Well done!

# 2MBA60 Analysis 2, Group 4-4

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## 15.12.1

Let $A : V \to W$ be a linear map from a finite-dimensional normed vector space $(V, \| \cdot \|_V)$ to a normed vector space $(W, \| \cdot \|_W)$.
Show that $A$ is differentiable on $V$.
It suffices to show that there exists a bounded linear map $L_a : V \to W$ such that, if we define $\mathrm{Err}_a(x) := A(x) - A(a) - L_a(x-a)$, it holds that $\lim_{x\to a} \frac{\|\mathrm{Err}_a(x)\|_W}{\|x-a\|_V} = 0$.
Choose $L_a := A$, then $L_a$ is a linear map which is bounded since $L_a : V \to W$ and $V$ is a finite dimensional normed vector space.
From this it follows that:

$$\begin{aligned}\lim_{x\to a} \frac{\| \mathrm{Err}_a(x) \|_W}{\| x-a \|_V} &= \lim_{x\to a} \frac{\| f(x) - f(a) - L_a(x-a) \|_W}{\| x-a \|_V} \\ &= \lim_{x\to a} \frac{\| f(x) - f(a) - (f(x) - f(a)) \|_W}{\| x-a \|_V} \\ &= \lim_{x\to a} \frac{\| 0 \|_W}{\| x-a \|_V} = 0\end{aligned}$$

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## 15.12.3

The function $\ln : (0, \infty) \to \mathbb{R}$ is the unique, differentiable function such that $\ln(1) = 0$ and $\ln'(x) = \frac{1}{x}$. Show that for all $x \in (-1, \infty)$, it holds that

$$\ln(x+1) \leq x$$

with equality if and only if $x = 0$.

Define $f : (-1, \infty) \to \mathbb{R} : f(x) = \ln(1+x) - x$.

### We begin by showing equality:

$x = 0 \implies \ln(1+0) = \ln(1) = 0 = x$
Assume $\exists p \neq 0 : f(p) = 0$, then by Rolle's theorem there has to be a $k \in (0,p) : f'(k) = 0$.
Since $\forall s \in \mathbb{R} : \frac{1}{s} \neq 0$ we know there is no such $k$.
Hence the only solution to $f(x) = 0$ is 0 and thus

$$f(x) = 0 \iff \ln(1+x) - x \iff \ln(1+x) = \iff x = 0.$$

### We follow by showing inequality:

By the Sum Rule, since $\ln(1+x)$ and $-x$ are differentiable, we know $f$ is differentiable and hence continuous.
Let $p \in (-1, \infty) \setminus \{0\}$.

We have two cases: $p > 0 \vee p < 0$.
Case 1: $p > 0$
By the Mean Value Theorem, since $f$ is continuous on it's domain and $[0, p] \subset (-1, \infty)$, we know there exists a $c \in (0, p)$ such that $f'(c) = \frac{f(p)-f(0)}{p-0} = \frac{f(p)}{p}$.
Since $c > 0$ we know $\frac{1}{1+c} < 1$ and thus

$$\begin{aligned}
& f'(c) = \frac{1}{c+1} - 1 < 0 \\
\iff & \frac{1}{c+1} - 1 = \frac{\ln(1+p) - p}{p} < 0 \\
\iff & \frac{\ln(1+p)}{p} - 1 < 0 \\
\iff & \frac{\ln(1+p)}{p} < 1 \\
\iff & \ln(1+p) < p
\end{aligned}$$

Case 2: $M < 0$
By the Mean Value Theorem, since $f$ is continuous on it's domain and $[p, 0] \subset (-1, \infty)$, we know there exists a $c \in (p, 0)$ such that $f'(c) = \frac{f(0)-f(p)}{0-p} = \frac{f(p)}{p}$.
Since $-1 < c < 0$ we know $\frac{1}{1+c} > 1$

$$\begin{aligned}
& f'(c) = \frac{1}{c+1} - 1 > 0 \\
\iff & \frac{1}{c+1} - 1 = \frac{\ln(1+p) - p}{p} > 0 \\
\iff & \frac{\ln(1+p)}{p} - 1 > 0 \\
\iff & \frac{\ln(1+p)}{p} > 1 \\
& \text{Since } p < 0 \\
\iff & \ln(1+p) < p
\end{aligned}$$

Since we checked every case, we have proven

$$\forall x \in (-1, \infty) : f(x) \leq 0 \iff \ln(1+x) - 1 \leq 0 \iff \ln(1+x) \leq x$$

with equality if and only if $x = 0$.

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## 15.12.4

Proposition 15.1.5. Let $\Omega \subset \mathbb{R}$ be open and consider a function $f : \Omega \to W$ interpreted as a function from the subset $\Omega$ of the normed vector space $(\mathbb{R}, |\cdot|)$ to a normed vector space $(W, \|\cdot\|_W)$. Let $a \in \Omega$. Then $f$ is differentiable in $a$ if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Then, if this limit exists we denote it by $f'(a)$, and then for all $h \in \mathbb{R}$,

$$f'(a) \cdot h = (Df)_a(h).$$

Prove Proposition 15.1.5. You may assume that $W$ is finite-dimensional.
We show both directions.

- We need to show that $f$ is differentiable in $a$ $\implies$

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

Since $f$ is differentiable in $a$, it holds that
there exists a bounded linear map $L_a : \mathbb{R} \to W$ such that, if we define the error function $\mathrm{Err}_a :$ $\Omega \to W$ by

$$\mathrm{Err}_a := f(x) - f(a) - L_a(x-a)$$

it holds that

$$\lim_{x\to a} \frac{\|\mathrm{Err}_a(x)\|_W}{|x-a|} = 0.$$

Obtain such a bounded linear map $L_a : \mathbb{R} \to W$.
It holds that

$$\lim_{x\to a} \frac{\|f(x)-f(a)-L_a(x-a)\|_W}{|x-a|} = 0.$$

Since this limit exists, it holds that $\lim_{x\to a} \|f(x)-f(a)-L_a(x-a)\|_W$ exists and $\lim_{x\to a} |x-a|$ exists ($\lim_{x\to a} |x-a| = 0$) exist and $\forall a \in \mathbb{R} : |x-a| > \|f(x)-f(a)-L_a(x-a)\|_W$ (Since the limit of their quotient is 0).
Since $\lim_{x\to a} \|f(x)-f(a)-L_a(x-a)\|_W$ exists and $L_a$ is bounded, it holds that $\lim_{x\to a} \|f(x)-f(a)\|$ also exists.
Since $\lim_{x\to a} \|f(x)-f(a)\|_W$ $f$ and $\lim_{x\to a} |x-a|$ both exist and $\lim_{x\to a} |x-a| > \lim_{x\to a} \|f(x)-f(a)-L_a(x-a)\|_W$ with $L_a$ bounded, it holds that

Use

$(-1)$

$$\lim_{x\to a} \frac{\|f(x)-f(a)\|_W}{|x-a|} \text{ exists.}$$

$\left\| \frac{f(x)-f(a)}{x-a} - f'(a) \right\|_W$

as below

Because of this, we conclude that

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ also exists.}$$

- We need to show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists $\implies$ $f$ is differentiable in $a$.
Since it holds that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, there exists some value $d \in \mathbb{R}, d = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.
Obtain such a $d$.
Choose $L_a : \Omega \to \mathbb{R}, L_a(x) := (x) \cdot d$, then $L_a$ is a bounded linear map since $\Omega$ is a finite dimensional normed vector space
for all $x, y, \lambda \in \mathbb{R}, L_a(x+y) = x \cdot d + y \cdot d = L_a(x) + L_a(y)$
$\lambda \cdot L_a(x) = \lambda \cdot x \cdot d = L_a(\lambda \cdot x)$
$L_a(0) = 0.$
Then:

$$\begin{aligned}
\lim_{x\to a} \frac{\|f(x)-f(a)-L_a(x-a)\|_W}{|x-a|} &= \lim_{x\to a} \frac{\|f(x)-f(a)-(x-a)\cdot d\|_W}{|x-a|} \\
&= \lim_{x\to a} \frac{\left\|f(x)-f(a)-(x-a)\cdot \frac{f(x)-f(a)}{x-a}\right\|_W}{|x-a|} \\
&= \lim_{x\to a} \frac{\|f(x)-f(a)-f(x)+f(a)\|_W}{|x-a|} = \lim_{x\to a} \frac{\|0\|_W}{|x-a|} = 0
\end{aligned}$$

$(-1)$

Limit inside a limit

We conclude that $f$ is differentiable in $a$.

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## 15.12.5

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two two-dimensional vector spaces with bases $v_1, v_2$ and $w_1, w_2$ respectively. Assume that a function $f : V \to W$ is differentiable in 0 with

$$(Df)_0(v_1 + v_2) = w_1$$

and

$$(Df)_0(v_1 - 2v_2) = w_1 - w_2.$$

Give the matrix representation of the linear map $(Df)_0 : V \to W$ with respect to the bases $v_1, v_2$ and $w_1, w_2$.
Define the coordinate map for the basis $\{w_1, w_2\}$ as $\Psi : W \to \mathbb{R}^2 : \{\Psi(w_1) = (1, 0), \Psi(w_2) = (0, 1)\}$.
Then $([Df]_0)_{ij}) = (\Psi((Df)_0(v_j))_i$.
Since $(Df)_0 \in \mathrm{Lin}(V, W)$ it holds that

$$(Df)_0(v_1 + v_2) = (Df)_0(v_1) + (Df)_0(v_2) = w_1$$

and

$$(Df)_0(v_1 - 2v_2) = (Df)_0(v_1) - 2(Df)_0(v_2) = w_1 - w_2.$$

So

$$(Df)_0(v_2) = \frac{1}{3} w_2$$

and

$$\begin{aligned} 2(Df)_0(v_1 + v_2) + (Df)_0(v_1 - 2v_2) &= 2w_1 + w_1 - w_2 \\ \iff 2(Df)_0(3v_1) &= 3w_1 \\ \iff 2(Df)_0(v_1) &= w_1 \end{aligned}$$

Thus we get

$$[Df]_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$-\frac{1}{3}$

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## 16.4.4

i. Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) := (\cos(t), \sin(t), \arctan(t)).$$

Show that $f$ is differentiable and give an expression for the function $f' : \mathbb{R} \to \mathbb{R}^3$ and for the derivative $(Df) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, \mathbb{R}^3)$.
The component functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ and $f_3 : \mathbb{R} \to \mathbb{R}$ are given by

$$\begin{aligned} f_1(t) &= \cos(t) \\ f_2(t) &= \sin(t) \\ f_3(t) &= \arctan(t) \end{aligned}$$

Since these component functions are differentiable standard functions, we find by a proposition in the lecture notes that $f$ is differentiable as well and

$$f'(t) = (f_1'(t), f_2'(t), f_3'(t)) = (-\sin(t), \cos(t), \frac{1}{1+t^2})$$

From a proposition in the lecture notes it follows that, for all $h \in \mathbb{R}$, $(Df) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, \mathbb{R}^3)$ is given by

$$t \mapsto (h \mapsto h \cdot (-\sin(t), \cos(t), \frac{1}{1+t^2}))$$

ii. Let $w_1$ and $w_2$ be two vectors in a finite-dimensional normed vector space $(W, \|\cdot\|_W)$. Consider the function $g : \mathbb{R} \to W$ given by

$$g(t) = \cosh(t)w_1 + \sinh(t)w_2.$$

Show that $g$ is differentiable and give an expression for the function $g' : \mathbb{R} \to W$ and for the derivative $(Dg) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, W)$.

We define

$$\begin{aligned} f(t) &= \cosh(t) \cdot w_1 \\ h(t) &= \sinh(t) \cdot w_2 \end{aligned}$$

It holds that $\cosh(t)$ and $\sinh(t)$ are both differentiable in a point $a \in \mathbb{R}$ with derivative $\sinh(t)$ and $\cosh(t)$ respectively.
Since $w_1, w_2$ are constants the derivatives of $f(t) = \cosh(t)w_1$ and $h(t) = \sinh(t)w_2$ are $f'(t) = \sinh(t)w_1$ and $h'(t) = \cosh(t)w_2$.
Since $f(t)$ and $h(t)$ are both differentiable on $\mathbb{R}$ and $g = f + h$, by the Sum Rule, the function $g : \mathbb{R} \to W$ is also differentiable on $\mathbb{R}$ with derivative

$$g'(t) = f'(t) + h'(t) = \sinh(t)w_1 + \cosh(t)w_2$$

From a proposition in the lecture notes it follows that, for all $h \in \mathbb{R}$, $(Dg) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, W)$ is given by

$$t \mapsto (h \mapsto h \cdot (\sinh(t)w_1 + \cosh(t)w_2))$$