# Homework: Week 2

Jiaqi Wang

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## 1 Exercise 6.8.1

**Problem 1.1** Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below. Prove it.

*Proof.* We prove both directions of the if and only if statement. $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below.

1. We prove the forward direction.

Suppose $a : \mathbb{N} \to \mathbb{R}$ is bounded.
Then

$$\begin{aligned}&\text{there exists } M_0 > 0,\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad |a_n| \le M_0.\end{aligned}$$

Obtain such $M_0$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad -M_0 \le a_n \le M_0.\end{aligned}$$

Choose $m = -M_0$, then $m \in \mathbb{R}$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad m \le a_n.\end{aligned}$$

We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded from below.

Choose $M = M_0$, then $M \in \mathbb{R}$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad a_n \le M.\end{aligned}$$

We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded from above.

2. Now we prove the reverse direction.

Assume $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and bounded from below.
By definition of bounded from above, we have

$$\begin{aligned}&\text{there exists } M_1 \in \mathbb{R},\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad a_n \le M_1.\end{aligned}$$

By definition of bounded from below, we have

$$
\begin{aligned}
&\text{there exists } M_2 \in \mathbb{R}, \\
&\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad M_2 \leq a_n.
\end{aligned}
$$

We need to show that

$$
\begin{aligned}
&\text{there exists } M_0 > 0, \\
&\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad |a_n| \leq M_0.
\end{aligned}
$$

Choose $M_0 = \max\{|M_1|, |M_2|\}$, then $M_0 \in \mathbb{R}$.
It holds that

$$
\begin{aligned}
&\text{for all } n \in \mathbb{N}, \\
&\quad -M_0 \leq a_n \text{ and } a_n \leq M_0.
\end{aligned}
$$

Then it holds that $|a_n| \leq M_0$.
We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded.

Since both directions hold, we conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below. □

## 2 Exercise 6.8.2

**Problem 2.1** Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be real-valued sequences. Prove that

$$\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (-a_n) = -\infty$$

*Proof.* We show both directions of the if and only if statement.

$$\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (-a_n) = -\infty$$

1. First we prove the forward direction.

Suppose $\lim_{n\to\infty} a_n = \infty$.
Then

$$
\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \text{there exists } N_0 \in \mathbb{N}, \\
&\quad\quad \text{for all } n \geq N_0, \\
&\quad\quad\quad a_n > M.
\end{aligned}
$$

We need to show that

$$
\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \text{there exists } N \in \mathbb{N}, \\
&\quad\quad \text{for all } n \geq N, \\
&\quad\quad\quad -a_n \leq M.
\end{aligned}
$$

Take $M \in \mathbb{R}$ Choose $N = N_0$, then $N \in \mathbb{N}$.
It holds that

$$
\begin{aligned}
&\text{for all } n \geq N, \\
&\quad a_n > M.
\end{aligned}
$$

□

# 3 Exercise 6.8.3

**Problem 3.1** Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be real-valued sequences. Prove that

$$\lim_{n\to\infty} b_n = \infty \iff \lim_{n\to\infty} \frac{1}{b_n} = 0$$

*Proof.* $\square$

# 4 Exercise 6.8.5

**Problem 4.1** Define the sequence $x : \mathbb{N} \to \mathbb{R}$ recursively by

$$x_{n+1} := \frac{2 + x_n^2}{2x_n}$$

for $n \in \mathbb{N}$ while $x_0 = 2$. Prove that the sequence $x : \mathbb{N} \to \mathbb{R}$ converges and determine its limit.

*Proof.* $\square$

# 5 Exercise 6.8.6

**Problem 5.1** Determine whether the following sequences converge, diverge to $+\infty$, diverge to $-\infty$, or diverge in a different way. In case the sequence converges, determine the limit.

- $a_n := \frac{1}{n^3} - 3$
- $b_n := \frac{5n^5+2n^2}{3n^5+7n^3+4}$
- $c_n := n - \sqrt{n}$
- $d_n := \frac{2^n}{n^100}$
- $e_n := \sqrt{n^2+n} - n$
- $f_n := \sqrt[n]{3n^2}$
- $g_n := \frac{2^n+5n^200}{3^n+n^10}$
- $h_n := (-1)^n 3^n$
- $i_n := \sqrt[n]{5^n + n^2}$

*Proof.* By limit laws and standard limits, we have

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{1}{n^3} - 3\right) \\
&= \lim_{n\to\infty} \frac{1}{n^3} - \lim_{n\to\infty} 3 \\
&= \left(\lim_{n\to\infty} \frac{1}{n}\right)^3 - 3 \\
&= 0^3 - 3 \\
&= -3
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \left(\frac{5n^5 + 2n^2}{3n^5 + 7n^3 + 4}\right) \\
&= \lim_{n\to\infty} \frac{5 + \frac{2}{n^3}}{3 + \frac{7}{n^2} + \frac{4}{n^5}} \\
&= \frac{\lim_{n\to\infty}(5 + \frac{2}{n^3})}{\lim_{n\to\infty}(3 + \frac{7}{n^2} + \frac{4}{n^5})} \\
&= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{2}{n^3}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{7}{n^2} + \lim_{n\to\infty} \frac{4}{n^5}} \\
&= \frac{5}{3}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} (n - \sqrt{n}) \\
&= \lim_{n\to\infty} \left( \frac{(n-\sqrt{n})(n+\sqrt{n})}{n+\sqrt{n}} \right) \\
&= \lim_{n\to\infty} \left( \frac{n^2 - n}{n+\sqrt{n}} \right) \\
&= \lim_{n\to\infty} \left( \frac{n(n-1)}{n(1+\frac{1}{\sqrt{n}})} \right) \\
&= \lim_{n\to\infty} \left( \frac{n-1}{1+\frac{1}{\sqrt{n}}} \right) \\
&= \frac{\lim_{n\to\infty}(n-1)}{\lim_{n\to\infty}(1+\frac{1}{\sqrt{n}})} \\
&= \frac{\lim_{n\to\infty} n - \lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{\sqrt{n}}} \\
&= \frac{\lim_{n\to\infty} n - 1}{1 + \lim_{n\to\infty} \frac{1}{\sqrt{n}}} \\
&= \frac{\lim_{n\to\infty} n - 1}{1+0} \\
&= \lim_{n\to\infty} n - 1 \\
&= \infty
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} d_n &= \lim_{n\to\infty} (\frac{2^n}{n^100}) \\
&= \lim_{n\to\infty} \left( \frac{2^n}{n^{100}} \right)
\end{aligned}
$$

$\square$