Assignment 7

Jiaqi Wang

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1 Exercise 10.7.6

Problem 1.1 Let $P : \mathbb{N} \to \{\text{blue}, \text{orange}\}$ be a sequence taking values in the set with exactly the two elements blue and orange. Assume that

$$\begin{aligned} &\text{for all } k \in \mathbb{N}, \\ &\quad \text{there exists } m \geq k, \\ &\quad\quad P_m = \text{blue}. \end{aligned} \tag{*}$$

Show that there is a subsequence of $P : \mathbb{N} \to \{\text{blue}, \text{orange}\}$ for which every term equal blue.

Proof. We construct a index sequence $n : \mathbb{N} \to \mathbb{N}$ inductively such that for all $\ell \in \mathbb{N}$, $P_{n_\ell} = \text{blue}$ and $n_\ell < n_{\ell+1}$.

Base step:
Choose $k = 0$ in (*), then there exists $m \geq 0$, such that $P_m = \text{blue}$.
Obtain such m.
Set $n_0 = m$.

Inductive step:
Suppose we have defined $n_0, \ldots, n_\ell$ for some $\ell \in \mathbb{N}$
such that $P_{n_0} = \text{blue}, \ldots, P_{n_\ell} = \text{blue}$ and $n_0 < \cdots < n_\ell$.
Choose $k = n_\ell + 1$ in (*), then there exists $m \geq n_\ell + 1 > n_\ell$ such that $P_m = \text{blue}$.
Obtain such m.
Choose $n_{\ell+1} = m$.
Then $P_{n_{\ell+1}} = \text{blue}$ and $n_{\ell+1} > n_\ell$.

By induction, we have defined $n : \mathbb{N} \to \mathbb{N}$ such that for all $\ell \in \mathbb{N}$, $P_{n_\ell} = \text{blue}$ and $n_\ell < n_{\ell+1}$.
Then $P_{n_\ell} = \text{blue}$ for all $\ell \in \mathbb{N}$.

□

2 Exercise 11.6.1

Problem 2.1 Let $V, \|\cdot\|)$ be a normed linear space and let A be the closed ball of radius 1 aroudn the origin, i.e.

$$A := \{v \in V \mid \|v\| \leq 1\}.$$

Show that the set A is closed

Proof. Need to show that A is closed, i.e. $V \setminus A$ is open. □

3 Exercise 11.6.2

Problem 3.1 Show that the interval $[0, 1)$ is neither open nor closed (seen as a subset of the normed linear space $(\mathbb{R}, |\cdot|)$).

Proof. □

4 Exercise 11.6.4

Problem 4.1 Consider the following line $\mathbb{R}^2$

$$L := \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 1\}.$$

Show that L is a closed subset of $\mathbb{R}^2$ and that L is complete.

Proof. □