2IT80 Discrete Structures

2023-24 Q2

Lecture 6: Counting II

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Recap

Basic counting

Summary

Map objects you want to count to (combinations of) mathematical objects (tuples/functions/permutations etc.) and use known results.

Choosing, order matters:

may choose objects more then once $\rightarrow$ functions

may choose each object only once $\rightarrow$ injective functions

Choosing, order irrelevant:

may choose objects only once $\rightarrow$ binomial coefficients

may choose objects multiple times $\rightarrow$ balls into bins

Properties of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Theorem (Binomial theorem):
For every integer $n \geq 0$ it holds that

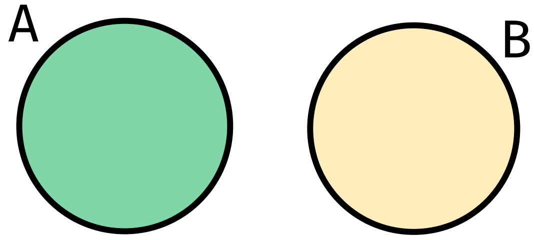
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} .$$

Counting with non-disjoint sets

Non-disjoint counting

We sometimes separate sets into subsets and count them independently:

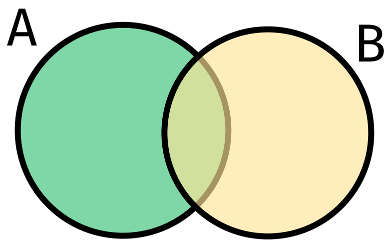
$X = A \cup B$, and $A \cap B = \emptyset$, then $|X| = |A| + |B|$



What if $X = A \cup B$ but $A \cap B \neq \emptyset$?

Can we still compute $|X|$ from $|A|$ and $|B|$?

Generally no.



But $|X| = |A| + |B| - |A \cap B|$

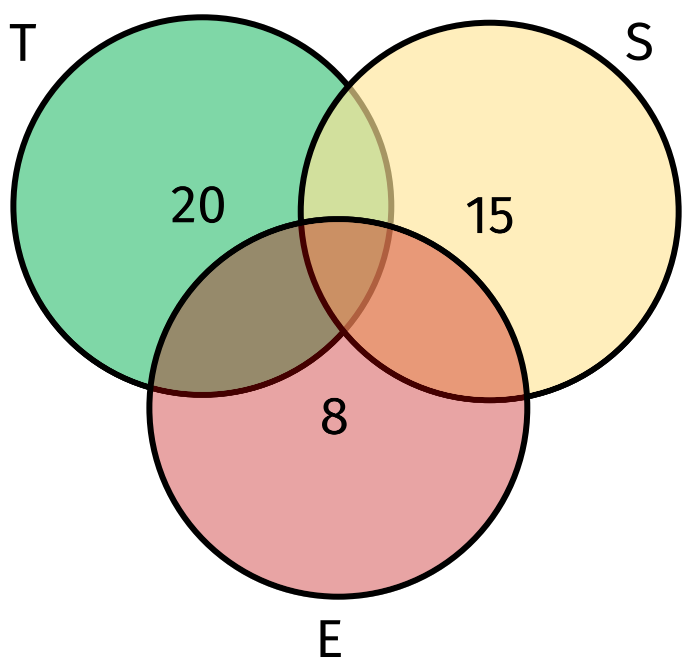
So yes, if we know also $|A \cap B|$

The inclusion-exclusion principle

Too much – too little – again too much...

Inclusion-exclusion

Example. In the City E. there are three clubs. The tennis club has 20 members, the stamp collectors have 15 and the Egyptology club has 8. Among the Egyptologists there are two of the tennis players and three of the stamp collectors. Six people play both tennis and collect stamps. One eager person participates in all three clubs. How many people are engaged in the club life of E.?

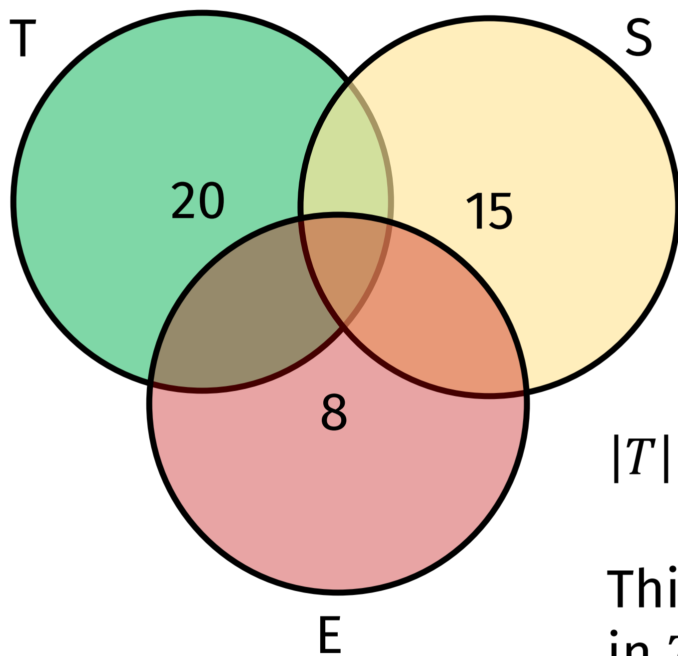


$$|T| = 20$$
$$|S| = 15$$
$$|E| = 8$$

$$|T \cup E \cup S| = ?$$

Inclusion-exclusion

Example. In the City E. there are three clubs. The tennis club has 20 members, the stamp collectors have 15 and the Egyptology club has 8. Among the Egyptologists there are two of the tennis players and three of the stamp collectors. Six people play both tennis and collect stamps. One eager person participates in all three clubs. How many people are engaged in the club life of E.?



$$|T| = 20$$
$$|S| = 15$$
$$|E| = 8$$

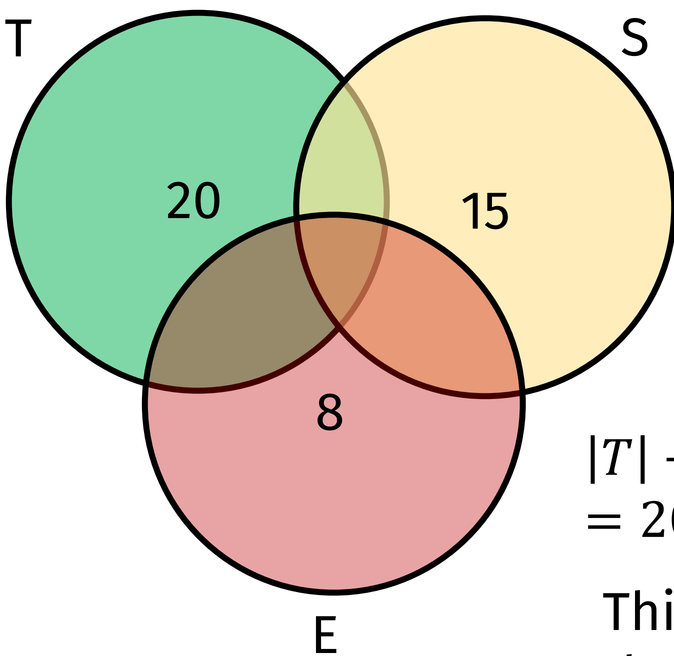
$$|T \cup E \cup S| = ?$$

$$|T| + |S| + |E| = 20 + 15 + 8 = 43$$

This is too much, we count elements in $T \cap S, T \cap E, S \cap E$ twice, elements in $T \cap \mathrm{S} \cap \mathrm{E}$ even three times!

Inclusion-exclusion

Example. In the City E. there are three clubs. The tennis club has 20 members, the stamp collectors have 15 and the Egyptology club has 8. Among the Egyptologists there are two of the tennis players and three of the stamp collectors. Six people play both tennis and collect stamps. One eager person participates in all three clubs. How many people are engaged in the club life of E.?



$$|T| = 20 \qquad |T \cap S| = 6$$
$$|S| = 15 \qquad |T \cap E| = 2$$
$$|E| = 8 \qquad |S \cap E| = 3$$

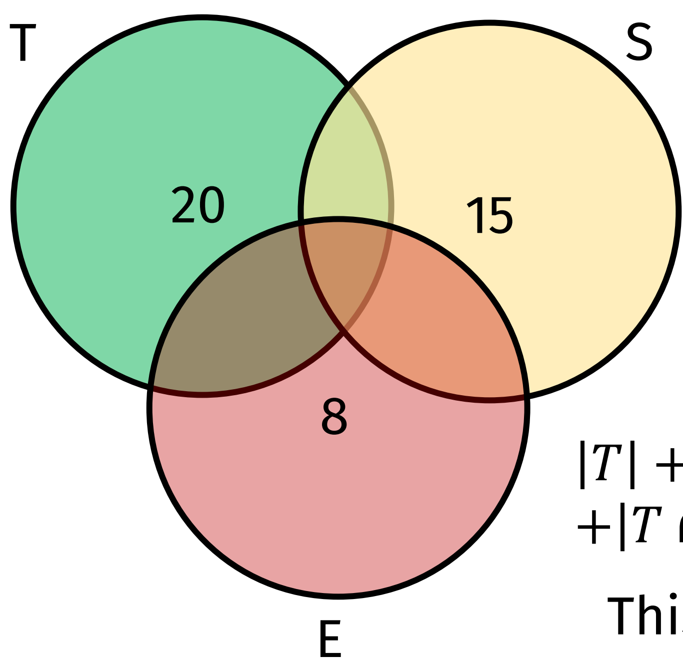
$$|T \cup E \cup S| = ?$$

$$|T| + |S| + |E| - |T \cap S| - |T \cap E| - |S \cap \mathrm{E}|$$
$$= 20 + 15 + 8 - 6 - 2 - 3 = 32$$

This is too little, now we do not count the elements in $T \cap \mathrm{S} \cap \mathrm{E}$.

Inclusion-exclusion

Example. In the City E. there are three clubs. The tennis club has 20 members, the stamp collectors have 15 and the Egyptology club has 8. Among the Egyptologists there are two of the tennis players and three of the stamp collectors. Six people play both tennis and collect stamps. One eager person participates in all three clubs. How many people are engaged in the club life of E.?



$$|T| = 20 \quad |S| = 15 \quad |E| = 8$$

$$|T \cap S| = 6 \quad |T \cap E| = 2 \quad |S \cap E| = 3$$

$$|T \cap S \cap E| = 1$$

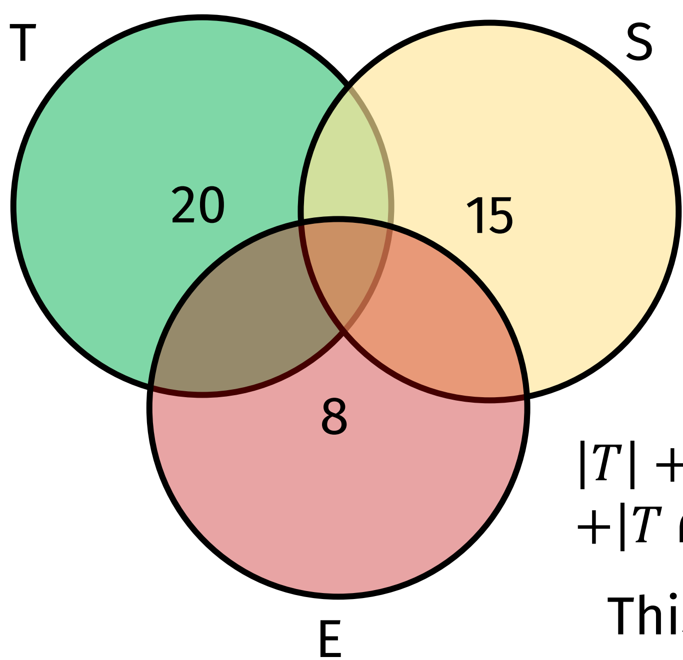
$$|T \cup E \cup S| = ?$$

$$|T| + |S| + |E| - |T \cap S| - |T \cap E| - |S \cap \mathrm{E}| + |T \cap S \cap E| = 20 + 15 + 8 - 6 - 2 - 3 + 1 = 33$$

This is just right, each element in $T \cup S \cup E$ is counted exactly once.

Inclusion-exclusion

Example. In the City E. there are three clubs. The tennis club has 20 members, the stamp collectors have 15 and the Egyptology club has 8. Among the Egyptologists there are two of the tennis players and three of the stamp collectors. Six people play both tennis and collect stamps. One eager person participates in all three clubs. How many people are engaged in the club life of E.?



$$|T| = 20 \qquad |T \cap S| = 6$$
$$|S| = 15 \qquad |T \cap E| = 2 \qquad |T \cap S \cap E| = 1$$
$$|E| = 8 \qquad |S \cap E| = 3$$

$$|T \cup E \cup S| = 33$$

$$|T| + |S| + |E| - |T \cap S| - |T \cap E| - |S \cap \mathrm{E}|$$
$$+|T \cap S \cap E| = 20 + 15 + 8 - 6 - 2 - 3 + 1 = 33$$

This is just right, each element in $T \cup S \cup E$ is counted exactly once.

Inclusion-exclusion

How to compute $|A_1 \cup A_2 \cup \cdots \cup A_n|$?
Add all sizes
Subtract sizes of all pairwise intersections
Add sizes of all intersections of three sets
Subtract sizes of all intersections of four sets
...

How to express this formally?

$$|A_1 \cup \mathrm{A}_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i_1 < i_2 \le n} \left|A_{i_1} \cap A_{i_2}\right| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Inclusion-exclusion principle

Theorem: For finite sets $A_1, A_2, \dots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right|.$$

Proof. Consider an arbitrary element $x \in A_1 \cup \cdots \cup A_n$.

x contributes 1 on the left side of the equation

How much does it contribute on the right side?

- Let j denote the number of sets A_i that contain x.
- without loss of generality, $x \in A_1, \dots, A_j$ and $x \notin A_{j+1}, \dots, A_n$
- x is contained in the intersection of any k of the sets $A_1, \dots, A_j$ and in no other intersections, i.e., x is contained in $\binom{j}{k}$ intersections of k-sets, which are counted with sign $(-1)^{k-1}$.

Nr. of times x is counted

$$\binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \cdots + (-1)^{j-1}\binom{j}{j} = ???$$

Calculation in one slide:

$$\binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \cdots + (-1)^{j-1}\binom{j}{j}$$

Rewrite into sum $= \sum_{k=1}^{j} (-1)^{k-1}\binom{j}{k}$

Add $k = 0$ to sum $= \sum_{k=0}^{j} \left((-1)^{k-1}\binom{j}{k}\right) + 1$

Take factor -1 out $= -1 \cdot \sum_{k=0}^{j} \left((-1)^{k}\binom{j}{k}\right) + 1$

Re-order $= -1 \cdot \sum_{k=0}^{j} \left(\binom{j}{k}(-1)^{k}\right) + 1$

$1^x = 1$ for any x $= -1 \cdot \sum_{k=0}^{j} \left(\binom{j}{k}(-1)^{k}1^{j-k}\right) + 1$

Binomial Theorem ($x = -1, y = 1$) $= -1 \cdot (-1+1)^j + 1 = 1$

Inclusion-Exclusion Principle

For finite sets $A_1, A_2, \ldots, A_n$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right|.$$

Bonferroni Inequalities:

For every even q:

$$\sum_{k=1}^{q} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right| \leq \left|\bigcup_{i=1}^{n} A_i\right|$$

For every odd q:

$$\sum_{k=1}^{q} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right| \geq \left|\bigcup_{i=1}^{n} A_i\right|$$

Examples

Numbers with square divisors

How many numbers in $\{1,2,\ldots,99\}$ are not divisible by a square of any integer greater than 1?

9 is divisible by $9 = 3^2$ should not be counted

28 is divisible by $4 \ = \ 2^2$ should not be counted

$70 = 2 \cdot 5 \cdot 7$ is not divisible by any square greater than 1 and should be counted

Numbers with square divisors

How many numbers in $\{1,2,\ldots,99\}$ are not divisible by a square of any integer greater than 1?

Which squares could be divisors?

4, 9, 16, 25, 36, 49, 64, 81

Numbers with square divisors

How many numbers in $\{1,2,\ldots,99\}$ are not divisible by a square of any integer greater than 1?

Which squares could be divisors?

4, 9, 16, 25, 36, 49, 64, 81

If x is divided by a red number, then it is also divided by a black one.

Suffices to count numbers in $\{1,2,\ldots,99\}$ not divisible by 4,9,25,49.

Numbers with square divisors

How many numbers in $\{1,2,\dots,99\}$ are not divisible by 4,9,25,49?

Define: $A_i = \{x \in \{1,2,\dots,99\}: x \text{ is divisible by } i\}$

Want to compute: $99 - |A_4 \cup A_9 \cup A_{25} \cup A_{49}|$

What is $|A_4 \cup A_9 \cup A_{25} \cup A_{49}|$? – Use Inclusion-exclusion Principle!

$$\left|\bigcup_{i\in\{4,9,25,49\}} A_i\right| = \sum_{k=1}^{4} (-1)^{k-1} \sum_{I \in \binom{\{4,9,25,49\}}{k}} \left|\bigcap_{i\in I} A_i\right|$$

Intuition:

$|A_4 \cup A_9 \cup A_{25} \cup A_{49}| =$ Sizes of sets individual sets – Intersections of 2 sets + intersections of 3 sets – intersection of 4 sets

We need: Size of intersections of A_4, A_9, A_{25}, A_{49}

Numbers with square divisors

Define: $A_i = \{x \in \{1,2, \dots, 99\}: x \text{ is divisible by } i\}$

We need: Size of intersections of A_4, A_9, A_{25}, A_{49}

$$|\mathrm{A_i}| = \left\lfloor \frac{99}{i} \right\rfloor$$

$$|\mathrm{A_4}| = 24, |\mathrm{A_9}| = 11, |\mathrm{A_{25}}| = 3, |\mathrm{A_{49}}| = 2,$$

What about intersections of 2 or more sets?

$$\left|\mathrm{A_i} \cap A_j\right| = \left\lfloor \frac{99}{i * j} \right\rfloor$$

If i, j are co-prime

Observation: only intersection of A_4, A_9, A_{25}, A_{49} that is non-empty is $A_4 \cap A_9$. It is $|A_4 \cap A_9| = \left\lfloor \frac{99}{36} \right\rfloor = 2.$

Numbers with square divisors

What is $|A_4 \cup A_9 \cup A_{25} \cup A_{49}|$? – Use Inclusion-exclusion Principle!

$|A_4 \cup A_9 \cup A_{25} \cup A_{49}| =$ Sizes of sets individual sets – Intersections of 2 sets + intersections of 3 sets – intersection of 4 sets.

$$
\begin{aligned}
&|A_4 \cup A_9 \cup A_{25} \cup A_{49}| \\
&= |A_4| + |A_9| + |A_{25}| + |A_{49}| - |A_4 \cap A_9| \\
&= 24 + 11 + 3 + 2 - 2 = 38.
\end{aligned}
$$

Numbers with square divisors

How many numbers in $\{1,2,\dots,99\}$ are not divisible by a square of any integer greater than 1?

We showed that it suffices to count numbers in $\{1,2,\dots,99\}$ not divisible by 4,9,25,49.

We define: $A_i = \{x \in \{1,2,\dots,99\}: x \text{ is divisible by } i\}$

Want to compute: $99 - |A_4 \cup A_9 \cup A_{25} \cup A_{49}|$

We calculated using inclusion-exclusion principle that
$|A_4 \cup A_9 \cup A_{25} \cup A_{49}| = 38$

So the answer is $99 - 38 = 61$

Intermediate summary

- Inclusion exclusion lets you compute union of sets using intersection of sets.

- Useful when computing intersections is easy and unions are hard. (Often due to overlapping sets.)

- When using to prove something make sure to
  - Define your sets
  - Explain how to calculate sizes of intersections
  - Use inclusion-exclusion to calculate the union
  - Use that to find the actual answer

Back to functions

Counting functions: overview

Let A be an n-set and B be an m-set:

Number of functions $f : A \to B$? m^n

Number of injective functions $f : A \to B$? $\prod_{i=0}^{n-1}(m-i)$

Number of surjective functions?

... later: harder to compute, no nice formula!

Number of surjective functions

Let X be an n-set and Y an m-set.

How many surjective functions are there?

Without loss of generality: $Y = \{1,2,\dots,m\}$.

$m = 1$: There is only one function, and it is surjective.

Answer: 1

Number of surjective functions

Let X be an n-set and Y an m-set.

How many surjective functions are there?

Without loss of generality: $Y = \{1,2, \dots, m\}$.

$m = 2$:

$A_1 = \{f : X \to Y : 1 \notin f(X)\}$,

$A_2 = \{f : X \to Y : 2 \notin f(X)\}$

$A = A_1 \cup A_2$ is the set of non-surjective functions.

So there are $2^n - |A|$ surjective functions.

$$|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 1 + 1 - 0 = 2$$

There are $2^n - 2$ surjective functions from an n-set to a 2-set.

Number of surjective functions

Let X be an n-set and Y an m-set.
How many surjective functions are there?

Without loss of generality: $Y = \{1,2,\ldots,m\}$.

$m = 3$:

$A_i = \{f : X \to Y : i \notin f(X)\}$ for $i \in \{1,2,3\}$

$A = A_1 \cup A_2 \cup A_3$ is the set of non-surjective functions.

$|A| = |A_1 \cup A_2 \cup A_3|$ the number of non-surjective function

So there are $3^n - |A|$ surjective functions.

$$
\begin{aligned}
|A| &= |A_1 \cup A_2 \cup A_3| \\
&= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \\
&= 2^n + 2^n + 2^n - 1 - 1 - 1 + 0 = 3 \cdot 2^n - 3
\end{aligned}
$$

There are $3^n - 3 \cdot 2^n + 3$ surjective functions from n-set to a 3-set.

Number of surjective functions

Let X be an n-set and Y an m-set.
How many surjective functions are there?

Without loss of generality: $Y = \{1,2,\dots,m\}$.

General case, arbitrary $m \geq 1$: $A_i = \{f\colon X \to Y \mid i \notin f(X)\}$.

What is $\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right|$?
In $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ are functions $f : X \to Y$ with $i_1, i_2, \dots, i_k \notin f(X)$.
So functions $f : X \to (Y \setminus \{i_1, i_2, \dots, i_k\})$.
Hence $\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right| = (m-k)^n$

$$\left|\bigcup_{i=1}^{m} A_i\right| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,m\}}{k}} \left|\bigcap_{i \in I} A_i\right| = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (m-k)^n$$

Number of surjective functions

Let X be an n-set and Y an m-set.

How many surjective functions are there?

Without loss of generality: $Y = \{1,2,\ldots,m\}$.

$$m^n \;-\; \left|\bigcup_{i=1}^{m} A_i\right|$$

$$= m^n - \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} (m-k)^n$$

Back to permutations

The hatcheck problem

There is a reception where n people are invited, and they leave their jackets at the cloakroom. Unfortunately, the person operating the cloakroom is completely overstrained and just hands out the jackets randomly. What is the probability that none of the attendance get their own jacket back?

Formalization:

The persons correspond to numbers $1, \ldots, n$

Their jackets correspond to numbers $1, \ldots, n$

Let $\pi(i)$ denote the number of the jacket handed to person i.
π is a random permutation.

Question: What is the probability that $\pi(i) \neq i$ for all $i \in \{1, \ldots, n\}$?

An index i with $\pi(i) = i$ is called a fixpoint of the permutation π.

Fixpoints in random permutations

What is the probability that a random permutation has no fixpoint?

Let $D(n)$ denote the number of fixpoint-free permutations on an n-set. Then the probability is ${}^{D(n)}/_{n!}$

We need to compute $D(n)$, i.e., count the number of fixpoint-free permutations!

Fixpoints in random permutations

We need to compute $D(n)$, i.e., count the number of fixpoint-free permutations!

Step 1: Define sets:

$$\boxed{S_n = \{f : \{1, \dots, n\} \to \{1, \dots, n\} \text{ bijective } \}}$$

Let $A_i = \{\pi \in S_n \mid \pi(i) = i\}$ the permutations having i as fixpoint.

$A_1 \cup A_2 \cup \cdots \cup A_n$ are the "bad" permutations having a fixpoint.

$$D(n) = n! - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

Need to compute $|A_1 \cup A_2 \cup \cdots \cup A_n|$. Use inclusion-exclusion!

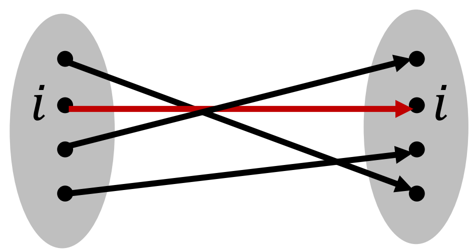
Counting permutations with fixpoints

Step 2: Compute size of intersections

Let $A_i = \{\pi \in S_n \mid \pi(i) = i\}$ the permutations having i as fixpoint.

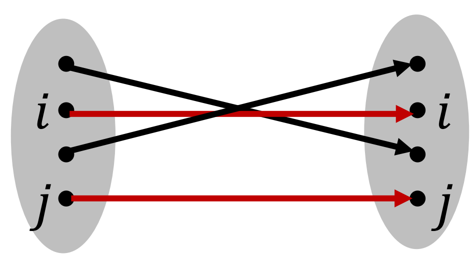
How many permutations are in A_i?

$\pi(i) = i$, permute remaining elements arbitrarily $\Rightarrow |A_i| = (n-1)!$



How many permutations are in $A_i \cap A_j$ for $i \neq j$?

$\pi(i) = i, \pi(j) = j$, permute rest arbitrarily $\Rightarrow \left|A_i \cap A_j\right| = (n-2)!$



In general: $\left|A_{i_1} \cap \mathrm{A}_{\mathrm{i}_2} \cap A_{i_3} \cap \cdots \cap A_{i_k}\right| = (n-k)!$

Applying inclusion-exclusion

Step 3:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} \left|\bigcap_{i \in I} A_i\right|$$

Fill in $|\bigcap_{i \in I} A_i|$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\dots,n\}}{k}} (n-k)!$$

$(n-k)!$ does not depend on I

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$

Def. of bin. coef.

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!\,(n-k)!} (n-k)!$$

Math

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

Applying inclusion-exclusion

Step 4:

$$D(n) = n! - \left|\bigcup_{i=1}^{n} A_i\right|$$

$$= n! - n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right)$$

$$\to \frac{n!}{e} \ (n \to \infty)$$

So with probability $\approx 1/e$ nobody gets their own jacket back.

Summary

Use the Inclusion-Exclusion principle when

- you want to compute the size of a union of sets, and
- the sizes of the intersections of sets are easy to compute

When using, make sure to:

- Define your sets
- Explain how to calculate sizes of intersections
- Use inclusion-exclusion to calculate the union
- Use that to find the actual answer

Remark: There is also a “dual” version for computing sizes of intersections if unions are easy to compute.

(This is informational only, you will not need this for the assignments/exam!)

Organizational

- Practice set:
  - Ex. 9: Basic application of Incl / Excl
  - Ex. 10: Slightly more mathy
  - No discussion group on this topic, make sure to practice!

- Test on A2 on Monday

- Next week there is the roundtable discussion for CS.