will be used in the following sections a lot***:

> **Theorem 1.3.15 (Effect of change of basis).** *Choose in a finite-dimensional space V two bases α and β, and suppose $\mathcal{A} : V \to V$ is linear. Then*
>
> $$A_\beta = {}_\beta S_\alpha \, A_\alpha \, {}_\alpha S_\beta \, .$$

In other words: To compute how the β–coordinates of a vector $\underline{v}$ are mapped to the β–coordinates of $\mathcal{A}\underline{v}$, we can

- first transform the β–coordinates of $\underline{v}$ into α–coordinates;
- compute the α–coordinates of $\mathcal{A}\underline{v}$ using the matrix A_α (we'll see in example 1.3.16 below that this can sometimes be easier than computing A_β directly); and
- lastly, transform the result back to β–coordinates.

Proof of Theorem 1.3.15. Per definition, A_β is the matrix for the composed map $\beta\mathcal{A}\beta^{-1}$ and A_α is the matrix for the composed map $\alpha\mathcal{A}\alpha^{-1}$. We now use an 'insert-1'-trick to relate the two composed maps to each other: We have

$$\beta\mathcal{A}\beta^{-1} \overset{(\star)}{=} \beta(\alpha^{-1}\alpha)\mathcal{A}(\alpha^{-1}\alpha)\beta^{-1} = (\beta\alpha^{-1})(\alpha\mathcal{A}\alpha^{-1})(\alpha\beta^{-1}) \ ,$$

where $(\star)$ used that the factors α^{-1} and α cancel (so we inserted a 'matrix-1'), and we afterwards reordered the brackets. We now get our claim by identifying $\beta\alpha^{-1}$ and $\alpha\beta^{-1}$ with their matrices ${}_\beta S_\alpha$ and ${}_\alpha S_\beta$ and using that the matrix of a composition is the matrix product of its factors' matrices (Theorem 1.2.6). □

Example 1.3.16 ***(Representation matrix for an orthogonal projection)*** We once more revisit the projection example 1.1.2, but this time we make it less abstract by picking $V := \mathbb{R}^2$ with the standard inner product, and letting $\mathcal{P}$ be the orthogonal projection unto the line ℓ with equation $2x - 3y = 0$.

Our goal is to determine the matrix P_ε of $\mathcal{P}$ w.r.t. the standard basis ε. Determining the coordinates of $\mathcal{P}\underline{e}_1$ and $\mathcal{P}\underline{e}_2$ directly, however, is a bit unwieldy. Therefore, we ***start by picking a basis $\alpha = \{\underline{a}_1, \underline{a}_2\}$ such that $\mathcal{P}$ has an easy description*** in terms of α-coordinates. We take $\underline{a}_1 = (3,2) \in \ell$ and $\underline{a}_2 = (2,-3) \perp \ell$ and write the projection images in terms of the α-vectors: $\mathcal{P}\underline{a}_1 = \underline{a}_1 = 1 \cdot \underline{a}_1 + 0 \cdot \underline{a}_2$ and $\mathcal{P}\underline{a}_2 = \underline{0} = 0 \cdot \underline{a}_1 + 0 \cdot \underline{a}_2$. So the matrix P_α is

$$P_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Second, we 'translate P_α into ε-coordinates': Per definition, α maps vectors in V to their coordinate vectors w.r.t. α. In our special case where $V = \mathbb{R}^2$, α is the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ that simply translates ε-coordinates to α-coordinates (so, e.g., $(3,2)$ is mapped to $(1,0)$), and we can identify its matrix with the basis transition matrix ${}_\alpha S_\varepsilon$. Accordingly, the matrix for α^{-1} is ${}_\varepsilon S_\alpha$. Like we already observed in example 1.3.10, computing these transition matrices is quite easy:

$${}_\varepsilon S_\alpha = \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \quad \text{and} \quad {}_\alpha S_\varepsilon = {}_\varepsilon S_\alpha{}^{-1} = \frac{1}{13}\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}.$$

Using Theorem 1.3.15, we get

$$P_\varepsilon = {}_\varepsilon S_\alpha \, P_\alpha \, {}_\alpha S_\varepsilon = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Alternatively, the result could be obtained with the technique in remark 1.2.2: Put the information $\mathcal{P}\underline{a}_1 = \underline{a}_1$ and $\mathcal{P}\underline{a}_2 = \underline{0}$ in ε-coordinates in the rows of a matrix

$$\left(\begin{array}{cc|cc} 3 & 2 & 3 & 2 \\ 2 & -3 & 0 & 0 \end{array}\right) \xrightarrow{\text{row reduction}} \left(\begin{array}{cc|cc} 1 & 0 & \frac{9}{13} & \frac{6}{13} \\ 0 & 1 & \frac{6}{13} & \frac{4}{13} \end{array}\right)$$

yielding

$$P_\varepsilon = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Example 1.3.17 ***(Representation matrix for differentiation.)*** We look back to example 1.3.14, where we gave the representation matrix D_α w.r.t. basis $\alpha = \{1, x, x^2\}$. We will take a different basis, $\beta = \{x^2 - x, x^2 + 3, x^2 - 1\}$, and determine the matrix D_β of $\mathcal{D}$. We can now use:

$$D_\beta = {}_\beta S_\alpha \, D_\alpha \, {}_\alpha S_\beta \, .$$

Using $x^2 - x = 0 \cdot 1 - 1 \cdot x + 1 \cdot x^2$ and so on, we get

$$ {}_\alpha S_\beta = \begin{pmatrix} 0 & 3 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, $$

so

$$ {}_\beta S_\alpha = {}_\alpha S_\beta{}^{-1} = \frac{1}{4} \begin{pmatrix} 0 & -4 & 0 \\ 1 & 1 & 1 \\ -1 & 3 & 3 \end{pmatrix} $$

and

$$ D_\beta = {}_\beta S_\alpha \, D_\alpha \, {}_\alpha S_\beta = \frac{1}{4} \begin{pmatrix} -8 & -8 & -8 \\ 1 & 2 & 2 \\ 7 & 6 & 6 \end{pmatrix}. $$

As an exemplary sanity check, we look at $p(x) = 2x^2 - 3x + 5$: Since $2x^2 - 3x + 5 = 3(x^2 - x) + (x^2 + 3) - 2(x^2 - 1)$, p has β–coordinates $(3, 1, -2)$, and

$$ D_\beta \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -8 & -8 & -8 \\ 1 & 2 & 2 \\ 7 & 6 & 6 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -16 \\ 1 \\ 15 \end{pmatrix}. $$

These now should be the β–coordinates of the derivative of $p(x)$: $-4(x^2 - x) + \frac{1}{4}(x^2 + 3) + \frac{15}{4}(x^2 + 1) = 4x - 3 = p'(x)$. ✓

Chapter 2

'Simpler descriptions' for maps/matrices

What happens in Chapter 2?

A major reason why linear maps between finite-dimensional vector spaces are well-understood is because they can be analysed using specific values, called eigenvalues. The theory of eigenvalues allows to describe such maps via matrices that have a pleasantly simple form. Eigenvalue theory was first developed to study the rotational motion of rigid bodies, but as it turns out, they have much wider range of applications: you can find them, e.g., in vibration analysis to detect equipment faults, stability analysis, atomic theory, quantum mechanics, and facial recognition.

In Section 2.1, we introduce the theory of eigenvalues which is central to the course and will accompany us throughout. We proceed by slightly generalising the theory in Section 2.2, where we look at subspaces that are mapped into itself (so-called 'invariant' subspaces) and see how they also help simplify matrices. Based on the knowledge we gathered in these two sections, we'll develop some criteria in Section 2.3 that let us decide whether a quadratic matrix is diagonalisable. We finish this chapter with Section 2.4 that discusses the 'next best thing' to being diagonal, to deal with non-diagonalisable matrices.

Learning Goals of Chapter 2: When we are finished with Chapter 2, you can

- work with eigenvalues: you can
  - restate the definition of eigenvalues/vectors/spaces;
  - state their fundamental properties;
  - demonstrate your understanding of these concepts, e.g., by giving (counter)examples and deciding whether a particular value/vector is an eigenvalue/vector;
  - apply the discussed 'algorithm' to compute eigenvalues/vectors/spaces;
- work with invariant subspaces: you can
  - restate the definition of invariant subspaces;
  - demonstrate your understanding of the definition, e.g., by giving (counter)examples and deciding whether a particular subspace is invariant;

  - compute the restriction of a map unto such a subspace;
  - explain how invariant subspaces help to simplify the matrix of a map;
  - compute invariant subspaces for both linear maps and square matrices over real vector spaces;
- decide if a quadratic matrix is diagonisable;
- deal with the Jordan Normal Form of a matrix; and
- prove (simple) statements about all involved definitions/concepts.

Why do we care? In conclusion, all sections help us bring matrices into simpler form. As mentioned before, you might encounter the respective techniques throughout your studies, e.g., when studying numerical methods or differential equations. Chapters 3 and 4 serve as examples that illustrate how what we can use the techniques to simplify several mathematical problems.

2.1. Diagonalisation via eigenvalues

What will we do? Let $\mathcal{A} : V \to V$ be a linear map, mapping a finite-dimensional $\mathbb{K}$- vector space V into itself. So far, we saw that for every choice of a basis α for V, the map $\mathcal{A}$ is determined by a matrix A_α. We also saw the connection between the two matrices A_α and A_β for different bases α and β (in Theorem 1.3.15). In the two examples 1.3.14 and 1.3.16, we have already seen that the matrix A_α sometimes has a pleasantly simple form for a suitably chosen basis α. It turns out that there exist tricks to find such simple forms systematically, and this section develops these tricks/techniques. In more detail, we'll learn

- the most central concepts of the whole course, called eigenvalues, eigenvectors and eigenspaces;
- how these concepts help us bring maps into a 'pleasantly simple' form; and
- how to compute eigenvalues/vectors/spaces for any given linear map via an 'algorithm'.

Why do we care? In general, eigenvectors/-values characterise transformations and hence play an important role in all areas that apply linear algebra. Often, a system is represented by a linear transformation whose outputs are then re-purposed as new input. In this case, the largest eigenvalue has particular importance because it governs the long-term behaviour of the system (when applying the transformation many times), and the associated eigenvector is the steady state of the system.

We start by ***making precise what we mean by 'simple form'***:

Definition 2.1.1. *A square matrix A has* **diagonal form** *(or shorter,* **is diagonal***) if all elements a_{ij} with $i \neq j$ are equal to zero.*

Diagonality of a matrix can also be expressed through how it acts on bases:

Theorem 2.1.2. *Let $\mathcal{A} : V \to V$ be a linear map and let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for V. The matrix*

A_α has the diagonal form

$$A_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

if and only if $\mathcal{A}\underline{a}_i = \lambda_i \underline{a}_i$ for $i = 1, \dots, n$.

Proof. We consider both directions:

- $\Rightarrow$: If A_α has the diagonal form above, this means that $\mathcal{A}$ maps the α–coordinates of $\underline{a}_i$ (so $\underline{e}_i$) to $\lambda_i \underline{e}_i$, the α–coordinates of $\lambda_i \underline{a}_i$. In other words, $\mathcal{A}$ maps $\underline{a}_i$ to $\lambda_i \underline{a}_i$.
- $\Leftarrow$: The argument is quite similar to the $\Rightarrow$ direction, just the other way around: the i-th column of A_α is the image of $\underline{a}_i$, in α-coordinates. If $\mathcal{A}$ maps $\underline{a}_i$ to $\lambda_i \underline{a}_i$, then the i-th column of A_α is $\lambda_i \underline{e}_i$.

□

We are now ready to give the ***central definition of this course: eigenvalues and eigenvectors***.

Definition 2.1.3 (Eigenvector and eigenvalue). *Let $\mathcal{A} : V \to V$ be a linear map from a $\mathbb{K}$-vector space V to itself. A vector $\underline{v} \neq \underline{0}$ is called* **eigenvector** *of $\mathcal{A}$ with* **eigenvalue** *$\lambda \in \mathbb{K}$ if $\mathcal{A}\underline{v} = \lambda \underline{v}$. We denote the set of all eigenvalues of $\mathcal{A}$ by* $\mathrm{spec}(\mathcal{A})$ *and call it the* **spectrum** *of $\mathcal{A}$.*

The prefix 'eigen' is adopted from the German word for 'characteristic'/'own'. So 'eigenvalues/vectors' roughly means 'own' values/vectors. To explain this naming convention: the definition expresses that non-zero vectors are mapped unto a multiple of themselves (with the multiple being the eigenvalue), meaning they do not change direction. Or, in other words: ***they are mapped into their own linear span***.

You will re-encounter the spectrum in a more general setting, e.g., in the course on ordinary differential equations.

To get used to this definition, we first reformulate Theorem 2.1.2 in eigenvalue/-vector terms:

Theorem 2.1.4. *Let $\mathcal{A} : V \to V$ be a linear map with representation matrix A_α for basis α. A_α is in diagonal form if and only if α is a basis of eigenvectors. In that case, the diagonal elements are the eigenvalues.*

Example 2.1.5 ***(Example 1.3.16, cont'd)*** In example 1.3.16, we looked at the projection unto a line. The vectors $\underline{a}_1$ and $\underline{a}_2$ we defined there are eigenvectors: $\underline{a}_1$ has eigenvalue 1 (since line elements are not changed at all by projections) $\underline{a}_2$ has eigenvalue 0 (since the orthogonal complement is mapped to 0). The matrix w.r.t. the basis $\{\underline{a}_1, \underline{a}_2\}$ is therefore

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

a diagonal matrix with the eigenvalues on the diagonal. ***Notice the order of the two eigenvalues 1 and 0 on the diagonal***: this corresponds to the order of the eigenvectors.

Example 2.1.6 Consider in the euclidean plane E^2 a rotation around the origin by 90° . No vector different from $\underline{0}$ is mapped to a multiple of itself. So this linear map has no eigenvectors (let alone a basis of such). There is hence no choice of basis for which the rotation matrix could possibly be diagonal.

We give another ***central definition of this course: eigenspaces***, i.e., the spaces of eigenvectors for an eigenvalue.

Definition 2.1.7 (Eigenspace). *Let $\mathcal{A} : V \to V$ be a linear map from a $\mathbb{K}$-vector space V to itself. For any scalar $\lambda \in \mathbb{K}$, we denote*

$$E_\lambda := \mathcal{N}(\mathcal{A} - \lambda\mathcal{I}) \ .$$

Since null spaces are subspaces, E_λ is a subspace, called the **eigenspace** *of $\mathcal{A}$ for λ.*

Eigenspaces ***indeed are spaces of eigenvectors for a given eigenvalue***: E_λ is the null space of the linear map $\mathcal{A} - \lambda\mathcal{I}$. So any vector $\underline{v}$ lies in E_λ if and only if $(\mathcal{A} - \lambda\mathcal{I})\underline{v} = \underline{0}$, which is equivalent to $\mathcal{A}v - \lambda\underline{v} = \underline{0}$. Hence $\underline{v} \in E_\lambda$ if and only if $\mathcal{A}\underline{v} = \lambda\underline{v}$, so if and only if $\underline{v}$ is eigenvector for eigenvalue λ or $\underline{v} = \underline{0}$. (Note that we ruled out $\underline{0}$ as a possible eigenvector by definition).

A few more remarks on the definition. E_λ only is interesting if λ is an eigenvalue in the other case, E_λ only contains $\underline{0}$. The other way around, we say that λ is an eigenvalue if and only if E_λ contains a non-trivial vector, that is, a vector $\underline{v} \neq \underline{0}$. (So if and only if $\dim(E_\lambda) > 0$.) Some authors only use the word eigenspace if λ is an eigenvalue.

We can also write the null space of $\mathcal{A}$ as an eigenspace: E_0 consists of vectors that are mapped to 0 times itself, so on $\underline{0}$. So E_0 is the null space of $\mathcal{A}$.

2.1.1 Computing eigenvalues and -spaces

We will need to compute eigenvalues, eigenvectors and eigenspaces a lot during this course. So how do we do this? Assume you're given a map $\mathcal{A}$, and you're tasked with the following two things:

- determine the eigenvalues, i.e., all values λ such that $\dim(E_\lambda) > 0$; and
- find the eigenvectors for such a λ, i.e., the vectors in E_λ different from $\underline{0}$.

To tackle this problem, we'll use the following theorem:

Theorem 2.1.8. *λ is an eigenvalue if and only if* $\det(A - \lambda\mathcal{I}) = 0$. *Let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for V, and let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \ldots & \ldots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. this basis. Then the eigenvectors for eigenvalue λ, in α-coordinates, are the non-zero solutions of the system

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} . \tag{2.1}$$

Proof. We fix a value $\lambda \in \mathbb{K}$. When looking for eigenvectors for λ, we are concerned with solutions to the system $(\mathcal{A}-\lambda\mathcal{I})\underline{v} = \underline{0}$. The matrix of $\mathcal{A}-\lambda\mathcal{I}$ w.r.t. basis α is

$$A-\lambda\mathcal{I} = \begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn}-\lambda \end{pmatrix} .$$

et $\underline{v}$ be a vector in V, written in α-coordinates as $\underline{v} = v_1\underline{a}_1 + \dots + v_n\underline{a}_n$. Per definition, $\underline{v}$ is eigenvector for eigenvalue λ if and only if $\underline{v} \neq \underline{0}$ and $(\mathcal{A}-\lambda\mathcal{I})\underline{v} = \underline{0}$, so if and only if $(v_1,\dots,v_n) \neq (0,\dots,0)$ and

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

This means that there exist eigenvectors for λ if and only the dimension of the solution space of the homogeneous system $(A-\lambda\mathcal{I})\underline{x} = \underline{0}$ is bigger than 0. This is the case if and only if the rank of $A-\lambda\mathcal{I}$ is less than n, which happens if and only if $\det(A-\lambda\mathcal{I}) = 0$.

If λ is an eigenvalue, then the solutions of the system Eq. (2.1) give us the coordinate vectors of the elements of E_λ. □

Example 2.1.9 ***(Example 1.3.16, cont'd)*** In example 2.1.5, the continuation of 1.3.16, we saw that the vectors $\underline{a}_1$ and $\underline{a}_2$ are eigenvectors of the considered projection $\mathcal{P}$ belonging to eigenvalues 1 and 0, and that the matrix for $\mathcal{P}$ w.r.t. the basis $\{\underline{a}_1, \underline{a}_2\}$ is

$$P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

We'll now use Theorem 2.1.8 to determine all eigenvalues of the projection and their respective eigenspaces. We first use the determinant criterion to find the eigenvalue candidates:

$$\det(P-\lambda\mathcal{I}) = \begin{pmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{pmatrix} = -\lambda(1-\lambda) ,$$

which is 0 if and only if λ is 0 or 1, meaning the only possible eigenvalues are the ones we already found. To determine E_1 in α-coordinates, we solve the corresponding homogeneous

solution, plugging in 1 for λ:

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ -v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So the eigenvectors for eigenvalue 1 are the vectors of the form $\underline{v} = v_1 \cdot \underline{a}_1 + 0 \cdot \underline{a}_2$ for arbitrary scalars v_1. We can put this shorter (and get rid of the coordinate description): $E_1 = \text{span}(\underline{a}_1)$.
To determine E_0 in α-coordinates, we do essentially the same thing, plugging in 0 for λ:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

yields $\underline{v} = 0 \cdot \underline{a}_1 + v_2 \cdot \underline{a}_2$ for arbitrary scalars v_2, so $E_0 = \text{span}(\underline{a}_2)$.

Example 2.1.10 ***(Example 2.1.9, generalised)*** If you feel nerdy, you can try to generalise the result of example 2.1.9 by proving the following claim: If $\mathcal{P}$ is a projection unto a line ℓ in $\mathbb{R}^2$, then $\mathcal{P}$ has exactly the two eigenvalues 0 and 1, and the eigenspaces are $E_1 = \ell$ and $E_0 = \ell^\perp$. (The general case works because you always can get a basis like in example 2.1.9 by picking one vector from ℓ and one from its complement.)

We translate the observation made in Theorem 2.1.8 into a method to compute eigenspaces from a given eigenvalue, by means of an 'algorithm':

Algorithm 2.1.11 [Computing the eigenspace for a given eigenvalue, as a linear span]

Input:	Map $\mathcal{A}$, eigenvalue λ, basis α
Output:	Eigenspace $E_\lambda(A)$ in span() form
Step 1:	Compute matrix A_α of $\mathcal{A}$ w.r.t α
Step 2:	Compute space of solutions to the equation $(A_\alpha - \lambda\mathcal{I})\underline{v} = 0$
	// Solution to $(A_\alpha - \lambda\mathcal{I})\underline{v} = 0$ = coordinates of a vector in E_λ.
	Row-reduce matrix $(A_\alpha - \lambda\mathcal{I})$ (if necessary)
	Solve $(A_\alpha - \lambda\mathcal{I})\underline{v} = 0$
	Find $\text{span}(\underline{w}_1, \ldots, \underline{w}_n)$ description of solutions
	// (like in ex. 2.1.9)
Step 3:	Compute eigenspace $E_\lambda(\mathcal{A})$ from coordinate vectors $\{\underline{w}_1, \ldots, \underline{w}_n\}$
	// $\{\underline{w}_1, \ldots, \underline{w}_n\}$ are still coordinate vectors.
	for $i = 1$ to n
	$\underline{v}_i := \alpha^{-1}(\underline{w}_i)$ // Translate $\underline{w}_i$ into vector $\underline{v}_i$ in V
Step 4:	output '$E_\lambda(\mathcal{A}) = \text{span}(\underline{v}_1, \ldots, \underline{v}_n)$'

In Theorem 2.1.8, the polynomial $\det(A - \lambda\mathcal{I})$ played a central role in determining possible eigenvalues. As we will develop and use many nice results about this function, it gets its own name:

Definition 2.1.12 (Characteristic polynomial). *Let $\mathcal{A} : V \to V$ be a linear map and let A_α be the matrix of $\mathcal{A}$ w.r.t. a basis α. We call the equation $\det(A_\alpha - \lambda\mathcal{I}) = 0$ the* **characteristic equation** *of A_α, and the left-hand side of this equation, $\det(A_\alpha - \lambda\mathcal{I})$, the* **characteristic polynomial** *of A_α.*

We will also call these objects the characteristic equation/polynomial of $\mathcal{A}$, and denote the characteristic polynomial by $\chi_{\mathcal{A}}$.

Looking at this definition, you might ask yourself whether the term $\chi_{\mathcal{A}}$ is well-defined - the way we defined it is relative to a basis. ***The following theorem states that our naming convention is okay because $\chi_{\mathcal{A}}$ actually doesn't depend on the choice of basis.***

Theorem 2.1.13. *Let $\mathcal{A} : V \to V$ be a linear map, let α and β be two bases for V, and let A_α/A_β be the matrix of $\mathcal{A}$ w.r.t. α/β. Then* $\det(A_\alpha - \lambda\mathcal{I}) = \det(A_\beta - \lambda\mathcal{I})$.

Proof. We prove the equation by using the 'coordinate-switching relation' between A_α and A_β, rewriting $\mathcal{I}$, and using that the determinant is multiplicative:

$$\begin{aligned}
\det(A_\beta - \lambda\mathcal{I}) &= \det({}_\beta S_\alpha A_{\alpha\alpha} S_\beta - \lambda {}_\beta S_{\alpha\alpha} S_\beta) \\
&= \det({}_\beta S_\alpha (A_\alpha - \lambda\mathcal{I})_\alpha S_\beta) && = \det({}_\beta S_\alpha)\det(A_\alpha - \lambda\mathcal{I})\det({}_\alpha S_\beta) \\
&= \det(A_\alpha - \lambda\mathcal{I})\det({}_\beta S_{\alpha\alpha} S_\beta) && = \det(A_\alpha - \lambda\mathcal{I})\det(\mathcal{I}) \\
&= \det(A_\alpha - \lambda\mathcal{I}) \ .
\end{aligned}$$

□

We now make some useful observations about how the characteristic polynomial looks:

Theorem 2.1.14. *Let $\mathcal{A} : V \to V$ be a linear map on a vector space V of dimension n, and let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. some basis. Then the characteristic polynomial $\chi_{\mathcal{A}}$ is a polynomial of degree (exactly) n, and of the following shape:

$$\chi_{\mathcal{A}} = (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

for some coefficients $c_0, c_1, \ldots \in \mathbb{K}$.

Proof. We know that the characteristic polynomial $\chi_{\mathcal{A}}$ is the determinant

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} - \lambda \end{vmatrix} .$$

This determinant is the sum of $n!$ many terms, with every summand being a product of n many matrix elements (one from each row and one from each column). We first reason that the degree of

$\chi_{\mathcal{A}}$ is at most n: In each summand, λ can pop up at most n many times. Therefore, each summand in this sum is a polynomial in λ of degree at most n.

Now, we look at the terms in $\chi_{\mathcal{A}}$ that have the highest degree. When we look at the product of the elements on the main diagonal, λ pops up in every single factor :

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(a_{11}+a_{22}+\cdots+a_{nn})\lambda^{n-1}+\cdots \ .$$

The remaining summands in $\chi_{\mathcal{A}}$ all contain an element a_{ij} with $i \neq j$. In these summands, the diagonal elements $(a_{ii}-\lambda)$ and $(a_{jj}-\lambda)$ therefore do not occur, so what is left are polynomials of degree at most $n-2$.The terms in $\chi_{\mathcal{A}}$ that have degree n and $n-1$ hence are the terms from the main diagonal product, and we see that $\chi_{\mathcal{A}}$ has the claimed shape. □

Definition 2.1.15 (Trace). *The sum of the diagonal elements of a square matrix A is called the* **trace** *of the matrix A. We denote it by* $\mathrm{tr}(A)$.

Using Theorem 2.1.14, we now show that trace and determinant are invariant under the choice of basis, and that we can read out trace and determinant of $\mathcal{A}$ from the characteristic polynomial $\chi_{\mathcal{A}}$:

Definition/theorem 2.1.16. *Let $\mathcal{A} : V \to V$ be a linear map with $\dim(V) < \infty$. For every basis α, the matrix A_α*

1. *has the same trace, which we therefore also call the* **trace of** *$\mathcal{A}$ and denote by* $\mathrm{tr}(\mathcal{A})$. *We have the identity $\mathrm{tr}(\mathcal{A}) = c_{n-1}$, where c_{n-1} is the second-highest coefficient in the characteristic polynomial $\chi_{\mathcal{A}}$.*
2. *has the same determinant, which we therefore also call the* **determinant of** *$\mathcal{A}$. We have the identity $\det(\mathcal{A}) = c_0$, where c_0 is the constant coefficient in the characteristic polynomial $\chi_{\mathcal{A}}$.*

Proof. To prove 1., we recall that the trace of a map's matrix A_α is exactly c_{n-1}, the second-highest coefficient of $\chi_{\mathcal{A}}$ (according to Theorem 2.1.14). But Theorem 2.1.13 already told us that $\chi_{\mathcal{A}}$ does not depend on the choice of basis, so this applies in particular to c_{n-1}.

To prove 2., we plug 0 into $\chi_{\mathcal{A}}$ and get

$$\chi_{\mathcal{A}}(0) = c_0 \ .$$

At the same time,

$$\chi_{\mathcal{A}}(0) = \det(\mathcal{A} - 0\mathcal{I}) = \det(\mathcal{A}) \ .$$

□

To make the previous observation more useful, let's look at the special case where V is a $\mathbb{C}$- or $\mathbb{R}$-vector space. In that case, the characteristic polynomial $\chi_{\mathcal{A}}$ is a polynomial with coefficients in $\mathbb{C}$. Due to the fundamental theorem of algebra, we know that the characteristic polynomial $\chi_{\mathcal{A}}(\lambda)$ then – like any other polynomial – completely factors into linear terms $\lambda - \lambda_i$, where the constant terms λ_i are the zeros (or 'roots') of $\chi_{\mathcal{A}}$. (See Theorem B.1.6 and the subsequent discussion if you do not recall this.) We can relate these roots to the trace and the determinant of matrices:

Theorem 2.1.17. *Let A be a square matrix with entries in $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, with characteristic polynomial χ_A. Then the*

- *trace of the matrix is the sum of the roots of χ_A; and the*
- *determinant of the matrix is the product of the roots of χ_A.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the n roots (in $\mathbb{C}$, with multiplicity) of the characteristic polynomial χ_A. Then we have the following identity:

$$\begin{aligned}\chi_A &= (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \\ &= (-1)^n\lambda^n + (-1)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n)\lambda^{n-1} + \cdots + \lambda_1\lambda_2\cdots\lambda_n \ .\end{aligned}$$

□

Why is this useful? Because it tells us that we can compute eigenvalues of a map $\mathcal{A}$ by simply looking for the roots of $\chi_{\mathcal{A}}$, and taking the roots that are in the right field. We summarise the techniques we saw so far in the 'algorithm' below.

Algorithm 2.1.18 [Computing the eigenvalues and -spaces of a map when $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$]

Input:	Map $\mathcal{A}$
Output:	Eigenvalues $\lambda_1, \ldots, \lambda_n$, eigenspaces $E_{\lambda_1}(A), \ldots, E_{\lambda_n}(A)$
Step 1:	Pick a basis α
Step 2:	Compute matrix A_α of $\mathcal{A}$ w.r.t α
Step 3:	Determine eigenvalues: Compute characteristic polynomial $\chi_{\mathcal{A}} = \det(A_\alpha - \lambda\mathcal{I})$. $\lambda_1, \ldots, \lambda_n :=$ the zeros of polynomial $\chi_{\mathcal{A}}$ that are in $\mathbb{K}$. *// In a real vector space, only the real roots are eigenvalues.* *// In a complex space, every root is an eigenvalue.*
Step 4:	for $1 \le i \le n$, compute eigenspace $E_{\lambda_i}(\mathcal{A})$: Algorithm 2.1.11 yields description of $E_{\lambda_i}(\mathcal{A})$.
Step 5:	output $\lambda_1, \ldots, \lambda_n$ and $E_{\lambda_1}(\mathcal{A}), \ldots, E_{\lambda_n}(\mathcal{A})$

⚠ There will sometimes be smarter ways to compute eigenvalues/-spaces. Algorithm 2.1.18 simply is a 'sledgehammer' solution that always works.

We will proceed with some application examples for algorithm 2.1.18. But first, we briefly recall why we care about finding eigenvalues/spaces: as seen in Theorem 2.1.2, a basis of eigenvectors allows to describe the map via a matrix that is pleasantly simple – the respective matrix is a diagonal matrix that has the eigenvalues on the diagonal. (In the same order as the corresponding eigenvectors are ordered in the basis). This will be reflected in examples 2.1.19, 2.1.21 and 2.1.22 below.

Computing eigenvalues and -spaces: examples

Example 2.1.19 ***(Revisiting projection example 1.3.16)*** We once more consider the projection unto the line $\ell : 2x-3y=0$ in $\mathbb{R}^2$ with the standard inner product. In example 2.1.9, we already determined the eigenvalues, as well as the eigenspaces in coordinates relative to an eigenvector basis. We could now simply translate the coordinates back into standard basis terms using the basis transition matrix ${}_\varepsilon S_\alpha$.
But this approach cheated in a way - we already started from an eigenvector basis to get the eigenspaces! We will now practice with algorithm 2.1.18 to practice determining eigenvalues/-spaces without knowing a nice eigenvector basis. Let's pick the standard basis.
In example 1.3.16, we saw that the matrix of this projection (in standard coordinates) is

$$P_\varepsilon = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \ ,$$

the characteristic equation therefore is

$$\begin{vmatrix} \frac{9}{13}-\lambda & \frac{6}{13} \\ \frac{6}{13} & \frac{4}{13}-\lambda \end{vmatrix} = \frac{1}{169}\begin{vmatrix} 9-13\lambda & 6 \\ 6 & 4-13\lambda \end{vmatrix}$$

$$= \frac{1}{169}\Big((9-13\lambda)(4-13\lambda)-36\Big) = \lambda^2-\lambda = 0\,.$$

The characteristic equation has two roots: $\lambda_1 = 1$ and $\lambda_2 = 0$. (Nerdy as we are, we make the side observation that the trace of the matrix is $1 = \lambda_1+\lambda_2$ and that the determinant is $0 = \lambda_1\cdot\lambda_2$, exactly as promised by Theorem 2.1.17.)
We find the eigenspace E_1 for $\lambda_1 = 1$ by analysing the system of equations with coefficient matrix $P_\varepsilon - 1\cdot\mathcal{I}$:

$$\left(\begin{array}{cc|c} \frac{9}{13}-1 & \frac{6}{13} & 0 \\ \frac{6}{13} & \frac{4}{13}-1 & 0 \end{array}\right) \sim \begin{pmatrix} -4 & 6 \\ 6 & -9 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}\,.$$

So, the eigenspace E_1 satifies the equation $2x-3y=0$ (the equation of ℓ, no big surprise). All solutions (x,y) are therefore multiples of $(3,2)$, so

$$E_1 = \operatorname{span}\big(3\underline{e}_1+2\underline{e}_2\big) \quad (=\ell)\,.$$

The eigenspace E_0 for $\lambda = 0$ can be found from the system

$$\begin{pmatrix} \frac{9}{13}-0 & \frac{6}{13} \\ \frac{6}{13} & \frac{4}{13}-0 \end{pmatrix} \sim \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}.$$

So, the eigenspace E_0 satifies the equation $3x+2y=0$, meaning all solutions (x,y) are multiples of $(2,-3)$, so

$$E_0 = \operatorname{span}\big(2\underline{e}_1-3\underline{e}_2\big) \quad (=\ell^\perp)\,.$$

We can now build an eigenvector basis α by taking $\underline{a}_1 = 3\underline{e}_1+2\underline{e}_2$ and $\underline{a}_2 = 2\underline{e}_1-3\underline{e}_2$, and setting $\alpha := \{\underline{a}_1,\underline{a}_2\}$. W.r.t. the eigenvector basis α, the projection's matrix has the diagonal form

$$P_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

that has the eigenvalues on the diagonal. Just like the matrix P_ε, matrix P_α has trace 1 and determinant 0 (in accordance with theorems 2.1.16 and 2.1.17).

Example 2.1.20 ***(Rotation)*** We choose in the euclidean plane E^2 an orthonormal basis $\{\underline{e}_1, \underline{e}_2\}$ and consider a rotation by 90°, sending $\underline{e}_1$ to $\underline{e}_2$ and $\underline{e}_2$ to $-\underline{e}_1$ (so $\mathcal{A}\underline{e}_1 = \underline{e}_2$ and $\mathcal{A}\underline{e}_2 = -\underline{e}_1$). W.r.t. basis $\{\underline{e}_1, \underline{e}_2\}$, the respective matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The characteristic equation is

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \ ,$$

which has the two roots $\lambda = i$ and $\lambda = -i$. (We furthermore note that the trace is 0 and the determinant is 1.) Since E^2 is a real vector space, neither i nor $-i$ can be an eigenvalue. So we have shown that this rotation has no eigenvectors (with an alternative method to example 2.1.6).

The next example covers the case where we the map does have an eigenvalue, but not enough linearly independent eigenvectors to be brought into diagonal shape.

Example 2.1.21 ***(Polynomial differentiation)*** Let V be the vector space of real polynomials of degree at most two, and let $\mathcal{D}$ be the differentiation map, so $\mathcal{D} : V \to V$ is defined by $\mathcal{A}p = p'$. We pick the basis $\alpha = \{1, x, x^2\}$. We saw in example 1.3.14 that w.r.t. α, $\mathcal{D}$ has the matrix

$$D_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} .$$

The characteristic equation is therefore

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 = 0 \, .$$

The only root of the characteristic polynomial is $\lambda = 0$ with multiplicity 3; the trace is 0 and the determinant is 0. We find the eigenspace E_0 by solving the homogeneous system with coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} .$$

All solutions to this system of equations (in α-coordinates) are multiples of $(1,0,0)$, so

$$E_0 = \operatorname{span}\left(1 \cdot 1 + 0 \cdot x + 0 \cdot x^2\right) = \operatorname{span}(1).$$

E_0 therefore consists of the constant polynomials, and indeed, constant polynomials have the zero polynomial as derivative. We note that $\dim(E_0) = 1$, even though $\lambda = 0$ has multiplicity 3. Since $\lambda = 0$ is the only root of the characteristic polynomial (and therefore the only eigenvalue candidate), we cannot even find two linearly independent eigenvectors (let alone 3). Since we do not get an eigenvector basis, the differentiation has no diagonal form.

Example 2.1.22 We define a linear map $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ via the matrix

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

In this example, we work w.r.t. the standard basis $\varepsilon = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$. In this case, we can identify vectors and coordinates. The characteristic equation is:

$$\begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0, \quad \text{so} \quad -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0.$$

This polyomial factors as $-(\lambda-9)(\lambda-2)^2$. So there are two eigenvalues, $\lambda = 9$ and $\lambda = 2$, the last one with multiplicity 2. (The trace is indeed $9+2+2 = 13$ and the determinant is 36. Check this.)
We find the eigenspace E_9 from the system of equations with matrix

$$\begin{pmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has as solutions the multiples of $(1,1,1)$. So $E_9 = \text{span}\,((1,1,1))$.
The eigenspace E_2 we find from the equations with matrix

$$\begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix},$$

so E_2 is the plane $2x - y + 6z = 0$, i.e., $E_2 = \text{span}\,((1,2,0),(0,6,1))$.
We can collect all base vectors into one basis α: $\alpha = \{(1,1,1),(1,2,0),(0,6,1)\}$ is a basis of eigenvectors. W.r.t. eigenvector basis α, we find that the respective matrix is

$$A_\alpha = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so a diagonal matrix with the eigenvalues on the diagonal (and of course with trace 13 and determinant 36).

2.1.2 Linear independence of eigenvectors

In examples 2.1.19 and 2.1.22, the maps' matrices were diagonalisable because – going over the different eigenvalues – we found a complete basis of eigenvectors. We will now answer the question whether this always works, assuming the eigenspaces gift us with enough vectors (unlike in example 2.1.21).

More concretely, say the dimensions d_λ of the different eigenspaces E_λ add up to the dimension d of the vector space. Then collecting d_λ many linearly independent vectors per eigenspace E_λ would yield a generating system of size d. This is the right size for us to hope we found a complete eigenvector basis. But maybe eigenvectors belonging to different eigenvalues could be linearly dependent, meaning the generating system has a span of dimension smaller than d and thus does not span the whole vector space?

Luckily, the following theorem tells us that we do not have to worry about this: it shows that ***systems of eigenvectors for mutually different eigenvalues*** $\lambda_1,\dots,\lambda_n$ ***are always linearly independent, meaning they always form an eigenvector basis (for a subspace of dimension*** n***).*** We will use this result in Section 2.3.1 to give a diagonalisability criterion which allows us to decide when certain maps are diagonalisable.

> **Theorem 2.1.23.** *Let $\mathcal{A}: V \to V$ be a linear map and let $\underline{v}_1,\dots,\underline{v}_n$ be eigenvectors of $\mathcal{A}$ for* **mutually different** *eigenvalues $\lambda_1,\dots,\lambda_n$. Then $\underline{v}_1,\dots,\underline{v}_n$ are independent.*

Proof. Suppose $\underline{v}_1,\dots,\underline{v}_n$ are not independent, meaning there is a vector depending on the others. We may assume that $\underline{v}_1$ is dependent of $\underline{v}_2,\dots,\underline{v}_n$ (if necessary, we simply renumber the vectors).

We will now use the eigenvector properties to show that $\underline{v}_1$ must be $\underline{0}$. This yields a contradiction (because $\underline{v}_1$ is an eigenvector and thus cannot be $\underline{0}$), hence there cannot be any dependent vector and we're done with the proof (as soon as we have shown that $\underline{v}_1$ indeed must be $\underline{0}$).

We first prune $\underline{v}_2,\dots,\underline{v}_n$ to an independent system $\underline{v}_2,\dots,\underline{v}_p$ (to accomplish this, we might have to renumber the vectors again). After this, we have $\operatorname{span}\left(\underline{v}_2,\dots,\underline{v}_p\right) = \operatorname{span}\left(\underline{v}_2,\dots,\underline{v}_n\right)$.

Since $\underline{v}_1$ is dependent of $\underline{v}_2,\dots,\underline{v}_n$ and since $\operatorname{span}\left(\underline{v}_2,\dots,\underline{v}_p\right) = \operatorname{span}\left(\underline{v}_2,\dots,\underline{v}_n\right)$, we know that there exists some linear combination such that

$$\underline{v}_1 = \sum_{i=2}^{p} \alpha_i \underline{v}_i \ .$$

We now multiply by eigenvalue λ_1:

$$\lambda_1 \underline{v}_1 = \sum_{i=2}^{p} \alpha_i \lambda_1 \underline{v}_i \ \text{, but at the same time,}$$
$$\lambda_1 \underline{v}_1 = \mathcal{A}\underline{v}_1 = \mathcal{A}(\sum_{i=2}^{p} \alpha_i \underline{v}_i) = \sum_{i=2}^{p} \alpha_i \mathcal{A}\underline{v}_i = \sum_{i=2}^{p} \alpha_i \lambda_i \underline{v}_i \ ,$$

where the last line used the eigenvector properties of the different vectors. Subtracting the two equations from another, we see that

$$\sum_{i=2}^{p} \alpha_i (\lambda_i - \lambda_1) \underline{v}_i = \underline{0}.$$

Since $\underline{v}_2, \ldots, \underline{v}_p$ are independent and $\lambda_i - \lambda_1 \neq 0$ for $i = 2, \ldots, p$, it follows that $\alpha_i = 0$ for $i = 2, \ldots, p$, meaning $\underline{v}_1 = \underline{0}$. □

2.1.3 Diagonalisation of a square matrix

So far, we have developed methods to represent a linear map by a diagonal matrix, with the starting point being a linear map. But we can also start from a square matrix A, by viewing A as the matrix of a linear map $\mathbb{K}^n \to \mathbb{K}^n$ (using the standard basis ε) and looking for a basis α of eigenvectors (provided such a basis exists). Collecting the eigenvalues in a diagonal matrix D, we obtain a diagonal form of the matrix A: We have

$$D = {}_\alpha S_\varepsilon \, A_\varepsilon S_\alpha \quad \text{, or, equivalently,}$$
$$A = {}_\varepsilon S_\alpha \, D_\alpha S_\varepsilon \quad .$$

This procedure is called ***diagonalising the matrix*** A .

2.2. Using subspaces that are stable under the mapping (invariant subspaces)

What will we do? In the previous section, we saw that if $\alpha = \{\underline{v}_1, \ldots, \underline{v}_n\}$ is an eigenvector basis for map $\mathcal{A}$, then A_α has the nice shape

$$A_\alpha = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} ,$$

where λ_i is the eigenvalue for $\underline{v}_i$, and we learnt how we can find such an eigenvector basis if it exists. But we also learnt from the examples in Section 2.1.1 that finding an eigenvector basis is not always possible, meaning we cannot always represent the map by a diagonal matrix. Nonetheless, it is often possible to at least break down the matrix into simpler blocks, which will be the subject of this section.

Intuitively, this section generalises the concept of an eigenspace by making its defining feature a bit more general: per definition, eigenvectors are mapped to a multiple of themselves. (Taking a geometrical perspective, the line spanned by an eigenvector is mapped into itself.) We will generalise this by looking at subspaces whose vectors might not necessarily be mapped unto multiples of ***themselves***, but rather stay in the subspace. We call such subspaces 'invariant' because the map will not move them outside of themselves.

As an example, imagine a rotation about an axis in $\mathbb{R}^3$ – neither will the axis be changed by the rotation, nor will the plane which is perpendicular to the rotation axis. Each of them is hence an invariant subspace. Looking ahead, we will get back to this important example in more detail in Chapter 3, and you will re-encounter invariant subspaces in the course on ordinary differential equations.

In more detail, we'll learn

- what invariant subspaces are;
- how we can use them to bring matrices into a simpler form; and
- for linear maps $\mathcal{A} : V \to V$ where V is real and the characteristic equation has a non–real root: how to get an invariant subspace from the root.

Why do we care? If a map has no (or not enough) eigenvectors, it might still have invariant subspaces, which will lead to simpler matrix forms. More concretely, we will see later in the course that we can use this section's techniques to bring certain types of linear maps into diagonal block shape, which will allow us to determine their geometric meaning even if the initial matrix looked super unwieldy.

Like in the section before, ***all maps considered in this section are maps from a finite-dimensional vector space into itself***, i.e., they are maps $\mathcal{A} : V \to V$ with $\dim(V) < \infty$.

We start with the ***central definition of this section: invariant subspaces***.

Definition 2.2.1 (Invariant subspace). *Let W be a subspace of V. W is called* **invariant under linear map** $\mathcal{A} : V \to V$ *if $\mathcal{A}\underline{w} \in W$ for all $\underline{w} \in W$.*

Example 2.2.2 ***(Null space and range)***

- The null space $\mathcal{N}$ of a linear map $\mathcal{A}$ is always invariant: if $\underline{x} \in \mathcal{N}$, then $\mathcal{A}\underline{x} = \underline{0}$, and $\underline{0}$ again belongs to $\mathcal{N}$.
- The range $\mathcal{R}$ is also always invariant: if $\underline{y} \in \mathcal{R}$, then $\mathcal{A}\underline{y}$ is obviously again contained in $\mathcal{R}$.

Example 2.2.3 ***(Counterexample: rotation in two-dimension space)*** Like in example 2.1.20, we take as map $\mathcal{A}$ a rotation by 90° in the euclidean plane E^2, so $\mathcal{A}\underline{e}_1 = \underline{e}_2$ and $\mathcal{A}\underline{e}_2 = -\underline{e}_1$. We set $W = \operatorname{span}(\underline{e}_1)$. Since $\mathcal{A}\underline{e}_1 = \underline{e}_2 \notin W$, W is not invariant.

Example 2.2.4 ***(Eigenspaces)*** We will now show eigenspaces are invariant. Say λ is an eigenvalue of a linear map $\mathcal{A}$, and let $\underline{v} \in E_\lambda$. We need to show that $A\underline{v}$ is also in E_λ, which is true because $A(A\underline{v}) = A(\lambda\underline{v}) = \lambda A\underline{v}$.

Fortunately, ***we do not need to check the invariance criterion for each single vector of*** W – it is enough if we check it on a basis of W:

Theorem 2.2.5. *Let $\mathcal{A} : V \to V$ be linear and let $W = \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_n)$. W is invariant under $\mathcal{A}$ if and only if $\mathcal{A}\underline{a}_i \in \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_n)$ for $i = 1, \ldots, n$.*

Proof. If W is invariant, then $\mathcal{A}\underline{w} \in W = \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_n)$ for all $\underline{w} \in W$, so this is true in particular for $\underline{w} = \underline{a}_1, \ldots, \underline{a}_n$.

Conversely, suppose $\mathcal{A}\underline{a}_i \in W$ for $i = 1, \ldots, n$. Now take an arbitrary $\underline{w} \in W$, which we can write as $\underline{w} = w_1\underline{a}_1 + \cdots + w_n\underline{a}_n$. Then $\mathcal{A}\underline{w} = w_1\mathcal{A}\underline{a}_1 + \cdots + w_n\mathcal{A}\underline{a}_n$. Since every $\mathcal{A}\underline{a}_i \in W$ and W is a linear subspace, $\mathcal{A}\underline{w} \in W$. □

The next definition reflects that for invariant subspaces, we can essentially 'forget' about the rest of the vector space. We call this forgetting about the rest 'restricting':

Definition 2.2.6 (Restriction unto an invariant subspace). *If W is invariant under $\mathcal{A}$, then all image vectors $\mathcal{A}\underline{w}$ with $\underline{w} \in W$ are again in W. So if we restrict $\mathcal{A}$ to W, we obtain a well-defined linear map $W \to W$, the* **restriction of the map** *$\mathcal{A}$ unto W, which we sometimes denote by $\mathcal{A}_{|W}$.*

Invariant spaces give us a simpler matrix shape, because the matrix contains a block for the restriction:

Theorem 2.2.7. *Suppose $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ is a basis for V such that $W = \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_m)$ is invariant under $\mathcal{A}$. Then the matrix A_α has the following form*

$$A_\alpha = \begin{pmatrix} & * & \ldots & * \\ M_1 & \vdots & & \vdots \\ 0 \ldots 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 \ldots 0 & * & \ldots & * \end{pmatrix},$$

The $m \times m$–matrix M_1 is the matrix of the restriction $\mathcal{A}_{|W} : W \to W$ w.r.t. the basis $\{\underline{a}_1, \ldots, \underline{a}_m\}$.

Proof. For each $i = 1, \ldots, m$, we have that $\mathcal{A}\underline{a}_i \in W = \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_m)$, so for each $i = 1, \ldots, m$, $\mathcal{A}\underline{a}_i$ can be written as the linear combination

$$\mathcal{A}\underline{a}_i = a_{i1}\underline{a}_1 + \cdots + a_{im}\underline{a}_m + 0\underline{a}_{m+1} + \cdots + 0\underline{a}_n \ .$$

Thinking back to what this tells us about the matrix A_α (see Section 1.3.3, the i–th column of A_α consists of the α–coordinates of $\mathcal{A}\underline{a}_i$), we know that the i–th column of A_α is the (vertical) coordinate vector $(a_{i1}, \cdots a_{im}, 0, \cdots, 0)$, so the first m many columns of A_α have only 0s below the m–th row.

The restriction matrix M_1 is exactly the m-by-m-matrix that only concerns the m-dimensional subspace W. It therefore consists of the aforementioned columns, when removing the 0s below the m–th entry (because the vectors $\underline{a}_{m+1}, \ldots, \underline{a}_n$ are not in W and hence do not show up). □

Example 2.2.8 ***(Proving invariance and analysing a map without even knowing its full map description)***
Consider in $\mathbb{R}^4$ the (independent) vectors

$$\underline{a} = (1, -1, 1, -1) \text{ and } \underline{b} = (1, 1, 1, 1).$$

Say we have a linear map $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^4$ of which we only know that

$$\mathcal{A}\underline{a} = (4, -6, 4, -6) \text{ and } \mathcal{A}\underline{b} = (4, 6, 4, 6) \ .$$

Even without knowing the full description of $\mathcal{A}$, we will now show that $W = \operatorname{span}(\underline{a}, \underline{b})$ is invariant and determine a matrix of the restriction unto W $\mathcal{A}_{|W} : W \to W$.

To show the invariance of $\operatorname{span}(\underline{a},\underline{b})$, we must verify that $\mathcal{A}\underline{a}$ and $\mathcal{A}\underline{b}$ are linear combinations of $\underline{a}$ and $\underline{b}$. We do this by simultaneously solving the systems of equations with columns $\underline{a},\underline{b},\mathcal{A}\underline{a},\mathcal{A}\underline{b}$:

$$\left(\begin{array}{cc|cc} 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & 6 \\ 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & 6 \end{array}\right).$$

After row reduction and deleting zero rows, the system reduces to

$$\left(\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & -1 & 5 \end{array}\right),$$

which tells us that $\mathcal{A}\underline{a} = 5\underline{a} - \underline{b}$ and $\mathcal{A}\underline{b} = -\underline{a} + 5\underline{b}$. So $W = \operatorname{span}(\underline{a},\underline{b})$ is invariant under $\mathcal{A}$. This also tells us how the matrix of the restriction $\mathcal{A}_{|W} : W \to W$ w.r.t. the basis $\{\underline{a},\underline{b}\}$ looks (we can simply read it out from the system): the matrix is

$$\begin{pmatrix} 5 & 1 \\ -1 & 5 \end{pmatrix}.$$

At a first glance, this might look weird to you – the vector space has 4 dimensions, so how did we end up with a 2-dimensional matrix? This is because the subspace W has only two dimensions, and because the restriction matrix is given with respect to ***the coordinates*** of the two basis vectors $\underline{a}$ and $\underline{b}$. (Remember that the coordinate vector of $\underline{a}$ with respect to the basis $\{\underline{a},\underline{b}\}$ simply is $\underline{e}_1$, the first unity vector in $\mathbb{R}^2$.)
Using the restriction matrix, we can now even ***determine some eigenvectors without knowing the full map***: Using algorithm 2.1.18, we find that the matrix has eigenvalues 4 and 6. In coordinates, we compute the respective eigenspaces as $\operatorname{span}((1,1))$ (for eigenvalue 4) and $\operatorname{span}((1,-1))$ (for eigenvalue 6). In this basis, the restriction map is simply the diagonal map with the eigenvalues on the diagonal (as usual): it is

$$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

We transform the coordinate vectors back into elements of $\mathbb{R}^4$: $\underline{a}+\underline{b} = (2,0,2,0)$ and $\underline{a}-\underline{b} = (0,-2,0,-2)$. So the eigenvector basis of W is $\{(2,0,2,0),(0,2,0,2)\}$.
We now can simplify the representation of the full map: If we pick any basis α of $\mathbb{R}^4$ such that the first two basis vectors are the eigenvectors $(2,0,2,0)$ and $(0,2,0,2)$, then the full matrix has the shape

$$A_\alpha = \begin{pmatrix} & & * & \dots & * \\ \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} & & \vdots & & \vdots \\ & & & & \\ 0 \dots 0 & & \vdots & & \vdots \\ \vdots\!\vdots & & \vdots & & \vdots \\ 0 \dots 0 & & * & \dots & * \end{pmatrix},$$

Using Theorem 2.2.7, we see that ***the characteristic polynomial of a restriction always divides the***

characteristic polynomial of the larger map:

Theorem 2.2.9. *If W is an invariant subspace for the linear map $\mathcal{A}: V \to V$, then $\chi_{\mathcal{A}_{|W}}$, the characteristic polynomial of $\mathcal{A}$'s restriction unto W, $\mathcal{A}_{|W}: W \to W$, is a factor of $\chi_{\mathcal{A}}$, the characteristic polynomial of the map $\mathcal{A}: V \to V$.*

Proof. Let W be an invariant subspace for linear map $\mathcal{A}: V \to V$. Pick a basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ for V in a way such that $\{\underline{a}_1, \ldots, \underline{a}_m\}$ is a basis for W. Then Theorem 2.2.7 tells us how A_α looks, and subtracting $\lambda\mathcal{I}$ from A_α yields

$$A_\alpha - \lambda\mathcal{I} = \begin{pmatrix} & * & \ldots & * \\ M_1 - \lambda\mathcal{I}_m & \vdots & & \vdots \\ 0 \ldots 0 & \vdots & & \vdots \\ \vdots & \vdots & M_2 & \vdots \\ 0 \ldots 0 & * & \ldots & * \end{pmatrix},$$

where M_1 is the matrix of the restriction $\mathcal{A}_{|W}$ and $\mathcal{I}_m$ is the $m \times m$ identity matrix.

I now claim that

$$\det(A_\alpha - \lambda\mathcal{I}) = \det(M_1 - \lambda\mathcal{I}_m)\det(M_2). \tag{2.2}$$

On the left, we now have $\chi_{\mathcal{A}}$, the characteristic polynomial of $\mathcal{A}: V \to V$. The first factor on the right is $\chi_{\mathcal{A}_{|W}}$, the characteristic polynomial of the restriction $\mathcal{A}_{|W}: W \to W$.

The only thing left to do is to prove claim 2.2, which we do via the more general helper Lemma 2.2.10 below. □

Lemma 2.2.10. *Let A be a $p \times p$–matrix, and let B be a $q \times q$–matrix B. Then*

$$\det\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \det(A) \cdot \det(B).$$

Here, $*$ stands for an arbitrary $p \times q$–matrix, and O stands for the $q \times p$–zero matrix.

Proof. Let us first assume that A is not invertible. Since A does not have full rank (recall Theorem 4.2.25 from LinA 1), neither does the matrix given in the claim, meaning both sides are 0 and the claim is true.

Now let us assume that A is invertible. We will rewrite the matrix given in the claim as a matrix product, and then use that the determinant is multiplicative (recall Theorem 4.2.18 from LinA 1):

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} \mathcal{I} & A^{-1} \cdot * \\ 0 & \mathcal{I} \end{pmatrix} \text{ and } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & B \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ 0 & \mathcal{I} \end{pmatrix},$$

so

$$\det\begin{pmatrix} A & * \\ 0 & B \end{pmatrix} = \det\begin{pmatrix} \mathcal{I} & 0 \\ 0 & B \end{pmatrix} \cdot \det\begin{pmatrix} A & 0 \\ 0 & \mathcal{I} \end{pmatrix} \cdot \det\begin{pmatrix} \mathcal{I} & A^{-1} \cdot * \\ 0 & \mathcal{I} \end{pmatrix} = \det(B) \cdot \det(A) \cdot 1,$$

where the last step used that we can expand across the rows belonging to identity matrices. □

2.2.1 Nice results for combinations of invariant subspaces

The following theorem is a slight extension of Theorem 2.2.7. It tells us that ***if V can be broken down into two invariant subspaces for $\mathcal{A}$, we can find a representation matrix with a shape that is even nicer (block diagonal)***, and that ***determinants/characteristic polynomials become products of the respective restriction terms***:

Theorem 2.2.11. *Let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for V such that $W_1 = \operatorname{span}(\underline{a}_1, \ldots, \underline{a}_m)$ and $W_2 = \operatorname{span}(\underline{a}_{m+1}, \ldots, \underline{a}_n)$ are invariant under $\mathcal{A} : V \to V$. Then the matrix A_α has the form*

$$A_\alpha = \begin{pmatrix} & 0 & \ldots & 0 \\ M_1 & \vdots & & \vdots \\ & 0 & \ldots & 0 \\ 0 \ldots 0 & & & \\ \vdots & & M_2 & \\ 0 \ldots 0 & & & \end{pmatrix}.$$

Here M_1 and M_2 are the $m \times m$ and $(n-m) \times (n-m)$ matrices of the two restrictions $\mathcal{A}_{|W_1} : W_1 \xrightarrow{A} W_1$ and $\mathcal{A}_{|W_2} : W_2 \xrightarrow{A} W_2$. (With respect to their respective bases $\{\underline{a}_1, \ldots, \underline{a}_m\}$ and $\{\underline{a}_{m+1}, \ldots, \underline{a}_n\}$.)

In addition we have that

$$\det(A_\alpha) = \det(M_1)\det(M_2) \ ,$$

and that the characteristic polynomial of $\mathcal{A}$ is the product of the characteristic polynomials of the two restrictions:

$$\chi_{\mathcal{A}} = \chi_{\mathcal{A}_{|W_1}} \cdot \chi_{\mathcal{A}_{|W_2}} \ .$$

Proof. The claim about the matrix shape can be proven by adapting the proof of Theorem 2.2.7. We can now apply Lemma 2.2.10 to prove the determinant equation and adapt the proof of Theorem 2.2.9 to prove the claim about the characteristic polynomials. □

Remark 2.2.12 We remark that this result can be generalised further such that it holds for an arbitrary number of invariant subspaces: if V can be broken down into invariant subspaces $W_1, \ldots W_p$, we can pick a basis α whose i-th section is a basis for W_i. Let $\mathcal{A}_i : W_i \xrightarrow{A} W_i$ denote the restriction of $\mathcal{A}$ unto the subspace W_i.

Then the matrix A_α has the form

$$A_\alpha = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_p \end{bmatrix} \ .$$

where M_i is the matrix of the respective restriction $\mathcal{A}_i$ (with respect to the respective basis.)

In that case, the determinant of A_α is the product of all of the restriction matrix determinants, the same goes for the characteristic polynomial.

With this, we also get a result about spec($\mathcal{A}$), the set of all eigenvalues of $\mathcal{A}$: you get the eigenvalues of $\mathcal{A}$ by picking all roots of the characteristic polynomial that actually lie in the field. Since the characteristic polynomial of $\mathcal{A}$ is the product of the characteristic polynomials of the restrictions $\mathcal{A}_i$, you get $\mathcal{A}$'s spectrum by collecting the eigenvalues of the restrictions. In short:

$$\text{spec}(\mathcal{A}) = \bigcup_{i=1}^{p} \text{spec}(\mathcal{A}_i) \ .$$

You will re-encounter this in, e.g., Theory and Application of Ordinary Differential Equations. There, the family of restrictions $\{\mathcal{A}_1, \dots \mathcal{A}_p\}$ will be called a ***decomposition of $\mathcal{A}$ with respect to the invariant, independent subspaces*** $W_1, \dots W_p$.

2.2.2 How to get invariant subspaces from non-real roots

For real vector spaces, only the real roots of a characteristic polynomial $\chi_{\mathcal{A}}$ are eigenvalues. In consequence, its non-real roots have been useless so far, as they do not provide us with eigenvectors. We will now see that they are useful nonetheless: ***non-real roots give rise to a two-dimensional invariant subspace, which still helps simplifying the matrix.*** (Because we can at split the matrix into a block belonging to this subspace, and another block.) We will use this again in Section 3.1 to prove that linear maps belonging to a certain class will always be a combination of simple geometric building blocks (rotations and reflections).

Theorem 2.2.13 (Two-dimensional invariant subspaces). *Let $\mathcal{A}$ be a linear map $\mathcal{A} : V \to V$ of a* **real** *finite-dimensional vector space V unto itself, and let μ be a non-real root of the map's characteristic polynomial $\chi_{\mathcal{A}}$.*

Then μ gives rise to a two-dimensional invariant subspace W of V. We find a natural basis of W such that the matrix of the respective restriction is

$$\begin{pmatrix} \text{Re}(\mu) & \text{Im}(\mu) \\ -\text{Im}(\mu) & \text{Re}(\mu) \end{pmatrix} ,$$

The characteristic polynomial of the respective restriction, $\chi_{\mathcal{A}_{|W}}$, is

$$\chi_{\mathcal{A}_{|W}} = (\lambda - \mu) * (\lambda - \overline{\mu}) \ .$$

Proof. Fix any basis $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$ for V. We will now view the representation matrix A_α as the matrix of a map $\mathbb{C}^n \to \mathbb{C}^n$. (We can do this since A_α has real coefficients, meaning it can also be viewed as a complex matrix.) Then μ is an eigenvalue of that complex map, meaning we find a complex eigenvector $\underline{z}$, i.e., a vector $\underline{z} \neq \underline{0}$ such that

$$A_\alpha \underline{z} = \mu \underline{z} \ .$$

We now split $\underline{z}$ into a vector $\underline{x}$ for the real part of $\underline{z}$ (so '$\underline{x} = \text{Re}\underline{z}$') and a vector $\underline{y}$ for the complex part of $\underline{z}$ (so '$\underline{y} = \text{Im}\underline{z}$'):

$$\begin{aligned} (x_1, \dots, x_n) &:= (\text{Re}z_1, \dots, \text{Re}z_n) \\ (y_1, \dots, y_n) &:= (\text{Im}z_1, \dots, \text{Im}z_n) \ . \end{aligned}$$

Since $\underline{x}$ and $\underline{y}$ both only have real entries, we can view them as vectors in V. We will just write $\underline{x}$ and $\underline{y}$ for those, though strictly speaking, we translated α-coordinate vectors back into vectors in V. (So more formally, we are talking about $\alpha^{-1}(\underline{x}) = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n \in V$ and $\alpha^{-1}\underline{y} = y_1\underline{a}_1 + \cdots + y_n\underline{a}_n \in V$.)

We set $W := \text{span}\left(\underline{x}, \underline{y}\right)$ (so '$W = \text{span}\left(\text{Re}\underline{z}, \text{Im}\underline{z}\right)$'), and show that W is invariant: We only need to argue that $\mathcal{A}\underline{x}, \mathcal{A}\underline{y}$ can both be written as a (real) linear combination of $\underline{x}$ and $\underline{y}$ (due to Theorem 2.2.5).

We achieve this by looking at $A_\alpha\underline{z}$ in the complex space. We compute $A_\alpha\underline{z}$ in two different ways and compare the results: Per definition, $\underline{z} = \underline{x} + i\,\underline{y}$.

- The rules for matrix multiplication give
$$A_\alpha\underline{z} = A_\alpha(\underline{x} + i\,\underline{y}) = A_\alpha\underline{x} + i\,A_\alpha\underline{y} \ .$$
- At the same time, $\underline{z}$ is an eigenvector for eigenvalue μ, hence
$$A_\alpha\underline{z} = \mu\,\underline{z} = \text{Re}(\mu)\,\underline{x} - \text{Im}(\mu)\,\underline{y} + i\,\text{Im}(\mu)\,\underline{x} + i\,\text{Re}(\mu)\,\underline{y} \ ,$$
where the last step split μ into real and imaginary and dissolved the brackets.

Comparing the two results (and splitting them into real/imaginary), we get

$$A_\alpha\underline{x} = \text{Re}(\mu)\,\underline{x} - \text{Im}(\mu)\,\underline{y}, \qquad A_\alpha\underline{y} = \text{Im}(\mu)\,\underline{x} + \text{Re}(\mu)\,\underline{y}.$$

We have just shown that $\mathcal{A}\underline{x}, \mathcal{A}\underline{y}$ can both be written as a (real) linear combination of $\underline{x}$ and $\underline{y}$:

$$\begin{aligned}\mathcal{A}\underline{x} &= \text{Re}(\mu)\,\underline{x} - \text{Im}(\mu)\,\underline{y}, \text{ and}\\ \mathcal{A}\underline{y} &= \text{Im}(\mu)\,\underline{x} + \text{Re}(\mu)\,\underline{y}.\end{aligned}$$

So W is an invariant subspace that has dimension 2 over $\mathbb{R}$ (check this!). With respect to the basis $\{\underline{x}, \underline{y}\}$, the restriction of $\mathcal{A}$ unto W has the following matrix M_W:

$$M_W = \begin{pmatrix} \text{Re}(\mu) & \text{Im}(\mu) \\ -\text{Im}(\mu) & \text{Re}(\mu) \end{pmatrix} .$$

The restriction $\mathcal{A}_{|W}$ unto W has μ and $\overline{\mu}$ as roots of the characteristic polynomial $\chi_{\mathcal{A}_{|W}}$: We have

$$\chi_{\mathcal{A}_{|W}}(\lambda) = \det(M_W - \lambda\cdot\mathcal{I}) = \left(\text{Re}(\mu) - \lambda\right)^2 + \text{Im}(\mu)^2 \ .$$

For this to be 0, we need $\text{Re}(\mu) - \lambda \in \{\pm i\cdot\text{Im}(\mu)\}$. In other words, we need $\lambda \in \{\text{Re}(\mu) \pm i\cdot\text{Re}(\mu)\}$, so $\lambda \in \{\mu, \overline{\mu}\}$. □

Remark 2.2.14 ***(Conjugates raise the same subspace)*** If $\mathcal{A}$ and μ are like in Theorem 2.2.13, and the complex conjugate $\bar{\mu}$ of μ is also a root of the map's characteristic polynomial $\chi_{\mathcal{A}}$, then $\bar{\mu}$ gives rise to the same two-dimensional invariant subspace W of V as μ does:

We could do the same steps as in Theorem 2.2.13. The only difference between the two bases raised by μ vs. $\bar{\mu}$ is that for $\bar{\mu}$, the vector '$\underline{y} := \text{Im}\underline{z}$' is multiplied by -1. (Verify this by checking that the eigenvectors for $\bar{\lambda}$ are the conjugates of λ's eigenvectors, since A_α is real.) So while we get a slightly different basis, we still get the same subspace.

2.3. Diagonalisability and dealing with incomplete eigenvector bases

What will we do? This section offers new perspectives on diagonalisability - in Section 2.3.1, we relate diagonisability to a requirement made on the characteristic polynomial and the eigenvalues, and in Section 2.3.2, we look at diagonisability through our matrix glasses. We conclude the section with looking at the 'next best thing' to being diagonalised, which is being brought into a 'standardized' diagonal block shape called Jordan Normal Form.

Why do we care? The reasoning presented in Section 2.3.1 will save us some disappointment when doing computational work towards diagonalisation: it shows that our computations will turn out to be futile if a certain criterion is met. (Thus also giving us an argument to refuse doing any more work once we notice that the criterion holds for a given map/matrix.) The matrix perspective taken in Section 2.3.2 will serve us nicely in Chapter 3, when we start analysing particular kinds of maps/matrices. Lastly, the results in will a) give us an alternative way to reason about diagonalisability by simply looking at the Jordan Normal Form, and b) providing a 'fail-safe' alternative to diagonalisation which - unlike diagonalisation - will always work for complex maps.

Like in the sections before, ***all maps considered in this section are maps from a finite-dimensional vector space into itself***, i.e., they are maps $\mathcal{A} : V \to V$ with $\dim(V) < \infty$.

2.3.1 'How many vectors do we get?' - eigenvalue multiplicity as a diagonalisability criterion

We already saw examples (e.g., example 2.1.21) where we did not get enough linearly independent eigenvectors to diagonalise the matrix. The problem in cases like example 2.1.21 was an eigenvalue μ that gifted us with less eigenvectors than we would have hoped for by looking at how often eigenvalue μ pops up in the characteristic polynomial.

We make this difference formal by defining two different kinds of multiplicity: an 'algebraic' one to count how often you can fit the linear factor belonging to μ into the characteristic polynomial, and a 'geometric' one to count how many linearly independent eigenvectors you can get from μ.

Definition 2.3.1. *Let V be a finite-dimensional vector space, and let $\mathcal{A} : V \to V$ be a linear map with characteristic polynomial $\chi_{\mathcal{A}}$ and eigenvalue μ.*

We let $m_{\text{alg}}(\mu)$ denote the power to which $(\lambda - \mu)$ divides the characteristic polynomial, so

$$m_{\text{alg}}(\mu) := \max\{m \mid (\lambda - \mu)^m \text{ divides } \chi_{\mathcal{A}}\} \ ,$$

and call $m_{\text{alg}}(\mu)$ the **algebraic multiplicity of** μ.

We furthermore define

$$m_{\text{geo}}(\mu) := dim(E_\mu(\mathcal{A}))$$

and call $m_{\text{geo}}(\mu)$ the **geometric multiplicity of** μ.

Example 2.3.2 ***(Revisiting example 2.1.21)*** In example 2.1.21, we looked at the differentiation map $\mathcal{D}$ and computed that its characteristic polynomial is $\chi_{\mathcal{D}} = \lambda^3$.
The only root of the characteristic polynomial (and hence, the only eigenvalue) is $\lambda = 0$,

with algebraic multiplicity

$$m_{\text{alg}}(0) = 3 \ .$$

We also computed the eigenspace E_0 for eigenvalue 0 and found that is has dimension $\dim(E_0) = 1$, meaning we have geometric multiplicity

$$m_{\text{geo}}(0) = 1 \ .$$

As observed in example 2.1.21, we do not get an eigenvector basis (and hence the differentiation has no diagonal form) exactly because we get too few eigenvectors from the eigenspace E_0. In other words, we cannot diagonalise the differentiation map because the geometric multiplicity $m_{\text{geo}}(0)$ is too small.

In all examples so far, we had that $m_{\text{geo}}(\mu) \leq m_{\text{alg}}(\mu)$, meaning that ***the algebraic multiplicity*** m_{alg} ***(*** μ ***) always bounded how big the dimension of the corresponding eigenspace could possibly be.*** Indeed, this is always the case according to the following theorem:

Theorem 2.3.3. *Let V be a finite-dimensional vector space, and let $\mathcal{A} : V \to V$ be a linear map with characteristic polynomial $\chi_{\mathcal{A}}$ and eigenvalue μ.*

We always have that $m_{\text{geo}}(\mu) \leq m_{\text{alg}}(\mu)$.

Proof. In essence, the theorem follows from how the restriction unto an eigenspace looks (diagonal), and from the fact that characteristic polys of restrictions always divide the 'complete' characteristic polynomial.

In more detail, let $m := m_{\text{geo}}(\mu)$. Since we defined m_{geo} (μ) to be the dimension of the corresponding eigenspace $E_\mu(\mathcal{A})$, we know that we can always find an eigenvector basis α of size m for $E_\mu(\mathcal{A})$.

Also, we already saw (in example 2.2.4) that eigenspaces are invariant, we can hence take a look at the restriction $\mathcal{A}_{|E_\mu(\mathcal{A})}$ unto the eigenspace. With respect to the eigenvector basis α, the matrix of the restriction is $\mu \cdot \mathcal{I}_m$, and the restriction's characteristic polynomial is $\chi_{\mathcal{A}_{|W}} = (\lambda - \mu)^m$.

Now the 'restriction charpols always divide the charpol' Theorem 2.2.9 tells us that $(\lambda - \mu)^m$ divides $\chi_{\mathcal{A}}$. This means that at least $m := m_{\text{geo}}(\mu)$ many copies of $\lambda - \mu$ must fit into $\chi_{\mathcal{A}}$. We have just shown that $m_{\text{geo}}(\mu) \leq m_{\text{alg}}(\mu)$. □

We also get a ***diagonalisability criterion for maps***, provided that the characteristic polynomial has no roots which cannot be used as an eigenvalue due to living outside of the scalar field. (More formally: provided that their characteristic polynomial completely splits into linear factors $\lambda - \lambda_i$ belonging to the respective eigenvalues λ_i.) Remember that ***maps over complex vector spaces always satisfy this requirement*** (see the discussion above Theorem 2.1.17), but you might also encounter other examples during your studies.

Theorem 2.3.4. *Let V be a finite-dimensional $\mathbb{K}$-vector space, and let $\mathcal{A} : V \to V$ be a linear map such that its characteristic polynomial $\chi_{\mathcal{A}}$ completely factors over $\mathbb{K}$, i.e., a map such that*

$$\chi_{\mathcal{A}}(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{m_i} \ ,$$

where $\lambda_1, \ldots, \lambda_m$ are the (distinct) eigenvalues of $\mathcal{A}$ and $m_i = m_{\text{alg}}(\lambda_i)$ are the respective algebraic multiplicities.

Then map $\mathcal{A}$ has a basis of eigenvectors (in other words, it is diagonalisable) if no eigenvalue has geometric multiplicity smaller than its algebraic multiplicity, i.e., if $m_{\text{geo}}(\lambda_i) = m_i$ for each eigenvalue λ_i of $\mathcal{A}$.

Proof. We already know (due to Theorem 2.1.14) that the degree of $\chi_{\mathcal{A}}$ is the dimension of the vector space V, so we have

$$\dim(V) = \deg(\chi_{\mathcal{A}}) = \sum_{i=1}^{n} m_i \ .$$

Since we assume that geometric and algebraic multiplicity coincide for each eigenvalue λ_i, we get m_i many independent eigenvectors from eigenvalue λ_i. Collecting them all into a set α, we have a set of eigenvectors whose size is exactly the dimension of V. This set α always forms a basis since eigenvectors belonging to different eigenvalues are linearly independent (remember Theorem 2.1.23). □

2.3.2 A matrix view on diagonalisability

We will now ***translate diagonalisability into matrix terms***.

Definition 2.3.5. *Let V be a finite-dimensional vector space, and let $\mathcal{A} : V \to V$ be a linear map from V into itself.*

Once we have fixed a basis α, associated to $\mathcal{A}$ is its representation matrix $A = A_\alpha$. If $B = A_\beta$ is the matrix with respect to a second basis β, then A and B are related (remember Theorem 1.3.15) by

$$B = S^{-1}AS \ .$$

Here $S = {}_\alpha S_\beta$ is the basis transition matrix that belongs to the change from β– to α–coordinates.

When two matrices are related like this, we call the matrices A and B **conjugates of each other**.

⚠ This has nothing to do with complex conjugation.

Diagonalisation consists of switching from some basis α to an eigenvector basis β:

$$A_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} = {}_\alpha S_\beta{}^{-1} \ A_\alpha \ {}_\alpha S_\beta = {}_\beta S_\alpha \ A_\alpha \ {}_\alpha S_\beta \ .$$

In particular, we recall that if ε is the standard basis of $\mathbb{K}^n$ and β is an eigenvector basis of $\mathbb{K}^n$, then ${}_\varepsilon S_\beta$ is the matrix whose columns are the eigenvectors in β.

We can now express diagonalisability of a matrix in these terms (we omit the proof):

Theorem 2.3.6. *A matrix A is diagonalisable if it is conjugated with a diagonal matrix.*

We can also determine whether matrices are conjugated with each other (again, we omit the proofs):

Theorem 2.3.7. *Two diagonalisable matrices are conjugated if and only if they are conjugated with the same diagonal matrix.*

Two diagonal matrices are conjugated if and only if they have the same multiset of diagonal elements, i.e., if the only difference is the ordering in which the diagonal elements occur. (Or, more formally: if they have the same spectrum.)

As a result of Theorem 2.3.7, we get the following corollary:

Corollary 2.3.8. *Let A, B be two conjugated diagonalisable matrices. Then B has the same eigenvalues as A. (Or, more formally:* $\operatorname{spec}(A) = \operatorname{spec}(B)$*.)*

Remark 2.3.9 Being conjugates is an equivalence relation on the set of quadratic matrices of the same dimension. (If you feel nerdy, you can verify this yourself.)

2.4. The 'next best thing' to being diagonal - Jordan Normal Form

In Section 2.3.2, we saw that diagonalising square matrices via conjugation is the same as finding a 'nice' (eigenvector) basis. In this section, we look at the special case where V is a complex, finite-dimensional vector space.

- Finding eigenvalues works: the characteristic polynomial completely splits into linear terms over $\mathbb{C}$, so all roots lie in the vector space's scalar field $\mathbb{C}$ (and hence serve as eigenvalues).
- But diagonalising might still not work: it might be that there are too few linearly independent eigenvectors. (As seen in Section 2.3.1, diagonalisation of complex maps is possible exactly when algebraic and geometric multiplicity coincide for all eigenvalues.)

When diagonalisation doesn't work, we can still get something whose shape is 'close enough' to diagonal for many computational problems: we get at least diagonal block shape with very simple building blocks. This is called ***Jordan Normal form*** (or Jordan canonical form), and the building blocks are called Jordan blocks.

⚠ This section does not focus on proofs, as proving the necessary helper results is outside of the scope of this course. Instead, we focus on a) how we can apply what we learnt so far, and b) what the JNF means when analysing a matrix.

Definition 2.4.1 (Jordan blocks). *We denote by $J_n(\lambda)$ the $n \times n$ matrix*

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

that has λ's on the diagonal, 1's directly above the diagonal, and 0's elsewhere.

We call a matrix J a **Jordan block** *if it is of that shape, i.e., if $J = J_n(\lambda)$ for some field element $\lambda \in \mathbb{K}$ and some integer n.*

Example 2.4.2 (***Jordan block of dimension 3 for value 2.***) $J_3(2) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

Definition 2.4.3 (Jordan Blocks and Jordan Normal Form). *Let A be a quadratic matrix with entries in $\mathbb{K}$. We say that A is in* **Jordan Normal Form** *if it is of a diagonal block shape*

$$A = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix},$$

with each block J_i being a Jordan block, so each block being $J_i = J_{n_i}(\lambda_i)$ for some field element $\lambda_i \in \mathbb{K}$ and some integer n_i.

Example 2.4.4 (***Diagonal form as Jordan Normal Form.***) Diagonalised matrices are the special case of a Jordan Normal form where each Jordan block has dimension 1, and belongs to one of the eigenvalues.

We will now indicate how ***complex square matrices can be brought into Jordan Normal Form, with each Jordan block belonging to an eigenvalue*** λ_i. To get to this result, and to determine the number and size of the blocks, we need some helper results. We will first study how we can annihilate complex square matrices, using an important polynomial that is closely related to the characteristic polynomial. After this, we study a generalisation of eigenvectors. To arrive at the Jordan Normal Form, we conclude by combining the two results with our knowledge about invariant subspaces .

2.4.1 Annihilating matrices using minimal polynomials

We first remind ourselves that polynomials can be applied to linear maps and square matrices (see remark 1.2.8). For example, take the polynomial $p(X) = (X-2)^2 = X^2 - 4X + 4$, and the matrix $A = 2 \cdot \mathcal{I}_2$: we compute

$$p(A) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^2 - 4\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, we can plug a square matrix into its own characteristic polynomial. We now state (without a proof) an important theorem named after Arthur Cayley and William Rowan Hamilton that tells us that the result will always be the zero map/matrix:

Theorem 2.4.5 (Cayley-Hamilton theorem for $\mathbb{K} = \mathbb{C}$). *A complex quadratic matrix's characteristic polynomial always annihilates it. (In other words, a quadratic matrix always satisfies its own characteristic equation.)*

More formally: Let A be a complex quadratic matrix with characteristic polynomial χ_A, and let $\chi_A(A)$ be the matrix obtained by applying χ_A to A. Then $\chi_A(A)$ is the zero matrix.

Looking at the example, you can verify that the characteristic polynomial of $A = 2 \cdot \mathcal{I}_2$ is $\chi_A(X) = (X-2)^2$, so the result comes as no surprise once you are familiar with the Cayley-Hamilton theorem.

In many cases, however, A can already be annihilated by a (non-zero) polynomial whose degree is smaller than that of χ_A. This 'smallest annihilator' is called minimal polynomial.

Example 2.4.6 We again consider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We see that $A - 2 \cdot \mathcal{I}_2$ is already the zero matrix, so $\lambda - 2$ is an annihilating polynomial. $\lambda - 2$ is of the smallest possible degree because no constant non-zero polynomial could possibly annihilate A.

We make this formal with an important definition which will later help us compute the Jordan Normal Form:

Definition 2.4.7. *Let A be a complex square matrix. The* **minimal polynomial** *of A is the (unique) polynomial m_A for which*

- *m_A annihilates A, i.e., $m_A(A) = 0$ (the zero matrix);*
- *m_A is minimal, i.e., any other polynomial q with $q(A) = 0$ is a (polynomial) multiple of m_A; and*
- *m_A is monic, i.e., the highest coefficient of m_A is 1.*

We convince ourselves that we can equivalently replace the second bullet point with requiring minimal degree:

Theorem 2.4.8. *Let m be a monic polynomial that annihilates a complex square matrix A. Then the following two statements are equivalent:*

1. *m divides all other polynomials that annihilate A.*
2. *The degree of m is minimal, i.e., the degree of all other polynomials annihilating A is at least as high as m's degree.*

Proof. The $1 \Rightarrow 2$ implication is straightforward: If there were an annihilating polynomial p of degree smaller than the degree of m, than m could not divide p.

To verify the $2 \Rightarrow 1$ implication, we assume that p is a polynomial with $p(A) = 0$ that is not divided by m. Since the degree of m is minimal, it is at most that of p. So we can use polynomial division to write $p = q \cdot m + r$ for some remainder polynomial r (whose degree is smaller than that of m) and some polynomial q. Since $p(A) = 0 = m(A)$, applying this to A yields

$$0 = p(A) = q(A) \cdot m(A) + r(A) = r(A) \ ,$$

r is hence a non-zero polynomial that annihilates A, but has degree smaller than that of m, meaning m cannot have minimal degree. □

The Cayley-Hamilton theorem tells us that χ_A always annihilates the matrix A. Given that the minimal polynomial m_A is defined such that it divides any other annihilating polynomial, we can conclude:

Corollary 2.4.9. *The minimal polynomial m_A of a complex square matrix A always divides the characteristic polynomial χ_A.*

Before discussing what we can do with the minimal polynomial, we collect a helper result that helps us compute it. According to the corollary, the only candidates are the polynomial divisors of χ_A whose highest coefficient is 1. The next theorem tells us that we can further reduce the group of suspects:

> **Theorem 2.4.10.** *Let $\mathcal{A} : V \to V$ be a linear map on a finite-dimensional complex vector space V, with* **distinct** *eigenvalues $\lambda_1, ..., \lambda_r$.*
>
> *Then $(\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \cdot \cdots \cdot (\lambda - \lambda_r)$ divides the minimal polynomial m_A.*

Proof. To prove this, we show the following: If μ is an eigenvalue of A, then μ is a zero of m_A, so $\lambda - \mu$ divides the minimal polynomial m_A. (In conclusion, then so does the product for the distinct eigenvalues.)

Let $\underline{v}$ be an eigenvector for eigenvalue μ. Since $m_A(A)$ is the zero matrix, we have $m_A(A)\underline{v} = \underline{0}$. We will now show that at the same time, $m_A(A)\underline{v} = m_A(\mu)\underline{v}$. Since $\underline{v}$ is a non-zero vector, $m_A(\mu)$ must then be 0.

To show that $m_A(A)\underline{v} = m_A(\mu)\underline{v}$, we write $m_A(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0$ and apply this to A:

$$\begin{aligned} m_A(A)\underline{v} &= (A^m + a_{m-1}A^{m-1} + \cdots + a_1 A + a_0)\underline{v} \\ &= A^m\underline{v} + a_{m-1}A^{m-1}\underline{v} + \cdots + a_1 A\underline{v} + a_0\underline{v} \ . \end{aligned}$$

Since $\underline{v}$ is an eigenvector for eigenvalue μ, we get $A^k\underline{v} = A^{k-1}A\underline{v} = A^{k-1}\mu\underline{v} = \mu A^{k-2}A\underline{v} = \cdots = \mu^k\underline{v}$. We can hence rewrite this:

$$\begin{aligned} A^m\underline{v} + a_{m-1}A^{m-1}\underline{v} + \cdots + a_1 A\underline{v} + a_0\underline{v} &= \mu^m\underline{v} + a_{m-1}\mu^{m-1}\underline{v} + \cdots + a_1\mu\underline{v} + a_0\underline{v} \\ &= (\mu^m + a_{m-1}\mu^{m-1} + \cdots + a_1\mu + a_0)\underline{v} = m_A(\mu)\underline{v} \ . \end{aligned}$$

□

We thus can 'almost' read out the minimal polynomial from the characteristic polynomial:

Corollary 2.4.11. *Let $\mathcal{A} : V \to V$ be a linear map on a finite-dimensional complex vector space V, with* **distinct** *eigenvalues $\lambda_1, ..., \lambda_r$ and characteristic polynomial $\chi_{\mathcal{A}}(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{m_i}$. Then the minimal polynomial of $\mathcal{A}$ is*

$$m_{\mathcal{A}}(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{e_i} \ ,$$

with each multiplicity e_i being somewhere between 1 and m_i, the algebraic multiplicity of λ_i.

The following examples show that this relationship is quite useful when you have to determine minimal polynomials.

Example 2.4.12 ***(Revisiting example 2.4.6)*** As a simply example, we reconsider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} .$$

We can almost read out the minimal polynomial from its characteristic polynomial $\chi_A(\lambda) = (\lambda - 2)^2$: according to Corollary 2.4.11, the only possibilities for m_A would have been the characteristic polynomial itself, $(\lambda - 2)^2$, and its only linear divisor, $\lambda - 2$. We already saw that $\lambda - 2$ is sufficient to annihilate A, so this is the minimal polynomial.

Example 2.4.13 ***(Example with more than one eigenvalue)*** For the matrix

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & -1 \end{pmatrix} ,$$

we find the eigenvalues $0, -4$ and 6, so $\lambda(\lambda + 4)(\lambda - 6)$ must divide the minimal polynomial m_A. At the same time, $\lambda(\lambda + 4)(\lambda - 6)$ is the complete characteristic polynomial, and m_A must divide the characteristic polynomial.
So the two polynomials must be equal and the minimal polynomial hence is $\lambda(\lambda + 4)(\lambda - 6)$.

2.4.2 Generalised eigenvectors and -spaces

We proceed by generalising the concept of eigenvectors. This generalisation will allow to compute bases that bring a matrix into its Jordan Normal Form.

Definition 2.4.14. *Let $\mathcal{A} : V \to V$ be a linear map on a finite-dimensional vector space, and let μ be an eigenvalue of $\mathcal{A}$. We call $\underline{x}$ a* **generalized eigenvector of rank** *m of $\mathcal{A}$ for eigenvalue μ if*

- $(\mathcal{A} - \mu \cdot \mathcal{I})^m \underline{x} = \underline{0}$, *but*
- $(\mathcal{A} - \mu \cdot \mathcal{I})^{m-1} \underline{x} \neq \underline{0}$.

We denote the collection of all generalized eigenvectors of rank up to m by $E_\mu^{(m)}$, so

$$E_\mu^{(m)} := \mathcal{N}((\mathcal{A} - \mu \cdot \mathcal{I})^m) \ ,$$

and call it the **generalized eigenspace of rank** *m of $\mathcal{A}$ for eigenvalue μ .*

Example 2.4.15 ***(Finding generalised eigenspaces and -vectors)*** Let

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} .$$

We compute the characteristic polynomial as $\chi_A(\lambda) = (\lambda - 2)^3$, so 2 is the only eigenvalue. We now calculate the generalized eigenspaces for 2. The generalized eigenspace for 2 of rank 1 is simply the usual eigenspace:

$$E_\mu^{(1)} = \mathcal{N}(A - 2 \cdot \mathcal{I}) = \mathcal{N}\begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} ,$$

so $E_\mu^{(1)} = \text{span}\,((1,0,0))$.
The generalized eigenspace of rank 2 is

$$E_\mu^{(2)} = \mathcal{N}((A - 2 \cdot \mathcal{I})^2) = \mathcal{N}\begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

so $E_\mu^{(2)} = \text{span}\,((1,0,0),(0,1,0))$.
Note that ***we do not need to compute the generalised eigenspaces of rank larger than 2***: since $\chi_A(A) = (A - 2 \cdot \mathcal{I})^3 = 0$ (Theorem 2.4.5), we know that $(A - 2 \cdot \mathcal{I})^3$ is the zero matrix and the null space of $(A - 2 \cdot \mathcal{I})^3$ is hence the full vector space.

How to find a generalised eigenvector of rank 2: such a vector

- must be in $E_\mu^{(2)}$ (since we require $(\mathcal{A} - \mu \cdot \mathcal{I})^2 \underline{x} = \underline{0}$), but
- cannot be in $E_\mu^{(1)}$ (since we require $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{x} \neq \underline{0}$).

So to get a generalised eigenvector of rank 2, we can simply pick the base vector from $\{(1,0,0),(0,1,0)\}$ that wasn't already included in the basis for $E_\mu^{(1)}$, so $(0,1,0)$.

Why is this a generalisation of the 'usual' eigenspace? Because the 'usual' eigenspace

- is exactly $E_\mu^{(1)}$, the generalized eigenspace of rank 1; and it
- is contained in any generalized eigenspace of rank higher than 1. (If $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{v}$ is already the zero vector, then certainly also $(\mathcal{A} - \mu \cdot \mathcal{I})^m \underline{v}$.)

Our overall goal is to arrive at a block diagonal matrix. We know that we get block diagonal shape if the vector space completely splits into invariant subspaces, so let's check that generalised eigenspaces are suitable candidates:

Theorem 2.4.16. *Generalised eigenspaces are invariant.*

Proof. Let μ be the respective eigenvalue of $\mathcal{A}$ and let m be the respective rank. We need to show the following: if $\underline{v} \in E_\mu^{(m)}$, then also $\mathcal{A}\underline{v} \in E_\mu^{(m)}$. In other words: if $(\mathcal{A}-\mu\cdot\mathcal{I})^m\underline{v} = \underline{0}$, then also $(\mathcal{A}-\mu\cdot\mathcal{I})^m(\mathcal{A}\underline{v}) = \underline{0}$.

We will now assume that we can essentially switch $(\mathcal{A}-\mu\cdot\mathcal{I})^m$ and $\mathcal{A}$ around, so we assume that

$$(\mathcal{A}-\mu\cdot\mathcal{I})^m(\mathcal{A}\underline{v}) = \mathcal{A}((\mathcal{A}-\mu\cdot\mathcal{I})^m\underline{v}) \ .$$

Since $(\mathcal{A}-\mu\cdot\mathcal{I})^m\underline{v} = \underline{0}$, the assumption yields

$$(\mathcal{A}-\mu\cdot\mathcal{I})^m(\mathcal{A}\underline{v}) = \mathcal{A}(\underline{0}) = \underline{0} \ ,$$

it hence remains to prove that we can indeed switch $(\mathcal{A}-\mu\cdot\mathcal{I})^m$ and $\mathcal{A}$ around. In the following theorem, we will show that we can actually do this also for any other polynomial applied to $\mathcal{A}$. □

Theorem 2.4.17. *Let $\mathcal{A}: V \to V$ be a linear map. For any polynomial f with coefficients in V's scalar field, we have*

$$f(\mathcal{A})(\mathcal{A}\underline{v}) = \mathcal{A}(f(\mathcal{A})\underline{v})$$

Proof. To show that we can essentially switch $f(\mathcal{A})$ and $\mathcal{A}$ around, we write $f(\mathcal{A}) = a_m\mathcal{A}^m + a_{m-1}\mathcal{A}^{m-1} + \cdots + a_1\mathcal{A} + a_0$ and compute:

$$\begin{aligned} f(\mathcal{A})(\mathcal{A}\underline{v}) &= (a_m\mathcal{A}^m + a_{m-1}\mathcal{A}^{m-1} + \cdots + a_1\mathcal{A} + a_0)(\mathcal{A}\underline{v}) \\ &= (a_m\mathcal{A}^{m+1} + a_{m-1}\mathcal{A}^m + \cdots + a_1\mathcal{A}^2 + a_0\mathcal{A})(\underline{v}) \ . \end{aligned}$$

Since $\mathcal{A}$ is linear and since the coefficients are just linear factors, we can push the sum into the first application of $\mathcal{A}$:

$$\begin{aligned} (a_m\mathcal{A}^{m+1} + a_{m-1}\mathcal{A}^m + \cdots + a_1\mathcal{A}^2 + a_0\mathcal{A})(\underline{v}) &= \mathcal{A}(a_m\mathcal{A}^m + a_{m-1}\mathcal{A}^{m-1} + \cdots + a_1\mathcal{A} + a_0)(\underline{v}) \\ &= \mathcal{A}(f(\mathcal{A})\underline{v}) \ . \end{aligned}$$

□

So assuming that we could split the vector space into generalised eigenspaces, we would have found a decomposition into invariant subspaces, thus getting closer to the desired block diagonal shape. The following theorem (which we do not prove) states that the vector space can indeed be split as intended.

Theorem 2.4.18 (Primary Decomposition Theorem). *For any linear map $A: V \to V$ on a finite-dimensional complex vector space V, we can find a basis of generalized eigenvectors. V splits into the (invariant) generalised eigenspaces, with the ranks corresponding to the respective eigenvalue's degree in the minimal polynomial.*

More formally, let $A: V \to V$ be a linear map on a finite-dimensional complex vector space V, with **distinct** *eigenvalues $\lambda_1, ..., \lambda_r$ and minimal polynomial $m_A(\lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{e_i}$.*

Then for each i, we find a basis α_i of the generalised eigenspace $E_{\lambda_i}^{(e_i)}$ in a way such that

- *the bases $\alpha_1, \cdots, \alpha_r$, are all linearly independent; and such that*

- *combining the bases, so taking the union $\alpha := \bigcup_i \alpha_i$, spans the full vector space V.*

Remark 2.4.19 Since we are decomposing the vector space along the map's eigenvalues, and since the collection of the map's eigenvalues is called the spectrum, this is also sometimes called ***spectral decomposition***.

Example 2.4.20 ***(Decomposing the vector space according to Theorem 2.4.18.)*** Say we're given an $n \times n$ matrix and its minimal polynomial $m_A(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{e_i}$. For example, let's look at

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} .$$

We compute $\chi_A(\lambda) = (\lambda-1)^2(\lambda-5)$, so the only two candidates for m_A are $(\lambda-1)^2(\lambda-5)$ and $(\lambda-1)(\lambda-5)$. We start by ruling out the smaller-degree option:

$$(A - 1 \cdot \mathcal{I}_3)(A - 5 \cdot \mathcal{I}_3) = \begin{pmatrix} 0 & -8 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

which is not the zero matrix. Hence, the minimal polynomial can only be $(\lambda-1)^2(\lambda-5)$.
The first step towards the Jordan Normal Form is to split V into the respective invariant subspaces, by determining a basis of the generalised eigenspaces $E_{\lambda_i}^{(e_i)}$ of rank e_i.
The rank in question for eigenvalue 5 is 1, so the generalized eigenspace for 5 in question is simply the usual eigenspace:

$$E_5^{(1)} = \mathcal{N}(A - 5 \cdot \mathcal{I}) = \mathcal{N}\begin{pmatrix} -4 & 2 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\,((0,0,1)) \ .$$

The rank in question for eigenvalue 1 is 2, so the generalized eigenspace for 1 in question is

$$E_1^{(2)} = \mathcal{N}((A - \mathcal{I})^2) = \mathcal{N}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 16 \end{pmatrix} = \text{span}\,((1,0,0),(0,1,0)) \ .$$

By picking the standard basis, we thus already found a decomposition into the two generalised eigenspaces. This comes as no surprise since the matrix was already in diagonal block shape. With our new terminology, we can identify the entry in the lower right as the 1×1 diagonal block for the eigenspace belonging to eigenvalue 5, and the 2×2 diagonal block in the upper left as the block for the generalised eigenspace for eigenvalue 1.

⚠ As we can see in the example, however, the decomposition alone does not yet tell us how to find a basis for which the separate blocks have the desired Jordan shape.

2.4.3 Finding a basis that creates the Jordan Normal Form (Jordan bases)

We will not fully formalise how to compute the full basis, but rather describe the main idea of the procedure. As a helper result, we will use Theorem 2.4.21 below, according to which eigenvalues either give us

- an eigenspace whose dimension exactly matches the algebraic multiplicity, in which case the corresponding block can be fully diagonalised with our already known techniques (Section 2.3.1); or
- a generalised eigenvector of rank 2, and we will use this as an indication how to move towards a suitable Jordan basis.

Theorem 2.4.21. *Let $\mathcal{A} : V \to V$ be a linear map on a finite vector space, and let μ be an eigenvalue such that $m_{\text{geo}}(\mu) < m_{\text{alg}}(\mu)$.*

Then there exists a vector $\underline{x}$ such that $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{x} \neq \underline{0}$, but $(\mathcal{A} - \mu \cdot \mathcal{I})^2 \underline{x} = \underline{0}$. In other words, $\underline{x}$ is a generalised eigenvector of rank 2.

Proof. To prove existence of such a vector $\underline{x}$, we look at the map $\mathcal{A} - \mu \cdot \mathcal{I}$ and assume that the intersection of its range and its null space contains more than just the zero vector, so we assume that there exists a vector $\underline{x'} \neq \underline{0}$ in the intersection. Since $\underline{x'}$ lies in the range of $\mathcal{A} - \mu \cdot \mathcal{I}$, we find a preimage vector $\underline{x}$, i.e., a vector $\underline{x}$ such that $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{x} = \underline{x'}$. We verify that $\underline{x}$ satisfies both requirements:

- We have $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{x} \neq \underline{0}$ since $(\mathcal{A} - \mu \cdot \mathcal{I})\underline{x} = \underline{x'} \neq \underline{0}$.
- We have $(\mathcal{A} - \mu \cdot \mathcal{I})^2 \underline{x} = (\mathcal{A} - \mu \cdot \mathcal{I})\underline{x'} = \underline{0}$ since $\underline{x'}$ was taken from the null space of $\mathcal{A} - \mu \cdot \mathcal{I}$.

It remains to prove the assumption that range and null space of $\mathcal{A} - \mu \cdot \mathcal{I}$ share more than the zero vector, provided that $m_{\text{geo}}(\mu) < m_{\text{alg}}(\mu)$. For that, we apply the helper theorem below to the map $\mathcal{B} := \mathcal{A} - \mu \cdot \mathcal{I}$ (which has eigenvalue 0 if $\mathcal{A}$ has eigenvalue μ). □

Theorem 2.4.22 (Helper theorem for Theorem 2.4.21). *Let $\mathcal{A} : V \to V$ be a linear map on a finite-dimensional vector space. If the algebraic multiplicity of 0 is larger than its geometric multiplicity, then the intersection of $\mathcal{A}$'s range and null space contains more than the zero vector.*

Proof. Assume that range and null space only have the zero vector in common. Due to the dimension theorem, we can then pick a basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ of V such that $\{\underline{a}_1, \ldots, \underline{a}_m\}$ is a basis for the null space and that $\{\underline{a}_{m+1}, \ldots, \underline{a}_n\}$ is a basis for the range. (We note that m is the geometric multiplicity of 0.)

Since null spaces and ranges are always invariant (2.2.2), the matrix of $\mathcal{A}$ with respect to this basis is

$$A_\alpha = \begin{pmatrix} 0 & & & 0 & \dots & \dots & 0 \\ & \ddots & & \vdots & & & \vdots \\ & & 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & & & & \\ \vdots & & \vdots & & M & & \\ \vdots & & \vdots & & & & \\ 0 & \dots & 0 & & & & \end{pmatrix} ,$$

with M being the $(n-m)\times(n-m)$ matrix of $\mathcal{A}$'s restriction unto $\mathcal{R}$. We can now show that the algebraic multiplicity of 0 equals m: It follows from the shape of A_α that the characteristic equation of the full map $\mathcal{A}: V \to V$ is

$$(-\lambda)^m \cdot \det(M - \lambda \cdot \mathcal{I}_{n-m}) = 0 .$$

Since we assume that the range and the null space only share the zero vector, $\mathcal{A}$'s restriction unto the range cannot have 0 as an eigenvalue and $\det(M-\lambda\cdot\mathcal{I}_{n-m})$ hence has no linear factor λ. In conclusion, 0 is a m-fold root of $\mathcal{A}$'s characteristic equation. In other words, the algebraic multiplicity of 0 is m and thus matches the geometric multiplicity of 0.

We conclude: if the algebraic multiplicity of 0 is larger than the geometric multiplicity of 0, the intersection must contain more than the zero vector. □

So whenever an eigenvalue does not give rise to diagonal shape, we find a generalised eigenvector of rank 2. We can now analyse how this vector leads to a 2-dimensional Jordan block. We start by revisiting example 2.4.15.

Example 2.4.23 ***(2-dimensional Jordan from rank-2-eigenvector, for example 2.4.15)*** We already found $\underline{x} := (0,1,0)$ as a generalised eigenvector $\underline{x}$ of rank 2 for eigenvalue 2. We now set

$$\underline{x'} := (\mathcal{A} - 2\cdot\mathcal{I})\underline{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} .$$

By verifying that $(\mathcal{A}-2\cdot\mathcal{I})\underline{x'} = 0$ and that $\underline{x}$ and $\underline{x'}$ are linearly independent, we see that these two vectors create a 'mini chain' of vectors which span a two-dimensional subspace of the generalised eigenspace for eigenvalue 2. This 2-dimensional subspace is invariant: We have

- $A\underline{x'} = (4,0,0) = 2\underline{x'}$ and
- $A\underline{x} = (2,2,0) = \underline{x'} + 2\underline{x}$.

If we restrict the map unto this subspace and pick the basis $\{\underline{x'}, \underline{x}\}$, the respective representation matrix for this subspace is

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} .$$

Theorem 2.4.24. (2-dimensional Jordan form from rank-2-eigenvector, general case) *In general, taking a generalised eigenvector $\underline{x}$ of rank 2 for an eigenvalue μ and the vector $\underline{x'} := (\mathcal{A}-\mu\cdot\mathcal{I})\underline{x}$ always creates an invariant, two-dimensional subspace of $E_\mu^{(2)}$, and for the basis $\{\underline{x'}, \underline{x}\}$, the respective representation matrix for this subspace is the 2-dimensional Jordan block for eigenvalue μ, so*

$$J_2(\mu) = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} .$$

Proof. We can verify that the two vectors always are linearly independent by assuming that $a_1\underline{x} + a_2\underline{x'} = 0$ and multiplying this equation by $\mathcal{A} - \mu\cdot\mathcal{I}$ to see that a_1 and a_2 must be 0. (Since $(\mathcal{A}-\mu\cdot\mathcal{I})\underline{x'} = (\mathcal{A}-\mu\cdot\mathcal{I})^2\underline{x} = \underline{0}$ if $\underline{x}$ is of rank 2).

We now analyse what $\mathcal{A}$ does to the vectors:

- $A\underline{x'} = \mu\underline{x'}$ since $(A-\mu\mathcal{I})\underline{x'} = (A-\mu\mathcal{I})^2\underline{x} = \underline{0}$.
- $A\underline{x} = \underline{x'} + \mu\underline{x}$ since we defined $\underline{x'} = (A-\mu\mathcal{I})\underline{x}$.

So the two vectors indeed span an invariant subspace, and the respective matrix is

$$\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} .$$

□

As an intuition for higher ranks, it can be shown that we can always find a generalized eigenvector $\underline{x}$ of rank e if e is the multiplicity of μ in the minimal polynomial m_A. (We do not prove this.) Similar to the rank-2 case, this generates a 'Jordan chain' of generalized eigenvectors:

Definition 2.4.25 (Jordan chain). *Let μ be an eigenvalue of the matrix A whose degree in the minimal polynomial m_A is e, and let $\underline{x}$ be a generalized eigenvector of rank e for eigenvalue μ. We set*

- $\underline{x}_e := \underline{x}$
- $\underline{x}_{e-1} := (A-\mu\mathcal{I})\underline{x}_e$
- $\underline{x}_{e-2} := (A-\mu\mathcal{I})\underline{x}_{e-1} = (A-\mu\mathcal{I})^2\underline{x}$
- $\cdots$
- $\underline{x}_1 := (A-\mu\mathcal{I})\underline{x}_2 = (A-\mu\mathcal{I})^{e-1}\underline{x}$,

and call $(\underline{x}_1, \cdots, \underline{x}_e)$ a **Jordan chain**.

Theorem 2.4.26. (Jordan form from generalised eigenvectors of higher ranks) *A Jordan chain as defined Definition 2.4.25 creates an invariant, e-dimensional subspace of $E_\mu^{(e)}$, and when picking the Jordan chain as its basis, the respective representation matrix for this subspace is the e-dimensional Jordan block for eigenvalue μ.*

Proof. All vectors in the chain are contained in $E_\mu^{(e)}$, so they span a subspace of $E_\mu^{(e)}$. It can be shown that the vectors $\underline{x}_1, \cdots, \underline{x}_e$ are always linearly independent, so the subspace is e-dimensional. (This works similar to the rank-2-case, by multiplying the linear combination with suitably many copies of $\mathcal{A} - \mu \cdot \mathcal{I}$ to iteratively show that all coefficients must be 0.)

We verify that this subspace is invariant: Since we picked $\underline{x}$ such that $(A - \mu\mathcal{I})^e \underline{x} = \underline{0}$, we have for

- $\mathcal{A}\underline{x}_1$: $(A - \mu\mathcal{I})\underline{x}_1 = (A - \mu\mathcal{I})^e \underline{x} = \underline{0}$, so $\mathcal{A}\underline{x}_1 = \mu\underline{x}_1$.
- $\mathcal{A}\underline{x}_2$: Since $\underline{x}_1 = (A - \mu\mathcal{I})\underline{x}_2$, we have $A\underline{x}_2 = \underline{x}_1 + \mu\underline{x}_2$.
- $\cdots$
- $\mathcal{A}\underline{x}_{e-1}$: Since $\underline{x}_{e-2} = (A - \mu\mathcal{I})\underline{x}_{e-1}$, we have $A\underline{x}_{e-1} = \underline{x}_{e-2} + \mu\underline{x}_{e-1}$.
- $\mathcal{A}\underline{x}_e$: Since $\underline{x}_{e-1} = (A - \mu\mathcal{I})\underline{x}_e$, we have $A\underline{x}_e = \underline{x}_{e-1} + \mu\underline{x}_e$.

When we restrict the map unto the subspace spanned by $\underline{x}_1, \cdots, \underline{x}_e$ and pick them as the basis, we can again determine how the respective matrix looks by using what $\mathcal{A}$ does to the base vectors. This indeed results in the $e \times e$ matrix

$$J_e(\mu) = \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \mu & 1 \\ & & & \mu \end{bmatrix}.$$

□

We again do not prove the next theorem, but it can be shown that we cannot generate a Jordan chain that is longer than e. (To be more specific, we could of course define longer chains, but from the $e+1$-th position on, the chain then will only consist of the zero vector which is useless to create a basis.) This tells us that we can read out the size of an eigenvalue's largest Jordan block from the minimal polynomial.

Theorem 2.4.27. *Let μ be an eigenvalue of a matrix A, and let the degree of the linear term $\lambda - \mu$ in the matrix's minimal polynomial m_A be e.*

Then the length of the longest Jordan chain for μ is e.

The following theorem (which we again do not prove) tells us that we can also read out the dimension of the relevant generalised eigenspaces (the ones in Theorem 2.4.18) from the characteristic polynomial:

Theorem 2.4.28. *The dimension of a generalised eigenspace that appears in the vector space decomposition given in Theorem 2.4.18 matches the algebraic multiplicity of the respective eigenvalue.*

More formally, let μ be an eigenvalue of a linear map $\mathcal{A} : V \to V$ on a finite-dimensional vector space V, and let e denote the multiplicity of μ in $\mathcal{A}$'s minimal polynomial $m_\mathcal{A}$.

Then the dimension of the generalised eigenspace $E_\mu^{(e)}$ is equal to the algebraic multiplicity of μ, i.e., the multiplicity of μ in the characteristic polynomial.

⚠ Since the minimal polynomial and the characteristic polynomial aren't necessarily the same, the degree e of λ_μ can be smaller than μ's algebraic multiplicity $m_{\text{alg}}(\mu)$. So while Theorem 2.4.26 promises us a nice e-dimensional subspace for which we get the desired Jordan block shape, we are not necessarily done: we cannot safely assume that this is the only block belonging to eigenvalue μ, since there are still $m_{\text{alg}}(\mu) - e$ many vectors missing for the full basis of V. In fact, the next (also not proven) theorem specifies how many separate blocks we will find for an eigenvalue.

Theorem 2.4.29. *The number of Jordan blocks for an eigenvalue μ equals the dimension of the corresponding eigenspace E_μ, so the dimension of the null space $\mathcal{N}(A - \mu\mathcal{I})$.*

Remark 2.4.30 To find a complete basis of a generalised eigenspace is beyond the scope of this course. Essentially, we would need to

- start a new chain by looking for an additional generalised eigenvector that
  - does not live in the subspace generated by the Jordan chain we already found;
  - starts a new Jordan chain that is as long as it can be (so at most $m_{\text{alg}}(\mu) - e$, but it could also be smaller),
- repeat this process until we have enough vectors for a full basis of the generalised eigenspace.

The length of the new Jordan chains would also allow to determine the size of the other blocks.

Remark 2.4.31 ***(Takeaway: reading out the Jordan Normal Form from the minimal polynomial and the characteristic polynomial)***

- We compute the ***distinct*** eigenvalues $\lambda_1, ..., \lambda_r$ of the matrix.
- Per eigenvalue, we compute the following:
  - Number of Jordan blocks for λ_i: is equal to the number of linearly independent usual eigenvectors (so $m_{\text{geo}}(\lambda_i)$).
  - Dimension of the largest Jordan block belonging to λ_i: is equal to the degree of the factor $\lambda - \lambda_i$ in the minimal polynomial (so the multiplicity of λ_i in m_A).
  - Dimension of the generalised eigenspace for λ_i, so size of all λ_i-blocks combined: is equal to the degree of the factor $\lambda - \lambda_i$ in the characteristic polynomial (so $m_{\text{alg}}(\lambda_i)$).

By now, we know how to determine the number of Jordan blocks and the size of the biggest Jordan block belonging to an eigenvalue. For the others blocks, we only know that their dimensions (plus the one of the biggest block) add up to the eigenvalue's algebraic multiplicity, but we do not know the specific dimensions without computing a Jordan basis. Nevertheless, for matrices of dimension up to 6, we can already determine the Jordan Normal Form from the multiplicities in the minimal polynomial and the geometric multiplicities. Here are some examples to illustrate this.

Example 2.4.32 The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

has characteristic polynomial $\chi_A(\lambda) = \lambda^2$ and the single eigenvalue 0.
We compute the minimal polynomial: the only options are λ and λ^2. Since $A \neq 0$, the minimal polynomial must be λ^2.
This tells us that the biggest Jordan block for eigenvalue 0 must be of size 2 and hence already fills up the complete Jordan Normal Form:

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Alternatively, we can compute the eigenspace for eigenvalue 0 and see that the dimension is 1, meaning there can only be a single Jordan block (which hence must have size 2).

Example 2.4.33 ***(Revisiting example 2.4.15)*** We already found that 2 is the only eigenvalue, and that the eigenspace for 2 is spanned by the single vector $(1,0,0)$. So there can only be a single Jordan block and the Jordan Normal Form must be

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} .$$

Alternatively, we could repurpose the computations already made in example 2.4.15 to argue that $(\lambda-2)^3$ is the minimal polynomial of A, since the minimal polynomial must divide $\chi_A(\lambda) = (\lambda-2)^3$ and since $(A-2\cdot\mathcal{I})^2$ is not the zero matrix. We can hence alternatively read out from the minimal polynomial that the largest Jordan block for 2 must have size 3.

Example 2.4.34 ***(Revisiting example Theorem 2.4.20.)*** We again look at

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} ,$$

which we saw has the minimal polynomial $(\lambda-1)^2(\lambda-5)$. Since the dimensions of the blocks for eigenvalue 1 must add up to 2, the only options would two 1×1 blocks of the form (1) (diagonalisable case), or a single Jordan Block of size 2, so either

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} .$$

Computing $\mathcal{N}((A-\mathcal{I})^2)$, we see that the eigenspace for eigenvalue 1 has dimension 1, which a) is too small for the matrix to be diagonalisable and b) tells us that we get a single Jordan

block for 1. This leaves

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

as the only option for the Jordan Normal Form of A.

Example 2.4.35 ***(Giving the JNF without knowing the matrix.)*** Suppose A is a matrix with characteristic and minimal polynomials

$$\chi_A(\lambda) = (\lambda-2)^2(\lambda-4)^3$$
$$m_A(\lambda) = (\lambda-2)(\lambda-4).$$

A and its JNF must be a 5×5 matrix (due to the dimension being equal to the degree of χ_A). Looking at eigenvalue 2, the characteristic polynomial χ_A tells us that the sizes of all Jordan blocks taken together add up to 2, and the minimal polynomial m_A tells us that the size of the biggest block is 1. So there must be two 1×1 blocks of the form (2).
When looking at eigenvalue 4, χ_A tells us that the sizes of all Jordan blocks taken together add up to 3, and m_A tells us that the size of the biggest block is 2, leaving us with a second block of size 1.
We conclude that the JNF is

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Remark 2.4.36 It can be shown that $\dim\mathcal{N}(A-\lambda\mathcal{I})^2 - \dim\mathcal{N}(A-\lambda\mathcal{I})$ equals the the number of Jordan blocks for λ of size at least 2, and so on: $\dim\mathcal{N}(A-\lambda\mathcal{I})^k - \dim\mathcal{N}(A-\lambda\mathcal{I})^{k-1}$ is the number of Jordan blocks (for λ) of size at least k. This also implies that a matrix's Jordan Normal Form is uniquely determined (up to how we order the blocks).

Assuming the existence of a full Jordan basis, we now arrive at the following theorem.

Theorem 2.4.37. *Every $n\times n$ complex matrix A can be brought into Jordan normal form (Jordan canonical form) J by conjugation. In other words, there exists an invertible matrix S such that*

$$S^{-1}AS = J = A = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_p}(\lambda_p) \end{bmatrix},$$

with each block J_i being a Jordan block of size n_i for an eigenvalue λ_i.

In particular, $n = \sum_{i=1}^{p} n_i$.

⚠ The values λ_i in this theorem are not necessarily distinct: As seen before, we might end up in a

situation where we need to go through several Jordan chains. We can, however, always reorder the basis to at least ensure that all blocks belonging to a particular eigenvalue are adjacent.

To complete the link between the Jordan Normal Form and diagonalisability, we make some basic helper observations about Jordan blocks that are true regardless of whether the scalar field is $\mathbb{C}$ or any other field (convince yourself that these are true):

Theorem 2.4.38. *Let $J_n(\mu)$ be a Jordan block for some field element $\mu \in \mathbb{K}$ and some dimension n.*

- *The characteristic polynomial of $J_n(\mu)$ is $(\lambda - \mu)^n$. Hence, the only eigenvalue of $J_n(\mu)$ is μ (with algebraic multiplicity $m_{\text{alg}}(\mu) = n$).*
- *The dimension of the eigenspace for μ (i.e., the geometric multiplicity $m_{\text{geo}}(\mu)$), however, is only 1.*

Theorem 2.4.38 tells us the following:

Corollary 2.4.39. if a matrix is in Jordan Normal Form, and one of its Jordan blocks has dimension greater than 1, then the matrix cannot be diagonalised. *(Because at least one eigenvalue will not gift enough eigenvectors).*

Corollary 2.4.40. *A is diagonalisable if and only if the minimal polynomial has only simple roots, i.e. if*

$$m_{\mathcal{A}}(\lambda) = (\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_r) \ .$$

Proof. The $\Rightarrow$ direction directly follows from the corollary: if one of the terms appears with multiplicity greater than 1, there exists a Jordan block of dimension greater than 1 and the matrix can hence not be diagonalisable.

Conversely, assume that the minimal polynomial only contains linear terms with multiplicity 1. According to the Primary Decomposition Theorem 2.4.18, for each eigenvalue λ_i we find a basis α_i of the generalised eigenspace $E^{(1)}_{\lambda_i}$ of rank 1, in a way such that their combination yields a basis of the full vector space. Since generalised eigenspace of rank 1 are just the usual eigenspaces, we just found an eigenvector basis. □

Example 2.4.41 The matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

has characteristic polynomial $\chi_A(\lambda) = \lambda^2 - 5\lambda - 2$. With Corollary 2.4.40 we can already argue that A is diagonalisable without doing any further computations: χ_A is not a perfect square, so it must factor into two distant linear factors $\lambda - \lambda_1$ and $\lambda - \lambda_2$ with $\lambda_1 \neq \lambda_2$. Since the minimal polynomial divides the characteristic polynomial, it is the same product of distinct linear terms and the matrix is hence diagonalisable.

Remark 2.4.42 Some concluding remarks on the Jordan Normal Form:

- We indicated how to bring a complex matrix A into its Jordan Normal Form J, by finding a basis consisting of Jordan chains.
- This also told us that there exists an invertible matrix S such that $S^{-1}AS = J$. Note that the matrices S and J can be complex, even when A is real!
- The strategy that is used to argue existence of (and compute) J can be generalised beyond complex maps: it works for all fields that are 'algebraically closed'.
- While the omitted proofs in this section are beyond the scope of this course, this still might give an indication how nicely we sometimes can combine reasoning about polynomials (algebra) with reasoning about geometric objects, like, e.g., subspaces that are stable under matrix operations.
- While the JNF can be very useful in derivations/proofs, it is not well-suited for numerical algorithms because it can be very sensitive to small matrix perturbations. (See example 2.4.43 below.) There exist better alternatives that transform any matrix into upper triangular form (in a numerically stable way).

Example 2.4.43 ***(The JNF is pertubation-sensitive)*** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}.$$

We determine the Jordan Normal Form of A, depending on ε:

- If $\varepsilon = 0$, then A is a Jordan block and hence already has Jordan Normal Form:

$$J = A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

  In this case, A has only one eigenvalue ($\lambda = 1$), with algebraic multiplicity 2 and geometric multiplicity 1.

- If $\varepsilon \neq 0$, then A has two distinct eigenvalues, $\lambda_1 = 1 + \sqrt{\varepsilon}$ and $\lambda_2 = 1 - \sqrt{\varepsilon}$. In this case, A is diagonalisable, so the JNF J is the eigenvalue diagonal matrix:

$$J = \begin{pmatrix} 1+\sqrt{\varepsilon} & 0 \\ 0 & 1-\sqrt{\varepsilon} \end{pmatrix}.$$

We see that A is an example where the Jordan Normal Form can change a lot, even when the matrix only changes a little (because we only added a - possibly tiny - ε in the lower left corner).

Chapter 3

Neat tricks for special linear maps: Orthogonal and symmetric maps

What happens in Chapter 3?

In the previous chapters, we looked at linear maps $\mathcal{A} : V \to V$, mostly without using any special properties of the vector space V. In this chapter, we look at the case where V is a real inner product space as introduced in *Linear Algebra 1* (see the summary in Definition B.2.8). So we have a notion for length, angle and what it means to be perpendicular. We will study two classes of linear maps that have nice properties: the first class we'll look into is the class of length-preserving maps (called orthogonal maps), the second is called symmetric maps. In both cases, we first discuss the theoretical groundwork, and then proceed by looking at the role played by the matrices that describe these maps. We will see that this also gives us geometric applications.

Learning Goals of Chapter 3: You can

- work with symmetric and orthogonal maps;
- use the presented eigenvalue/eigenvector techniques to analyse orthogonal maps, in order to determine their (geometric) meaning; and
- solve quadratic equations by applying the presented eigenvalue/eigenvector techniques for symmetric maps.

⚠ These learning goals are still 

3.1. Orthogonal maps

In this section, we restrict ourselves to ***finite-dimensional, real*** inner product spaces. We'll learn

- what orthogonal linear maps are;
- what an orthogonal matrix is and how this connects to orthogonal maps; and
- that orthogonal maps can be classified as collections of rotations and reflections, and that we can bring their representation matrix into such a nice shape that we can easily read out what the map actually does on the space.

The concept 'orthogonal matrix' originates in the work of G.F. Frobenius (1849–1917). Orthogonality relations proved to be important in his work on group representations.

Why do we care? Because we'll see that we can do some pretty straightforward computations to fully understand the geometric meaning of a given orthogonal matrix (and bring it into a shape that consumes very little space).

We start with the central definition of this section:

Definition 3.1.1 (Orthogonal map). *Let V be a real inner product space. A linear map $\mathcal{A}: V \to V$ is called* **orthogonal** *if*

$$\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$$

for all vectors $\underline{x} \in V$. In other words: a linear map $\mathcal{A}: V \to V$ is orthogonal if the **length is invariant** *under $\mathcal{A}$.*

Example 3.1.2 ***('trivial' example: identity map $\mathcal{I}$, multiples of $\mathcal{I}$)*** For the identity map $\mathcal{I}: V \to V$, one has $\mathcal{I}\underline{x} = \underline{x}$ for every vector $\underline{x}$. So the identity map is orthogonal: $\|\mathcal{I}\underline{x}\| = \|\underline{x}\|$. In the same way, $-\mathcal{I}$ is orthogonal. For $\lambda \neq \pm 1$, however $\lambda\mathcal{I}$ is not orthogonal.

Example 3.1.3 The linear map $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathcal{A}(x,y) = \frac{1}{5}(-3x+4y, 4x+3y)$$

is orthogonal because

$$\begin{aligned} \|\mathcal{A}(x,y)\|^2 &= \tfrac{1}{25}((-3x+4y)^2 + (4x+3y)^2) \\ &= \tfrac{1}{25}(9x^2+16y^2-24xy+16x^2+9y^2+24xy) \\ &= x^2+y^2 = \|(x,y)\|^2. \end{aligned}$$

Example 3.1.4 ***(Orthogonal reflection)*** Let $\underline{a}$ be a vector of length 1 in the inner product space V. We look at the linear map $\mathcal{A}: V \to V$ given by the rule $\mathcal{A}\underline{x} = \underline{x} - 2(\underline{x},\underline{a})\underline{a}$.
To show that this map is orthogonal, we expand $\|\mathcal{A}\underline{x}\|^2$:

$$\begin{aligned} \|\mathcal{A}\underline{x}\|^2 = (\mathcal{A}\underline{x}, \mathcal{A}\underline{x}) &= (\underline{x} - 2(\underline{x},\underline{a})\underline{a}, \underline{x} - 2(\underline{x},\underline{a})\underline{a}) = \\ &= (\underline{x},\underline{x}) - 4(\underline{x},\underline{a})(\underline{x},\underline{a}) + 4(\underline{x},\underline{a})^2(\underline{a},\underline{a}) = (\underline{x},\underline{x}), \end{aligned}$$

from which the claim immediately follows.

Taking a geometric point of view, we will convince ourselves that $\mathcal{A}$ is a (perpendicular, or orthogonal) reflection in the subspace $\text{span}(\underline{a})^\perp$, i.e., the space that is orthogonal to the line spanned by $\underline{a}$. (So it is not surprising that this map is orthogonal.)
To see this, we take any vector $\underline{x}$ and define the vector $\underline{y}$ we get by moving from $\underline{x}$ in the direction of $\underline{a}$, until we reach the orthogonal complement $\text{span}(\underline{a})^\perp$. More formally, $\underline{y}$ is the intersection of the line $\underline{x}+\lambda\underline{a}$ with $\text{span}(\underline{a})^\perp$, so we get $\underline{y}$ by solving $(\underline{x}+\lambda\underline{a},\underline{a})=0$ for λ. This has the solution $\lambda=-(\underline{x},\underline{a})$, so $\underline{y}=\underline{x}-(\underline{x},\underline{a})\,\underline{a}$. So by subtracting $(\underline{x},\underline{a})\,\underline{a}$ from $\underline{x}$ twice, $\mathcal{A}$ moves $\underline{x}$ through $\text{span}(\underline{a})^\perp$ and maps it unto its reflection.

Example 3.1.5 ***(Orthogonal projection)*** We once more look at an orthogonal projection $\mathcal{P}$ unto a line $\ell=\text{span}(\underline{a})$, in a vector space V of dimension at least 2. Map $\mathcal{P}$ is not orthogonal: Since the dimension of V is bigger than 1, we can find a vector $\underline{b}$ that is perpendicular to $\underline{a}$ and different from the zero vector. Then $\mathcal{P}\underline{b}=\underline{0}$, so $0=\|\mathcal{P}\underline{b}\|\neq\|\underline{b}\|$.

We will soon use the map-matrix connection to study the properties of matrices that belong to orthogonal maps. For that, we'll need Theorem 3.1.9 below which shows that ***orthogonality can be determined in four equivalent ways, of which some might be more convenient depending on context***. (The usefulness of Theorem 3.1.9 will become clear further when we actually start looking at the respective matrices.) To get to this equivalent classification approaches, we will use three helper results about inner products (Theorems 3.1.6 - 3.1.8).

Theorem 3.1.6 (Polarisation formula). *In a real inner product space, we always have*

$$(\underline{x},\underline{y})=\frac{1}{2}\Big((\underline{x}+\underline{y},\underline{x}+\underline{y})-(\underline{x},\underline{x})-(\underline{y},\underline{y})\Big)\ .$$

As a consequence, we can express **inner products between vectors in terms of the vectors lengths:**

$$(\underline{x},\underline{y})=\frac{1}{2}(\|\underline{x}+\underline{y}\|^2-\|\underline{x}\|^2-\|\underline{y}\|^2)\ .$$

Proof. Using bilinearity of the inner product space, we get

$$(\underline{x}+\underline{y},\underline{x}+\underline{y})=(\underline{x},\underline{x})+2(\underline{x},\underline{y})+(\underline{y},\underline{y})\ ,$$

and we can isolate the term $(\underline{x},\underline{y})$ on the right-hand side. □

Next, we recall a helpful fact from *Linear Algebra I*: ***given an orthonormal basis, the inner product on V boils down to the standard inner product of the coordinate vectors.***

Theorem 3.1.7. *Let V be a real inner product space and let $\alpha=\{\underline{a}_1,\cdots,\underline{a}_n\}$ be an orthonormal basis of V. Let $\underline{x},\underline{y}$ be two vectors in V with respective coordinate vectors $(x_1,\cdots,x_n),(y_1,\cdots,y_n)$.*

Then $(\underline{x},\underline{y})=\sum_{i=1}^n x_i y_i$. In other words, **when using an orthonormal basis, inner product = inner product of coordinate vectors**.

Proof. Like any other vectors, $\underline{x}, \underline{y}$ are linear combinations of the base vectors in α, with their linear scalars being the elements of the coordinate vectors. We can now use that the inner product is bilinear, and that the base vectors are orthogonal to each other and have length 1:

$$(\underline{x}, \underline{y}) = (\sum_{i=1}^{n} x_i \underline{a}_i, \sum_{j=1}^{n} x_j \underline{a}_j) = \sum_{i,j=1}^{n} x_i x_j (\underline{a}_i, \underline{a}_j) = \sum_{i,j=1}^{n} x_i x_j .$$

□

Theorem 3.1.7 also has a useful consequence concerning vector lengths:

Theorem 3.1.8. *Let V, α be like in Theorem 3.1.7, and let $\underline{v}$ be a vector in V with respective coordinate vector $(v_1, \cdots, v_n)$. Then $\|\underline{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$.*

In other words, **when using an orthonormal basis, vector length = coordinate vector length**.

Proof. This follows directly from Theorem 3.1.7: Per definition of the length on V,

$$\|\underline{v}\| = \sqrt{(\underline{v}, \underline{v})} \overset{3.1.7}{=} \sqrt{\sum_{i=1}^{n} v_i^2} \ .$$

□

With these helper results at hand, we can now see that ***we can also define orthogonality of a map by the invariance of the inner product***; another consequence is that ***the inner product between vectors does not change if we replace them by their images***.

Theorem 3.1.9. *Let V be a finite-dimensional real inner product space, and $\mathcal{A} : V \to V$ be linear. The following statements are equivalent:*

1. *$\mathcal{A}$ is orthogonal, i.e., $\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$ for all vectors $\underline{x} \in V$. In other words,* **the length is invariant under $\mathcal{A}$**.
2. *$(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = (\underline{x}, \underline{y})$ for all vectors $\underline{x}, \underline{y} \in V$. In other words,* **the inner product is invariant under $\mathcal{A}$**.
3. *For every orthonormal system $\underline{a}_1, \ldots, \underline{a}_n$ in V, the system $\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_n$ is also orthonormal. In other words,* **orthonormal systems are mapped to orthonormal systems**.
4. *There exists an orthonormal basis $\{\underline{a}_1, \ldots, \underline{a}_m\}$ of V such that $\{\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_m\}$ also is an orthonormal basis of V.*

Proof. We will do a circular proof, meaning we will establish the following chain of implications: 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 1).

1) ⇒ 2): Assume statement 1 to be true, and take two arbitrary vectors $\underline{x}$ and $\underline{y}$ in V. We will now use the polarisation formula to show that $(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = (\underline{x}, \underline{y})$: According to the polarisation formula (Theorem 3.1.6), we have

$$(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = \frac{1}{2}(\|\mathcal{A}\underline{x} + \mathcal{A}\underline{y}\|^2 - \|\mathcal{A}\underline{x}\|^2 - \|\mathcal{A}\underline{y}\|^2) \ .$$

We can now apply statement 1 to the summands:

$$\|\mathcal{A}\underline{x}\|^2 = \|\underline{x}\|^2, \quad \|\mathcal{A}\underline{y}\|^2 = \|\underline{y}\|^2 \quad \text{and} \quad \|\mathcal{A}\underline{x} + \mathcal{A}\underline{y}\|^2 = \|\mathcal{A}(\underline{x}+\underline{y})\|^2 = \|\underline{x}+\underline{y}\|^2 \ ,$$

so

$$(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = \frac{1}{2}(\|\underline{x}+\underline{y}\|^2 - \|\underline{x}\|^2 - \|\underline{y}\|^2) = (\underline{x}, \underline{y}) \ ,$$

where the last equation again used the polarisation formula.

So if the length is invariant under $\mathcal{A}$, then also the inner product.

2) $\Rightarrow$ 3): Assume statement 2 to be true, and let $\underline{a}_1, \ldots, \underline{a}_n$ be an orthonormal system. We show that $\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_n$ also is an orthonormal system by verifying the inner product condition: $(\mathcal{A}\underline{a}_i, \mathcal{A}\underline{a}_j) = (\underline{a}_i, \underline{a}_j) = 1$ if $i = j$, and 0 if $i \neq j$.

3) $\Rightarrow$ 4): Assume statement 3 to be true, and set $m := \dim V$. Choose an orthonormal basis $\{\underline{a}_1, \ldots, \underline{a}_m\}$ of V. (Such a basis exists, e.g., due to the availability Gram–Schmidt procedure). What's left to show is that the set $\{\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_m\}$ also is an orthonormal basis of V. Orthonormality directly follows from statement 3, and the set in principle has the right size ($m = \dim V$). The only missing point is that the set's vectors are also linearly independent, which is true because orthonormal sets of vectors (such as this set) in real inner product spaces are always linearly independent. (This was shown in *Linear Algebra I*.)

4) $\Rightarrow$ 1): Assume statement 4 to be true, and let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_m\}$ be the respective orthonormal basis that exists according to statement 4. (So the set $\mathcal{A}\alpha := \{\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_m\}$ is again an orthonormal basis.) We will now show that $\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$ for all vectors $\underline{x} \in V$, so take an arbitrary vector $\underline{x} \in V$. Writing $\underline{x} = x_1\underline{a}_1 + \cdots + x_m\underline{a}_m$, we see that $\mathcal{A}\underline{x} = x_1\mathcal{A}\underline{a}_1 + \cdots + x_m\mathcal{A}\underline{a}_m$. We can now use our result that for orthonormal bases, the vector length always equals the coordinate vector length (Theorem 3.1.8, with basis α for $\|\underline{x}\|$ and basis $\mathcal{A}\alpha$ for $\|\mathcal{A}\underline{x}\|$):

$$\|\underline{x}\| = \sqrt{x_1^2 + \cdots + x_m^2} = \|\mathcal{A}\underline{x}\| \ .$$

□

The last thing we note before moving on to matrices is that ***compositions and inverses of orthogonal maps (which always exist) are also orthogonal.***

Theorem 3.1.10. *Let V be a finite-dimensional real inner product space.*

1. *If $\mathcal{A} : V \to V$ and $\mathcal{B} : V \to V$ are orthogonal, then the map $\mathcal{AB} : V \to V$ is also orthogonal. In other words,* **composing preserves orthogonality**.
2. *If $\mathcal{A} : V \to V$ is orthogonal, then $\mathcal{A}$ is invertible and the inverse $\mathcal{A}^{-1}$ is also orthogonal.*

Proof. 1) To show that $\|\mathcal{AB}\underline{x}\| = \|\underline{x}\|$ for all vectors $\underline{x} \in V$, we use that both $\mathcal{A}$ and $\mathcal{B}$ are orthogonal:

$$\|\mathcal{AB}\underline{x}\| = \|\mathcal{A}(\mathcal{B}\underline{x})\| = \|\mathcal{B}\underline{x}\| = \|\underline{x}\| \ .$$

2) We will first show that $\mathcal{A}$ is invertible, using the invertibility criterion 1.1.19 from the LinA I refresher: According to Theorem 1.1.19, $\mathcal{A}$ is invertible if its null space is trivial, i.e., if $\mathcal{N} = \{\underline{0}\}$. So

we only need to show that if a vector $\underline{x}$ lies in $\mathcal{N}$ (so $\mathcal{A}\underline{x} = \underline{0}$), then $\underline{x}$ must be $\underline{0}$. For this, we use that $\mathcal{A}$ is orthogonal: Since $\mathcal{A}$ is orthogonal, we have $\|\underline{x}\| = \|\mathcal{A}\underline{x}\| = \|\underline{0}\| = 0$, so $\underline{x} = \underline{0}$.

To show that $\mathcal{A}^{-1}$ is also orthogonal, we cleverly rewrite $\|\underline{x}\|$ in a way such that we can use orthogonality of $\mathcal{A}$: For all vectors $\underline{x} \in V$, we have $\underline{x} = \mathcal{A}\mathcal{A}^{-1}\underline{x}$. Since $\mathcal{A}$ is orthogonal, this implies $\|\underline{x}\| = \|\mathcal{A}\mathcal{A}^{-1}\underline{x}\| = \|\mathcal{A}^{-1}\underline{x}\|$. □

Remark 3.1.11 As a consequence, powers of orthogonal maps (like $\mathcal{A}^n$, $\mathcal{A}^{-n}$ for some n) are also orthogonal.

⚠ In inner product spaces of infinite dimension, there exist orthogonal maps that are not invertible.

3.1.1 Orthogonal matrices and their connection with orthogonal maps, $\mathbb{R}^n$ case

We will now concentrate on orthogonal maps on $\mathbb{R}^n$ and use the map-matrix connection to study the properties of their matrices. So this subsection addresses orthogonal maps $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$.

To start translating orthogonality into matrix terms, we will now make Theorem 3.1.9 useful by deriving a corollary:

Corollary 3.1.12. *We now consider $\mathbb{R}^n$ with the standard inner product. A linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if $\mathcal{A}\underline{e}_1, \ldots, \mathcal{A}\underline{e}_n$ is an orthonormal system.*

Proof. This follows from plugging the standard basis (which is orthonormal) into Theorem 3.1.9. □

This is interesting because we know that for maps $\mathcal{A}$ acting on $\mathbb{R}^n$, the columns of their representation matrix A are the images of the standard basis vectors (so $\mathcal{A}\underline{e}_1, \ldots, \mathcal{A}\underline{e}_n$). So, to check whether $\mathcal{A}$ is orthogonal, we can look at its representation matrix A and verify whether its columns form an orthonormal system. This observation motivates how we define orthogonal matrices:

Definition 3.1.13 (Orthogonal matrix). *A real $n \times n$ matrix A is called* **orthogonal** *if its columns form an orthonormal system in $\mathbb{R}^n$.*

We convince ourselves that this definition makes sense: according to Theorem 3.1.14, $\mathcal{A}$ is orthogonal iff its matrix is orthogonal. Actually, the criteria given in Theorem 3.1.14 allow several convenient ways to determine orthogonality by looking at a matrix:

Theorem 3.1.14. *Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with representation matrix A. The following statements are equivalent:*

1. *$\mathcal{A}$ is orthogonal.*
2. *The columns of A form an orthonormal system. In other words,* **$\mathcal{A}$'s matrix is orthogonal.**
3. *$A^\top A = \mathcal{I}_n$. In other words,* **the transpose is the inverse**.

4. *The rows of A form an orthonormal system.*

Proof. We already know that 1) $\Leftrightarrow$ 2): if $\mathcal{A}$ is orthogonal, then the columns of A form an orthonormal system, and vice versa.

2) $\Leftrightarrow$ 3): To check equivalence, we'll use the matrix multiplication rules: the element at the ij-th position of the matrix product $A^\top A$ equals the inner product of the i–th and the j–th column of A. This helps us with showing both directions:

We first show that the columns of A form an orthonormal system if $A^\top A = \mathcal{I}_n$: If $A^\top A = \mathcal{I}_n$, then the element at the ij-th position of $A^\top A$ is 0 if $i \neq j$ and 1 if $i = j$. Since this element is equal to the inner product of the i–th and the j–th column of A, we get that the columns are orthonormal.

Vice versa, we show that $A^\top A = \mathcal{I}_n$ if the columns of A form an orthonormal system: again, we use that the element at the ij-th position of $A^\top A$ equals the inner product of the i–th and the j–th column of A. So, provided the columns form an orthonormal system, the element is 0 if $i \neq j$ and 1 if $i = j$, so $A^\top A = \mathcal{I}_n$.

1) $\Leftrightarrow$ 4): We first show that also the rows of A form an orthonormal system if $\mathcal{A}$ is orthogonal: If $\mathcal{A}$ is orthogonal, then so is its inverse, which has the matrix $A^{-1} = A^\top$ due to statement 3. But then, the columns of $A^\top$ (so the rows of A) form an orthonormal system due to statement 2.

In a similar way, we can show that $\mathcal{A}$ is orthogonal if the rows of A form an orthonormal system: Then A^T is orthogonal, so statement 3 tells us that $\mathcal{I}_n = (A^\top)^\top A^\top = AA^\top$. So A is also orthogonal and hence $\mathcal{A}$. □

Example 3.1.15 We revisit Theorem 3.1.3: The linear map $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathcal{A}(x, y) = \frac{1}{5}(-3x + 4y, 4x + 3y)$$

has the matrix

$$A = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

To check for orthogonality, we could check

- with the definition as we did in example Theorem 3.1.3;
- that every column of A has length 1 and that the two different columns have inner product 0;
- that $A^\top A = \mathcal{I}_2$.

Statement 3 of Theorem 3.1.14 also tells us that computing the inverse is simple because we only need to transpose:

$$A^{-1} = A^\top = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

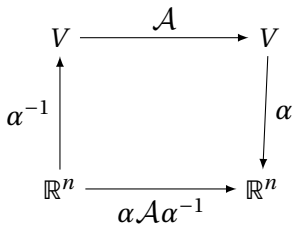
3.1.2 Orthogonal matrices and their connection with orthogonal maps, general case

We will now discuss how we can generalise Section 3.1.1 for general orthogonal maps, meaning we generalise our results concerning orthogonal maps $\mathcal{A}:\mathbb{R}^n \to \mathbb{R}^n$ for orthogonal maps $\mathcal{A}: V \to V$ in the case where V is still a real, finite-dimensional vector space, but not necessarily $\mathbb{R}^n$. On a high level, we'll once more apply the strategy we already used in previous sections: we'll use coordinates to bring everything back to the $\mathbb{R}^n$ case.

Towards that end, we'll use that coordinate switching plays nicely with orthogonal maps:

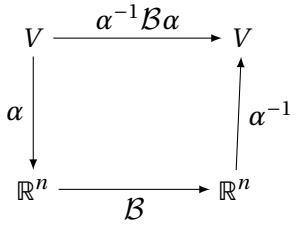
Lemma 3.1.16. *Let V be an n-dimensional real inner product space, with its inner product denoted as $(-,-)_V$, and α an orthonormal basis of V. Let $\mathcal{A}: V \to V$ be an orthogonal map. We denote by $\|-\|_{st}$ the standard length in $\mathbb{R}^n$ and by $\|-\|_V$ the length implied by V's inner product. Like in previous sections, we also denote by α the map that maps vectors to their coordinate vectors. Then*

1. *$\|\alpha(\mathcal{A}\underline{v})\|_{st} = \|\underline{v}\|_V$ for all vectors $\underline{v} \in V$, in other words,* **applying the map and then switching to coordinates preserves length***;*
2. *$\|\mathcal{A}\alpha^{-1}\underline{x}\|_V = \|\underline{x}\|_{st}$ for all vectors $\underline{x} \in \mathbb{R}^n$, in other words,* **switching from coordinates to vectors in V and then applying the map preserves length***;*
3. *the map $\alpha\mathcal{A}\alpha^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, in other words,* **translating $\mathcal{A}$ into an $\mathbb{R}^n$ map preserves orthogonality**.



Vice versa, we get the following:

4. *If a map $\mathcal{B}: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then so is the map $\alpha^{-1}\mathcal{B}\alpha: V \to V$. In other words,* **translating an $\mathbb{R}^n$ map into a V map also preserves orthogonality**.



Proof. For the proof, we will use that vector orthonormality is preserved when switching between the 'abstract vectors level' and the 'coordinate level' (as seen with Theorem 3.1.7 and Theorem 3.1.8).

We start by showing 1., so we start by showing that $\|\alpha(\mathcal{A}\underline{v})\|_{st} = \|\underline{v}\|_V$ for all vectors $\underline{v} \in V$: For any vector $\underline{v} \in V$, we have that

$$\|\alpha(\mathcal{A}\underline{v})\|_{st} \overset{3.1.8}{=} \|\mathcal{A}\underline{v}\|_V = \|\underline{v}\|_V \ .$$

Next, we show 2., so we show that $\|\mathcal{A}\alpha^{-1}\underline{x}\|_V = \|\underline{x}\|_{st}$ for all vectors $\underline{x} \in \mathbb{R}^n$: We have

$$\|\mathcal{A}\alpha^{-1}\underline{x}\|_V = \|\alpha^{-1}\underline{x}\|_V \overset{3.1.8}{=} \|\underline{x}\|_{st} \ ,$$

where the last step used that we can apply 3.1.8 to the vector $\underline{v} \in V$ which has $\underline{x}$ as its coordinate vector. (In more detail, we have $\underline{x} = \alpha(\underline{v})$. So $\|\alpha^{-1}\underline{x}\|_V = \|\alpha^{-1}(\alpha(\underline{v}))\|_V = \|\underline{v}\|_V \overset{3.1.8}{=} \|\alpha(\underline{v})\|_{st} = \|\underline{x}\|_{st}$.)

Next, we show 3., so we show that the map $\alpha\mathcal{A}\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal. To prove that indeed $\|\alpha\mathcal{A}\alpha^{-1}\underline{x}\|_{st} = |\underline{x}\|_{st}$ for any vector $\underline{x} \in \mathbb{R}^n$, we use the statements which we just proved: We have

$$\|\alpha\mathcal{A}\alpha^{-1}\underline{x}\|_{st} \overset{1.}{=} \|\mathcal{A}\alpha^{-1}\underline{x}\|_V \overset{2.}{=} |\underline{x}\|_{st} \ .$$

Lastly, we show that the map $\alpha^{-1}\mathcal{B}\alpha : V \to V$ is orthogonal for any orthogonal map $\mathcal{B} : \mathbb{R}^n \to \mathbb{R}^n$. For any $\underline{v} \in V$, we have

$$\|\alpha^{-1}\mathcal{B}\alpha\underline{v}\|_V \overset{1.}{=} \|\mathcal{B}\alpha\underline{v}\|_{st} = \|\alpha\underline{v}\|_{st} = \|\underline{v}\|_V \ .$$

where the first step used the first item (applied to $\alpha^{-1}\mathcal{B}\alpha\underline{v}$) and the second step used orthogonality of $\mathcal{B}$. □

We'll later use that we can switch between bases (so we'll also switch the coordinate system). We quickly convince ourselves that coordinate switching preserves orthogonality (and hence does not disrupt orthogonality checking):

> **Theorem 3.1.17.** *If α and β are two orthonormal bases in a real inner product space, then the transition matrix ${}_\beta S_\alpha$ is orthogonal.*

⚠ It is important that α and β are ***orthonormal*** bases – the theorem does not hold without this condition!

Proof. The base transition matrix ${}_\beta S_\alpha$ simply maps α-coordinate vectors to β-coordinate vectors, it is the matrix of the map $\beta\alpha^{-1}$. It is hence sufficient to check that its map $\beta\alpha^{-1}$ is an orthogonal map (due to Theorem 3.1.14), which we'll now do.

Given that α, β are orthonormal bases, the coordinate vector length is identical for both bases:

$$\|\alpha\underline{v}\|_{st} = \|\underline{v}\|_V = \|\beta\underline{v}\|_{st}$$

for all vectors $\underline{v} \in V$. Taking any vector $\underline{x} \in \mathbb{R}^n$, we can denote the corresponding 'abstract' vector with respect to α by $\underline{v}$ (so $\underline{v} := \alpha^{-1}\underline{x}$) and plug it into the length equation to get

$$\|\beta\alpha^{-1}\underline{x}\|_{st} = \|\beta\underline{v}\|_{st} = \|\alpha\underline{v}\|_{st} = \|\underline{x}\|_{st} \ .$$

□

Given that ${}_\beta S_\alpha$ is an orthogonal matrix, we can easily compute the transition matrix for the other direction:

Theorem 3.1.18. *If α and β are two orthonormal bases in a real inner product space, then*

$$ {}_\alpha S_\beta = {}_\beta S_\alpha{}^{-1} \overset{3.1.14}{=} {}_\beta S_\alpha{}^\top \ . $$

Proof. We know from Section 1.3.2 that the 'other direction' matrix ${}_\alpha S_\beta$ is the inverse of ${}_\beta S_\alpha$, and Theorem 3.1.14 tells us that the inverse of an orthogonal matrix is its transpose. □

Remark 3.1.19 In a nutshell: we can switch between

- the vector space level and the coordinate level or
- coordinate systems for different basis

without disrupting orthogonality (as long as were working with an ***orthonormal*** basis).

Since switching between vector space and coordinate level does not disrupt orthogonality, we are now ready to generalise the relation between orthogonal maps and matrices to the setting where V is not simply $\mathbb{R}^n$.

Theorem 3.1.20. *Let α be an orthonormal basis for a finite-dimensional real inner product space V, and let $\mathcal{A}: V \to V$ be a linear map and A_α the matrix of $\mathcal{A}$ (with respect to basis α).*

Then the map $\mathcal{A}$ is orthogonal if and only if its matrix A_α is orthogonal.

Proof. As promised, we'll translate the situation back to $\mathbb{R}^n$ (via coordinates), where we can then apply the map-matrix orthogonality equivalence (Theorem 3.1.14): We will use that the matrix A_α of $\mathcal{A}$ w.r.t. the basis α is defined to be the matrix of the linear map $\alpha\mathcal{A}\alpha^{-1}: \mathbb{R}^n \to \mathbb{R}^n$.

Assume that $\mathcal{A}: V \to V$ is orthogonal. Then we already know that we can switch to the coordinate level and get that $\alpha\mathcal{A}\alpha^{-1}$ (due to Lemma 3.1.16). For orthogonal maps acting on $\mathbb{R}^n$, we know that their matrix (in our case A_α) is an orthogonal matrix (due to Theorem 3.1.14).

Conversely, if the matrix A_α is orthogonal, then the map $\alpha\mathcal{A}\alpha^{-1}$ is orthogonal (also due to Theorem 3.1.14), and therefore we can switch back to the 'abstract vectors' level and see that $\mathcal{A} = \alpha^{-1}(\alpha\mathcal{A}\alpha^{-1})\alpha$ is orthogonal (due to Lemma 3.1.16). □

Example 3.1.21 ***(Orthogonal reflection (example 3.1.4), continued)*** To check orthogonality of map $\mathcal{A}$, we can alternatively check orthogonality of the representation matrix for a cleverly chosen orthonormal basis.
We choose an the orthonormal basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ whose first vector $\underline{a}_1$ is $\underline{a}$. Then $\mathcal{A}\underline{a}_1 = -\underline{a}_1$ (since $\mathcal{A}\underline{a} = \underline{a} - 2(\underline{a},\underline{a})\underline{a} = -\underline{a}$), and $\mathcal{A}\underline{a}_i = \underline{a}_i$ for $i = 2, \ldots, n$ (since $\mathcal{A}\underline{a}_i = \underline{a}_i - 2(\underline{a}_i,\underline{a})\underline{a} = \underline{a}_i - 0 \cdot \underline{a}$).
We see that this basis is not only orthonormal, but it also only consists of eigenvectors! (The eigenvalues are -1 and 1.) The matrix is therefore a diagonal matrix whose diagonal is -1, $1, \ldots, 1$, and for such diagonal matrices, checking orthogonality is easy (due to the transpose criterion).

3.1.3 Classification of orthogonal maps

In this remainder of our orthogonality Section 3.1, we study how real orthogonal maps could possibly look, in particular for dimensions up to 3. It will turn out that orthogonal maps always boil down to being rotations, reflections and combinations of the two. We will also see how to find a basis such that the representation matrix becomes easy to interpret.

On our way towards this result, we first make another useful observation about orthogonality and a case separation:

Definition/theorem 3.1.22. *Let α be an orthonormal basis for a finite-dimensional real inner product space V, and let $\mathcal{A} : V \to V$ be a linear map and A_α the matrix of $\mathcal{A}$ (with respect to basis α).*

Then $\det(A_\alpha) = \pm 1$.

We say that an orthogonal map $\mathcal{A}$ is

- **directly orthogonal** *if* $\det(A_\alpha) = 1$*; and*
- **indirectly orthogonal** *if* $\det(A_\alpha) = -1$.

Before we do the proof, we quickly note that we assign (in)direct orthogonality to the $\mathcal{A}$ (instead of just to the matrix A_α) because the determinant is independent of the choice of basis (Definition/theorem 2.1.16).

Proof. We will show the general result that all orthogonal matrices have determinant ± 1 – we just saw that matrices of orthogonal maps are orthogonal matrices (Theorem 3.1.20), so we can than apply the general result to A_α.

So let A be an orthogonal matrix. Then $A^\top A = \mathcal{I}$ (due to Theorem 3.1.14), and we can now use multiplicity of the determinant:

$$1 = \det(\mathcal{I}) = \det(A^\top) \cdot \det(A) = \det(A)^2 \ ,$$

where the last step used that the determinant of the transpose is equal to the determinant ('Linear Algebra 1', there currently Theorem 4.2.16). Since $\det(A)^2 = 1$, it follows that $\det(A) = \pm 1$. □

With this observation at hand, we'll now determine how the roots of the characteristic polynomial $\chi_\mathcal{A}$ look for orthogonal maps, which also tells us a lot about the eigenvalues.

Theorem 3.1.23. *Let α be an orthonormal basis for a real inner product space V of dimension n, and let $\mathcal{A} : V \to V$ be an orthogonal map, A_α the matrix of $\mathcal{A}$ (with respect to basis α), and $\chi_\mathcal{A} = \det(A_\alpha - \lambda\mathcal{I})$ the characteristic polynomial of $\mathcal{A}$.*

Then

- *every real root of $\chi_\mathcal{A}$ is either 1 or -1.*
- *for any non-real root μ of $\chi_\mathcal{A}$, its complex conjugate $\bar{\mu}$ is also a root of $\chi_\mathcal{A}$.*

- *if* $\det(A_\alpha) = -1$ *(so, if* $\mathcal{A}$ *is indirectly orthogonal), then -1 is an eigenvalue of* $\mathcal{A}$.

Proof. We discussed in Chapter 2 (below 2.1.16) that – like any other polynomial with real coefficients – the characteristic polynomial $\chi_\mathcal{A} = \det(A_\alpha - \lambda I)$ completely splits into linear terms, so it is a product of linear terms $\lambda - \lambda_i$, with each linear term belonging a complex root λ_i of $\chi_\mathcal{A}$.

To prove the first statement, let λ_i be one of the real roots of $\chi_\mathcal{A}$, meaning λ_i is also an eigenvalue. We now use orthogonality of $\mathcal{A}$ to argue that λ_i can only be ± 1: Pick an eigenvector $\underline{v}$ for λ_i, so $\mathcal{A}\underline{v} = \lambda_i \underline{v}$ and hence $\|\mathcal{A}\underline{v}\| = |\lambda_i| \|\underline{v}\|$. But at the same time, orthogonal maps preserve length, so $\|\mathcal{A}\underline{v}\| = \|\underline{v}\|$. Therefore, $|\lambda_i|$ must be 1, so $\lambda_i = \pm 1$.

To prove the second statement, we remind ourselves that the matrix's determinant $\det(A_\alpha)$ is the product of the roots (Theorem 2.1.17). At the same time, we just saw that $\det(A_\alpha) = \pm 1$. So the determinant (and hence the product of the roots) is real. Therefore, any non-real root λ_i must come with its complex conjugate $\bar{\lambda_n}$. (Otherwise, the product would not be a real value.)

To prove the third statement, we again use that $\det(A_\alpha)$ is the product of the roots and show that -1 must be amongst those roots. (Meaning -1 is an eigenvalue.) The product of the non-real roots is a positive real: it is a product of non-real roots that come together with their complex conjugates, and for non-real values a we have that $a\,\bar{a} = |a|^2 > 0$. So the determinant can only be -1 if the product of the real roots is negative. This is only possible if one of the real roots is negative, with the only possibility being -1 since all real roots are ± 1. □

We can now use our knowledge about the roots/eigenvalues to deconstruct/classify orthogonal maps. We start with the simplest case, the case where the dimension of V is 1.

Theorem 3.1.24 (Classification for dimension 1). *On real vector spaces of dimension 1,*

- *the only map with* $\det(A_\alpha) = 1$ *(so, the only directly orthogonal map) is* $\mathcal{I}$*, the identity; and*
- *the only map with* $\det(A_\alpha) = -1$ *(so, the only indirectly orthogonal map) is* $-\mathcal{I}$*. (So, the map defined by* $-\mathcal{I}\underline{v} = -\underline{v}$*. In other words: the reflection through the origin.)*

Proof. To prove the theorem, we first note that linear maps on real, one-dimensional vector spaces are always simply multiplication with some real value λ. (Check this, e.g., by noting that the matrix of such a linear map is a real 1×1 matrix whose only element is some λ.) In other words, any linear map $\mathcal{A}$ is of the form $\mathcal{A} = \lambda\mathcal{I}$ for some λ. (So the map defined by $\lambda\mathcal{I}(\underline{v}) = \lambda\underline{v}$.)

For $\mathcal{A}$ to be orthogonal, we need that it preserves length, which can only be the case if $\lambda = \pm 1$ (as seen in example 3.1.2). In the case where $\lambda = 1$, we have that $\mathcal{A} = \mathcal{I}$ (which is directly orthogonal). In the other case, we have that $\mathcal{A} = -\mathcal{I}$ (which is indirectly orthogonal). □

Theorem 3.1.25 (Classification for dimension 2). *If* V *is a real inner product space of dimension 2,*

- *any directly orthogonal map* $\mathcal{A}$ *on* V *is a rotation around the origin by some angle* φ*, and we can always find an orthonormal basis* α *such that the respective matrix* A_α *of* $\mathcal{A}$ *becomes*

the rotation matrix:

$$A_\alpha = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

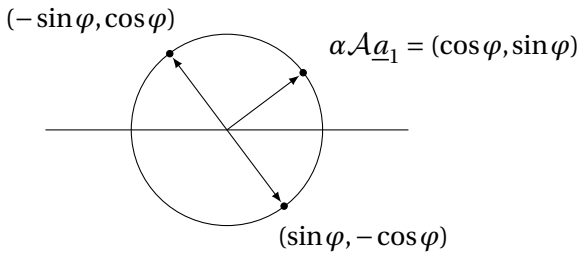
- *any indirectly orthogonal map $\mathcal{A}$ on V is a reflection with a line $\ell = \operatorname{span}(\underline{b})$ as its reflection axis, with eigenvalues ± 1 and eigenspaces $E_1 = \ell$ and $E_{-1} = \ell^\perp$, the orthogonal complement of ℓ. We can always find an orthonormal basis β such that*

$$A_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. We'll first analyse at how the matrix A_α of a general orthonormal map $\mathcal{A}$ would look for an orthonormal basis $\alpha = \{\underline{a}_1, \underline{a}_2\}$, by looking at the columns of A_α. In other words, we'll analyse the α-coordinate vectors of the images $\mathcal{A}\underline{a}_1$ and $\mathcal{A}\underline{a}_2$ (as these are the columns of A_α). (If $V = \mathbb{R}^2$, we can simply work with the standard basis.)

We first note that $\mathcal{A}\underline{a}_1, \mathcal{A}\underline{a}_2$ again form an orthonormal system (because $\mathcal{A}$ is orthogonal and α is orthonormal). Therefore,

- $\mathcal{A}\underline{a}_1$ has length 1, as does its coordinate vector with respect to basis α. So we can write $\alpha\mathcal{A}\underline{a}_1$, the coordinate vector of $\mathcal{A}\underline{a}_1$, as $(\cos\varphi, \sin\varphi)$ for some angle φ.
- the coordinate vector of $\mathcal{A}\underline{a}_2$ also has length 1 and is perpendicular to $(\cos\varphi, \sin\varphi)$. (Because $\mathcal{A}\underline{a}_2$ has length 1 and is perpendicular to $\mathcal{A}\underline{a}_1$.) This leaves us with two possibilities for the coordinate vector of $\mathcal{A}\underline{a}_2$: it is either $(-\sin\varphi, \cos\varphi)$ or $(\sin\varphi, -\cos\varphi)$.



In the first case, we get the matrix

$$A_\alpha = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

which has determinant +1 (so the map must be directly orthogonal). This is the matrix of a ***rotation around the origin by angle*** φ. (You can verify this by checking that the angle between any vector $\underline{v}$ and its image $\mathcal{A}\underline{v}$ is φ. E.g., take arbitrary vector $\underline{v} = v_1 \cdot \underline{a}_1 + v_2 \cdot \underline{a}_2$, apply A_α to the coordinate vector (v_1, v_2) to see that $\mathcal{A}\underline{v} = (v_1\cos\varphi - v_2\sin\varphi)\underline{a}_1 + (v_1\sin\varphi + v_2\cos\varphi)\underline{a}_2$, and then compute the angle between $\underline{v}$ and its $\mathcal{A}\underline{v}$.)

In the second case, we get the matrix

$$A_\alpha = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix},$$

which has determinant -1 (so the map must be indirectly orthogonal). To show existence of the basis β, we first note that $\mathcal{A}$ has eigenvalues ± 1:

- Eigenvalue -1 follows from $\mathcal{A}$ being indirectly orthogonal and Theorem 3.1.23.
- Why is 1 also an eigenvalue of $\mathcal{A}$? Because the characteristic polynomial factors into two linear terms $\lambda - \lambda_1$ and $\lambda - \lambda_2$ (since V is 2-dimensional), with eigenvalue -1 being among the two roots λ_1 and λ_2. The determinant is the product of these roots (Theorem 2.1.17) and at the same time -1, so $\lambda_1 \cdot \lambda_2 = -1$, hence the root that is not -1 must be +1.

Given that we have two different eigenvalues (which will always have some eigenvectors) and that the dimension of V is 2, we know that we can find a basis $\beta = \{\underline{b}_1, \underline{b}_2\}$ of V that consists of 2 eigenvectors, one for each of the two eigenvalues -1 and 1. With respect to this basis, the matrix of $\mathcal{A}$ is the eigenvalue diagonal matrix, so

$$A_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lastly, we look at the geometric meaning in the second case: we first note that the eigenvectors $\underline{b}_1$ and $\underline{b}_2$ are perpendicular. (Because

$$(\underline{b}_1, \underline{b}_2) = (\mathcal{A}\underline{b}_1, \mathcal{A}\underline{b}_2) = (-\underline{b}_1, \underline{b}_2) = -(\underline{b}_1, \underline{b}_2)$$

which can only be if $(\underline{b}_1, \underline{b}_2) = 0$.)

From the matrix A_β, we can now read out that the map $\mathcal{A}$ is a reflection with the line $\operatorname{span}(\underline{b}_2)$ as its reflection axis: Take any vector $\underline{v} = v_1\underline{b}_1 + v_2\underline{b}_2$. So $\underline{v}$ has the component $\lambda_1\underline{b}_1$ which is perpendicular to $\operatorname{span}(\underline{b}_2)$ and will be mapped to $-\lambda_1\underline{b}_1$, and the component $\lambda_2\underline{b}_2$ which is in the direction of $\operatorname{span}(\underline{b}_2)$ and will be mapped to itself. So $\underline{v}$ is mapped to $-\lambda_1\underline{b}_1 + \lambda_2\underline{b}_2$, the reflection of $\underline{v}$ along the axis $\ell = \operatorname{span}(\underline{b}_2)$. □

One major (possibly homework-relevant) takeaway of this theorem is: When looking for a basis that brings directly orthogonal maps on $\mathbb{R}^2$ into rotation matrix shape, ***any*** orthonormal basis will do. In fact, any orthonormal basis will lead to almost the same rotation matrix:

Remark 3.1.26 ***($\mathbb{R}^2$: Rotation matrix looks mostly the same for different orthonormal bases.)*** We are now looking at a directly orthogonal map $\mathcal{A}$ acting on $\mathbb{R}^2$, and take the standard basis ε. Following the proof of Theorem 3.1.25, we see that we always find an angle φ such that the respective matrix A_ε of $\mathcal{A}$ is

$$A_\varepsilon = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

A_α will look almost the same for any other orthonormal basis $\alpha = \{\underline{a}_1, \underline{a}_2\}$: Switching from ε to α is orthogonal, so this corresponds to either rotating or reflecting. The former is the case if the angle between the two base vectors is 90° (like between $(1,0)$ and $(0,1)$), the latter is the case if the angle between $\underline{a}_1, \underline{a}_2$ is $-90°$.

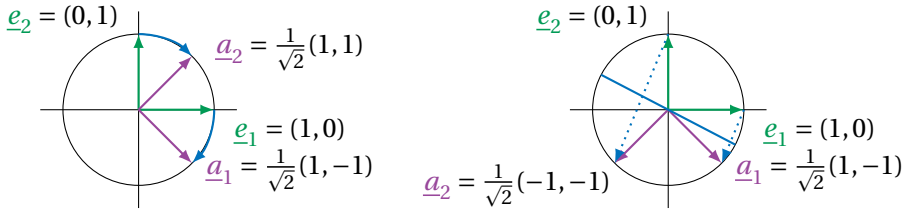


Figure 3.1: Angle case examples. Left: the angle between the two vectors in $\alpha = \{\underline{a}_1, \underline{a}_2\}$ is 90° – we get from $\varepsilon = \{(1,0),(0,1)\}$ to α by rotating. Right: −90° case – we get from ε to α by reflecting.

Geometrically, it is now easy to understand that

- all bases α with angle 90° have the representation matrix A_ε since the map simply rotates them by φ. (Tedious, but straightforward way to verify: also write α in sinus/cosinus terms like in the proof of Theorem 3.1.25, picking the angle 90° case, compute $A_\alpha = {}_\alpha S_\varepsilon \; A_\varepsilon \; {}_\varepsilon S_\alpha$, check that ${}_\alpha S_\varepsilon \; A_\varepsilon \; {}_\varepsilon S_\alpha = A_\varepsilon$.)
- for all bases α with angle −90°, the representation matrix is the inverse of A_ε because reflecting, rotating, and then reflecting back changes the rotation's direction. So

$$A_\alpha = \begin{pmatrix} \cos\varphi & -\sin -\varphi \\ \sin -\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} .$$

  (To verify, we could again compute $A_\alpha = {}_\alpha S_\varepsilon \; A_\varepsilon \; {}_\varepsilon S_\alpha$, after writing α in −90° case sinus/cosinus terms.)

To analyse orthogonal maps on spaces of dimension larger than two, we will simplify the task by using an additional trick: we will pick an invariant subspace W and split V into W and its orthogonal complement $W^\perp$. (The definition of $W^\perp$ is recalled as B.2.13.) Using the 'invariant subspaces tricks' described in Section 2.2, we can break down the matrix of $\mathcal{A}$ into smaller, easier-to-analyse building blocks. For that, however, we would need that the orthogonal complement is also invariant, which we show in the next theorem. ***The theorem below therefore prepares us to apply the 'invariant subspaces tricks', using W and $W^\perp$.***

> **Theorem 3.1.27.** *Let $\mathcal{A} : V \to V$ be an orthogonal map on a real finite-dimensional inner product space V, and let W be a linear subspace such that W is invariant under $\mathcal{A}$.*
>
> *Then $W^\perp$ is also invariant under $\mathcal{A}$.*

Proof. We need to show the following: if $\underline{v} \in W^\perp$, then also $\mathcal{A}\underline{v} \in W^\perp$. In other words, we need to show: if $(\underline{v}, \underline{w}) = 0$ for all vectors $\underline{w} \in W$, then also $(\mathcal{A}\underline{v}, \underline{w}) = 0$ for all vectors $\underline{w} \in W$.

To do that, let's fix a vector $\underline{v}$ such that $(\underline{v}, \underline{w}) = 0$ for all vectors $\underline{w}$. The main idea to argue that $(\mathcal{A}\underline{v}, \underline{w}) = 0$ is to write $\underline{w}$ in a more convenient form. Concretely, we'll show that we always find a vector $\underline{w}' \in W$ such that $\underline{w} = \mathcal{A}\underline{w}'$. Once we have shown this, we can use that $\mathcal{A}$ is orthogonal and hence preserves inner products:

$$(\mathcal{A}\underline{v}, \underline{w}) = (\mathcal{A}\underline{v}, \mathcal{A}\underline{w}') = (\underline{v}, \underline{w}') = 0 \ ,$$

where the last step used that we took $\underline{w}'$ from W and that $\underline{v}$ is perpendicular to ***any*** vector living in W.

So the only thing left to do is to show that any vector $\underline{w} \in W$ indeed has such a preimage $\underline{w}'$. In other words, we need to show that $W \subset \mathcal{A}(W)$. Since W is invariant, $\mathcal{A}(W) \subset W$ is a subspace of W that is spanned by the same vectors as W (Theorem 2.2.5). Hence, they could only differ if the dimension of $\mathcal{A}(W)$ is smaller than the one of W, which cannot be since $\mathcal{A}$ has an inverse (Theorem 3.1.10) and hence is injective. (To check this, we can use the dimension theorem (Theorem 1.1.17) for the map $\mathcal{A} : W \to \mathcal{A}W$:

- $dim(\mathcal{R}) = dim(\mathcal{A}W)$. (Because the range of that map is exactly $\mathcal{A}W$.)
- $dim(\mathcal{N}) = 0$: Since the larger map $\mathcal{A} : V \to V$ is injective, the only element mapped to $\underline{0}$ is $\underline{0}$, so $\mathcal{N} = \{\underline{0}\}$.

So $\dim(W) = 0 + dim(\mathcal{A}W)$.) □

Since W and $W^\perp$ always span the full vector space (*Linear Algebra 1*, there currently Theorem 5.2.14), taking an invariant subspace W and its orthogonal complement $W^\perp$ indeed gives us a a decomposition of V into two invariant subspaces. In other words, combining a basis of W with a basis of $W^\perp$ will always give us a basis of the full vector space V. We can now plug W and $W^\perp$ into our 'V decomposed into invariant subspaces → block shape' Theorem 2.2.11:

Theorem 3.1.28. *Let $\mathcal{A}$ be an orthogonal map in a finite-dimensional inner product space V, and W be an invariant subspace of V. Let α_W be an orthonormal basis of W, $\alpha_{W^\perp}$ be an orthonormal basis of $W^\perp$, and set $\alpha := \alpha_W \cup \alpha_{W^\perp}$ (which yields a basis of V). Then*

$$A_\alpha = \begin{pmatrix} M_1 & O_1 \\ O_2 & M_2 \end{pmatrix},$$

where M_1 and M_2 are the orthogonal matrices of the restrictions $\mathcal{A} : W \to W$ (so in short, $\mathcal{A}_{|W}$) and $\mathcal{A} : W^\perp \to W^\perp$ (so $\mathcal{A}_{|W^\perp}$), respectively; and O_1, O_2 are the zero matrices of the right size. In addition, $\det(A_\alpha) = \det(M_1) \cdot \det(M_2)$.

We are now ready to deal with dimension 3.

Theorem 3.1.29 (Classification for dimension 3). *If V is a real inner product space of dimension 3,*

- *any directly orthogonal map $\mathcal{A}$ on V is a* **rotation** *around a line $\ell = \operatorname{span}(\underline{a}_1)$ (called the* **axis (of rotation)***) by some angle φ. We can always find an orthonormal basis $\alpha = \{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ of V such that the respective matrix A_α of $\mathcal{A}$ has the following (rotation matrix) shape:*

$$A_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

  A_α has the trace $\operatorname{tr}(A_\alpha) = 1 + 2\cos\varphi$.

- *any indirectly orthogonal map $\mathcal{A}$ on V is a* **rotoreflection**, *i.e., a rotation around a line $\ell = \operatorname{span}(\underline{a}_1)$ by some angle φ, together with a reflection with mirror plane $\operatorname{span}(\underline{a}_1)^\perp$. We can always find an orthonormal basis$\alpha = \{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ of V such that the respective matrix A_α of $\mathcal{A}$ has the following (rotoreflection matrix) shape:*

$$A_\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

A_α has the trace $\operatorname{tr}(A_\alpha) = -1 + 2\cos\varphi$.

Proof. The main idea is to find an eigenvector $\underline{a}_1$ and to analyse the map by decomposing V into the subspace $\operatorname{span}(\underline{a}_1)$ and its two-dimensional complement $\operatorname{span}(\underline{a}_1)^\perp$.

We first show that ± 1 is an eigenvalue (with $\pm$depending on the case), which gifts us with $\underline{a}_1$:

- In the first case (directly orthogonal map, so the determinant is 1), we use that the determinant is the product of the roots of the characteristic polynomial $\chi_\mathcal{A}$ to show that 1 is among the roots (and hence eigenvalue).
  - If all roots are real, 1 must be among them: all real roots are ± 1 (Theorem 3.1.23), and the product of the (three) roots would be -1 if they all were -1.
  - If there there exists a non-real root λ of $\chi_\mathcal{A}$, its conjugate $\bar{\lambda}$ is also a root (Theorem 3.1.23). The product $\lambda\bar{\lambda}$ is a positive real, so the third root must be real (since the determinant is) and therefore 1 ($\lambda\bar{\lambda} > 0$ is positive).
- In the second case (indirectly orthogonal map, so the determinant is -1), Theorem 3.1.23 tells us directly that -1 is an eigenvalue.

In both cases, we get an eigenvector for the respective eigenvalue, which we normalise to obtain an eigenvector $\underline{a}_1$ of length 1. As a shorthand, we set $W := \operatorname{span}(\underline{a}_1)^\perp$. In both cases, Theorem 3.1.28 helps us get a diagonal block shape: Let's pick $\underline{a}_1$ as the first basis vector of a basis α and use some (not yet known) basis of W for the rest. Then

$$A_\alpha = \begin{pmatrix} \pm 1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where ± 1 is the case-dependent eigenvalue (the one-dimensional block belonging to $\operatorname{span}(\underline{a}_1)$), and M_2 is the (two-dimensional) matrix of $\mathcal{A}$'s restriction unto W.

We show that M_2 is a rotation matrix in both cases, using the dimension-2-classification:

- If the determinant of $\mathcal{A}$ is 1, the found eigenvalue is 1, so 3.1.28 tells us that the restriction $\mathcal{A}_{|W} : W \to W$ of $\mathcal{A}$ unto W (which is again orthogonal) also has determinant 1. According to Theorem 3.1.25, we find an orthonormal basis $\{\underline{a}_2, \underline{a}_3\}$ of W such that the matrix of $\mathcal{A}_{|W}$ w.r.t. $\{\underline{a}_2, \underline{a}_3\}$ is

$$M_2 = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

$\mathcal{A}$'s matrix w.r.t. the basis $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ hence is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

- If the determinant of $\mathcal{A}$ is -1, the found eigenvalue is -1, so the restriction $\mathcal{A}_{|W} : W \to W$ again has determinant 1. We therefore found an orthonormal basis $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ for which $\mathcal{A}$'s matrix is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

□

Remark 3.1.30 ***(Takeaway: Using Theorem 3.1.29 to interpret a concrete*** 3×3***-matrix*** A***)***

1. We compute $\det(A)$ to check whether A represents a rotation or a rotoreflection.
2. The determinant also tells us the sign of the eigenvalue ± 1.
3. We determine the rotation axis by computing an eigenvector $\underline{a}_1$ for the eigenvalue.
4. Once we know $\det(A)$, we can easily read out the rotation angle φ from the trace.
5. We determine the basis $\alpha = \{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$ for which the representation matrix has the shape given in Theorem 3.1.29: To enforce diagonal block shape, we split V into the line spanned by $\underline{a}_1$ and its orthogonal complement $\underline{a}_1^{\perp}$. Concretely:
   - We set the first basis vector to $\underline{a}_1$.
   - To complete the basis, we need to fill up $\{\underline{a}_1\}$ with some orthonormal vectors $\underline{a}_2, \underline{a}_3$ that are also orthogonal to $\underline{a}_1$. (Since we want them to span $\underline{a}_1^{\perp}$.)
   - We can find $\underline{a}_2, \underline{a}_3$, e.g., by applying Gram-Schmidt to the 'pre-basis' $\{\underline{a}_1, \underline{e}_2, \underline{e}_3\{$. This ensures that the new basis has the same orientation as the standard basis, and hence Remark 3.1.26 ensures that the resulting rotation matrix indeed looks as intended.
6. Rotoreflection case: the mirror plane is the plane which is orthogonal to the rotation axis, $\operatorname{span}(\underline{a}_1)$, so we can determine the mirror plane as the plane spanned by the two base vectors $\underline{a}_2, \underline{a}_3$.

Example 3.1.31 Consider the map $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ determined by the matrix

$$A = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

This matrix is orthogonal, so $\mathcal{A}$ is orthogonal as well.
We first compute the determinant of A and find it to be 1. $\mathcal{A}$ is therefore a rotation around the axis in the direction of an eigenvector $\underline{a}_1$ with eigenvalue 1. As usual, we find $\underline{a}_1$ by

solving $(\mathcal{A}-I)\underline{x}=\underline{0}$, so by row-reducing

$$\frac{1}{3}\begin{pmatrix}-1 & 1 & -2\\ 1 & -1 & 2\\ 2 & -2 & -1\end{pmatrix}\approx\begin{pmatrix}1 & -1 & 2\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

We hence get $E_1=\operatorname{span}((1,1,0))$, so $\underline{a}_1:=(1,1,0)$ is the axis of rotation.
We read out the angle of rotation from the trace: $1+2\cos\varphi=\frac{5}{3}$, so $\cos\varphi=\frac{1}{3}$, $\varphi=\arccos(\frac{1}{3})$.

To deal with dimensions higher than 3, we'll essentially repurpose the tricks for dimension 3: we'll identify invariant subspaces to break the matrix down into smaller building blocks. With the following theorem, we see that we can break everything down to a situation where the map's building blocks are either

- an eigenvalue diagonal matrix for
  - eigenvalue 1 (so on the respective subspace, the map does virtually nothing);
  - eigenvalue -1 (so on the respective subspace, the map could be seen a reflection, with the mirror plane being the orthogonal complement); or
- a 2-dimensional rotation matrix.

Theorem 3.1.32 (Classification for dimensions > 3). *If V is a real inner product space of dimension n and $\mathcal{A}:V\to V$ is an orthogonal map, we can always find an orthonormal basis α of V and a number $m\geq 0$ such that the respective matrix A_α of $\mathcal{A}$ has the following shape:*

$$A_\alpha=\begin{pmatrix}1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & 0 & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & \boxed{\mathrm{Rot}_1} & & \\ & & & 0 & & & & \ddots & \\ & & & & & & & & \boxed{\mathrm{Rot}_m}\end{pmatrix},$$

where $\mathrm{Rot}_1,\dots,\mathrm{Rot}_m$ *are* 2×2 *rotation matrices for some rotation angles* $\varphi_1,\cdots\varphi_m$.

(In the case where $m=0$, A_α is the diagonal matrix with diagonal $(1,\dots,1,-1,\dots,-1)$*, without any rotation matrices.)*

To summarise the proof, we'll take the subspace that combines the two eigenspaces for ± 1 and then split V into this subspace and its orthogonal complement. On the combined eigenspaces, we get the eigenvalue diagonal shape with diagonal $(1,\dots,1,-1,\dots,-1)$. On the complement, we show that the characteristic polynomial only has non-real roots. The complement therefore splits into 2-dimensional invariant subspaces whose 2×2 matrix can be brought into rotation matrix shape.

Proof. First, we choose orthonormal bases of the eigenspaces E_1 and E_{-1} for the map $\mathcal{A}$. Together, these bases span an invariant subspace W. The matrix of the restriction unto W with respect to this eigenvector basis is the eigenvalue diagonal matrix. Since $W^\perp$ again is also invariant, we can complete the eigenvector basis of W by some (yet-to-be-specified) orthonormal basis of $W^\perp$ to get a basis α of V, in which case A_α will have the shape

$$A_\alpha = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & & & & & & \boxed{???} \end{pmatrix},$$

where $\boxed{???}$ is the matrix of the restriction unto $W^\perp$. (In the case where W spans the full vector space, $W^\perp$ vanishes. So in this case, we're already done and A_α is just the eigenvalue diagonal matrix.)

In the case where W does not span the full vector space, $W^\perp$ is a subspace of dimension > 0. We now want to show that we can find a basis of $W^\perp$ such that

$$\boxed{???} = \begin{pmatrix} \boxed{\text{Rot}_1} & & \\ & \ddots & \\ & & \boxed{\text{Rot}_m} \end{pmatrix},$$

we restrict $\mathcal{A}$ unto $W^\perp$ and study the characteristic polynomial of the restriction.

This polynomial only has non-real roots:

- Any real root would be ± 1, so $W^\perp$ would share an eigenvector with W. But we already 'shaved off' all such eigenvectors - the only vector shared by subspaces and their orthogonal complement is $\underline{0}$ (no eigenvector).

Each of these non-real roots gives rise to a 2-dimensional invariant subspace (Theorem 2.2.13). (Their conjugate gives rise to the same subspace due to Remark 2.2.14, we therefore get m many pairs of non-real roots and their conjugates for some number m. We can even compute m: it is $(\dim(V) - \dim(W))/2$.) We know that the restrictions unto these 2-dimensional subspaces are either a reflection or a rotation (Theorem 3.1.25). They cannot be a reflection (they would have eigenvalue 1), so they all must be rotations. (Alternatively, you can check that the determinant of the matrix in Theorem 2.2.13 is positive and hence the restriction is a rotation due to Theorem 3.1.25.) We therefore always find a suitable basis which brings its matrix into rotation matrix shape, which we denote by Rot_i. We now use that we can fully decompose $W^\perp$ into these m many invariant subspaces:

- We can iteratively 'split off' one of the invariant subspaces from $W^\perp$, i.e. decompose $W^\perp$ into the invariant space and its orthogonal complement. We can then repeat the procedure on the remaining orthogonal complement, and so forth. (This works because the remaining

complement has the same invariant subspaces as before, only without the subspace we just removed. We can check this by looking at the characteristic polynomial: the complement's polynomial is the same as before, only divided by the determinant associated to the invariant subspace we just removed.)

With this, we have found a decomposition of $W^\perp$ into m many invariant subspaces, on which the map is a rotation. By combining the respective bases, we hence indeed get a basis of $W^\perp$ such that the matrix is

$$\boxed{???} = \begin{pmatrix} \boxed{\mathrm{Rot}_1} & & \\ & \ddots & \\ & & \boxed{\mathrm{Rot}_m} \end{pmatrix} .$$

□

3.2. Symmetric maps

In this section, we again restrict ourselves to ***finite-dimensional, real*** inner product spaces. We will now study a second important class of linear maps on such spaces, the class of symmetric maps. We'll learn

- what symmetric linear maps are;
- that they always have a basis of eigenvectors (and hence are always diagonalisable!);
- how to translate 'symmetric map' into matrix terms; and
- how we can classify and analyse quadratic curves/surfaces by applying the gathered results to their defining equations (and quadratic forms in general).

The concept 'symmetric matrix' can be tied to the work of Cauchy, who considered diagonalisability of symmetric matrices and the application to the study of quadratic forms.

Why do we care? Quadratic forms occur, e.g., in analysis, when we investigate the behaviour of a function in the neighbourhood of a stationary point. (Second-order approximations tell us whether the function has a local minimum or maximum, or a saddle point or worse.) As another example, systems of linear equations with symmetric coefficient matrix can be solved using numerical algorithms, in an efficient and stable way. Other examples can be found in graph theory, probability theory and statistics. (E.g., modelling 'linear' dependence between random variables, linear regression, or assessing the precision of estimators.)

We start with the central definition of this section:

Definition 3.2.1 (Symmetric map). *Let V be a real inner product space. A linear map $\mathcal{A} : V \to V$ is called* **symmetric** *if $(\mathcal{A}\underline{x}, \underline{y}) = (\underline{x}, \mathcal{A}\underline{y})$ for all $\underline{x}, \underline{y}$ in V.*

Example 3.2.2 ***('Trivial' example: multiplication by a scalar)*** Say $\mathcal{A}$ is defined by the rule $\mathcal{A}\underline{v} = \lambda\underline{v}$ for some real scalar λ. Since the inner product is bilinear, we have

$$(\mathcal{A}\underline{x}, \underline{y}) = (\lambda\underline{x}, \underline{y}) = \lambda(\underline{x}, \underline{y}) = (\underline{x}, \lambda\underline{y}) = (\underline{x}, \mathcal{A}\underline{y}) \quad .$$

for all $\underline{x}$ and $\underline{y}$.

Example 3.2.3 ***(Orthogonal projection*** The orthogonal projection $\mathcal{P}$ on a line ℓ is symmetric: If $\ell = \text{span}(\underline{a})$ and $\|\underline{a}\| = 1$, then $\mathcal{P}\underline{x} = (\underline{x}, \underline{a})\underline{a}$. For all $\underline{x}$ and $\underline{y}$, we have

$$\begin{aligned}(\mathcal{P}\underline{x}, \underline{y}) &= ((\underline{x}, \underline{a})\underline{a}, \underline{y}) = (\underline{x}, \underline{a})(\underline{a}, \underline{y}) \\ (\underline{x}, \mathcal{P}\underline{y}) &= (\underline{x}, (\underline{y}, \underline{a})\underline{a}) = (\underline{y}, \underline{a})(\underline{x}, \underline{a})\end{aligned},$$

meaning $(\mathcal{P}\underline{x}, \underline{y}) = (\underline{x}, \mathcal{P}\underline{y})$.
In a similar way, one can show that the orthogonal projection unto a subspace W is a symmetric linear map.

3.2.1 Symmetric matrices and their connection with symmetric maps

As usual, our next step will be to connect the concept 'symmetric map' to matrices. Like in Section 3.1, we ***prepare for studying the matrix by describing how symmetric maps act on orthonormal bases*** (which gives us information about the columns of the map's matrix):

Theorem 3.2.4. *For a linear map $\mathcal{A} : V \to V$, the following are equivalent:*

1. *$\mathcal{A} : V \to V$ is symmetric.*
2. *For every orthonormal system $\underline{a}_1, \ldots, \underline{a}_m$ in V, we have $(\mathcal{A}\underline{a}_i, \underline{a}_j) = (\underline{a}_i, \mathcal{A}\underline{a}_j)$ for all i, j.*
3. *There is an orthonormal basis $\{\underline{a}_1, \ldots, \underline{a}_n\}$ of V satisfying $(\mathcal{A}\underline{a}_i, \underline{a}_j) = (\underline{a}_i, \mathcal{A}\underline{a}_j)$ for all i, j.*

Proof. 1) $\Rightarrow$ 2) and 2) $\Rightarrow$ 3) are easy to check (2) $\Rightarrow$ 3): Gram-Schmidt).

3) $\Rightarrow$ 1): Any vector $\underline{x} \in V$ can be written as $\underline{x} = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n$, and any vector $\underline{y} \in V$ can be written as $\underline{y} = y_1\underline{a}_1 + \cdots + y_n\underline{a}_n$. Then

$$(\mathcal{A}\underline{x}, \underline{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j (\mathcal{A}\underline{a}_i, \underline{a}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j (\underline{a}_i, \mathcal{A}\underline{a}_j) = (\underline{x}, \mathcal{A}\underline{y}).$$

□

This is interesting (again) because we know that the columns of a representation matrix are the (coordinate vectors of) the coordinate vectors' images. The name 'symmetric' has to do with the matrix representation of $\mathcal{A}$ in the finite-dimensional case, because its entries are symmetric along the main diagonal:

Theorem 3.2.5. *Let V be a finite-dimensional real inner product space and α an orthonormal basis of V. The linear map $\mathcal{A} : V \to V$ is symmetric if and only if its matrix A_α w.r.t. α satisfies $A_\alpha^\top = A_\alpha$.*

Proof. Let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be an arbitrary ***orthonormal*** basis for V. The columns of A_α are the α–coordinates of the vectors $\mathcal{A}\underline{a}_1, \mathcal{A}\underline{a}_2, \ldots, \mathcal{A}\underline{a}_n$. On the ij–th position, we find the i–th coordinate of $\mathcal{A}\underline{a}_j$, which is $(\underline{a}_i, \mathcal{A}\underline{a}_j)$. On on the ji–th position, we find the j–th coordinate of $\mathcal{A}\underline{a}_i$, which is $(\mathcal{A}\underline{a}_i, \underline{a}_j)$. If $\mathcal{A}$ is symmetric, then these elements are equal, so the matrix is symmetric in the main diagonal. In other words: $A_\alpha^\top = A_\alpha$.

Conversely, if A_α satisfies $A_\alpha^\top = A_\alpha$, then the element on the ij–th position is equal to the element on the ji–th position, hence $(\underline{a}_i, \mathcal{A}\underline{a}_j) = (\underline{a}_j, \mathcal{A}\underline{a}_i)$. Using symmetry of the inner product, we get $(\underline{a}_i, \mathcal{A}\underline{a}_j) = (\mathcal{A}\underline{a}_i, \underline{a}_j)$, the map thus satisfies the third criterion in Theorem 3.2.4. □

This motivates how we define symmetric matrices:

Definition 3.2.6 (Symmetric matrix). *A real $n \times n$ matrix A is called* **symmetric** *if $A^\top = A$.*

Example 3.2.7 ***(Orthogonal projection, cnt'd*** We already know that orthogonal projections $\mathcal{P}_W : V \to V$ unto a subspace W are symmetric (example 3.2.3). Through Theorem 3.2.5, we get an alternative way to show that this: We choose an orthonormal basis $\{\underline{a}_1, \ldots, \underline{a}_n\}$ of V for which $\{\underline{a}_1, \ldots, \underline{a}_m\}$ is an orthonormal basis of W. Every vector from the basis is an eigenvector: the first m vectors for eigenvalue 1, the last $n-m$ many for eigenvalue 0. The matrix of $\mathcal{P}_W$ w.r.t. this orthonormal basis is therefore a diagonal matrix and hence symmetric. Due to Theorem 3.2.5, we can conclude that so is $\mathcal{P}_W$.

3.2.2 We can always diagonalise symmetric matrices

A central result of this section is that any symmetric map acting on a finite-dimensional real inner product space can be fully diagonalised. To prove this, we will need some helper theorems.

To diagonalise orthogonal maps as far as possible, we discussed how the roots of their characteristic polynomial could possibly look. ***As helper theorem 1, we now study the roots for symmetric maps and obtain a maybe somewhat surprising, but very nice result:***

Theorem 3.2.8. *Let $\mathcal{A} : V \to V$ be a symmetric map on a real finite-dimensional space V. Then all roots of the characteristic polynomial $\chi_\mathcal{A}$ are real (and hence an eigenvalue).*

Proof. We do a proof by contradiction: Suppose μ is a non-real root of the characteristic equation. According to 2.2.13, we then find a 2-dimensional invariant linear subspace W such that the restriction unto W has a characteristic equation with roots μ and $\bar{\mu}$ (which are not the same since μ is non-real). We choose an orthonormal basis α for W. Then

$$A_\alpha = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

is symmetric according to Theorem 3.2.5, with characteristic polynomial

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = \lambda^2 - (a+c)\lambda + ac - b^2.$$

The discriminant of this quadratic is $(a+c)^2 - 4ac + 4b^2 = (a-c)^2 + 4b^2 \geq 0$, so the two roots μ, $\bar{\mu}$ are real, which yields a contradiction. □

To classify orthogonal maps, we broke down their matrices into two smaller matrices, one for the restriction unto an invariant subspace and one for the restriction unto its orthogonal complement. For that, we used that spaces which are invariant under orthogonal maps have an orthogonal complement that itself is invariant. ***As helper theorem 2, we now show that orthogonal complements are also invariant when dealing with symmetric maps.***

> **Theorem 3.2.9.** *Let $\mathcal{A} : V \to V$ be a symmetric map on a real inner product space V, and let W be an invariant subspace.*
>
> *Then $W^\perp$ is also invariant under $\mathcal{A}$.*

Proof. Like in the proof of Theorem 3.1.27, we need to show that $(\mathcal{A}\underline{v}, \underline{w}) = 0$ for all vectors $\underline{w} \in W$ if $(\underline{v}, \underline{w}) = 0$ for all vectors $\underline{w} \in W$. To do that, we again fix $\underline{v} \in V, \underline{w} \in W$. Using symmetry of $\mathcal{A}$, we compute:

$$(\mathcal{A}\underline{v}, \underline{w}) = (\underline{v}, \mathcal{A}\underline{w}) = 0 \ ,$$

where the last step used W is invariant, meaning $\mathcal{A}\underline{w} \in W$, and that $\underline{v}$ is perpendicular to ***any*** vector living in W. □

Using the two helper theorems, we can now show that ***any symmetric map can be fully diagonalised because we always get an orthonormal basis of eigenvectors***:

> **Theorem 3.2.10.** *Let $\mathcal{A} : V \to V$ be a symmetric linear map on V with $\dim(V) < \infty$. Then there exists an orthonormal basis of eigenvectors of $\mathcal{A}$.*

Proof. Let $\dim(V) = n$. We will do the proof by 'splitting off' eigenspaces to successively reduce the dimension.

Since the characteristic equation of $\mathcal{A} : V \to V$ has only real roots, we find an eigenvalue λ_1. We let $\underline{v}_1$ be an eigenvector with $\|\underline{v}_1\| = 1$ (we can always normalise), and $W_1 := \operatorname{span}(\underline{v}_1)^\perp$. Then W_1 is invariant, $\dim(W_1) = n-1$, and the restriction of $\mathcal{A}$ unto W_1 is again symmetric.

We can now repeat the procedure: the restriction's characteristic equation has real roots only, meaning we find an eigenvalue λ_2 that has eigenvectors living in W_1. Let $\underline{v}_2$ be such an eigenvector in W_1 for λ_2 and $\|\underline{v}_2\| = 1$. Then $\underline{v}_1, \underline{v}_2$ form an orthonormal system and $\operatorname{span}(\underline{v}_1, \underline{v}_2)$ is invariant.

We now also split off the span of $\underline{v}_2$, by again taking the orthogonal complement: $W_2 = \operatorname{span}(\underline{v}_1, \underline{v}_2)^\perp$ is invariant, $\dim W_2 = n-2$, and the restriction of $\mathcal{A}$ unto W_2 is again symmetric and hence has an eigenvalue λ_3. We choose an eigenvector $\underline{v}_3 \in W_2$ with eigenvalue λ_3 and $\|\underline{v}_3\| = 1$. Then $\underline{v}_1, \underline{v}_2, \underline{v}_3$ is an orthonormal system, etc.

We repeat the procedure of taking an eigenvector from the remaining subspace, splitting off its span, and using that the orthogonal complement still has eigenvectors, until we arrive at the final subspace $W_{n-1} = \operatorname{span}(v_1, \ldots, v_{n-1})^\perp$, which has dimension 1 and spits out the last eigenvector.

Per construction, all vectors are of length 1 and orthogonal to all others. We hence found an orthonormal eigenvector basis of V. □

We saw in Section 2.1.2 that eigenvectors belonging to different eigenvalues are always independent, and the construction in the previous theorem gave us an orthonormal basis. For symmetric

maps, we can even show that eigenspaces for different eigenvalues will always be orthogonal (so ***the space completely splits into orthogonal eigenspaces***):

> **Theorem 3.2.11.** *Let $\mathcal{A} : V \to V$ be a symmetric linear map on V with $\dim(V) < \infty$, and λ_1, λ_2 be two mutually different eigenvalues (so $\lambda_1 \neq \lambda_2$). Then all vectors in the eigenspace $E_{\lambda_1}(A)$ are orthogonal to all vectors in the eigenspace $E_{\lambda_2}(A)$. Or, in short: $E_{\lambda_1}(A) \perp E_{\lambda_2}(A)$.*

Proof. Take $\underline{x} \in E_{\lambda_1}(A)$ and $\underline{y} \in E_{\lambda_2}(A)$. To show that $(\underline{x}, \underline{y}) = 0$, we use bilinearity of the inner product and symmetry of $\mathcal{A}$:

$$\lambda_1(\underline{x}, \underline{y}) = (\lambda_1 \underline{x}, \underline{y}) = (\mathcal{A}\underline{x}, \underline{y}) = (\underline{x}, \mathcal{A}\underline{y}) = (\underline{x}, \lambda_2 \underline{y}) = \lambda_2(\underline{x}, \underline{y}) \ ,$$

so $(\lambda_1 - \lambda_2)(\underline{x}, \underline{y}) = 0$, and $\lambda_1 - \lambda_2 \neq 0$. □

So any symmetric map is diagonalisable. As a last step, ***we now apply this to matrices by translating the result into 'diagonalisable matrix terms':***

Corollary 3.2.12. *A symmetric matrix is diagonalisable by changing to an orthonormal basis of eigenvectors, so by means of an orthogonal coordinate transformation and hence an orthogonal transition matrix.*

Proof. Let A be a symmetric $n \times n$ matrix, which we can view as the matrix of the (symmetric) map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ defined by the rule $\underline{e}_i \mapsto \mathcal{A}\underline{e}_i$, so with respect to the standard basis ε. We can always find an orthonormal eigenvector basis α (Theorem 3.2.10), meaning the matrix A_α has eigenvalue diagonal form, and (remember Theorem 1.3.15)

$$A_\alpha = {}_\alpha S_\varepsilon \, A_\varepsilon S_\alpha \ .$$

The respective coordinate transformations are orthogonal maps (they map between orthonormal basis ε and orthonormal basis α, so we can apply Theorem 3.1.9), hence so are their matrices. □

Remark 3.2.13 ***(Takeaway: Using Theorem 3.2.10 to diagonalise a concrete $n \times n$-matrix A)***

1. We use the characteristic polynomial $\det(A - \lambda\mathcal{I})$ (as explained in Section 2.1.1) to find
   - eigenvalues $\lambda_1, \cdots, \lambda_n$, together with
   - corresponding eigenvectors $\underline{a}_1, \cdots, \underline{a}_n$.
2. This gives us an eigenvector basis $\alpha := \{\underline{a}_1, \cdots, \underline{a}_n\}$.
3. We turn α into an ***orthonormal*** eigenvector basis by normalising all eigenvectors (= dividing them by their length). (Orthogonality we get for free due to Theorem 3.2.11.)
4. The representation matrix for basis α is the eigenvalue diagonal matrix:

$$A_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} .$$

5. To transform A into A_α via $A_\alpha = {}_\alpha S_\varepsilon \; A \; {}_\varepsilon S_\alpha$ (see the proof of Corollary 3.2.12), we need the basis transition matrices ${}_\alpha S_\varepsilon$ and ${}_\varepsilon S_\alpha$. We get them basically for free:

   - ${}_\varepsilon S_\alpha$ is the matrix whose columns are exactly the base vectors of α (as explained, e.g., in example 1.3.10, 'method with shortcut').
   - ${}_\alpha S_\varepsilon$ is the inverse of ${}_\varepsilon S_\alpha$ (see discussion between examples 1.3.7 and 1.3.8). Since ${}_\varepsilon S_\alpha$ is orthogonal (see Corollary 3.2.12), its inverse is simply the transpose (Theorem 3.1.14). We can hence compute
   $$ {}_\alpha S_\varepsilon = {}_\varepsilon S_\alpha{}^T \; . $$

Example 3.2.14 ***(Diagonalising a symmetric matrix)*** For the symmetric matrix

$$ A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 4 & -2 \\ -3 & -2 & -1 \end{pmatrix} , $$

we find the eigenvalues $0, -4$ and 6 with eigenspaces

$$ E_0 = \operatorname{span}((-1,1,1)) \,,\; E_{-4} = \operatorname{span}((1,0,1)) \,,\; E_6 = \operatorname{span}((1,2,-1)). $$

By normalising, we thus get the following orthonormal eigenvector basis:

$$ \alpha = \left\{ \frac{1}{\sqrt{3}}(-1,1,1) \,,\; \frac{1}{\sqrt{2}}(1,0,1) \,,\; \frac{1}{\sqrt{6}}(1,2,-1) \right\}. $$

Since the matrix A_α is simply the eigenvalue diagonal matrix, it can be written down without any further computations:

$$ A_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}. $$

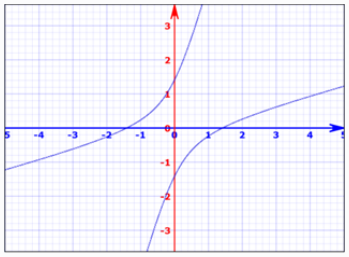
3.2.3 Application: quadratic forms and analysing curves

We will now study how we can apply our results on symmetric maps to the following geometric problem: Say we want to analyse a curve or a surface, but we only know its quadratic equation, which is given to us not in standard form. Then its not so easy to determine the curve's/surface's shape from the equation. This is where our diagonalisation result from Section 3.2.2 comes to the rescue: we will use that we can bring the equation into standard form by playing with the coordinate system, or, in other words: by bringing the curve/surface 'on principal axes'.

Example 3.2.15 As an example, we consider the curve in the plane defined by the equation

$$ x_1^2 - 4x_1x_2 + x_2^2 = 2 \; . $$

Using a curve plotter for the equation, we would get a graph showing a hyperbola.



One of our goals in this section will be to develop a strategy to determine the geometric meaning of such equations (including, e.g., symmetry axes). Luckily, we can throw our knowledge about symmetric matrices at this problem.

To link such quadratic equations to symmetric matrices, we first look at the following vector-matrix-vector product: We can compute

$$\begin{aligned}(x_1 \cdots x_n)\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 \\ &\quad + (a_{12}+a_{21})x_1x_2 + (a_{13}+a_{31})x_1x_3 + \cdots + (a_{n-1,n}+a_{n,n-1})x_{n-1}x_n \\ &= \sum_i a_{ii}x_i^2 + \sum_{i,j>i}(a_{ij}+a_{ji})x_ix_j \ .\end{aligned}$$

The result is a homogeneous polynomial of degree 2 in $x_1,\dots,x_n$, with the coefficients of $x_1^2,\dots,x_n^2$ being the matrice's diagonal elements and the coefficient of x_ix_j with $i \neq j$ being the sum of a_{ij} and a_{ji}.

We call such homogeneous polynomial of degree 2 'quadratic forms':

Definition 3.2.16. Quadratic forms over $\mathbb{R}$ *are homogeneous polynomials of degree 2 with coefficients in* $\mathbb{R}$*, i.e., polynomials* $p(x_1,x_2,,x_n)$ *of the form* $p(x_1,x_2,,x_n) = \sum_i a_{ii}x_i^2 + \sum_{i,j>i} b_{ij}x_ix_j$ *for some scalars* $a_1,\cdots,a_n,b_{12},\cdots,b_{n-1,n} \in \mathbb{R}$.

So any real matrix can be linked to a quadratic form, but we can also do it the other way around: As usual, we write $\underline{x}$ as shorthand for the vertical vector, and we additionally write $\underline{x}^T$ for its horizontal counterpart.

Remark 3.2.17 Any quadratic form $p(x_1,x_2,,x_n)$ over $\mathbb{R}$ can be written as a vector-matrix product $\underline{x}^T A\underline{x}$ for a ***symmetric*** matrix A:

We set the diagonal elements of A to the coefficients of $x_1^2,\dots,x_n^2$, and for the non-diagonal elements $i \neq j$, we set $a_{ij} = a_{ji}$ to the coefficient of x_ix_j, divided by 2.

Example 3.2.18 To practice with this, check that

$$x_1^2 - 4x_1x_2 + x_2^2 = (x_1,\ x_2)\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(So we linked example 3.2.15 to a symmetric matrix!)

$$2x^2-4xy+4xz-3y^2-3z^2=(x\ y\ z)\begin{pmatrix}2&-2&2\\-2&-3&0\\2&0&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Diagonal matrices are particular nice for quadratic forms, since they get rid of the mixed terms x_ix_j:

Remark 3.2.19 If D is a ***diagonal*** matrix with diagonal $(\lambda_1,\cdots,\lambda_n)$, then the corresponding quadratic form $p(x_1,x_2,,x_n)$ is

$$\underline{x}^T D\underline{x}=\underline{x}^T\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_1\end{pmatrix}\underline{x}=\sum_i\lambda_i x_i^2\ .$$

Over the reals, we can now throw our theory for symmetric matrices at quadratic forms over the real to show that ***any quadratic form can be brought into a simpler shape where the mixed products vanish***. (Which we'll then use right away to analyse the geometry behind the form.)

Theorem 3.2.20. *For any quadratic form $p(x_1,x_2,\dots,x_n)$ over the reals,* **there exists a substitution rule $\underline{x}\mapsto\underline{y}$ that brings $p(x_1,x_2,\dots,x_n)$ into a form without mixed products.**

More formally, there exists a basis transition map $\alpha:\mathbb{R}^n\to\mathbb{R}^n$ and a quadratic form $p'(y_1,y_2,\dots,y_n)$ over the reals such that

- *for $(y_1,y_2,\dots,y_n):=\alpha(x_1,x_2,\dots,x_n)$, we always get $p(x_1,x_2,\dots,x_n)=p'(y_1,y_2,\dots,y_n)$ (***same form up to coordinate substitution***); and*
- *$p'(y_1,y_2,\dots,y_n)=\sum_i\lambda_i y_i^2$ for some coefficients $\lambda_1,\dots,\lambda_n$ (***form has no mixed products***).*

Proof. We already know that

$$p(x_1,\dots,x_n)=(x_1\ \dots\ x_n)A\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$$

for some symmetric matrix A. The main idea now is to find a basis transition to some basis α that diagonalises A to an eigenvector diagonal basis D, which yields our substitution rule:

For a symmetric matrix like A, we always find an orthonormal eigenvector basis $\alpha=\{\underline{a}_1,\dots,\underline{a}_n\}$ for the respective eigenvalues $\lambda_1,\dots,\lambda_n$ (Theorem 3.2.10). We denote the coordinates of $(x_1,\dots,x_n)$ w.r.t. the basis α by $(y_1,\dots,y_n)$. So the connection is

$$\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}={}_\alpha S_\varepsilon\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}={}_\varepsilon S_\alpha\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}\ ,$$

with the columns of the matrix ${}_\varepsilon S_\alpha$ being the vectors $\underline{a}_1, \ldots, \underline{a}_n$.

To write the quadratic form p in terms of $\underline{y}$, we also translate $\underline{x}^T$ into $\underline{y}^T$ terms: after transposition, the relation above reads $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)_\varepsilon S_\alpha^\top$. Since the matrix ${}_\varepsilon S_\alpha$ is orthogonal, $({}_\varepsilon S_\alpha)^\top = ({}_\varepsilon S_\alpha)^{-1}$, which in turn equals ${}_\alpha S_\varepsilon$. We now plug these equalities into p, which yields

$$(y_1 \ \ldots \ y_n)\ {}_\alpha S_\varepsilon\ A\ {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = (y_1 \ \ldots \ y_n)\ A_\alpha \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

because A_α is the eigenvector diagonal matrix D with diagonal $(\lambda_1, \ldots, \lambda_n)$. After this coordinate transformation, all double products have vanished. □

Remark 3.2.21 As soon as we know the eigenvalues (including their multiplicities), we can already write down the mixed-term-free 'substituted' form $p'(y_1, y_2, \ldots, y_n) = \sum_i \lambda_i y_i^2$ without any further computations.

Using the theorem, we can hence bring the quadratic equation into an easier-to-analyse shape, albeit in a modified coordinate system which we still need to translate back after analysing. To make this more clear, lets look at the example from before:

Example 3.2.22 ***(Applying Theorem 3.2.20 to example 3.2.15)*** We again consider the curve in the plane defined by the equation

$$x_1^2 - 4x_1x_2 + x_2^2 = 2 \ .$$

where (x_1, x_2) are the coordinates of a vector w.r.t. an orthonormal basis. (Think of the standard basis ε.) We already saw that we can write the-left hand side as

$$(x_1 \ \ x_2) \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ .$$

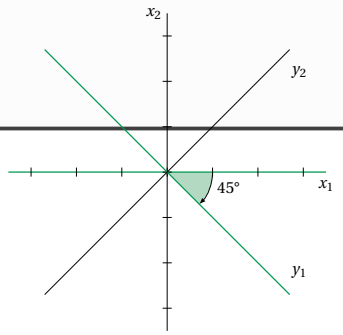
Following the recipe in remark 3.2.13, we now diagonalise the matrix: its eigenvalues are 3 with eigenvector $(1,-1)$ and -1 with eigenvector $(1,1)$. So we get the orthonormal eigenvector basis $\alpha = \{\frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1)\}$, with

$${}_\varepsilon S_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } {}_\alpha S_\varepsilon = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \ .$$

To substitute $\underline{x}$ by $\underline{y} := {}_\alpha S_\varepsilon \underline{x}$, the representation of $\underline{x}$ in α-coordinates, we would compute:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ .$$

Geometrically, this represents a rotation of the coordinate grid. (This is a ***passive*** rotation since we do not change vectors, only their coordinate system.) We see that this is a clockwise rotation by $\pi/4$, so by 45 degrees:



- Method 1: Sketch the coordinate grids.
  - y_1-axis: emerges from x_1-axis (so span$(1,0)$) as span of ${}_\varepsilon S_\alpha(1,0) = (1,-1)$.
  - y_2-axis: emerges from x_2-axis (so span$(0,1)$) as span of ${}_\varepsilon S_\alpha(1,0) = (1,1)$.
- Method 2: Extract the rotation angle from ${}_\alpha S_\varepsilon$: the rotation is passive, hence the respective rotation matrix is (see, e.g., the rotation matrix wikipedia, paragraph 4)

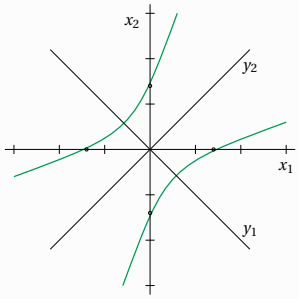
$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

  So $\cos\varphi = 1/\sqrt{2}$ and $\sin\varphi = -1/\sqrt{2}$, yielding (counter-clockwise) rotation angle $-\pi/4$.

When we now substitute this relation in the curve's equation, we get (without any further computations, according to remark 3.2.21)

$$3y_1^2 - y_2^2 = 2.$$

So indeed, the curve is a hyperbola. (See Appendix C.2 if it is not clear to you how to determine this from the equation.)
We determine the hyperbola's axes of symmetry (or 'principal axes') in the rotated coordinate grid:



- In terms of (y_1, y_2), the axes of symmetry are $y_1 = 0$ and $y_2 = 0$, so span$(0,1)$ and span$(1,0)$ in α-coordinates. (Also see Appendix C.2 if it's unclear to you how to determine the axes of symmetry from a hyperbola's standard equation.)
- To translate this back into standard coordinates, we rotate back the grid: we apply ${}_\varepsilon S_\alpha$ to the two axes' direction vectors $(0,1)$ and $(1,0)$ and obtain that the axes are in the direction of the eigenvectors $(1,1)$ and $(1,-1)$.

Dealing with the case where the equation additionally has a linear term

So far, we learnt how to deal with quadratic equations that are purely of degree 2, meaning all variables appear as a term of degree 2 (x_i^2 or mixed term $x_i x_j$), and the equation does not have any linear terms (like, e.g., $3x_2$). (In other words: the case only considered ***homogeneous*** polynomials.) This is a real limitation, since quadratic curves and surfaces in general also contain such linear terms. (In other words: the general case also needs to consider ***non-homogeneous*** polynomials.)

This subsection deals with the non-homogeneous case as follows: we bring it back to the previous homogeneous case, by shifting the coordinate grid's origin in a clever way. In other words, we will

do this by an additional translation of the coordinate system.

Example 3.2.23 As our example in this subsection, we'll look at the quadratic curve with equation

$$16x_1^2 - 24x_1x_2 + 9x_2^2 - 30x_1 - 40x_2 = 0 \ .$$

We can rewrite this as a vector-matrix-vector product (like in the no-linear-term case), plus an additional product for the linear terms:

$$(x_1 \ \ x_2) \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (-30 \ \ -40) \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = 0 \ .$$

We rewrite this in a more compact way:

$$\underline{x}^T \ A \, \underline{x} + \underline{b}^T \underline{x} = 0 \ ,$$

where $A = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}$ and $\underline{b}^T = (-30 \ \ -40)$.

We call such geometric shapes 'quadratic hyper-surfaces':

Definition 3.2.24. *A* **quadratic hyper-surface** H **in** $\mathbb{R}^n$ *is the set of solutions for some quadratic equation:*

$$H = \{\underline{x} \in \mathbb{R}^n \mid \underline{x}^T \ A \, \underline{x} + \underline{b}^T \underline{x} = 0\} \ ,$$

where A is a symmetric, real $n \times n$ matrix, $\underline{b} \in \mathbb{R}^n$, and d is some real number.

Over the reals, we can now throw our theory for symmetric matrices also at such quadratic hyper-surfaces to show that ***also any quadratic hyper-surface can be brought back to a simpler shape, i.e. a shape where each variable appears just once, either as a square or as a linear term.*** (Which we'll then again use right away to analyse the geometry behind the form.) The main difference between the theorem below and Theorem 3.2.20 is that the homogeneous case in Theorem 3.2.20 was simpler and led to a shape does not contain linear terms.

Theorem 3.2.25. *For any quadratic hyper-surface H in $\mathbb{R}^n$,* **there exists a substitution rule $\underline{x} \mapsto \underline{z}$ that brings the hyper-surface's equation into a form containing all variables just once, and without mixed products.**

More formally: let $\underline{x}^T \ A \, \underline{x} + \underline{b}^T \underline{x} = 0$ be the defining equation of H. Then there exists a 'coordinate substitution map' $T : \mathbb{R}^n \to \mathbb{R}^n$ and a quadratic form $p'(z_1, z_2, \ldots, z_n)$ over the reals such that

- *for $(z_1, z_2, \ldots, z_n) := T(x_1, x_2, \ldots, x_n)$, we always get $\underline{x}^T \ A \, \underline{x} + \underline{b}^T \underline{x} = p'(z_1, z_2, \ldots, z_n)$ (***same form up to coordinate substitution***); and*
- *$p'(z_1, z_2, \ldots, z_n)$ only contains each z_i once, either as a linear term z_i or as a quadratic term z_i^2 (***variables appear only once and not in mixed products***).*

Proof. The main idea is to

1. first, repeat the steps in Theorem 3.2.20 (diagonalise A to get rid of mixed products); then

2. get rid of variable double appearances (so, appearances as both linear and quadratic) by finding an additional suitable substitution rule.

Step 1. In more detail, we again find an orthonormal eigenvector basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ of A with respective eigenvalues $\lambda_1, \ldots, \lambda_n$. Again, we denote the coordinates of a vector $\underline{x} = (x_1, \ldots, x_n)$ w.r.t. α by $\underline{y} = (y_1, \ldots, y_n)$. So again, the relation is

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} ,$$

where the columns of ${}_\varepsilon S_\alpha$ again are the basis vectors $\underline{a}_1, \ldots, \underline{a}_n$. After substituting this in the equation, the equation reads

$$(y_1 \ \ldots \ y_n)\, {}_\varepsilon S_\alpha^\top \, A \, {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + (b_1 \ \ldots \ b_n)\, {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + d = 0 \ .$$

Since ${}_\varepsilon S_\alpha^\top \, A \, {}_\varepsilon S_\alpha$ again is the eigenvalue diagonal matrix, we can shorten this to

$$\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 + c_1 y_1 + \cdots + c_n y_n + d = 0 \ ,$$

where

$$(c_1 \ \ldots \ c_n) \coloneqq (b_1 \ \ldots \ b_n)\, {}_\varepsilon S_\alpha \text{, so } \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = {}_\varepsilon S_\alpha{}^\top \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} .$$

In other words, $(c_1, \ldots, c_n)$ are the the α–coordinates of the vector $\underline{b}$.

Step 2. We now go over the different variables to eradicate double appearances: if $\lambda_i = 0$, the variable x_i appears only as a linear term (if at all), so no additional step needs to be taken in this case. If $\lambda_i \neq 0$, we complete the square to get our substitution rule: we write

$$\lambda_i y_i^2 + c_i y_i = \lambda_i \left(y_i^2 + \frac{c_i}{\lambda_i} y_i \right) = \lambda_i \left(y_i + \frac{c_i}{2\lambda_i} \right)^2 - \frac{c_i^2}{4\lambda_i} .$$

We now substitute

$$z_i \coloneqq \begin{cases} y_i & \lambda_i = 0 \\ y_i + \frac{c_i}{2\lambda_i} & \lambda_i \neq 0 \end{cases} .$$

The result is a polynomial in which each of the variables (z_i) occurs in exactly one term, either linear or quadratic. On the way, the constant term d changes to d, minus $\frac{c_i^2}{4\lambda_i}$ for all i where $\lambda_i \neq 0$. □

Example 3.2.26 ***(Continuation of example Theorem 3.2.23)*** We again consider the quadratic curve with equation

$$16x_1^2 - 24x_1x_2 + 9x_2^2 - 30x_1 - 40x_2 = 0 \ ,$$

which we rewrote as

$$(x_1 \;\; x_2)\begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (-30 \;\; -40)\begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = 0 \;\; .$$

The eigenvalues of the matrix are 25 and 0 with eigenspaces $E_{25} = \text{span}\,((4,-3))$ and $E_0 = \text{span}\,((3,4))$, so we get the orthonormal eigenvector basis $\{\frac{1}{5}(4,-3),\ \frac{1}{5}(3,4)\ \}$. The relation between the coordinates is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

When substituting, we again get the sum over the quadratics for free (without computation) because the scalars are just the eigenvalues. After substitution, we thus get

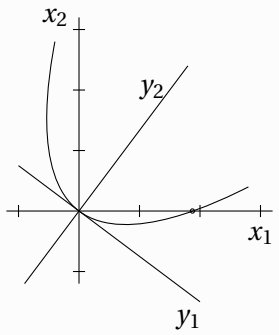
$$25y_1^2 + 0y_2^2 - 30 \cdot \tfrac{1}{5}(4y_1 + 3y_2) - 40 \cdot \tfrac{1}{5}(-3y_1 + 4y_2) = 0,$$

which simplifies to $25y_1^2 - 50y_2 = 0$, so $y_2 = \frac{1}{2}y_1^2$.
The curve is hence a parabola, with equation $y_2 = \frac{1}{2}y_1^2$. The center of the parabola, in α-coordinates, is $(0,0)$, and the axis of symmetry is given by the equation $y_1 = 0$. So the axis of symmetry can be written (in α-coordinates) as $\text{span}\,((0,1))$. We translate back into standard coordinates:

- Since we did not change the coordinate grid's origin in this example (we only rotated), the center's standard coordinates are still $(0,0)$.
- The standard coordinates of $(0,1)$ are

  $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \;\; ,$$

  so the symmetry axis is $\text{span}\,((3,4))$.



Chapter 4

Another application example: linear differential equations

What happens in Chapter 4?

As the last part of these lecture notes, we look at another application of the knowledge we gathered: we apply our diagonalisation technique and our knowledge about eigenvalues/-vectors to solve certain systems of differential equations. The functions that appear in these systems are functions of a real variable t. We will denote their derivative by either a dot above the function symbol (so for example $\dot{x}$ for function $x = x(t)$), or with the usual prime (so x').

Learning Goals of Chapter 4: You can

- explain the connection between differential equations and eigenvalues/-vectors; and
- compute solutions for simple examples, using the techniques provided in this chapter.

Why do we care? Systems of differential equations describe situations where several quantities influence one another. The functions usually represent some physical quantities, the derivatives represent their rates of change, and the differential equation defines a relationship between the two. In this context, eigenvalues/-vectors were introduced to decouple the dependencies between the different quantities. The field of applications ranges from engineering and physics over economics to biology. You will also re-encounter the techniques developed in this section in *Theory and Application of Ordinary Differential Equations* (2MBC20).

⚠ In practice, we usually not only have a system of differential equations, but also several (boundary) conditions which the solutions need to satisfy. In this case, we can first determine the general solutions, and then pick out of those the ones which satisfy the boundary conditions.

We start with a simple case, in which we have a system of just a single equation in a single unknown, with no other additive terms. So the system looks as follows:

$$\dot{x} = ax\,,$$

where x is a complex-valued function of a real variable t, and a is a (complex) constant. As the set

of solutions, we find

$$x(t) = ce^{at}, \quad c \in \mathbb{C} \ .$$

4.1. A solving recipe for linear differential equations

What about systems that have more than a single equation? To motivate our approach, we look at the following example:

Example 4.1.1 ***(System with 2 equations, written in matrix terms)*** We look at the system

$$\begin{aligned} \dot{x_1}(t) &= x_1(t) + 2x_2 + e^t, \\ \dot{x_2}(t) &= 12x_1(t) - x_2(t) + 2e^t. \end{aligned}$$

We write this system in matrix form and get:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t \ .$$

Picking matrix A and function vector $\underline{f}$ accordingly, we get as a short-hand:

$$\dot{\underline{x}} = A\underline{x} + \underline{f} \ .$$

We will now translate the left-hand side, $\dot{\underline{x}}$, into 'linear map terms', and additionally separate systems into ones where no additive term disturbs our reasoning and systems where there are such additive terms:

Definition 4.1.2. *We recall from Linear Algebra 1 that taking the derivative is linear. We can hence define the derivative map on the space of n many differentiable functions $\mathbb{R} \to \mathbb{C}$, by defining*

$$\mathcal{D}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \dot{x_1} \\ \vdots \\ \dot{x_n} \end{pmatrix} \ ,$$

or, shorter,

$$\mathcal{D}\underline{x} = \dot{\underline{x}} \ .$$

A **system of n linear differential equations with constant coefficients** *and in n many unknown functions $x_1, \ldots, x_n$ of a real variable t has the form*

$$\begin{pmatrix} \dot{x_1} \\ \vdots \\ \dot{x_n} \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \ ,$$

where A is an $n \times n$ matrix and $f_1, \ldots, f_n$ are functions of t.

We call the system **homogeneous** *if $\underline{f} = \underline{0}$ and* **inhomogeneous** *otherwise.*

So the system in example 4.1.1 is an example for an inhomogeneous system.

To work our way towards a 'solving recipe', we recall the recipe for general vector equations given by the third item of Theorem 1.1.14: the set of solutions to a vector equation is found by a) finding a particular solution, and b) adding to that particular solution all solutions of the corresponding homogeneous equation.

We relate this solving approach to our application at hand: Consider the set V whose elements are vectors of n many functions $x_1, \dots, x_n : \mathbb{R} \to \mathbb{C}$. For convenience, we assume that these functions can be differentiated as often as we want. It is easy to check that V is a (complex) vector space. For every vector $\underline{x} = (x_1, \dots, x_n) \in V$, we can define

$$\mathcal{A}\underline{x} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} .$$

We now use that $\mathcal{A}$ and $\mathcal{D}$ are linear maps on V and that the system of differential equations can be rewritten as

$$(\mathcal{D} - \mathcal{A}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} .$$

Remark 4.1.3 ***(Takeaway: Linear differential equations solving recipe)*** We find all solutions to the system by

1. ***computing the homogeneous solutions***, i.e., computing

$$\mathcal{N}(\mathcal{D} - \mathcal{A}) = \{\underline{x} \in V \mid (\mathcal{D} - \mathcal{A})\underline{x} = \underline{0}\} = \{\underline{x} \in V \mid \mathcal{D}\underline{x} = \mathcal{A}\underline{x}\} \ ;$$

2. ***finding a particular solution***, i.e. a vector $\underline{p}$ satisfying $(\mathcal{D} - \mathcal{A})\underline{p} = \underline{f}$, so a vector $\underline{p}$ of functions satisfying

$$\begin{pmatrix} \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{pmatrix} = A \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \ ;$$

3. ***adding the solutions***: the set of all solutions is $\underline{p} + \mathcal{N}(\mathcal{D} - \mathcal{A})$.

4.2. Linear differential equations solving recipe, step 1: using diagonalisation

We proceed by discussing step 1. As usual, life is simpler when dealing with a diagonal matrix:

Example 4.2.1 ***(Step 1 for a diagonal matrix)*** Say we want to compute the homogeneous solutions for a system with matrix

$$A = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} .$$

In other words, we want to solve

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

So the system is decoupled (the quantities x_1, x_2 do not influence each other's derivative):

$$x_1'(t) = 5x_1(t) \text{ and } x_2'(t) = -5x_2(t) .$$

We find the solutions like in the single-equation case:

$$x_1(t) = c_1 e^{5t} \text{ and } x_2(t) = c_2 e^{-5t}$$

for any complex scalars c_1, c_2. Rewriting this as a vector yields

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t} ,$$

so the set of solutions is the (complex) linear span spanned by $\underline{e}_1 e^{5t}$ and $\underline{e}_2 e^{-5t}$.
Notice that the exponent coefficients in the two exponential functions (± 5) are exactly the diagonal elements of A.

The solution would work if we replaced 5 and -5 with other values, so we generalise this by the following theorem:

Theorem 4.2.2. *If A is an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \cdots, \lambda_n$, then the set of solutions for the respective homogeneous system $\dot{\underline{x}} = A\underline{x}$ is*

$$\{\underline{x} \in V \mid \dot{\underline{x}} = \mathcal{A}\underline{x}\} = \{c_1 e^{\lambda_1 t}\underline{e}_1 + \cdots + c_n e^{\lambda_n t}\underline{e}_n \mid c_1, \ldots, c_n \in \mathbb{C}\} .$$

Proof. It is easy to verify that elements of the right-hand side set satisfy the equation $\dot{\underline{x}} = \mathcal{A}\underline{x}$:

$$\dot{\underline{x}} = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} .$$

To check that all solutions indeed are in the right-hand side set, we assume that $\underline{x}$ satisfies $\dot{\underline{x}} = A\underline{x}$, so $x_i' = \lambda_i x_i$ for each index i. To show that $\underline{x}$ is in the right-hand side set, we show that $x_i(t) = c_i e^{\lambda_i t}$ for each index i: We define the function $y_i(t) = e^{-\lambda_i t} \cdot x_i(t)$ and prove that its derivative is 0.

$$y_i'(t) = -\lambda_i e^{-\lambda_i t} \cdot x_i(t) + e^{-\lambda_i t} \cdot x_i'(t) = e^{-\lambda_i t} \cdot (-\lambda_i x_i(t) + x_i'(t)) = e^{-\lambda_i t} \cdot 0 ,$$

where the last step used that $x_i' = \lambda_i x_i$ for each index i. If the derivative of a function is 0, then it must be constant, so there exists a constant c_i such that $e^{-\lambda_i t} \cdot x_i(t) = c_i$. □

This result allows us to continue with example 4.1.1, using (once again) the diagonalisation approach:

Example 4.2.3 ***(Diagonalising example 4.1.1)*** We now want to compute the homogeneous solutions of the system in 4.1.1, i.e., we want to solve

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

We bring the coefficient matrix A in diagonal form by switching to an eigenvector basis: the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 12 & -1-\lambda \end{vmatrix} = \lambda^2 - 25 = 0 .$$

So we have two real eigenvalues, 5 with eigenvector $(1,2)$ and -5 with eigenvector $(-1,3)$, which yields the eigenvector basis $\alpha = \{(1,2),\ (-1,3)\}$. Then

$${}_\varepsilon S_\alpha = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}, \quad {}_\alpha S_\varepsilon = {}_\varepsilon S_\alpha{}^{-1} = \tfrac{1}{5} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix},$$

and we get the diagonal matrix

$$A_\alpha = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}.$$

We now transform coordinates to the basis α:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = {}_\alpha S_\varepsilon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} 3x_1 + x_2 \\ -2x_1 + x_2 \end{pmatrix}.$$

Then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = {}_\varepsilon S_\alpha \begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix}$ and the system hence changes into

$${}_\varepsilon S_\alpha \begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = A_\varepsilon {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} .$$

Multiplying on the left with ${}_\alpha S_\varepsilon$ yields

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = {}_\alpha S_\varepsilon A_\varepsilon {}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

So we end up with the system in example 4.2.1 and find the solutions

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t} .$$

To switch back to $\underline{x}$, we multiplying on the left with ${}_\varepsilon S_\alpha$:

$${}_\varepsilon S_\alpha \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 {}_\varepsilon S_\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{5t} + c_2 {}_\varepsilon S_\alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t},$$

and we get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-5t}.$$

Notice that the "vector coefficients" of e^{5t}, e^{-5t} are exactly the eigenvectors for 5 and -5.

This gives us a general recipe for diagonalisable matrices:

Theorem 4.2.4. *If A can be brought into real diagonal form via an eigenvector basis $\{\underline{a}_1, \underline{a}_n \ldots\}$, then the set of solutions for the respective homogeneous system $\dot{\underline{x}} = A\underline{x}$ is*

$$\{\underline{x} \in V \mid \dot{\underline{x}} = \mathcal{A}\underline{x}\} = \{c_1 e^{\lambda_1 t}\underline{a}_1 + \cdots + c_n e^{\lambda_n t}\underline{a}_n \mid c_1, \ldots, c_n \in \mathbb{C}\} \ ,$$

where λ_i is the eigenvalue for respective Eigenvector $\underline{a}_i$.

Proof. This can be verified by generalising the reasoning in Theorem 4.2.3: Exactly as in example Theorem 4.2.3, we bring the matrix A in diagonal form by transforming (setting $\underline{y} := {}_\alpha S_\varepsilon \underline{x}$), which leaves us with a decoupled (diagonal) system whose coefficients are the eigenvalues. We solve the diagonal system, which yields

$$\underline{y} = c_1 e^{\lambda_1 t}\underline{e}_1 + \cdots + c_n e^{\lambda_n t}\underline{e}_n \ ,$$

where $c_1, \ldots, c_n \in \mathbb{C}$, and transform back, which yields

$$\begin{aligned}\underline{x} = {}_\varepsilon S_\alpha \underline{y} &= c_1 e^{\lambda_1 t}\, {}_\varepsilon S_\alpha \underline{e}_1 + \cdots + c_n e^{\lambda_n t}\, {}_\varepsilon S_\alpha \underline{e}_n \\ &= c_1 e^{\lambda_1 t}\underline{a}_1 + \cdots + c_n e^{\lambda_n t}\underline{a}_n \ .\end{aligned}$$

□

Example 4.2.5 ***(Giving solutions in real terms for non-real eigenvalues/-vectors)*** What if the eigenvalues/-vectors are non-real, but the matrix is? This example indicates how to non-real solution sets can be brought into real shape.
Say A is a real 2×2 matrix with eigenvalue i and eigenvector $(i, 1)$. Then $-i$ is also an eigenvalue, with eigenvector $(-i, 1)$:

$$A\begin{pmatrix} -i \\ 1 \end{pmatrix} = \overline{A\begin{pmatrix} i \\ 1 \end{pmatrix}} = \overline{A\begin{pmatrix} i \\ 1 \end{pmatrix}} = \overline{i\begin{pmatrix} i \\ 1 \end{pmatrix}} = -i\begin{pmatrix} -i \\ 1 \end{pmatrix} \ .$$

⚠ This does not work if A contains complex entries. (Because then $\overline{A} \neq A$.)
Our solution recipe tells us that the solutions are

$$\{\underline{x} \in V \mid \dot{\underline{x}} = \mathcal{A}\underline{x}\} = \{c_1 e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C}\} \ ,$$

in other words, the (complex) span of $e^{it}(i, 1)$ and $e^{-it}(-i, 1)$, which we now rewrite: $e^{-it}(-i, 1)$ is the complex conjugate of $e^{it}(i, 1)$. (Check this!) So we can linearly combine the two vectors such that

$$\operatorname{span}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}, e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right) = \operatorname{span}\left(\operatorname{Re}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right), \operatorname{Im}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}\right)\right)$$

Since $e^{it} = \cos(t) + i\sin(t)$ (see B.1.5), we have

$$e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = (\cos(t) + i\sin(t))\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \ .$$

So the solution set can be rewritten as

$$\{\underline{x} \in V \mid \dot{\underline{x}} = \mathcal{A}\underline{x}\} = \{c_1\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + c_2\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \mid c_1, c_2 \in \mathbb{C}\} \ .$$

4.3. Linear differential equations solving recipe, step 2: tricks to find particular solutions

For diagonalisable matrices, we now know how to solve the homogeneous system, so we know how to do step 1 of our solving recipe. Step 2 has no one-size-fits all solution, but there are some standard tricks we pick up by looking at examples.

Example 4.3.1 ***(Particular solution and final solution set for example 4.1.1)*** We now want to compute a particular solution for the system in 4.1.1, i.e., we want to find a vector such that

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t \, .$$

Since the difference between the derivative and the matrix-vector product is a 'vector multiple' of e^t, the solution should probably also be a vector multiple of e^t.
We will hence try out $\underline{p} = \underline{u}e^t$, by plugging this into the equation and checking whether this works out for some vector $\underline{u} = (u_1, u_2) \in \mathbb{C}^2$:
Since $\underline{u}$ is a constant vector, we get

$$\dot{\underline{p}} = \underline{u}e^t = \underline{p}.$$

So for our choice of $\underline{p}$, the equation is equivalent to

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^t = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t \, .$$

We get rid of the common factor e^t: if the two sides are supposed to be equal for any input t, the equality must also be true without the factor e^t. We hence get a plain equation system for $\underline{u}$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 12 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} .$$

Solving this yields $\underline{u} = -1/4(2,1)$, so $\underline{p} = -1/4(2,1)e^t$ is a particular solution to the system.
We can now determine the set of all solutions to the system by adding to $\underline{p}$ the set of homogeneous solutions that we determined in Example 4.2.3: it is

$$\{-1/4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} e^{-5t} . \mid c_1, c_2 \in \mathbb{C}\} \, .$$

Take-away: if the additional term $\underline{f}$ is a a vector multiple of e^{at}, try $\underline{p} = \underline{u}e^{at}$.

Example 4.3.2 We look for a particular solution of

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & -8 & 8 \\ 1 & 4 & -1 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \left(t \begin{pmatrix} -22 \\ 3 \\ 10 \end{pmatrix} + \begin{pmatrix} 21 \\ -6 \\ -4 \end{pmatrix} \right) e^{2t} \, .$$

Trying $\underline{p} = \underline{a}e^{at}$ doesn't yield a solution, so we next try a solution of the form $\underline{x} = (\underline{u}t + \underline{v})e^{2t}$.

Let A be the coefficient matrix, $\underline{a} = (-22, 3, 10)$, $\underline{b} = (21, -6, -4)$. Substitution in the equation then gives

$$(\underline{u} + 2\underline{u}t + 2\underline{v})e^{2t} = (\mathcal{A}\underline{u}\, t + \mathcal{A}\underline{v} + \underline{a}t + \underline{b})e^{2t}\,,$$

$$\mathcal{A}\underline{u} + \underline{a} = 2\underline{u},\ \underline{u} + 2\underline{v} = \mathcal{A}\underline{v} + \underline{b}\,,$$

$$(\mathcal{A} - 2\mathcal{I})\underline{u} = -\underline{a},\ (\mathcal{A} - 2\mathcal{I})\underline{v} = \underline{u} - \underline{b}\,.$$

We start by solving the equation for $\underline{u}$:

$$\left(\begin{array}{ccc|c} -2 & -8 & 8 & 22 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -4 & -10 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right),$$

we find $\underline{u} = (1, -1, 2)$. Next, with this $\underline{u}$ we solve the equation $\underline{v}$:

$$\left(\begin{array}{ccc|c} -2 & -8 & 8 & -20 \\ 1 & 2 & -1 & 5 \\ 0 & 2 & -4 & 6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

So $\underline{v} = (2, 1, -1)$ and a particular solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(t\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right)e^{2t}\,.$$

Take-away: if the additional term $\underline{f}$ is a a vector multiple of e^{at}, but $\underline{p} = \underline{u}e^{at}$ did not work, try bigger linear combinations. $((\underline{u} + \underline{v}t)e^{at}, (\underline{u} + \underline{v}t + \underline{w}t^2)e^{at}, \ldots)$

Example 4.3.3 ***(Sinus/cosinus example)*** Consider the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

We first solve the homogeneous system: the characteristic equation is $\lambda^2 + 9 = 0$, so the eigenvalues are $3i$ and $-3i$. We find the eigenspaces $E_{3i} = \langle(1+3i, 5)\rangle$ and $E_{-3i} = \langle(1-3i, 5)\rangle$. The solutions to the homogeneous equation are therefore

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} 1+3i \\ 5 \end{pmatrix}e^{3it} + c_2\begin{pmatrix} 1-3i \\ 5 \end{pmatrix}e^{-3it}\,,$$

or, in real form (we again rearrange the span, using $e^{it} = \cos(t) + i\sin(t)$):

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1'\begin{pmatrix} \cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2'\begin{pmatrix} 3\cos 3t + \sin 3t \\ 5\sin 3t \end{pmatrix}.$$

It remains to find a particular solution. Since the difference between the derivative and the matrix-vector product is a combination of sinus and cosinus, the solution should probably also be a combination of sinus and cosinus, we hence try $\underline{p} = \underline{u}\cos t + \underline{v}\sin t$.

Differentiation of the vector entries yields

$$\dot{\underline{p}} = \underline{u}(-\sin(t)) + \underline{v}\cos t.$$

So for our choice of $\underline{p}$, the equation is equivalent to

$$\underline{u}(-\sin(t)) + \underline{v}\cos t = \begin{pmatrix} 1 & -2 \\ 5 & -1 \end{pmatrix} \underline{u}\cos t + \underline{v}\sin t + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Rearranging the system yields

$$\cos t \left(\underline{v} - A\underline{u} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) - \sin t \left(\underline{u} + A\underline{v} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = 0 \ .$$

Since cos and sin are different functions, and since the terms in the brackets yield constant vectors, both brackets must equal 0. It hence remains to solve

$$\underline{v} - A\underline{u} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{0} \text{ and } \underline{u} + A\underline{v} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{0} \ .$$

This is solved by

$$\underline{u} = -\frac{1}{8} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \underline{v} = \frac{1}{8} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \ ,$$

we hence found a particular solution:

$$\underline{p} = \frac{1}{8} \begin{pmatrix} -\cos t + \sin t \\ -\cos t + 4\sin t \end{pmatrix} \ ,$$

and the set of all real solutions to the system is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -\cos t + \sin t \\ -\cos t + 4\sin t \end{pmatrix} +$$

$$c_1 \begin{pmatrix} \cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2 \begin{pmatrix} 3\cos 3t + \sin 3t \\ 5\sin 3t \end{pmatrix} \ ,$$

with c_1 and $c_2 \in \mathbb{R}$.

Appendix A

Notation used in this course

A.1. Table of frequently used notation

Since we use a fair amount of notation, I thought it would be handy to include a reference table where you can look up the most prominently used symbols. Additionally, you can find set- and map- related notions and notation from previous courses in Appendix A.2.

Notation	What does it denote?	Definition in...
$\mathbb{K}$	an arbitrary field	B.1.1 on page 113
$\mathbb{C}$	field of complex numbers	B.1.2 on page 114
$\exp(x+yi)$ or e^{x+yi}	exponentiation by a complex number	B.1.5 on page 115
V	usually a vector space	B.2.1 on page 116
$\mathbb{K}^n$	vector space: n copies of $\mathbb{K}$	B.2.2 on page 117
$\underline{v}$	underlining indicates a vector	
$\text{span}(\underline{v}_1,\ldots,\underline{v}_m)$	subspace spanned by $\underline{v}_1,\ldots,\underline{v}_m$	B.2.3 on page 117
$(\underline{v},\underline{w})$	the inner product of two vectors $\underline{v}$,$\underline{w}$	B.2.7 on page 117 (standard inn. prod., $\mathbb{K}^n$) B.2.8 on page 117 (general inn. prod.)
$\underline{v} \perp \underline{w}$	$\underline{v}$, $\underline{w}$ are orthogonal	B.2.8 on page 117
$W^\perp$	orthogonal complement of a subspace W	B.2.13 on page 118
V/U	quotient space of V modulo subspace U	B.2.15 on page 119
$\text{Mat}_{\mathbb{K}}(m,n)$	vector space of n-by$m$-matrices	B.3.2 on page 119
${}_\beta S_\alpha$	matrix that transforms α-coords into β-coords	1.3.5 on page 17
$E_\lambda(\mathcal{A})$	Eigenspace of map $\mathcal{A}$ for eigenvalue λ	2.1.7 on page 28
$\chi_\mathcal{A}$	characteristic polynomial: $det(\mathcal{A}-\lambda\mathcal{I})$	2.1.12 on page 30
$\mathcal{A}_{\|W}$	restriction of map $\mathcal{A}: V \to V$ unto a subspace W of V so $\mathcal{A}_{\|W}$ is defined by $w \mapsto Aw$ for $w \in W$	2.2.6 on page 40
$m_{\text{alg}}(\mu)$, $m_{\text{geo}}(\mu)$	algebraic/geometric multiplicity: $m_{\text{alg}}(\mu)$: how often does $(\lambda-\mu)$ fit into $\chi_\mathcal{A}(\lambda)$? $m_{\text{geo}}(\mu)$: dimension of $E_\mu(\mathcal{A})$	2.3.1 on page 46

A.2. Sets and maps

A.2.1 Sets

Sets are usually specified in one of the following ways:

- ***Listing the elements between curly braces.*** For example:

$$\{1,2,3,5\}, \quad \{1,2,3,\ldots\}, \quad \{1,2,3,5,3\}, \quad \{2,\sqrt{3},x^2-1\}.$$

  The dots in the second example mean that we expect the reader to recognize the pattern and complete it: so 4, 5 etc. also belong to this set. Two sets are equal if they contain the same elements, so the first and the third set are equal.

- ***Using a defining property.*** Examples:

$$\{x \mid x \text{ is an even integer}\}, \quad \{y \mid y \text{ is real and } y<0\}.$$

  To highlight in which set (universe) our elements live, we can alternatively write

$$\{x \in \mathbf{Z} \mid x \text{ even}\}, \quad \{y \in \mathbf{R} \mid y<0\} \ .$$

As a reminder, we list some frequently used no(ta)tions.

$\emptyset$	the empty set
$a \in A$	a is an element of the set A
$a \notin A$	a is ***not an element*** of A
$A \subset B$ (sometimes: $B \supset A$ or $A \subseteq B$)	A is a subset of B (or: A is contained in B) i.e. if $a \in A$ then $a \in B$
$A \not\subset B$	A is not a subset of B
$A \cap B := \{x \mid x \in A \text{ and } x \in B\}$	the ***intersection*** of A and B
$A \cup B := \{x \mid x \in A \text{ or } x \in B\}$	the ***union*** of A and B
$A - B := \{x \mid x \in A \text{ and } x \notin B\}$ (sometimes: $A \setminus B$)	the ***complement*** or ***set-theoretic difference*** of A and B
$A \times B := \{(a,b) \mid a \in A, b \in B\}$	the ***(Cartesian) product*** of A and B
$A_1 \times A_2 \times \cdots \times A_n :=$ $\{(a_1,a_2,\ldots,a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$	the ***product*** of A_1, A_2, …, A_n
$A^n := \{(a_1,a_2,\ldots,a_n) \mid a_1,\ldots,a_n \in A\}$	$A^n = A \times A \times \cdots \times A$

A.2.2 Maps

For sets A and B, a ***map*** from A to B is a rule associating to each element of A exactly one element of B, denoted by $f : A \to B$. The set A is called the ***domain*** of the map, B the ***codomain***. If the elements of B are numbers, then one often uses the more common word ***function*** instead of map. Two functions are equal if they have the same domain, the same codomain and the same value for every element of the domain.

We again list some frequently used notation and definitions:

$f : A \to B$	map with domain A and codomain B
	(other letters are also allowed of course!)
$f(a)$	the element in B that is associated to a
	(called value of f in a or ***image*** of a (under f)
$f : a \mapsto b$	f maps a to b
$f(D) := \{f(d) \mid d \in D\}$	the ***image*** of D, for a subset D of A
$f(A)$	***image*** or ***range*** of f
$f^{-1}(E) := \{a \in A \mid f(a) \in E\}$	the (complete) ***inverse image*** or ***pre-image*** of a subset E of B
(sometimes: $f^{\leftarrow}(E)$)	
$f^{-1}(b) := f^{-1}(\{b\})$	set of all elements that are mapped to b
$f : A \to B$ ***injective***	for all $a, a' \in A$: $f(a) = f(a') \Rightarrow a = a'$;
	equivalently: for all $a, a' \in A$: if $a \neq a'$ then $f(a) \neq f(a')$
$f : A \to B$ ***surjective***	for all $b \in B$ there is a pre-image $a \in A$, i.e. an $a \in A$ with $f(a) = b$
	in other words: $f(A) = B$
$f : A \to B$ ***bijective***	f is injective and surjective
	So it's one-to-one: for all $b \in B$ there is a unique $a \in A$ with $f(a) = b$)

If $f : A \to B$ is a bijection, then for all $b \in B$ there is a unique $a \in A$ with $f(a) = b$. In this case, we can define a map from B to A by the rule: $b \mapsto a$ if $f(a) = b$. This map (which only exists if f is a bijection) is called the ***inverse*** of f and is denoted by f^{-1}. Be aware that the same symbol sometimes also refers to the inverse image (see the table above), but it will always be clear from context how f^{-1} is used.

Appendix B

Prerequisites: Vector spaces as seen in *Linear Algebra 1*

This chapter recaps the most important definitions you saw in *Linear Algebra 1*.

B.1. Fields and complex numbers

Definition B.1.1 (fields). *A field is a set $\mathbb{K}$ that contains two special elements, 0 and 1, together with two binary operations, addition and multiplication $+, \cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, such that addition and multiplication obey the following rules (we omit most of the $\forall$-quantifiers, and mostly write ab for $a \cdot b$):*

i)	$a + b = b + a$	*addition is commutative*
ii)	$(a + b) + c = a + (b + c)$	*addition is associative*
iii)	$a + 0 = a$	0 *acts as the neutral element for addition*
iv)	$\forall a : \exists(-a) : a + (-a) = 0$	*existence of an inverse for addition*
	In short, i-iv express that $(\mathbb{K}, +, 0)$ *is an abelian group.*	
v)	$ab = ba$	*multiplication is commutative*
vi)	$(ab)c = a(bc)$	*multiplication is associative*
vii)	$1 \cdot a = a$	1 *acts as the neutral element for multiplication*
viii)	$\forall a \neq 0 : \exists a^{-1} : a \cdot a^{-1} = 1$	*existence of an inverse for multiplication*
	In short, v-viii express that $(\mathbb{K} \setminus \{0\}, \cdot, 1)$ *is an abelian group.*	
ix)	$(a + b)c = ac + bc$	*multiplication is distributive over addition*

Examples. The fields we know from high school are $\mathbb{Q}$ (the rational numbers) and $\mathbb{R}$ (the real numbers). In *Linear Algebra 1*, you also encountered $\mathbb{C}$, the complex numbers, that were constructed as follows:

Definition B.1.2 (complex numbers). *We turn the vector space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ into a field, called $\mathbb{C}$, by defining $0 := (0,0)$ and $1 := (1,0)$ and using the following addition and multiplication:*

- $(a,b) + (c,d) := (a+c, b+d)$*, the usual addition in* $\mathbb{R}^2$.
- $(a,b) \cdot (c,d) := (ac - bd, ad + bc)$.

You saw in *Linear Algebra 1* that $\mathbb{C}$ indeed is a field. To briefly recap, it's easy to check that we get additive inverses via $-(a,b) := (-a,-b)$. Less obviously, we also get multiplicative inverses via $(a,b)^{-1} := (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$.
Setting $i := (0,1)$, we can use that $1 = (1,0)$ to rewrite $(a,b) = a + bi$. We briefly verify that this alternative notation is compatible with how we defined addition and multiplication:

- $(a+bi) + (c+di) = (a+c) + (b+d)i$ and
- $(a+bi)(c+di) = ac + adi + bic + bidi = ac - bd + (ad+bc)i$, where we used that $i^2 = -1$.

We recall some notation that was introduced for complex numbers:

Definition B.1.3. *Let $z = x + yi$, $x, y \in \mathbb{R}$ be a complex number. We denote*

- $\mathrm{Re}\, z := x$ — *the real part of z*
- $\mathrm{Im}\, z := y$ — *the imaginary part of z*
- $\bar{z} := x - yi$ — *the complex conjugate of z*
- $|z| := \|(x,y)\| := \sqrt{x^2+y^2}$ — *the absolute value (also 'length' or 'modulus') of z*

It can be easily checked that $\bar{0} = 0$, $\bar{1} = 1$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1 z_2}$.

Conjugation does not do anything to a complex number z exactly when z itself is a real number, i.e., when its imaginary part is 0: $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$. Using the last lines, it can also be argued that the conjugation map $z \mapsto \bar{z}$ is an $\mathbb{R}$-automorphism of $\mathbb{C}$.

We furthermore have: $z + \bar{z} = 2\,\mathrm{Re}\, z$, $z - \bar{z} = 2i\,\mathrm{Im}\, z$, $z\bar{z} = |z|^2$, and $z^{-1} = \bar{z}/|z|^2$.

We saw that complex numbers can be expressed in polar coordinates:

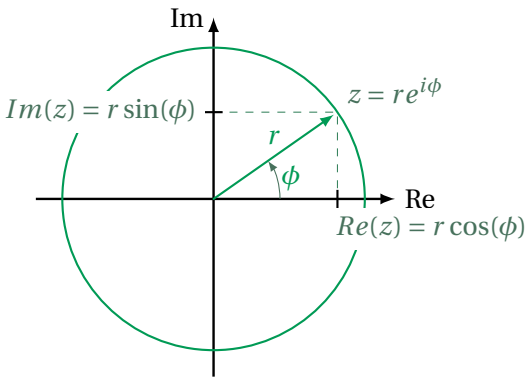
Definition B.1.4 (polar representation). *Every $z \in \mathbb{C}$ has a representation of the form*

$$z = r(\cos\phi + i\sin\phi) =: r\ cis\,\phi$$

for some $\phi \in \mathbb{R}$ and r being the absolute value of z, i.e., $r := |z| \geq 0$.

For $z = 0$, ϕ can be arbitrary. If $z \neq 0$, then ϕ represents the angle between the positive x-axis and the vector $(x,y) \in \mathbb{R}^2$. In that case, ϕ hence is determined up to a multiple of 2π.

We call ϕ the argument *of z and denote it by $\arg z$. When we pick a fixed interval of length 2π (usually $[0, 2\pi)$ or $(-\pi, \pi]$), exactly one argument ϕ lies in this interval. We call this uniquely determined argument the* principal argument.



The polar representation allows us to derive two important rules:

- $|z_1 z_2| = |z_1||z_2|$ the modulus of the product is the product of the moduli
- $\arg\ z_1 z_2 = \arg\ z_1 + \arg\ z_2$ the argument of the product is the sum of the arguments

(Proof: Say that $z_1 = r_1$ cis α and $z_2 = r_2$ cis β. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2\ ((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)) = \\ &= r_1 r_2\ (\cos(\alpha+\beta) + i\sin(\alpha+\beta)) = r_1 r_2\ \text{cis}(\alpha+\beta)\ , \end{aligned}$$

where we used the addition formulas for cosine and sine.)

The function cis behaves very much like the function exp: cis $\alpha \cdot$ cis $\beta =$ cis $(\alpha+\beta)$ and $\exp x \cdot \exp y = \exp(x+y)$. This property allows us to extend the definition of the exponential function to the field $\mathbb{C}$ (and forget about cis altogether).

Definition B.1.5 (exponential representation). *Let $\alpha \in \mathbb{R}$, and let $z = x + yi$ for $x, y \in \mathbb{R}$. We define:*

$$\begin{aligned} \exp(i\alpha) &:= e^{i\alpha} := cis\ \alpha = \cos\alpha + i\sin\alpha \\ \exp(z) &:= e^z = e^{x+yi} := e^x \cdot e^{iy} = e^x \cdot cis\ y = e^x(\cos y + i\sin y). \end{aligned}$$

Since cis y is of length one, we have that $|e^z| = e^x = e^{\text{Re } z}$. We also have that $\arg(e^z) = y = \text{Im } z$. If z has modulus r and argument ϕ, we can identify $z = r\ e^{i\phi}$.

An important reason why complex numbers play a useful role in the theory of real vector spaces is the following theorem.

> **Theorem B.1.6 (fundamental theorem of algebra).** *Every non-constant polynomial over $\mathbb{C}$ has a zero (or 'root') in $\mathbb{C}$.*

As the name suggests, this theorem is of fundamental importance. One of its consequences is that any polynomial over $\mathbb{C}$ can be neatly written as a product of linear factors: If f is a non-constant polynomial over $\mathbb{C}$, it has a zero $\alpha \in \mathbb{C}$. Then the linear polynomial $z - \alpha$ divides f, i.e. $f(z) = (z-\alpha)g(z)$ for some polynomial g. If g is neither a linear polynomial nor a constant, we can apply

the theorem to g, and so forth. Since the degree of the remaining polynomial decreases by one with each repetition, we will at some point end up with a complete factorization into linear factors and one constant factor $\beta \in \mathbb{C}$: If f has degree n, we will end up with a factorisation $f(z) = \beta(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$.

Definition/theorem B.1.7 (Conjugates of polynomials). *If $p(z) = p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0$ is polynomial with real coefficients, i.e., if $p_i \in \mathbb{R}$ for all i, then we let*

$$\begin{aligned} \overline{p(z)} &:= \overline{p_n z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0} \\ &\overset{(\star)}{=} p_n \bar{z}^n + p_{n-1} \bar{z}^{n-1} + \cdots + p_1 \bar{z} + p_0 = p(\bar{z}) \ , \end{aligned}$$

where $(\star)$ used that conjugation is an automorphism of $\mathbb{C}$ that leaves elements of $\mathbb{R}$ unchanged.

As a consequence, we get that any polynomial p with real coefficients can be written as a product of real polynomials that are either linear or quadratic, and a real factor $\beta \in \mathbb{R}$: We know that p splits into the product of linear complex polynomials $(z-\alpha_i)$, where $\alpha_1, \cdots \alpha_n \in \mathbb{C}$ are the zeros of p, and some factor $\beta \in \mathbb{C}$. If a particular zero α is real, the corresponding factor $(z-\alpha)$ is a linear real polynomial, which matches our claim. Let's also take a closer look at the zeros α that are complex, but not real: Given that $p(\alpha) = 0$, we can defer that $\bar{\alpha}$ also is a zero of p since $p(\bar{\alpha}) \overset{B.1.7}{=} \overline{p(\alpha)} = 0$. In other words, we can sort the non-real zeros of p into tuplets $(\alpha, \bar{\alpha})$ of conjugates. For each such tuplet, we obtain that the product $(z-a)(z-\bar{a}) = z^2 - (a+\bar{a})z + a\bar{a} = z^2 - 2\,\mathrm{Re}\, az + |a|^2$ is a quadratic real polynomial, thus matching our claim. We finish by observing that β is real since the highest coefficient of p is real.

B.2. Vector spaces

Definition B.2.1 (vector space). *Let $\mathbb{K}$ be a field. A $\mathbb{K}$-vector space is a set V that contains a special element, $\underline{0}$, called the zero vector, together with two operations, addition $+ : V \times V \to V$, $(\underline{v}, \underline{w}) \mapsto \underline{v} + \underline{w}$ and scalar multiplication $\cdot : \mathbb{K} \times V \to V$, $(\lambda, \underline{v}) \mapsto \lambda \cdot \underline{v}$ (or simply $\lambda \underline{v}$), such that addition and multiplication obey the following rules (we omit the $\forall$-quantifiers):*

i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ — *addition is commutative*

ii) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ — *addition is associative*

iii) $\underline{a} + \underline{0} = \underline{a}$ — *$\underline{0}$ acts as the neutral element for addition*

iv) $\forall \underline{a} : \exists (-\underline{a}) : \underline{a} + (-\underline{a}) = \underline{0}$ — *existence of an inverse for addition*

In short, i-iv express that $(V, +, \underline{0})$ is an abelian group.

v) $\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$ — *multiplication is distributive over vector addition*

vi) $(\lambda + \mu)\underline{a} = \lambda \underline{a} + \mu \underline{a}$ — *multiplication is distributive over scalar addition*

vii) $(\lambda \mu)\underline{a} = \lambda(\mu \underline{a})$ — *associativity of combining field multiplication with scalar multiplication*

In $\mathbb{K}$-vector spaces, we will always have (due to the rules) that $1 \cdot \underline{a} = \underline{a}$, $(-1) \cdot \underline{a} = -\underline{a}$, and $0 \cdot \underline{a} = \underline{0}$.

Definition B.2.2 (the vector space $\mathbb{K}^n$). *By $\mathbb{K}^n$ we denote the set of (ordered) n-tuples of elements of a field $\mathbb{K}$, so vectors $\underline{v} \in \mathbb{K}^n$ are tuples $\underline{v} = (v_1, \ldots, v_n)$ with $v_1, \ldots, v_n \in \mathbb{K}$. We define $\mathbb{K}^n$ as a $\mathbb{K}$-vector space by defining the following rules for addition and scalar multiplication: $\underline{v} + \underline{w} := (v_1 + w_1, \ldots, v_n + w_n)$ and $\lambda \underline{v} := (\lambda v_1, \ldots, \lambda v_n)$. The zero vector in this vector space is $\underline{0} = (0, \ldots, 0)$.*

Definition B.2.3 (subspace and span). *A subset W of V is a (linear) subspace if (i) $\underline{0} \in W$ and (ii) W is closed under addition and scalar multiplication.*

If $\{\underline{v}_1, \ldots, \underline{v}_m\}$ is a set of vectors in V, then we define as the span of (or also the subspace spanned by) $\{\underline{v}_1, \ldots, \underline{v}_m\}$ the set of all linear combinations $\lambda_1 \underline{v}_1 + \cdots + \lambda_m \underline{v}_m$. We denote it by $\operatorname{span}(\underline{v}_1, \ldots, \underline{v}_m)$. *(Another common notation is $< \underline{v}_1, \ldots, \underline{v}_m >$.)*

We define the span of the empty set to be $\{\underline{0}\}$, and the span of an infinite set as the collection of all finite *linear combinations, meaning we allow only a finite number of non-zero coefficients λ.*

Definition B.2.4 (linear (in)dependence). *A linear combination is called* non-trivial *if at least one of its coefficients is non-zero. A set A of vectors in a vector space V is called* dependent *if there is a non-trivial linear combination of vectors in A that equals the zero vector, otherwise the set is called* independent.

On a high level, the usual way to prove independence is this: Start by supposing that $\lambda_1 \underline{v}_1 + \cdots + \lambda_m \underline{v}_m = \underline{0}$, then do some reasoning to show that $\lambda_1 = \cdots = \lambda_m = 0$.

Definition B.2.5 (basis). *If a set of vectors α spans V and is linearly independent, we say that α is a basis for V.*

An important property of vector spaces is that every vector space has a basis (if the basis is infinite, we need some version of the axiom of choice to prove this).

> **Definition/theorem B.2.6 (dimension).** *All bases of a vector space have the same size (cardinality), called the* dimension *of V and denoted by* $\dim(V)$.

B.2.1 Inner product spaces

Definition B.2.7 (standard inner product of $\mathbb{K}^n$). *Let $\mathbb{K}$ be a field. For two vectors $\underline{a}, \underline{b} \in \mathbb{K}^n$, we denote the standard inner product by $(\underline{a}, \underline{b})$:*

$$(\underline{a}, \underline{b}) = a_1 b_1 + \cdots + a_n b_n.$$

(Other popular notations are $\underline{a} \bullet \underline{b}$ or $\langle \underline{a}, \underline{b} \rangle$.)

Definition B.2.8. *[inner product, orthogonality, length of a vector, angle between vectors] More generally, let V be a vector space. We say that V together with a map $(.,.) : V \times V \to \mathbb{R}$ is an inner product space if the following properties are satisfied:*

i) $(\underline{a}, \underline{b}) = (\underline{b}, \underline{a})$ — *inner product is symmetric*

ii) inner product is linear:

1. $(\underline{a}+\underline{b},\underline{c}) = (\underline{a},\underline{c}) + (\underline{b},\underline{c})$, *and*
2. $(\lambda\cdot\underline{a},\underline{b}) = \lambda\cdot(\underline{a},\underline{b})$

iii) $(\underline{a},\underline{a}) \geq 0$ *with equality iff* $\underline{a} = \underline{0}$ *inner product is positive definite*

We say that $\underline{a}$ *and* $\underline{b}$ *are* **orthogonal** *(or* **perpendicular***) and write* $\underline{a} \perp \underline{b}$ *if* $(\underline{a},\underline{b}) = 0$.

The **length** *of* $\underline{a}$ *is given by* $\|\underline{a}\| = \sqrt{(\underline{a},\underline{a})}$*, and the angle between non-zero vectors* $\angle(\underline{a},\underline{b})$ *is given by* $\cos\angle(\underline{a},\underline{b}) := \frac{(\underline{a},\underline{b})}{\|\underline{a}\|\|\underline{b}\|}$.

Theorem B.2.9 (theorem of Cauchy-Schwarz).

$$(\underline{a},\underline{b})^2 \leq (\underline{a},\underline{a})(\underline{b},\underline{b}),$$

with equality if and only if the set $\{\underline{a},\underline{b}\}$ *is dependent (so* $\underline{a} = \underline{0}$*, or* $\underline{b} = \lambda\underline{a}$ *for some* λ*).*

Definition B.2.10 (orthonormal basis). *A basis* α *for an inner product space* V *is called* orthonormal *if the basis vectors are mutually orthogonal unit vectors, i.e., if*

$$(\underline{a}_i,\underline{a}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

If $\{\underline{a}_1,\ldots,\underline{a}_m\}$ is an orthonormal basis of a subspace W, then the orthogonal projection of a vector $\underline{v} \in V$ on the subspace W is given by $\mathcal{P}_W(\underline{v}) = (\underline{v},\underline{a}_1)\underline{a}_1 + \cdots + (\underline{v},\underline{a}_m)\underline{a}_m$.

Additionally, we know that orthonormal systems are always linearly independent:

Theorem B.2.11. *Orthonormal sets of vectors in real inner product spaces are linearly independent.*

We can turn any basis into one that is orthonormal:

Definition B.2.12 (Gram–Schmidt). *Given a basis* $\alpha = \{\underline{a}_1,\ldots,\underline{a}_n\}$ *for an inner product space* V*, an orthonormal basis* β *can be computed efficiently as follows: Let* $B_i = \operatorname{span}(\underline{b}_1,\ldots,\underline{b}_i)$ *(so we start with* $B_0 = \{\underline{0}\}$*). For* $i = 1,\ldots,n$*, define*

- $\underline{v}_i := \underline{a}_i - \mathcal{P}_{B_i}(\underline{a}_i)$*, then*
- $\underline{b}_i := \underline{v}_i / \|\underline{v}_i\|$.

In practice, it is more reasonable to work with the projection formula for mutually orthogonal basis vectors and only normalise in the end, so: $\underline{v}_1 = \underline{a}_1$, $\underline{v}_2 = \underline{a}_2 - \dfrac{(\underline{a}_2,\underline{v}_1)}{(\underline{v}_1,\underline{v}_1)}\underline{v}_1$, $\underline{v}_3 = \underline{a}_3 - (\cdots)\underline{v}_1 - (\cdots)\underline{v}_2$*, etc..*

Definition B.2.13 (Orthogonal complement). *Let* W *be a linear subspace of an inner product space* V*. The* **orthogonal complement of** W *is the set* $W^\perp := \{\underline{v} \in V \mid\}(\underline{v},\underline{w}) = 0$ for all $\underline{w} \in W$.

The orthogonal complement is sometimes also called **orthoplement**.

We also recall some useful properties of the orthogonal complement $W^\perp$:

Theorem B.2.14. *Let V be a real inner product space and let W be a linear subspace of V. Then*

- *$W^\perp$ is a linear subspace of V.*
- *$W \cap W^\perp = \{\underline{0}\}$, i.e., a linear subspace W and its orthogonal complement $W^\perp$ only have the zero vector in common.*

B.2.2 Quotient spaces

From set theory we know that, given an equivalence relation $\sim$ on a set S, the quotient set $S/\sim$ can be formed consisting of the equivalence classes with respect to $\sim$. In the setting of vector spaces, we saw that a linear subspace U of a vector space V can be used to define an equivalence relation and construct a quotient set: We defined the equivalence relation $\sim$ on V as follows:

$$\underline{v}_1 \sim \underline{v}_2 \Leftrightarrow \underline{v}_1 - \underline{v}_2 \in U \ ,$$

and denoted the equivalence class of a vector $v \in V$ by $[v]$.

We saw that with this relation, the quotient set $V/\sim$ turns out to be a vector space:

Definition B.2.15 (Quotient space modulo a subspace). *Let V be a vector space with subspace U. We denote the quotient set $V/\sim$ by V/U.*

V/U becomes a vector space by introducing the following vector space operations: For vectors $\underline{v}_1, \underline{v}_2, \underline{v} \in V/U$ and scalars $\lambda \in \mathbb{K}$, we define

$$\begin{aligned} + &: [\underline{v}_1] + [\underline{v}_2] := [\underline{v}_1 + \underline{v}_2] \ , \\ \cdot &: \lambda \cdot [\underline{v}] := [\lambda \underline{v}] \ . \end{aligned}$$

V/U is called it the **quotient space**.

B.3. Matrices

Definition B.3.1 (matrices). *An $m \times n$ matrix A is a rectangular block with m many rows and n many columns, consisting of numbers $a_{ij} \in \mathbb{K}$.*

Abstract and more general: if R and C are sets, then an $R \times C$ matrix A is a map $R \times C \to \mathbb{K}$. (Note the little difference: in the first definition, rows and columns are ordered.)

Definition B.3.2 (addition and scalar multiplication, vector space of matrices). *If A and B are $m \times n$ matrices, then we can define a matrix $S = A + B$ by setting $s_{ij} = a_{ij} + b_{ij}$. For scalars $\lambda \in \mathbb{K}$, we can also define λA as the matrix with ij-entry λa_{ij}.*

For the more abstract definition, where A and B are $R \times C$ matrices, we define $S = A + B$ by setting $S(r,c) = A(r,c) + B(r,c)$ for all $r \in R$ and $c \in C$ and λA as the matrix with (r,c)-entry $\lambda A(r,c)$.

Together with the zero matrix, these operations turn the set of $m \times n$ matrices into a vector space, which we denote by $Mat_{m,n}(\mathbb{K})$ or $Mat_{\mathbb{K}}(m,n)$ or also $\mathbb{K}^{m \times n}$.

Definition B.3.3 (matrix multiplication). *If A is an $m \times k$ matrix, and B is a $k \times n$ matrix, then the product $P = A \cdot B$ is the $m \times n$ matrix defined by*

$$p_{ij} = \sum_{*=1}^{k} a_{i*} b_{*j},$$

More general: if A is an $R \times K$ matrix and B is a $K \times C$ matrix, then $P = A \cdot B$ is the $R \times C$ matrix defined by

$$p(r,c) = \sum_{* \in K} a(r,*) b(*,c).$$

We also saw that matrix multiplication is *associative*: $(AB)C = A(BC)$.

(Proof: $ABC(i,j) = \sum_{kl} a_{ik} b_{kl} c_{li}$.)

Products of vectors and matrices. We usually consider a vector $\underline{v} \in \mathbb{K}^n$ as a column vector, so we identify it with an $n \times 1$ matrix, again denoted $\underline{v}$. In case we want to view it as a row vector (a $1 \times n$ matrix), we write $\underline{v}^\top$. For example, $\underline{v} = (x,y) \in \mathbb{K}^2$ corresponds to the matrix $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, with $\underline{v}^\top = (x \;\; y)$.

This way, we can also define products of vectors and an $m \times n$ matrix A: we can define $\underline{u}^\top A$ and $A\underline{v}$ for $\underline{u} \in \mathbb{K}^m$, $\underline{v} \in \mathbb{K}^n$.

Appendix C

Additional helpers

C.1. Identifying Greek letters

In mathematics we use many letters from the Greek alphabet. If you feel uncertain which one is which, you can look up the letter in the table below that lists the Greek alphabet. (Letters that are frequently used in Linear Algebra are highlighted in green and indicated with a *.)

name	**m**inuscule (= lowercase letter)	**c**apital (= uppercase letter)
alpha	α *	A
beta	β *	B
gamma	γ *	Γ
delta	δ *	Δ
epsilon	ε	E
zeta	ζ	Z
eta	η	H
theta	θ or ϑ	Θ
iota	ι	I
kappa	κ	K
lambda	λ *	Λ
mu	μ *	M
nu	ν	N
xi	ξ	Ξ
omikron	o	O
pi	π	Π
rho	ρ *	R
sigma	σ *	Σ
tau	τ *	T
upsilon	υ	Υ
phi	ϕ or φ *	Φ
chi	χ	X
psi	ψ *	Ψ
omega	ω *	Ω

C.2. Geometric shapes: ellipse, hyperbola, parabola

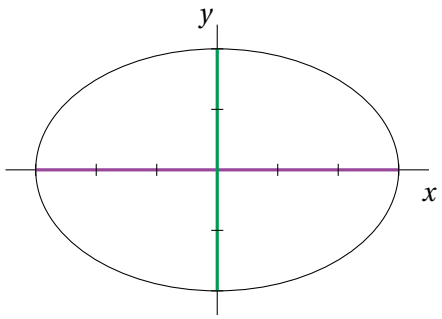
Ellipse

The standard equation for an ***ellipse*** with center (c_1, c_2) is $\frac{(x-c_1)^2}{a^2} + \frac{(y-c_2)^2}{b^2} = 1$, where the denominators a, b are real (positive) numbers. Every ellipse has two axes of symmetry, the longer axis one is called ***major axis*** and the shorter one is called ***minor axis***. The center is the midpoint of both the major and minor axes. The denominators a, b express the distance between the center and the outmost points of the ellipse on the two axes.

Reading out the axis lengths from the standard equation: Since a, b express the distance between the center and the outmost points of the ellipse on the two axes, the length of the major axis is $2 \cdot \max\{a, b\}$ and the length of the minor axis is $2 \cdot \min\{a, b\}$.

Determining major and minor axis, origin center case: If the ellipsis' center is the origin, the two axes of symmetry simply are the x-axis and the y-axis. We can determine which one is major/minor by intersecting the two axes with the ellipse, i.e., by plugging either $x = 0$ or $y = 0$ into the left-hand side of the standard equation, and seeing in which case the remaining denominator is bigger/smaller, thus indicating bigger/smaller axis length.

E.g., take the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$: Plugging 0 in for x yields $\frac{1}{4}y^2$ and plugging in 0 for y yields $\frac{1}{9}x^2$. This means that the outmost points on the y-axis are $(0,-2)$ and $(0,2)$, and that the outmost points on the x-axis are $(-3,0)$ and $(3,0)$. Thus, the major axis is $y = 0$ (the x-axis) and the minor axis is $x = 0$ (the y-axis).



Ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ with center $(0,0)$, major axis $y = 0$ and minor axis $x = 0$.

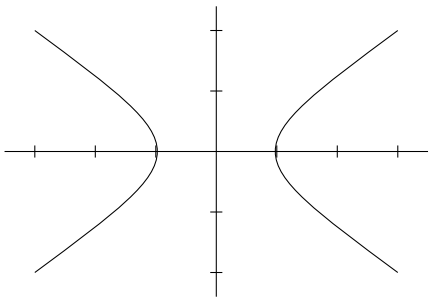
Determining major and minor axis, general case: The case where the center isn't the origin back can be brought back to the origin center case by substituting x and y by $x' := x - c_1$ and $y' := y - c_2$. The minor/major axis can then be determined in (x', y')-coordinates using the reasoning above, and afterwards can be translated to (x, y)-coordinates by substituting back.

E.g., assume you have an ellipse $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$, so the center is $(1,2)$. Say you substituted $x' := x - 1$ and $y' := y - 2$ and determined that the major axis in (x', y')-coordinates is $y' = 0$ since a is bigger than b. Then re-substituting yields the major axis in the orginal coordinates: $y - 2 = y' = 0$, so the major axis is $y = 2$.

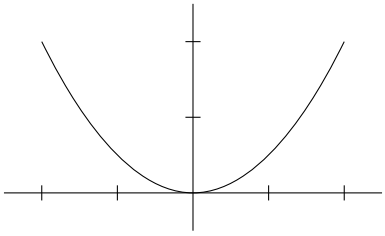
Hyperbola and parabola

The standard equation for a ***hyperbola*** is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where a, b are real (positive) numbers. For hyperbolas in standard shape, the axes symmetry are the x- and the y-axis.

The standard equation for a ***parabola*** is $y = ax^2$, where a is a non-zero real number. For parabolas in standard shape, the axis of symmetry is the y-axis.



The hyperbola $x^2 - 2y^2 = 1$.



The parabola with equation $y = \frac{1}{2}x^2$.

C.3. Trigonometric formulas

This section recalls some useful identities concerning the trigonometric functions:

- $\cos^2(x) + \sin^2(x) = 1$;
- $\sin(x + 2\pi) = \sin(x)$ and $\cos(x + 2\pi) = \cos(x)$;
- $\sin(\pi - x) = \sin(x)$ and $\cos(\pi - x) = -\cos(x)$;
- $\sin(\pi + x) = -\sin(x)$ and $\cos(\pi + x) = -\cos(x)$;
- $\sin(\pi/2 - x) = \cos(x)$ and $\cos(\pi/2 - x) = \sin(x)$;
- $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$;
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

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