Linear Algebre 2

Jiaqi Wang

December 14, 2023

1 From Linear Algebra 1

1.1 Field

Definition 1.1.1 – **Field** Sets in which we can compute like in $\mathbb{R}$.

1.2 Vector Space and Subspace

Definition 1.2.1 – **Vector Space** A set V with two operations, addition and scalar multiplication, such that

1. V is an abelian group under addition.
2. $\forall \alpha, \beta \in F, \forall \underline{u}, \underline{v} \in V$, we have
   (a) $\alpha(\underline{u} + \underline{v}) = \alpha\underline{u} + \alpha\underline{v}$.
   (b) $(\alpha + \beta)\underline{u} = \alpha\underline{u} + \beta\underline{u}$.
   (c) $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$.
   (d) $1\underline{u} = \underline{u}$.

Definition 1.2.2 – **Subspace** A subset W of a vector space V is a subspace of V if W is a vector space under the same operations as V.

Definition 1.2.3 – **Bases** A set of vectors $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ is a basis of a vector space V if

1. $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ is linearly independent.
2. $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ spans V.

1.3 Linear Map

Definition 1.3.1 – **Linear Map** A map $\mathcal{A} : V \to W$ (V, W vector spaces) is linear if

1. $\mathcal{A}(\underline{u} + \underline{v}) = \mathcal{A}(\underline{u}) + \mathcal{A}(\underline{v})$.
2. $\mathcal{A}(\lambda\underline{u}) = \lambda\mathcal{A}(\underline{u})$.

Combined we have $\mathcal{A}(\lambda\underline{u} + \mu\underline{v}) = \lambda\mathcal{A}\underline{u} + \mu\mathcal{A}\underline{v}$.

Example 1.3.2
Reflections, rotations, projections, identity map, zero map, etc.

1.4 Multiplication with Matrices

$\mathcal{A}(\underline{v})) = A \cdot \underline{v}$, where A is a matrix.

1.5 Orthogonal projection

Definition 1.5.1 – Orthogonal projection Let V be a vector space with inner product $\langle \cdot, \cdot \rangle$. Let W be a subspace of V. The orthogonal projection of V onto W is the linear map $\mathcal{P}_W : V \to W$ such that

1. $\mathcal{P}_W(\underline{v}) \in W$.
2. $\underline{v} - \mathcal{P}_W(\underline{v}) \in W^{\perp}$.

Theorem 1.5.2 – Addition For $\mathcal{A}, \mathcal{B}$ linear maps, we define $(\mathcal{A} + \mathcal{B})(\underline{v}) = \mathcal{A}(\underline{v}) + \mathcal{B}(\underline{v})$.

Theorem 1.5.3 – Scalar multiplication For $\mathcal{A}$ linear map, we define $(\lambda\mathcal{A})(\underline{v}) = \lambda\mathcal{A}(\underline{v})$.

Theorem 1.5.4 – Composition For $\mathcal{A}, \mathcal{B}$ linear maps, we define $(\mathcal{A} \circ \mathcal{B})(\underline{v}) = \mathcal{A}(\mathcal{B}(\underline{v}))$.

Theorem 1.5.5 – Inverse For $\mathcal{A}$ linear map, we define $\mathcal{A}^{-1}$ such that $\mathcal{A}^{-1} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{A}^{-1} = \mathcal{I}$.

1.5.6 Powers of maps

1. $\mathcal{A}^2 = \mathcal{A}\mathcal{A}$
2. $\mathcal{A}^n = \mathcal{A}\mathcal{A}^{n-1}$
3. $\mathcal{A}^{-n} = (\mathcal{A}^{-1})^n$

1.5.7 Null space and range

Definition 1.5.8 – Null space The null space of a linear map $\mathcal{A} : V \to W$ is the set of vectors $\underline{v} \in V$ such that $\mathcal{A}(\underline{v}) = \underline{0}$.

Definition 1.5.9 – Range The range of a linear map $\mathcal{A} : V \to W$ is the set of vectors $\underline{w} \in W$ such that $\exists \underline{v} \in V$ such that $\mathcal{A}(\underline{v}) = \underline{w}$.

Theorem 1.5.10 – Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is injective if and only if $\mathcal{N}(\mathcal{A}) = \{\underline{0}\}$.

Theorem 1.5.11 – Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is surjective if and only if $\mathcal{R}(\mathcal{A}) = W$.

Theorem 1.5.12 – Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is bijective if and only if $\mathcal{N}(\mathcal{A}) = \{\underline{0}\}$ and $\mathcal{R}(\mathcal{A}) = W$.

1.5.13 Null space / Range for matrix multiplication

Theorem 1.5.14 – Let A be an $m \times n$ matrix. Then $\mathcal{N}(A) = \{\underline{v} \in V \mid A\underline{v} = \underline{0}\}$ and $\mathcal{R}(A) = \{\underline{w} \in V \mid \exists \underline{v} \in V \text{ such that } A\underline{v} = \underline{w}\}$.

1.5.15 Quotient spaces

Definition 1.5.16 – **Quotient space** Let V be a vector space and W a subspace of V. The quotient space V/W is the set of cosets of W in V. I.e. $V/W = \{\underline{v} + W \mid \underline{v} \in V\}$.

Theorem 1.5.17 – **Noether's fundamental theorem on homomorphisms** For any linear map $\mathcal{A} : V \to W$, there exists a linear bijection between its range $\mathcal{R}$ and the quottient space $V/\mathcal{N}$.

1.5.18 Example

Take $P : \mathbb{R}^3 \to \mathbb{R}^3, (x, a, b) \mapsto (0, a, b)$ and $\mathcal{R}(P) =< (0, a, 0), (0, 0, b) >$

Proof. ... $\bar{\mathcal{A}} : v/\mathcal{N}(\mathcal{A}) \to W, \underline{v} \mapsto \mathcal{A}\underline{v}$ Restrict target space of $\bar{\mathcal{A}}$ to $\mathcal{R}(A)$. Homework : show that $\bar{\mathcal{A}}$ is linear and injective. So $\bar{\mathcal{A}}$ is a linear bijection. □

$\bar{\mathcal{A}}^{-1}$:

2 Transition Matrices

We can find matrices for linear maps $\mathbb{K}^n \to \mathbb{K}^m$ by using the standard basis of $\mathbb{K}^n$ and $\mathbb{K}^m$. However, working with abstract vector spaces, we do not have a standard basis. For that reason we look at transition matrices.

2.1 Coordinates

Definition 2.1.1 – **coordinates** Let V be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$. Every vector $\underline{v} \in V$ can be expressed as a linear combination of the basis vectors in exactly one way:

$$\underline{v} = \sum_{i=1}^{n} \lambda_i \underline{a}_i. \tag{1}$$

The coordinates of $\underline{v}$ with respect to α are the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Remark 2.1.2
Clearly, the coordinates depend on the choice of the basis α.

Theorem 2.1.3 – Let V be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$. We will denote the map sending each vector $\underline{v}$ to its coordinates with respect to α by α.
Then α is an invertible linear map from V to $\mathbb{K}^n$.

With this notation, $\alpha(\underline{v})$ is the coordinate vector of the vector $\underline{v} \in V$ with respect to the basis α.

Definition 2.1.4 – Coordinate transformation (map) Let α and β be bases of an n-dimensional vector space V. The map $\beta\alpha^{-1} : \mathbb{K}^n \to \mathbb{K}^n$ is called the coordinate transformation (map) from α to β.

2.2 Basis transition matrix

Definition 2.2.1 – Transition matrix Let α and β be bases of an n-dimensional vector space V. We call the $n \times n$-matrix associated to the linear map $\beta\alpha^{-1}$ the transition matrix from basis α to basis β and denote it by ${}_\beta S_\alpha$.

The following theorem states that multiplication with the matrix ${}_\beta S_\alpha$ translates α- into β-coordinates, and gives a direct description of how the matrix looks, entry-wise.

Theorem 2.2.2 – Let α and β be bases of an n-dimensional vector space V and let ${}_\beta S_\alpha$ be the basis transition matrix, i.e. the matrix of $\beta\alpha^{-1}$. Let $\underline{x} := \alpha(\underline{v})$ be the α-coordinate vector of a vector $\underline{v} \in V$. Then the β-coordinate vector of $\underline{v}$ is equal to the product ${}_\beta S_\alpha \underline{x}$. Furthermore, the columns of matrix ${}_\beta S\alpha$ are the β-coordinate vectors of the α-basis vectors.

Remark 2.2.3

$${}_\alpha S_{\beta\beta} S_\alpha = I, \text{ so } {}_\alpha S_\beta =_\beta S_\alpha^{-1}$$

Theorem 2.2.4 – Let α, β and γ be bases of an n-dimensional vector space V, with respective basis transition matrices ${}_\beta S_\alpha$ and ${}_\gamma S_\beta$. Then the basis transition matrix from α to γ is ${}_\gamma S_\alpha = {}_\gamma S_{\beta\beta} S_\alpha$.

Remark 2.2.5
It is important to distinguish between calculating with vectors (so elements of the vectors space V) and calculating with coordinates (so sequences of elements from $\mathbb{K}^n$).

2.3 Generalizing the map-matrix connection for spaces that aren't $\mathbb{K}^n$

Definition 2.3.1 – Matrix of a linear map Let V and W be vector spaces with bases α and β respectively. Let $\mathscr{A} : V \to W$ be a linear map. We denote the matrix of the linear map $\beta\mathscr{A}\alpha^{-1}$ by ${}_\beta A_\alpha$ and call it the matrix of $\mathscr{A}$ with respect to the bases α and β.

Remark 2.3.2
If $V = W$ and $\alpha = \beta$, then we simplify notation by denoting the corresponding matrix by A_α. We call it the matrix of $\mathscr{A}$ with respect to the basis α.

Remark 2.3.3 (How does the matrix look?)
The columns of ${}_\beta A\alpha$ are

$$(\beta\mathscr{A}\alpha^{-1})(\underline{e}_i) = \beta(\mathscr{A}\underline{a}_i), \qquad i = 1, \ldots, n,$$

meaning the i-th column consists of the β-coordinates of the image $\mathscr{A}\underline{a}_i$ of the i-th basis vector $\underline{a}_i$.

Remark 2.3.4
To find the image of a vector $\underline{v} \in V$, we can:

1. Determine the coordinate vector $\alpha(\underline{v})$ of $\underline{v}$;

2. Multiply $\alpha(\underline{v})$ with the representation matrix ${}_{\beta}A_{\alpha}$, yielding the coordinate vector of $\mathscr{A}\underline{v}$;
3. Translate the coordinate vector of $\mathscr{A}\underline{v}$ back to the corresponding vector in W.

2.4 How do base changes affect the matrix of a linear map?

Theorem 2.4.1 – **Effect of change of basis** Choose in a finite-dimensional space V two bases α and β, and suppose $\mathscr{A} : V \to V$ is linear. Then

$$A_{\beta} = {}_{\beta}S_{\alpha}A_{\alpha\alpha}S_{\beta}.$$

3 Eigenvalues and Eigenvectors

3.1 Diagonalization of matrices

Definition 3.1.1 – A square matrix A has *diagonal form* if all elements a_{ij} with $i \neq j$ are zero.

Theorem 3.1.2 – Let $\mathcal{A} : V \to V$ be a linear map and let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis of V. The matrix A_{α} has a diagonal form

$$A_{\alpha} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}$$

if and only if $\mathcal{A}\underline{a}_1 = \lambda_i \underline{a}_i$ for all $i \in \{1, \ldots, n\}$.

3.2 Eigenvalues and eigenvectors

Definition 3.2.1 – **Eigenvector and eigenvalue** Let $\mathcal{A} : V \to V$ be a linear map from a $\mathbb{K}$-vector space V to itself. A vector $\underline{v} \neq \underline{0} \in V$ is called an *eigenvector* of $\mathcal{A}$ with *eigenvalue* λ if $\mathcal{A}\underline{v} = \lambda\underline{v}$. We denote the set of all eigenvalues of $\mathcal{A}$ by $\operatorname{spec}(\mathcal{A})$ and call it the *spectrum* of $\mathcal{A}$.

Theorem 3.2.2 – Let $\mathcal{A} : V \to V$ be a linear map with representation matrix A_{α} for a basis α. Then A_{α} is in diagonal form if and only if α is a basis of eigenvectors of $\mathcal{A}$. In this case, the diagonal entries of A_{α} are the eigenvalues of $\mathcal{A}$.

Definition 3.2.3 – **Eigenspace** Let $\mathcal{A} : V \to V$ be a linear map. For any scalar $\lambda \in \mathbb{K}$, we denote

$$E_{\lambda} := \mathcal{N}(\mathcal{A} - \lambda\mathcal{I})$$

Since *null spaces* are subspaces, E_{λ} is a subspace, called *the eigenspace* of $\mathcal{A}$ for λ.

Remark 3.2.4
Eigenspaces *indeed are spaces of eigenvectors for a given eigenvalue*: E_{λ} is the null space of the linear map $\mathcal{A} - \lambda\mathcal{I}$.

- $\underline{v} \in E_\lambda \iff (\mathcal{A} - \lambda\mathcal{I})\underline{v} = \underline{0}$.
- $\underline{v} \in E_\lambda \iff \mathcal{A}\underline{v} - \lambda\underline{v} = \underline{0}$.

So any vector $\underline{v}$ lies in E_λ if and only if $(\mathcal{A} - \lambda\mathcal{I})\underline{v} = \underline{0}$, which is equivalent to $\mathcal{A}\underline{v} - \lambda\underline{v} = \underline{0}$.

Remark 3.2.5 (null space as eigenspace)
We can also write the null space of $\mathcal{A}$ as an eigenspace: E_0 consists of vectors that are mapped to 0 times itself, so on $\underline{0}$.

3.3 Computing eigenvalues and eigenspaces

Theorem 3.3.1 – λ is an eigenvalue if and only if $det(A - \lambda\mathcal{I}) = 0$. Let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for V, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \ldots & \ldots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. this basis. Then the eigenvectors for eigenvalue λ, in α-coordinates, are the non-zero solutions of the system

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \ldots & \ldots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

3.4 Characteristic polynomial

Definition 3.4.1 – **Characteristic polynomial** Let $\mathcal{A} : V \to V$ be a linear map and let A_α be the matrix of $\mathcal{A}$ w.r.t. a basis α. We call the equation $\det(A_\alpha - \lambda\mathcal{I}) = 0$ the *characteristic equation* of A_α, and the left-hand side off this equation, $\det(A_\alpha - \lambda\mathcal{I})$, the *characteristic polynomial* of A_α.

We also call them characteristic equation/polynomial of $\mathcal{A}$, and denote the characteristic polynomial by $\chi_{\mathcal{A}}$.

Theorem 3.4.2 – Let $\mathcal{A} : V \to V$ be a linear map, α and β be two bases for V and let A_α/A_β be the matrix of $\mathcal{A}$ w.r.t. a basis α/β. Then $\det(A_\alpha - \lambda\mathcal{I}) = \det(A_\beta - \lambda\mathcal{I})$.

Remark 3.4.3
The characteristic polynomial is independent of the choice of basis.

Theorem 3.4.4 – Let $\mathcal{A} : V \to V$ be a linear map on a vector space V of dimension n, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. a basis α. Then the characteristic polynomial $\chi_{\mathcal{A}}$ is a polynomial of degree (exactly) n, and of the following shape:

$$\chi_{\mathcal{A}}(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \dots + a_{nn})\lambda^{n-1} + \dots + c_1\lambda + c_0$$

for some coefficients $c_0, c_1, \dots \in \mathbb{K}$.

Definition 3.4.5 – **Trace** The sum of the diagonal elements of a square matrix A is called the *trace* of the matrix A. We denote it by $\operatorname{tr}(A)$.

Theorem 3.4.6 – Let $\mathcal{A} : V \to V$ be a linear map with $\dim(V) < \infty$. For every basis α, the matrix A_α

1. has the same trace, which we therefore also call the *trace* of $\mathcal{A}$, and denote it by $\operatorname{tr}(\mathcal{A})$.
2. has the same determinant, which we therefore also call the *determinant* of $\mathcal{A}$, and denote it by $\det(\mathcal{A})$. We have the identity $\det(\mathcal{A}) = c_0$, where c_0 is the constant coefficient of the characteristic equation.

Theorem 3.4.7 – Let A be a square matrix with entries in $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{C}\}$, with characteristic polynomial $\chi_A(\lambda)$. Then the

- trace of the matrix is the sum of the roots of χ_A; and the
- determinant of the matrix is the product of the roots of χ_A.

3.5 Linear independence of eigenvectors

Theorem 3.5.1 – Let $\mathcal{A} : V \to V$ be a linear map and let $\underline{v}_1, \dots, \underline{v}_n$ be eigenvectors of $\mathcal{A}$ for *mutually different* eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\underline{v}_1, \dots, \underline{v}_n$ are linearly independent.

4 Invariant subspaces

4.1 Invariant subspace

Definition 4.1.1 – **Invariant subspace** Let W be a subspace of V. W is called *invariant under linear map* $\mathcal{A} : V \to V$ if $\mathcal{A}\underline{w} \in W$ for all $\underline{w} \in W$.

Example 4.1.2 (Null space and range)

- The null space $\mathcal{N}$ pf a linear map $\mathcal{A}$ is always invariant: if $\underline{x} \in \mathcal{N}$, then $\mathcal{A}\underline{x} = \underline{0}$, and $\underline{0} \in \mathcal{N}$.
- The range $\mathcal{R}$ of a linear map $\mathcal{A}$ is invariant if and only if $\mathcal{A}$ is surjective: if $\underline{y} \in \mathcal{R}$, then $\underline{y} = \mathcal{A}\underline{x}$ for some $\underline{x} \in V$, and $\mathcal{A}\underline{x} \in \mathcal{R}$.

Example 4.1.3 (Counterexample, rotation in two-dimension space)
Let $\mathcal{A}$ be a $90°$ rotation map. Then let $W =< e_1 >$. W is not invariant

Theorem 4.1.4 – Let $\mathcal{A} : V \to V$ be linear and let $W =< \underline{a}_1, \dots, \underline{a}_n >$. W is invariant under $\mathcal{A}$ if and only if $\mathcal{A}\underline{a}_i \in W$ for $i = 1, \dots, n$.

4.2 Restriction unto an invariant subspace

Definition 4.2.1 – **Restriction unto an invariant subspace** If W is invariant under $\mathcal{A}$, then all image vectors $\mathcal{A}\underline{w}$ with $\underline{w} \in W$ are again in W. So if we restrict $\mathcal{A}$ to W, we obtain a well-defined linear map $W \to W$, the *restriction of the map* $\mathcal{A}$ unto W, which we denote by $\mathcal{A}|_W$.

Invariant spaces give us a simpler matrix shape, because the matrix contains a block of the restriction:

Theorem 4.2.2 – Suppose $\alpha = \{\underline{a}_1, \dots, \underline{a}_2\}$ is a basis for V such that $W =< \underline{a}_1, \dots, \underline{a}_m >$ is invariant under $\mathcal{A}$. Then the matrix A_α has the following form:

$$\begin{pmatrix} & & & * & \dots & * \\ & M_1 & & \vdots & & \vdots \\ 0 & \dots & 0 & \vdots & & \vdots \\ & \vdots\vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & * & \dots & * \end{pmatrix},$$

The $m \times m$-matrix M_1 is the matrix of the restriction $\mathcal{A}_{|W} : W \to W$ w.r.t. the basis $\{\underline{a}_1, \dots, \underline{a}_m\}$.

Example 4.2.3 (Proving invariance and analysing a map without even knowing its full map description)
Consider in $\mathbb{R}^4$ the (independent) vectors

$$\underline{a} = (1, -1, 1, -1) \text{ and } \underline{b} = (1, 1, 1, 1).$$

Say we have a linear map $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^4$ of which we only know that

$$\mathcal{A}\underline{a} = (4, -6, 4, -6) \text{ and } \mathcal{A}\underline{b} = (4, 6, 4, 6).$$

Even without knowing the full description of $\mathcal{A}$, we will now show that $W =< \underline{a}, \underline{b} >$ is invariant and determine a matrix of the restriction unto W - $\mathcal{A}_{|W} : W \to W$.

To show the invariance of $< \underline{a}, \underline{b} >$, we must verify that $\mathcal{A}\underline{a}$ and $\mathcal{A}\underline{b}$ are linear combinations of $\underline{a}$ and $\underline{b}$. We do this by simultaneously solving the systems of equations with columns $\underline{a}, \underline{b}, \mathcal{A}\underline{a}$ and $\mathcal{A}\underline{b}$:

$$\left(\begin{array}{cc|cc} 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & -6 \\ 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & -6 \end{array}\right)$$

After row reduction and deleting zero rows, the system reduces to

$$\left(\begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & -1 & 5 \end{array}\right),$$

which tells us that $\mathcal{A}\underline{a} = 5\underline{a} - \underline{b}$ and $\mathcal{A}\underline{b} = -\underline{a} + 5\underline{b}$, so W is invariant under $\mathcal{A}$. This also tells us how the matrix of the restriction $A_{|W}W \to W$ w.r.t. the basis $\{\underline{a}, \underline{b}\}$ looks like:

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}.$$

Using the restriction matrix, we can now even *determine some eigenvectors without knowing the full map*. The characteristic polynomial of the restriction is $\chi_{A_{|W}}(\lambda) = (5-\lambda)^2 - 1 = 25 - 10\lambda + \lambda^2 - 1 = (\lambda - 4)(\lambda - 6)$. We find that the matrix has eigenvalues 4 and 6. In coordinates, we compute the respective $E_4 =< (1,1) >$ and $E_6 =< (1,-1 >$. In this basis, the restriction map is simply the diagonal map with the eigenvalues on the diagonal:

$$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

We transform the coordinate vectors back into elements of $\mathbb{R}^4 : \underline{a} + \underline{b} = (2,0,2,0)$ and $\underline{a} - \underline{b} = (0,2,0,2)$. So the eigenvector basis of W is $\{(2,0,2,0),(0,2,0,2)\}$.

We now can simplify the representation of the full map: if we pick any basis α of $\mathbb{R}^4$ such that the first two basis vectors are the eigenvectors $(2,0,2,0)$ and $(0,2,0,2)$, then the full matrix has the shape

$$A_\alpha = \begin{pmatrix} & * & \dots & * \\ \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} & \vdots & & \vdots \\ 0 \dots 0 & \vdots & & \vdots \\ \vdots\vdots & \vdots & & \vdots \\ 0 \dots 0 & * & \dots & * \end{pmatrix}$$

Remark 4.2.4
The characteristic polynomial of a restriction always divides the characteristic polynomial of the larger map.

Theorem 4.2.5 – If W is an invariant subspace for the linear map $\mathcal{A}V \to V$, then $\chi_{A_{|W}}$, the characteristic polynomial of $\mathcal{A}$'s restriction unto W, $\mathcal{A}_{|W} : W \to W$, is a factor of $\chi_{\mathcal{A}}$, the characteristic polynomial of the map $\mathcal{A} : V \to V$.

Lemma 4.2.6 – Let A be a $p \times p$-matrix, and let B be a $q \times q$-matrix. Then

$$\det \begin{pmatrix} A & * \\ O & B \end{pmatrix} = \det(A) \cdot \det(B),$$

where $*$ stands for na arbitrary $(p \times q)$-matrix and O for the $q \times p$-zero matrix.

4.3 Nice results for combinations of invariant subspaces

Theorem 4.3.1 – Let $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$ be a basis for V such that $W_1 =< \underline{a}_1, \dots, \underline{a}_m >$ and $W_2 =< \underline{a}_{m+1}, \dots, \underline{a}_n$ are invariant under $\mathcal{A} : V \to V$. Then the matrix A_α has the following form:

$$A_\alpha = \begin{pmatrix} \begin{matrix} & \\ M_1 & \\ & \end{matrix} & \begin{matrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix} \\ \begin{matrix} 0\dots 0 \\ \vdots\vdots \\ 0\dots 0 \end{matrix} & \begin{matrix} & \\ M_2 & \\ & \end{matrix} \end{pmatrix}$$

Here M_1 and M_2 are the $m \times m$ and $(n-m) \times (n-m)$ matrices of the two restrictions $\mathcal{A}_{|W_1} : W_1 \to W_1$ and $\mathcal{A}_{|W_2} : W_2 \to W_2$.
In addition we have that

$$\det(A_\alpha) = \det(M_1)\det(M_2),$$

and that the characteristic polynomial of $\mathcal{A}$ is the product of the characteristic polynomials of the two restrictions:

$$\chi_{\mathcal{A}} = \chi_{A_{|W_1}} \chi_{A_{|W_2}}.$$

Remark 4.3.2
We remark that this result can be generalised further such that it holds for an arbitrary number of invariant subspaces: if V can be broken down into invariant subspaces $W_1, \dots, W_p$, we can pick a basis α whose i-th section is a basis for W_i. Let $A_i : W_i \to W_i$ denote the restriction of $\mathcal{A}$ unto the subspace W_i.

Then the matrix A_α has the form.

$$A_\alpha = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_p \end{pmatrix},$$

where M_i is the matrix of the respective restriction $\mathcal{A}_i$ w.r.t. the respective basis.

5 Orthogonal and symmetric maps

5.1 Orthogonal maps

Definition 5.1.1 – **Orthogonal map** Let v be a real inner product space. A linear map $\mathcal{A} : V \to V$ is called *orthogonal* if

$$\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$$

for all vectors $\underline{x} \in V$. In other words, a linear map $\mathcal{A} : V \to V$ is orthogonal if the *length is invariant* under $\mathcal{A}$.

Theorem 5.1.2 – **Polarization formula** In a real inner product space V, we always have

$$(\underline{x}, \underline{y}) = \frac{1}{2}\left((\underline{x}+\underline{y}, \underline{x}+\underline{y}) - (\underline{x}, \underline{x}) - (\underline{y}, \underline{y})\right)$$

As a consequence, we can express *inner products between vectors in terms of vector lengths*:

$$(\underline{x}, \underline{y}) = \frac{1}{2}\left(\|\underline{x}+\underline{y}\| - \|x\| - \|y\|\right).$$

Theorem 5.1.3 – Let V ba a finite real inner product space, and let $\mathcal{A} : V \to V$ be linear. Then the following are equivalent:

1. $\mathcal{A}$ is orthogonal.
2. $\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$ for all $\underline{x} \in V$.
3. $(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = (\underline{x}, \underline{y})$ for all $\underline{x}, \underline{y} \in V$.
4. For every orthonormal system $\underline{a}_1, \dots, \underline{a}_n$ in V, the system $\mathcal{A}\underline{a}_1, \dots, \mathcal{A}\underline{a}_n$ is again orthonormal.
5. For every orthonormal basis α of V, the basis $\mathcal{A}\alpha$ is again orthonormal.

Theorem 5.1.4 – Let V be a finite real inner product space, and let $\mathcal{A} : V \to V$ and $\mathcal{B} : v \to V$ be orthogonal linear maps.

1. The composition $\mathcal{AB} : V \to V$ is orthogonal.
2. $\mathcal{A}$ is invertible and $\mathcal{A}^{-1}$ is orthogonal.

Remark 5.1.5
As a consequence, powers of orthogonal maps are orthogonal. However, in infinite dimensional spaces, there are exist orthogonal maps that are not invertible.

5.2 Orthogonal matrices

Corollary 5.2.1 – We now consider $\mathbb{R}^n$ with the standard inner product. A linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if the matrix $\mathcal{A}e_1, \dots, \mathcal{A}e_n$ is an orthonormal system.

Definition 5.2.2 – **Orthogonal matrix** A real $n \times n$-matrix A is called *orthogonal* if the columns of A form an orthonormal system in $\mathbb{R}^n$.

Theorem 5.2.3 – Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with representation matrix A. The following statements are equivalent:

1. $\mathcal{A}$ is orthogonal.
2. A is orthogonal.
3. $A^\top A = \mathcal{I}_n$. In other words, the transpose the inverse.
4. Thr rows of A form an orthonormal system in $\mathbb{R}^n$.

Lemma 5.2.4 – Let V be an n-dimensional real inner product space, with its inner product denoted as $(\cdot, \cdot)_V$, and α an orthonormal basis of V. Let $\mathcal{A} : V \to V$ be an orthogonal map. We denote $\|\cdot\|_{\text{st}}$ the standard length in $\mathbb{R}^n$ and by $\|\cdot\|_V$ the length implied by V's inner product.

1. $\|\alpha\mathcal{A}\underline{v}\|_{\text{st}} = \|\underline{v}\|_V$
2. $\left\|\mathcal{A}\alpha^{-1}\underline{x}\right\| = \|\underline{x}\|_{\text{st}}$
3. $\alpha\mathcal{A}\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal.
4. If $\mathcal{B} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, the so is $\alpha^{-1}\mathcal{B}\alpha : V \to V$.

Theorem 5.2.5 – If α, β are two *orthonormal* bases of in a real inner product space, then the transition matrix $\beta S\alpha$ is orthogonal.

Theorem 5.2.6 – If α and β are two orthonormal bases in a real inner product space, then

$$\alpha S\beta = \beta S\alpha^{-1} = \beta S\alpha^{\top}$$

Theorem 5.2.7 – Let α be an orthonormal basis for a finite-dimensional real inner product space V, and let $\mathcal{A} : V \to V$ be a linear map and A_α the matrix of $\mathcal{A}$ (with respect to basis α). Then the map $\mathcal{A}$ is orthogonal if and only if its matrix A_α is orthogonal.

5.3 Symmetric maps