Assignment 7

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1 Exercise 10.7.6

Problem 1.0.1 Let $P : \mathbb{N} \to \{\text{blue}, \text{orange}\}$ be a sequence taking values in the set with exactly the two elements blue and orange. Assume that

$$\begin{aligned}&\text{for all } k \in \mathbb{N},\\ &\quad \text{there exists } m \geq k,\\ &\qquad P_m = \text{blue}.\end{aligned} \tag{*}$$

Show that there is a subsequence of $P : \mathbb{N} \to \{\text{blue}, \text{orange}\}$ for which every term equal blue.

Proof. We construct a index sequence $n : \mathbb{N} \to \mathbb{N}$ inductively such that for all $\ell \in \mathbb{N}$, $P_{n_\ell} = \text{blue}$ and $n_\ell < n_{\ell+1}$.

Base step:
Choose $k = 0$ in (*), then there exists $m \geq 0$, such that $P_m = \text{blue}$.
Obtain such m.
Set $n_0 = m$.

Inductive step:
Suppose we have defined $n_0, \ldots, n_\ell$ for some $\ell \in \mathbb{N}$
such that $P_{n_0} = \text{blue}, \ldots, P_{n_\ell} = \text{blue}$ and $n_0 < \cdots < n_\ell$.
Choose $k = n_\ell + 1$ in (*), then there exists $m \geq n_\ell + 1 > n_\ell$ such that $P_m = \text{blue}$.
Obtain such m.
Choose $n_{\ell+1} = m$.
Then $P_{n_{\ell+1}} = \text{blue}$ and $n_{\ell+1} > n_\ell$.

By induction, we have defined $n : \mathbb{N} \to \mathbb{N}$ such that for all $\ell \in \mathbb{N}$, $P_{n_\ell} = \text{blue}$ and $n_\ell < n_{\ell+1}$.
Then $P_{n_\ell} = \text{blue}$ for all $\ell \in \mathbb{N}$.

□

2 Exercise 11.6.1

Problem 2.0.1 Let $(V, \|\cdot\|)$ be a normed linear space and let A be the closed ball of radius 1 around the origin, i.e.
$$A := \{v \in V \mid \|v\| \leq 1\}.$$
Show that the set A is closed

Proof. Need to show that A is closed, i.e. $V \setminus A$ is open.
I.e. $\overline{A} = V \setminus A$, defined by
$$\overline{A} := \{v \in V \mid \|v\| > 1\}$$
is open.
We need to show that for all $q \in \overline{A}$, q is an interior point of $\overline{A}$.
I.e. for all $q \in \overline{A}$, there exists $r > 0$ such that $B(q, r) \subseteq \overline{A}$.
Let $q \in \overline{A}$.
Then $\|q\| > 1$.
Choose $r = \frac{\|q\|-1}{2} > 0$.
Need to show that $B(q, r) \subseteq \overline{A}$, i.e. for all $p \in B(q, r)$, $p \in \overline{A}$.
Let $p \in B(q, r)$.
Need to show that $\|p\| > 1$
Note: $\|p\| > \inf\{\|v\| \mid v \in B(q, r)\} = \|q\| - r = \|q\| - \frac{\|q\|-1}{2} = \frac{\|q\|+1}{2} > 1$
Since $\|p\| > 1$
Then $p \in \overline{A}$.
So $B(q, r) \subseteq \overline{A}$.
And $\overline{A}$ is open.
Therefore A is closed.

□

3 Exercise 11.6.2

Problem 3.0.1 Show that the interval $[0, 1)$ is neither open nor closed (seen as a subset of the normed linear space $(\mathbb{R}, |\cdot|)$).

Proof. We first show that $[0, 1)$ is not open.
i.e. there exists $q \in [0, 1)$ such that for all $r > 0$, $B(q, r) \not\subseteq [0, 1)$.
Choose $q = 0$, then $q \in [0, 1)$
Let $r > 0$,
We need to show that $B(q, r) \not\subseteq [0, 1)$, i.e. there exists $p \in B(q, r)$ such that $p \notin [0, 1)$.
Let $p = -\frac{r}{2}$.
Then $p \in B(q, r)$.
But $p \notin [0, 1)$.
So $B(q, r) \not\subseteq [0, 1)$.
Therefore $[0, 1)$ is not open.

Now we show that $[0, 1)$ is not closed.
i.e. there exists $q \in \mathbb{R} \setminus [0, 1)$ such that for all $r > 0$, $B(q, r) \not\subseteq \mathbb{R} \setminus [0, 1)$.
Choose $q = 1$, then $q \in \mathbb{R} \setminus [0, 1)$
Let $r > 0$,
Need to show that $B(q, r) \not\subseteq \mathbb{R} \setminus [0, 1)$, i.e. there exists $p \in B(q, r)$ such that $p \notin \mathbb{R} \setminus [0, 1)$.
Let $p = 1 + \frac{r}{2}$.
Then $p \in B(q, r)$.
But $p \notin \mathbb{R} \setminus [0, 1)$.
So $B(q, r) \not\subseteq \mathbb{R} \setminus [0, 1)$.

Therefore $[0, 1)$ is not closed.

□

4 Exercise 11.6.4

Problem 4.0.1 Consider the following line $\mathbb{R}^2$

$$L := \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 1\}.$$

Show that L is a closed subset of $\mathbb{R}^2$ and that L is complete.

Proof. First we show that L is closed.
I.e. $\overline{L} := \mathbb{R}^2 \setminus L = \{(x, y) \in \mathbb{R}^2 \mid x + 2y \neq 1\}$ is open.
We need to show that for all $q \in \overline{L}$, q is an interior point of $\overline{L}$.
I.e. for all $q \in \overline{L}$, there exists $r > 0$ such that $B(q, r) \subseteq \overline{L}$.
Let $q \in \overline{L}$.
Choose $r = |\mathcal{P}(q) - q|$, where $\mathcal{P}(q)$ is the orthogonal projection of q onto L.
Then $r > 0$.
and $B(q, r) \subseteq \overline{L}$, i.e. for all $p \in B(q, r)$, $p \in \overline{L}$.
Thus L is closed.

Now we show that L is complete.
By proposition 11.4.3, we have that $\mathbb{R}^d$ is complete, in particular $\mathbb{R}^2$
Since L is a closed subset of $\mathbb{R}^2$, by proposition 11.4.5, we have that L is complete.

□