Homework: Week 2

Jiaqi Wang

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1 Exercise 4.9.1

Problem 1.1 Show that for all $a, b \in \mathbb{R}$, if $a < b$ then

$$\inf[a, b) = a$$

Proof. To show that a is an infima we need to show that a is a lower bound and that

$$\text{for all } \epsilon > 0,$$
$$\text{there exists } x \in [a, b),$$
$$x < a + \epsilon.$$

First we show that a is a lower bound.
Take $x \in [a, b)$.
It holds that $a \leq x$.
We conclude that a is a lower bound.

Now we show that for all $\epsilon > 0$, there exists $x \in [a, b)$, $x < a + \epsilon$.
Take $\epsilon > 0$.
Choose $x = a$. Then $x \in [a, b)$
It holds that $x = a < a + \epsilon$.
We conclude that $\inf[a, b) = a$. □

2 Exercise 4.9.2

Problem 2.1 Let A be a subset of $\mathbb{R}$. Prove that if $\sup A \in A$ then A has a maximum and $\max A = \sup A$.

Proof. We need to show that $\max A = \sup A$.
Assume $\sup A \in A$.
Choose $x = \sup A$.
It holds that $x \in A$.
Take $a \in A$.
It holds that $a \leq x$.
We conclude that $a = \sup A = \max A$. □

3 Exercise 4.9.3

Problem 3.1 Show that
$$\sup[0,4) = 4$$

Proof. We need to show that 4 is an upper bound and that

$$\text{for all } \epsilon > 0,$$
$$\text{there exists } x \in [0,4),$$
$$x > 4 - \epsilon.$$

First we show that 4 is an upper bound.
Take $x \in [0,4)$.
It holds that $x \leq 4$.
We conclude that 4 is an upper bound.

Now we show that for all $\epsilon > 0$, there exists $x \in [0,4)$, $x > 4 - \epsilon$.
Take $\epsilon > 0$.
Choose $x = \max(0, 4 - \frac{\epsilon}{2})$. Then $x \in [0,4)$
It holds that $x > 4 - \epsilon$.
We conclude that $\sup[0,4) = 4$. □

4 Exercise 4.9.5

> **Problem 4.1** Let A be a non-empty subset of $\mathbb{R}$. Assume that A is bounded from above. Prove that for all $\lambda \geq 0, \sup(\lambda A) = \lambda \sup A$.

Proof. Take $A \subseteq \mathbb{R}$. Suppose A is bounded from above.
It holds that $\sup A$ exists.
Take $\lambda \geq 0$.
By the alternative characterization of the supremum we need to show that $\lambda \sup A$ is an upper bound for λA and that

$$\begin{gathered}\text{for all } \epsilon > 0,\\ \text{there exists } a \in \lambda A,\\ a > \lambda \sup A - \epsilon.\end{gathered}$$

First we show that $\lambda \sup A$ is an upper bound for λA.
Since $\sup A$ is a supremum for A, it is an upper bound for A.
It holds that $\sup A \geq a$ for all $a \in A$.
It holds that $\lambda \sup A \geq \lambda a$ for all $a \in A$ and $\lambda a \in \lambda A$
We conclude that $\lambda \sup A$ is an upper bound for λA.

Now we show that for all $\epsilon > 0$, there exists $a \in \lambda A$, $a > \lambda \sup A - \epsilon$.
Take $\epsilon > 0$.
Since $\sup A$ is a supremum for A, it is an upper bound for A.
By the alternative characterization of the supremum,

$$\begin{gathered}\text{for all } \epsilon_1 > 0,\\ \text{there exists } a_1 \in A,\\ a_1 > \sup A - \epsilon_1.\end{gathered}$$

Either $\lambda = 0$ or $\lambda > 0$.

- Case $\lambda = 0$, then $\lambda \sup A - \epsilon = -\epsilon < 0$.
Choose $a = 0$ and $a \in \lambda A$.
It holds that $a = 0 > \lambda \sup A - \epsilon = -\epsilon$.

- Case $\lambda > 0$
Choose $\epsilon_1 = \dfrac{\epsilon}{\lambda}$. Then $\epsilon_1 > 0$ and there exists $a_1 \in A$, such that $a_1 > \sup A - \epsilon_1 = \sup A - \frac{\epsilon}{\lambda}$
Obtain such a_1.
Choose $a = \lambda a_1$.
It holds that $a = \lambda a_1 > \lambda \sup A - \epsilon$.
We have shown that for all $\epsilon > 0$, there exists $a = \lambda a_1 \in \lambda A$, $a > \lambda \sup A - \epsilon$.
We conclude that $\sup(\lambda A) = \lambda \sup A$. □

5 Exercise 4.9.6

Problem 5.1 Let A be a non-empty subset of $\mathbb{R}$. Assume that A is bounded from below. Prove that for all $\lambda \geq 0, \inf(\lambda A) = \lambda \inf A$.

Proof. We need to show that $\lambda \inf A$ is a lower bound for λA and that

$$\text{for all } \epsilon > 0,$$
$$\text{there exists } a \in \lambda A,$$
$$a < \lambda \inf A + \epsilon.$$

First we show that $\lambda \inf A$ is a lower bound for λA.
Since $\inf A$ is a infimum for A, it is a lower bound for A.
It holds that $\inf A \leq a$ for all $a \in A$.
It holds that $\lambda \inf A \leq \lambda a$ for all $a \in A$ and $\lambda a \in \lambda A$
We conclude that $\lambda \inf A$ is a lower bound for λA.

Now we show that for all $\epsilon > 0$, there exists $x \in \lambda A$, $x < \lambda \inf A + \epsilon$.
Take $\epsilon > 0$.
Since $\inf A$ is a infimum for A, by the alternative characterization of the infimum it holds that

$$\text{for all } \epsilon_1 > 0,$$
$$\text{there exists } a_1 \in A,$$
$$a_1 < \inf A + \epsilon_1.$$

Either $\lambda = 0$ or $\lambda > 0$.

- Case $\lambda = 0$, then $\lambda \inf A + \epsilon = \epsilon > 0$.
Choose $a = 0$ and $a \in \lambda A$.
It holds that $a = 0 < \lambda \inf A + \epsilon = \epsilon$.

- Case $\lambda > 0$
Choose $\epsilon_1 = \frac{\epsilon}{\lambda}$. Then $\epsilon_1 > 0$ and there exists $a_1 \in A$, such that $a_1 < \inf A + \epsilon_1 = \inf A + \frac{\epsilon}{\lambda}$
Obtain such a_1.
It holds that $a_1 < \inf A + \frac{\epsilon}{\lambda}$.
It holds that $a = \lambda a_1 < \lambda \inf A + \epsilon$.
We conclude that $\inf(\lambda A) = \lambda \inf A$ $\square$