Lecture notes analysis

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January 8, 2024

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Chapter 1

Introduction

This course aims to help you develop an understanding of mathematical analysis, and help you learn how to prove statements in analysis.

We will use a certain amount of abstraction in the course. For instance, we will set up Analysis 1 around the concept of a *metric space*. The reason is that I believe that this level of abstraction will make things *simpler*.

The lecture notes are still under construction. Any feedback is more than welcome. It also means that the lecture notes will keep getting updated as the course progresses.

Chapter 2

Sets, spaces and functions

2.1 What analysis is about

Mathematical analysis is for a large part the rigorous study of the approximate behavior of functions. Consider for instance the function $b : \mathbb{N} \to \mathbb{Q}$ given by

$$b(n) := \frac{1}{n+1}.$$

As n becomes very large, the values $b(n)$ get very close to zero. Another example can be given for the function $f : \mathbb{Q} \to \mathbb{Q}$ given by

$$f(x) = x^2 + 2.$$

As x becomes very close to 1, the values $f(x)$ get very close to 3. But the terms that I used such as "very large" and "very close" do not have a precise meaning. In analysis, we introduce precise concepts such as limits, that allow you to make precise statements and give rigorous proofs.

2.2 Functions between sets

The main characters in analysis are therefore functions. We use the notation $f : X \to Y$ to indicate that f is a function from a set X to a set Y. I like to think of a function f as a machine, with input from a set X and output in a set Y. If you have ever written a computer program before, then it can also help to see the analogy with functions in computer science. They take in data of some type (such as the type of integers) and produce data of another type (such as strings).

The most important example in this course for a set Y is the set $\mathbb{R}$ of the real numbers. In the next chapter, we are going to be more precise about what the real numbers are, but for now we assume that you have a working knowledge.

Other examples of sets are the set of natural numbers $\mathbb{N}$, the empty set $\emptyset$ (which has no elements at all), the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, and of course any subsets of those, such as the following subset of $\mathbb{Z}$,

$$\{2k \mid k \in \mathbb{Z}\}$$

which is the set of all *even* integers. More complicated examples of sets also exist: we can look at the set of all polynomials of degree at most 5, or the set of all possible grammatically correct sentences in the English language, or the set of all movies on Netflix. Whenever you see the word "set", you can imagine any of these options. I also encourage you to think of options yourself, it can be helpful to be creative.

Here are some examples of functions in mathematics.

Example 2.2.1. The function $b : \mathbb{N} \to \mathbb{R}$ defined by

$$b(n) := \frac{1}{1+n^2}, \quad \text{for } n \in \mathbb{N}$$

is an example of what we call a *real-valued sequence*, or a *sequence of real numbers*. In general, a *real-valued sequence* is a function from the natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$.

Example 2.2.2. The function $b : \mathbb{N} \to \mathbb{N}$ defined by

$$b(n) := n^3, \quad \text{for } n \in \mathbb{N}$$

is an example of a *sequence of natural numbers*. In general, a *sequence of natural numbers* is a function from the natural numbers $\mathbb{N}$ to the natural numbers $\mathbb{N}$.

Example 2.2.3. If M is an $(n \times m)$ matrix, with n rows and m columns, then $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$L(x) := Mx$$

is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

2.3 Set-theoretic definition of functions

Just so you have seen it before, let us also mention the strict mathematical definition of functions in set theory. Within set theory, a function $f : X \to Y$ between two sets X and Y is defined as a subset (let's call it F) of the Cartesian product $X \times Y$, such that for every $x \in X$, there is exactly one $y \in Y$ such that $(x, y) \in F$. We usually write this unique value y as $f(x)$.

2.4 Metric spaces

In analysis, we don't just work with any sets. It is crucial that we can make sense of what it means for two points in a set to be close, or far away. We can make this precise by the concept of a *distance*. A distance is really nothing more than a function, that takes a pair of points in X as input, and produces a real number as output, satisfying the properties in the definition listed below.

Definition 2.4.1 (distance)**.** Let X be a set. A function $d : X \times X \to \mathbb{R}$ is called a *distance* on X if

i. (positivity) For all $a, b \in X$, it holds that $d(a, b) \geq 0$.

ii. (non-degeneracy) For all $a, b \in X$, if $d(a, b) = 0$ then $a = b$.

iii. (symmetry) For all $a, b \in X$, $d(a, b) = d(b, a)$.

iv. (triangle inequality) For all $a, b, c \in X$,

$$d(a, c) \leq d(a, b) + d(b, c).$$

v. (reflexivity) For all $a \in X$, $d(a, a) = 0$.

Usually conditions (ii) and (v) are combined to the condition that for all $a, b \in X$, $d(a, b) = 0$ if and only if $a = b$.

Definition 2.4.2 (metric space)**.** A metric space (X, dist) is a pair of a set X and a distance function $\mathsf{dist} : X \times X \to \mathbb{R}$ on X.

Example 2.4.3 (example of a metric space)**.** Let $X := \{\mathsf{red}, \mathsf{yellow}, \mathsf{blue}\}$ be the set of (traditional) primary colors. The following table describes a function $d : X \times X \to \mathbb{R}$ as follows: in the row corresponding to a color x and a column corresponding to a color y the value $d(x, y)$. So for instance $d(\mathsf{yellow}, \mathsf{blue}) = 3$.

	red	yellow	blue
red	0	1	2
yellow	1	0	3
blue	2	3	0

We claim that $d : X \times X \to \mathbb{R}$ is indeed a distance on X. We can loosely verify the properties in Definition 2.4.1 quite easily by inspecting the table. For instance, because all elements in the table are positive, the positivity property holds for the function d.[a] Similarly, we can verify the non-degeneracy, symmetry, the triangle inequality and the reflexivity. We conclude that d is a distance on X, the pair (X, d) is a metric space.

[a] this isn't quite a rigorous proof. A full proof is a bit cumbersome because we really need to go through all possible cases, i.e. through all possible combinations of colors.

It will be very useful to give a definition to all points within a certain distance of a given point in a metric space.

Definition 2.4.4 ((open) ball)**.** Let (X, dist) be a metric space. The (open) ball around a point $p \in X$ with radius $r > 0$ is denoted by $B(p,r)$ and is defined as the set

$$B(p,r) := \{q \in X \mid \mathsf{dist}(q,p) < r\}.$$

Example 2.4.5. In 2.4.3, the open ball $B(\mathsf{yellow}, 3/2)$ is the set $\{\mathsf{yellow}, \mathsf{red}\}$, because $d(\mathsf{yellow}, \mathsf{yellow}) = 0$ which is strictly less than $3/2$, and $d(\mathsf{yellow}, \mathsf{red}) = 1 < 3/2$, but $d(\mathsf{yellow}, \mathsf{blue}) = 3 > 3/2$.

2.5 Normed vector spaces

Great examples of metric spaces are (constructed from) normed vector spaces. If you have had a course in linear algebra, you will most likely have seen the following definition of a *vector space* over a field $\mathbb{K}$. In these lecture notes, we will mostly only consider vector spaces over $\mathbb{R}$, unless we explicitly indicate otherwise. It is not necessary to learn this definition by heart. What is important is to take away that a vector space over a field is a set, together with operations called scalar multiplication and addition, that together satisfy some properties.

Definition 2.5.1 (vector space)**.** Let $\mathbb{K}$ be a field (or if you don't know what a field is, let $\mathbb{K}$ be the real or complex numbers or the rational numbers). A vector space $(V, \cdot, +, 0)$ over the field $\mathbb{K}$ is a set V, together with functions $\cdot : \mathbb{K} \times V \to V$ and $+ : V \times V \to V$ called scalar multiplication and addition, and a particular element $0 \in V$ such that the following properties are satisfied.

i. (commutativity of addition) for all $v, w \in V$,

$$v + w = w + v$$

ii. (associativity of addition) for all $u, v, w \in V$,

$$u + (v + w) = (u + v) + w$$

iii. (0 is the unit for addition) for all $v \in V$,

$$v + 0 = 0 + v = v$$

iv. every vector v has an opposite $(-v) \in V$, such that

$$v + (-v) = 0.$$

v. ($1 \in \mathbb{K}$ is the unit for scalar multiplication) for all $v \in V$,

$$1 \cdot v = v.$$

vi. (associativity of scalar multiplication) for all $\lambda, \mu \in \mathbb{K}$ and all $v \in V$,

$$\lambda \cdot (\mu \cdot v) = (\lambda\mu) \cdot v.$$

vii. for all $v, w \in V$ and $\lambda \in \mathbb{K}$,

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w.$$

viii. for all $\lambda, \mu \in \mathbb{K}$ and all $v \in V$,

$$(\lambda + \mu) \cdot v = \lambda v + \mu v.$$

One of the reasons that vector spaces are so important in analysis is that for functions defined on vector spaces, we can introduce the concept of *differentiation*, while this is in general problematic for functions defined on metric spaces.

Definition 2.5.2 (norm)**.** Let V be a vector space. We say that a function $\| \cdot \| : V \to \mathbb{R}$ is a *norm* if

i. (positivity) For all $v \in V$, it holds that $\|v\| \geq 0$.

ii. (non-degeneracy) For all $v \in V$, if $\|v\| = 0$ then $v = 0$.

iii. (absolute homogeneity) For all $\lambda \in \mathbb{R}$ and all $v \in V$ it holds that $\|\lambda v\| = |\lambda|\|v\|$.

iv. (triangle inequality) For all $v, w \in V$ it holds that

$$\|v + w\| \leq \|v\| + \|w\|.$$

Example 2.5.3. Denote by $\mathbb{R}^d$ the vector space of column vectors of length d with values in $\mathbb{R}$. For a vector $x \in \mathbb{R}^d$, denote its components by $x_1, \dots, x_d$. The Euclidean norm is defined as

$$\|x\|_2 := \sqrt{\sum_{i=1}^{d} x_i^2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

In fact, the norm $\| \cdot \|_2$ comes from an inner product: If we define the standard inner product

$$(x, y) := \sum_{i=1}^{d} x_i y_i$$

then $\|x\|_2^2 = (x, x)$. The triangle inequality follows from the Cauchy-Schwarz inequality in linear algebra. Let $a, b \in \mathbb{R}^d$. Then

$$\begin{aligned} \|a + b\|_2^2 &= \|a\|_2^2 + 2(a, b) + \|b\|_2^2 \\ &\leq \|a\|_2^2 + 2\|a\|_2\|b\|_2 + \|b\|_2^2 \\ &= (\|a\|_2 + \|b\|_2)^2. \end{aligned}$$

Example 2.5.4. We now specify the above example to the case $d = 1$. Then the vector space is $\mathbb{R}$ itself. The norm is given by

$$\|x\|_2 = \sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

which is the absolute value of x. We will denote the norm by $|\cdot| : \mathbb{R} \to \mathbb{R}$.

The following proposition tells us that if we have a normed vector space $(V, \|\cdot\|)$, we can also interpret it as a metric space.

Proposition 2.5.5. Let $(V, \|\cdot\|)$ be a normed vector space. Define the function $d_{\|\cdot\|} : V \times V \to \mathbb{R}$ by

$$d_{\|\cdot\|}(x, y) := \|x - y\|.$$

Then $d_{\|\cdot\|}$ is a distance on V.

Proof. We need to show that $d_{\|\cdot\|}$ satisfies all properties of a distance on V. We will check these properties one by one.

First we check positivity. We need to show that

$$\begin{aligned} &\text{for all } v, w \in V \\ &\quad d_{\|\cdot\|}(v, w) \geq 0. \end{aligned}$$

Since the statement we need to show starts with "for all $v, w \in V$", we start by writing:

Let $v, w \in V$. Then

$$d_{\|\cdot\|}(v, w) = \|v - w\| \geq 0$$

by positivity of the norm.

We will now show non-degeneracy. We need to show that

$$\begin{aligned} &\text{for all } v, w \in V \\ &\quad \text{if } d_{\|\cdot\|}(v, w) = 0 \text{ then } v = w. \end{aligned}$$

Let $v, w \in V$. Then

$$d_{\|\cdot\|}(v, w) = \|v - w\| = 0.$$

It follows by non-degeneracy of the norm that $v - w = 0$. We conclude that $v = w$.

We will show symmetry. We need to show that

$$\text{for all } v, w \in V \quad d_{\|\cdot\|}(v, w) = d_{\|\cdot\|}(w, v).$$

Let $v, w \in V$. Then we use absolute homogeneity of the norm to find

$$\begin{aligned} d_{\|\cdot\|}(v, w) &= \|v - w\| \\ &= \|(-1) \cdot (w - v)\| \\ &= |-1| \|w - v\| \\ &= \|w - v\| = d_{\|\cdot\|}(w, v). \end{aligned}$$

We now show the triangle inequality. We need to show that

$$\text{for all } u, v, w \in V \quad d_{\|\cdot\|}(u, w) \leq d_{\|\cdot\|}(u, v) + d_{\|\cdot\|}(v, w).$$

Let $u, v, w \in V$. Then

$$\begin{aligned} d_{\|\cdot\|}(u, w) &= \|u - w\| \\ &= \|(u - v) + (v - w)\| \\ &\leq \|u - v\| + \|v - w\| \\ &= d_{\|\cdot\|}(u, v) + d_{\|\cdot\|}(v, w). \end{aligned}$$

We finally show reflexivity. We need to show that

$$\text{for all } u \in V, \quad d_{\|\cdot\|}(u, u) = 0.$$

Let $u \in V$. Then by the absolute homogeneity of the norm,

$$\begin{aligned} d_{\|\cdot\|}(u,u) &= \|u-u\| \\ &= \|0 \cdot u\| \\ &= |0| \cdot \|u\| \\ &= 0 \cdot \|u\| \\ &= 0. \end{aligned}$$

□

Although strictly speaking, the metric space $(V, d_{\|\cdot\|})$ and $(V, \|\cdot\|)$ are different objects (one is a metric space and the other is a normed vector space), we will usually be a bit sloppy about this difference.

Remark 2.5.6 (Notation for Euclidean distance on $\mathbb{R}^d$ and $\mathbb{R}$)**.** We will usually write $\mathsf{dist}_{\mathbb{R}^d}$ instead of $\mathsf{dist}_{\|\cdot\|_2}$ for the standard (Euclidean) distance on $\mathbb{R}^d$. In particular, if $d \geq 2$, we have

$$\mathsf{dist}_{\mathbb{R}^d}(v,w) = \|v-w\|_2 = \sqrt{\sum_{i=1}^{d}(v_i - w_i)^2}$$

and if $d = 1$ we just have

$$\mathsf{dist}_{\mathbb{R}} = |v-w|.$$

And if there is no room for confusion, we will just leave out the subscript altogether.

2.6 The reverse triangle inequality

Lemma 2.6.1 (Reverse triangle inequality)**.** Let $(V, \|\cdot\|)$ be a normed vector space. Then for all $v, w \in V$,

$$\big| \|v\| - \|w\| \big| \leq \|v - w\|.$$

Proof. Let $v, w \in V$. Define $a := v - w$ and $b := w$. It follows by the triangle inequality that

$$\|a + b\| \leq \|a\| + \|b\|$$

so that

$$\|v\| \leq \|v - w\| + \|w\|$$

and

$$\|v\| - \|w\| \leq \|v - w\|.$$

Similarly,

$$\|w\| - \|v\| \leq \|w - v\| = \|v - w\|.$$

We conclude that

$$\big| \|v\| - \|w\| \big| \leq \|v - w\|.$$

□

2.7 Exercises

Please read also the following chapter and the best practices in Appendix A **before** attempting the next exercises. In particular, please follow the best practices in writing down your solutions.

2.7.1 Blue exercises

The following exercises are also available in Waterproof.

Exercise 2.7.1. Let (Y, dist_Y) be a metric space. Let X be a set. Let $f : X \to Y$ be injective. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, z) := \mathsf{dist}_Y(f(x), f(z)), \qquad \text{for all } x, z \in X.$$

Show that the function d is a distance on X.

Exercise 2.7.2. Let (X, dist) be a metric space. Let $A \subset X$ be a subset. Define the function $\mathsf{dist}|_A : A \times A \to \mathbb{R}$ by

$$\mathsf{dist}|_A(x,y) := \mathsf{dist}(x,y), \qquad \text{for all } x, y \in A.$$

Then $\mathsf{dist}|_A$ is called the restriction of dist to A. Show that $\mathsf{dist}|_A$ is a distance on A.

Exercise 2.7.3. Consider the function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by

$$d(a,b) = \begin{cases} 0, & \text{if } a = b, \\ 3, & \text{if } a \neq b. \end{cases}$$

Show that d is a distance function on $\mathbb{Z}$.

2.7.2 Orange exercises

Exercise 2.7.4. Let (X, dist) be a metric space. Define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sqrt{\mathsf{dist}(x,y)}.$$

Show that d is a distance on X.

Exercise 2.7.5. Let $(V, \|\cdot\|)$ be a normed vector space. We say a subset $U \subset V$ is convex if

$$\begin{aligned} &\text{for all } x, y \in U, \\ &\quad \text{for all } \lambda \in (0,1), \\ &\quad\quad \lambda x + (1-\lambda) y \in U. \end{aligned}$$

Let $z \in V$ and $r > 0$. Recall Definition 2.4.4 of the open ball $B(z,r)$. Show that $B(z,r)$ is convex.

Chapter 3

Proofs in analysis

3.1 What is a proof?

I would like to give two perspectives on what is a mathematical proof.

The first perspective is that a proof is a rigorous, precise argumentation why a certain mathematical statement is true. The argumentation uses statements that are just assumed to be true (they are called axioms), uses statements that are already proven before (lemmas, propositions, theorems), and then follows very specific rules to deduce new statements, rules such as if $A \implies B$ and $B \implies C$ then $A \implies C$. These rules are the rules of *logic*, and to a certain extent they agree with common sense. However, some of the rules of logic actually take some getting used to, especially if they involve many quantifiers.

My personal experience is that especially in the initial stages of learning how to prove mathematical statements, instead of thinking in terms of truth, i.e. instead of trying to use common sense to figure out why an argument shows that a statement is true, it is helpful to learn by heart the expectations on what is a proof, and to try to satisfy those expectations when you write proofs yourself.

This brings me to the second perspective on mathematical proofs. From this perspective, a proof as written down in mathematical books can be seen as *pseudocode* that explains how one could construct a *formal proof*. A *formal proof* is in itself a mathematical object, in fact it can be stored as (represented by) a computer program. For every mathematical statement, there are extremely precise rules that a *formal proof* should satisfy. These are again the rules of logic. As a consequence, the same rules dictate structure for the pseudocode. One of your main tasks in this course is therefore to learn what are the expectations on the pseudocode, i.e. to given a certain mathematical statement, learn what kind of pseudocode passes as a proof of the statement.

In the rest of this chapter, I will try to briefly explain the expectations on proofs, the expectations on pseudocode. I will do so alongside an example. A more structured guide for proving mathematical statements is given as a list of "best practices" in Appendix A.

Best practices. Just like big software companies have best practices for writing code, with the aim of having well-maintainable code with a minimal amount of bugs, in this course we have a set of best practices for writing proofs. These best practices are recorded in Appendix A. If we all try to adhere to the best practices, authors are helped in structuring their proofs, and readers, reviewers and authors are all helped in understanding the proofs.

3.2 Expectations on proofs

I believe that one of the main difficulties of Analysis 1 and 2 is learning to deal with a large number of quantifiers (that is, larger than 1). Imagine that we need to show the following statement:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, |1/n - 0| < \epsilon.$$

To clarify the nested structure of one mathematical statement inside the other, I like to use indentation, and I prefer to use words over quantifier-symbols

for all $\epsilon > 0$,
there exists $n_0 \in \mathbb{N}$,
for all $n \geq n_0$,
$|1/n - 0| < \epsilon$.

Such a statement looks complicated, and you may not know how to start a proof or how to continue.

But one of the main messages of this chapter is that the statement itself tells you how to start, and how to continue. In fact, the statement gives you a template, where there are only a few things left for you to fill in.

3.3 Prove statements block by block

The key to proving a mathematical statement involving many quantifiers is to prove it *block by block*. I will explain how to directly prove "for all" statements and "there exists" statements (that is, without using a contradiction argument). As a running example, we will prove that

for all $\epsilon > 0$,
there exists $n_0 \in \mathbb{N}$,
for all $n \geq n_0$,
$|1/n - 0| < \epsilon$

by peeling layer after layer of the statement.

3.4 Directly proving "for all" statements

If you want to directly (i.e. without a contradiction argument) show a statement such as

> for all $a \in A$,
>
> $\quad \dots$

you need to do the following. You first need to introduce (i.e. define) the variable $a \in A$ by writing something like

> Let $a \in A$.

or

> Take $a \in A$ arbitrary.

Next, you continue to prove the indented block on the next line.

In our example, we encounter this situation twice: in

> for all $\epsilon > 0$,
>
> $\quad \dots$

and in

> for all $n \geq n_0$,
>
> $\quad \dots$

For the statement

> for all $\epsilon > 0$,
>
> $\quad \dots$

this comes down to writing

> Let $\epsilon > 0$.

3.5 Directly proving "there exists" statements

If you want to directly (i.e. without using a contradiction argument) show a statement of the form

> there exists $b \in B$,
> $\quad \ldots$

you need to do the following. You need to make a choice for b, e.g. by writing

> Choose $b := \ldots$

after which you continue to show the indented statement $(\ldots)$ with that choice of b.

In our example we encounter this situation with

> there exists $n_0 \in \mathbb{N}$ such that $\{\ldots\}$

We could for instance write:

> Choose $n_0 := 10$.

and then continue with the proof of the block $\{\ldots\}$, with now n_0 fixed as 10. Making choices is hard. With a bad choice, you won't be able to prove whatever is inside the block $\{\ldots\}$. In many of the proofs that you will write, this is probably the step that requires the most thinking, the most creativity.

3.6 Trying to finish the proof

Where are we now? Our proof so far consists of

Let $\epsilon > 0$.
  Choose $n_0 := 10$.

and now we need to prove the statement

for all $n \geq n_0$,
  $|1/n - 0| < \epsilon$.

with the only knowledge about ϵ that it is a (real) number larger than 0, and that $n_0 = 10$.

Let us stick to the recipe. We need to show a statement of the form

for all $n \geq n_0$

so we define n by writing

Let $n \geq n_0$

and continue proving the block $\{|1/n - 0| < \epsilon\}$.

Of course, now we are in big trouble. Because we chose $n_0 = 10$, we can merely guarantee that

$|1/n - 0| = 1/n \leq 1/n_0 = 1/10$.

But we do not know whether ϵ is larger than $1/10$ or not! All we know is that it is a positive real number.

We cannot prove that

$|1/n - 0| < \epsilon$

because we made a bad choice for n_0. This happens. It is almost impossible to figure out the proof 'linearly', in a prearranged order of steps. To know how to choose n_0, you need to know your endgame, you already need to know how you will finish the proof. This means, you need to do a lot of scratchwork.

Let us do some of that scratchwork. We can figure out what choice of n_0 *would* lead to a proof. For instance if we choose instead

$$n_0 := \lceil 1/\epsilon \rceil + 1$$

then n_0 is a natural number strictly larger than $1/\epsilon$ (here $\lceil 1/\epsilon \rceil$ is $1/\epsilon$ rounded up to a natural number). In that case

$$|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon.$$

It works!

But you need to present the proof following the above steps. You just make better choices.

3.7 Let's try again

We need to show

for all $\epsilon > 0$,
  there exists $n_0 \in \mathbb{N}$,
    for all $n \geq n_0$,
      $|1/n - 0| < \epsilon$

so we write

Let $\epsilon > 0$.

and continue with the proof of the indented statement

> there exists an $n_0 \in \mathbb{N}$ such that
> for all $n \geq n_0$
> $|1/n - 0| < \epsilon$.

We are wiser now, and write

> Choose $n_0 := \lceil 1/\epsilon \rceil + 1$

and continue with the proof of

> for all $n \geq n_0$,
> $|1/n - 0| < \epsilon$.

Next, we write

> Let $n \geq n_0$.

and we are ready to, once again, try to prove that

> $|1/n - 0| < \epsilon$.

Indeed, now it is time to insert our calculation

> $|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon$.

This finishes the proof. If we write everything in one go it works as follows.

> Let $\epsilon > 0$.
> Choose $n_0 := \lceil 1/\epsilon \rceil + 1$.
> Let $n \geq n_0$.
> Then $|1/n - 0| = 1/n \leq 1/n_0 < 1/(1/\epsilon) = \epsilon$.

The statement we proved already suggested to us the template

> Let $\epsilon > 0$.
>   Choose $n_0 := \dots$
>   Let $n \geq n_0$.
>   Then **show desired estimate**.

We filled it in by choosing n_0 appropriately, and making a correct estimate.

3.8 (Natural) induction

To show a statement

> for all $n \in \mathbb{N}$,
> $P(n)$

you can use natural induction. Here, $P(n)$ is a statement depending on n. The template is as follows:

> We use induction on $n \in \mathbb{N}$.
>   We first show the base case, i.e. that $P(0)$ holds.
>   *... insert here a proof of* $P(0)$ *...*
>   We now show the induction step.
>   Let $k \in \mathbb{N}$ and assume that $P(k)$ holds.
>   We need to show that $P(k+1)$ holds.
>   *... insert here a proof of* $P(k+1)$ *...*

3.9 Negations and quantifiers

We write the negation of a mathematical statement $(\dots)$ as $\neg(\dots)$. The so-called *De Morgan's laws* specify how quantifiers behave under negation.

The statement

$\neg$ (for all $a \in A$, $(\dots)$)

is equivalent to

there exists $a \in A$, $(\neg(\dots))$.

Similarly, the statement

$\neg$ (there exists $a \in A$, $(\dots)$)

is equivalent to

for all $a \in A$, $(\neg(\dots))$.

3.10 Proofs by contradiction

When you are stuck proving something directly, it is a good idea to try to give a proof by contradiction: You assume that whatever you want to show is not true, and derive a contradiction from there. Some statements in analysis are (almost?) impossible to show without using a contradiction argument somewhere.

You can use the following template for a contradiction argument.

We argue by contradiction. Suppose $\neg(\dots)$.
... *derivation that leads to a contradiction* ...
Contradiction. We conclude that $(\dots)$ holds.

3.11 Exercises

3.11.1 Blue exercises

Exercise 3.11.1. Show that

$$\begin{aligned}
&\text{there exists } M \in \mathbb{R},\\
&\quad \text{for all } x \in [0,5],\\
&\qquad x \leq M.
\end{aligned}$$

Exercise 3.11.2. Show that

$$\begin{aligned}
&\text{for all } x \in \mathbb{R},\\
&\quad \text{there exists } y \in \mathbb{R},\\
&\qquad \text{for all } u \in \mathbb{R},\\
&\qquad\quad \text{if } u > 0 \text{ then}\\
&\qquad\qquad \text{there exists } v \in \mathbb{R},\\
&\qquad\qquad\quad v > 0 \text{ and } x + u < y + v.
\end{aligned}$$

3.11.2 Orange exercises

Exercise 3.11.3. Prove that

$$\begin{aligned}
&\text{for all } x \in \mathbb{R},\\
&\quad \text{if } \left(\text{for all } \epsilon > 0, \quad x > 1 - \epsilon\right)\\
&\qquad \text{then } x \geq 1.
\end{aligned}$$

Chapter 4

Real numbers

The functions that we will study will most often have the real numbers as a target space. And if the target space is not the space of real numbers, then it will most often be a normed vector space or a metric space with properties very similar to the real numbers.

In this chapter we will therefore take a careful look at the properties of the real numbers. Although most of the properties may seem obvious, there is one property that you usually don't encounter so explicitly. The name of this property is the *completeness axiom*.

The absolute main message of this chapter is the importance of the completeness axiom: at the innermost core of almost every proof in these notes is the completeness axiom. It is the motor of analysis.

4.1 What are the real numbers?

The real numbers are a set, together with two operations called addition $(+)$ and multiplication $(\times)$ and an order $(\leq)$, that satisfy a whole list of properties. These properties are called *axioms*.

The axioms can be summarized as follows: the real numbers are a complete, totally ordered field. You may remember the concept of a *field* from Set Theory and Algebra. A *totally ordered* field is a field together with a total order such that the order and the field operations are compatible. The rational numbers with the standard addition, multiplication and order are an example of a totally ordered field. We will come to the exact meaning of 'complete' in the next section, but let's already mention that a totally ordered field is called *complete* if every non-empty, bounded-from-above subset of the totally ordered field has a *least upper bound*.

At this stage we assume that complete, totally ordered fields exist. We choose one such complete ordered field and call it the real numbers and denote it by $\mathbb{R}$. Are there other possibilities then? Yes and not really. Yes, there are *different* complete totally ordered fields satisfying all these axioms, but they are all *essentially the same* in the sense that they are isomorphic as totally ordered fields. We will not show these statements in these notes. For the purpose of the notes it's enough to just make a choice of a complete ordered field and call it $\mathbb{R}$.

Between various books there may be some slight difference in choice of axioms. Although the lists are different, they do specify (essentially) the same complete totally ordered field. In Abbott's book [A$^+$15], you can find the axioms of the real numbers in Section 8.6.

In proof assistants such as Waterproof, the axioms for the real numbers are also introduced and they are used as the fundamental building blocks on which the rest of the theory is formally built up. You can find a full list of possible axioms for the real numbers as they are used in Waterproof on the website:

coq.inria.fr/library/Coq.Reals.Raxioms.html.

4.2 The completeness axiom

In this section, we will one by one introduce the concepts that occur in the completeness axiom. All these concepts make sense for any totally ordered field. Throughout this section, we will denote by $\mathbf{R}$ an arbitrary totally ordered field. We use the slight typographic difference between '$\mathbb{R}$' and '$\mathbf{R}$' to allow you to think of $\mathbf{R}$ as the real line, but in principle it could be another totally ordered field as well, such as the rational numbers $\mathbb{Q}$. However, as soon as you assume that a totally ordered field $\mathbf{R}$ satisfies the completeness axiom, then it is just the same as the real numbers $\mathbb{R}$.

We first define what we mean by upper and lower bounds of subsets of a totally ordered field.

Definition 4.2.1 (Upper bound and lower bound)**.** We say a number $M \in \mathbf{R}$ is an *upper bound* for a subset $A \subset \mathbf{R}$ if

$$\text{for all } a \in A, \\ a \leq M.$$

We say a number $m \in \mathbf{R}$ is a *lower bound* for a subset $A \subset \mathbf{R}$ if

$$\text{for all } a \in A \\ m \leq a.$$

Example 4.2.2. Let $A := (0,2) \cup (3,6)$ be a subset of $\mathbb{R}$. Then 8 is an upper bound for A. Let us prove this according to best practices.

We need to show that for all $a \in A$, $a \leq M$.

Let $a \in A$. We need to show that $a \leq 8$. Since $a \in (0,2) \cup (3,6)$, it indeed holds that $a \leq 8$.

Given the definition of upper and lower bounds, we define what it means for a set to be bounded from above, bounded from below and just bounded.

Definition 4.2.3 (bounded from above, bounded from below, bounded)**.** We say that a subset $A \subset \mathbf{R}$ is *bounded from above* if there exists an $M \in \mathbf{R}$ such that M is an upper bound for A.

We say that a subset $A \subset \mathbf{R}$ is *bounded from below* if there exists an $m \in \mathbf{R}$ such that m is a lower bound for A.

We say that a subset $A \subset \mathbf{R}$ is *bounded* if A is both bounded from above and bounded from below.

A *least upper bound* is an upper bound that is smaller than or equal to any other upper bound.

Definition 4.2.4 (least upper bound a.k.a. supremum)**.** Precisely, M is a *least upper bound* of a subset A if both

i. M is an upper bound

ii. For every upper bound $L \in \mathbf{R}$ of A, it holds that $M \leq L$.

Proposition 4.2.5. Suppose both M and W are a least upper bound of a subset $A \subset \mathbf{R}$. Then $M = W$. In other words, the least upper bound is unique.

Proof. Since M is a least upper bound, and W is an upper bound, we know by item (ii) in the characterization of the least upper bound that $M \leq W$. Similarly, W is a least upper bound, and M is an upper bound, so $W \leq M$. It follows that $M = W$. $\square$

Definition 4.2.6 (The supremum)**.** A different name for the least upper bound of a set $A \subset \mathbf{R}$ is the *supremum* of A, which we also write as $\sup A$.

Given all the terminology defined above, we can now state the completeness axiom.

Axiom 4.2.7 (Completeness axiom)**.** We say that a totally ordered field $\mathbf{R}$ satisfies the *completeness axiom* if every non-empty subset of $\mathbf{R}$ that is bounded from above has a *least upper bound*.

As soon as the completeness axiom is satisfied, we can also derive that every nonempty subset that is bounded from below has a largest lower bound.

Lemma 4.2.8. Every non-empty subset of the real line that is bounded from below has a *largest lower bound*.

Proof. Let A be a non-empty subset of the real line that is bounded from below. We need to show that there exists an $m \in \mathbf{R}$ such that

i. m is a lower bound of A.

ii. for every lower bound l of A, it holds that $l \leq m$.

We define $B := -A$, where by $-A$ we mean

$$-A := \{-a \mid a \in A\}.$$

Then B is nonempty, and B is bounded from above. Therefore, B has a smallest upper bound $M = \sup B$. We choose $m := -M$. We will show (i) and (ii).

We first show (i), in other words we need to show that

$$\text{for all } a \in A \\ m \leq a.$$

Let $a \in A$. Because M is the supremum of B, we know that for all $b \in B$, it holds that $b \leq M$. In particular, since $-a \in -A = B$, we know that $-a \leq M = -m$. Therefore indeed $m \leq a$.

We now show (ii). Let l be a lower bound for A. Then $-l$ is an upper bound for $-A = B$. Because M is the supremum of B, we know that $M \leq -l$. Recall that $M = -m$. Therefore $-m \leq -l$ and $l \leq m$. □

Definition 4.2.9. We usually call the largest lower bound of a non-empty set $A \subset \mathbf{R}$ that is bounded from below the *infimum* of A, and we denote it by $\inf A$.

4.3 Alternative characterizations of suprema and infima

In this section we provide alternative characterizations of suprema and infima. These alternative characterizations are usually easier to use in proofs than the definitions. We will give an example at the end of this section.

Proposition 4.3.1 (alternative characterization of supremum)**.** Let $A \subset \mathbb{R}$ be non-empty and bounded from above. Let $M \in \mathbb{R}$. Then M is the supremum of A if and only if

i. M is an upper bound for A, and

ii.
$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } a \in A, \\ &\quad\quad a > M - \epsilon. \end{aligned}$$

The proof of Proposition 4.3.1 is the content of Exercise 4.9.4.

Proposition 4.3.2 (alternative characterization of infimum)**.** Let $A \subset \mathbb{R}$ be non-empty and bounded from below. Let $m \in \mathbb{R}$. Then m is the infimum of A if and only if

i. m is a lower bound for A, and

ii.
$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } a \in A, \\ &\quad\quad a < m + \epsilon. \end{aligned}$$

These alternative characterizations of the supremum and infimum really provide a standard way to determining the supremum and infimum of subsets of the real line. Sometimes there may be a more creative argument for determining the supremum or infimum, but the alternative characterization usually provides a decent and solid route to the argument.

Example 4.3.3. Consider the following set

$$A := (1, 4) \cup (5, 7) \cup (8, 9)$$

We claim that $\inf A = 1$.

We will use the alternative characterization of the infimum. Hence we need to show that

i. 1 is a lower bound for A

ii.
$$\begin{aligned} &\text{for all } \epsilon > 0 \\ &\quad \text{there exists } a \in A \\ &\quad\quad a < 1 + \epsilon. \end{aligned}$$

We first show (i). We need to show that for all $a \in A$, $1 \leq a$. Because we need to show a *for-all* statement, we start with:

Let $a \in A$. Then $a \in (1, 4)$ or $a \in (5, 7)$ or $a \in (8, 9)$, and in all cases, $1 \leq a$.

We now show (ii). Because we need to show a *for-all* statement, we start with: Let $\epsilon > 0$. Now we need to show that there exists $a \in A$ such that

$$a < 1 + \epsilon$$

Therefore the next step is to choose an $a \in A$, in the hope that we can then show that $a < 1 + \epsilon$. This is often the trickiest part in the proof. It is very important that the choice for a that we make, is actually an element of A.

Here's the danger: We often think of ϵ as small, and if ϵ *is* strictly less than 6, the number

$$1 + \epsilon/2$$

is indeed in the interval $(1,4)$, and therefore it is an element of A. However, ϵ could also be very large, for instance $\epsilon = 1000$, and then

$$1 + \epsilon/2 = 501 \notin A.$$

Therefore in general

$$1 + \epsilon/2 \notin A.$$

This happens a lot, that there is an initial idea which is almost ok, but it doesn't quite work. In that case we can adapt. One way is as follows:

Choose

$$a := \min(1 + \epsilon/2, 2)$$

In that case, we know that a is always between 1 and 2, and therefore $a \in A$. What we now need to do is show that $a < 1 + \epsilon$.

For this we write a small chain of inequalities. Indeed, it holds that

$$a = \min(1 + \epsilon/2, 2) \leq 1 + \epsilon/2 < 1 + \epsilon.$$

4.4 Maxima and minima

In this section we say something about the relationship between the supremum and the maximum, and between the infimum and the minimum.

Definition 4.4.1 (maximum and minimum)**.** Let $A \subset \mathbb{R}$ be a subset of the real numbers. We say that $y \in A$ is the *maximum* of A, and write $y = \max A$, if

$$\begin{aligned} &\text{for all } a \in A, \\ &\quad a \leq y. \end{aligned}$$

We say that $x \in A$ is the *minimum* of A, and write $x = \min A$, if

$$\begin{aligned} &\text{for all } a \in A, \\ &\quad x \leq a. \end{aligned}$$

One of the very important aspects of the above definition is that the minima and maxima are always elements of the set itself.

Warning: even if a set $A \subset \mathbb{R}$ is non-empty and bounded from above, a maximum may not always exist. Similarly, even if a set $A \subset \mathbb{R}$ is non-empty and bounded from below, a minimum may not always exist.

Proposition 4.4.2. Let A be a subset of $\mathbb{R}$. If A has a maximum, then A is non-empty and bounded from above, and $\sup A = \max A$. If A has a minimum, then A is non-empty and bounded from below, and $\inf A = \min A$.

Proposition 4.4.3. Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from above. If $\sup A \in A$ then A has a maximum and $\max A = \sup A$.

Proposition 4.4.4. Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from below. If $\inf A \in A$ then A has a minimum and $\min A = \inf A$.

4.5 The Archimedean property

We also state another property of the real numbers. This one sounds very logical, but is sometimes needed in mathematical proofs.

Proposition 4.5.1 (Archimedean property)**.** For every real number $x \in \mathbb{R}$ there exists a natural number $m \in \mathbb{N}$ such that $x < m$.

Proof. We argue by contradiction. Suppose therefore that there exists an $x \in \mathbb{R}$ such that for all $m \in \mathbb{N}$, it holds that $m \leq x$. That means that $\mathbb{N}$ is bounded from above. Since $\mathbb{N}$ is also nonempty, we know that the supremum $\sup \mathbb{N}$ would exist. Then there exists a natural number $m \in \mathbb{N}$ such that

$$m > \sup \mathbb{N} - 1/2.$$

Now $m+1$ is a natural number as well, and

$$m+1 > \sup \mathbb{N} - 1/2 + 1 > \sup \mathbb{N}$$

which is a contradiction. $\square$

Given the previous proposition, we can define the ceiling function (that we actually have already used in the running example in the previous chapter).

Definition 4.5.2. (ceiling function) The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is defined as follows. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer $z \in \mathbb{Z}$ such that $x \leq z$.

The Archimedean property implies that between every two real numbers you can find a rational number. That is the content of the following proposition.

Proposition 4.5.3. For every two real numbers $a, b \in \mathbb{R}$ with $a < b$, there exists a $q \in \mathbb{Q}$ with $a < q < b$.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Define

$$M := \left\lceil \frac{3}{b-a} \right\rceil \in \mathbb{N}.$$

Then

$$Mb - Ma = M(b-a) = \left\lceil \frac{3}{b-a} \right\rceil (b-a) \geq \frac{3}{b-a}(b-a) = 3.$$

Therefore

$$Ma < \lceil Ma \rceil + 1 < Ma + 2 < Mb. \tag{4.5.1}$$

Choose $q := \frac{\lceil Ma \rceil + 1}{M}$, which is indeed an element of $\mathbb{Q}$. After dividing both sides of (4.5.1) by M, we conclude that

$$a < \frac{\lceil Ma \rceil + 1}{M} < b.$$

□

The next proposition loosely speaking says that the $\sqrt{2}$ is irrational. If we would be very precise, though, at this stage we wouldn't even know how to define $\sqrt{2}$. Sure, we could define it as a real number $x \in \mathbb{R}$ such that $x^2 = 2$, but who says such a real number exists?

Proposition 4.5.4. There does not exist a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$.

Proof. We argue by contradiction. Suppose there exists a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$. We can therefore choose such an r, and we can even choose it such that $r > 0$. Then $r = p/q$ with p and q nonzero natural numbers such that their greatest common divisor is one. We then have

$$\frac{p^2}{q^2} = 2,$$

so that

$$p^2 = 2q^2$$

Since the right-hand side is divisible by 2 (i.e. it is even), the left-hand side is divisible by 2 as well. Recall that if a product of positive natural numbers ab is divisible by 2, then at least one of a or b is divisible by 2. Therefore p is divisible by 2 and p^2 is divisible by 4. However, because the greatest common divisor of p and q is 1, we have that q is not divisible by 2. Therefore $2q^2$ is not divisible by 4, which gives a contradiction. □

The next proposition really defines $\sqrt{2}$.

Proposition 4.5.5. Consider the set

$$A := \{a \in \mathbb{Q} \mid a^2 \leq 2 \text{ and } a > 0\}.$$

Then A is non-empty, bounded above and $(\sup A)^2 = 2$. In other words, $\sqrt{2}$ exists as a real number and equals $\sup A$.

Proof. We first show that A is non-empty. This holds because $1 \in A$.

We will now show that A is bounded above. We need to show that there exists an $M \in \mathbb{R}$ such that M is an upper bound for A.

We choose $M := 2$.

We now have to show that 2 is an upper bound for A. We need to show that for all $a \in A$, $a \leq 2$. We argue by contradiction. Suppose $\neg(a \leq 2)$. Then $a > 2$ and therefore $a^2 > 4 > 2$, which is a contradiction.

Therefore we can conclude that $\sup A$ exists and that $1 \leq \sup A \leq 2$.

We will now show that $(\sup A)^2 = 2$. We again argue by contradiction. Suppose $(\sup A)^2 \neq 2$. Then either $(\sup A)^2 < 2$ or $(\sup A)^2 > 2$.

We first consider the case $(\sup A)^2 < 2$. We also know that $\sup A \geq 1$. By Proposition 4.5.3 we may choose a $q \in \mathbb{Q}$ such that

$$\sup A < q < \sup A + \frac{2 - (\sup A)^2}{4 \sup A}.$$

We define $\epsilon := q - \sup A$, and note that

$$0 < \epsilon < \frac{2 - (\sup A)^2}{4 \sup A} < 1.$$

Therefore

$$\begin{aligned} q^2 &= (\sup A + \epsilon)^2 \\ &= (\sup A)^2 + 2\epsilon \sup A + \epsilon^2 \\ &< (\sup A)^2 + 2\epsilon \sup A + \epsilon \\ &< (\sup A)^2 + 2\epsilon \sup A + 2\epsilon \sup A \\ &= (\sup A)^2 + 4\epsilon \sup A < 2, \end{aligned}$$

where in the first inequality we used that $\epsilon < 1$. In other words, we have found a $q \in A$ such that $q > \sup A$. This is a contradiction since $\sup A$ is an upper bound for A.

We now consider the case that $(\sup A)^2 > 2$. Define

$$\delta := \frac{(\sup A)^2 - 2}{2 \sup A}.$$

By the alternative characterization of the supremum, there exists an $r \in A$ such that

$$r > \sup A - \frac{(\sup A)^2 - 2}{2 \sup A}.$$

Choose such an r. Then

$$\begin{aligned} r^2 &> (\sup A - \delta)^2 \\ &= (\sup A)^2 - 2\delta \sup A + \delta^2 \\ &> (\sup A)^2 - 2\delta \sup A \\ &= (\sup A)^2 - 2\frac{(\sup A)^2 - 2}{2 \sup A} \sup A = 2, \end{aligned}$$

which is also a contradiction.

We conclude that $(\sup A)^2 = 2$. □

Corollary 4.5.6. For every two real numbers $a, b \in \mathbb{R}$ with $a < b$, there exists an irrational number $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

Proof. Let $a, b \in \mathbb{R}$. By the Proposition 4.5.3 there exists a $q \in \mathbb{Q}$ such that $a < q < b$. Choose such a q. Set

$$N := \lceil 1/(b-q) \rceil + 1.$$

Choose

$$r := q + \frac{\sqrt{2}}{2N}.$$

Then r is irrational and

$$a < r = q + \frac{\sqrt{2}}{2N} \leq q + \frac{1}{N} < q + (b - q) = b.$$

□

4.6 Sets can be complicated

Subsets of the real line can be incredibly complicated monstrous objects. What do I mean by this and why is it relevant?

If you think of examples of sets, you might think of an interval such as $(2,4]$, or a subset that consists of a single point $\{5\}$, or if you go crazy an example may be $(2,4] \cup \{37\} \cup [40,\infty)$. But none of these examples are very representative: they are much simpler than an 'arbitrary' set.

Why is this relevant? In this course, the aim is to prove statements such as *for all nonempty, bounded subsets* $A, B \subset \mathbb{R}$*, it holds that* $\sup(A+B) = \sup A + \sup B$. If you need to prove such a statement, you really need to show that *for every* possible subsets A and B, and you can easily fool yourself by considering examples that are too simple.

So, I encourage you, to every once in a while think about whether the examples you think of are representative. What is the most complicated subset of the real line you can think of?

4.7 Computation rules for suprema

In the proposition below, we use the definitions

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for subsets $A, B \subset \mathbb{R}$ and a scalar $\lambda \in \mathbb{R}$.

Proposition 4.7.1. Let A, B, C, D be nonempty subsets of $\mathbb{R}$. Assume that A and B are bounded from above and C and D are bounded from below. Then

i. $\sup(A + B) = \sup A + \sup B$

ii. $\inf(C + D) = \inf C + \inf D$

iii. for all $\lambda \geq 0$, $\sup(\lambda A) = \lambda \sup A$

iv. for all $\lambda \geq 0$, $\inf(\lambda C) = \lambda \inf C$

v. $\sup(-C) = -\inf C$

vi. $\inf(-A) = -\sup A$

Proof. We first show (i) in the list above. We set $M := \sup A + \sup B$ and will show that M is indeed the supremum of the set $A + B$ by showing items (i) and (ii) of the alternative characterization of the supremum in Proposition 4.3.1.

We first show item (i) of Proposition 4.3.1, namely that M is an upper bound. We need to show that

$$\begin{aligned} &\text{for all } c \in A + B, \\ &\quad c \leq M. \end{aligned}$$

Let $c \in A + B$. Then there exists an $a \in A$ and a $b \in B$ such that $c = a + b$. We also know that $a \leq \sup A$ and $b \leq \sup B$. Therefore

$$c = a + b \leq \sup A + \sup B = M$$

which was what we wanted to show.

We will now show item (ii) of Proposition 4.3.1 for M, namely that

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } c \in A + B, \\ &\qquad c > M - \epsilon. \end{aligned}$$

Let $\epsilon > 0$. By item (ii) of the characterization of $\sup A$ in Proposition 4.3.1, we know that

$$\begin{aligned} &\text{for all } \epsilon_1 > 0, \\ &\quad \text{there exists } a \in A, \\ &\quad\quad a > \sup A - \epsilon_1. \end{aligned} \tag{4.7.1}$$

Choose $\epsilon_1 := \epsilon/2$ in (4.7.1). Then

we find that there exists an $a \in A$ such that $a > \sup A - \epsilon/2$. Similarly, there exists a $b \in B$ such that $b > \sup B - \epsilon/2$.

Choose $c := a + b$. Then

$$c = a + b > \sup A - \epsilon/2 + \sup B - \epsilon/2 = \sup A + \sup B - \epsilon = M - \epsilon.$$

Item (v) was shown in the proof of Lemma 4.2.8.

The other items are left as exercises. □

A remark about the presentation of the proof above: The text in the lighter gray inset around statement (4.7.1) is optional. Once we get more skilled in proving, we will usually omit it. The omission makes the proof a bit shorter and perhaps easier to read, while if you have seen this type of argument a few times, you will know what lines to insert to make the proof more detailed.

4.8 Bernoulli's inequality

You may recall that for all $a, b \in \mathbb{R}$, the power $(a+b)^n$ can be expanded using Newton's binomial coefficients

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

If a and b are positive, we can get some useful inequalities by just leaving out some terms on the right hand side. We will use this technique repeatedly in the lecture notes.

We can even get some inequalities if $b = 1$ and $a \geq -1$. One such inequality is Bernoulli's inequality.

Proposition 4.8.1 (Bernoulli's inequality)**.** For all $a \geq -1$, and all $n \in \mathbb{N}$,

$$(1+a)^n \geq 1 + na.$$

Proof. Let $a \geq -1$. We prove Bernoulli's inequality by induction on n. For $n = 0$, we have

$$(1+a)^0 = 1 \geq 1 = 1 + 0 \cdot a$$

so the inequality holds.

Suppose the inequality holds for $n = k$ for some $k \in \mathbb{N}$. Then we would like to show the inequality for $n = k+1$. We find

$$\begin{aligned}(1+a)^{k+1} &= (1+a)^k(1+a) \\ &\geq (1+ka)(1+a) \\ &= 1 + (k+1)a + ka^2 \\ &\geq 1 + (k+1)a.\end{aligned}$$

□

4.9 Exercises

4.9.1 Blue exercises

Exercise 4.9.1. Show that for all $a, b \in \mathbb{R}$, if $a < b$ then

$$\inf[a,b) = a.$$

Exercise 4.9.2. Prove Proposition 4.4.3.

Exercise 4.9.3. Show that

$$\sup[0,4) = 4$$

4.9.2 Orange exercises

Exercise 4.9.4. Prove Proposition 4.3.1.

Exercise 4.9.5. Prove item (iii) of Proposition 4.7.1.

Exercise 4.9.6. Prove item (vi) of Proposition 4.7.1.

Chapter 5

Sequences

This chapter will introduce *sequences*. Sequences are extra important since they can be used to determine whether metric spaces or functions between metric spaces satisfy certain properties.

5.1 A sequence is a function from the natural numbers

Let X be a set, for instance $X = \mathbb{R}$. A sequence in X is just a function from the natural numbers $\mathbb{N}$ to X. We will use the convention that the natural numbers $\mathbb{N}$ include 0.

Definition 5.1.1. Let X be a set. A *sequence* $a : \mathbb{N} \to X$ in X is a function from the natural numbers $\mathbb{N}$ to X.

Example 5.1.2. Consider the set $X := \{\mathsf{red}, \mathsf{yellow}, \mathsf{blue}\}$ of primary colors. The function $a : \mathbb{N} \to X$ defined by

$$a(n) := \begin{cases} \mathsf{blue}, & \text{if } n \text{ is odd,} \\ \mathsf{red}, & \text{if } n \text{ is even,} \end{cases}$$

is an example of a sequence in X.

We really want to stress the point of view here that a sequence, for instance a sequence $a : \mathbb{N} \to \mathbb{R}$ of real numbers, is really a function. In practice, we often write $(a_n)_{n\in\mathbb{N}}$, (a_n), $(a^{(n)})$, or

$$a_0, a_1, a_2, a_3, \ldots$$

For $k \in \mathbb{N}$, the term a_k is called an *element* of the sequence. It is also referred to as the kth element. Moreover, k is called the *index* of the element a_k.

5.2 Terminology around sequences

Definition 5.2.1 (bounded sequences)**.** Let (X, dist) be a metric space. We say a sequence $a : \mathbb{N} \to X$ is *bounded* if

$$\begin{aligned}
&\text{there exists } q \in X, \\
&\quad \text{there exists } M > 0, \\
&\quad\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad\quad \mathsf{dist}(a_n, q) \leq M.
\end{aligned}$$

In normed linear spaces, we can use a simpler criterion to check whether a sequence is bounded. That is the content of the following proposition.

Proposition 5.2.2. Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. The sequence a is bounded if and only if

$$\begin{aligned}
&\text{there exists } M > 0, \\
&\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad \|a_n\| \leq M.
\end{aligned}$$

The proof is not difficult, but it is an excellent opportunity to go through it slowly, carefully following the proof expectations formulated in Chapter 3.

Proof. We first show the "if" part of the statement. So we assume that

$$\begin{aligned}&\text{there exists } M_1 > 0,\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad \|a_n\| \leq M_1\end{aligned}$$

and we need to show that

$$\begin{aligned}&\text{there exists } q \in V,\\ &\quad \text{there exists } M > 0,\\ &\qquad \text{for all } n \in \mathbb{N},\\ &\qquad\quad \|a_n - q\| \leq M.\end{aligned}$$

Because the statement we need to show is of the form "there exists a $q \in V$ such that ...", we need to choose an appropriate $q \in V$.

We choose $q := 0$.

Next, we need to show "there exists an $M > 0$ such that ...", so we need to identify an appropriate M.

For this, we first conclude by our assumption that there exists an $M_1 > 0$ such that for all $n \in \mathbb{N}$, it holds that $\|a_n\| \leq M_1$.

We choose $M := M_1$.

We need to show "for all $n \in \mathbb{N}$, ...". Therefore the next line of the proof is:

Let $n \in \mathbb{N}$.

Finally we need to show that $\|a_n - q\| \leq M$.

But $\|a_n - q\| = \|a_n - 0\| = \|a_n\|$ and we already know that $\|a_n\| \leq M_1$ and we chose $M := M_1$ so that indeed $\|a_n\| \leq M$.

Now we will show the "only if" part of the statement. We assume that

$$\begin{aligned}&\text{there exists } q \in V,\\ &\quad \text{there exists } M > 0,\\ &\quad\quad \text{for all } n \in \mathbb{N},\\ &\quad\quad\quad \|a_n - q\| \leq M\end{aligned}$$

and we need to show that

$$\begin{aligned}&\text{there exists } M_1 > 0,\\ &\quad \text{for all } n \in \mathbb{N},\\ &\quad\quad \|a_n\| \leq M_1.\end{aligned}$$

Can you see the template we should be using?

Choose $M_1 := M + \|q\|$.

Let $n \in \mathbb{N}$.

Then by the triangle inequality

$$\begin{aligned}\|a_n\| &= \|a_n - q + q\|\\ &\leq \|a_n - q\| + \|q\|\\ &\leq M + \|q\| = M_1.\end{aligned}$$

□

5.3 Convergence of sequences

Remember that analysis is for a large part about making rigorous statements about the approximate behavior of functions? The following definition is a rigorous statement about the behavior of a sequence $a : \mathbb{N} \to X$, where (X, dist) a metric space. The definition is a precise version of the following approximate statement: for large n, the distance between a_n and p is small.

Definition 5.3.1. Let (X, dist) be a metric space. We say that a sequence

$a : \mathbb{N} \to X$ converges to a point $p \in X$ if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad \mathsf{dist}(a_n, p) < \epsilon.\end{aligned}$$

We sometimes write

$$\lim_{n\to\infty} a_n = p$$

to express that the sequence (a_n) converges to p.

Example 5.3.2. Let's see what this definition looks like when the metric space (X, dist) is $(\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$, where by $\mathsf{dist}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ we always mean the standard distance on $\mathbb{R}$ given by

$$\mathsf{dist}_{\mathbb{R}}(x, y) = |x - y|.$$

A sequence $a : \mathbb{N} \to \mathbb{R}$ then converges to $L \in \mathbb{R}$ if and only if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad |a_n - L| < \epsilon.\end{aligned}$$

Definition 5.3.3. Let (X, dist) be a metric space. A sequence $a : \mathbb{N} \to X$ is called *divergent* if it is not convergent.

5.4 Examples and limits of simple sequences

Proposition 5.4.1 (The constant sequence)**.** Let (X, dist) be a metric space. Let $p \in X$ and assume that the sequence (a_n) is given by $a_n = p$

for every $n \in \mathbb{N}$. We also say that (a_n) is a constant sequence. Then $\lim_{n\to\infty} a_n = p$.

The proof of Proposition 5.4.1 is the content of Blue Exercise 5.9.1.

Example 5.4.2 (a standard limit)**.** Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence such that $a_n = 1/n$ for $n \geq 1$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to 0.

Proof. Let $\epsilon > 0$. Choose $N := \lceil 1/\epsilon \rceil + 1$. Take $n \geq N$. Then

$$\mathsf{dist}_{\mathbb{R}}(a_n, 0) = |a_n - 0| = |1/n| = 1/n \leq 1/N < \epsilon.$$

□

5.5 Uniqueness of limits

Proposition 5.5.1 (uniqueness of limits)**.** Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence in X. Assume that $p, q \in X$ and assume that

$$\lim_{n\to\infty} a_n = p \quad \text{and} \quad \lim_{n\to\infty} a_n = q.$$

Then $p = q$.

Proof. We argue by contradiction. Suppose $p \neq q$.

Set $\epsilon := \mathsf{dist}(p, q)/2 > 0$. Since (a_n) converges to p, we know by the definition of convergence that

$$\begin{aligned}
&\text{for all } \epsilon_1 > 0, \\
&\quad \text{there exists } N_1 \in \mathbb{N}, \\
&\quad\quad \text{for all } n \geq N_1, \\
&\quad\quad\quad \mathsf{dist}(a_n, p) < \epsilon_1.
\end{aligned} \tag{5.5.1}$$

Choose $\epsilon_1 := \epsilon$. Then we know that

there exists an $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$,

$$\mathsf{dist}(a_n, p) < \epsilon.$$

Since (a_n) converges to q, there exists an $N_2 \in \mathbb{N}$ such that for every $n \geq N_2$,

$$\mathsf{dist}(a_n, q) < \epsilon.$$

Choose $N := \max(N_1, N_2)$. Then

$$\mathsf{dist}(p, q) \leq \mathsf{dist}(p, a_N) + \mathsf{dist}(a_N, q) < 2\epsilon = \mathsf{dist}(p, q)$$

which is a contradiction. □

Again, the lighter inset around statement (5.5.1) above denotes optional text. As we progress in these notes, we will more and more often omit it, but in the early stages it shows the argumentation a bit more clearly.

5.6 More properties of convergent sequences

Proposition 5.6.1. Let (X, dist) be a metric space and suppose that $a : \mathbb{N} \to X$ is a sequence. Let $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the real-valued sequence

$$n \mapsto \mathsf{dist}(a_n, p)$$

converges to 0 in $\mathbb{R}$.

Proof. We define the real-valued sequence (b_n) by

$$b_n := \mathsf{dist}(a_n, p).$$

We need to show that (a_n) converges to p if and only if (b_n) converges to 0.

We first show "only if". So assume that the sequence $a : \mathbb{N} \to X$ converges to p.

Let $\epsilon > 0$. Since (a_n) converges to p, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\mathsf{dist}(a_n, p) < \epsilon.$$

Choose $N := N_0$. Let $n \geq N$. Then,

$$0 \leq \mathsf{dist}(a_n, p) < \epsilon$$

so that indeed

$$\mathsf{dist}_{\mathbb{R}}(\mathsf{dist}(a_n, p), 0) = |\mathsf{dist}(a_n, p)| < \epsilon.$$

We now show the "if" part of the statement. We assume that (b_n) converges to 0 and we need to show that (a_n) converges to p.

Let $\epsilon > 0$.

Since (b_n) converges to zero, there exists an N_1 such that for all $n \geq N_1$,

$$b_n = \mathsf{dist}_{\mathbb{R}}(b_n, 0) < \epsilon$$

where the first equality holds because $b_n \geq 0$ for all $n \in \mathbb{N}$.

Choose $N_0 := N_1$.

Let $n \geq N_0$. Then

$$\mathsf{dist}(a_n, p) = b_n < \epsilon.$$

□

Proposition 5.6.2 (Convergent sequences are bounded)**.** Let (X, dist) be a metric space. Let $a : \mathbb{N} \to X$ be a sequence in X converging to

$p \in X$. Then the sequence $a : \mathbb{N} \to X$ is bounded.

Proof. We need to show that

$$\begin{aligned}&\text{there exists } q \in X,\\ &\quad \text{there exists } M > 0,\\ &\quad\quad \text{for all } n \in \mathbb{N},\\ &\quad\quad\quad \mathsf{dist}(a_n, q) \leq M.\end{aligned}$$

Choose $q := p$. Because the sequence (a_n) converges to p,

we know that

$$\begin{aligned}&\text{for all } \epsilon_1 > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad \mathsf{dist}(a_n, p) < \epsilon_1.\end{aligned} \tag{5.6.1}$$

Choose $\epsilon_1 := 1$ in (5.6.1). Then

there exists an $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\mathsf{dist}(a_n, p) < 1.$$

Choose

$$M := \max(\mathsf{dist}(a_1, p), \dots, \mathsf{dist}(a_{N-1}, p), 1).$$

Let $n \in \mathbb{N}$. We need to show that $\mathsf{dist}(a_n, p) \leq M$. We make a case distinction.

In the case $n \leq N - 1$, then

$$\mathsf{dist}(a_n, p) \leq \max(\mathsf{dist}(a_1, p), \dots, \mathsf{dist}(a_{N-1}, p), 1) = M.$$

In the case $n \geq N$, then

$$\operatorname{dist}(a_n, p) < 1 \leq M.$$

□

The following proposition is one of the strongest statements in this chapter, and if you look carefully you can show a few other propositions in this chapter by just appealing to the next proposition.

Proposition 5.6.3. Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences. Let $p \in X$ and suppose that $\lim_{n\to\infty} a_n = p$. Then $\lim_{n\to\infty} b_n = p$ if and only if

$$\lim_{n\to\infty} \operatorname{dist}(a_n, b_n) = 0.$$

Proof. We first show the "only if" direction. Assume $\lim_{n\to\infty} b_n = p$. We need to show that

$$\lim_{n\to\infty} \operatorname{dist}(a_n, b_n) = 0.$$

Let $\epsilon > 0$.

Because $\lim_{n\to\infty} a_n = p$, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\operatorname{dist}(a_n, p) < \frac{\epsilon}{2}.$$

Because $\lim_{n\to\infty} b_n = p$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\operatorname{dist}(b_n, p) < \frac{\epsilon}{2}.$$

Choose $N := \max(N_0, N_1)$.

Let $n \geq N$. Because then $n \geq N_0$ and $n \geq N_1$, we know

$$\operatorname{dist}(a_n, b_n) \leq \operatorname{dist}(a_n, p) + \operatorname{dist}(p, b_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We now show the "if" direction. Assume $\lim_{n\to\infty} \mathsf{dist}(a_n, b_n) = 0$. We need to show that $\lim_{n\to\infty} \mathsf{dist}(b_n, p) = 0$.

Let $\epsilon > 0$.

Because $\lim_{n\to\infty} a_n = p$, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\mathsf{dist}(a_n, p) < \frac{\epsilon}{2}.$$

Because $\lim_{n\to\infty} \mathsf{dist}(a_n, b_n) = 0$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$\mathsf{dist}(a_n, b_n) = \mathsf{dist}_{\mathbb{R}}(\mathsf{dist}(a_n, b_n), 0) < \frac{\epsilon}{2}$$

Choose $N := \max(N_0, N_2)$.

Let $n \geq N$.

Because $n \geq N_0$ and $n \geq N_2$, we find

$$\begin{aligned}
\mathsf{dist}(b_n, p) &\leq \mathsf{dist}(b_n, a_n) + \mathsf{dist}(a_n, p) \\
&= \mathsf{dist}(a_n, b_n) + \mathsf{dist}(a_n, p) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

I've added the next corollary later in the year 2021-2022 just for your convenience, just to highlight a consequence of the previous proposition.

Proposition 5.6.4 (Eventually equal sequences have the same limit)**.** Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences such that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n = b_n.$$

Then the sequence $a : \mathbb{N} \to X$ converges if and only if the sequence

$b : \mathbb{N} \to X$ converges. If the sequences converge, they have the same limit.

5.7 Limit theorems for sequences taking values in a normed vector space

If we want to show that a sequence converges, or if we want to compute its limit, we don't always want to go back to the formal definition of a limit. Instead, we can use a whole collection of theorems such as the next. The first part of the next theorem says that the sum of two convergent sequences (taking values in normed vector spaces, after all we need to have a possibility to add elements) is itself convergent. Theorems like these are called *limit theorems* or *limit laws*.

Theorem 5.7.1. Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Assume that the limit $\lim_{n\to\infty} a_n$ exists and is equal to $p \in V$ and that the limit $\lim_{n\to\infty} b_n$ exists and is equal to $q \in V$. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $\mu \in \mathbb{R}$. Assume that $\lim_{n\to\infty} \lambda_n = \mu$. Then

i. The limit $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $p + q$.

ii. The limit $\lim_{n\to\infty}(\lambda_n a_n)$ exists and is equal to μp.

Proof. We leave the proof of (i) as an exercise and prove (ii), which is a bit more difficult.

We need to show that $\lim_{n\to\infty}(\lambda_n a_n) = \mu p$, i.e. we need to show that

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad\text{there exists } N \in \mathbb{N},\\
&\quad\quad\text{for all } n \geq N,\\
&\quad\quad\quad \mathsf{dist}_{\|\cdot\|}(\lambda_n a_n, \mu p) < \epsilon.
\end{aligned}$$

Let $\epsilon > 0$.

Since the sequence $\lambda : \mathbb{N} \to \mathbb{R}$ is convergent, it is bounded. Therefore there exists an $M > 0$ such that for all $n \in \mathbb{N}$,

$$|\lambda_n| \leq M.$$

Since $\lim_{n\to\infty} \lambda_n = \mu$, there exists an $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$,

$$\mathsf{dist}_{\mathbb{R}}(\lambda_n, \mu) = |\lambda_n - \mu| < \frac{\epsilon}{2(\|p\| + 1)}.$$

We have divided here by $(\|p\| + 1)$ rather than $\|p\|$ to not run into trouble (i.e. to not divide by zero) when $\|p\| = 0$.

Since $\lim_{n\to\infty} a_n = p$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\mathsf{dist}_{\|\cdot\|}(a_n, p) = \|a_n - p\| < \frac{\epsilon}{2M}.$$

Choose $N := \max(N_0, N_1)$.

Let $n \geq N$. Then

$$\begin{aligned}
\|\lambda_n a_n - \mu p\| &= \|\lambda_n a_n - \lambda_n p + \lambda_n p - \mu p\| \\
&= \|\lambda_n(a_n - p) + (\lambda_n - \mu)p\| \\
&\leq \|\lambda_n(a_n - p)\| + \|(\lambda_n - \mu)p\| \\
&= |\lambda_n|\|a_n - p\| + |\lambda_n - \mu|\|p\| \\
&\leq M\|a_n - p\| + |\lambda_n - \mu|\|p\| \\
&< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2(\|p\| + 1)} \cdot \|p\| \\
&< \epsilon.
\end{aligned}$$

□

5.8 Index shift

The next proposition is another example of a theorem that allows you to conclude the existence of a certain limit without going back to the formal definition. You can use it when you know that a sequence converges, to conclude that the same sequence but with index shifted is also convergent.

Proposition 5.8.1 (Index shift)**.** Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence in X. Let $k \in \mathbb{N}$ and $p \in X$. Then the sequence (a_n) converges to p if and only if the sequence $(a_{n+k})_n$ (i.e. the sequence $n \mapsto a_{n+k}$) converges to p.

The proof of Proposition 5.8.1 is the topic of Blue Exercise 5.9.2.

5.9 Exercises

5.9.1 Blue exercises

Exercise 5.9.1. Prove Proposition 5.4.1.

Exercise 5.9.2. Prove Proposition 5.8.1.

5.9.2 Orange exercises

Exercise 5.9.3. Prove item (i) of Theorem 5.7.1.

Exercise 5.9.4. Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a bounded sequence in X. Let $p \in X$. Define also the sequence $s : \mathbb{N} \to \mathbb{R}$ by

$$s_k := \sup\{\mathsf{dist}(a_l, p) \mid l \in \mathbb{N},\ l \geq k\}.$$

Show that $\lim_{n\to\infty} a_n = p$ if and only if

$$\inf_{k\in\mathbb{N}} s_k = 0.$$

Here, $\inf_{k\in\mathbb{N}} s_k$ is shorthand for

$$\inf_{k\in\mathbb{N}} s_k := \inf\{s_k \mid k \in \mathbb{N}\}.$$

Hint: Due to notation, this exercise may look intimidating. However, if you let yourself be guided by the best practices and if you use the alternative characterization of the infimum, it becomes quite a bit easier than it looks.

Chapter 6

Real-valued sequences

In this chapter, we specify to sequences $a : \mathbb{N} \to \mathbb{R}$ that take values in $\mathbb{R}$. The main additional aspect with respect to the previous chapter is the fact that $\mathbb{R}$ has an order $(\leq)$. The most important result of the chapter is that monotone bounded sequences are always convergent.

6.1 Terminology

Because the real numbers come with an order $(\leq)$, we can define increasing, decreasing and monotone sequences.

Definition 6.1.1 (increasing, decreasing and monotone sequences)**.** We say a sequence (a_n) is *increasing* if for every $n \in \mathbb{N}$, $a_{n+1} \geq a_n$. We say it is *strictly increasing* if for every $n \in \mathbb{N}$, $a_{n+1} > a_n$. Similarly, we say a sequence a_n is *decreasing* if for every $n \in \mathbb{N}$, $a_{n+1} \leq a_n$ and we say it is *strictly decreasing* if for every $n \in \mathbb{N}$, $a_{n+1} < a_n$. We finally say a sequence is (strictly) *monotone* if it is either (strictly) increasing or (strictly) decreasing.

The main result of this chapter is that monotone, bounded sequences are convergent. In order to introduce what it means for a sequence to be bounded, we first introduce upper and lower bounds.

Definition 6.1.2 (upper bound and lower bound for a sequence)**.** We say that a number $M \in \mathbb{R}$ is an *upper bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}, \\ a_n \leq M.$$

We say a number $m \in \mathbb{R}$ is a *lower bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}, \\ m \leq a_n.$$

Definition 6.1.3. We say a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that M is an upper bound for a.

We say a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded below* if there exists an $m \in \mathbb{R}$ such that m is a lower bound for a.

In the previous chapter, we have already defined what it means for a sequence to be bounded. The next proposition relates the two definitions to each other.

Proposition 6.1.4. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded (in the sense of Definition 5.2.1) if and only if it is both bounded above and bounded below (according to Definition 6.1.3).

6.2 Monotone, bounded sequences are convergent

Theorem 6.2.1. Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) is convergent and

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad \Big(= \sup \{a_n \mid n \in \mathbb{N}\} \Big)$$

Proof. Because the sequence (a_n) is bounded from above, we know that the supremum

$$\sup_{n \in \mathbb{N}} a_n$$

exists. To not get too lengthy expressions, we write

$$L := \sup_{n \in \mathbb{N}} a_n.$$

We need to show that for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - L| < \epsilon.$$

Let $\epsilon > 0$. Then, by the definition of the supremum, there exists a $k \in \mathbb{N}$ such that

$$L - \epsilon < a_k.$$

Choose $N := k$. Let $n \geq N$. Because the sequence (a_ℓ) is increasing, we find that

$$a_n \geq a_N = a_k > L - \epsilon.$$

Because of the definition of L, we also know that $a_n \leq L < L + \epsilon$. Summarizing,

$$|a_n - L| < \epsilon.$$

□

Theorem 6.2.2. Let (a_n) be a decreasing sequence that is bounded from below. Then (a_n) is convergent and

$$\lim_{n \to \infty} a_n = \inf_{n \in \mathbb{N}} a_n.$$

6.3 Limit theorems

If you want to show that more complicated limits exist, and if you want to compute their value, you wouldn't want to have to use the definition all the time. Instead there are much more efficient methods to show that limits exist. They are called limit theorems.

Theorem 6.3.1 (Limit theorems for real-valued sequences)**.** Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be two converging sequences, and let $c, d \in \mathbb{R}$ be real numbers such that

$$\lim_{n\to\infty} a_n = c \quad \text{and} \quad \lim_{n\to\infty} b_n = d.$$

Then

i. The limit $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $c + d$.

ii. The limit $\lim_{n\to\infty}(a_n b_n)$ exists and is equal to $c \cdot d$.

iii. If $d \neq 0$, then the limit $\lim_{n\to\infty}(a_n / b_n)$ exists and is equal to c/d.

iv. For every nonnegative integer $m \in \mathbb{N}$, the limit $\lim_{n\to\infty}(a_n)^m$ exists and is equal to c^m.

v. If for every $n \in \mathbb{N}$, the number a_n is nonnegative, then for every positive integer $k \in \mathbb{N} \setminus \{0\}$, the limit $\lim_{n\to\infty}(a_n)^{1/k}$ exists and is equal to $c^{1/k}$.

Proof. Let us aim to prove item (v). We first show the statement for $c = 1$, then for every $k \in \mathbb{N}_+$, the limit $\lim_{n\to\infty} a_n^{1/k}$ exists and is equal to 1.

Let $\epsilon > 0$. Define $\epsilon_0 := \min(\epsilon, 1/2)$. (We will prefer to work with ϵ_0 over ϵ because we know that $\epsilon_0 \leq 1/2$, which will be convenient below when we want to take the kth root of $(1 - \epsilon_0)$.) Since $a_n \to c$ by assumption, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$,

$$|a_n - 1| < \epsilon_0.$$

Let $n \geq n_0$. Then

$$1 - \epsilon_0 < a_n < 1 + \epsilon_0$$

and therefore

$$1 - \epsilon_0 < (1 - \epsilon_0)^{1/k} < (a_n)^{1/k} < (1 + \epsilon_0)^{1/k} < 1 + \epsilon_0.$$

Hence,

$$\left|(a_n)^{1/k} - 1\right| < \epsilon_0 \leq \epsilon.$$

Now suppose $c > 0$. Then we define a new sequence $\tilde{a} : \mathbb{N} \to \mathbb{R}$ by $\tilde{a}_n = a_n/c$. By item (iii) it holds that the sequence $\tilde{a} : \mathbb{N} \to \mathbb{R}$ converges to 1. By the previous part of the proof, we find that

$$\lim_{n\to\infty} (\tilde{a}_n)^{1/k} = 1.$$

Note that $a_n = \tilde{a}_n \cdot c$, so that item (ii) implies that

$$\lim_{n\to\infty} (a_n)^{1/k} = \lim_{n\to\infty} \left((\tilde{a}_n)^{1/k} c^{1/k}\right) = \left(\lim_{n\to\infty} (\tilde{a}_n)^{1/k}\right) \cdot \left(\lim_{n\to\infty} c^{1/k}\right) = c^{1/k}.$$

Finally, we consider the case $c = 0$. Let $\epsilon > 0$. Then we may choose an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n| < \epsilon^k.$$

Let $n \geq N$. Then,

$$|(a_n)^{1/k} - 0^{1/k}| = |a_n|^{1/k} < \epsilon.$$

□

Example 6.3.2. Consider the sequence $a : \mathbb{N} \to \mathbb{R}$ defined (for $n \geq 1$), by

$$a_n := 3 + \frac{1}{n^2}$$

We claim that the sequence $a : \mathbb{N} \to \mathbb{R}$ converges and that the limit equals 3.

We will use limit theorems to prove this claim.

We know that the limit of the sequence $n \mapsto 1/n$ exists as this is a standard limit (see Example 5.4.2).

The text here is optional, as on the one hand it is really required for a rigorous proof but on the other hand the amount to write down would be way too much for more involved limits.

By the limit theorem for powers, Theorem 6.3.1, item (iv), it follows that the sequence $n \mapsto (1/n)^2$ also converges.

We also know that the sequence $n \mapsto 3$ converges, as this is a constant sequence, see Proposition 5.4.1.

By the limit theorem for the sum and the power, we conclude that the sequence $a : \mathbb{N} \to \mathbb{R}$ also converges and

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(3 + \frac{1}{n^2}\right)\\ &= \lim_{n\to\infty} 3 + \lim_{n\to\infty}\left(\frac{1}{n}\right)^2\\ &= 3 + \left(\lim_{n\to\infty}\frac{1}{n}\right)^2\\ &= 3 + 0^2 = 3.\end{aligned}$$

(What one really needs to do, when leaving out the optional text, is to read the above chain of equalities from back to front, and making sure that all steps are justified. In particular, it is extremely important to verify that all involved limits exist.)

Example 6.3.3. Consider the sequence $a : \mathbb{N} \to \mathbb{R}$ defined by

$$a_n := \frac{3n^2 + 5n + 9}{2n^2 + 3n + 7}$$

We claim that the sequence $a : \mathbb{N} \to \mathbb{R}$ converges, and that the limit equals $3/2$.

When confronted with a sequence that is given as a fraction of two terms, the first thing to do is to divide numerator and denominator by the fastest growing term in n. In this case, we need to divide by n^2. We get

$$a_n := \frac{3 + 5\frac{1}{n} + 9\frac{1}{n^2}}{2 + 3\frac{1}{n} + 7\frac{1}{n^2}}.$$

We would like to use the limit theorem for quotients, namely Theorem 6.3.1, item (iii). However, to apply this limit theorem, we should really make sure that the limit of numerator and denominator exist, and that the limit of the denominator is not equal to 0. Whereas the previous example had optional text to justify all the steps, here we leave it out. We will use the strategy described at the end of the previous example, to read chains of equalities backwards while making sure all involved limits exist.

Note that the limit

$$\lim_{n\to\infty} \frac{1}{n}$$

exists, and equals 0, as this is the standard limit from Example 5.4.2.

By the limit theorems it follows that

$$\begin{aligned}\lim_{n\to\infty} \left(2 + 3\frac{1}{n} + 7\frac{1}{n^2}\right) &= \lim_{n\to\infty} 2 + 3 \lim_{n\to\infty} \frac{1}{n} + 7 \left(\lim_{n\to\infty} \frac{1}{n}\right)^2 \\ &= 2 + 0 + 0 = 2.\end{aligned}$$

(Here, we read the chain of equalities backwards to make sure every step is justified.) Because $2 \neq 0$, we may now apply the limit theorem for the quotient, and find

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{3+5\frac{1}{n}+9\frac{1}{n^2}}{2+3\frac{1}{n}+7\frac{1}{n^2}} \\
&= \frac{\lim_{n\to\infty}\left(3+5\frac{1}{n}+9\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(2+3\frac{1}{n}+7\frac{1}{n^2}\right)} \\
&= \frac{\lim_{n\to\infty} 3 + 5\lim_{n\to\infty}\frac{1}{n} + 9\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}{2} \\
&= \frac{3+0+0}{2} = 3/2.
\end{aligned}$$

(Again we read the chain of equalities backwards to make sure every step is justified.)

6.4 The squeeze theorem

Theorem 6.4.1 (Squeeze theorem)**.** Let $a, b, c : \mathbb{N} \to \mathbb{R}$ be three sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n \leq b_n \leq c_n$$

and assume $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ for some $L \in \mathbb{R}$. Then the limit $\lim_{n\to\infty} b_n$ exists and is equal to L.

Proof. Take three arbitrary sequences $a, b, c : \mathbb{N} \to \mathbb{R}$, and assume that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n \leq b_n \leq c_n$$

and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ for some $L \in \mathbb{R}$. We need to show that

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N_0 \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N_0,\\ &\quad\quad\quad |b_n - L| < \epsilon.\end{aligned}$$

Take $\epsilon > 0$ arbitrary. Since $\lim_{n\to\infty} a_n = L$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - L| < \epsilon$. Since $\lim_{n\to\infty} c_n = L$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|c_n - L| < \epsilon$. Now define $N_0 := \max(N, N_1, N_2)$. Let $n \geq N_0$. Then

$$L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon$$

so that indeed, $|b_n - L| < \epsilon$. □

The squeeze theorem is a great tool to show existence of limits and to compute limits for a sequences that can easily be compared to other sequences as in the next example.

Example 6.4.2. Consider the sequence $b : \mathbb{N} \to \mathbb{R}$ defined by

$$b_n := \frac{\sin(n)}{n+1}.$$

We can use the squeeze theorem to show that

$$\lim_{n\to\infty} b_n = 0.$$

Because for every $n \in \mathbb{N}$, it holds that

$$-1 \leq \sin(n) \leq 1,$$

we know that

$$-\frac{1}{n+1} \leq \frac{\sin(n)}{n+1} \leq \frac{1}{n+1}.$$

We know by the standard limit in Example 5.4.2 and by index shift (Proposition 5.8.1) that

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

and we know by the limit theorems that then also

$$\lim_{n\to\infty} -\frac{1}{n+1} = 0.$$

It follows by the squeeze theorem that

$$\lim_{n\to\infty} b_n = 0$$

as well.

6.5 Divergence to ∞ and $-\infty$

Definition 6.5.1. We say a sequence (a_n) *diverges to* ∞, and write

$$\lim_{n\to\infty} a_n = \infty$$

if

$$\begin{aligned} &\text{for all } M \in \mathbb{R}, \\ &\quad \text{there exists } N \in \mathbb{N}, \\ &\quad\quad \text{for all } n \geq N, \\ &\quad\quad\quad a_n > M. \end{aligned}$$

Similarly, we say a sequence (a_n) *diverges to* $-\infty$, and write

$$\lim_{n\to\infty} a_n = -\infty$$

if

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad a_n < M.\end{aligned}$$

Proposition 6.5.2. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} a_n = \infty.$$

Then the sequence (a_n) is bounded from below.

Similarly, let $b : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} b_n = -\infty.$$

Then the sequence (b_n) is bounded from above.

6.6 Limit theorems for improper limits

Theorem 6.6.1. Let $a, b, c, d : \mathbb{N} \to \mathbb{R}$ be four sequences such that

$$\lim_{n\to\infty} a_n = \infty \quad \text{and} \quad \lim_{n\to\infty} c_n = -\infty,$$

the sequence (b_n) is bounded from below and the sequence (d_n) is bounded from above. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence bounded below by some $\mu > 0$. Then

i. $\lim_{n\to\infty}(a_n + b_n) = \infty$.

ii. $\lim_{n\to\infty}(c_n + d_n) = -\infty$.

iii. $\lim_{n\to\infty}(\lambda_n a_n) = \infty$.

iv. $\lim_{n\to\infty}(\lambda_n c_n) = -\infty$.

Proposition 6.6.2. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $b : \mathbb{N} \to (0, \infty)$ be a real-valued sequence taking on only strictly positive values. Then

i. $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}(-a_n) = -\infty$.

ii. $\lim_{n\to\infty} b_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

6.7 Standard sequences

6.7.1 Geometric sequence

Proposition 6.7.1 (Standard limit of geometric sequence)**.** Let $q \in \mathbb{R}$. The sequence (a_n) defined by $a_n := q^n$ for $n \in \mathbb{N}$

- converges to 0 if $q \in (-1, 1)$
- converges to 1 if $q = 1$
- diverges to ∞ if $q > 1$
- diverges, but not to ∞ or $-\infty$ if $q \leq -1$.

Proof. If $q = 0$ then it is clear that the sequence $n \mapsto q^n$ converges to 0. If $q = 1$ then it is clear that the sequence $n \mapsto q^n$ converges to 1.

If $q \in (0, 1)$, then the sequence (a_n) is decreasing and bounded from below by 0. Therefore, the sequence (a_n) is convergent. Moreover,

$$s = \lim_{n\to\infty} a_{n+1} = q \lim_{n\to\infty} a_n = qs$$

so $s = 0$.

If $q \in (-1, 0)$, then
$$-|q|^n \leq q^n \leq |q|^n$$
and it follows from the squeeze theorem and the previous part of the proof that $\lim_{n\to\infty} q^n = 0$.

Now assume $q > 1$. We will show that the sequence $n \mapsto q^n$ diverges to ∞. Let $M \in \mathbb{R}$.

Then we can write $q := 1 + b$ for some $b > 0$.

Choose $N := M\lceil 1/b \rceil$. Let $n \geq N$. By the Bernoulli inequality, it follows that

$$q^n = (1+b)^n \geq 1 + nb \geq 1 + Nb \geq 1 + M\frac{1}{b}b > M.$$

Finally, we consider the case $q \leq -1$. Suppose the sequence $n \mapsto (-q)^n$ converges to some $r \geq 0$. Then we may choose an $N \in \mathbb{N}$ such that for all $n \geq N$, $|(-q)^n - r| < 1/2$. Therefore,

$$r - (-q)^{2N+1} < 1/2$$

which is a contradiction because $(-q)^{2N+1} = -q^{2N+1} \leq -1$.

In a similar way, we can rule out that $n \mapsto (-q)^n$ converges to some $r < 0$, or diverges to ∞ or diverges to $-\infty$. □

6.7.2 The nth root of n

Proposition 6.7.2 (Standard limit of $(\sqrt[n]{n})$)**.** The sequence (b_n) defined as $b_n := \sqrt[n]{n}$ converges to 1.

Proof. We write $b_n = 1 + d_n$, where $d_n \geq 0$. By the limit theorems, it suffices to show that $\lim_{n\to\infty} d_n = 0$. Note that $b_n^n = (1 + d_n)^n = n$. Let $n \geq 2$. Then

$$1 + \binom{n}{2} d_n^2 \leq \sum_{k=0}^{n} \binom{n}{k} d_n^k = (1 + d_n)^n = n.$$

Therefore,

$$0 < d_n^2 \leq \frac{2}{n}$$

and

$$0 < d_n \leq \sqrt{\frac{2}{n}}.$$

The limit of the left-hand side is zero, and by the limit theorems, we know that the limit of the right-hand-side is 0 as well. It follows by the squeeze theorem that $\lim_{n\to\infty} d_n = 0$. $\square$

Corollary 6.7.3. Let $a > 0$. Then the sequence (b_n) defined by $b_n := \sqrt[n]{a}$ converges to 1.

6.7.3 The number e

In this chapter we are going to introduce the number e as follows. We will first define the sequence

$$a_n := \left(1 + \frac{1}{n}\right)^n$$

We will show that this sequence is bounded and increasing. It therefore has a limit value, and that limit value is called e.

Lemma 6.7.4. The sequence (a_n) defined by $a_n := (1 + 1/n)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is strictly increasing.

Proof. We need to show that for all $n \in \mathbb{N} \setminus \{0\}$, $a_n < a_{n+1}$. Let $n \in \mathbb{N}$ be larger than or equal to 1. We can just write out

$$a_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \frac{1}{k!} \frac{n!}{(n-k)!} \left(\frac{1}{n}\right)^k$$

whereas

$$a_{n+1} = \sum_{k=0}^{n+1} \frac{1}{k!} \frac{(n+1)!}{(n+1-k)!} \left(\frac{1}{n+1}\right)^k$$

How to compare these and show that $a_n < a_{n+1}$? First, because all terms in the sum are positive, we can estimate a_{n+1} from below by forgetting the last term

$$a_{n+1} > \sum_{k=0}^{n} \frac{1}{k!} \frac{(n+1)!}{(n+1-k)!} \left(\frac{1}{n+1}\right)^k$$

Next, we will show that each of the term in this sum is larger than the corresponding sum for a_n. We can see this better if we rewrite

$$\begin{aligned} a_{n+1} &> \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{n+1}{n+1}\right) \left(\frac{n+1-1}{n+1}\right) \cdots \left(\frac{n+1-(k-1)}{n+1}\right) \\ &> \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-(k-1)}{n}\right) \\ &= a_n \end{aligned}$$

□

Lemma 6.7.5. The sequence (a_n) defined by $a_n = (1+1/n)^n$ for $n \in \mathbb{N} \setminus \{0\}$ (and $a_0 = 1$) is bounded from above by 3.

Proof. Again we write

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k \\ &= \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-(k-1)}{n}\right) \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \\ &\leq 1 + 2 = 3. \end{aligned}$$

□

By the previous lemmas, the sequence

$$n \mapsto \left(1 + \frac{1}{n}\right)^n$$

converges. Let's record in the next definition that we call the limit e.

Definition 6.7.6 (Standard limit corresponding to the number e)**.** We define

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

6.7.4 Exponentials beat powers

Proposition 6.7.7. Let $a \in (1, \infty)$ and let $p \in (0, \infty)$. Then

$$\lim_{n\to\infty} \frac{n^p}{a^n} = 0.$$

Proof. Define $b := a - 1 > 0$, so that $a = 1 + b$. By the Archimedean property there exists an $M \in \mathbb{N}$ such that $M > p + 1$. Define $N := 2M$.

We now claim that for all $n \geq N$,

$$a^n \geq \frac{n^M}{2^M M!} b^M.$$

Indeed, let $n \geq N$. First note that because $n \geq 2M$, we know

$$n - M \geq \frac{n}{2}. \tag{6.7.1}$$

We now compute

$$\begin{aligned} a^n = (1+b)^n &= \sum_{k=0}^{n} \binom{n}{k} b^k \\ &\geq \binom{n}{M} b^M \\ &= n(n-1)\cdots(n-M+1)\frac{1}{M!}b^M \\ &\geq \left(\frac{n}{2}\right)^M \frac{1}{M!} b^M \end{aligned}$$

where for the last inequality we used (6.7.1). This proves our claim.

Since $M > p+1$ we find that for all $n \geq N$

$$0 < \frac{n^p}{a^n} < 2^M M! \frac{1}{b^M} \frac{1}{n}.$$

We know that

$$\lim_{n\to\infty} 2^M M! \frac{1}{b^M} \frac{1}{n} = 0$$

by limit theorems and the standard limit $\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore, it holds by the squeeze theorem (Theorem 6.4.1) that

$$\lim_{n\to\infty} \frac{n^p}{a^n} = 0.$$

□

Sequences with values in $\mathbb{R}^d$

(Note: this topic originally occurred further down the lecture notes, but I have moved it forward so we may get more concrete examples of sequences)

Proposition 6.7.8. Consider the metric space $(\mathbb{R}^d, \|\cdot\|_2)$. Let $z \in \mathbb{R}^d$ and let $x : \mathbb{N} \to \mathbb{R}^d$ be a sequence (we are going to denote this sequence also as $(x^{(n)})$). Denote by y_i the ith component of a vector $y \in \mathbb{R}^d$. Then the sequence $(x^{(n)})$ converges to z if and only if for all $i \in \{1, \ldots, d\}$, the sequence $(x_i^{(n)})$ converges to z_i.

Example 6.7.9. Consider the sequence $x : \mathbb{N} \to \mathbb{R}^2$ taking values in the normed vector space $(\mathbb{R}^d, \|\cdot\|_2)$, defined by

$$x^{(n)} := \left(\frac{1}{n}, \left(\frac{1}{2}\right)^n\right)$$

for $n \geq 1$. We use a *super*script for the index (n) of the sequence, so that we can use *sub*scripts for the components of the sequence, i.e. the first component sequence $(a_1^{(n)})$ is given by

$$x_1^{(n)} = \frac{1}{n}$$

and the second component sequence $(a_2^{(n)})$ is given by

$$x_2^{(n)} = \left(\frac{1}{2}\right)^n$$

By standard limits, we know that both

$$\lim_{n\to\infty} x_1^{(n)} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\lim_{n\to\infty} x_2^{(n)} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0.$$

By Proposition 6.7.8 it follows that the sequence $x : \mathbb{N} \to \mathbb{R}^2$ converges to

$$\left(\lim_{n\to\infty} \frac{1}{n}, \lim_{n\to\infty} \left(\frac{1}{2}\right)^n\right) = (0,0) = 0.$$

Note how in the last term we use the notation 0 for the 0-vector in the vector space $\mathbb{R}^2$.

6.8 Exercises

6.8.1 Blue exercises

Exercise 6.8.1. Prove Proposition 6.1.4.

Exercise 6.8.2. Prove item (i) of Proposition 6.6.2.

6.8.2 Orange exercises

Exercise 6.8.3. Prove item (ii) of Proposition 6.6.2.

Exercise 6.8.4. Prove the statement about the sequence (b_n) in Proposition 6.5.2.

Exercise 6.8.5. Define the sequence $x : \mathbb{N} \to \mathbb{R}$ recursively by

$$x_{n+1} := \frac{2 + x_n^2}{2x_n}$$

for $n \in \mathbb{N}$ while $x_0 = 2$. Prove that the sequence $x : \mathbb{N} \to \mathbb{R}$ converges and determine its limit.

Exercise 6.8.6. Determine whether the following sequences converge, diverge to ∞, diverge to $-\infty$ or diverge in a different way. In case the sequence converges, determine the limit.

$$\begin{aligned} a_n &:= \frac{1}{n^3} - 3 & b_n &:= \frac{5n^5 + 2n^2}{3n^5 + 7n^3 + 4} & c_n &:= n - \sqrt{n} \\ d_n &:= \frac{2^n}{n^{100}} & e_n &:= \sqrt{n^2 + n} - n & f_n &:= \sqrt[n]{3n^2} \\ g_n &:= \frac{2^n + 5n^{200}}{3^n + n^{10}} & h_n &:= (-1)^n 3^n & i_n &:= \sqrt[n]{5^n + n^2} \end{aligned}$$

Chapter 7

Series

7.1 Definitions

Definition 7.1.1. Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Let $K \in \mathbb{N}$. We say that a series

$$\sum_{n=K}^{\infty} a_n$$

is *convergent* if the associated *sequence of partial sums* $S_K : \mathbb{N} \to V$, i.e. the sequence $(S_K^n)_{n\in\mathbb{N}}$ converges. The term S_K^n is, for $n \in \mathbb{N}$, defined as

$$S_K^n := \sum_{k=K}^{n} a_k$$

If $K = 0$, we usually just write S^n or even S_n instead of S_0^n.

If the series $\sum_{n=K}^{\infty} a_n$ is convergent, the *value* of the series is by definition equal to the limit of the sequence of partial sums, i.e.

$$\sum_{k=K}^{\infty} a_k := \lim_{n\to\infty} S_K^n = \lim_{n\to\infty} \sum_{k=K}^{n} a_k.$$

7.2 Geometric series

In this and the next section we will give some examples of sums and series taking values in the real line (or to be specific the normed vector space $(\mathbb{R}, |\cdot|)$).

Proposition 7.2.1. Let $a \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Proof. We consider

$$\begin{aligned}
(1-a)\sum_{k=0}^{n} a^k &= \sum_{k=0}^{n} a^k - a\sum_{k=0}^{n} a^k \\
&= \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} \\
&= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k \\
&= 1 - a^{n+1}.
\end{aligned}$$

□

Proposition 7.2.2. Let $a \in (-1, 1)$. Then the series

$$\sum_{k=0}^{\infty} a^k$$

is convergent and has the value

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

Proof. By Proposition 7.2.1 it follows for the partial sums that

$$S_n := \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Because

$$\lim_{n \to \infty} a^{n+1} = 0$$

by index shift and Proposition 6.7.1, we find with the limit laws that $\lim_{n\to\infty} S_n$ exists as well and equals

$$\sum_{k=0}^{\infty} a^k := \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - a^{n+1}}{1 - a} = \frac{1}{1 - a}.$$

□

7.3 The harmonic series

Example 7.3.1 (Harmonic series)**.** The series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

Proof. Consider for every $\ell \in \mathbb{N}$ the partial sum

$$S_{2^\ell} = \sum_{k=1}^{2^\ell} \frac{1}{k}.$$

Note that for $k \in \{2^\ell + 1, \dots, 2^{\ell+1}\}$ we have that

$$\frac{1}{k} \geq \frac{1}{2^{\ell+1}}.$$

We can conclude that

$$S_{2^{\ell+1}} - S_{2^\ell} \geq 2^\ell \times \frac{1}{2^{\ell+1}} = \frac{1}{2}.$$

We can show by induction that

$$S_{2^\ell} \geq \frac{\ell}{2}.$$

Note also that the sequence of partial sums (S_n) is increasing. Therefore, the sequence of partial sums (S_n) diverges to infinity. $\square$

7.4 The hyperharmonic series

Example 7.4.1 (Hyperharmonic series)**.** Let $p > 1$. Then the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges.

Proof. For $\ell \in \mathbb{N} \setminus \{0\}$ we now consider the partial sums

$$S_{2^\ell - 1} = \sum_{k=1}^{2^\ell - 1} \frac{1}{k^p}.$$

For every $m \in \mathbb{N} \setminus \{0\}$ and for $k \in \{2^{m-1}, \dots, 2^m - 1\}$ we have that

$$\frac{1}{k^p} \leq \left(\frac{1}{2^{m-1}}\right)^p = \frac{1}{2^{p(m-1)}}.$$

Since there are 2^{m-1} such terms, we find that

$$S_{2^m - 1} - S_{2^{m-1} - 1} \leq 2^{m-1} \times \frac{1}{2^{p(m-1)}} = \left(\frac{1}{2^{p-1}}\right)^{m-1}.$$

Therefore

$$\begin{aligned} S_{2^\ell - 1} &= \sum_{m=1}^{\ell} \sum_{k=2^{m-1}}^{2^m - 1} \frac{1}{k^p} \\ &\leq \sum_{m=1}^{\ell} \left(\frac{1}{2^{p-1}} \right)^{m-1}. \end{aligned}$$

We recognize the last sum as a geometric sum, and conclude that

$$S_{2^\ell - 1} \leq \frac{1}{1 - \frac{1}{2^{p-1}}}.$$

The last bound is independent of ℓ. We then know that the sequence (S_n) is increasing and bounded from above, and therefore convergent. □

Example 7.4.2. Here is an example of a series taking values in the normed vector space $(\mathbb{R}^2, \| \cdot \|_2)$:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2}, \left(\frac{1}{2} \right)^k \right)$$

7.5 Only the tail matters for convergence

Lemma 7.5.1. Let $(V, \| \cdot \|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence taking values in V. Let $K, L \in \mathbb{N}$. The series

$$\sum_{n=K}^{\infty} a_n$$

is convergent if and only if the series

$$\sum_{n=L}^{\infty} a_n$$

is convergent. Moreover, if either of the series converges, and $K < L$, then

$$\sum_{n=K}^{\infty} a_n = \sum_{n=K}^{L-1} a_n + \sum_{n=L}^{\infty} a_n \tag{7.5.1}$$

Proof. Without loss of generality, we may assume that $K < L$. We then know that for all $n \geq L$,

$$S_K^n = \sum_{k=K}^{L-1} a_k + S_L^n.$$

Suppose that the series

$$\sum_{n=L}^{\infty} a_n$$

is convergent. By definition, this means that the sequence $(S_L^n)_n$ converges. By limit theorems, it follows that S_K^n converges as well, and

$$\begin{aligned}\sum_{k=K}^{\infty} a_k &= \lim_{n\to\infty} \sum_{k=K}^{n} a_k \\ &= \sum_{k=K}^{L-1} a_k + \lim_{n\to\infty} S_L^n \\ &= \sum_{k=K}^{L-1} a_k + \sum_{k=L}^{\infty} a_k.\end{aligned}$$

which shows the equality (7.5.1).

Similarly, suppose that the series

$$\sum_{n=K}^{\infty} a_n$$

is convergent. By definition, this means that the sequence $(S_K^n)_n$ converges. Since

$$S_L^n = S_K^n - \sum_{k=K}^{L-1} a_n$$

it follows again by limit theorems that the sequence $(S_L^n)_n$ converges. $\square$

Proposition 7.5.2. Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and assume that the series

$$\sum_{k=M}^{\infty} a_k$$

is convergent. Then

$$\lim_{m\to\infty} \sum_{k=m}^{\infty} a_k = 0.$$

Proof. The sequence of partial sums $n \mapsto S_M^n$ is convergent, with limit

$$L := \sum_{k=M}^{\infty} a_k.$$

We know by Lemma 7.5.1 that for $m > M$

$$\sum_{k=M}^{\infty} a_k = \sum_{k=m}^{\infty} a_k + \sum_{k=M}^{m-1} a_k.$$

Rearranging terms, we find

$$\sum_{k=m}^{\infty} a_k = \sum_{k=M}^{\infty} a_k - \sum_{k=M}^{m-1} a_k.$$

By using limit theorems and index shift we find that the right-hand side converges to 0 as $m \to \infty$. $\square$

Proposition 7.5.3 (Index shift for series)**.** Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and let $\ell \in \mathbb{N}$. Then the series

$$\sum_{k=M}^{\infty} a_k$$

converges if and only if

$$\sum_{k=M}^{\infty} a_{k+\ell}$$

converges. Moreover, if either series converges,

$$\sum_{k=M}^{\infty} a_{k+\ell} = \sum_{k=M+\ell}^{\infty} a_k.$$

7.6 Divergence test

Proposition 7.6.1. Let $(V, \|\cdot\|)$ be a normed vector space, and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Suppose $\sum_{n=0}^{\infty} a_n$ is convergent to $L \in V$. Then

$$a_n = S_n - S_{n-1}$$

where S_n denotes the partial sum $\sum_{k=0}^{n} a_k$. Because S_n and S_{n-1} are both convergent to L, the sequence (a_n) is convergent as well and converges to $L - L = 0$. □

The following is a very simple, but often useful test for divergence.

Theorem 7.6.2 (Divergence test)**.** Let $(V, \|\cdot\|)$ be a normed vector space, and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the limit $\lim_{n\to\infty} a_n$ does not exist or is not equal to 0. Then the series

$$\sum_{n=0}^{\infty} a_n$$

is divergent.

7.7 Limit laws for series

Theorem 7.7.1 (Limit laws series)**.** Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Suppose that the series

$$\sum_{n=0}^{\infty} a_n \qquad \text{and} \qquad \sum_{n=0}^{\infty} b_n$$

are convergent. Suppose $\lambda \in \mathbb{R}$. Then

i. The series
$$\sum_{n=0}^{\infty} (a_n + b_n)$$
is convergent and converges to
$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

ii. The series
$$\sum_{n=0}^{\infty} (\lambda a_n)$$
is convergent and converges to
$$\lambda \sum_{n=0}^{\infty} a_n$$

7.8 Exercises

7.8.1 Blue exercises

Exercise 7.8.1. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Define the sequence $b : \mathbb{N} \to \mathbb{R}$ by

$$b_n := a_{n+1} - a_n, \qquad \text{for } n \in \mathbb{N}.$$

i. Show that the series
$$\sum_{n=0}^{\infty} b_n$$
converges if and only if the sequence a converges.

ii. Show that if the sequence $a : \mathbb{N} \to \mathbb{R}$ converges, then
$$\lim_{n\to\infty} a_n = a_0 + \sum_{n=0}^{\infty} b_n$$

Exercise 7.8.2. Show part (i) of Theorem 7.7.1.

7.8.2 Orange exercises

Exercise 7.8.3. We consider in this exercise sequences taking values in the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ (recall that this is $\mathbb{R}^2$ with the standard Euclidean norm). Give an example of a sequence $a : \mathbb{N} \to \mathbb{R}^2$ such that

i. $\lim_{n\to\infty} a_n = 0$,

ii. $\sum_{n=1}^{\infty} a_n$ diverges.

(As always, prove that your example satisfies these properties).

Exercise 7.8.4. Determine whether the following series converge or diverge. As always, give a proof of your statement.

(a)	$\displaystyle\sum_{k=3}^{\infty} \frac{2}{k^3}$	(b)	$\displaystyle\sum_{k=1}^{\infty} k$	(c)	$\displaystyle\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k$
(d)	$\displaystyle\sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{3}\right)^{2k}$	(e)	$\displaystyle\sum_{k=0}^{\infty} \frac{2k+3}{(k+1)^2(k+2)^2}$	(f)	$\displaystyle\sum_{k=1}^{\infty} \sqrt[k]{2^{-k}+3}$

Chapter 8

Series with positive terms

In this chapter, we will consider a very special, but very important type of series. These are series with real, positive, terms.

The chapter gives tools for answering the question: Does a series of positive terms converge or not, i.e. does it converge or does it diverge? So far, we only know this for very specific series: we have seen that the harmonic series diverges, the hyperharmonic series converges and geometric series $\sum_{k=0}^{\infty} q^k$ converges if and only if $q \in (-1, 1)$. With the tools in this chapter, however, we can conclude for many more series that they converge or diverge.

As an example, consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}$$

For large k, the terms in this series, namely $k/(k^2 - 1)$ are very close to $1/k$. We may therefore expect that the series diverges, just like the harmonic series does. In this chapter, we will see various theorems that allow you to rigorously derive this conclusion.

8.1 Comparison test

Theorem 8.1.1 (Comparison test)**.** Let $a : \mathbb{N} \to [0, \infty)$ and $b : \mathbb{N} \to [0, \infty)$ be two sequences. Assume that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \leq b_n$.

i. Suppose the series $\sum b_n$ converges, then the series $\sum a_n$ converges as well.

ii. Suppose the series $\sum a_n$ diverges, then the series $\sum b_n$ diverges as well.

Proof. We first show (i). Suppose the series $\sum b_n$ converges. Denote

$$S_n := \sum_{k=N}^{n} a_k \qquad T_n := \sum_{k=N}^{n} b_k.$$

Then we know that for every $n \geq N$ that

$$S_n \leq T_n \leq \sum_{k=N}^{\infty} b_n.$$

The sequence (S_n) is therefore bounded and increasing, thus convergent.

We now show (ii). Suppose the series $\sum a_n$ diverges. Let $M \in \mathbb{N}$. Then there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sum_{k=N}^{n} a_k > M.$$

Then also

$$\sum_{k=N}^{n} b_k > M$$

Therefore the series $\sum_{k=N}^{\infty} b_k$ diverges. Then also the series $\sum_{k=0}^{\infty} b_k$ diverges. $\square$

Example 8.1.2. Consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}.$$

We would like to determine whether this series diverges or converges. There are usually two stages to reaching a conclusion. First is building intuition, the second is setting up a rigorous argumentation.

Let us discuss the intuition first. For that, it is helpful to squint your eyes and get a feeling for the approximate behavior of the terms when k is large. Since the terms

$$\frac{k}{k^2 - 1}$$

are very close to $1/k$ for large k, we may expect that this series diverges, as the harmonic series $\sum \frac{1}{k}$ diverges as well.

The previous theorem allows us to turn this intuition into a precise argument.

The precise argument looks as follows. We first observe that for all $k \geq 2$,

$$\frac{k}{k^2 - 1} \geq \frac{k}{k^2} = \frac{1}{k}.$$

Because the series

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

diverges, the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}$$

diverges as well by the comparison test, Theorem 8.1.1.

Warning: Whenever you want to apply the Comparison Test, as with all theorems, you first need to check the conditions. If you want to show that a series $\sum_{k=N}^{\infty} a_k$ converges, by comparing it to a series $\sum_{k=N}^{\infty} b_k$, you need to show that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$,

$a_n \le b_n$, and you need to show that the series $\sum_{k=N}^{\infty} b_k$ indeed converges.

In particular, do **not** write

$$\sum_{k=N}^{\infty} a_k \le \sum_{k=N}^{\infty} b_k$$

before you applied the Comparison Test, because before you concluded the convergence of the left-hand side, the statement does not make sense.

8.2 Limit comparison test

As a motivation for the next theorem, consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2+1}$$

Just as in the previous example, the terms

$$\frac{k}{k^2+1}$$

are very close to $1/k$ for large k, so we might still expect that the series diverges, because the standard harmonic series diverges as well. However, in contrast to the previous example, the terms

$$\frac{1}{k}$$

are *larger* than the terms $\frac{k}{k^2+1}$. That means we cannot apply the Comparison Test directly. There is however a very convenient way around it: this way is industrialized by the following theorem.

Theorem 8.2.1 (Limit comparison test)**.** Let $a : \mathbb{N} \to [0, \infty)$ and $b : \mathbb{N} \to (0, \infty)$ be two sequences.

i. Assume the series $\sum b_k$ **converges** and assume the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists (i.e. the sequence $n \mapsto a_n/b_n$ converges). Then the series $\sum a_k$ converges as well.

ii. Assume the series $\sum b_k$ **diverges** and assume that either the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists and is **strictly larger than zero**, or that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

Then the series $\sum a_k$ diverges as well.

Proof. We show item (i). Assume $\sum b_k$ converges and that the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

exists. Let's call the limit $L \in [0, \infty)$.

Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

we have that

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } N \in \mathbb{N}, \\ &\quad\quad \text{for all } n \geq N, \\ &\quad\quad\quad \left| \frac{a_n}{b_n} - L \right| < \epsilon. \end{aligned} \tag{8.2.1}$$

Choose $\epsilon := 1$ in (8.2.1). Then

there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \frac{a_n}{b_n} - L \right| < 1.$$

Choose such an N. We claim for all $n \geq N$,

$$a_n \leq b_n(L+1).$$

Indeed let $n \geq N$. Then

$$\frac{a_n}{b_n} - L < 1$$

so that

$$a_n < b_n(L+1).$$

Since the series $\sum b_k$ converges, by the limit laws for series, Theorem 7.7.1, the series

$$\sum_{k=N}^{\infty} b_k(L+1)$$

converges as well.

Therefore by the comparison test, Theorem 8.1.1, we find that the series

$$\sum_{k=N}^{\infty} a_k$$

converges as well. □

Example 8.2.2. Let us see how we can use the limit comparison test to conclude that the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2+1}$$

diverges.

For this, we will apply part (ii) of the Limit Comparison Test, Theorem 8.2.1.

We use sequences $a : \mathbb{N} \to (0, \infty)$ and $b : \mathbb{N} \to (0, \infty)$ defined for

$k \geq 2$ by

$$a_k := \frac{k}{k^2+1}$$

and

$$b_k := \frac{1}{k}.$$

(In general, for the comparison sequence b_k it is good to try a sequence for which you understand well whether the corresponding series diverges or converges, while at the same time you believe, have the intuition, the inkling or the guess that a_k and b_k are close for k large.) Then

$$\frac{a_k}{b_k} = \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \frac{1}{1+\frac{1}{k^2}}.$$

By limit laws, we find that the limit of the denominator is 1, i.e.

$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\right) = \lim_{k\to\infty} 1 + \lim_{k\to\infty}\frac{1}{k^2} = 1+0 = 1.$$

Therefore, we may apply the limit law for the quotient and conclude that

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \frac{1}{\lim_{k\to\infty}\left(1+\frac{1}{k^2}\right)} = \frac{1}{1} = 1.$$

The series $\sum_{k=2}^{\infty}\frac{1}{k}$ diverges and therefore it follows from the Limit Comparison Test that the series

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty}\frac{k}{k^2+1}$$

diverges as well.

Let me also say a word about a crucial technique we used in Theorem 8.2.1: we used that because the sequence a_n/b_n converges to L, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$L-1 < \frac{a_n}{b_n} < L+1$$

This expresses that we have some pretty good control on the terms $\frac{a_n}{b_n}$

when n is larger than or equal to N.

Similarly, it is maybe good to ponder about the fact that if a sequence $c : \mathbb{N} \to \mathbb{R}$ converges to some $L \in \mathbb{R}$, that then there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$L - \frac{1}{58249104762} < c_n < L + \frac{1}{58249104762}.$$

The number $\frac{1}{58249104762}$ was a random result of fingers hitting the keyboard. It is true that the existence of such an N_1 is a fairly direct consequence of the definition of convergence of a sequence, yet sometimes it takes some time getting used to what such a definition can actually do for you.

8.3 Ratio test

The next test, called the ratio test, is very convenient for determining that a series such as

$$\sum_{k=0}^{\infty} \frac{2^k}{k!}$$

converges. Interestingly, such series occur very often 'in the wild'. We are not ready to show this yet, but at some point we will see that the value of the series is actually equal to e^2, where e was introduced in Section 6.7.3 as the limit

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

Theorem 8.3.1 (Ratio Test)**.** Let $a : \mathbb{N} \to (0, \infty)$ be a sequence only taking on strictly positive values.

i. If there exists an $N \in \mathbb{N}$ and a $q \in (0, 1)$ such that for all $n \geq N$, it holds that

$$\frac{a_{n+1}}{a_n} \leq q,$$

then the series $\sum a_k$ converges.

ii. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that

$$\frac{a_{n+1}}{a_n} \geq 1,$$

then the series $\sum a_k$ diverges.

Proof. We first show (i). So assume there exists an $N \in \mathbb{N}$ and a $q \in (0,1)$ such that for all $n \geq N$, it holds that

$$\frac{a_{n+1}}{a_n} \leq q.$$

Then it holds for all $k \in \mathbb{N}$ that

$$0 < a_{N+k} \leq q^k a_N$$

Note that the series

$$\sum_{k=0}^{\infty} q^k$$

is convergent as it is a standard geometric series with $|q| < 1$. By Theorem 7.7.1, part (ii), the series

$$\sum_{k=0}^{\infty} q^k a_N$$

is convergent as well. Therefore, we find by the Comparison Test that the series

$$\sum_{k=N}^{\infty} a_k$$

is convergent as well.

We now show (ii). Assume there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that

$$\frac{a_{n+1}}{a_n} \geq 1.$$

Then for all $n \geq N$, $a_n \geq a_N$, so that a_n does not converge to zero. By the divergence test (Theorem 7.6.2), we find that the series

$$\sum_{k=0}^{\infty} a_k$$

is divergent. □

Corollary 8.3.2 (Ratio Test, limit version)**.** Let (a_n) be a sequence of *strictly positive* real numbers.

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q \in [0,1)$, then the series $\sum_k a_k$ converges.
- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q \in (1,\infty)$, or if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_k a_k$ diverges.

Warning: We cannot conclude anything about the convergence of a series $\sum_k a_k$ when

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

8.4 Root test

Theorem 8.4.1 (Root Test)**.** Let (a_n) be a sequence of nonnegative real numbers.

i. If there exists an $N \in \mathbb{N}$ and a $q \in (0,1)$ such that for all $n \geq N$, $\sqrt[n]{a_n} \leq q$, then the series $\sum a_n$ converges.

ii. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $\sqrt[n]{a_n} \geq 1$, then the series $\sum a_n$ diverges.

Proof. Suppose there exists an $N \in \mathbb{N}$ and a $q \in (0,1)$ such that for all $n \geq N$, it holds that

$$\sqrt[n]{a_n} \leq q.$$

Then for all $n \geq N$, it holds that

$$0 \leq a_n \leq q^n.$$

The series

$$\sum_{n=N}^{\infty} q^n$$

converges as it is a standard geometric series and $q \in (0, 1)$.

Therefore, the series

$$\sum_{n=N}^{\infty} a_n$$

converges by the comparison test, Theorem 8.1.1. Finally, then the sequence

$$\sum_{n=0}^{\infty} a_n$$

converges as well by Lemma 7.5.1. □

Corollary 8.4.2 (Root Test, limit version)**.** Let (a_n) be a sequence of non-negative real numbers.

- If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ and $q \in [0, 1)$, then the series $\sum_k a_k$ converges.
- If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q \in (1, \infty)$ or if $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_k a_k$ diverges.

Warning We cannot conclude anything about the convergence of a series $\sum_k a_k$ if

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1.$$

8.5 Exercises

8.5.1 Blue Exercises

Exercise 8.5.1. Determine whether the following series converge or diverge.

$$(a) \quad \sum_{k=1}^{\infty} \frac{3^k k}{(2k+1)!} \qquad\qquad (b) \quad \sum_{k=10}^{\infty} \frac{k+2}{k^3-6}$$

$$(c) \quad \sum_{k=2}^{\infty} \frac{1}{\sqrt{k+1}} \qquad\qquad (d) \quad \sum_{k=3}^{\infty} \frac{1}{k^{100}} 3^k$$

8.5.2 Orange Exercises

Exercise 8.5.2. Let $c : \mathbb{N} \to (0, \infty)$ be a sequence taking on only strictly positive values, and assume that $c : \mathbb{N} \to (0, \infty)$ converges to $3/2$. Determine whether the following series diverges or converges

$$\sum_{k=1}^{\infty} \frac{1}{k^{(c_k)}}.$$

Chapter 9

Series with general terms

Whereas in the previous section, we have considered techniques for concluding the convergence or divergence of very special types of series (series with positive, real, terms), we will in this chapter go back to general series. How can we conclude convergence or divergence of those?

In the next section, we will first consider alternating series of real terms. There is a nice convergence test for such series, called the Leibniz Test.

In addition, we will borrow a theorem from a later chapter that works as follows for sequences taking values in the real numbers. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence of real numbers. We can now make the following series of positive terms (which brings us back in the realm of the previous chapter)

$$\sum_{k=0}^{\infty} |a_k|.$$

Suppose this series of absolute values converges (we will say that the series $\sum_{k=0}^{\infty} a_k$ converges *absolutely*). Then the theorem will allow us to conclude that the series

$$\sum_{k=0}^{\infty} a_k$$

converges as well.

9.1 Series with real terms: the Leibniz test

Theorem 9.1.1 (Leibniz Test, a.k.a. Alternating Series Test)**.** Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequence such that for all $k \in \mathbb{N}$, $b_k = (-1)^k a_k$. Assume that there exists a $K \in \mathbb{N}$ such that

i. $a_k \geq 0$ for every $k \geq K$

ii. $a_k \geq a_{k+1}$ for every $k \geq K$

iii. $\lim_{k\to\infty} a_k = 0$

Then, the series

$$\sum_{k=K}^{\infty} b_k = \sum_{k=K}^{\infty} (-1)^k a_k$$

is convergent. In addition, the following estimate holds for every $N \geq K$,

$$\left| S_N - \sum_{k=K}^{\infty} b_k \right| < a_{N+1}$$

where for all $n \in \mathbb{N}$, $S_n := \sum_{k=K}^{N} b_k$.

Proof. We only prove the case in which $K = 0$.

We note that (S_{2n}) is a decreasing sequence, because for all $n \in \mathbb{N}$,

$$S_{2n+2} = S_{2n} - a_{2n+1} + a_{2n+2} \leq S_{2n}.$$

Similarly, (S_{2n+1}) is an increasing sequence, because for all $n \in \mathbb{N}$,

$$S_{2n+3} = S_{2n+1} + a_{2n+2} - a_{2n+3} \geq S_{2n+1}.$$

Finally, we note that for all $n \in \mathbb{N}$, it holds that

$$S_1 \leq S_{2n+1} = S_{2n} - a_{2n+1} \leq S_{2n} \leq S_0.$$

As a consequence, the sequence (S_{2n}) is bounded from below by S_1, and the sequence (S_{2n+1}) is bounded above by S_0. Therefore, both sequences are convergent.

Because the sequence (a_n) converges to zero, we can show that the sequence (a_{2n+1}) converges to zero as well. By the limit laws, we find that

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} S_{2n}.$$

In words, the sequences (S_{2n+1}) and (S_{2n}) converge to the same limit. Let's call this limit s.

Finally,

$$S_{2n} > s > S_{2n+1} = S_{2n} - a_{2n+1}$$

so that $|S_{2n} - s| < a_{2n+1}$.

Similarly,

$$S_{2n+1} < s < S_{2n+2} = S_{2n+1} + a_{2n+2}$$

so that $|S_{2n+1} - s| < a_{2n+2}$.

In conclusion, in such an alternating series we have the estimate for all $n \in \mathbb{N}$

$$|S_n - s| \leq a_{n+1}$$

Therefore the sequence (S_n) converges to s. □

Example 9.1.2. We claim that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

We would like to apply the Alternating Series Test. To do so, we need to check its conditions.

We define the sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_k := \frac{1}{k}$$

for $k \geq 1$ (and $a_0 = a_1 = 1$).

We now check the conditions for the Alternating Series Test.

i. We need to show that $a_k \geq 0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then

$$a_k = \frac{1}{k} \geq 0.$$

ii. We need to show that $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then

$$a_k = \frac{1}{k} \geq \frac{1}{k+1} = a_{k+1}.$$

iii. We need to show that

$$\lim_{k\to\infty} a_k = 0.$$

This follows as this is a standard limit.

It follows from the Alternating Series Test that the series

$$\sum_{k=1}^{\infty} (-1)^k a_k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

9.2 Series characterization of completeness in normed vector spaces

Definition 9.2.1. Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence in V. We say the series

$$\sum_{k=0}^{\infty} a_k$$

converges *absolutely* if

$$\sum_{k=0}^{\infty} \|a_k\|$$

converges.

Definition 9.2.2 (Series characterization of completeness)**.** We say a normed vector space $(V, \|\cdot\|)$ satisfies the *series characterization of completeness* if every series in V that is absolutely convergent is also convergent.

In a later chapter, we will prove the following proposition.

Proposition 9.2.3. Every finite-dimensional normed vector space satisfies the series characterization of completeness.

In particular, $\mathbb{R}^d$ endowed with the standard Euclidean norm satisfies the series characterization of completeness, and $(\mathbb{R}, |\cdot|)$ satisfies the series characterization of completeness.

Example 9.2.4. Consider the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}.$$

Since this is not an alternating series, we cannot apply the Leibniz test. However, for every $k \in \mathbb{N} \setminus \{0\}$, we have

$$\left| \frac{\sin(k)}{k^2} \right| \leq \frac{1}{k^2}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a standard hyperharmonic series, of which we know that it converges. By the Comparison Test, we conclude that the series

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right|$$

converges as well.

Therefore, the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges **absolutely**. Because $(\mathbb{R}, |\cdot|)$ is complete, we find that

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges.

Definition 9.2.5. Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence. We say that a series

$$\sum_{k=0}^{\infty} a_k$$

converges *conditionally* if it converges, but doesn't converge absolutely.

9.3 The Cauchy product

Intuitively, this section is about the multiplication of two real-valued series. The precise statement is covered in the next theorem.

Theorem 9.3.1 (Cauchy product)**.** Let (A_k) and (B_k) be two real-valued sequences, and assume that the series

$$\sum_{k=0}^{\infty} A_k \qquad \text{and} \qquad \sum_{k=0}^{\infty} B_k$$

both converge *absolutely*. Then the series

$$\sum_{k=0}^{\infty} C_k$$

converges absolutely as well, where $C_k := \sum_{\ell=0}^{k} A_\ell B_{k-\ell}$, and

$$\sum_{k=0}^{\infty} C_k = \left(\sum_{k=0}^{\infty} A_k\right)\left(\sum_{k=0}^{\infty} B_k\right).$$

Proof. We will first show that the series

$$\sum_{k=0}^{\infty} C_k$$

converges absolutely. Note that

$$\begin{aligned} \sum_{k=0}^{n} C_k &= \sum_{k=0}^{n} \sum_{\ell=0}^{k} A_\ell B_{k-\ell} \\ &= \sum_{\ell=0}^{n} \sum_{k=\ell}^{n} A_\ell B_{k-\ell} \\ &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} A_\ell B_m. \end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{n} |C_k| &\leq \sum_{\ell=0}^{n} |A_\ell| \sum_{m=0}^{n-\ell} |B_m| \\ &\leq \left(\sum_{\ell=0}^{n} |A_\ell|\right)\left(\sum_{m=0}^{n} |B_m|\right) \\ &\leq \left(\sum_{\ell=0}^{\infty} |A_\ell|\right)\left(\sum_{m=0}^{\infty} |B_m|\right)\end{aligned}$$

It follows that the sequence of partial sums

$$n \mapsto \sum_{k=0}^{n} |C_k|$$

is bounded from above. It is also increasing, and therefore it converges.

Now let $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{k=0}^{2n} C_k - \sum_{\ell=0}^{2n} A_\ell \sum_{m=0}^{2n} B_m &= \sum_{\ell=0}^{2n} \sum_{m=2n-\ell+1}^{2n} A_\ell B_m \\ &= \sum_{\ell=0}^{2n} A_\ell \sum_{m=2n-\ell+1}^{2n} B_m \\ &= \sum_{\ell=0}^{n} A_\ell \sum_{m=2n-\ell+1}^{2n} B_m + \sum_{\ell=n+1}^{2n} A_\ell \sum_{m=2n-\ell+1}^{2n} B_m\end{aligned}$$

It follows that

$$\left|\sum_{k=0}^{2n} C_k - \sum_{\ell=0}^{2n} A_\ell \sum_{m=0}^{2n} B_m\right| \leq \sum_{\ell=0}^{n} |A_\ell| \sum_{m=n}^{\infty} |B_m| + \sum_{\ell=0}^{n} |B_\ell| \sum_{m=n}^{\infty} |A_n|. \tag{9.3.1}$$

Now note that because the series

$$\sum_{\ell=0}^{\infty} A_\ell$$

is absolutely convergent, the series

$$\sum_{\ell=0}^{\infty} |A_\ell|$$

is convergent, which exactly means that

$$\lim_{n\to\infty} \sum_{\ell=0}^{\infty} |A_\ell|$$

exists. Similarly,

$$\lim_{n\to\infty} \sum_{\ell=0}^{\infty} |B_\ell|$$

exists.

Moreover, because the series

$$\sum_{m=0}^{\infty} |B_m|$$

converges, it follows by Proposition 7.5.2 that

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} |B_m| = 0.$$

Similarly,

$$\lim_{n\to\infty} \sum_{m=n}^{\infty} |A_m| = 0.$$

It follows by limit laws that the right-hand side of (9.3.1) converges to 0 as $n \to \infty$.

It follows that the sequence

$$n \mapsto \sum_{k=0}^{n} C_k$$

converges to

$$\sum_{\ell=0}^{\infty} A_\ell \sum_{m=0}^{\infty} B_m.$$

□

9.4 Exercises

9.4.1 Blue exercises

Exercise 9.4.1. Determine whether the following series converge or diverge.

$$(a) \quad \sum_{k=0}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k+1}} \qquad\qquad (b) \quad \sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{6}\right) \frac{1}{k^4}$$

9.4.2 Orange exercises

Exercise 9.4.2. Give an example of a sequence $a : \mathbb{N} \to \mathbb{R}$ such that

- for all $k \in \mathbb{N}$, it holds that $a_k > 0$,
- $\lim_{k\to\infty} a_k = 0$,

yet the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

diverges.

Chapter 10

Subsequences, lim sup and lim inf

Why are subsequences useful?

For many students, the topic of *subsequences* is initially difficult to grasp. I believe that what makes it easier, is to keep reminding yourself that sequences (and subsequences too!), are functions from the natural numbers. This is one of the main reasons we spent so much time on this aspect before.

10.1 Index sequences and subsequences

Subsequences are made by precomposing a sequence by a very special type of sequence: an index sequence.

Definition 10.1.1 (index sequence)**.** We say a sequence $n : \mathbb{N} \to \mathbb{N}$ is an *index sequence* if n is strictly increasing.

There are two important elements to this definition: first of all, index sequences are sequences taking values in the natural numbers (as opposed to just an arbitrary space). Secondly, an index sequence is strictly increasing, so for every $k \in \mathbb{N}$, $n_{k+1} > n_k$.

We often write $(n_k)_{k\in\mathbb{N}}$ or just (n_k) to denote an index sequence.

Example 10.1.2. The sequence $n : \mathbb{N} \to \mathbb{N}$ defined by

$$n_k := 2k$$

is a is strictly increasing sequence of natural numbers. In other words, it is an index sequence.

The next definition describes how to make a subsequences by precomposing a sequence with an index sequence.

Definition 10.1.3 (subsequence)**.** Let $a : \mathbb{N} \to X$ be a sequence. A sequence $b : \mathbb{N} \to X$ is called a *subsequence* of a if there exists an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that $b = a \circ n$.

Just as we often write $(a_n)_{n\in\mathbb{N}}$ for a sequence called a, we often write $(a_{n_k})_{k\in\mathbb{N}}$ for the subsequence $a \circ n$.

Example 10.1.4. Let's see what happens in an example. Let $a : \mathbb{N} \to \mathbb{R}$ be defined by $a_\ell := \frac{1}{\ell+1}$ for $\ell \in \mathbb{N}$. Let $n : \mathbb{N} \to \mathbb{N}$ be the index sequence defined by $n_\ell := (2\ell + 1)$.

Here is a table that indicates for this example the first terms of the sequences, $(a_\ell)_\ell$, $(a_{n_k})_k$ and $(n_k)_k$.

ℓ	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
a_ℓ	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	
a_ℓ	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{1}{14}$	
a_{n_k}		$\frac{1}{2}$		$\frac{1}{4}$		$\frac{1}{6}$		$\frac{1}{8}$		$\frac{1}{10}$		$\frac{1}{12}$		$\frac{1}{14}$	
a_{n_k}		a_1		a_3		a_5		a_7		a_9		a_{11}		a_{13}	
n_k		1		3		5		7		9		11		13	
k		0		1		2		3		4		5		6	

It could really help to think of an example sequence yourself, and an example index sequence, and create a similar table for your own example.

Warning: the following subtle point requires some quiet overthinking. The subsequences $(a_{n_k})_{k\in\mathbb{N}}$ and $(a_{n_\ell})_{\ell\in\mathbb{N}}$ are exactly the same! Both notations represent the subsequence $a \circ n$.

10.2 (Sequential) accumulation points

As a motivation for the following definition, let us consider the sequence $a_\ell = (-1)^\ell$. A term a_ℓ of this sequence equals 1 if its index ℓ is even, and equals (-1) if its index ℓ is odd. This sequence does not converge, but if we just consider the subsequence $(a_{n_k})_k$ with $n_k := 2k$, then this subsequence is a constant sequence, every term equals 1 and therefore it does converge.

In general then, we will find it interesting to know if a subsequence of a sequence converges. And the limit of such a subsequence is special too. We call a limit of a subsequence a (sequential) accumulation point.

Definition 10.2.1 ((Sequential) accumulation points)**.** Let (X, dist) be a metric space. A point $p \in X$ is called an *accumulation point* of a sequence $a : \mathbb{N} \to X$ if there is a subsequence $a \circ n$ of a such that $a \circ n$ converges to p.

10.3 Subsequences of a converging sequence

Proposition 10.3.1. Let (X, dist) be a metric space. Let (a_n) be a sequence in X converging to $p \in X$. Then every subsequence of (a_n) is convergent to p.

Proof. Let (a_n) be a sequence converging to $p \in X$ and let (a_{n_k}) be a subsequence of a. We need to show that for all $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\mathsf{dist}(a_{n_k}, p) < \epsilon.$$

Let $\epsilon > 0$. Because (a_m) converges to p, there exists an $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,

$$\mathsf{dist}(a_m, p) < \epsilon. \tag{10.3.1}$$

Choose $k_0 := m_0$. Let $k \geq k_0$. Then

$$n_k \geq n_{k_0} \geq k_0 = m_0$$

where in the last inequality we made use of the fact that the index sequence $n : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Because $n_k \geq m_0$, it follows by (10.3.1) that

$$\mathsf{dist}(a_{n_k}, p) < \epsilon.$$

□

10.4 lim sup

In this section we are going to define a function (called the lim sup) that takes in a real-valued sequence and outputs either

- the symbol "∞" if the sequence is not bounded from above
- the symbol "$-\infty$", if the sequence diverges to $-\infty$
- a real number otherwise.

Consider a real-valued sequence (a_n) that is bounded from above and does not diverge to $-\infty$. We can then define a new sequence

$$k \mapsto \sup_{n \geq k} a_n.$$

Note that this sequence is decreasing, because for larger k the supremum is taken over a smaller set. We will show in the lemma below that then the sequence $k \mapsto \sup_{n \geq k} a_n$ is also bounded from below.

Therefore, the sequence $k \mapsto \sup_{n \geq k} a_n$ has a limit, and the limit is in fact equal to the infimum of the sequence. This limit is called the lim sup

$$\begin{aligned} \limsup_{n \to \infty} a_n &:= \inf_{k \in \mathbb{N}} \sup_{n \geq k} a_n \\ &= \lim_{k \to \infty} \left(\sup_{n \geq k} a_n \right). \end{aligned}$$

We still need to show the announced lemma.

Lemma 10.4.1. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence that is bounded from above and does not diverge to $-\infty$. Then the sequence $k \mapsto \sup_{n \geq k} a_n$ is bounded from below.

Proof. We argue by contradiction. Suppose therefore that the sequence $k \mapsto \sup_{n \geq k} a_n$ is not bounded from below. We are going to show that the sequence a_n diverges to $-\infty$, which would indeed be a contradiction.

We will show that

$$\begin{aligned} &\text{for all } M \in \mathbb{R} \\ &\quad \text{there exists } N \in \mathbb{N} \\ &\quad\quad \text{for all } n \geq N, \\ &\quad\quad\quad a_n < M. \end{aligned}$$

Let $M \in \mathbb{R}$. Since the sequence $k \mapsto \sup_{n \geq k} a_n$ is not bounded from below, there exists an $m \in \mathbb{N}$, such that

$$\sup_{n \geq m} a_n < M$$

Choose $N := m$. Let $n \geq N$. Then

$$a_n \leq \sup_{\ell \geq m} a_\ell < M.$$

This finishes our proof that (a_n) diverges to $-\infty$, and we have derived a contradiction. Hence the sequence $k \mapsto \sup_{n \geq k}(a_n)$ is in fact bounded from below. $\square$

Proposition 10.4.2 (Alternative characterization of lim sup)**.** Let (a_n) be a real-valued sequence. Let $M \in \mathbb{R}$. Then $M \in \mathbb{R}$ equals $\limsup_{\ell \to \infty} a_\ell$ if and only if the following two conditions hold:

i.
$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } \ell \geq N,\\ &\quad\quad\quad a_\ell < M + \epsilon.\end{aligned}$$

ii.
$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{for all } k \in \mathbb{N},\\ &\quad\quad \text{there exists } m \geq k,\\ &\quad\quad\quad a_m > M - \epsilon.\end{aligned}$$

Proof. Let $M \in \mathbb{R}$. We first show that if $M = \limsup_{n \to \infty} a_n$, then the conditions (i) and (ii) hold. Assume $M = \limsup_{n \to \infty} a_n$. Then by definition of the lim sup, it follows that $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$.

We will now show (i). Let $\epsilon > 0$. We need to show that there exists a $N \in \mathbb{N}$ such that for all $\ell \geq N$, it holds that $a_\ell < M + \epsilon$.

By the definition of $\limsup_{\ell \to \infty} a_\ell$ as $\inf_{\ell \in \mathbb{N}} \sup_{k \geq \ell} a_k$, there exists an

$\ell_0 \in \mathbb{N}$ such that

$$\sup_{k \geq \ell_0} a_k < M + \epsilon.$$

Choose $N := \ell_0$. Let $\ell \geq N$. Then by the previous inequality we conclude that

$$a_\ell < M + \epsilon.$$

We will now show (ii). Let $\epsilon > 0$. Let $k \in \mathbb{N}$. We need to show that there exists an $m \geq k$ such that $a_m > M - \epsilon$. By the definition of $\limsup_{\ell \to \infty} a_\ell$ as $\inf_{\ell \in \mathbb{N}} \sup_{k \geq \ell} a_k$, we know that

$$\sup_{n \geq k} a_n \geq M.$$

Therefore, there exists an $m \geq k$ such that

$$a_m > \sup_{n \geq k} a_n - \epsilon \geq M - \epsilon.$$

We will now show that if $M \in \mathbb{R}$ satisfies conditions (i) and (ii), that then M equals $\limsup_{\ell \to \infty} a_\ell$.

Assume M satisfies conditions (i) and (ii).

We first need to settle that $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$.

We will show that $a : \mathbb{N} \to \mathbb{R}$ is bounded from above. By (i), we may obtain an $N \in \mathbb{N}$ such that for all $\ell \geq N$, $a_\ell < M + 1$. Choose $L := \max(a_0, a_1, \dots, a_{N-1}, M + 1)$. Then for all $\ell \in \mathbb{N}$, it holds that $a_\ell \leq L$. Hence L is an upper bound for $a : \mathbb{N} \to \mathbb{R}$ and $a : \mathbb{N} \to \mathbb{R}$ is indeed bounded.

We will now show that $a : \mathbb{N} \to \mathbb{R}$ does not diverge to $-\infty$. We argue by contradiction. Suppose $a : \mathbb{N} \to \mathbb{R}$ diverges to $-\infty$. Then there exists an $N_0 \in \mathbb{N}$ such that for all $\ell \geq N_0$, $a_\ell < M - 1$. However, by (ii) there exists an $m \geq N_0$ such that $a_m > M - 1$. This is a contradiction.

It now follows that $\limsup_{\ell\to\infty} a_\ell = \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n$. We therefore need to show that

$$M = \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n.$$

We first show that for every $k \in \mathbb{N}$,

$$M \leq \sup_{n\geq k} a_n.$$

Let $k \in \mathbb{N}$. Note that it suffices to show that for every $\epsilon > 0$

$$M - \epsilon < \sup_{n\geq k} a_n$$

Let $\epsilon > 0$. By (ii) we know that there exists an $m \geq k$ such that

$$M - \epsilon < a_m.$$

Therefore also

$$M - \epsilon < \sup_{n\geq k} a_n$$

Finally, we show that for every $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that

$$\sup_{n\geq k} a_n < M + \epsilon$$

Let $\epsilon > 0$. By condition (i) we know that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$a_n < M + \epsilon/2$$

Choose $k := n_0$. Then

$$\sup_{n\geq k} a_n \leq M + \epsilon/2 < M + \epsilon.$$

$\square$

Theorem 10.4.3. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded from above and does not diverge to $-\infty$. Then $\limsup_{\ell\to\infty} a_\ell$

is a (sequential) accumulation point of the sequence a, i.e. there exists a subsequence of a that converges to $\limsup_{\ell\to\infty} a_\ell$.

Proof. We denote $\limsup_{k\to\infty} a_k$ by M.

We need to find an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{k\to\infty} a_{n_k} = M.$$

We do this inductively. We first know that there exists an m_0 such that for all $m \geq m_0$,

$$a_m \leq M + 1/1.$$

Then we know there exists an n_0 such that $n_0 > m_0$ and

$$M - 1/1 < a_{n_0}$$

Because $n_0 > m_0$, we also know that

$$a_{n_0} < M + 1/1$$

Suppose now that $n_{\ell-1}$ is defined for some $\ell \in \mathbb{N} \setminus \{0\}$. We are going to define n_ℓ. We know that there exists an $m_\ell \in \mathbb{N}$ such that for every $m \geq m_\ell$,

$$a_m \leq M + \frac{1}{\ell+1}.$$

Now, there exists an $n_\ell \geq \max(n_{\ell-1}, m_\ell) + 1$ such that

$$M - \frac{1}{\ell+1} < a_{n_\ell}$$

and because $n_\ell > m_\ell$, we also know that

$$a_{n_\ell} < M + \frac{1}{\ell+1}.$$

By construction, we know that $n : \mathbb{N} \to \mathbb{N}$ is strictly increasing. In other words, we know that it is an index sequence. Also by construction, we know that for all $\ell \in \mathbb{N}$,

$$M - \frac{1}{\ell + 1} < a_{n_\ell} < M + \frac{1}{\ell + 1}.$$

By the squeeze theorem, we know that the $\lim_{\ell \to \infty} a_{n_\ell}$ exists and equals M. □

The previous theorem has the following consequence, which is a key fact in analysis.

Corollary 10.4.4 (Bolzano-Weierstrass)**.** Every bounded, real-valued sequence has a subsequence that converges in $(\mathbb{R}, \mathsf{dist}_\mathbb{R})$.

Theorem 10.4.3 shows that if a sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to infinity, the number $\limsup_{\ell \to \infty} a_\ell$ is a sequential accumulation point. However, we can derive more: in fact it is the maximum of the set of accumulation points.

Theorem 10.4.5. Suppose a sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$. Then

$$\limsup_{\ell \to \infty} a_\ell$$

is the maximum of the set of sequential accumulation points.

10.5 lim inf

Similarly to the lim sup, we can also define the lim inf. In some sense,

$$\liminf_{\ell \to \infty} a_\ell = -\limsup_{\ell \to \infty} (-a_\ell)$$

More precisely, the lim inf is a function that takes in a real-valued sequence $a : \mathbb{N} \to \mathbb{R}$ and outputs

- The symbol "$-\infty$" if the sequence is not bounded from below
- The symbol "∞ if the sequence diverges to ∞, and otherwise
- the real number

$$\begin{aligned}\liminf_{\ell\to\infty} a_\ell &:= \sup_{\ell\in\mathbb{N}} \inf_{k\geq\ell} a_k \\ &= \lim_{\ell\to\infty}\left(\inf_{k\geq\ell} a_k\right).\end{aligned}$$

Let us also record an alternative characterization of the lim inf.

Proposition 10.5.1 (Alternative characterization lim inf)**.** Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $M \in \mathbb{R}$. Then M equals $\liminf_{\ell\to\infty} a_\ell$ if and only if the following two conditions hold

i.
$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\qquad \text{for all } \ell \geq N,\\ &\qquad\quad a_\ell > M - \epsilon.\end{aligned}$$

ii.
$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{for all } K \in \mathbb{N},\\ &\qquad \text{there exists } m \geq K,\\ &\qquad\quad a_m < M + \epsilon.\end{aligned}$$

Theorem 10.5.2. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is a sequential accumulation point of the sequence a, i.e. there is a subsequence of a that converges to $\liminf_{\ell\to\infty} a_\ell$.

We can in fact show a bit more, namely that if $a : \mathbb{N} \to \mathbb{R}$ is bounded below and does not diverge ∞, the number $\liminf_{\ell\to\infty}$ is the smallest sequential accumulation point of the sequence $a : \mathbb{N} \to \mathbb{R}$.

Theorem 10.5.3. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is the minimum of the set of sequential accumulation points of the sequence $a : \mathbb{N} \to \mathbb{R}$.

10.6 Relations between lim, lim inf and lim sup

A bounded sequence may not always converge. In other words, its limit may not always exist. But the lim sup and lim inf always do exist for bounded sequences. The following proposition tells us moreover that the sequence converges if and only if the lim sup and the lim inf are the same.

Proposition 10.6.1. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} a_\ell = L.$$

So far we haven't seen a statement that said that if two convergent sequences $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ are ordered as in $a_\ell \leq b_\ell$ for all ℓ, then $\lim_{\ell\to\infty} a_\ell \leq \lim_{\ell\to\infty} b_\ell$. Part of the reason is that the assumption of convergence of both sequences is a bit unsatisfactory. The next proposition is a generalization that can be much more useful, especially when it is combined with the previous proposition.

Proposition 10.6.2. Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences, such that there exists an $N \in \mathbb{N}$ such that for all $\ell \geq N$, $a_\ell \leq b_\ell$. Then

$$\limsup_{\ell\to\infty} a_\ell \leq \limsup_{\ell\to\infty} b_\ell$$

and

$$\liminf_{\ell\to\infty} a_\ell \leq \liminf_{\ell\to\infty} b_\ell.$$

10.7 Exercises

10.7.1 Blue exercises

Exercise 10.7.1. Consider the set $A := \{a, b, \cdots, z\}$ of letters in the english alphabet and let $\alpha : \mathbb{N} \to A$ be a sequence of which the first terms are (in the order in which you would normally read)

r	j	z	b	a	g	w	q	o	r
x	o	l	b	d	x	s	l	e	e
u	h	g	c	e	c	k	v	n	i
v	c	n	i	l	t	j	n	h	c
e	e	i	u	s	u	m	e	c	t
c	r	y	o	b	v	n	d	f	d
b	d	h	z	f	a	z	s	l	i
h	f	k	s	o	x	c	f	n	o
a	x	c	n	d	i	a	d	c	l
e	u	y	i	j	s	c	v	i	k
m	s	g	n	o	c	d	n	f	g

Let $\nu : \mathbb{N} \to \mathbb{N}$ be the index sequence defined by $\nu_\kappa := \kappa + 5$ and let $\mu : \mathbb{N} \to \mathbb{N}$ be the index sequence defined by $\mu_\kappa := 3\kappa$.

Write down the first 33 terms of the sub-subsequence $(\alpha_{\nu_{\mu_\kappa}})_\kappa$ of the sequence $(\alpha_\kappa)_\kappa$.

Exercise 10.7.2. Let (X, dist) be a metric space. Let $a : \mathbb{N} \to X$ be a sequence, and let $n : \mathbb{N} \to \mathbb{N}$ be an index sequence. Suppose that the subsequence $a \circ n$ converges. Show that every subsequence of $a \circ n$ is convergent.

Exercise 10.7.3. Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences, such that $a : \mathbb{N} \to X$ converges to some $p \in X$.

Now consider the following sequence $c : \mathbb{N} \to X$, defined by

$$c_k := \begin{cases} a_k & \text{if } k \text{ even} \\ b_k & \text{if } k \text{ odd.} \end{cases}$$

Show that p is an accumulation point of $c : \mathbb{N} \to X$. (See Definition 10.2.1).

10.7.2 Orange exercises

Exercise 10.7.4. Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence with values in X. Let $p \in X$. Suppose that every subsequence of $a : \mathbb{N} \to X$ has itself a subsequence that converges to p. Show that $a : \mathbb{N} \to X$ itself converges to p as well.

Hint: Argue by contradiction, and use a similar proof technique as Theorem 10.4.3.

Exercise 10.7.5. Prove Proposition 10.6.1.

Exercise 10.7.6. Let $P : \mathbb{N} \to \{\mathsf{blue}, \mathsf{orange}\}$ be a sequence taking values in the set with exactly the two elements blue and orange. Assume that

$$\begin{aligned}&\text{for all } k \in \mathbb{N},\\ &\quad \text{there exists } m \geq k,\\ &\quad\quad P_m = \mathsf{blue}.\end{aligned}$$

Show that there is a subsequence of $P : \mathbb{N} \to \{\mathsf{blue}, \mathsf{orange}\}$ for which every term equals blue by going through the following steps:

i. Inductively define an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$, $P_{n_k} = \mathsf{blue}$ following the template in the Best Practices item (xiii):

   (a) First define $n_0 \in \mathbb{N}$ appropriately and prove that $P_{n_0} = \mathsf{blue}$.

   (b) For $k \in \mathbb{N}$, with $n_0, \ldots, n_k$ defined, define n_{k+1} appropriately, and prove that $n_{k+1} > n_k$ and
   $$P_{n_{k+1}} = \mathsf{blue}.$$

ii. Conclude your proof by saying that the sequence $P \circ n$ is a subsequence of P and that by construction, for all $k \in \mathbb{N}$,
$$P_{n_k} = \mathsf{blue}.$$

Exercise 10.7.7. Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence with (at least) two sequential accumulation points $p, q \in \mathbb{R}$ (with $p \neq q$). Prove that the sequence $a : \mathbb{N} \to \mathbb{R}$ does not converge.

Chapter 11

Point-set topology of metric spaces

The main purpose of the current section and the next is to introduce three stronger and stronger properties for subsets of a metric space: **closedness**, **completeness** and **compactness**. Here 'stronger and stronger' means that every compact set is complete, and every complete set is closed. However, not every closed set is complete, and not every complete set is compact.

If we know that a subset of K of a metric space is compact, we get a lot of amazing properties for free.

11.1 Open sets

Before we can define closed sets in the next section (according to the standard definition), we first need to introduce *open sets*. First let us recall the definition of an (open) ball $B(p,r)$ around a point p with radius r from definition 2.4.4

$$B(p,r) := \{x \in X \mid \mathsf{dist}(x,p) < r\}.$$

The reason for the parentheses around 'open' is that yes, soon we will prove that this set is indeed open, however so far we have not defined what 'open' really is!

Before we define what it means for a set to be open, we define when a point in a subset is an *interior point*.

Definition 11.1.1. Let (X, dist) be a metric space and let A be a subset of X. A point $a \in A$ is called an *interior point* of A if

$$\text{there exists } r > 0, \\ B(a, r) \subset A.$$

Open sets are subsets for which every point in the subset is an interior point.

Definition 11.1.2. Let (X, dist) be a metric space. We say that a subset $O \subset X$ is *open* if every $x \in O$ is an interior point of O.

Having defined what it means for a set to be open, we can now prove that the (open) ball is indeed open.

Proposition 11.1.3. Let (X, dist) be a metric space. The ball

$$B(p, r) := \{x \in X \mid \mathsf{dist}(x, p) < r\}$$

is indeed open.

Before giving the proof of the proposition, I'd like to say the following. If by this point, you have the blue exercises and the best practices down, the proof of the proposition may come to you very easily. This is one of the reasons that I stress the best practices so much: whereas without them it may be difficult to even see where to start, with them the proof can be written down almost mechanically.

If you still would have difficulties giving such a proof yourself, don't worry, it takes time to get used to proving mathematical statements. If you're still struggling with following the best practices, the proofs in this chapter may help you get further in your understanding. It is especially helpful to see if you can recognize the various components of the best practices in the proofs that are given.

Proof. We need to show that every $x \in B(p,r)$ is an interior point.

Let $x \in B(p,r)$. We need to show that x is an interior point, i.e. we need to show that there exists a $\rho > 0$ such that $B(x,\rho) \subset B(p,r)$. Judging from what we need to show, we now need to prepare ourselves for choosing such a $\rho > 0$.

Since $x \in B(p,r)$, we know that $\mathsf{dist}(x,p) < r$. Then

$$r - \mathsf{dist}(x,p) > 0.$$

Choose $\rho := r - \mathsf{dist}(x,p) > 0$.

We need to show that $B(x,\rho) \subset B(p,r)$. The standard proof of such a set inclusion is by showing that for all $z \in B(x,\rho)$ it holds that $z \in B(p,r)$. So let $z \in B(x,\rho)$. We need to show that $z \in B(p,r)$, i.e. that $\mathsf{dist}(z,p) < r$.

Because $z \in B(x,\rho)$, it holds that $\mathsf{dist}(z,x) < \rho$. It now follows by the triangle inequality that

$$\begin{aligned} \mathsf{dist}(z,p) &\leq \mathsf{dist}(z,x) + \mathsf{dist}(x,p) \\ &< \rho + \mathsf{dist}(x,p) \\ &= r - \mathsf{dist}(x,p) + \mathsf{dist}(x,p) = r. \end{aligned}$$

which was what we needed to show. □

The following proposition characterizes which intervals are open.

Proposition 11.1.4 ('Open' intervals are open)**.** Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals (a,b), $(-\infty, b)$ and (a, ∞) are all open subsets of $\mathbb{R}$ (i.e. of the normed vector space $(\mathbb{R}, |\cdot|)$).

The second part of the proof gives another good example of a proof that shows that a set is open.

Proof. Note that the interval (a,b) is exactly equal to the (open) ball

$$B\left(\frac{a+b}{2}, \frac{b-a}{2}\right) = \left\{x \in \mathbb{R} \mid \left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}\right\}.$$

We therefore know that it is open by Proposition 11.1.3.

Let us now prove that $(-\infty, b)$ is open. Let $x \in (-\infty, b)$. We need to show that x is an interior point of $(-\infty, b)$. That is, we need to show that there exists an $r > 0$ such that $B(x,r) \subset (-\infty, b)$. Choose $r := b - x$, which is indeed strictly positive ($r > 0$) because $b > x$. We now need to show that $B(x,r) \subset (-\infty, b)$. Let $y \in B(x,r)$. Then $|y - x| < r$. In particular

$$y < x + r \leq x + (b - x) = b$$

so indeed $y \in (-\infty, b)$.

In a similar way, we can prove that (a, ∞) is open. □

Proposition 11.1.5. Let (X, dist) be a metric space. Then both the empty set $\emptyset$ and the set X itself (both of these are subsets of X) are open.

Proof. We will first show that the empty set is open. The argument is a bit silly (yet logically correct). We argue by contradiction. Suppose there exists a point $x \in \emptyset$ such that x is not an interior point of X. Then we have a contradiction, because the empty set has no elements.

We will now show that X is open. Let $x \in X$. We will show that x is an interior point, i.e. we will show that there exists an $r > 0$ such that $B(x,r) \subset X$.

Choose $r := 1$. Then $B(x,r) = B(x,1) \subset X$. □

The set of all interior points of a subset $A \subset X$ is called the *interior* of the set A.

Definition 11.1.6 (The interior of a set)**.** Let (X, dist) be a metric space and let $A \subset X$ be a subset of X. Then the *interior* of the set A, denoted by int A is the set of all interior points of A, i.e. int A is defined as

$$\text{int } A := \{x \in A \mid x \text{ is an interior point of } A\}.$$

Example 11.1.7. The interior of the interval $[2, 5)$ (viewed as a subset of $(\mathbb{R}, |\cdot|)$, is the interval $(2, 5)$.

Proof. We already know that $(2, 5)$ is an open subset of $\mathbb{R}$. Therefore, for all $x \in (2, 5)$, there exists an $r > 0$ such that $B(x, r) \subset (2, 5)$. Since $(2, 5) \subset [2, 5)$, we can easily show that for all $x \in (2, 5)$ there exists an $r > 0$ such that $B(x, r) \subset [2, 5)$. We conclude that the interval $(2, 5)$ is at least contained in $\text{int}[2, 5)$.

Since the interior of a set is by definition a subset of the set, the only other possible interior point is 2.

Now we will show that 2 is not an interior point of the interval $[2, 5)$. We argue by contradiction. Suppose 2 is an interior point. Then there exists an $r > 0$ such that $B(2, r) \subset [2, 5)$. Choose such an r. Then $y := (2 - r/2) \in B(2, r)$ but $y \notin [2, 5)$. This is a contradiction. Therefore 2 is not an interior point of the interval $[2, 5)$. $\square$

The interior of a set is always open.

Proposition 11.1.8. Let (X, dist) be a metric space and let $A \subset X$. Then int A is open.

At the end of this section we provide a few ways to create new open sets out of sets about which you already know that they are open.

The union of open sets is always open

Unions of open sets are always open. You may recall that if $\mathcal{I}$ is some set, and if for every $\alpha \in \mathcal{I}$ we have a subset $A_\alpha \subset X$, then the union

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha \subset X$$

is defined as

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha := \{x \in X \mid \text{there exists } \alpha \in \mathcal{I} \text{ such that } x \in A_\alpha\}.$$

Proposition 11.1.9. Let (X, dist) be a metric space, let $\mathcal{I}$ be some set and assume that for every $\alpha \in \mathcal{I}$, we have a subset $O_\alpha \subset X$. Suppose moreover that for all $\alpha \in \mathcal{I}$, the set O_α is open. Then also the union

$$\bigcup_{\alpha \in \mathcal{I}} O_\alpha$$

is open.

Example 11.1.10. We already know that for every $n \in \mathbb{N}$, the interval $(2n, 2n+1)$ is an open subset of $(\mathbb{R}, |\cdot|)$. Therefore, (choosing $\mathcal{I} = \mathbb{N}$ and $O_\alpha = (2\alpha, 2\alpha + 1)$ in the previous proposition,) we also know that the set

$$\bigcup_{n \in \mathbb{N}} (2n, 2n+1)$$

is an open subset of $(\mathbb{R}, |\cdot|)$ as well.

Finite intersections of open sets are open

Proposition 11.1.11. Let (X, dist) be a metric space and let $O_1, \dots, O_N$ be open subsets of X. Then the intersection

$$O_1 \cap \dots \cap O_N$$

is also open.

Cartesian products of open sets

Proposition 11.1.12. Let $O_1, \cdots, O_d$ be open subsets of $\mathbb{R}$. Then

$$O_1 \times \cdots \times O_d (= \{(o_1, \cdots, o_d) \mid o_i \in O_i\})$$

is an open subset of $(\mathbb{R}^d, \|\cdot\|_2)$.

11.2 Closed sets

We are now ready to give a definition of a closed set.

Definition 11.2.1. Let (X, dist) be a metric space. We say a set $C \subset X$ is *closed* if its complement $X \setminus C$ is open.

Both the empty set and the full set are closed.

Proposition 11.2.2. Let (X, dist) be a metric space. Then both the empty set $\emptyset$ and the set X itself are closed.

Proof. The empty set is closed because its complement, $X \setminus \emptyset = X$ is open by Proposition 11.1.5.

The set X is closed because its complement, $X \setminus X = \emptyset$ is open by Proposition 11.1.5. $\square$

Warning: The following two facts may conflict your expectations when you use intuition for the meaning of 'open' and 'closed' from daily life:

i. Combining Propositions 11.1.5 and 11.2.2 we see that both the empty set and the full set X are both open and closed.

ii. In Exercise 11.6.2 we will see that there is a set (and many more) that are neither open nor closed.

What does it mean in practice? If you want to show that a set is closed, **it is not enough** to show that the set is not open.

Proposition 11.2.3 (Sequence characterization of closedness)**.** A set $C \subset X$ is closed if and only if for every sequence (c_n) in C converging to some $x \in X$, it holds that $x \in C$.

Proof. Assume C is closed. Let (c_n) be a sequence in C converging to some $x \in X$. We need to show that $x \in C$, and we will argue by contradiction. So suppose that $x \in O := X \setminus C$. Since C is closed, we have (by definition of 'closed') that O is open. Since $x \in O$, and O is open, x is (by the definition of 'open') an interior point of O. Therefore, there exists an $r > 0$ such that $B(x, r) \subset O$.

Since the sequence (c_n) converges to x, we may obtain an $N \in \mathbb{N}$ such that for all $n \geq N$, $\mathsf{dist}(c_n, x) < r$. In particular, $\mathsf{dist}(c_N, x) < r$ which means that $c_N \in B(x, r) \subset O$, which is a contradiction because $c_N \in C$.

Now assume that for every sequence (c_n) in C converging to some $x \in X$, it holds that $x \in C$. We want to show that $O := X \setminus C$ is open, i.e. that every $p \in O$ is an interior point. Let $p \in O$. We need to show that p is an interior point. We argue by contradiction. Suppose $p \in O$ is not an interior point, then

$$\begin{aligned} &\text{for all } r > 0, \\ &\quad B(p, r) \not\subset O. \end{aligned} \tag{11.2.1}$$

In other words

$$\begin{aligned} &\text{for all } r > 0, \\ &\quad \text{there exists } a \in B(p, r) \\ &\qquad a \notin O. \end{aligned} \tag{11.2.2}$$

Then there exists a point $p \in X \setminus C$ which is not an interior point.

We claim that there exists a sequence $y : \mathbb{N} \to C$ with values in C converging to p. For $n \in \mathbb{N}$ we choose $r := 2^{-n}$ in (11.2.2). Then there exists an $a_n \in B(p, 2^{-n})$ such that $a_n \notin O$. Choose such an a_n and define $y_n := a_n$. Note that $y_n \in C$ and

$$0 \leq \mathsf{dist}(y_n, p) < 2^{-n}.$$

It now follows by the squeeze theorem (Theorem 6.4.1) that

$$\lim_{n\to\infty} \mathsf{dist}(y_n, p) = 0$$

and therefore (by Proposition 5.6.1) that

$$\lim_{n\to\infty} y_n = p.$$

Since (y_n) converges to p, we know by assumption that $p \in C$, which is a contradiction. □

Here is a typical example of how you can show that set is closed.

Example 11.2.4. Consider the subset A of the metric space $(\mathbb{R}^2, \|\cdot\|_2)$ defined by

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq (x_2)^2\}$$

Proof. By the sequence characterization of closedness, it suffices to show that for all sequences $y : \mathbb{N} \to A$, if the sequence y converges to some point $z \in \mathbb{R}^2$, then actually $z \in A$.

Let therefore $y : \mathbb{N} \to A$ be a sequence in A. Assume that the sequence $y : \mathbb{N} \to A$ converges to some point $z \in X$. We need to show that actually $z \in A$.

By Proposition 6.7.8, we know that the component sequences of the sequence y converge as well to the components of $z \in \mathbb{R}^2$, namely

$$\lim_{n\to\infty} y_1^{(n)} = z_1$$

and

$$\lim_{n\to\infty} y_2^{(n)} = z_2.$$

By limit theorems, we know that the limit of the sequence $n \mapsto (y_2^{(n)})^2$ also exists and

$$\lim_{n\to\infty} \left(y_2^{(n)}\right)^2 = (z_2)^2.$$

Since for all $n \in \mathbb{N}$, $y^{(n)} \in A$, we also know that for all $n \in \mathbb{N}$, $y_1^{(n)} \leq (y_2^{(n)})^2$. Therefore,

$$z_1 = \lim_{n\to\infty} y_1^{(n)} \leq \lim_{n\to\infty} \left(y_2^{(n)}\right)^2 = (z_2)^2.$$

We conclude that indeed $z \in A$. □

Proposition 11.2.5. Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals $[a, b]$, $(-\infty, b]$ and $[a, \infty)$ are all closed.

We now provide a few ways to create new closed sets out of sets about which you already know that they are closed.

Intersections of closed sets are always closed

Let (X, dist) be a metric space. If $\mathcal{I}$ is a set, and for every $\alpha \in \mathcal{I}$, we have a subset A_α of X, then the intersection

$$\bigcap_{\alpha \in I} A_\alpha$$

is defined as

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in X \mid \text{ for all } \alpha \in \mathcal{I}, x \in A_\alpha\}.$$

Proposition 11.2.6. Let (X, dist) be a metric space. Let $\mathcal{I}$ be a set and suppose for every $\alpha \in \mathcal{I}$ we have a subset $C_\alpha \subset X$. Assume that for every $\alpha \in \mathcal{I}$ the set C_α is closed. Then the intersection

$$\bigcap_{\alpha \in \mathcal{I}} C_\alpha$$

is closed as well.

Finite unions of closed sets are closed

Proposition 11.2.7. Let (X, dist) be a metric space. Let $C_1, \ldots C_N$ be closed subsets of X. Then the finite union

$$C_1 \cup \cdots \cup C_N$$

is also closed.

Products of closed sets

Proposition 11.2.8. Let $C_1, \ldots, C_d$ be closed subsets of $\mathbb{R}$. Then the Cartesian product

$$C_1 \times \cdots \times C_d (= \{(c_1, \cdots, c_d) \mid c_i \in C_i\})$$

is a closed subset of $(\mathbb{R}^d, \|\cdot\|_2)$.

The topological boundary of a set

We now give the definition of the topological boundary of a subset A of a metric space (X, dist).

Although for some sets, the topological boundary may coincide with what you intuitively think of as a ‘boundary’ of a set, for many sets the topological boundary is a very counter-intuitive set!

Definition 11.2.9 (The topological boundary)**.** Let (X, dist) be a metric space and let $A \subset X$. The *topological boundary* of a set A is denoted by ∂A and defined as

$$\partial A := X \setminus \big((\mathrm{int}\, A) \cup (\mathrm{int}(X \setminus A))\big)$$

Example 11.2.10. The topological boundary of the interval $[2, 5)$ (viewed as a subset of the normed vector space $(\mathbb{R}, |\cdot|)$) is the set $\{2, 5\}$ that exactly consists of the points 2 and 5.

Proof. In a previous example we have already shown that

$$\mathrm{int}[2, 5) = (2, 5).$$

Moreover, $\mathbb{R} \setminus [2, 5) = (-\infty, 2) \cup [5, \infty)$.

With a similar argument, we can show that

$$\mathrm{int}[5, \infty) = (5, \infty).$$

and

$$\mathrm{int}(\mathbb{R} \setminus [2, 5)) = (-\infty, 2) \cup (5, \infty).$$

Therefore

$$\partial([2, 5)) = \mathbb{R} \setminus ((2, 5)) \cup ((-\infty, 2) \cup (5, \infty)) = \{2, 5\}.$$

$\square$

11.3 Cauchy sequences

Recall that the aim of this chapter and the next is to define subsequently stronger properties for subsets of a metric space: closedness, completeness and compactness. In the previous section, we have defined what it means for a subset of a metric space to be closed, and now we are slowly going to make our way towards the definition of *complete* subsets. For that, we need the concept of *Cauchy sequences*.

Definition 11.3.1 (Cauchy sequence)**.** Let (X, dist) be a metric space. We say that a sequence $a : \mathbb{N} \to X$ is a Cauchy sequence if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } m, n \geq N,\\ &\quad\quad\quad \mathsf{dist}(a_m, a_n) < \epsilon.\end{aligned}$$

Proposition 11.3.2. Every Cauchy sequence is bounded.

Proof. We need to show that every Cauchy sequence is bounded. So let $a : \mathbb{N} \to X$ be a Cauchy sequence. We need to show that a is bounded, i.e. we need to show that

$$\begin{aligned}&\text{there exists } p \in X,\\ &\quad \text{there exists } M > 0,\\ &\quad\quad \text{for all } n \in \mathbb{N},\\ &\quad\quad\quad \mathsf{dist}(a_n, p) \leq M.\end{aligned}$$

Because (a_n) is a Cauchy sequence, we know that there exists an $N \in \mathbb{N}$ such that for every $m, n \geq N$, it holds that $\mathsf{dist}(a_m, a_n) < 1$. Choose $p := a_N$ and choose

$$M := \max(\mathsf{dist}(a_0, p), \mathsf{dist}(a_1, p), \dots, \mathsf{dist}(a_{N-1}, p), 1).$$

Let $n \in \mathbb{N}$. Then

$$\mathsf{dist}(a_n, p) \leq M.$$

□

Proposition 11.3.3. Let $a : \mathbb{N} \to X$ be a Cauchy sequence and assume that a has a subsequence converging to $p \in X$. Then the sequence a itself converges to p.

Proof. Let $a : \mathbb{N} \to X$ be a Cauchy sequence and assume that a has a subsequence $a \circ n$ converging to $p \in X$, where $n : \mathbb{N} \to \mathbb{N}$ is an index sequence. We need to show that a converges to p, i.e. we need to show that

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } m \geq N,\\ &\quad\quad\quad \mathsf{dist}(a_m, p) < \epsilon.\end{aligned}$$

Let $\epsilon > 0$. Because a is a Cauchy sequence, there exists an ℓ_0 such that for all $\ell, m \geq \ell_0$,

$$\mathsf{dist}(a_\ell, a_m) < \epsilon/2.$$

Choose $N := \ell_0$. Let $m \geq N$. Because $a \circ n$ converges to p, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\mathsf{dist}(a_{n_k}, p) < \epsilon/2.$$

Because n is an index sequence, there exists a $k \in \mathbb{N}$ such that $n_k \geq \ell_0$. We find by the triangle inequality that indeed

$$\mathsf{dist}(a_m, p) \leq \mathsf{dist}(a_m, a_{n_k}) + \mathsf{dist}(a_{n_k}, p) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Proposition 11.3.4. Let (X, dist) be a metric space. Let (x_n) be a converging sequence in X. Then (x_n) is a Cauchy sequence.

Proof. Assume that (x_n) is a converging sequence, converging to a point $p \in X$, say. Let $\epsilon > 0$. Because (x_n) converges to p, there exists an $N_0 \in \mathbb{N}$ such that for all $m \geq N_0$,

$$\mathsf{dist}(x_m, p) < \epsilon/2.$$

Choose such an N_0 and choose $N := N_0$. Let $m, n \geq N$. Then

$$\mathsf{dist}(x_m, x_n) \leq \mathsf{dist}(x_m, p) + \mathsf{dist}(p, x_n) < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

11.4 Completeness

We first now give a definition of completeness for a *metric space*.

Definition 11.4.1. Let (X, dist) be a metric space. We say that a subset $A \subset X$ is *complete* (in (X, dist)) if every Cauchy sequence in A is convergent, with limit in A.

We also say the metric space (X, dist) itself is complete if X is a complete subset of X in (X, dist).

Note that we have used the term 'complete' various times in the lecture notes: completeness of a *totally ordered field*, the series characterization of completeness in *normed vector spaces* and now completeness of a *metric space*. In the next section we will see that a normed vector space satisfies the series characterization of completeness if and only if the corresponding metric space is complete. What we will do next is show that the metric space $(\mathbb{R}, \mathsf{dist}_\mathbb{R})$ is complete (as a metric space). Under the hood, we really use the Completeness Axiom 4.2.7 for this: that axiom is really what makes everything work.

Theorem 11.4.2. The metric space $(\mathbb{R}, \mathsf{dist}_\mathbb{R})$ is complete.

Proof. Let $a : \mathbb{N} \to \mathbb{R}$ be a Cauchy sequence. Because a is a Cauchy sequence, it is in particular bounded. As a consequence, by Theorem

10.4.3, there is a subsequence $a \circ n$ such that $a \circ n$ converges to

$$\limsup_{k\to\infty} a_k.$$

Finally, we know from Proposition 11.3.3 that if a subsequence of a Cauchy sequence converges, that then the whole sequence converges. Therefore, the sequence $a : \mathbb{N} \to \mathbb{R}$ is convergent. $\square$

Proposition 11.4.3. The metric space $(\mathbb{R}^d, \mathsf{dist}_{\|\cdot\|_2})$ is complete, where $\|\cdot\|_2$ is the Euclidean norm.

The proof of Proposition 11.4.3 is the topic of Orange Exercise 11.6.3.

Proposition 11.4.4. Let (X, dist) be a metric space. Suppose $A \subset X$ is complete. Then A is closed.

Proof. Let (x_n) be a sequence in A, converging to a point $x^* \in X$. Then (x_n) is a Cauchy sequence. Since A is complete, the sequence (x_n) is converging to a point $p \in A$. By uniqueness of limits, we know that $x^* = p$. We conclude that $x^* \in A$. $\square$

The following proposition says that a subset of a complete set is complete if and only if it is closed.

Proposition 11.4.5. Let (X, dist) be a metric space and let $C \subset X$ be a complete subset. Let $A \subset C$ be a subset of C. Then, A is complete if and only if A is closed.

Proof. The "only if" side of this proposition follows from Proposition 11.4.4.

We will now show the "if" part of the proposition. Suppose A is closed. Let (x_n) be a Cauchy sequence in A. Then (x_n) is also a sequence in C.

Since C is complete, there exists a point $p \in C$ such that (x_n) converges to p. Because A is closed, in fact $p \in A$. □

11.5 Series characterization of completeness in normed vector spaces

We will now show that a normed vector space satisfies the series characterization of completeness if and only if the corresponding metric space is complete. We already know that $(\mathbb{R}^d, \|\cdot\|)$ is complete, therefore after we have proved the theorem we know that if a series in $(\mathbb{R}^d, \|\cdot\|)$ converges absolutely then it also converges.

Theorem 11.5.1. Let $(V, \|\cdot\|)$ be a normed vector space. Then $(V, \|\cdot\|)$ is complete if and only if every absolutely converging series is convergent.

Proof. We first show that 'only if' direction. Suppose $(V, \|\cdot\|)$ is complete. Let $a : \mathbb{N} \to V$ be a sequence and suppose the series

$$\sum_{k=0}^{\infty} \|a_k\|$$

is convergent. Consider also the sequence of partial sums

$$S^n := \sum_{k=0}^{n} a_k.$$

We are going to show that (S^n) is a Cauchy sequence. Let $\epsilon > 0$. Since the series

$$\sum_{k=0}^{\infty} \|a_k\|$$

converges, we know by Proposition 7.5.2 that

$$\lim_{N\to\infty} \sum_{k=N}^{\infty} \|a_k\| = 0.$$

Choose $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \|a_k\| < \epsilon.$$

Let $m, n \geq N$. Assume without loss of generality that $n > m$. Then

$$\|S^n - S^m\| = \left\| \sum_{k=m+1}^{n} a_k \right\| \leq \sum_{k=m+1}^{n} \|a_k\| \leq \sum_{k=m+1}^{\infty} \|a_k\| < \epsilon.$$

We have shown that (S^n) is a Cauchy sequence. Since $(V, \|\cdot\|)$ was assumed to be complete, the sequence (S^n) is convergent.

We now show the 'if' direction. Suppose that every absolutely converging series in V is convergent. Let $a : \mathbb{N} \to V$ be a Cauchy sequence. We need to show that $a : \mathbb{N} \to V$ is convergent.

We can construct a subsequence such that

$$\|a_{n_{k+1}} - a_{n_k}\| \leq 2^{-k}.$$

Define the sequence $b : \mathbb{N} \to V$ by

$$b_k := a_{n_{k+1}} - a_{n_k}.$$

Then the series

$$\sum_{k=0}^{\infty} \|b_k\|$$

is convergent by comparison with the converging series

$$\sum_{k=0}^{\infty} 2^{-k}.$$

It follows that the series

$$\sum_{k=0}^{\infty} b_k$$

is convergent. Note that the partial sums corresponding to this series are

$$S^{\ell} = \sum_{k=0}^{\ell} b_k = a_{n_{\ell+1}} - a_{n_0}.$$

It follows by limit theorems that the sequence (a_{n_ℓ}) is convergent. Then it follows by Proposition 11.3.3 that the sequence (a_m) converges as well. □

Corollary 11.5.2. Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Suppose the series

$$\sum_{n=0}^{\infty} a_n$$

converges absolutely, i.e. the series

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Then also the series

$$\sum_{n=0}^{\infty} a_n$$

converges.

Example 11.5.3. The series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$$

converges, because it converges absolutely. Indeed,

$$\sum_{k=1}^{\infty} \left| (-1)^k \frac{1}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a standard converging hyperharmonic series.

11.6 Exercises

11.6.1 Blue exercises

Exercise 11.6.1. Let $(V, \|\cdot\|)$ be a normed linear space and let A be the closed ball of radius 1 around the origin, i.e.

$$A := \{v \in V \mid \|v\| \leq 1\}.$$

Show that the set A is closed.

Exercise 11.6.2. Show that the interval $[0, 1)$ is neither open nor closed (seen as a subset of the normed linear space $(\mathbb{R}, |\cdot|)$).

Note the moral of the previous exercise: there are sets that are neither open nor closed.

11.6.2 Orange exercises

Exercise 11.6.3. Prove Proposition 11.4.3.

Exercise 11.6.4. Consider the following line in $\mathbb{R}^2$

$$L := \{(x, y) \in \mathbb{R}^2 \mid x + 2y = 1\}.$$

Show that L is a closed subset of $\mathbb{R}^2$ and that L is complete.

Exercise 11.6.5. Give an example of a metric space (X, dist) that is not complete (as always, actually prove that (X, dist) is indeed not complete).

Exercise 11.6.6. Consider the following subset A of $\mathbb{R}^2$

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \mid 4(x_1)^2 + (x_2)^2 \leq 25\}.$$

Prove that the set A is a closed and bounded subset of $(\mathbb{R}^2, \|\cdot\|_2)$.

Chapter 12

Compactness

In this chapter, we are going to define what it means for a subset of a metric space to be *compact*. Compactness is a strong property: Every compact subset is complete, and every complete subset is closed. We will in this chapter also give an alternative characterization of compactness: We will define what it means for a subset to be *totally bounded* and will use this concept to show that a subset is compact if and only if it is complete and totally bounded. In $(\mathbb{R}^d, \|\cdot\|_2)$ however, we will see that a subset is compact if and only if it is closed and bounded.

12.1 Definition of (sequential) compactness

In this short section, we define what it means for a subset of a metric space to be (sequentially) compact. We usually leave out the word 'sequentially'.

Definition 12.1.1 ((sequential) compactness)**.** Let (X, dist) be a metric space. We say a subset $K \subset X$ is *(sequentially) compact* if every sequence $x : \mathbb{N} \to K$ in K has a converging subsequence $x \circ n$, converging to a point $z \in K$.

The rest of the chapter will be devoted to deriving alternative characterizations of compactness. Especially in $(\mathbb{R}^d, \|\cdot\|)$, these alternative characterizations are a bit easier to deal with.

12.2 Boundedness and total boundedness

We first define what it means for a subset of a metric space to be bounded. This definition has many similarities with the definition of boundedness for sequences.

Definition 12.2.1 (bounded sets)**.** Let (X, dist) be a metric space. We say that a subset $A \subset X$ is *bounded* if

$$\begin{aligned}&\text{there exists } q \in X,\\ &\quad \text{there exists } M > 0,\\ &\qquad \text{for all } p \in A,\\ &\qquad\quad \mathsf{dist}(p, q) \leq M.\end{aligned}$$

Just as with the concept of boundedness for sequences, in normed vector spaces boundedness has a somewhat easier alternative characterization.

Proposition 12.2.2. Let $(V, \|\cdot\|)$ be a normed linear space. A subset $A \subset V$ is bounded if and only if

$$\begin{aligned}&\text{there exists } M > 0,\\ &\quad \text{for all } v \in A,\\ &\qquad \|v\| \leq M.\end{aligned}$$

We will now define what it means for a subset to be totally bounded. Intuitively, it means that for every radius $r > 0$ (which could be extremely small) the subset can be covered with only a *finite* number of balls with radius r.

Definition 12.2.3 (totally bounded sets)**.** Let (X, dist) be a metric space.

We say that a subset $A \subset X$ is *totally bounded* if

$$
\begin{aligned}
&\text{for all } r > 0, \\
&\quad \text{there exists } N \in \mathbb{N}, \\
&\quad\quad \text{there exists } p_1, \dots, p_N \in X, \\
&\quad\quad\quad A \subset \bigcup_{i=1}^{N} B(p_i, r).
\end{aligned}
$$

In the next proposition we will see that "total boundedness" is a stronger property than just "boundedness".

Proposition 12.2.4. Let (X, dist) be a metric space and let A be a subset of X. If A is totally bounded, it is bounded.

Proof. Assume A is totally bounded. We need to show that A is bounded, i.e. we need to show that

$$
\begin{aligned}
&\text{there exists } q \in X, \\
&\quad \text{there exists } M > 0, \\
&\quad\quad \text{for all } p \in A, \\
&\quad\quad\quad \mathsf{dist}(p, q) \leq M.
\end{aligned}
$$

Because A is totally bounded, we may obtain an $N \in \mathbb{N}$ and points $p_1, \dots, p_N \in X$ such that

$$A \subset \bigcup_{i=1}^{N} B(p_i, 1).$$

Choose $q := p_1$.

Choose $M := \max(\mathsf{dist}(p_2, p_1), \dots, \mathsf{dist}(p_N, p_1)) + 1$.

Let $p \in A$. Then there exists an $i \in \{1, \dots, N\}$ such that $p \in B(p_i, 1)$.

It follows that

$$\begin{aligned}
\mathsf{dist}(p,q) &= \mathsf{dist}(p,p_1) \\
&\le \mathsf{dist}(p,p_i) + \mathsf{dist}(p_i,p_1) \\
&\le 1 + \max(\mathsf{dist}(p_2,p_1),\dots,\mathsf{dist}(p_N,p_1)) \\
&= M.
\end{aligned}$$

□

In the special case of the normed vector space $(\mathbb{R}^d, \|\cdot\|_2)$, however, a subset is totally bounded if and only if it is bounded.

Proposition 12.2.5. Consider now the normed vector space $(\mathbb{R}^d, \|\cdot\|_2)$. A subset $A \subset \mathbb{R}^d$ is bounded in $(\mathbb{R}^d, \|\cdot\|_2)$ if and only if it is totally bounded.

Proof. The "if" direction follows from the previous theorem.

Let us prove the "only if" direction. Suppose $A \subset \mathbb{R}^d$ is bounded. Then we may obtain an $M > 0$ such that for all $x \in A$, $\|x\|_2 \le M$. We need to show that

for all $r > 0$,
  there exists $N \in \mathbb{N}$,
    there exists $p_1, \dots, p_N \in \mathbb{R}^d$,

$$A \subset \bigcup_{i=1}^{N} B(p_i, r).$$

Let $r > 0$. Define $\delta := r/(2\sqrt{d})$. To choose $N \in \mathbb{N}$ and the points $p_1, \dots, p_N$ we are going to make a large grid. We define the set

$$G := [-M, M]^d \cap (\delta\mathbb{Z})^d.$$

These are all points in $\mathbb{R}^d$ of which the coordinates are an integer multiple of δ, and lie between $-M$ and M. Note that G is a finite set, so we can choose an $N \in \mathbb{N}$ and points $p_1, \dots, p_N$ such that

$$G = \{p_1, \dots, p_N\}.$$

Now let $a \in A$. It suffices to show that there exists a point $g \in G$ such that

$$\mathsf{dist}(a, g) < r.$$

We can express a in its components in $\mathbb{R}^d$ as $a = (a_1, \dots, a_d)$. We can then define the point

$$g := (\delta\lceil a_1/\delta\rceil, \dots, \delta\lceil a_d/\delta\rceil)$$

in G. Note that for all $i \in \{1, \dots, d\}$, $|g_i - a_i| \leq \delta$. Therefore

$$\|g - a\|_2 = \sqrt{\sum_{i=1}^{d}(g_i - a_i)^2} \leq \sqrt{\sum_{i=1}^{d}\delta^2} = \sqrt{d}\delta \leq \frac{r}{2} < r.$$

□

12.3 Alternative characterization of compactness

We are now ready to show that a subset $K \subset X$ is compact if and only if it is complete and totally bounded. The proof is one of the most beautiful, but also one of the most complicated in these lecture notes. You may want to skip it on first reading.

Theorem 12.3.1. A subset $K \subset X$ is compact if and only if it is complete and totally bounded.

Proof. We first show the "only if" direction. Suppose $K \subset X$ is compact.

We are going to show that K is totally bounded. We argue by contradiction. So suppose the set K is not totally bounded. Then

$$\begin{aligned}&\text{there exists } r > 0,\\ &\quad \text{for all } N \in \mathbb{N},\\ &\qquad \text{for all } p_1, \dots, p_N \in X,\\ &\qquad\quad K \not\subset \bigcup_{i=1}^{N} B(p_i, r).\end{aligned}$$

Choose such an $r > 0$. We can now inductively construct a sequence of points $p_0, p_1, p_2, \dots$ in K such that

$$\mathsf{dist}(p_i, p_j) \geq r$$

for every $i \neq j$. Note that we can also phrase this last property as that for all $k \in \mathbb{N}$, and all $i < k$, $\mathsf{dist}(p_i, p_k) \geq r$.

We first just take some point $p_0 \in K$. Now let $k \in \mathbb{N}$ and assume the points $p_0, \dots, p_k$ have already been defined, and $\mathsf{dist}(p_i, p_k) \geq r$ for $i \in \{0, \dots, k-1\}$. Then we know that

$$K \not\subset \bigcup_{i=0}^{k} B(p_i, r).$$

In other words, there exists a point

$$q \in K \setminus \left(\bigcup_{i=0}^{k} B(p_i, r) \right).$$

Now define $p_{k+1} := q$. Then indeed for all $i = 0, \dots, k$, it holds that $\mathsf{dist}(p_{k+1}, p_i) \geq r$.

Because K is compact, there is a converging subsequence (p_{n_i}) in K. In particular, the sequence (p_{n_i}) is a Cauchy sequence, so that for k, ℓ large enough in fact

$$\mathsf{dist}(p_{n_k}, p_{n_\ell}) < r.$$

This is a contradiction.

We will now show that K is complete. Let (x_n) be a Cauchy sequence in K. Since K is compact, there is a converging subsequence (x_{n_k}) of (x_n), converging to a point $z \in K$. But then the original sequence (x_n) converges to z as well by Proposition 11.3.3.

We will now show the "if" direction. Assume that $K \subset X$ is complete and totally bounded. We are going to show that K is compact.

Let (x_n) be an arbitrary sequence in K. We are going to construct a limit point $x^* \in K$ by what is called a *diagonal argument*. The preparation for this is as follows. We are going to define subsequences of (x_n) inductively. Precisely, we will use induction to, for every $k \in \mathbb{N}$, construct an index sequence $n^{(k)} : \mathbb{N} \to \mathbb{N}$. Said differently, we are going to construct a sequence of index sequences. Moreover, we will construct this sequence of index sequences in such a way that for every $\tilde{k}, k \in \mathbb{N}$, if $\tilde{k} \geq k$ then the index sequence $n^{(\tilde{k})}$ is a subsequence of the index sequence $n^{(k)}$, and such that for all $k \in \mathbb{N}$, and all $\ell, j \in \mathbb{N}$,

$$\mathsf{dist}\left(x_{n_\ell^{(k)}}, x_{n_j^{(k)}}\right) < \frac{2}{k}. \tag{12.3.1}$$

We will now tackle the base case of the inductive definition. For this, we let the index sequence $n^{(0)} : \mathbb{N} \to \mathbb{N}$ be just the identity function, i.e.

$$n_\ell^{(0)} := \ell.$$

We now continue with the inductive step of the inductive definition. Let $k \in \mathbb{N}$ and assume that the index sequence $n^{(k-1)} : \mathbb{N} \to \mathbb{N}$ is defined for some $k \in \mathbb{N} \setminus \{0\}$. We are going to define the index sequence $n^{(k)} : \mathbb{N} \to \mathbb{N}$ as a subsequence of $n^{(k-1)}$. Note that there exists an $N_k \in \mathbb{N}$ and points $p_1^{(k)}, \dots, p_{N_k}^{(k)} \in K$ such that

$$K \subset \bigcup_{i=1}^{N_k} B\left(p_i^{(k)}, 1/k\right).$$

Said differently, the set K is covered by only *finitely* many balls of radius $1/k$. Hence, there exists a point $p_{i_k}^{(k)}$ such that $x_{n_\ell^{(k-1)}} \in B(p_{i_k}^{(k)}, 1/k)$

for *infinitely* many $\ell \in \mathbb{N}$. Therefore, there exists an index sequence $n^{(k)} : \mathbb{N} \to \mathbb{N}$, itself a subsequence of $n^{(k-1)}$, such that

$$x_{n_\ell^{(k)}} \in B\left(p_{i_k}^{(k)}, 1/k\right)$$

for all $\ell \in \mathbb{N}$. In particular, for all $\ell, j \in \mathbb{N}$,

$$\mathsf{dist}\left(x_{n_\ell^{(k)}}, x_{n_j^{(k)}}\right) \leq \mathsf{dist}\left(x_{n_\ell^{(k)}}, p_{i_k}^{(k)}\right) + \mathsf{dist}\left(p_{i_k}^{(k)}, x_{n_j^{(k)}}\right) < \frac{2}{k}.$$

Now define the sequence $m : \mathbb{N} \to \mathbb{N}$

$$m_\ell := n_\ell^{(\ell)}.$$

We claim that $m : \mathbb{N} \to \mathbb{N}$ is an index sequence. To prove the claim, we need to show that $m : \mathbb{N} \to \mathbb{N}$ is strictly increasing. This follows because for every $\ell \in \mathbb{N}$,

$$m_\ell = n_\ell^{(\ell)} < n_{\ell+1}^{(\ell)} \leq n_{\ell+1}^{(\ell+1)} = m_{\ell+1}$$

where for the strict inequality we used that $n^{(\ell)}$ is an index sequence and therefore strictly increasing, while the second inequality follows because for every two index sequences $a, b : \mathbb{N} \to \mathbb{N}$ and every $i \in \mathbb{N}$, $a_{b_i} \geq a_i$ (this can be applied to the case where $a \circ b$ equals $n^{(\ell+1)}$ and a equals $n^{(\ell)}$). This finishes the proof of the claim that $m : \mathbb{N} \to \mathbb{N}$ is an index sequence.

We now claim that the sequence $(x_{m_\ell})_\ell$ is a Cauchy sequence. Let $\epsilon > 0$. Choose $M := \lceil 2/\epsilon \rceil + 1$. Let $\tilde{\ell}, \tilde{j} \geq M$. Then, because $n^{(\tilde{\ell})}$ is a subsequence of $n^{(M)}$, we can find an $\ell \in \mathbb{N}$ such that

$$m_{\tilde{\ell}} = n_{\tilde{\ell}}^{(\tilde{\ell})} = n_\ell^{(M)}.$$

For the same reason, we may find a $j \in \mathbb{N}$ such that $m_{\tilde{j}} = n_j^{(M)}$. It follows by (12.3.1) that

$$\text{dist}\left(x_{m_{\tilde{\ell}}}, x_{m_{\tilde{j}}}\right) = \text{dist}\left(x_{n_\ell^{(M)}}, x_{n_j^{(M)}}\right) < \frac{2}{M} < \epsilon.$$

Since K is complete, the sequences $(x_{m_\ell})_\ell$ is convergent with a limit $x^* \in K$. This was what we needed to show. □

In the special case of $(\mathbb{R}^d, \|\cdot\|_2)$ we have an easier alternative characterization of compactness.

Theorem 12.3.2 (Heine-Borel Theorem)**.** A subset of $(\mathbb{R}^d, \|\cdot\|_2)$ is compact if and only if it is closed and bounded.

Proof. We first show the "only if" direction. Suppose $A \subset \mathbb{R}^d$ is compact. We will first show that A is closed. By the alternative characterization of compactness in Theorem 12.3.1, we know that the set A is complete. By Proposition 11.4.4, which says that every complete set is closed, it follows that the set A is closed.

We will now show that A is bounded. By Theorem 12.3.1 we know that A is totally bounded. By Proposition 12.2.4, which says that every totally bounded set is bounded, we know that A is bounded.

Suppose now that $A \subset \mathbb{R}^d$ is closed and bounded. Because $\mathbb{R}^d$ is complete, the closed set A is a subset of a complete set, and (by Proposition 11.4.5) therefore itself complete as well. Moreover, by Proposition 12.2.5, we know that every bounded subset of $(\mathbb{R}^d, \|\cdot\|_2)$ is also totally bounded. In particular, the set A is totally bounded. We conclude that A is complete and totally bounded, and therefore compact by the alternative characterization of compactness in Theorem 12.3.1. □

12.4 Exercises

12.4.1 Blue exercises

Exercise 12.4.1. Let $a, b \in \mathbb{R}$ be two real numbers such that $a < b$. Prove that the interval $[a, b]$ is a compact subset of the normed vector space $(\mathbb{R}, |\cdot|)$.

Exercise 12.4.2. Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence in X. Show that the sequence $a : \mathbb{N} \to X$ is *bounded* (according to Definition 5.2.1) if and only if the set

$$A := \{a_n \mid n \in \mathbb{N}\}$$

is bounded (according to Definition 12.2.1).

12.4.2 Orange exercises

Exercise 12.4.3. Consider the metric space $((0,1), \mathsf{dist})$, where $(0,1)$ denotes the interval from 0 to 1 and $\mathsf{dist}(x, y) = |x - y|$. Prove that $(0,1)$ is a closed and bounded subset of the metric space $((0,1), \mathsf{dist})$. Also prove that $(0,1)$ is not a *compact* subset of the metric space $((0,1), \mathsf{dist})$.

The moral of this exercise is that the Heine-Borel theorem (Theorem 12.3.2) is an alternative characterization of compactness in $(\mathbb{R}^d, \|\cdot\|_2)$: for other metric spaces or normed vector spaces it does not hold that subsets are compact if and only if they are closed and bounded.

Exercise 12.4.4. Let (X, dist) be a metric space and let $K \subset X$ be a compact subset. Let $a : \mathbb{N} \to X$ be a sequence with values in X, such that

$$\begin{aligned} &\text{for all } N \in \mathbb{N}, \\ &\quad \text{there exists } \ell \geq N, \\ &\quad\quad a_\ell \in K. \end{aligned} \tag{12.4.1}$$

(this is formal equivalent way of saying that there are infinitely many $\ell \in \mathbb{N}$ such that $a_\ell \in K$).

The exercise consists of two parts:

i. Use (12.4.1) to inductively define an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, $a_{n_k} \in K$. See the best practices on how to set up such an inductive definition.

ii. Use the fact that K is compact to show that there is a point $p \in K$ and a subsequence of $a : \mathbb{N} \to X$ converging to p.

Exercise 12.4.5. Consider the sets

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 1\}$$

and

$$B := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 \leq 1\}$$

Prove that the set $A \cap B$ is compact (as a subset of the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$).

Chapter 13

Limits and continuity

We are finally ready to treat perhaps the most important target of Analysis 1: the concepts of *limits* and *continuity*. These concepts embody the adagio of this course: to make rigorous statements about the approximate behavior of functions.

The setting is as follows: We will consider functions $f : D \to Y$ mapping from a subset $D \subset X$ of a metric space (X, dist_X) to a metric space (Y, dist_Y). These are quite some actors: an input metric space (X, dist_X), a subset D of the metric space, and an output metric space (Y, dist_Y), and the concept of *limits* and *continuity* depend on all these actors. That makes it a bit tricky.

On the coarsest level, if $p \in X$ and $q \in Y$ then the statement that

$$\lim_{x \to p} f(x) = q$$

will mean that if the distance between x and p is small, but not zero, the distance between $f(x)$ and q will be small. Using a similar approach as we took with sequences, we will make this vague statement completely rigorous.

There is, however, one critical, tricky point. This concept of limits only behaves nicely if p satisfies a special property with respect to the set D on which f is defined. We will discuss this property in the next section.

13.1 Accumulation points

To get a useful concept of a limit in a point $p \in X$, the point p needs to be an *accumulation point* of the domain D of the function.

Definition 13.1.1 (Accumulation points)**.** Let (X, dist_X) be a metric space and let $D \subset X$ be a subset of X. We say a point $p \in X$ is an *accumulation point* of the set D if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } x \in D,\\ &\qquad 0 < \mathsf{dist}_X(x, p) < \epsilon.\end{aligned}$$

We denote the set of accumulation points of a set D by D'.

Note that accumulation points of a set D do not have to lie in the set D themselves. If a point does lie in D, but is not an accumulation point, then we call it an isolated point.

Definition 13.1.2 (Isolated points)**.** Let (X, dist) be a metric space and let $D \subset X$ be a subset of X. We say a point $a \in D$ is an *isolated point* if it is not an accumulation point, i.e. if $a \in D \setminus D'$.

13.2 Limit in an accumulation point

We can now define limits in accumulation points of D.

Definition 13.2.1 (Limit in an accumulation point)**.** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$ be a subset of X. Let $f : D \to Y$ be a function and let $q \in Y$ be a point in Y. Let $a \in D'$ be an accumulation point of D. Then we say f converges to q as x goes to a, and write

$$\lim_{x \to a} f(x) = q$$

if

for all $\epsilon > 0$,
there exists $\delta > 0$,
for all $x \in D$,
if $0 < \mathsf{dist}_X(x, a) < \delta$, then $\mathsf{dist}_Y(f(x), q) < \epsilon$.

13.3 Uniqueness of limits

Proposition 13.3.1. Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subset X$ be a subset of X. Let $f : D \to Y$ be a function on D. Let $a \in D'$ and assume

$$\lim_{x \to a} f(x) = p \qquad \text{and} \qquad \lim_{x \to a} f(x) = q$$

for points $p, q \in Y$. Then $p = q$.

Proof. Suppose $p \neq q$. Choose $\epsilon := \mathsf{dist}_Y(p, q)/2$. Then there exists a $\delta_1 > 0$ such that for all $x \in D$, if $0 < \mathsf{dist}_X(x, a) < \delta_1$ then

$$\mathsf{dist}_Y(f(x), p) < \epsilon = \frac{\mathsf{dist}_Y(p, q)}{2}$$

and there exists a $\delta_2 > 0$ such that for all $x \in D$, if $0 < \mathsf{dist}_X(x, a) < \delta_2$ then

$$\mathsf{dist}_Y(f(x), q) < \epsilon = \frac{\mathsf{dist}_Y(p, q)}{2}.$$

Choose such $\delta_1 > 0$ and $\delta_2 > 0$.

Now define $\delta := \min(\delta_1, \delta_2) > 0$. Because a is an accumulation point of D, there exists a point $b \in B(a, \delta)$. Then

$$\mathsf{dist}_Y(f(b), p) < \frac{\mathsf{dist}_Y(p, q)}{2}$$

and

$$\mathsf{dist}_Y(f(b), q) < \frac{\mathsf{dist}_Y(p, q)}{2}.$$

Therefore by the triangle inequality

$$\begin{aligned}\mathsf{dist}_Y(p, q) &\leq \mathsf{dist}_Y(p, f(b)) + \mathsf{dist}_Y(f(b), q)\\ &< \frac{\mathsf{dist}_Y(p, q)}{2} + \frac{\mathsf{dist}_Y(p, q)}{2}\\ &= \mathsf{dist}_Y(p, q).\end{aligned}$$

which gives a contradiction. □

13.4 Sequence characterization of limits

Theorem 13.4.1 (Sequence characterization of limits)**.** Let (X, dist_X) and (Y, dist_Y) be two metric spaces. Let $D \subset X$. Let $f : D \to Y$ and let $a \in D'$. Let $q \in Y$. Then

$$\lim_{x \to a} f(x) = q$$

if and only if

$$\begin{aligned}&\text{for all sequences } (x^n) \text{ in } D \setminus \{a\} \text{ converging to } a,\\ &\quad \lim_{n \to \infty} f(x^n) = q.\end{aligned}$$

Proof. We will first show the "only if" direction.

So assume that $\lim_{x \to a} f(x) = q$. We need to show that

$$\begin{aligned}&\text{for all sequences } (x^n) \text{ in } D \setminus \{a\} \text{ such that } \lim_{n \to \infty} x^n = a,\\ &\quad \lim_{n \to \infty} f(x^n) = q.\end{aligned}$$

Take therefore a sequence (x^n) in $D \setminus \{a\}$ such that $\lim_{n \to \infty} x^n = a$.

We now need to show that $\lim_{n \to \infty} f(x^n) = q$, i.e. we need to show that

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad \mathsf{dist}_Y(f(x^n), q) < \epsilon.\end{aligned}$$

Let $\epsilon > 0$. Because $\lim_{x\to a} f(x) = q$, there exists a $\delta > 0$ such that for all $x \in D$, if $0 < \mathsf{dist}_X(x, a) < \delta$ then

$$\mathsf{dist}_Y(f(x), q) < \epsilon.$$

Choose such a $\delta > 0$.

Because $\lim_{n\to\infty} x^n = a$, we know that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\mathsf{dist}_X(x^n, a) < \delta.$$

Choose such an $N \in \mathbb{N}$.

Let $n \geq N$. Then because $\mathsf{dist}_X(x^n, a) < \delta$ indeed

$$\mathsf{dist}_Y(f(x^n), q) < \epsilon.$$

This finishes the proof of the "only if" direction.

We will now show the "if" direction. Assume that

$$\begin{aligned}&\text{for all sequences } (x^n) \text{ in } D \setminus \{a\} \text{ such that } \lim_{n\to\infty} x^n = a,\\ &\quad \lim_{n\to\infty} f(x^n) = q.\end{aligned}$$

We will show that

$$\lim_{x\to a} f(x) = q.$$

We argue by contradiction. So assume there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists a point $x^* \in D$ such that $0 < \mathsf{dist}_X(x^*, a) < \delta$

but $\mathsf{dist}_Y(f(x^*), q) \geq \epsilon$. Choose such an $\epsilon > 0$. Then

$$\begin{aligned} &\text{for all } \delta > 0, \\ &\quad \text{there exists } x^* \in D, \\ &\qquad (0 < \mathsf{dist}_X(x^*, a) < \delta) \text{ and } \mathsf{dist}_Y(f(x^*), q) \geq \epsilon. \end{aligned} \tag{13.4.1}$$

We are now going to define a sequence (x^n) in $D \setminus \{a\}$, converging to a. Let $n \in \mathbb{N}$. Choose $\delta := 2^{-n}$ in (13.4.1). Then we may obtain an $x^* \in D$ such that $0 < \mathsf{dist}_X(x^*, a) < 2^{-n}$ while $\mathsf{dist}_Y(f(x^*), q) \geq \epsilon$. Define $x^n := x^*$.

Then the sequence (x^n) is a sequence in $D \setminus \{a\}$ converging to a, however it does not hold that $\lim_{n\to\infty} f(x^{(n)}) = q$, which is a contradiction. □

13.5 Limit laws

Just as with limits of sequences, if we need to show that a certain limit exists or if we need to determine its value, we usually want to avoid going back to the formal definition. Instead, we would like to rely on limit laws like the following.

Theorem 13.5.1. Let (X, dist_X) be a metric space and let $(V, \|\cdot\|)$ be a normed vector space. Let $D \subset X$ and let $f : D \to V$ and $g : D \to V$ be two functions. Let $a \in D'$. Moreover, assume that the limit $\lim_{x\to a} f(x)$ exists and equals $p \in V$ and the limit $\lim_{x\to a} g(x)$ exists and equals $q \in V$. Let $\lambda \in \mathbb{R}$. Then

i. The limit $\lim_{x\to a}(f(x) + g(x))$ exists and equals $p + q$.

ii. The limit $\lim_{x\to a}(\lambda f(x))$ exists and equals λp.

13.6 Continuity

Definition 13.6.1 (Continuity in a point)**.** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$ be a subset of X. We say a function $f : D \to Y$ is *continuous* in a point $a \in D \cap D'$ if

$$\lim_{x \to a} f(x) = f(a).$$

If $a \in D$ is an isolated point, i.e. if $a \in D \setminus D'$, then we also say that f is continuous in a.

We say a function is continuous if it is continuous in every point in its domain.

Definition 13.6.2 (Continuity on the domain)**.** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$ be a subset of X. We say a function $f : D \to Y$ is *continuous on* D if f is continuous in a for every $a \in D$.

Sometimes it is a bit cumbersome to make the distinction between isolated points and accumulation points. The following alternative characterization of continuity in a point circumvents this issue.

Proposition 13.6.3 (Alternative $\epsilon - \delta$ characterization of a continuity in a point)**.** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$ be a subset of X. Let $a \in D$. Then the function f is continuous in a if and only if

for all $\epsilon > 0$,
there exists $\delta > 0$,
for all $x \in D$,
if $0 < \mathsf{dist}_X(x, a) < \delta$, then $\mathsf{dist}_Y(f(x), f(a)) < \epsilon$.

13.7 Sequence characterization of continuity

As with many concepts in the course, continuity is conveniently probed with sequences.

Theorem 13.7.1 (Sequence characterization of continuity)**.** Let (X, dist_X) and (Y, dist_Y) be metric spaces. Let $D \subset X$ and let $f : D \to Y$ be a function. Let $a \in D$. The function f is continuous in a if and only if

$$\begin{aligned} &\text{for all sequences } (x^n) \text{ in } D \text{ converging to } a, \\ &\quad \lim_{n\to\infty} f(x^n) = f(a). \end{aligned}$$

13.8 Rules for continuous functions

The following proposition implies that the composition of two continuous functions is also continuous.

Proposition 13.8.1. Let (X, dist_X), (Y, dist_Y) and (Z, dist_Z) be metric spaces, let $D \subset X$ and $E \subset Y$. Let $f : D \to Y$ and $g : E \to Z$ be two functions, and assume that $f(D) \subset E$. Let $a \in D$. If f is continuous in a and g is continuous in $f(a)$ then $g \circ f$ is continuous in a.

Proof. We use the sequence characterization of continuity. We need to show that for every sequence (x^n) in D converging to a, in fact

$$\lim_{n\to\infty} (g \circ f)(x^n) = (g \circ f)(a)$$

or written differently that

$$\lim_{n\to\infty} g(f(x^n)) = g(f(a)).$$

Let (x^n) be a sequence in D converging to a. Since f is continuous, we know that

$$\lim_{n\to\infty} f(x^n) = f(a)$$

or in words the sequence $(f(x^n))$ converges to $f(a)$. Because g is continuous in $f(a)$, it follows that

$$\lim_{n\to\infty} g(f(x^n)) = g(f(a))$$

which was what we needed to show. □

13.9 Images of compact sets under continuous functions are compact

Proposition 13.9.1. Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $K \subset X$ be a compact subset of X. Let $f : K \to Y$ be continuous on K. Then $f(K)$ is a compact subset of Y.

Proof. It suffices to show that for every sequence (y_n) in $f(K)$ there is a subsequence (y_{n_k}) and a point $p \in f(K)$ such that $y_{n_k} \to p$ as $k \to \infty$.

Let (y_n) be a sequence in $f(K)$. Then there exists a sequence (x_n) such that $y_n = f(x_n)$ for every $n \in \mathbb{N}$. Because K is compact, there exists a point $z \in K$ and a subsequence (x_{n_k}) such that the subsequence converges to z. Because f is continuous, we find

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(z) \in f(K).$$

Therefore the subsequence (y_{n_k}) of y_n converges to the point $f(z)$ in $f(K)$, which shows that $f(K)$ is compact. □

13.10 Uniform continuity

Definition 13.10.1. Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subset X$ be a non-empty subset. We say that $f : D \to Y$ is *uniformly continuous* on D if

for all $\epsilon > 0$,
  there exists $\delta > 0$,
    for all $p, q \in D$,
      $0 < \mathsf{dist}_X(p, q) < \delta \implies \mathsf{dist}_Y(f(p), f(q)) < \epsilon.$

The following proposition shows that *uniform continuity* is a stronger property than continuity.

Proposition 13.10.2. Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subset X$ be a non-empty subset. Let $f : D \to Y$ be uniformly continuous on D. Then f is continuous on D.

Proof. We need to show that for all $a \in D$ the function f is continuous in a.

If $a \in D \setminus D'$, then f is continuous in a by definition. Let $a \in D'$. We need to show that

$$\lim_{x \to a} f(x) = f(a).$$

Let $\epsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that for all $p, q \in D$,

$$0 < \mathsf{dist}_X(p, q) < \delta \implies \mathsf{dist}_Y(f(p), f(q)) < \epsilon.$$

Choose such a $\delta > 0$.

Let $p \in D$. Assume that $0 < \mathsf{dist}_X(p, a) < \delta$. Then

$$\mathsf{dist}_Y(f(p), f(a)) < \epsilon.$$

Therefore, f is continuous in a. $\square$

Although uniform continuity is stronger than continuity, if a function is continuous on a compact set, it is even uniformly continuous.

Theorem 13.10.3. Let (X, dist_X) and (Y, dist_Y) be metric spaces, let $K \subset X$ be compact and let $f : K \to Y$ be continuous on K. Then f is uniformly continuous on K.

Proof. We argue by contradiction. Suppose f is not uniformly continuous. Then there exists an $\epsilon > 0$ such that for every $\delta > 0$, there exist points $x, y \in K$ with $0 < \mathsf{dist}_X(x, y) < \delta$ yet $\mathsf{dist}_Y(f(x), f(y)) \geq \epsilon$. We will now use this to construct sequences (p_n) and (q_n) in K.

We define these as follows. For $n \in \mathbb{N}$, we know that there exist points $p_n, q_n \in K$ with

$$0 < \mathsf{dist}_X(p_n, q_n) < \frac{1}{n}$$

yet $\mathsf{dist}_Y(f(p_n), f(q_n)) \geq \epsilon$.

Because K is compact, there exists a subsequence (p_{n_k}) of (p_n) converging to some point $a \in K$. Since $0 < \mathsf{dist}_X(p_n, q_n) < \frac{1}{n}$, it follows by the triangle inequality that

$$\begin{aligned} 0 \leq \mathsf{dist}_X(a, q_{n_k}) &\leq \mathsf{dist}_X(a, p_{n_k}) + \mathsf{dist}_X(p_{n_k}, q_{n_k}) \\ &< \mathsf{dist}_X(a, p_{n_k}) + \frac{1}{n_k} \end{aligned}$$

so that by the squeeze theorem and Proposition 5.6.1 we conclude that (q_{n_k}) converges to a as well. Since f is continuous in a, we know that

$$\lim_{k\to\infty} \mathsf{dist}_Y(f(p_{n_k}), f(a)) = \lim_{k\to\infty} \mathsf{dist}_Y(f(q_{n_k}), f(a)) = 0.$$

As a consequence, there exists a k such that

$$\mathsf{dist}_Y\left(f(p_{n_k}), f(q_{n_k})\right) < \epsilon,$$

which is a contradiction. □

13.11 Exercises

13.11.1 Blue exercises

Exercise 13.11.1. Let (X, dist_X) be $(\mathbb{R}^2, \mathsf{dist}_{\|\cdot\|_2})$ (i.e. the metric space associated to the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$) and let (Y, dist_Y) be $(\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$. Let $D = B(0,1) \subset \mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} x_1^2 + x_2^2 & \text{if } x \neq (0,0) \\ 185 & \text{if } x = (0,0). \end{cases}$$

Show that
$$\lim_{x \to (0,0)} f(x) \left(= \lim_{x \to (0,0)} x_1^2 + x_2^2 \right) = 0.$$

Exercise 13.11.2. Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x) = x \qquad \text{for } x \in \mathbb{R}$$

where $D = \mathbb{R}$. Prove that for every $a \in D$, the function f is continuous in a (when viewed as a function from $(\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$ to $(\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$).

Exercise 13.11.3. Let $(X, \mathsf{dist}_X) := (\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$ and set $D := \mathbb{N} \subset \mathbb{R}$. Let (Y, dist_Y) be a metric space and let $a : \mathbb{N} \to Y$ be a function. Show that $a : \mathbb{N} \to Y$ is continuous (when viewed as a function defined on $D := \mathbb{N}$ as a subset of the metric space (X, dist_X) mapping to the metric space (Y, dist_Y)).

13.11.2 Orange exercises

Exercise 13.11.4. Let (X, dist_X) and (Y, dist_Y) be two metric spaces, and let $D \subset X$ be a subset of X. Let $f : D \to Y$ be a bounded function (i.e. $f(D)$ is a bounded subset of (Y, dist_Y)) and let $a \in D \cap D'$. Define $\omega : (0, \infty) \to [0, \infty)$ by

$$\omega(r) := \sup\{\mathsf{dist}_Y(f(x), f(a)) \mid x \in D \text{ and } \mathsf{dist}_X(x, a) < r\}.$$

Suppose
$$\inf\{\omega(r) \mid r \in (0, \infty)\} = 0.$$
Show that $f : D \to Y$ is continuous in $a \in D \cap D'$.

Exercise 13.11.5. Let (X, dist_X) and (Y, dist_Y) be metric spaces, let $D \subset X$ and let $f : D \to Y$. Assume that $f : D \to Y$ is *Lipschitz continuous*, that means that there exists a constant $M > 0$ such that for all $x, z \in D$,

$$\mathsf{dist}_Y(f(x), f(z)) \leq M \, \mathsf{dist}_X(x, z).$$

Show that $f : D \to Y$ is uniformly continuous on D.

Chapter 14

Real-valued functions

14.1 More limit laws

If we need to show that a limit exists, or if we need to compute its value, we usually try to avoid going back to the formal definition of a limit. Instead, just as we did when we were working with sequences, we prefer to rely on limit laws.

The following theorem presents some limit laws for real-valued functions.

Theorem 14.1.1 (Limit laws for real-valued functions)**.** Let (X, dist) be a metric space, let D be a subset of X and assume that $a \in D'$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two real-valued functions and assume that $\lim_{x\to a} f(x)$ exists and equals $M \in \mathbb{R}$ and $\lim_{x\to a} g(x)$ exists and equals $L \in \mathbb{R}$. Then

i. For every $m \in \mathbb{N}$, the limit $\lim_{x\to a}(f(x))^m$ exists and equals M^m.

ii. The limit $\lim_{x\to a}(f(x)g(x))$ exists and equals ML.

iii. If $L \neq 0$, the limit
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$
exists and equals M/L.

iv. If for all $x \in D$, $f(x) \geq 0$, then for every $k \in \mathbb{N} \setminus \{0\}$,

$$\lim_{x \to a} \sqrt[k]{f(x)} = \sqrt[k]{M}.$$

The proof of this theorem follows from the sequence characterization of limits (Theorem 13.4.1), and from limit laws for sequences (Theorem 6.3.1). The proof of part (ii) will be the aim of Exercise 14.12.1.

14.2 Building new continuous functions

The following theorem translates the limit laws of Section 14.1 into statements about continuity.

Theorem 14.2.1. Let (X, dist) be a metric space, let D be a subset of X and assume $a \in D$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two real-valued functions that are continuous in $a \in D$.

i. For every $m \in \mathbb{N}$, the function f^m is continuous in a.

ii. The function $f + g$ is continuous in a.

iii. The function $f \cdot g$ is continuous in a.

iv. If $g(a) \neq 0$, the function f/g is continuous in a.

v. If for all $x \in D$, $f(x) \geq 0$, then for every $k \in \mathbb{N} \setminus \{0\}$, the function $\sqrt[k]{f}$ is continuous in a.

14.3 Continuity of standard functions

Proposition 14.3.1. Every (possibly multivariate) polynomial is continuous as a function from $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$.

Proposition 14.3.2. Every (possibly multivariate) rational function is continuous on its domain of definition (viewed as a function from a domain in $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$).

In some sense, we're not ready for the next proposition: not only do we not yet have the tools to prove it, worse, we are not even ready to define the functions involved. Most likely, however, you have seen these functions in high school or in Calculus. I think it's good to mention the proposition now anyway, as it plays a central role when you want to show that some more complicated functions are continuous.

Proposition 14.3.3 (Continuity of some standard functions)**.** The functions

$$\begin{aligned} \exp &: \mathbb{R} \to \mathbb{R} & \ln &: (0,\infty) \to \mathbb{R} \\ \sin &: \mathbb{R} \to \mathbb{R} & \arcsin &: [-1,1] \to \mathbb{R} \\ \cos &: \mathbb{R} \to \mathbb{R} & \arccos &: [-1,1] \to \mathbb{R} \\ \tan &: (-\pi/2, \pi/2) \to \mathbb{R} & \arctan &: \mathbb{R} \to \mathbb{R} \end{aligned}$$

are all continuous.

14.4 Limits from the left and from the right

Definition 14.4.1 (Limit from the left)**.** Let (Y, dist_Y) be a metric space, and let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$. Let $f : D \to Y$ be a function. Let $a \in \mathbb{R}$ be such that $a \in ((-\infty, a) \cap D)'$, i.e. such that a is an accumulation point of the set $(-\infty, a) \cap D$ in the metric space $(\mathbb{R}, \mathsf{dist}_\mathbb{R})$. Let $q \in Y$. We say that $f(x)$ *converges to q as x approaches a from the left (or from below)*, and write

$$\lim_{x \uparrow a} f(x) = q \quad \left(\text{or sometimes } \lim_{x \to a^-} f(x) = q \right)$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D \cap (-\infty, a),\\
&\quad\quad\quad 0 < \mathsf{dist}_{\mathbb{R}}(x, a) < \delta \implies \mathsf{dist}_Y(f(x), q) < \epsilon.
\end{aligned}$$

Definition 14.4.2 (Limit from the right)**.** Let (Y, dist_Y) be a metric space, and let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$. Let $f : D \to Y$ be a function. Let $a \in \mathbb{R}$ be such that $a \in ((a, \infty) \cap D)'$, i.e. such that a is an accumulation point of the set $(a, \infty) \cap D$ in the metric space $(\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$. Let $q \in Y$. We say that $f(x)$ *converges to q as x approaches a from the right (or from above)*, and write

$$\lim_{x \downarrow a} f(x) = q \quad \left(\text{or sometimes } \lim_{x \to a^+} f(x) = q\right)$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D \cap (a, \infty),\\
&\quad\quad\quad 0 < \mathsf{dist}_{\mathbb{R}}(x, a) < \delta \implies \mathsf{dist}_Y(f(x), q) < \epsilon.
\end{aligned}$$

14.5 The extended real line

In the next few sections, we will discuss several limits involving infinity. There are many combinations of such limits possible, all of them with their own types of limit theorems and alternative characterizations, and because of this it tends to get a bit out of hand. One remedy is to organize the information, definitions and arguments by introducing the *extended real line*. With that tool, we can use previous limit laws and sequence characterizations for the analysis of limits involving infinity.

We first discuss the extended real line as a *set*.

Definition 14.5.1 (The extended real line)**.** The *extended real line* $\mathbb{R}_{\mathsf{ext}}$ is the union of the set $\mathbb{R}$ and two symbols, "∞" and "$-\infty$". That is $\mathbb{R}_{\mathsf{ext}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

We now want to turn the extended real line into a metric space. For that, we need to define a distance on the extended real line. We do this by first defining the map $\iota : \mathbb{R}_{\mathsf{ext}} \to [-1, 1]$ by

$$\iota(x) := \begin{cases} -1 & \text{if } x = -\infty, \\ \frac{x}{1+x} & \text{if } x \in \mathbb{R} \text{ and } x \geq 0, \\ \frac{x}{1-x} & \text{if } x \in \mathbb{R} \text{ and } x < 0, \\ 1 & \text{if } x = \infty. \end{cases}$$

Because this function is injective, we can now use Exercise 2.7.1 to build a distance on $\mathbb{R}_{\mathsf{ext}}$.

Definition 14.5.2 (Distance on extended real line)**.** Given the definition of the injective function $\iota : \mathbb{R}_{\mathsf{ext}} \to [-1, 1]$ above, we define the distance on $\mathbb{R}_{\mathsf{ext}}$ by

$$\mathsf{dist}_{\mathbb{R}_{\mathsf{ext}}}(x, y) := \mathsf{dist}_{\mathbb{R}}(\iota(x), \iota(y)), \qquad \text{for } x, y \in \mathbb{R}_{\mathsf{ext}},$$

where $\mathsf{dist}_{\mathbb{R}}$ denotes the standard Euclidean distance on $\mathbb{R}$.

14.6 Limits to ∞ or $-\infty$

Definition 14.6.1 (divergence to ∞)**.** Let (X, dist_X) be a metric space, let D be a subset of X and assume $a \in D'$. Let $f : D \to \mathbb{R}$. We say that

f diverges to ∞ in a if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \text{there exists } \delta > 0, \\
&\quad\quad \text{for all } x \in D, \\
&\quad\quad\quad 0 < \mathsf{dist}_X(x, a) < \delta \implies f(x) > M.
\end{aligned}$$

Definition 14.6.2 (divergence to $-\infty$)**.** Let (X, dist_X) be a metric space, let D be a subset of X and assume $a \in D'$. Let $f : D \to \mathbb{R}$. We say that f diverges to $-\infty$ in a if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \text{there exists } \delta > 0, \\
&\quad\quad \text{for all } x \in D, \\
&\quad\quad\quad 0 < \mathsf{dist}_X(x, a) < \delta \implies f(x) < M.
\end{aligned}$$

Using the extended real line, we can give an alternative characterization of divergence to ∞. This alternative characterization brings us back to the 'usual' limits of functions between metric spaces as introduced in Definition 13.2.1.

Proposition 14.6.3 (Alternative characterization of divergence to ∞)**.** Let (X, dist_X) be a metric space, let D be a subset of X and assume $a \in D'$. Let $f : D \to \mathbb{R}$. Then f diverges to ∞ in a (as described by Definition 14.6.1) if and only if f *converges* in a to the element $\infty \in \mathbb{R}_{\mathsf{ext}}$ when viewed as a function mapping from D as a subset of (X, dist_X) to the extended real line $(\mathbb{R}_{\mathsf{ext}}, \mathsf{dist}_{\mathbb{R}_{\mathsf{ext}}})$.

14.7 Limits at ∞ and $-\infty$

Definition 14.7.1. Let (Y, dist_Y) be a metric space and let D be a subset of $\mathbb{R}$ that is unbounded from above. Let $q \in Y$. Let $f : D \to Y$ be a function. We say that $f(x)$ converges to q as $x \to \infty$, and write

$$\lim_{x \to \infty} f(x) = q$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists } z \in \mathbb{R}, \\
&\quad\quad \text{for all } x \in D, \\
&\quad\quad\quad x > z \implies \mathsf{dist}_Y(f(x), q) < \epsilon.
\end{aligned}$$

Definition 14.7.2. Let (Y, dist_Y) be a metric space and let D be a subset of $\mathbb{R}$ that is unbounded from below. Let $q \in Y$. Let $f : D \to Y$ be a function. We say that $f(x)$ converges to q as $x \to -\infty$, and write

$$\lim_{x \to -\infty} f(x) = q$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists } z \in \mathbb{R}, \\
&\quad\quad \text{for all } x \in D, \\
&\quad\quad\quad x < z \implies \mathsf{dist}_Y(f(x), q) < \epsilon.
\end{aligned}$$

We can also combine divergence to and at infinity.

Definition 14.7.3. Let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$ that is unbounded from above. Let $f : D \to \mathbb{R}$ be a function. We say that $f(x)$ diverges to ∞ as x approaches ∞ and write

$$\lim_{x \to \infty} f(x) = \infty$$

if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } z \in \mathbb{R},\\
&\qquad \text{for all } x \in D,\\
&\qquad\quad x > z \implies f(x) > M.
\end{aligned}$$

Definition 14.7.4. Let $D \subset \mathbb{R}$ be a subset of $\mathbb{R}$ that is unbounded from above. Let $f : D \to \mathbb{R}$ be a function. We say that $f(x)$ diverges to $-\infty$ as x approaches ∞ and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } z \in \mathbb{R},\\
&\qquad \text{for all } x \in D,\\
&\qquad\quad x > z \implies f(x) < M.
\end{aligned}$$

Overview of limit statements

To help get an overview of all the possible formal limit statements, we have created an overview in Tables 14.1 and 14.2. The first column in the tables indicate the general form of the limit definition, whereas the second column indicates what lines in the formal limit definition correspond to this format. The last column in the table indicates whether the domain or target actually needs to be the real line for the limit statement to make sense.

In total, the tables give rise to 15 possible limit statements. The following example gives an indication on how to use them.

Limit statement of form:	Lines in formal definition:	Needs $X = \mathbb{R}$?
$\lim_{x \to a} f(x) = \dots$	$\dots,$ there exists $\delta > 0$, for all $x \in D$, if $0 < \mathsf{dist}_X(x, a) < \delta$, $\dots$	No
$\lim_{x \uparrow a} f(x) = \dots$	$\dots,$ there exists $\delta > 0$, for all $x \in D \cap (-\infty, a)$, if $0 < \mathsf{dist}_X(x, a) < \delta$, $\dots$	Yes
$\lim_{x \downarrow a} f(x) = \dots$	$\dots,$ there exists $\delta > 0$, for all $x \in D \cap (a, \infty)$, if $0 < \mathsf{dist}_X(x, a) < \delta$, $\dots$	Yes
$\lim_{x \leftarrow -\infty} f(x) = \dots$	$\dots,$ there exists $z \in \mathbb{R}$, for all $x \in D$, if $x < z$, $\dots$	Yes
$\lim_{x \to \infty} f(x) = \dots$	$\dots,$ there exists $z \in \mathbb{R}$, for all $x \in D$, if $x > z$, $\dots$	Yes

Table 14.1: Possible patterns for limit statements regarding the domain, and the lines that correspond to them in the formal definition

Limit statement of form:	Lines in formal definition:	Needs $Y = \mathbb{R}$?
$\lim_{\dots} f(x) = q$	for all $\epsilon > 0$, $\quad \dots,$ $\qquad \dots,$ $\qquad\quad \dots,$ $\qquad\qquad \mathsf{dist}_Y(f(x), q) < \epsilon$	No
$\lim_{\dots} f(x) = -\infty$	for all $M \in \mathbb{R}$, $\quad \dots,$ $\qquad \dots,$ $\qquad\quad \dots,$ $\qquad\qquad f(x) < M$	Yes
$\lim_{\dots} f(x) = \infty$	for all $M \in \mathbb{R}$, $\quad \dots,$ $\qquad \dots,$ $\qquad\quad \dots,$ $\qquad\qquad f(x) > M$	Yes

Table 14.2: Possible patterns for limit statements regarding the target space, and the lines that correspond to them in the formal definition

Example 14.7.5. If we are interested in the formal limit definition for

$$\lim_{x \uparrow a} f(x) = \infty,$$

we can look in Table 14.1 to find that the pattern

$$\lim_{x \uparrow a} f(x) = \dots$$

corresponds to the lines

$$\begin{aligned}
&\dots, \\
&\quad \text{there exists } \delta > 0, \\
&\qquad \text{for all } x \in D \cap (-\infty, a), \\
&\qquad\quad \text{if } 0 < \mathsf{dist}_X(x, a) < \delta, \\
&\qquad\qquad \dots
\end{aligned}$$

and the pattern

$$\lim_{\dots} f(x) = \infty$$

corresponds to the lines

$$\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \dots, \\
&\qquad \dots, \\
&\qquad\quad \dots, \\
&\qquad\qquad f(x) > M.
\end{aligned}$$

Combining these, we get as the formal definition

$$\begin{aligned}
&\text{for all } M \in \mathbb{R}, \\
&\quad \text{there exists } \delta > 0, \\
&\qquad \text{for all } x \in D \cap (-\infty, a), \\
&\qquad\quad \text{if } 0 < \mathsf{dist}_X(x, a) < \delta, \\
&\qquad\qquad f(x) > M
\end{aligned}$$

14.8 The Intermediate Value Theorem

Theorem 14.8.1 (Intermediate Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $c \in \mathbb{R}$ be a value between $f(a)$ and $f(b)$. Then, there exists an $x \in [a, b]$ such that $f(x) = c$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $c \in \mathbb{R}$ be a value between $f(a)$ and $f(b)$. We need to find an $x \in [a, b]$ such that $f(x) = c$. Without loss of generality, assume that $f(a) < c$ (otherwise, if $f(a) = c$, then you are already done, or if $f(a) > c$ first consider $g = -f$). Now define

$$x := \sup\{y \in [a, b] \mid f(y) < c\}.$$

We will show that $f(x) = c$. First, we note that by properties of the supremum, there exists a sequence (x_n) in $[a, x]$ such that $x_n \to x$ and $f(x_n) < c$ for every $n \in \mathbb{N}$. By continuity of f, it holds that

$$f(x) = \lim_{n\to\infty} f(x_n) \leq c.$$

Similarly, there is a sequence (y_n) in $[x, b]$ converging to x such that $f(y_n) \geq c$. By continuity of f, it holds that

$$f(x) = \lim_{n\to\infty} f(y_n) \geq c.$$

In conclusion, $f(x) = c$. $\square$

14.9 The Extreme Value Theorem

The *Extreme Value Theorem* states that a continuous, real-valued function defined on a non-empty, compact domain K always attains both a maximum and minimum on K. This is quite special, because the concepts of maxima and minima come with large alarm bells: maxima and minima may not always exist and one usually needs to be very careful about this fact. And indeed, there are *discontinuous* functions defined on *compact* domains that neither attain a maximum nor a minimum. And there are *continuous* functions defined on *non-compact* domains that neither attain a maximum nor a minimum. However, continuous functions that are defined on a compact set always attain a maximum and a minimum.

Theorem 14.9.1 (The Extreme Value Theorem)**.** Let (X, dist) be a metric space, let $K \subset X$ be a nonempty, compact subset and let $f : K \to \mathbb{R}$ be continuous. Then f attains a maximum and a minimum on K.

Proof. Since f is continuous and K is compact, the image $f(K)$ is compact as well. Therefore, the set $f(K)$ is bounded, and $M := \sup f(K)$ is well-defined. We can construct a sequence (x_n) in K such that

$$\lim_{n\to\infty} f(x_n) = M.$$

Because K is compact, the sequence (x_n) has a converging subsequence (x_{n_k}), to $y \in K$. Because f is continuous,

$$\lim_{k\to\infty} f(x_{n_k}) = f(y).$$

On the other hand, the sequence $(f(x_{n_k}))$ is a subsequence of the sequence $f(x_n)$, so that

$$M = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k}) = f(y).$$

In conclusion, f attains its maximum in y.

The fact that f attains a minimum on K now follows from the fact that the (continuous) function $-f : K \to \mathbb{R}$ attains a maximum on K. $\square$

14.10 Equivalence of norms

Using the Extreme Value Theorem, we can finally prove a beautiful statement that implies that for many purposes, on finite-dimensional vector spaces, the choice of norm is not so important. Precisely, we will show that any two norms are *equivalent*, which is a concept specified precisely in the following definition.

Definition 14.10.1 (Equivalent norms)**.** Let V be a vector space and let $\| \cdot \|_A$ and $\| \cdot \|_B$ be two different norms on V. We say that the norms $\| \cdot \|_A$ and $\| \cdot \|_B$ are *equivalent* if there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $x \in V$

$$c_1 \|x\|_A \leq \|x\|_B \leq c_2 \|x\|_A.$$

We will now show that any two norms on a *finite-dimensional* vector space are equivalent.

Theorem 14.10.2 (Equivalence of norms on finite-dimensional vector spaces)**.** Let V be a finite-dimensional vector space and let $\| \cdot \|_A$ and $\| \cdot \|_B$ be two norms on V. Then the norms $\| \cdot \|_A$ and $\| \cdot \|_B$ are equivalent.

Proof. Because V is assumed to be finite-dimensional, there exists a basis $v_1, \dots v_d$ in V where $d \in \mathbb{N}$ is the dimension of V. Define the linear map $L : \mathbb{R}^d \to V$ by

$$L(x) := x_1 v_1 + \cdots + x_d v_d.$$

We claim that L is continuous. To show this, we will use Exercise 13.11.5 and show that L is even Lipschitz continuous, therefore uniformly continuous and therefore (by Proposition 13.10.2) continuous. Indeed, let $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned}
\|L(y - x)\|_A &= \|(y_1 - x_1)v_1 + \cdots + (y_d - x_d)v_d\|_A \\
&\leq \sum_{i=1}^{d} |y_i - x_i| \|v_i\|_A \\
&\leq \left(\max_{i=1,\dots,d} \|v_i\|_A \right) \sum_{i=1}^{d} 1 \cdot |y_i - x_i| \\
&\leq \left(\max_{i=1,\dots,d} \|v_i\|_A \right) \|(1, \dots, 1)\|_2 \|y - x\|_2 \\
&= \left(\max_{i=1,\dots,d} \|v_i\|_A \right) \sqrt{d} \|y - x\|_2
\end{aligned}$$

where in the last inequality we used the Cauchy-Schwarz inequality. We have shown that L is Lipschitz continuous.

Note that the unit sphere

$$S = \{(x_1, \dots, x_d)^T \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 = 1\}$$

in $\mathbb{R}^d$ is compact by the Heine-Borel theorem, because it is a bounded and closed subset of $\mathbb{R}^d$.

The function $f : \mathbb{R}^d \to \mathbb{R}$ given by

$$f(x) = (\|\cdot\|_A \circ L)(x) = \|L(x)\|_A$$

is continuous, as it is a composition of two continuous functions.

By the Extreme Value Theorem, the function f attains a maximum and a minimum on S.

We claim that $\min_{x \in S} f(x)$ is strictly positive. It is clear that it is larger than or equal to zero. Suppose it is equal to zero. Then there exists an $x \in S$ such that

$$\|x_1 v_1 + \cdots + x_d v_d\|_A = 0$$

By the property of the norm, it follows that

$$x_1 v_1 + \cdots + x_d v_d = 0$$

But since $v_1, \cdots, v_d$ is a basis, it follows that

$$x_1 = \cdots = x_d = 0.$$

This is a contradiction, because x was supposed to be a point on the unit sphere S.

We conclude that for all $s \in S$,

$$0 < \min_{x \in S} f(x) \leq \|L(s)\|_A \leq \max_{x \in S} f(x)$$

Now let $y \in \mathbb{R}^d \setminus \{0\}$. Then the point $y/\|y\|_2$ is in S. Therefore

$$\min_{x \in S} f(x) \leq \left\| L\left(\frac{y}{\|y\|_2}\right) \right\|_A \leq \max_{x \in S} f(x).$$

By multiplying these inequalities by $\|y\|_2$ and using the homogeneity of the norm, we find that

$$\left(\min_{x \in S} f(x)\right) \|y\|_2 \leq \|L(y)\|_A \leq \left(\max_{x \in S} f(x)\right) \|y\|_2.$$

Because $v_1, \dots, v_d$ is a basis, the map L is bijective. It follows that there exist constants $c_3 > 0$ and $c_4 > 0$ such that for all $v \in V$,

$$c_3 \|L^{-1}(v)\|_2 \leq \|v\|_A \leq c_4 \|L^{-1}(v)\|_2.$$

Similarly, there exist constants $c_5, c_6 > 0$ such that for all $v \in V$

$$c_5 \|L^{-1}(v)\|_2 \leq \|v\|_B \leq c_6 \|L^{-1}(v)\|_2.$$

We conclude that there exist constants $c_1, c_2 > 0$ such that for all $v \in V$,

$$c_1 \|v\|_A \leq \|v\|_B \leq c_2 \|v\|_A.$$

□

The equivalence of norms on finite-dimensional vector spaces has many important consequences. Let us mention a few. The first is that every finite-dimensional normed vector space is complete.

Theorem 14.10.3. Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space. Then $(V, \|\cdot\|)$ is complete.

From this theorem it also follows (by the series characterization of completeness in Theorem 11.5.1) that every absolutely converging series in a finite-dimensional normed vector space is converging: a statement that we announced as Proposition 9.2.3.

Another consequence of the equivalence of norms on finite-dimensional vector spaces is that in a finite-dimensional normed-vector space, a subset is compact if and only if it is closed and bounded.

Theorem 14.10.4 (Heine-Borel Theorem for finite-dimensional normed vector spaces)**.** Let $(V, \|\cdot\|)$ be a finite-dimensional vector space. Then a subset $A \subset V$ is compact if and only if it is closed and bounded.

14.11 Bounded linear maps and operator norms

We close this chapter with a section about linear maps from one normed vector space to another. We will show that linear maps are continuous if and only if they are bounded. Moreover, we will see that a linear map defined on a *finite-dimensional* vector space is always bounded, and therefore always continuous.

Let us first give the definition of a linear map.

Definition 14.11.1 (Linear map)**.** Let V and W be two vector spaces. A function $L : V \to W$ is called a *linear map* if both

i. for all $a, b \in V$,
$$L(a + b) = L(a) + L(b)$$

ii. for all $\lambda \in \mathbb{R}$ and $a \in V$,
$$L(\lambda a) = \lambda L(a).$$

Now we will define what it means for a linear map to be bounded.

Definition 14.11.2 (Bounded linear map)**.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. We say that a linear map $L : V \to W$ is *bounded* if the image under L of the closed unit ball

$$\bar{B}_V(0, 1) = \{v \in V \mid \|v\|_V \leq 1\}$$

is a bounded subset of $(W, \|\cdot\|_W)$, i.e. if

$$L(\bar{B}_V(0,1))$$

is a bounded subset of $(W, \|\cdot\|_W)$.

The following is an alternative characterization of boundedness of linear maps. It is usually a bit easier to deal with.

Proposition 14.11.3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. A linear map $L : V \to W$ is bounded if and only if there exists an $M > 0$ such that for all $v \in V$,

$$\|L(v)\|_W \leq M\|v\|_V.$$

Proposition 14.11.4. The space of bounded linear maps between one normed vector space to another is itself again a vector space, that we denote by $\mathsf{BLin}(V, W)$. Addition and scalar multiplication are defined pointwise, that means that if $L : V \to W$ and $K : V \to W$ are two linear maps and $\lambda \in \mathbb{R}$ is a scalar, then the linear map $L + K : V \to W$ is defined by

$$(L + K)(v) = L(v) + K(v)$$

and the map $\lambda L : V \to W$ is defined by

$$(\lambda L)(v) = \lambda(L(v)).$$

The zero-element in this vector space $\mathsf{BLin}(V, W)$ is the map that maps every vector to the zero-element of W.

We would like to be able to talk about the *norm* of such a linear map. We now introduce one such norm, called the operator norm.

Proposition 14.11.5. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Consider the vector space $\mathsf{BLin}(V, W)$ of bounded linear maps $L : V \to W$. Then the function $\|\cdot\|_{V \to W} : \mathsf{BLin}(V, W) \to \mathbb{R}$

defined by

$$\|L\|_{V\to W} := \sup_{x\in \bar{B}_V(0,1)} \|L(x)\|_W$$

is a norm on $\mathsf{BLin}(V, W)$.

The proof of Proposition 14.11.5 is the topic of Exercise 14.12.8.

Definition 14.11.6. The norm $\|\cdot\|_{V\to W}$ on the vector space $\mathsf{BLin}(V, W)$ is called the *operator norm*.

Proposition 14.11.7. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $L : V \to W$ be a bounded linear map. Then for all $v \in V$,

$$\|L(v)\|_W \leq \|L\|_{V\to W}\|v\|_V$$

and in fact

$$\|L\|_{V\to W} = \min\{C \geq 0 \mid \text{for all } v \in V, \|L(v)\|_W \leq C\|v\|_V\}. \tag{14.11.1}$$

Proof. Let $v \in V$. If $v = 0$, then indeed

$$\|L(v)\|_W = \|0\|_W = 0 \leq \|L\|_{V\to W}\|0\|_V = \|L\|_{V\to W}\|v\|_V.$$

If $v \neq 0$, then define $x := v/\|v\|_V$. Then $x \in \bar{B}_V(0,1)$ because

$$\|x\|_V = \left\| \frac{v}{\|v\|_V} \right\|_V = \frac{1}{\|v\|_V}\|v\|_V = 1.$$

It follows that

$$\|L(x)\|_W \leq \sup_{z\in \bar{B}_V(0,1)} \|L(z)\|_W = \|L\|_{V\to W}.$$

Therefore

$$\|L(v)\|_W = \|L(\|v\|_V x)\|_W = \|v\|_V \, \|L(x)\|_V \leq \|L\|_{V\to W}\|v\|_V.$$

To show (14.11.1), let $C \geq 0$ be a constant such that for all $v \in V$, $\|L(v)\|_W \leq C\|v\|_V$. Then for all $x \in \bar{B}_V(0,1)$, since $\|x\|_V \leq 1$, it holds that $\|L(x)\|_W \leq C\|x\|_V \leq C$. Therefore C is an upper bound for the set

$$\{\|L(x)\|_W \mid x \in \bar{B}_V(0,1)\}$$

and since $\|L\|_{V\to W}$ is the smallest upper bound of this set, we conclude that $\|L\|_{V\to W} \leq C$. □

Theorem 14.11.8. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces and assume that V is finite-dimensional. Let $L : V \to W$ be a linear map. Then L is bounded.

Proof. Since V is finite-dimensional, we may select a basis $v_1, \dots, v_d$ of V, where d is the dimension of V. Define the map $\iota : \mathbb{R}^d \to V$ by

$$\iota(x) = x_1 v_1 + \cdots + x_d v_d.$$

The map ι is a bijective linear map, and therefore its inverse ι^{-1} is a bijective linear map as well: it is the map that assigns to a vector v its components $x_1, \dots, x_d$ with respect to the basis $v_1, \dots, v_d$. Because ι^{-1} is injective, the function

$$\|\cdot\|_2 \circ \iota^{-1}$$

is a norm on V. By the equivalence of norms, there exists a constant $C > 0$ such that for every $v \in V$,

$$\|\iota^{-1}(v)\|_2 \leq C\|v\|_V.$$

Let $v \in \bar{B}(0,1)$. Define

$$x := \iota^{-1}(v).$$

Then

$$\begin{aligned}
\|L(v)\|_W &= \|L(x_1 v_1 + \cdots + x_d v_d)\|_W \\
&\leq |x_1| \|L(v_1)\|_W + \cdots + |x_d| \|L(v_d)\|_W \\
&\leq \left(\sum_{i=1}^{d} |x_i| \right) \max_{j=1,\ldots,d} \|L(v_j)\|_W \\
&\leq \left(\max_{j=1,\ldots,d} \|L(v_j)\|_W \right) \sqrt{d} \|x\|_2 \\
&\leq \left(\max_{j=1,\ldots,d} \|L(v_j)\|_W \right) \sqrt{d} C \|v\|_V
\end{aligned}$$

□

Theorem 14.11.9. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $L : V \to W$ be a linear map. The function L is continuous if and only if it is bounded.

Proof. We first show the "only if" direction. Assume therefore that $L : V \to W$ is continuous. Then L is in particular continuous in $0 \in V$. Therefore, there exists a $\delta > 0$ such that for all $v \in V$, if $0 < \|v\|_V < \delta$, then $\|Lv\|_W < 1$. Choose such a $\delta > 0$.

Choose $M := 2/\delta$. Let $v \in \bar{B}(0, 1)$. If $v = 0$, then $\|Lv\|_W = 0 < M$. Suppose now $v \neq 0$. We also know that $\|v\|_V < 2$. It follows that

$$\|Lv\|_W = \frac{2\delta}{2\delta} \|L(v)\|_W = \frac{2}{\delta} \|L(\delta v/2)\|_W < \frac{2}{\delta} \cdot 1 = \frac{2}{\delta} = M.$$

We now show the "if" direction. Because L is bounded, there exists an $M > 0$ such that for all $v \in V$,

$$\|L(v)\|_W \leq M \|v\|_V.$$

Now let $v \in V$ and let $u : \mathbb{N} \to V$ be a sequence in V converging to v.

Then

$$\begin{aligned} 0 &\leq \|L(u_n) - L(v)\|_W \\ &= \|L(u_n - v)\|_W \\ &\leq M\|u_n - v\|_V \end{aligned}$$

Since the sequence $u : \mathbb{N} \to V$ converges to v, it follows that

$$\lim_{n\to\infty} \|u_n - v\|_W = 0,$$

and by a limit law it holds that also

$$\lim_{n\to\infty} M\|u_n - v\|_W = 0.$$

Since we also know that $\lim_{n\to\infty} 0 = 0$, it follows by the squeeze theorem that

$$\lim_{n\to\infty} \|L(u_n) - L(v)\|_W = 0.$$

Therefore, the sequence (Lu_n) converges to Lv. We have shown that L is continuous in v. □

14.12 Exercises

14.12.1 Blue exercises

Exercise 14.12.1. Prove Part (ii) of Theorem 14.1.1.

Exercise 14.12.2. Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by

$$f(x) = \frac{\exp(x_1^2 - 3x_2)}{x_1^2 + x_2^2}.$$

Prove that $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is continuous considered as a function mapping from the domain $\mathbb{R}^2 \setminus \{0\}$ in the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$. **Hint:** Argue very precisely, using the results in this chapter, but avoid going back to the definition.

Exercise 14.12.3. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Prove that

$$\lim_{x\to\infty} f(x) = \infty.$$

14.12.2 Orange exercises

Exercise 14.12.4. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x_1^4+2x_2^4}{x_1^2+x_2^2} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

is continuous as a function from the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ to the normed vector space $(\mathbb{R}, |\cdot|)$.

Exercise 14.12.5. Let $f : (0,\infty) \to \mathbb{R}$ be a continuous function (viewed as a function from the domain $(0,\infty)$ in the normed vector space $(\mathbb{R}, |\cdot|)$) such that

$$\lim_{x\downarrow 0} f(x) = 1$$

and

$$\lim_{x\to\infty} f(x) = -\infty.$$

Show that there exists a $c \in (0,\infty)$ such that $f(c) = 0$.

Exercise 14.12.6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Let $L \in \mathbb{R}$. Show that

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\downarrow a} f(x) = \lim_{x\uparrow a} f(x) = L.$$

Exercise 14.12.7. Let $(V, \|\cdot\|_A)$ be a **finite-dimensional** normed vector space, let $D \subset V$ be a subset of V and consider a function $f : D \to \mathbb{R}$. Let $a \in D$ and suppose f is continuous in a in the normed vector space $(V, \|\cdot\|_A)$. Let $\|\cdot\|_B : V \to \mathbb{R}$ be another norm on V. Show that f is also continuous in a in the normed vector space $(V, \|\cdot\|_B)$.

Exercise 14.12.8. Prove Proposition 14.11.5.

Exercise 14.12.9. Let $f : (-\infty, 3) \to \mathbb{R}$ be a continuous function (viewed as a function from the domain $(-\infty, 3)$ in the normed vector space $(\mathbb{R}, |\cdot|)$). Assume that

$$\lim_{x \to -\infty} f(x) = \infty$$

and

$$\lim_{x \uparrow 3} f(x) = \infty.$$

Show that f attains a minimum on the interval $(-\infty, 3)$. I.e., show that there exists a $c \in (-\infty, 3)$ such that for all $x \in (-\infty, 3)$,

$$f(c) \leq f(x).$$

Exercise 14.12.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} \frac{(x_1)^4 - 2(x_2)^2}{(x_1)^4 + (x_2)^4} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0). \end{cases}$$

Either prove that the function f is continuous or prove that it is not continuous (where f is viewed as a function from the domain $\mathbb{R}^2$ in the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$ to the normed vector space $(\mathbb{R}, |\cdot|)$).

Chapter 15

Differentiability

If a function $f : \Omega \to Y$ is *continuous* in a point $a \in \Omega$, then close to a, the function f is reasonably well approximated by the constant function $x \mapsto f(a)$. In some sense, the constant function $x \mapsto f(a)$ is a basic, a zeroth, approximation of the function f around a. The good thing about this approximation is that it is very simple. The bad thing is that it is maybe too simple, and therefore the approximation may not be very good. Can we do better?

Differentiability is all about approximating functions by *affine* functions, i.e. functions that are the sum of a constant and a linear map. The good thing is that affine functions are still rather simple and the approximation with an affine function will usually be better than the approximation with just a constant function. For all this to make sense though, we will need to start restricting the context to functions mapping from (a domain in) one normed vector space to another normed vector space.

The approach that I follow in these chapters on differentiability is very close to the approach followed by Rodney Coleman in his book *Calculus on normed vector spaces* [Col12].

15.1 Definition of differentiability

The following is the definition of differentiability in a point. This definition most likely differs from what you have seen in Calculus. The reason for this deviation is that we really want a concept that works for maps between normed vector spaces. After we give the definition, we will provide a first indication on how the definition relates to the one you are more familiar with.

Definition 15.1.1 (Differentiability in a point)**.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two (if you prefer, but not necessarily finite-dimensional) vector spaces. Let $\Omega \subset V$ be an open subset of V. Let $f : \Omega \to W$ be a function and let $a \in \Omega$. We say that f is differentiable in a if there exists a bounded linear map $L_a : V \to W$ such that, if we define the error function $\mathsf{Err}_a : \Omega \to W$ by

$$\mathsf{Err}_a(x) := f(x) - f(a) - L_a(x - a)$$

it holds that

$$\lim_{x \to a} \frac{\|\mathsf{Err}_a(x)\|_W}{\|x - a\|_V} = 0.$$

We call L_a the derivative of f in a, and instead of L_a we often write $(Df)_a$.

Definition 15.1.2 (Differentiability on an open set)**.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $\Omega \subset V$ be open. We say that a function $f : \Omega \to W$ is differentiable on Ω if for every $a \in \Omega$, the function f is differentiable in a.

The following proposition relates the derivative to the derivative you are used to from Calculus.

Proposition 15.1.3. Let $\Omega \subset \mathbb{R}$ be an open subset of $\mathbb{R}$ and consider a function $f : \Omega \to \mathbb{R}$ interpreted as a function from the subset Ω of the normed vector space $(\mathbb{R}, |\cdot|)$ to the normed vector space $(\mathbb{R}, |\cdot|)$. Let

$a \in \Omega$. Then f is differentiable in a if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Moreover, if this limit exists, we call it $f'(a)$, and then for all $h \in \mathbb{R}$,

$$f'(a) \cdot h = (Df)_a(h). \tag{15.1.1}$$

Warning: It is good to take a moment and internalize the difference between the left-hand side and the right-hand side of (15.1.1). The left-hand side is a product of two real numbers, the number $f'(a) \in \mathbb{R}$ and the number $h \in \mathbb{R}$. The right-hand side is a linear map $(Df)_a : \mathbb{R} \to \mathbb{R}$ applied to the real number $h \in \mathbb{R}$.

Example 15.1.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x.$$

You have probably learned for every $a \in \mathbb{R}$, that $f'(a) = 1$, with the same limit definition of $f'(x)$ as given above, and indeed

$$\begin{aligned} \lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{x - a}{x - a} \\ &= 1. \end{aligned}$$

We can then also write down the derivative $(Df)_a : \mathbb{R} \to \mathbb{R}$ which is a linear map from $\mathbb{R}$ to $\mathbb{R}$. To describe $(Df)_a$, we need to specify what it does to an element $h \in \mathbb{R}$, and again with the previous proposition we know that

$$(Df)_a(h) = f'(a) \cdot h = h.$$

The previous proposition can be generalized to the case in which the target is an arbitrary normed vector space.

Proposition 15.1.5. Let $\Omega \subset \mathbb{R}$ be open and consider a function $f : \Omega \to W$ interpreted as a function from the subset Ω of the normed vector space $(\mathbb{R}, |\cdot|)$ to a normed vector space $(W, \|\cdot\|_W)$. Let $a \in \Omega$. Then f is differentiable in a if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{15.1.2}$$

exists. Moreover, if this limit exists we denote it by $f'(a)$, and then for all $h \in \mathbb{R}$,

$$f'(a) \cdot h = (Df)_a(h).$$

The limit in (15.1.2) exists if and only if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, and then they have the same value.

As an alternative notation, we will sometimes write

$$\left.\frac{d}{dt} f(t)\right|_{t=a}$$

instead of $f'(a)$.

15.2 The derivative as a function

Definition 15.2.1 (The derivative as a function)**.** Let $f : \Omega \to W$ be a function from an open domain Ω in a finite-dimensional normed vector space $(V, \|\cdot\|_V)$ to a finite-dimensional normed vector space $(W, \|\cdot\|_W)$. Suppose that f is differentiable on Ω, (i.e. suppose that for every $a \in \Omega$, the function f is differentiable in a). Then we define the *derivative of* f as the function

$$Df : \Omega \to \mathsf{Lin}(V, W)$$

that maps every $a \in \Omega$ to the derivative of f in a, i.e. to $(Df)_a \in$

$\mathrm{Lin}(V, W)$.

15.3 Constant and linear maps are differentiable

Proposition 15.3.1 (Constant maps are differentiable)**.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $b \in W$ and consider the constant function $f : V \to W$ given by $f(v) = b$ for all $v \in V$. Then f is differentiable and for all $a \in V$, $(Df)_a = 0$, i.e. it is the (linear) function that maps every element to 0.

We now give a first example of a differentiable functions: linear functions are always differentiable.

Proposition 15.3.2 (Linear maps are differentiable)**.** Let $A : V \to W$ be a linear map between the finite-dimensional normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. Then the function $A : V \to W$ is differentiable on V and for every $a \in V$ the derivative $(DA)_a \in \mathrm{Lin}(V, W)$ is just equal to A. Hence, the derivative of A is the constant function $DA : V \to \mathrm{Lin}(V, W)$ given by

$$a \mapsto A.$$

The proof of this proposition is the topic of Exercise 15.12.1.

15.4 Bases and coordinates

In this section we will give many examples of linear maps that are at the same time related to the choice of coordinates in the spaces V and W.

We first consider standard coordinate projections in $\mathbb{R}^m$.

Definition 15.4.1 (Coordinate projections)**.** Let $i \in \{1, \ldots, m\}$, and consider the map $P^i : \mathbb{R}^m \to \mathbb{R}$ given by

$$P^i(x) = x_i.$$

The map P^i is called the projection to the ith coordinate.

Proposition 15.4.2. The coordinate projections P^i in the above definition are linear.

Proof. Indeed, for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$,

$$P^i(\lambda x) = \lambda x_i = \lambda P^i(x)$$

and for all $x, y \in \mathbb{R}^d$

$$P^i(x + y) = x_i + y_i = P^i(x) + P^i(y).$$

$\square$

Since P^i is linear, it follows that the map P^i is differentiable and $DP^i : \mathbb{R}^d \to \mathsf{Lin}(\mathbb{R}^d, \mathbb{R})$ is the constant map $a \mapsto P^i$.

We will now discuss coordinates in arbitrary finite-dimensional vector spaces.

Recall that $w_1, \dots, w_m$ forms a basis of a vector space W if and only if for every $v \in W$, there are unique constants $x_1, \dots, x_m$ such that

$$v = x_1 w_1 + \cdots + x_m w_m$$

The numbers x_i are called the *coordinates* of the vector v with respect to the basis $w_1, \dots w_m$. The map that assigns to every element v in W the coordinate vector $(x_1, \dots, x_m)$ with respect to the basis $w_1, \dots, w_m$ is in fact a linear map and is called the coordinate map.

Definition 15.4.3 (Coordinate map)**.** Let W be a finite-dimensional vector space and assume that $w_1, \dots, w_m$ is a basis of W. The map $\Psi : W \to \mathbb{R}^m$ that assigns to every $v \in W$ its coordinates with respect to the basis $w_1, \dots, w_m$ is called the *coordinate map* with respect to the basis $w_1, \ldots, w_m$.

Proposition 15.4.4. The coordinate map $\Psi : W \to \mathbb{R}^m$ with respect to a basis $w_1, \ldots, w_m$ is linear.

As a consequence, the derivative $D\Psi : W \to \mathsf{Lin}(W, \mathbb{R}^m)$ is given by

$$(D\Psi)_a = \Psi$$

for all $a \in W$.

The component functions $\Psi_1, \ldots \Psi_m$ of Ψ are together sometimes called the dual basis of $w_1, \ldots, w_m$. Here, by component functions we mean the functions $\Psi_i : W \to \mathbb{R}$ that are defined by $\Psi_i := P^i \circ \Psi$, i.e.

$$\Psi = (\Psi_1, \ldots, \Psi_m).$$

Proposition 15.4.5 (Dual basis)**.** If W is a finite-dimensional normed vector space, and $w_1, \ldots, w_m$ is a basis of W, then there exist linear maps $\Psi_i : W \to \mathbb{R}$ for $i = 1, \ldots, m$ such that for all $v \in W$,

$$v = \Psi_1(v)w_1 + \cdots + \Psi_m(v)w_m = \sum_{i=1}^{m} \Psi_i(v)w_i.$$

Together, the functions $\Psi_1, \ldots, \Psi_m$ form a basis of the vector space $\mathsf{Lin}(W, \mathbb{R})$ and they are called *the dual basis* of $w_1, \ldots, w_m$.

Every Ψ_i is a linear map from W to $\mathbb{R}$, and

$$\Psi_i(w_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof. See e.g. Theorem 1.6.7 in the Linear Algebra 2 lecture notes □

Since Ψ_i is linear, it is differentiable and $D\Psi_i : W \to \mathsf{Lin}(W, \mathbb{R})$ is the constant map $a \mapsto \Psi_i$.

15.5 The matrix representation

We now briefly review *matrix representations*, a concept from linear algebra. Let $L : V \to W$ be a linear map between two finite-dimensional vector spaces with bases $v_1, \cdots, v_d$ and $w_1, \dots w_m$ respectively. Let $\Psi : W \to \mathbb{R}^m$ denote the coordinate map for the basis $w_1, \dots, w_m$. Then the *matrix (representation)* of L is the $m \times d$ matrix M such that for all $i = 1, \dots, m$ and $j = 1, \dots d$, the element of M in the ith row and jth column is

$$(M)_{ij} = (\Psi(Lv_j))_i = \Psi_i(Lv_j),$$

which in words means that $(M)_{ij}$ is the ith coordinate of the vector $L(v_j)$ expressed in the basis $w_1, \dots, w_m$.

The matrix M is precisely that matrix such that for all $x \in \mathbb{R}^d$, with $y = Mx$ it holds that

$$L(x_1 v_1 + \cdots + x_d v_d) = y_1 w_1 + \cdots + y_m w_m.$$

In other words, for all $x \in \mathbb{R}^d$,

$$\Psi \circ L(x_1 v_1 + \cdots + x_d v_d). = Mx$$

Definition 15.5.1 (Jacobian with respect to bases)**.** We will sometimes call the matrix representation of a derivative $(Df)_a : V \to W$ the *Jacobian* of f (with respect to the bases $v_1, \dots, v_d$ and $w_1, \dots w_m$) in the point a, and we will denote it by $[Df]_a$.

As a preview, I'd already like to mention that if a function $f : \Omega \to W$ is differentiable in a point $a \in \Omega$, then you can easily find $[Df]_a$ with the rules of calculus: to determine $([Df]_a)_{ij}$, i.e. the element in the ith row and jth column, you first compute the coordinate representation of f, namely

$$\bar{f} : \Phi(\Omega) \to \mathbb{R}^m$$

defined by

$$\bar{f}(x) = \Psi \circ f \circ \Phi^{-1}(x).$$

Then you view $\bar{f}$ as a function of x_j only, keeping the other x_k for $k \neq j$, and take the derivative of the ith component of $\bar{f}$ with respect to x_j, which is denoted by

$$\frac{\partial \bar{f}_i}{\partial x_j}$$

and is called the partial derivative of $\bar{f}_i$ with respect to x_j. Then

$$([Df]_a)_{ij} = \frac{\partial \bar{f}_i}{\partial x_j}(\Phi(a)).$$

Only later can we give a proper definition of the partial derivative.

For instance, if

$$\bar{f}(x_1, x_2) = \left(x_1^2 + x_2^3, x_1^4 + x_2^5\right)$$

then

$$[Df]_a = \begin{pmatrix} 2b_1 & 3(b_2)^2 \\ 4(b_1)^3 & 5(b_2)^4 \end{pmatrix}.$$

where $b := \Phi(a)$.

But this was just a preview. Before we can show this, we need computation rules such as the chain rule, the sum, product and quotient rules.

15.6 The chain rule

Theorem 15.6.1 (Chain rule)**.** Let $(U, \|\cdot\|_U)$, $(V, \|\cdot\|_V)$, and $(W, \|\cdot\|_W)$ be normed vector spaces.

Let $\Omega \subset U$ and $E \subset V$ both be open. Let $f : \Omega \to V$ be such that $f(\Omega) \subset E$. Let $g : E \to W$.

If f is differentiable in a point $a \in \Omega$, and g is differentiable in the point $f(a)$, then the function $g \circ f$ is differentiable in the point a.

Moreover,

$$(D(g \circ f))_a = (Dg)_{f(a)} \circ (Df)_a.$$

Proof. We need to show that there exists a linear map $A : U \to W$ such that, if we define the error function $\mathsf{Err}_a^{g \circ f} : D \to W$ by

$$\mathsf{Err}_a^{g \circ f}(x) := (g \circ f)(x) - (g \circ f)(a) - A(x - a)$$

that then

$$\lim_{x \to a} \frac{\|\mathsf{Err}_a^{g \circ f}(x)\|_W}{\|x - a\|_U} = 0.$$

We choose $A := (Dg)_{f(a)} \circ (Df)_a$. This is indeed a bounded linear operator from U to W.

We need to show that

$$\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists } \delta > 0, \\
&\qquad \text{for all } x \in \Omega, \\
&\qquad\quad \text{if } 0 < \|x - a\|_U < \delta, \\
&\qquad\qquad \text{then } \frac{\|\mathsf{Err}_a^{g \circ f}(x)\|_W}{\|x - a\|_U} < \epsilon.
\end{aligned}$$

Let $\epsilon > 0$.

According to the template, the next step would be to find a $\delta > 0$, but for this step we need quite some preparation. First of all it is helpful to define the error functions

$$\mathsf{Err}_a^f(x) := f(x) - f(a) - (Df)_a(x - a)$$

and

$$\mathsf{Err}_{f(a)}^g(y) := g(y) - g(f(a)) - (Dg)_{f(a)}(y - f(a))$$

We can then make for $x \in \Omega$ the following computation

$$\begin{aligned}
\mathsf{Err}_a^{g\circ f}(x) &= g(f(x)) - g(f(a)) - ((Dg)_{f(a)} \circ (Df)_a)(x-a) \\
&= (Dg)_{f(a)}(f(x) - f(a)) + \mathsf{Err}_{f(a)}^g(f(x)) \\
&\quad - (Dg)_{f(a)}((Df)_a(x-a)) \\
&= \mathsf{Err}_{f(a)}^g(f(x)) \\
&\quad + (Dg)_{f(a)}\big(f(x) - f(a) - (Df)_a(x-a)\big) \\
&= \mathsf{Err}_{f(a)}^g(f(x)) + (Dg)_{f(a)}(\mathsf{Err}_a^f(x)).
\end{aligned}$$

Using the triangle inequality, we can then estimate

$$\begin{aligned}
\|\mathsf{Err}_a^{g\circ f}(x)\|_W &= \|\mathsf{Err}_{f(a)}^g(f(x)) + (Dg)_{f(a)}(\mathsf{Err}_a^f(x))\|_W \\
&\leq \|\mathsf{Err}_{f(a)}^g(f(x))\|_W + \|(Dg)_{f(a)}(\mathsf{Err}_a^f(x))\|_W \\
&\leq \|\mathsf{Err}_{f(a)}^g(f(x))\|_W + \|(Dg)_{f(a)}\|_{V\to W}\|\mathsf{Err}_a^f(x)\|_V
\end{aligned} \tag{15.6.1}$$

Our strategy will be to find a $\delta > 0$ such that for all $x \in \Omega$, if $0 < \|x - a\|_U < \delta$, the right-hand-side of (15.6.1), and therefore also the left-hand-side, is less than ϵ. Let's see how we can find such a $\delta > 0$.

Because g is differentiable in $f(a)$ with derivative $(Dg)_{f(a)}$, it holds that

$$\lim_{y \to f(a)} \frac{\|\mathsf{Err}_{f(a)}^g(y)\|_W}{\|y - f(a)\|_V} = 0.$$

Therefore, there exists a $\rho > 0$ such that for all $y \in E$, if $0 < \|y - f(a)\|_V < \rho$, then

$$\frac{\|\mathsf{Err}_{f(a)}^g(y)\|_W}{\|y - f(a)\|_V} < \frac{\epsilon}{2(\|(Df)_a\|_{U\to V} + 1)}. \tag{15.6.2}$$

Choose such a $\rho > 0$.

Now define

$$\epsilon_2 := \min\left(1, \frac{\epsilon}{2\|(Dg)_{f(a)}\|_{V\to W} + 1}\right).$$

Because f is differentiable in a with derivative $(Df)_a$, it holds that

$$\lim_{x\to a} \frac{\|\mathsf{Err}_a^f(x)\|_V}{\|x-a\|_U} = 0.$$

Therefore there exists a $\delta_1 > 0$ such that for all $x \in \Omega$, if $0 < \|x - a\|_U < \delta_1$ then

$$\|\mathsf{Err}_a^f(x)\|_V < \epsilon_2 \|x-a\|_U.$$

Choose such a $\delta_1 > 0$.

Then, for $x \in \Omega$, if $0 < \|x-a\|_U < \delta_1$ also

$$\begin{aligned} \|f(x) - f(a)\|_V &= \|(Df)_a(x-a) + \mathsf{Err}_a^f(x)\|_V \\ &\leq \|(Df)_a(x-a)\|_V + \|\mathsf{Err}_a^f(x)\|_V \\ &\leq \|(Df)_a\|_{U\to V}\|x-a\|_U + \epsilon_2\|x-a\|_U. \end{aligned} \tag{15.6.3}$$

Define

$$\delta_2 := \frac{\rho}{1 + \|(Df)_a\|_{U\to V}}.$$

Choose $\delta := \min(\delta_1, \delta_2)$. Let $x \in \Omega$. Assume $0 < \|x-a\|_U < \delta$. Then, it follows by (15.6.3) and the fact that $\epsilon_2 \leq 1$ that

$$\begin{aligned} \|f(x) - f(a)\|_V &\leq \|(Df)_a\|_{U\to V}\|x-a\|_U + \epsilon_2\|x-a\|_U \\ &\leq (1 + \|(Df)_a\|_{U\to V})\|x-a\|_U \\ &< (1 + \|(Df)_a\|_{U\to V})\delta \leq \rho \end{aligned}$$

so that by estimate (15.6.2) it follows that

$$\begin{aligned}\|\mathsf{Err}^{g}_{f(a)}(f(x))\|_W &< \frac{\epsilon}{2(\|(Df)_a\|_{U\to V}+1)}\|f(x)-f(a)\|_V \\ &< \frac{\epsilon}{2}\|x-a\|_U.\end{aligned}$$

Therefore, using the result of our earlier computation (15.6.1)

$$\begin{aligned}\|\mathsf{Err}^{g\circ f}_a(x)\|_W &\leq \|\mathsf{Err}^{g}_{f(a)}(f(x))\|_W + \|(Dg)_{f(a)}\|_{V\to W}\|\mathsf{Err}^{f}_a(x)\|_V \\ &< \frac{\epsilon}{2}\|x-a\|_U + \epsilon_2\|(Dg)_{f(a)}\|_{V\to W}\|x-a\|_U \\ &\leq \frac{\epsilon}{2}\|x-a\|_U \\ &\qquad + \frac{\epsilon}{2\|(Dg)_{f(a)}\|_{V\to W}+1}\|(Dg)_{f(a)}\|_{V\to W}\|x-a\|_U \\ &< \epsilon\|x-a\|_U.\end{aligned}$$

□

15.7 Sum, product and quotient rules

The following sum, product and quotient rules will be familiar from Calculus, but now they are formulated in the language of linear maps.

Theorem 15.7.1. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $\Omega \subset V$ be open and let $f : \Omega \to W$ and $g : \Omega \to W$ be two functions, that are both differentiable in a point $a \in \Omega$, with derivative $(Df)_a : V \to W$ and $(Dg)_a : V \to W$ respectively.

Then the function $f + g : \Omega \to W$ is also differentiable in a with derivative

$$(D(f+g))_a = (Df)_a + (Dg)_a$$

Theorem 15.7.2. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces. Let $\Omega \subset V$ be open and let $f : \Omega \to W$ and $g : \Omega \to \mathbb{R}$ be two functions mapping to the normed vector space $(\mathbb{R}, |\cdot|)$. Assume both f and g are differentiable in the point $a \in \Omega$, with derivative $(Df)_a : V \to W$ and $(Dg)_a : V \to \mathbb{R}$ respectively. Then

i. (product rule) The function $f \cdot g$ is differentiable in a as well, with derivative given by

$$(D(f \cdot g))_a(h) = f(a)(Dg)_a(h) + g(a)(Df)_a(h)$$

for all $h \in V$.

ii. (quotient rule) If $g(a) \neq 0$, the function f/g is differentiable in a as well, with derivative given by

$$(D(f/g))_a(h) = \frac{1}{(g(a))^2}\big(g(a)(Df)_a(h) - f(a)(Dg)_a(h)\big)$$

for all $h \in V$.

15.8 Differentiability of components

Proposition 15.8.1. Let $w_1, \dots, w_m$ be a basis of W and let $\Psi_1, \dots, \Psi_m$ be the dual basis.

Then a function $f : \Omega \to W$ is differentiable in a point $a \in \Omega$ if and only if for every $i \in \{1, \dots, m\}$, the function

$$\Psi_i \circ f$$

is differentiable in $a \in \Omega$. Moreover, if the function f is differentiable in $a \in \Omega$, then for every $v \in V$,

$$(Df)_a(v) = \sum_{i=1}^{m} w_i D(\Psi_i \circ f)_a(v)$$

The proof of this proposition is the topic of Exercise 16.4.2.

Corollary 15.8.2. A function $f : \Omega \to \mathbb{R}^m$ is differentiable in a point $a \in \Omega$ if and only if for $i = 1, \dots, m$ the component function $f_i : \Omega \to \mathbb{R}$ given by $f_i = P^i \circ f$ is differentiable. Moreover, if f is differentiable in a, then for all $v \in V$,

$$(Df)_a(v) = \sum_{i=1}^{m} e_i (Df_i)_a(v) = \big((Df_1)_a(v), \cdots, (Df_m)_a(v)\big)$$

where e_i denote the standard unit vectors.

If in fact Ω is a subset of $\mathbb{R}$, then

$$f'(a) = \big(f_1'(a), \cdots, f_m'(a)\big).$$

15.9 Differentiability implies continuity

The next theorem tells us that differentiability in a point is a *stronger* condition than continuity in a point: whenever a function is differentiable in a point, it is also continuous in that point.

Theorem 15.9.1. Let $\Omega \subset V$ be open and suppose a function $f : D \to W$ is differentiable in a point $a \in \Omega$. Then f is continuous in a.

Proof. Suppose $f : \Omega \to W$ is differentiable in a point $a \in \Omega$. Then there exists a $\delta > 0$ such that for all $x \in \Omega$, if $0 < \|x - a\|_V < \delta$, then

$$\frac{\|\mathsf{Err}_a(x)\|_W}{\|x - a\|_V} < 1.$$

Now let $y : \mathbb{N} \to \Omega$ be a sequence in Ω converging to a. Then, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\|y_n - a\|_V < \delta.$$

Choose such an $N \in \mathbb{N}$ and let $n \geq N$. Then

$$\|f(y_n) - f(a) - L_a(y_n - a)\|_W = \|\mathsf{Err}_a(y_n)\|_W < \|y_n - a\|_V.$$

By the reverse triangle inequality (see Lemma 2.6.1), and by Proposition 14.11.7 we find

$$\begin{aligned}\|f(y_n) - f(a)\|_W &\leq \|y_n - a\|_V + \|L_a(y_n - a)\|_W \\ &\leq \|y_n - a\|_V + \|L_a\|_{V \to W}\|(y_n - a)\|_V \\ &= (1 + \|L_a\|_{V \to W})\|y_n - a\|_V.\end{aligned}$$

It follows by Proposition 5.6.1 and the squeeze theorem that the sequence $n \mapsto \|f(y_n) - f(a)\|_W$ converges to zero, and we conclude by Proposition 5.6.1 that the sequence $(f(y_n))$ converges to $f(a)$. □

15.10 Derivative vanishes in local maxima and minima

Theorem 15.10.1. Let Ω be an open subset of a normed vector space V. Suppose $f : \Omega \to \mathbb{R}$ is differentiable in $a \in \Omega$. Suppose that $f(a)$ is a local maximum or minimum, i.e. suppose there exists an $r > 0$ such that either

$$\text{for all } x \in B(a, r), \\ f(x) \leq f(a)$$

or

$$\text{for all } x \in B(a, r), \\ f(x) \geq f(a).$$

Then $(Df)_a = 0$.

Proof. We will show the statement for the case in which f attains a local maximum in a. In that case, there exists an $r > 0$ such that $f(x) \leq f(a)$ for all $x \in B(a, r)$. Because f is differentiable in a,

$$f(x) = f(a) + (Df)_a(x - a) + \mathsf{Err}^f_a(x)$$

where

$$\lim_{x \to a} \frac{|\mathsf{Err}_a^f(x)|}{\|x-a\|_V} = 0. \tag{15.10.1}$$

We argue by contradiction, so suppose $(Df)_a \neq 0$, i.e. suppose $(Df)_a$ is not the zero map. Then there exists a vector $u \in V$ such that

$$(Df)_a(u) \neq 0.$$

We choose such a u and define v to be either equal to $u/\|u\|_V$ or to $-u/\|u\|_V$, in such a way that

$$(Df)_a(v) > 0.$$

Note also that by the homogeneity of the norm,

$$\|v\|_V = \left\| \frac{u}{\|u\|_V} \right\|_V = \frac{1}{\|u\|_V} \|u\|_V = 1.$$

The intuition behind the rest of the proof is that f evaluated in points in the direction of v, close enough to a, is larger than $f(a)$.

By (15.10.1) there exists an $\delta > 0$ such that for all $x \in \Omega$, if $0 < \|x - a\|_V < \delta$ then

$$\frac{|\mathsf{Err}_a^f(x)|}{\|x-a\|_V} < \frac{1}{2} |(Df)_a(v)| \tag{15.10.2}$$

Now choose $\rho := \frac{1}{2}\min(r, \delta)$ and define

$$y := a + \rho v.$$

Then by positive homogeneity of the norm and the fact that $\|v\|_V = 1$,

$$0 < \|y - a\|_V = \|\rho v\|_V = \rho \|v\|_V = \rho = \frac{1}{2}\min(r, \delta).$$

Therefore on the one hand $\|y-a\|_V < r$ and thus $f(y) \leq f(a)$, but on the other hand $\|y-a\|_V < \delta$ and thus

$$\begin{aligned} f(y) &= f(a) + (Df)_a(y-a) + \mathsf{Err}_a^f(y) \\ &= f(a) + (Df)_a(\rho v) + \mathsf{Err}_a^f(y) \\ &= f(a) + \rho (Df)_a(v) + \mathsf{Err}_a^f(y) \\ &\geq f(a) + \rho (Df)_a(v) - |\mathsf{Err}_a^f(y)|. \end{aligned}$$

We now use (15.10.2) to find

$$\begin{aligned} f(y) &\geq f(a) + \rho (Df)_a(v) - \frac{1}{2}|(Df)_a(v)|\|y-a\|_V \\ &= f(a) + \rho (Df)_a(v) - \frac{\rho}{2}|(Df)_a(v)| \\ &= f(a) + \frac{\rho}{2}(Df)_a(v) > f(a) \end{aligned}$$

which is a contradiction. □

15.11 The mean-value theorem

Theorem 15.11.1 (Rolle's theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous, assume that f is differentiable on (a,b) and that $f(a) = f(b)$. Then there exists a $c \in (a,b)$ such that $f'(c) = 0$.

Proof. Since f is continuous, it achieves both a maximum and a minimum on $[a,b]$ by the Extreme Value Theorem. Since $f(a) = f(b)$ either the minimum or the maximum is achieved in some $c \in (a,b)$. By Theorem 15.10.1 it holds that $f'(c) = 0$. □

Theorem 15.11.2 (Mean-value theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous, and assume that f is differentiable on (a,b). Then there exists a

$c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define the function $g : [a, b] \to \mathbb{R}$ by
$$g(x) = f(x) - \frac{x - a}{b - a} f(b) - \frac{b - x}{b - a} f(a).$$

By the sum and product rules, the function g is differentiable on (a, b). By the rules for continuous functions, the function g is also continuous on $[a, b]$. Moreover,
$$g(a) = g(b) = 0.$$

It follows by Rolle's theorem that there exists a $c \in (a, b)$ such that
$$g'(c) = 0.$$

Then
$$0 = g'(c) = f'(c) - \frac{1}{b - a} f(b) + \frac{1}{b - a} f(a)$$
so that indeed
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
□

15.12 Exercises

15.12.1 Blue exercises

Exercise 15.12.1. Let $A : V \to W$ be a linear map from a finite-dimensional normed vector space $(V, \| \cdot \|_V)$ to a normed vector space $(W, \| \cdot \|_W)$. Show that A is differentiable on V.

15.12.2 Orange exercises

Exercise 15.12.2. Let $f : V \to \mathbb{R}$ be a differentiable function from a finite-dimensional normed vector space $(V, \|\cdot\|_V)$ to the normed vector space $(\mathbb{R}, |\cdot|)$. Assume that for all $a \in V$, $(Df)_a = 0$. Let $v \in V$. Show that $f(v) = f(0)$. (This would essentially show that f is constant on V).

i. Define the function $\ell_v : \mathbb{R} \to V$ by $\ell(t) = tv$. Show that ℓ_v is differentiable.

ii. Show that the function $g := f \circ \ell_v$ is differentiable and compute its derivative.

iii. Conclude that $f(v) = g(1) = g(0) = f(0)$.

Exercise 15.12.3. The function $\ln : (0, \infty) \to \mathbb{R}$ is the unique, differentiable function such that $\ln(1) = 0$ and $\ln'(x) = 1/x$. Show that for all $x \in (-1, \infty)$, it holds that

$$\ln(1 + x) \leq x$$

with equality if and only if $x = 0$.

Exercise 15.12.4. Prove Proposition 15.1.5. You may assume that W is finite-dimensional.

Exercise 15.12.5. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two two-dimensional vector spaces with bases v_1, v_2 and w_1, w_2 respectively. Assume that a function $f : V \to W$ is differentiable in 0 with

$$(Df)_0(v_1 + v_2) = w_1$$

and

$$(Df)_0(v_1 - 2v_2) = w_1 - w_2.$$

Give the matrix representation of the linear map $(Df)_0 : V \to W$ with respect to the bases v_1, v_2 and w_1, w_2.

Chapter 16

Differentiability of standard functions

Which functions are differentiable? We would like to give examples of large classes of functions that *are* differentiable. How can we find such classes? For now, we can combine the following observations:

- The constant function is differentiable
- Linear functions between finite dimensional normed vector spaces are always differentiable
- Sums, products, compositions and at times quotients of differentiable functions are differentiable. The precise statements are given by the sum rule, the product rule, the chain rule and the quotient rule in the previous chapter.

With these observations we can get pretty far, and conclude that polynomials and rational functions are differentiable.

16.1 Global context

Before going on with the lecture notes, I'd like to introduce the global context that I will use most often, so that we don't have to introduce it again for every definition, lemma etc. I will usually not reintroduce these variables, but only indicate deviations from it.

We will consider two normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ and a function $f : \Omega \to W$ where $\Omega \subset V$ is an open subset of V. We will from now assume that V and W are finite-dimensional, and we will denote by $v_1, \dots, v_d$ a basis in V, with corresponding coordinate map Φ, and by $w_1, \dots, w_m$ a basis in W with corresponding coordinate map Ψ.

16.2 Polynomials and rational functions are differentiable

Proposition 16.2.1 (Differentiability of polynomials in one variable)**.** For every $n \in \mathbb{N}$, it holds that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^n$$

is differentiable with

$$f'(x) = nx^{n-1}.$$

In other words, the derivative of f, i.e. $(Df) : \mathbb{R} \to \mathsf{Lin}(\mathbb{R}, \mathbb{R})$ is given by

$$x \mapsto \left(h \mapsto nx^{n-1}h\right)$$

Proposition 16.2.2 (Every polynomial is differentiable)**.** Every polynomial on $\mathbb{R}^d$ is differentiable.

Proposition 16.2.3 (Every rational function is differentiable on its domain)**.** Let $p : \mathbb{R}^d \to \mathbb{R}$ and $q : \mathbb{R}^d \to \mathbb{R}$ be two polynomials. Let

$$D := \{x \in \mathbb{R}^d \mid q(x) \neq 0\}.$$

Then the function $f : D \to \mathbb{R}$ given by

$$f(x) = \frac{p(x)}{q(x)}$$

is differentiable.

In other words, every rational function is differentiable on its domain of definition.

16.3 Differentiability of other standard functions

The following functions, that you may know from Calculus, are also differentiable. Just like when we introduced Proposition 14.3.3, we are not even ready to define these functions, but I think it's useful to mention the result here anyways.

Proposition 16.3.1. The functions

$$\begin{aligned} \exp &: \mathbb{R} \to \mathbb{R} & \ln &: (0, \infty) \to \mathbb{R} \\ \sin &: \mathbb{R} \to \mathbb{R} & \cos &: \mathbb{R} \to \mathbb{R} \\ \tan &: (-\pi/2, \pi/2) \to \mathbb{R} & \arctan &: \mathbb{R} \to \mathbb{R} \end{aligned}$$

are all differentiable on their domain, while the functions

$$\arcsin : [-1, 1] \to \mathbb{R} \qquad \arccos : [-1, 1] \to \mathbb{R}$$

are both differentiable on the interval $(-1, 1)$.

The derivatives are given by:

$$\begin{aligned} \exp'(t) &= \exp(t) & \ln'(t) &= 1/t \\ \sin'(t) &= \cos(t) & \cos'(t) &= -\sin(t) \\ \tan'(t) &= \frac{1}{\cos^2(t)} & \arctan'(t) &= \frac{1}{1+t^2} \\ \arcsin'(t) &= \frac{1}{\sqrt{1-t^2}} & \arccos'(t) &= -\frac{1}{\sqrt{1-t^2}} \end{aligned}$$

Example 16.3.2. Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(t) = \big(t^2, \sin(t)\big)$$

The component functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$f_1(t) = t^2$$

and

$$f_2(t) = \sin(t)$$

Since these component functions are differentiable standard functions, we find by Corollary 15.8.2 that f is differentiable as well and

$$f'(t) = \big(f_1'(t), f_2'(t)\big) = \big(2t, \cos(t)\big)$$

16.4 Exercises

Exercise 16.4.1. Consider the polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = 3x_1^m x_2^n + x_1^k$$

for some nonnegative integers m, n and k. Since f is a polynomial, it is differentiable on $\mathbb{R}^2$. Give $(Df) : \mathbb{R}^2 \to \mathsf{Lin}(\mathbb{R}^2, \mathbb{R})$ and justify your answer.

Exercise 16.4.2. Prove Proposition 15.8.1.

Exercise 16.4.3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x_1, x_2) = \begin{cases} \left(\frac{x_1 x_2^3}{x_1^2 + x_2^2},\ 5x_2\right) & \text{if } (x_1, x_2) \neq (0, 0) \\ (0, 0) & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Prove that f is differentiable on $\mathbb{R}^2$.

Exercise 16.4.4. i. Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) := \big(\cos(t), \sin(t), \arctan(t)\big).$$

Show that f is differentiable and give an expression for the function $f' : \mathbb{R} \to \mathbb{R}^3$ and for the derivative $(Df) : \mathbb{R} \to \mathsf{Lin}(\mathbb{R}, \mathbb{R}^3)$.

ii. Let w_1 and w_2 be two vectors in a finite-dimensional normed vector space $(W, \|\cdot\|_W)$. Consider the function $g : \mathbb{R} \to W$ given by

$$g(t) = \cosh(t)w_1 + \sinh(t)w_2.$$

Show that g is differentiable and give an expression for the function $g' : \mathbb{R} \to W$ and for the derivative $(Dg) : \mathbb{R} \to \mathsf{Lin}(\mathbb{R}, W)$.

Chapter 17

Directional and partial derivatives

17.1 A recurring and very important construction

The following construction is so important, that it really pays off to go through the following text very slowly and/or several times, making sure you understand every step.

To analyze the behavior of a function $f : \Omega \to W$, where Ω is some subset of V, we will often study how f behaves on lines. Let's make this precise. We will often select a point $a \in \Omega$, a direction $v \in V$, and consider the composition $f \circ \ell_{a,v}$, where the function $\ell_{a,v}$ maps from a small interval $(-\delta, \delta)$ around $0 \in \mathbb{R}$ to Ω and is given by

$$\ell_{a,v}(t) := a + tv.$$

Note that $\ell_{a,v}$ is an affine map, i.e. it is the sum of a constant and a linear map. It is therefore differentiable on $(-\delta, \delta)$, and for every $t \in (-\delta, \delta)$, its derivative $(D\ell_{a,v})_t$ in t is a linear map from $\mathbb{R}$ to V. In fact, for all $h \in \mathbb{R}$,

$$(D\ell_{a,v})_t(h) = hv.$$

If f is differentiable in a, then it follows by the chain rule that $f \circ \ell_{a,v}$ is differentiable in 0, and,

$$(D(f \circ \ell_{a,v}))_0 = (Df)_a \circ (D\ell_{a,v})_0.$$

To figure out what this means, we realize that $(D(f \circ \ell_{a,v}))_0$ is a linear map from $\mathbb{R}$ to W. So let's see its output when the input is $h \in \mathbb{R}$:

$$\begin{aligned}(D(f \circ \ell_{a,v}))_0(h) &= \big((Df)_a \circ (D\ell_{a,v})_0\big)(h) \\ &= (Df)_a\big((D\ell_{a,v})_0(h)\big) \\ &= (Df)_a\big(hv\big) \\ &= h(Df)_a(v)\end{aligned}$$

17.2 Directional derivative

We will now introduce the concept of the *directional derivative*, which can be viewed as the rate of change of a function when varying the input in a certain direction.

Definition 17.2.1 (Directional derivative)**.** Let $f : \Omega \to W$ be a function from an open domain Ω in a finite-dimensional normed vector space V to a finite-dimensional normed vector space W. Let $a \in \Omega$ and $v \in V$.

Then we say the *directional derivative* in the direction of v of f exists in the point $a \in \Omega$ if there exists a $\delta > 0$ such that the function

$$g := f \circ \ell_{a,v} : (-\delta, \delta) \to W$$

is differentiable in 0, where the function $\ell_{a,v} : (-\delta, \delta) \to V$ is defined by

$$\ell_{a,v}(t) := a + tv.$$

Moreover, if it exists, we define the directional derivative in the direction of v of f in the point a as

$$(D_v f)_a := g'(0) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}.$$

How does the *directional derivative* relate to the *derivative* of a function? The answer is subtle. If the derivative exists in a point a, then for all $v \in V$, the directional derivative in the direction of v of f in the point a exists as well and

$$(D_v f)_a = (Df)_a(v).$$

The last equality tells us that in this case the directional derivative in the direction of v at a point a, namely $(D_v f)_a$, is just the derivative of f in the point a (which is a linear map!) applied to the vector v, namely $(Df)_a(v)$. The precise statement is given by the next proposition. After the proposition, we will give a warning about the reverse direction: existence of directional derivatives does not say anything about existence of the derivative.

Proposition 17.2.2. Suppose $f : \Omega \to W$ is differentiable in a point $a \in \Omega$. Then for all $v \in V$, the directional derivative of f at a in the direction of v

$$(D_v f)_a$$

exists and is equal to the derivative of f at the point a (which is a linear map) applied to the vector v

$$(Df)_a(v).$$

Proof. This proposition follows from the Chain Rule, Theorem 15.6.1.

Indeed, let $v \in V$.

Because Ω is open, there exists a $\delta_1 > 0$ such that $B(a, \delta_1) \subset \Omega$.

Consider now the function $g := f \circ \ell_{a,v}$, which is a function from $(-\delta, \delta) \to W$ where $\delta := \delta_1 / \|v\|_V$.

The function $\ell_{a,v}$ is an affine function, and therefore it is differentiable in 0 with derivative

$$(D\ell_{a,v})_0 := \Big(h \mapsto hv\Big).$$

By the chain rule, Theorem 15.6.1, g is differentiable and the derivative of g, which is a linear map from $\mathbb{R}$ to W, is given by

$$(Dg)_0 = (D(f \circ \ell_{a,v}))_0 = (Df)_{\ell_{a,v}(0)} \circ (D\ell_{a,v})_0 \tag{17.2.1}$$

Since g is differentiable, by definition of the directional derivative, the directional derivative in the direction of v of f in a exists.

To find the value of the derivative, we now apply the left hand side and the right-hand side in (17.2.1) to the vector $1 \in \mathbb{R}$

$$(Dg)_0(1) = (Df)_{\ell_{a,v}(0)}\big((D\ell_{a,v})_0(1)\big) = (Df)_a(v)$$

On the other hand, by Proposition 15.1.5,

$$(D_v f)_a = g'(0) \cdot 1 = (Dg)_0(1) = (Df)_a(v)$$

which is what we wanted to show. □

There are functions $f : \Omega \to W$ that are **not differentiable in a point** a even though for every $v \in V$, the directional derivative $(D_v f)_a$ exists. See for instance the next example.

Example 17.2.3. Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x_1, x_2) := \begin{cases} x_1, & x_2 \neq 0 \\ 0, & x_2 = 0. \end{cases}$$

Let us verify that for all $v \in \mathbb{R}^2$, the directional derivative $(D_v f)_0$ exists. Let $v \in \mathbb{R}^2$. If $v_2 \neq 0$, then

$$\begin{aligned} (D_v f)_0 &= \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{tv_1 - 0}{t} \\ &= \lim_{t \to 0} v_1 \\ &= v_1. \end{aligned}$$

while if $v_2 = 0$ then

$$\begin{aligned} (D_v f)_0 &= \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{0 - 0}{t} \\ &= 0. \end{aligned}$$

In both cases, the directional derivative exists.

We now claim that f is **not** differentiable in 0. Indeed, *if* f would be differentiable in 0, then the derivative $(Df)_0$ would be a linear map from $\mathbb{R}^2$ to $\mathbb{R}$. Since $(Df)_0(e_1) = 0$ and $(Df)_0(e_2) = 0$, in fact $(Df)_0$ maps every vector to zero. In particular $(D_{(1,1)}f)_0 = 0$. However, our computation above shows that $(D_{(1,1)}f)_0 = 1$. This is a contradiction.

17.3 Partial derivatives

Partial derivatives are special types of directional derivatives, for functions that are defined on the vector space $\mathbb{R}^d$.

Definition 17.3.1. Let $f : \Omega \to W$ be a function defined on an open domain $\Omega \subset \mathbb{R}^d$. The ith partial derivative in a point $a \in \Omega$, denoted by

$$\frac{\partial f}{\partial x_i}(a),$$

is the directional derivative in the direction of the ith unit vector e_i

$$\frac{\partial f}{\partial x_i}(a) := (D_{e_i}f)_a = \frac{d}{dt}f(a + te_i)\bigg|_{t=0} = \lim_{h \to 0} \frac{f(a + he_i) - f(a)}{h}.$$

Here

$$e_i := (\underbrace{0, \ldots, 0, 1, 0 \ldots, 0}_{1 \text{ in } i\text{th position}}).$$

Example 17.3.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^4.$$

Let us determine whether the partial derivative

$$\frac{\partial f}{\partial x_2}.$$

exists in the point $a := (a_1, a_2)$.

To do so, by definition, we need to see if the directional derivative of f in the direction of e_2 in the point a, namely

$$(D_{e_2} f)_a$$

exists.

By Definition 17.2.1, we need to verify whether the derivative of the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} g(t) &:= (f \circ \ell_{a,e_2})(t) = f(a + te_2) \\ &= f\big((a_1, a_2 + t)\big) = a_1^2 + 2a_1(a_2 + t) + 3(a_2 + t)^4. \end{aligned}$$

exists in the point $t = 0$. Since g is a polynomial in one variable, it is indeed differentiable, and the derivative in the point $t = 0$ exists and

$$g'(0) = 2a_1 + 12a_2^3.$$

Therefore, according to Definition 17.2.1, the partial derivative of f in the point (a_1, a_2) exists and equals

$$\frac{\partial f}{\partial x_2}(a) = (D_{e_2} f)_a = 2a_1 + 12a_2^3.$$

In general, there are many different expressions for the partial derivative of a function in some point a. Here are a few of them

$$\begin{aligned} \frac{\partial f}{\partial x_i}(a) := (D_{e_i} f)_a &= \frac{d}{dt} f(a + te_i)\Big|_{t=0} \\ &= \frac{d}{dt} f(a_1, \dots, a_{i-1}, a_i + t, a_{i+1}, \dots, a_d)\Big|_{t=0} \\ &= \frac{d}{ds} f(a_1, \dots, a_{i-1}, s, a_{i+1}, \dots, a_d)\Big|_{s=a_i}. \end{aligned}$$

The moral of the last expression is very nice: to determine the partial derivative of f in a point a, you keep all coordinates fixed except for the ith coordinate, and you then view the function as a function of only that

ith coordinate. It then is a function of only one variable, and you can differentiate according to the one-variable definition in calculus.

Let us record the statement in a proposition.

Proposition 17.3.3. Let $f : \Omega \to W$ be a function from an open domain Ω in $\mathbb{R}^d$ to a (finite-dimensional) normed vector space $(W, \|\cdot\|_W)$. Let $a \in \Omega$.

The ith partial derivative of f in the point a exists if and only if the function

$$t \mapsto f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_d)$$

is differentiable in the point a_i, and in this case

$$\frac{\partial f}{\partial x_i}(a) = \frac{d}{dt} f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_d)\bigg|_{t=a_i}$$

Example 17.3.4. Consider again the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^4.$$

By the previous proposition, to determine whether the partial derivative

$$\frac{\partial f}{\partial x_2}(x_1, x_2)$$

exists in a point (x_1, x_2) and to determine its value, we just verify that

$$\frac{d}{dt} f(x_1, t)\bigg|_{t=x_2} = \frac{d}{dt}(x_1^2 + 2x_1 t + 3t^4)\bigg|_{t=x_2} = 2x_1 + 12x_2^3.$$

We conclude as before that

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 2x_1 + 12x_2^3.$$

17.4 The Jacobian of a map

In Section 15.5 and in particular in Definition 15.5.1 we introduced the Jacobian of a map (with respect to some bases). The Jacobian $[Df]_a$ of f in a is the matrix (representation) of the linear map $(Df)_a$ with respect to the bases $v_1, \dots, v_d$ and $w_1, \dots, w_m$. We will now give a way to compute the Jacobian, that was already announced in Section 15.5 but back then we didn't have the means to prove our statements.

First we start with the particular case when $f : \Omega \to \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^d$, and we choose the standard bases of unit vectors in $\mathbb{R}^d$ and $\mathbb{R}^m$.

Proposition 17.4.1. Suppose $f : \Omega \to \mathbb{R}^m$ is a function defined on an open domain $\Omega \subset \mathbb{R}^d$, and suppose f is differentiable in a point $a \in \Omega$. Then the Jacobian matrix of f (with respect to the standard bases) is given by

$$[Df]_a := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_d}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_d}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_d}(a) \end{pmatrix}$$

In other words, for all $x \in \mathbb{R}^d$, it holds that

$$(Df)_a(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_d}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_d}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_d}(a) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

Proof. Since $(Df)_a$ is a linear map, it follows from linear algebra that we just need to check that for every $j = 1, \dots, m$, the jth column of the matrix corresponds to the image of the standard unit vector e_j, i.e. to $(Df)_a(e_j)$. However, since f is differentiable, this last expression corresponds to the jth partial derivative of f:

$$(Df)_a(e_j) = \frac{\partial f}{\partial x_j}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_j}(a) \\ \frac{\partial f_2}{\partial x_j}(a) \\ \vdots \\ \frac{\partial f_m}{\partial x_j}(a) \end{pmatrix}$$

so indeed this corresponds with the jth column of the matrix. □

In the more general case of a map $f : \Omega \to W$ from a subset Ω in a finite-dimensional vector space V to a finite dimensional vector space W with basis $v_1, \dots, v_d$ and $w_1, \dots, w_m$ and coordinate maps Φ and Ψ respectively, we can compute $[Df]_a$ from the coordinate representation $\bar{f} = \Psi \circ f \circ \Phi^{-1}$ of f.

Proposition 17.4.2. Let $f : \Omega \to W$ with $\Omega \subset V$ open, and let $v_1, \dots, v_d$ be a basis of V with coordinate map Φ and let $w_1, \dots, w_m$ be a basis of W with coordinate map Ψ. Let $a \in \Omega$. Then the Jacobian of f with respect to these bases is given by

$$[Df]_a = [D\bar{f}]_{\Phi(a)}$$

where $\bar{f} := \Psi \circ f \circ \Phi^{-1}$ is the coordinate representation of f.

17.5 Linearization and tangent planes

If a function $f : \Omega \to W$ is differentiable in $a \in \Omega$, then by definition it can be well approximated by an affine function. This affine function is also called the *linearization* of f.

Definition 17.5.1 (Linearization)**.** Let $f : \Omega \to W$ be differentiable in a point $a \in \Omega$. Then the *linearization* of f is the function $L_a : V \to W$ given by

$$L_a(x) = f(a) + (Df)_a(x - a)$$

Recall that the *graph* of a function $f : \Omega \to \mathbb{R}$ is the following subset of $\Omega \times \mathbb{R}$:

$$\mathsf{Graph}(f) := \{(x, f(x)) \mid x \in \Omega\}.$$

Definition 17.5.2. Let $f : \Omega \to \mathbb{R}$, where Ω is a subset of a normed vector space V. Assume f is differentiable in $a \in \Omega$. Then the *tangent plane to the graph of f* at a is the graph of the linearization L_a of f, i.e.

$$T_a := \{(v, L_a(v)) \mid v \in V\}$$

Definition 17.5.3. Let $f : \Omega \to \mathbb{R}$ where Ω is a subset of a normed vector space V. Let $a \in \Omega$, and set $c := f(a)$. Assume f is differentiable in a with $(Df)_a \neq 0$. Then the *tangent plane to the level set*

$$f^{-1}(c) = \{x \in V \mid f(x) = c\}$$

at a is given by

$$\{x \in V \mid L_a(x) = c\}.$$

17.6 The gradient of a function

Definition 17.6.1. Let $f : \Omega \to \mathbb{R}$ be a function from an open domain Ω in $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$ and suppose f is differentiable in the point

$a \in \Omega$. Then we call the vector

$$\nabla f(a) := \begin{pmatrix} \frac{\partial f}{\partial x_1}(a) \\ \vdots \\ \frac{\partial f}{\partial x_d}(a) \end{pmatrix}$$

the *gradient* of f in the point a.

If a function $f : \Omega \to \mathbb{R}$ is differentiable in a point a, then the derivative $(Df)_a$ relates to the gradient $\nabla f(a)$ as follows.

Proposition 17.6.2. Let $f : \Omega \to \mathbb{R}$ be a function from an open domain Ω in $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$ and suppose f is differentiable in the point a. Then for all $v \in \mathbb{R}^d$,

$$(Df)_a(v) = (\nabla f(a), v) = (\nabla f(a))^T v = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(a) v_i$$

where $(\cdot, \cdot)$ denotes the standard inner product on $\mathbb{R}^d$.

Proposition 17.6.3. Let $f : \Omega \to \mathbb{R}$ from an open domain Ω in $(\mathbb{R}^d, \|\cdot\|_2)$. Assume f is differentiable in a point $a \in \Omega$ with $(Df)_a \neq 0$. Set $c := f(a)$. Then the tangent plane to the level set $f^{-1}(c)$ at a is given by

$$a + \{x \in \mathbb{R}^d \mid (\nabla f(a), x) = 0\}.$$

17.7 Exercises

Exercise 17.7.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) := \begin{cases} \frac{(x_2)^2}{x_1} & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

(a). Show that for all $v \in \mathbb{R}^2$, the directional derivative $(D_v f)_0$ (i.e. the directional derivative at 0 in the direction of v) exists and compute its value.

(b). Show that f is *not* continuous in $0 \in \mathbb{R}^2$.

Exercise 17.7.2. Consider the map $g : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$g((r, \phi)) := (r \cos \phi, r \sin \phi).$$

(a). Show that g is differentiable.

(b). Compute for every $(r, \phi) \in (0, \infty) \times \mathbb{R}$ the Jacobian $[Dg]_{(r,\phi)}$.

Exercise 17.7.3. Let $f : \Omega \to W$ be differentiable in a point $a \in \Omega$ and let $v_1, \dots, v_d$ and $w_1, \dots, w_m$ be bases of V and W respectively. Show that the matrix (representation) $[Df]_a$ of $(Df)_a$ with respect to the bases $\{v_j\}$ and $\{w_i\}$ is a matrix $[b_{ij}]$ of which the element in the ith row and jth column equals

$$b_{ij} = (D_{v_j}(\Psi_i \circ f))_a$$

where $\Psi_1, \dots, \Psi_m$ is the dual basis to $w_1, \dots, w_m$.

Hint: It may help to re-read Section 15.5 and to apply the Chain rule.

Exercise 17.7.4. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f((x_1, x_2, x_3)) = \sin\left((x_1)^2 + (x_2)^3 + (\cos(x_3))^4\right)$$

(a). Prove that for all $a \in \mathbb{R}^3$ the partial derivatives

$$\frac{\partial f}{\partial x_1}(a), \quad \frac{\partial f}{\partial x_2}(a), \quad \frac{\partial f}{\partial x_3}(a)$$

exist and compute their values.

(b). Compute for every $a \in \mathbb{R}^3$, the gradient $\nabla f(a)$.

Chapter 18

The Mean-Value Inequality

In this chapter, we prove what is perhaps the most important inequality of Analysis. It is a statement about functions $f : [a,b] \to W$, that are continuous on $[a,b]$ and differentiable on (a,b). In one version, the inequality says that

$$\|f(b) - f(a)\|_W \leq \sup_{t \in (a,b)} \|f'(t)\|_W (b-a).$$

To get the most out of this inequality, you often don't apply it to a function directly, but rather to the difference of two functions. This difference could for instance be a difference of the function you are interested in, and a linear function. By picking a good difference of functions, the right-hand side in the inequality actually becomes small, so that you can conclude that the left-hand side becomes small too.

18.1 The mean-value inequality for functions defined on an interval

Lemma 18.1.1 (Mean-value inequality (v0)). Let $f : [a,b] \to W$ be continuous on $[a, b]$ and differentiable on (a, b). Then

$$\|f(b) - f(a)\|_W \leq \sup_{t \in (a,b)} \|f'(t)\|_W (b - a).$$

Proof. Denote

$$K := \sup_{t \in (a,b)} \|f'(t)\|_W.$$

In this first part of the proof we will a few times use an "It suffices to show that-" construction to reduce what we need to show to an easier statement.

We first claim that it suffices to show that for all $\bar{a} \in (a, b)$

$$\|f(b) - f(\bar{a})\|_W \leq K(b - \bar{a}). \tag{18.1.1}$$

To see why this suffices, note that the left-hand side and right-hand side can be viewed as continuous functions of $\bar{a}$. Therefore, if we know that (18.1.1) holds, we can take the limit as $\bar{a} \to a$ on both sides, and conclude that also

$$\|f(b) - f(a)\|_W \leq K(b - a).$$

Let therefore $\bar{a} \in (a, b)$. We aim to show (18.1.1).

We now claim that it suffices to show that for all $\epsilon > 0$, all $s \in [\bar{a}, b]$,

$$\|f(s) - f(\bar{a})\|_W \leq (K + \epsilon)(s - \bar{a}). \tag{18.1.2}$$

To prove the claim, we aim to show (18.1.1) from (18.1.2). First note that if (18.1.2) holds for all $\epsilon > 0$ and all $s \in [\bar{a}, b]$, then it also holds for all $\epsilon > 0$ and $s = b$. We now argue by contradiction. Suppose

$$\|f(b) - f(\bar{a})\|_W > K(b - \bar{a})$$

Then, we may define

$$\epsilon_1 := \frac{1}{2}\left(\frac{\|f(b) - f(\bar{a})\|_W}{b - \bar{a}} - K\right) > 0$$

so that

$$\|f(b) - f(\bar{a})\|_W > (K + \epsilon_1)(b - \bar{a}).$$

Yet when we choose $\epsilon = \epsilon_1$ in (18.1.2), we obtain

$$\|f(b) - f(\bar{a})\|_W \leq (K + \epsilon_1)(b - \bar{a}).$$

which is a contradiction. We conclude that (18.1.1) holds.

After all these reductions, we are left with showing that for all $\epsilon > 0$ and all $s \in [\bar{a}, b]$, indeed (18.1.2) holds.

Let $\epsilon > 0$.

We will now show three claims.

Our first claim is that inequality (18.1.2) holds for $s = \bar{a}$. This holds because

$$\|f(a) - f(\bar{a})\|_W = 0 \leq 0 = (K + \epsilon)(\bar{a} - \bar{a}).$$

Our second claim is that whenever inequality (18.1.2) holds for all $s \in [\bar{a}, c)$ for some $c \in [\bar{a}, b]$, it also holds for $s = c$. This follows since the left-hand side and the right-hand of the inequality (18.1.2) are continuous when interpreted as functions in s.

Our third claim is that whenever the inequality holds for all $s \in [\bar{a}, c]$ for some $c \in [\bar{a}, b)$, there exists a $\delta > 0$ such that the inequality holds for all $s \in [\bar{a}, c + \delta)$.

To prove this third claim, let $c \in [\bar{a}, b)$ and assume the inequality holds for all $s \in [\bar{a}, c]$. Since f is differentiable in c, there exists a $\delta > 0$ such that for all $s \in [c, \delta)$,

$$\|f(s) - f(c) - f'(c)(s - c)\|_W = \|\mathsf{Err}_c^f(s)\|_W \leq \epsilon|s - c| = \epsilon(s - c)$$

Therefore by the triangle inequality for all $s \in [c, \delta)$,

$$\begin{aligned}\|f(s) - f(c)\|_W &= \|\mathsf{Err}_c^f(s) + f'(c)(s-c)\|_W \\ &\leq \|f'(c)(s-c)\|_W + \epsilon(s-c) \leq (K+\epsilon)(s-c)\end{aligned}$$

As a consequence,

$$\begin{aligned}\|f(s) - f(\bar{a})\|_W &\leq \|f(s) - f(c)\|_W + \|f(c) - f(\bar{a})\|_W \\ &\leq (K+\epsilon)(s-c) + (K+\epsilon)(c-\bar{a}) = (K+\epsilon)(s-\bar{a}).\end{aligned}$$

which shows that indeed inequality (18.1.2) is also satisfied for all $s \in [\bar{a}, c+\delta)$. Hence we have proved the third claim.

We now define the set S as those $s \in [\bar{a}, b]$ such that for all $\sigma \in [\bar{a}, s]$, inequality (18.1.2) is satisfied. In other words,

$$S := \{s \in [\bar{a}, b] \mid \|f(\sigma) - f(\bar{a})\|_W \leq (K+\epsilon)(\sigma - \bar{a}) \text{ for all } \sigma \in [\bar{a}, s]\}.$$

Note that just from its definition, it follows that S is either the empty set, or just the point $\{\bar{a}\}$ or it is an interval that is closed on the left with $\bar{a}$ as the left endpoint. From the first claim, we know that $\bar{a} \in S$, so S is not empty. The second claim tells us that S is closed, so it is either $\{\bar{a}\}$ or it is a closed interval of the form $[\bar{a}, c]$ with $c \in (\bar{a}, b]$. The third claim gives a contradiction when $S = \{\bar{a}\}$ or $S = [\bar{a}, c]$ with $c < b$. Therefore $S = [\bar{a}, b]$ and inequality (18.1.2) is satisfied in $s = b$. □

18.2 The mean-value inequality for functions on general domains

Before we can state the mean-value inequality for functions defined on general domains, let's have a small recap about the operator norm

$$\|\cdot\|_{V \to W}$$

defined on the space of linear maps $\mathsf{Lin}(V, W)$. What we need to remember about this norm is the following. Given a linear map $L : V \to W$, the norm $\|L\|_{V \to W}$ is the smallest constant $K \in \mathbb{R}$ such that for all $v \in V$,

$$\|L(v)\|_W \leq K\|v\|_V.$$

In other words,

i. for all $v \in V$, and every $L \in \mathsf{Lin}(V, W)$,

$$\|L(v)\|_W \leq \|L\|_{V \to W} \|v\|_V$$

ii. for every $L \in \mathsf{Lin}(V, W)$ we have the following. If $K \in \mathbb{R}$ is a constant such that for all $v \in V$

$$\|L(v)\|_W \leq K \|v\|_V,$$

then

$$\|L\|_{V \to W} \leq K.$$

If we combine the mean-value inequality from the previous section with the chain rule, we obtain the following version of the mean value-inequality.

Corollary 18.2.1 (Mean-value inequality)**.** Let $f : \Omega \to W$ be differentiable on an open domain $\Omega \subset V$. Then, for all $a, b \in \Omega$, if for every $\tau \in (0, 1)$, also

$$(1 - \tau)a + \tau b \in \Omega$$

then

$$\|f(b) - f(a)\|_W \leq \sup_{\tau \in (0,1)} \|(Df)_{(1-\tau)a+\tau b}\|_{V \to W} \|b - a\|_V.$$

The derivation of Corollary 18.2.1 from Lemma 18.1.1 is the topic of Exercise 18.4.1.

Lemma 18.2.2. Suppose $f : \Omega \to W$ is differentiable on Ω, and suppose its derivative function $Df : \Omega \to \mathsf{Lin}(V, W)$ is bounded. Let $a \in \Omega$ and assume $r > 0$ is such that $B(a, r) \subset \Omega$. Then for all $x \in B(a, r)$,

$$\|\mathsf{Err}_a^f(x)\|_W \leq \sup_{z \in B(a,r)} \|(Df)_z - (Df)_a\|_{V \to W} \|x - a\|_V$$

18.3 Continuous partial derivatives implies differentiability

As a first consequence of the Mean-Value Inequality, let us show that functions with continuous partial derivatives are continuously differentiable. This is quite useful, because this way, in order to conclude that a function is differentiable, it suffices to show that the partial derivatives exist and that they are continuous.

Let's first define what it means for a function to be continuously differentiable.

Definition 18.3.1. We say a function $f : \Omega \to W$ is continuously differentiable if it is differentiable and its derivative $(Df) : \Omega \to \mathsf{Lin}(V, W)$ is a continuous function on Ω.

We are now ready to state the proposition.

Proposition 18.3.2. Let $f : \Omega \to W$ be a function defined on some open set $\Omega \subset \mathbb{R}^d$ and let $a \in \Omega$. Assume that there exists a radius $r > 0$ such that for all $x \in B(a, r)$, and for all $i \in \{1, \dots, d\}$, the partial derivative

$$\frac{\partial f}{\partial x_i}(x)$$

exists and the function

$$\frac{\partial f}{\partial x_i} : \Omega \to W$$

is continuous on $B(a, r)$.

Then the function f is continuously differentiable on $B(a, r)$.

Proof. We need to show that for all $b \in B(a, r)$, the function f is differentiable in b. Define $\rho := r - \|b - a\|_2$. Then $B(b, \rho) \subset B(a, r)$ and therefore the partial derivatives exist and are continuous on $B(b, \rho)$.

As a possible candidate for the derivative, we define the linear map

$L_b : \mathbb{R}^d \to W$ by

$$L_b(v) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(b) v_i.$$

Now define the error function $\mathsf{Err}_b^f : \Omega \to W$ by

$$\mathsf{Err}_b^f(x) = f(x) - (f(b) + L_b(x - b)).$$

According to the definition of differentiability, in order to show that the linear map L_b is the derivative of f in $b \in \Omega$, we need to show that

$$\lim_{x \to b} \frac{\|\mathsf{Err}_b^f(x)\|_W}{\|x - b\|_2} = 0.$$

Let $\epsilon > 0$.

By assumption, for every $i \in \{1, \dots, d\}$, the partial derivative

$$\frac{\partial f}{\partial x_i} : \Omega \to W$$

is continuous. Therefore, by the definition of continuity, for every $i \in \{1, \dots, d\}$ there exists a $\delta_i > 0$ such that for all $z \in B(b, \delta_i)$,

$$\left\| \frac{\partial f}{\partial x_i}(z) - \frac{\partial f}{\partial x_i}(b) \right\|_W < \frac{\epsilon}{d}.$$

Choose such δ_i and choose

$$\delta := \min(\delta_1, \dots, \delta_d, \rho).$$

Now let $x \in B(b, \delta)$. To show that $\mathsf{Err}_b^f(x)$ is small, we are going to apply the Mean-Value Inequality (a few times), on paths that are parallel to the axes in $\mathbb{R}^d$.

We define the points

$$\begin{aligned} y^0 &:= (b_1, \dots, b_d) \\ y^j &:= (x_1, \dots, x_j, b_{j+1}, b_{j+2}, \dots, b_d) \\ y^d &:= (x_1, \dots, x_d) \end{aligned}$$

and write

$$\begin{aligned} \mathsf{Err}_b^f(x) &= f(x) - f(b) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(b)(x_i - b_i) \\ &= \sum_{i=1}^d \Big(f(y^i) - f(y^{i-1}) \Big) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(b)(x_i - b_i). \end{aligned}$$

We now apply the Mean-Value Inequality to the functions g_i given by

$$g_i(t) = f(x_1, \dots, x_{i-1}, t, b_{i+1}, \dots, b_d) - \frac{\partial f}{\partial x_i}(b)(t - b_i)$$

and find that

$$\begin{aligned} \|\mathsf{Err}_b^f(x)\| &\leq \sum_{i=1}^d \sup_{z \in B(b,\delta)} \left\| \frac{\partial f}{\partial x_i}(z) - \frac{\partial f}{\partial x_i}(b) \right\|_W |x_i - b_i| \\ &< \frac{\epsilon}{d} \sum_{i=1}^d |x_i - b_i| \\ &\leq \frac{\epsilon}{d} d \|x - b\|_2 \\ &= \epsilon \|x - b\|_2. \end{aligned}$$

Now that we know that the function f is differentiable, we also know that

$$(Df)_b\big((x_1, \dots, x_d)\big) = \frac{\partial f}{\partial x_1}(b) x_1 + \cdots + \frac{\partial f}{\partial x_d}(b) x_d.$$

and therefore, for $c \in B(a, r)$, using first the triangle inequality and then the Cauchy-Schwarz inequality

$$\begin{aligned}\|((Df)_c - (Df)_b)(x_1, \dots, x_d)\|_W &\leq \sum_{i=1}^{d} \left\| \frac{\partial f}{\partial x_i}(c) - \frac{\partial f}{\partial x_i}(b) \right\|_W |x_i| \\ &\leq \sqrt{\sum_{i=1}^{d} \left\| \frac{\partial f}{\partial x_i}(c) - \frac{\partial f}{\partial x_i}(b) \right\|_W^2} \|x\|_2\end{aligned}$$

It follows that

$$\|(Df)_c - (Df)_b\|_{V \to W} \leq \sqrt{\sum_{i=1}^{d} \left\| \frac{\partial f}{\partial x_i}(c) - \frac{\partial f}{\partial x_i}(b) \right\|_W^2}$$

so that the continuity of (Df) follows from the continuity of the partial derivatives. □

18.4 Exercises

Exercise 18.4.1. Prove Corollary 18.2.1.

Exercise 18.4.2. Give a proof of Lemma 18.2.2.

Exercise 18.4.3. Define the subset $\Omega \subset \mathbb{R}^2$ as follows

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid \|(x_1, x_2)\|_2 > 1.9\}.$$

Let $f : \Omega \to W$ be a differentiable function and assume that for all $a \in \Omega$,

$$\|(Df)_a\|_{\mathbb{R}^2 \to W} \leq 5.$$

Prove that

$$\|f((2,0)) - f((-2,0))\|_W \leq 10\pi.$$

Exercise 18.4.4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) := \begin{cases} \frac{(x_1)^2 (x_2)^7}{(x_1)^2 + (x_2)^2} & \text{if } (x_1, x_2) \neq (0,0) \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

a. Show that f is differentiable on $\mathbb{R}^2$ by showing that the partial derivatives exist and are continuous.

b. For $a \in \mathbb{R}^2$, compute $\nabla f(a)$.

Chapter 19

Higher order derivatives

The second order derivative is the derivative of the derivative, the third order derivative is the derivative of the second order derivative, the fourth order derivative is the derivative of the third order derivative, etc.. This way, we create quite complicated objects, and I would therefore like to encourage you to, at least at first, study what the statements in this chapter are, and what the mathematical objects are, rather than the proofs of the statements.

As a bit of help, here's a list of most important messages for this chapter:

- That the $(n+1)$st derivative is the derivative of the nth derivative
- The interpretation of the nth order derivative in terms of iterated directional derivatives.
- Concluding higher-order differentiability from continuity of higher order derivatives
- The symmetry of nth order derivatives

19.1 Definition of higher order derivatives

Higher order derivatives are defined inductively. If $f : \Omega \to W$ is differentiable on a domain $\Omega \subset V$, then the derivative *itself* can be interpreted as a function

$$(Df) : \Omega \to \mathsf{Lin}(V, W)$$

i.e. it is a function from Ω to $\mathsf{Lin}(V, W)$. Because $\mathsf{Lin}(V, W)$ is a finite-dimensional vector space again, we can use the definition of differentiability to check whether the function $(Df) : \Omega \to \mathsf{Lin}(V, W)$ is differentiable in a point a. If so, we denote the derivative of (Df) in the point a by $(D(Df))_a$.

If (Df) is differentiable in every point a in V, then we say f is twice differentiable, and the second derivative is a function

$$(D(Df)) : \Omega \to \mathsf{Lin}(V, \mathsf{Lin}(V, W)).$$

Similarly, the third derivative is a function

$$(D(D(Df))) : \Omega \to \mathsf{Lin}(V, \mathsf{Lin}(V, \mathsf{Lin}(V, W))).$$

The general definition can be given by induction. We first define the space $\mathsf{Lin}_n(V, W)$ inductively.

Definition 19.1.1. We set $\mathsf{Lin}_1(V, W) := \mathsf{Lin}(V, W)$ and for every $n \in \mathbb{N} \setminus \{0\}$, we define $\mathsf{Lin}_{n+1}(V, W) := \mathsf{Lin}(V, \mathsf{Lin}_n(V, W))$.

Definition 19.1.2 (Higher-order derivatives)**.** Let $n \in \mathbb{N} \setminus \{0, 1\}$. Suppose $f : \Omega \to W$ is n times differentiable on a ball $B(a, r) \subset \Omega$. We then say that f is $(n + 1)$ times differentiable in the point a if the function

$$D^n f : B(a, r) \to \mathsf{Lin}_n(V, W)$$

is differentiable in a. The $(n + 1)$th derivative in the point a is then defined as

$$(D^{n+1} f)_a := (D(D^n f))_a \in \mathsf{Lin}_{n+1}(V, W).$$

19.2 Multi-linear maps

The space $\mathsf{Lin}_n(V, W)$ is a bit cumbersome to work with, but we may equivalently interpret elements from $\mathsf{Lin}_n(V, W)$ as o-called *multi-linear* maps from the n-fold Cartesian product

$$V^{\times n} = \underbrace{V \times \cdots \times V}_{n \text{ times}}$$

to W.

Definition 19.2.1 (Multi-linear maps)**.** A map $L : V^{\times n} \to W$ is called multi-linear, or n-linear, if for every $i \in \{1, \dots, n\}$ and every $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n \in V$, the map

$$u \mapsto L(v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_n)$$

(which is a map from the vector space V to the vector space W) is linear. We will denote the vector space of n-linear maps from $V^{\times n}$ to W by $\mathsf{MLin}(V^{\times n}, W)$.

The statement that we may equivalently interpret elements from $\mathsf{Lin}_n(V, W)$ as multi-linear maps, precisely means that there is an invertible linear map from $\mathsf{Lin}_n(V, W)$ to $\mathsf{MLin}(V^{\times n}, W)$ that preserves norm (with the choice of norm on $\mathsf{MLin}(V^{\times n}, W)$ that we will give later). Intuitively, this has as a consequence that for all accounts and purposes these two spaces are the same.

The linear map $\mathcal{J}_n$ that brings elements in $\mathsf{Lin}_n(V, W)$ to multilinear maps in $\mathsf{MLin}(V^{\times n}, W)$ is given by

$$(\mathcal{J}_n A)(v_1, \dots, v_n) = A(v_1)(v_2) \cdots (v_n).$$

More precisely, we define the map $\mathcal{J}_n$ inductively, namely

$$\mathcal{J}_1 A := A$$

while for $n \in \mathbb{N} \setminus \{0\}$,

$$(\mathcal{J}_{n+1} A)(v_1, \cdots, v_{n+1}) := \mathcal{J}_n(A(v_1))(v_2, \cdots, v_{n+1})$$

We define the following norm on the space $\mathsf{MLin}(V^{\times n}, W)$ of multi-linear maps from $V^{\times n}$ to W:

$$\|L\|_{\mathsf{MLin}(V^{\times n}, W)} = \sup_{\|v_1\|_V, \dots, \|v_n\|_V \leq 1} \|L(v_1, \dots, v_n)\|_W \tag{19.2.1}$$

We also define inductively the maps $\mathcal{K}_n : \mathsf{MLin}(V^{\times n}, W) \to \mathsf{Lin}_n(V, W)$, which will turn out to be the inverses of the map $\mathcal{J}_n$. We define

$$\mathcal{K}_1 B = B$$

and

$$(\mathcal{K}_{n+1} B)(v_1) = \mathcal{K}_n(B(v_1, \cdots)).$$

Proposition 19.2.2. Then $\mathcal{J}_n$ is invertible with inverse $\mathcal{K}_n$, and it preserves the norm.

Proof. We are going to prove this by induction.

For $n = 1$, the maps $\mathcal{J}_n$ and $\mathcal{K}_n$ are both just the identity map. Therefore it is clear that they are each others inverses. They also preserve the norm, because the norm $\| \cdot \|_{V \to W}$ and $\| \cdot \|_{\mathsf{Mlin}(V^{\times 1}, W)}$ are actually the same.

Now let $n \in \mathbb{N} \setminus \{0\}$ and assume the statement is proven for $\mathcal{J}_n$. We will first show that $\mathcal{K}_{n+1} \circ \mathcal{J}_{n+1}$ is the identity.

Let $A \in \mathsf{Lin}_{n+1}(V, W)$ and let $v_1 \in V$. Then

$$\begin{aligned}
(\mathcal{K}_{n+1} \circ \mathcal{J}_{n+1}(A))(v_1) &= \Big(\mathcal{K}_{n+1}(\mathcal{J}_{n+1}(A))\Big)(v_1) \\
&= \mathcal{K}_n\Big(\mathcal{J}_{n+1}(A)(v_1, \cdots)\Big) \\
&= \mathcal{K}_n(\mathcal{J}_n(A(v_1))) \\
&= A(v_1).
\end{aligned}$$

We will now show that $\mathcal{J}_{n+1} \circ \mathcal{K}_{n+1}$ is the identity. Let $v_1 \in V$ and let $B \in \mathsf{MLin}(V^{\times(n+1)}, W)$. Then

$$\begin{aligned}
(\mathcal{J}_{n+1}(\mathcal{K}_{n+1}(B)))(v_1, \cdots) &= \mathcal{J}_n\left((\mathcal{K}_{n+1} B)(v_1)\right) \\
&= \mathcal{J}_n\left(\mathcal{K}_n(B(v_1, \cdots))\right) \\
&= B(v_1, \cdots).
\end{aligned}$$

Finally, $\mathcal{J}_{n+1}$ preserves the norm since

$$\begin{aligned}
\|A\|_{\mathsf{Lin}_{n+1}(V,W)} &= \sup_{\|v_1\|_V \leq 1} \|A(v_1)\|_{\mathsf{Lin}_n(V,W)} \\
&= \sup_{\|v_1\|_V \leq 1} \|\mathcal{J}_n(A(v_1))\|_{\mathsf{MLin}(V^{\times n},W)} \\
&= \sup_{\|v_1\|_V \leq 1} \sup_{\|v_2\|_V \leq 1,\dots,\|v_{n+1}\|_V \leq 1} \|\mathcal{J}_n(A(v_1))(v_2,\dots,v_{n+1})\|_W \\
&= \sup_{\|v_1\|_V \leq 1} \sup_{\|v_2\|_V \leq 1,\dots,\|v_{n+1}\|_V \leq 1} \|\mathcal{J}_{n+1}A(v_1,v_2,\dots,v_{n+1})\|_W \\
&= \|\mathcal{J}_{n+1}A\|_{\mathsf{MLin}(V^{\times(n+1)},W)}.
\end{aligned}$$

□

As a consequence of the previous proposition, it doesn't really matter whether we view a map as an element of $\mathsf{Lin}_n(V,W)$ or as an element of $\mathsf{MLin}(V^{\times n},W)$.

We will therefore just leave out the explicit application of $\mathcal{J}_n$, i.e. we write A instead of $\mathcal{J}_n(A)$ and use notation

$$A(v_1,\dots,v_n)$$

and

$$A(v_1)\cdots(v_n)$$

interchangeably.

19.3 Relation to n-fold directional derivatives

It is a bit difficult sometimes to interpret nth order derivatives, but it gets easier if we relate them to directional derivatives. If $f : \Omega \to W$ is a function, defined on an open subset Ω of V, if $v_1 \in V$ and the directional derivative $(D_{v_1}f)_a$ exists in every point $a \in \Omega$, then we can build the function

$$(D_{v_1}f) : \Omega \to W, \qquad a \mapsto (D_{v_1}f)_a$$

that maps $a \in \Omega$ to $(D_{v_1}f)_a \in W$. Now we can continue and take a new direction $v_2 \in V$ and check if the directional derivatives of the function

$(D_{v_1} f)$ in the direction of v_2 exists in the point a. In notation, we can check whether the directional derivative

$$(D_{v_2}(D_{v_1} f))_a$$

exists. We call this a two-fold directional derivative. If it exists for every $a \in \Omega$, then we can consider the function

$$(D_{v_2}(D_{v_1} f)) : \Omega \to W, \qquad a \mapsto (D_{v_2}(D_{v_1} f))_a$$

and see if directional derivatives of this function exist to get three-fold directional derivatives. Continuing in this way, we generally obtain n-fold directional derivatives.

The following proposition states that if a function f is n times differentiable, then also all n-fold directional derivatives exist and the nth derivative and the n-fold directional derivatives have a simple relationship to each other.

Proposition 19.3.1. Suppose a function $f : \Omega \to W$ is n times differentiable in a point $a \in \Omega$. Then all directional n-fold derivatives exist in a and for all $v_1, \dots, v_n \in V$,

$$(D^n f)_a(v_n, v_{n-1}, \cdots, v_2, v_1) = (D_{v_n}(D_{v_{n-1}} \cdots (D_{v_2}(D_{v_1} f)) \cdots))_a.$$

For functions defined on a subset of $\mathbb{R}^d$, we can relate nth order derivatives to partial derivatives of partial derivatives of partial derivatives (n times).

In particular if a function $f : \Omega \to W$, with $\Omega \subset \mathbb{R}^d$, is n times differentiable, then all partial derivatives up to order n exist. For instance, if f is 3 times differentiable in a point $a \in \Omega$, then

$$(D^3 f)_a(e_1, e_5, e_2) = \left(\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_5} \frac{\partial}{\partial x_2} f \right)(a).$$

19.4 A criterion for higher differentiability

The following theorem is often useful in practice to verify that a function is n times differentiable.

Theorem 19.4.1. Let $f : \Omega \to W$ where Ω is an open subset of $\mathbb{R}^d$. If all partial derivatives of f of order less than or equal to n exist, and if all partial derivatives of order n are continuous on Ω, then f is n times differentiable on Ω.

19.5 Symmetry of second order derivatives

Lemma 19.5.1. Let $f : \Omega \to W$ be a function defined on an open domain $\Omega \subset V$. Let $a \in \Omega$ and assume that f is twice differentiable in a. Then for all $u, v \in V$,

$$(D^2 f)_a(u, v) = (D^2 f)_a(v, u).$$

Proof. We will show that for all $u, v \in V$, the limit

$$\lim_{t \to 0} \frac{1}{t^2} \left(f(a + tu + tv) - f(a + tu) - f(a + tv) + f(a) \right) \tag{19.5.1}$$

exists and is equal to

$$(D^2 f)_a(u, v)$$

Note that the expression in (19.5.1) remains the same when u and v are interchanged, so that the limit is also equal to

$$(D^2 f)_a(v, u).$$

From there, we conclude that $(D^2 f)_a(u, v) = (D^2 f)_a(v, u)$.

Consider now for s, t real numbers (that are small enough so that $a + su + tv \in \Omega$) the expression

$$\begin{aligned} &\| f(a + su + tv) - f(a + su) - f(a + tv) + f(a) - st(D^2 f)_a(u, v) \|_W \\ &= \| f(a + su + tv) - st(D^2 f)_a(u, v) - f(a + su) - f(a + tv) + f(a) \|_W \end{aligned}$$

By the Mean-Value Inequality, applied to the function

$$g(t) := f(a+su+tv) - st(D^2 f)_a(u,v) - f(a+tv)$$

we find that

$$\begin{aligned}
&\|f(a+su+tv) - st(D^2 f)_a(u,v) - f(a+su) - f(a+tv) + f(a)\|_W \\
&= \|g(t) - g(0)\|_W \\
&\leq \sup_{\tau \in (-|t|,|t|)} \|(Df)_{a+su+\tau v}(v) - s(D^2 f)_a(u,v) - (Df)_{a+\tau v}(v)\|_W |t|
\end{aligned} \tag{19.5.2}$$

We now use the differentiability of the function $(Df) : \Omega \to \mathsf{Lin}(V,W)$ in the point a so that

$$(Df)_{a+su+\tau v} = (Df)_a + (D(Df))_a(su+\tau v) + \mathsf{Err}_a^{Df}(a+su+\tau v)$$

and

$$(Df)_{a+\tau v} = (Df)_a + (D(Df))_a(\tau v) + \mathsf{Err}_a^{Df}(a+\tau v)$$

Note that the left-hand side and right hand side of these equations are linear maps. We apply these linear maps to the vector $v \in V$ and find

$$\begin{aligned}
(Df)_{a+su+\tau v}(v) &:= (Df)_a(v) + (D(Df))_a(su+\tau v)(v) \\
&\qquad + \mathsf{Err}_a^{Df}(a+su+\tau v)(v) \\
&= (Df_a)(v) + s(D^2 f)_a(u,v) + \tau(D^2 f)_a(v,v) \\
&\qquad + \mathsf{Err}_a^{Df}(a+su+\tau v)(v)
\end{aligned}$$

and

$$\begin{aligned}
(Df)_{a+\tau v}(v) &:= (Df)_a(v) + (D(Df))_a(\tau v)(v) + \mathsf{Err}_a^{Df}(a+\tau v)(v) \\
&:= (Df)_a(v) + \tau(D^2 f)_a(v,v) + \mathsf{Err}_a^{Df}(a+\tau v)(v).
\end{aligned}$$

We now continue from the final line in the computation (19.5.2)

$$
\begin{aligned}
&\sup_{\tau \in (-|t|,|t|)} \|(Df)_{a+su+\tau v}(v) - s(D^2 f)_a(u,v) - (Df)_{a+\tau v}(v)\|_W |t| \\
&= \sup_{\tau \in (-|t|,|t|)} \|\mathsf{Err}_a^{Df}(a+su+\tau v)(v) - \mathsf{Err}_a^{Df}(a+\tau v)(v)\|_W |t| \\
&\leq \sup_{\tau \in (-|t|,|t|)} \left(\left\|\mathsf{Err}_a^{Df}(a+su+\tau v)\right\|_{V\to W} + \left\|\mathsf{Err}_a^{Df}(a+\tau v)\right\|_{V\to W} \right) |t| \|v\|_V.
\end{aligned}
$$

It follows that for $t \neq 0$,

$$
\begin{aligned}
&\left\| \frac{1}{t^2} \left(f(a+su+tv) - f(a+su) - f(a+tv) + f(a) \right) - (D^2 f)_a(u,v) \right\|_W \\
&\leq \frac{1}{|t|} \sup_{\tau \in (-|t|,|t|)} \left(\left\|\mathsf{Err}_a^{Df}(a+tu+\tau v)\right\|_{V\to W} + \left\|\mathsf{Err}_a^{Df}(a+\tau v)\right\|_{V\to W} \right) \|v\|_V
\end{aligned}
$$

and observe that the limit of the right-hand side as $t \to 0$ is indeed 0. □

19.6 Symmetry of higher-order derivatives

The previous section also has an immediate consequence for the symmetry of higher-order derivatives. In fact, if f is n times differentiable in a point $a \in \Omega$, then for every permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ we have

$$
(D^n f)_a(v_1, v_2, \cdots, v_n) = (D^n f)_a(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(n)})
$$

With the following definition, we can just express this by saying that if f is n times differentiable in a point a, that then $D^n f$ can be viewed as an element in the space of n-linear, symmetric maps $\mathsf{Sym}_n(V, W)$, where the space $\mathsf{Sym}_n(V, W)$ is defined as follows.

Definition 19.6.1. We denote by $\mathsf{Sym}_n(V, W)$ the collection of symmetric, n-linear maps from $V^{\times n}$ to W. That is, a map $\mathcal{S} : V^{\times n} \to W$ is in $\mathsf{Sym}_n(V, W)$ if and only if it is linear in every argument and if for every permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ it holds that

$$\mathcal{S}(v_1, \cdots, v_n) = \mathcal{S}(v_{\sigma(1)}, \cdots, v_{\sigma(n)}).$$

In conclusion, we may as well write

$$D^n f : \Omega \to \mathsf{Sym}_n(V, W)$$

19.7 Exercises

The following exercises are mostly meant as practice to get familiar with the concepts and theorems in this chapter.

Exercise 19.7.1. Consider the function $f : \Omega \to \mathbb{R}$ where $\Omega = \mathbb{R}^2$ given by

$$f((x_1, x_2)) = \exp((x_1)^2 - (x_2)^3)$$

Show that f is twice differentiable on $\mathbb{R}^2$ by going through the following steps:

a. Show that the partial derivative functions of f, namely

$$\frac{\partial f}{\partial x_1} : \Omega \to \mathbb{R} \qquad \text{and} \qquad \frac{\partial f}{\partial x_2} : \Omega \to \mathbb{R}$$

exist and compute them.

b. Show that the second order partial derivative functions

$$\begin{array}{ll} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} : \Omega \to \mathbb{R} & \qquad \dfrac{\partial^2 f}{\partial x_2 \partial x_1} : \Omega \to \mathbb{R} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} : \Omega \to \mathbb{R} & \qquad \dfrac{\partial^2 f}{\partial x_2 \partial x_2} : \Omega \to \mathbb{R} \end{array}$$

exist and compute them.

c. Now show that the second order partial derivatives are continuous and conclude, by quoting the right theorem, that f is twice differentiable.

Exercise 19.7.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) = (x_1)^5 (x_2)^8$$

a. For arbitrary $u \in \mathbb{R}^2$, give the function

$$D_u f : \mathbb{R}^2 \to \mathbb{R}$$

b. For arbitrary $v \in \mathbb{R}^2$, give the function

$$(D_v(D_u f)) : \mathbb{R}^2 \to \mathbb{R}$$

c. Define the vectors $u := (1, 3)$ and $v := (7, 2)$. Let $a = (1, 1) \in \mathbb{R}^2$. Give

$$(D^2 f)_a(v, u)$$

d. For the same choice of u, v and a, also give

$$(D^2 f)_a(u, v)$$

Exercise 19.7.3. About a certain function $f : \mathbb{R}^5 \to \mathbb{R}^2$ the following is known in a point $a \in \mathbb{R}^5$. The function is 4 times differentiable in a, and

$$\begin{aligned}
(D^4 f)_a(e_2, e_3, e_3, e_5) &= (5, 0) \\
(D^4 f)_a(e_2, e_3, e_5, e_5) &= (2, 3) \\
\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_3 \partial x_3}(a) &= (0, 1) \\
\frac{\partial^4 f}{\partial x_5 \partial x_3 \partial x_5 \partial x_3}(a) &= (1, 2)
\end{aligned}$$

Give

$$(D^4 f)_a(e_3 - 2e_2, 6e_5, e_3 + e_5, e_3).$$

Chapter 20

Polynomials and approximation by polynomials

20.1 Homogeneous polynomials

A homogeneous polynomial in d variables of degree k is a polynomial of which every term (i.e. every monomial) has degree precisely k. For instance, the function

$$(x_1, x_2) \mapsto 3x_1^4 x_2^1 + 2x_1^3 x_2^2 + x_2^5$$

is a homogeneous polynomial in two variables of degree 5.

Definition 20.1.1 (multi-index)**.** A d-dimensional multi-index α of order $k \in \mathbb{N}$ is a map $\{1, \dots, d\} \to \mathbb{N}$ such that

$$\alpha_1 + \cdots + \alpha_d = k$$

We write $|\alpha|$ for the order of a multi-index α.

If α is a multi-index, we use the notation

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$$

Similarly, for a function $f : \Omega \to W$ where Ω is a subset of $\mathbb{R}^d$, we will use the notation

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} f$$

We also define

$$\alpha! := \alpha_1!\alpha_2! \cdots \alpha_d!$$

Note that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} x^\alpha = \alpha!$$

This is maybe easier to appreciate in an example:

$$\frac{\partial^{14}}{\partial x_1^3 \partial x_2^7 \partial x_3^4} \left((x_1)^3 (x_2)^7 (x_3)^4\right) = 3!7!4!$$

This observation brings us to the following proposition.

Proposition 20.1.2. Every homogeneous polynomial $f : \mathbb{R}^d \to \mathbb{R}$ of degree n can be written as

$$\sum_{|\alpha|=n} \frac{1}{\alpha!} s_\alpha x^\alpha$$

for some coefficients $s_\alpha \in \mathbb{R}$. Moreover, the coefficients s_α are precisely determined by

$$s_\alpha = \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0).$$

Lemma 20.1.3. Given a basis $v_1, \dots, v_d$ of the vector space V, there is a one-to-one correspondence between homogeneous polynomials of degree n and $\mathsf{Sym}_n(V, \mathbb{R})$. More precisely there is an invertible linear map $\mathcal{F}$ from $\mathsf{Sym}_n(V, \mathbb{R})$ to the vector space of homogeneous polyomials in d variables of degree n. With the linear map $\iota : \mathbb{R}^d \to V$ defined as

$$\iota(x) = x_1 v_1 + \cdots + x_d v_d,$$

the n-linear symmetric map $\mathcal{S} \in \mathsf{Sym}_n(V, \mathbb{R})$ gets mapped to the homogeneous polynomial $F := \mathcal{F}(\mathcal{S}) : \mathbb{R}^d \to \mathbb{R}$ given by

$$\mathcal{F}(\mathcal{S})(x) = \frac{1}{n!}\mathcal{S}(\iota(x), \cdots, \iota(x))$$

Then the following equality holds for all $x \in \mathbb{R}^d$

$$\mathcal{F}(\mathcal{S})(x) = \frac{1}{n!}\mathcal{S}(\iota(x), \cdots, \iota(x)) = \sum_{|\alpha|=n} \frac{1}{\alpha!} x^\alpha S^{(\alpha)}$$

where

$$S^{(\alpha)} = S(v_{i_1}, v_{i_2}, \dots, v_{i_n})$$

where $i_1, \cdots, i_n \in \{1, \dots, d\}$ are such that v_1 appears α_1 times, v_2 appears α_2 times etc. In particular,

$$S^{(\alpha)} = \frac{\partial^{|\alpha|} F}{\partial x^\alpha}(0).$$

In particular, an element of $\mathsf{Sym}_n(V, \mathbb{R})$ is completely determined by the values on the diagonal, i.e. if $\mathcal{S}, \mathcal{T} \in \mathsf{Sym}_n(V, \mathbb{R})$, then $\mathcal{S} = \mathcal{T}$ if and only if for all $v \in V$,

$$\mathcal{S}(v, \cdots, v) = \mathcal{T}(v, \cdots, v).$$

Proof. Let $v_1, \dots, v_d$ be a basis of V. By n-linearity of $\mathcal{S}$, the map $\mathcal{S}$ is completely determined by how $\mathcal{S}$ evaluates on basis vectors, i.e. by the values of

$$\mathcal{S}\Big(v_{i_1}, \dots, v_{i_n}\Big)$$

where $i_1, \dots, i_n$ are indices in $\{1, \dots, d\}$. Moreover, because $\mathcal{S}$ is symmetric, it does not matter in which order the basis vectors appear. Therefore we may introduce for α a multi-index of order n the notation

$$\mathcal{S}^{(\alpha)} := \mathcal{S}\big(v_{i_1}, \cdots, v_{i_n}\big)$$

where $i_1, \dots i_n$ are such that v_1 appears α_1 times, v_2 appears α_2 times, etc..

If we now compute
$$\mathcal{S}(u, \cdots, u)$$
where $u := \sum_{j=1}^{d} x_j v_j$, we find

$$\begin{aligned}
\frac{1}{n!}\mathcal{S}(u, \cdots, u) &= \frac{1}{n!}\mathcal{S}\left(\sum_{j_1=1}^{d} x_{j_1} v_{j_1}, \cdots, \sum_{j_n=1}^{d} x_{j_n} v_{j_n}\right) \\
&= \frac{1}{n!}\sum_{j_1=1}^{d} \cdots \sum_{j_n=1}^{d} x_{j_1} \cdots x_{j_n} \mathcal{S}\left(v_{j_1}, \cdots, v_{j_n}\right) \\
&= \frac{1}{n!}\sum_{|\alpha|=n} \binom{n}{\alpha} x^\alpha \mathcal{S}^{(\alpha)} \\
&= \sum_{|\alpha|=n} \frac{1}{\alpha!} x^\alpha \mathcal{S}^{(\alpha)}.
\end{aligned}$$

Therefore the coefficients $\mathcal{S}^{(\alpha)}$ can be read off by inspecting the polynomial $F : \mathbb{R}^d \to \mathbb{R}$ given by

$$F(x) := \frac{1}{n!}\mathcal{S}(\iota(x), \cdots, \iota(x)) = \sum_{|\alpha|=n} \frac{1}{\alpha!} x^\alpha \mathcal{S}^{(\alpha)},$$

or alternatively,
$$\mathcal{S}^{(\alpha)} = \frac{1}{n!}\frac{\partial^{|\alpha|}}{\partial x^\alpha} F(0)$$

□

20.2 Taylor's theorem

When a function f is n times differentiable in a point, it can be approximated well by its Taylor expansion.

Definition 20.2.1 (Taylor expansion)**.** Let $f : \Omega \to W$ be n times differentiable in a point $a \in \Omega$. Then the function $T_{a,n} : V \to W$ given by

$$T_{a,n}(x) := f(a) + \sum_{k=1}^{n} \frac{1}{k!}(D^k f)_a(x-a, \cdots, x-a)$$

is called the Taylor expansion of f around a.

Taylor's theorem says that the Taylor expansion provides a good approximation.

Theorem 20.2.2. Let $\Omega \subset V$ be open, let $a \in \Omega$ and suppose $f : \Omega \to W$ is n times differentiable in a point $a \in \Omega$. Then there exists a function $\mathsf{Err}_{a,n} : \Omega \to W$ such that

$$f(v) = f(a) + \sum_{k=1}^{n} \frac{1}{k!}(D^k f)_a(v-a, \cdots, v-a) + \mathsf{Err}_{a,n}(v).$$

and such that

$$\lim_{v \to a} \frac{\|\mathsf{Err}_{a,n}(v)\|_W}{\|v-a\|_V^n} = 0.$$

Before we prove Taylor's theorem, let's record the following proposition. It allows us to determine the derivatives of the Taylor expansion of f around a, and in particular it allows us to conclude that the derivatives up to order n of the Taylor expansion $T_{a,n}$ of f of order n in the point a are exactly the same as those of f in a, while the derivatives of order higher than n all vanish.

Proposition 20.2.3. Let $a \in V$, let $k \in \mathbb{N} \setminus \{0\}$, let $\mathcal{S} \in \mathsf{Sym}_k(V, W)$ and consider the function $f : V \to W$ defined by

$$f(x) := \frac{1}{k!}\mathcal{S}(x-a, \ldots, x-a)$$

Then

i. for all $b \in V$,

$$(D^k f)_b = \mathcal{S}$$

ii. for all $b \in V$ and all $j > k$,

$$(D^j f)_b = 0,$$

iii. for all $b \in V$ and all $j < k$ and all $u_1, \dots, u_j \in V$,

$$(D^j f)_b(u_1, \cdots, u_j) = \frac{1}{(k-j)!}\mathcal{S}(u_1, \cdots, u_j, b-a, \cdots, b-a).$$

We can prove the above proposition for instance with help of the correspondence between homogeneous polynomials and symmetric multilinear forms of Lemma 20.1.3. The first approach is illustrated by Exercise 20.4.4, while the latter approach is illustrated by Exercise 20.4.2.

We will now give a sketch of the proof of Taylor's theorem.

Proof of Theorem 20.2.2. We give a sketch of the proof. First of all, by using the previous proposition we may without loss of generality assume that $f(a) = 0$ and that all derivatives of f up to and including order n vanish (because otherwise we just consider the function $g := f - T_{a,n}$). By repeatedly applying the Mean-Value Inequality we then find that

$$\begin{aligned}
\|f(v) - f(a)\|_W &\leq \sup_{\tau \in (0,1)} \|(Df)_{(1-\tau)a+\tau v}\|_{V \to W} \|v - a\|_V \\
&\leq \sup_{\tau \in (0,1)} \|(D^2 f)_{(1-\tau)a+\tau v}\|_{\mathsf{Sym}_2(V,W)} \|v - a\|_V^2 \\
&\leq \cdots \\
&\leq \sup_{\tau \in (0,1)} \|(D^{n-1} f)_{(1-\tau)a+\tau v}\|_{\mathsf{Sym}_{n-1}(V,W)} \|v - a\|_V^{n-1}
\end{aligned}$$

and therefore

$$\|f(v) - f(a)\|_W \leq \sup_{\tau \in (0,1)} \|\mathsf{Err}_a^{D^{n-1}f}((1-\tau)a + \tau v)\|_{\mathsf{Sym}_{n-1}(V,W)} \|v - a\|_V^{n-1}.$$

□

The following proposition is in some sense a uniqueness statement about the Taylor expansion.

Proposition 20.2.4. Suppose $f : \Omega \to W$ and $g : \Omega \to W$ are both n times differentiable in $a \in \Omega$ and

$$\lim_{x \to a} \frac{\|f(x) - g(x)\|_W}{\|x - a\|_V^n} = 0.$$

Then for all $k = 0, \dots, n$,

$$(D^k f)_a = (D^k g)_a.$$

We would now like to give a version of Taylor's theorem in coordinates. For that, we first need the following proposition.

Proposition 20.2.5. Let $\Omega \subset \mathbb{R}^d$ be open, let $a \in \Omega$ and suppose $f : \Omega \to \mathbb{R}^m$ is n times differentiable in $a \in \Omega$. then for all $k = 1, \dots, n$ and all $x \in \mathbb{R}^d$,

$$\frac{1}{k!}(D^k f)_a(x, \cdots, x) = \sum_{|\alpha|=k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) x^\alpha.$$

The previous proposition also implies that when q is a homogeneous polynomial of degree k, that then

$$q(x) = \frac{1}{k!}(D^k q)_0(x, \cdots, x).$$

Example 20.2.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$, let $a \in \mathbb{R}^2$ and suppose we want to find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for all $u \in \mathbb{R}^2$,

$$(D^3 f)_a(u, u, u) = 2(u_1)^2(u_2).$$

Note that it is necessary for the right-hand side to be a homogenous polynomial of degree three, otherwise such an f cannot be found. We call this homogeneous polynomial $q : \mathbb{R}^2 \to \mathbb{R}$.

By an earlier proposition, we know that

$$q(u) = \sum_{|\alpha|=3} \frac{1}{\alpha!} s_\alpha u^\alpha$$

where

$$s_\alpha = \frac{\partial^3 q}{\partial x^\alpha}(0).$$

We know by the previous proposition that if such a function f exist that then for all $u \in \mathbb{R}^2$,

$$\sum_{|\alpha|=3} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a) u^\alpha = \frac{1}{3!}(D^3 f)_a(u, u, u) = \frac{2}{3!}(u_1)^2(u_2).$$

If we compare the left-hand side and right-hand side this may suggest us to find a function such that

$$\frac{1}{3!} \frac{\partial^3 f}{(\partial x_1)^2 \partial x_2}(a) = \frac{2}{3!}.$$

and all other partial derivatives in a vanish. Now the polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) \mapsto \frac{1}{3}(x_1 - a_1)^2(x_2 - a_2)$$

is such a polynomial.

Theorem 20.2.7 (Taylor's theorem in coordinates)**.** Let $\Omega \subset \mathbb{R}^d$ be open, let $a \in \Omega$ and suppose $f : \Omega \to \mathbb{R}^m$ is n times differentiable in the point $a \in \Omega$. Then, defining the function $\mathsf{Err}_{a,n} : \Omega \to \mathbb{R}^m$ by

$$\mathsf{Err}_{a,n}(x) := f(x) - \left(f(a) + \sum_{1 \leq |\alpha| \leq n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a)(x-a)^\alpha \right)$$

we have that
$$\lim_{x \to a} \frac{\|\mathsf{Err}_{a,n}(x)\|_2}{\|x-a\|_2^n} = 0.$$

Definition 20.2.8. Let $\Omega \subset \mathbb{R}^d$ be open, let $a \in \Omega$ and suppose $f : \Omega \to \mathbb{R}$ is n times differentiable in the point $a \in \Omega$. Then the polynomial

$$f(a) + \sum_{1 \le |\alpha| \le n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(a)(x-a)^\alpha$$

is called the nth order Taylor polynomial of f around the point a.

In the context of Proposition 20.2.4, if g is an at most nth degree polynomial such that
$$\lim_{x \to a} \frac{|f(x) - g(x)|}{\|f(x) - g(x)\|_2^n} = 0$$
then it is necessarily the Taylor polynomial of f.

The approximation formula simplifies a bit if f is just a function of one variable.

Corollary 20.2.9 (Taylor's theorem for functions of one variable)**.** Let $\Omega \subset \mathbb{R}$ be open and let $f : \Omega \to \mathbb{R}$ be a function such that f is n times differentiable in a point $a \in \Omega$. Then there exists a function $\mathsf{Err}_{a,n} : \Omega \to \mathbb{R}$ such that
$$f(x) = f(a) + \sum_{k=1}^{n} \frac{1}{k!} f^{(k)}(a) \cdot (x-a)^k + \mathsf{Err}_{a,n}(x)$$
and such that
$$\lim_{x \to a} \frac{|\mathsf{Err}_{a,n}(x)|}{|x-a|^n} = 0.$$

Here $f^{(1)}(a)$ is notation for $f'(a)$, $f^{(2)}(a)$ is notation for $f''(a)$ etc..

Finally, we note that we can give an explicit expression for the error term if f maps to $\mathbb{R}$ and f has higher differentiability than the order of the Taylor polynomial.

Theorem 20.2.10 (Taylor's theorem with Lagrange remainder)**.** Let $f : \Omega \to \mathbb{R}$ be $(n+1)$ times differentiable on Ω. Let $a \in \Omega$. Then there exists a $\theta \in (0,1)$ such that

$$\begin{aligned} f(x) = f(a) + \sum_{1 \leq |\alpha| \leq n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha} (x-a)^\alpha \\ + \frac{1}{(n+1)!} (D^{n+1} f)_{a+\theta(x-a)} (x-a, \dots, x-a). \end{aligned}$$

20.3 Taylor approximations of standard functions

When we apply the theorems of the previous sections to the standard functions, we get the following approximations.

Corollary 20.3.1. For every $n \in \mathbb{N}$, it holds that

$$\begin{aligned} \exp(x) &= \sum_{k=0}^{n} \frac{x^k}{k!} + O(|x|^{n+1}) \\ \sin(x) &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + O(|x|^{2n+3}) \\ \cos(x) &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + O(|x|^{2n+2}) \\ \ln(1+x) &= \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} x^k + O(|x|^{n+1}) \end{aligned}$$

The notation

$$f(x) = g(x) + O(|x|^N)$$

should be read as that there exists a $C \geq 0$ and a $\delta > 0$ such that for all $x \in (-\delta, \delta)$,

$$|f(x) - g(x)| \leq C|x|^N.$$

20.4 Exercises

Exercise 20.4.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) := \sin\left(\frac{\pi}{2}\left((x_1)^2 + (x_2)\right)\right).$$

a. Show that f is differentiable on $\mathbb{R}^2$ and compute the partial derivative functions

$$\frac{\partial f}{\partial x_1} : \mathbb{R}^2 \to \mathbb{R} \quad \text{and} \quad \frac{\partial f}{\partial x_2} : \mathbb{R}^2 \to \mathbb{R}.$$

b. Show that all second order partial derivatives of f exist, compute them, and show that they are continuous.

c. Give the second-order Taylor polynomial $T_2 : \mathbb{R}^2 \to \mathbb{R}$ of f around the point $(1, 2)$.

d. Show that

$$\lim_{(x_1,x_2)\to(1,2)} \frac{|T_2(x) - f(x)|}{\|(x_1, x_2) - (1, 2)\|_2^2} = 0.$$

Exercise 20.4.2. The aim of this exercise is essentially to provide a proof of Proposition 20.2.3 in case $k = 2$. You can therefore not use this proposition in this exercise.

Let $\mathcal{S}$ be a symmetric, 2-linear map, from $V^{\times 2}$ to W. Now consider the map $f : V \to W$ given by

$$f(v) := \mathcal{S}(v, v).$$

a. Show that f is differentiable on V, and that for all $a \in V$, and all $u \in V$,

$$(Df)_a(u) = 2\mathcal{S}(u, a)$$

b. Show that f is twice differentiable on V, and that for all $a \in V$, and all $u \in V$,

$$(D^2 f)_a(u, u) = 2\mathcal{S}(u, u).$$

Note: You can use that there exists a constant $K \geq 0$ such that for all $u, v \in V$ it holds that

$$\|\mathcal{S}(u,v)\|_W \leq K\|u\|_V\|v\|_V$$

Some extra information: Such a constant exists because we assume that V and W are finite-dimensional, and the smallest such constant is actually the norm $\|\mathcal{S}\|_{\mathsf{MLin}(V^{\times 2},W)}$ of $\mathcal{S}$, that was introduced in (19.2.1).

Exercise 20.4.3. Determine whether the following limit exists, and if so, determine its value:

$$\lim_{(x_1,x_2)\to(0,0)} \frac{\exp((x_1)^2+(x_2)^2)-1}{\sin((x_1)^2+(x_2)^2)}$$

Exercise 20.4.4. Let α and β be two d-dimensional multi-indices.

a. Suppose that for all $i \in \{1,\dots,d\}$, it holds that $\alpha_i \leq \beta_i$. Give an expression for

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} x^\beta$$

and prove that your expression is correct. **Hint:** Make an induction argument on the order of α.

b. Suppose that there is an $i \in \{1,\dots,d\}$ such that $\alpha_i > \beta_i$. Show that

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} x^\beta = 0.$$

Chapter 21

Banach fixed point theorem

21.1 The Banach fixed point theorem

Definition 21.1.1 (Contraction)**.** Let (X, dist_X) and (Y, dist_Y) be metric spaces. Let $D \subset X$ be a subset of X and let $f : D \to Y$ be a function. We say that f is a contraction if there exists a $\kappa \in [0, 1)$ such that for all $x, z \in D$, it holds that

$$\mathsf{dist}_Y(f(x), f(z)) \leq \kappa \mathsf{dist}_X(x, z).$$

If f satisfies this inequality for all x, z, and a constant $\kappa \in [0, 1)$, then we will also sometimes say that f is a κ-contraction.

We first formulate the theorem for arbitrary metric spaces. Afterwards, we give a version for $\mathbb{R}^d$.

Theorem 21.1.2 (Banach fixed point theorem, metric space version)**.** Let (X, dist_X) be a metric space. Let $D \subset X$ be a non-empty and complete subset of X. Let $f : D \to D$ be a function, let $\kappa \in [0, 1)$ and assume that for all $x, z \in D$, it holds that

$$\mathsf{dist}_X(f(x), f(z)) \leq \kappa \, \mathsf{dist}_X(x, z).$$

Then there exists a unique point $p \in D$ such that $f(p) = p$.

Moreover, for all $q \in D$, if we define the sequence $(x^{(n)})_n$ by

$$\begin{aligned} x^{(0)} &:= q \\ x^{(n+1)} &:= f(x^{(n)}) \qquad \text{for } n \in \mathbb{N}. \end{aligned} \tag{21.1.1}$$

Then the sequence $(x^{(n)})_n$ converges to p, and for all $n \in \mathbb{N}$,

$$\mathsf{dist}_X(x^{(n)}, p) \leq \frac{\kappa^n}{1-\kappa} \mathsf{dist}_X(x^{(1)}, x^{(0)}). \tag{21.1.2}$$

Before given the proof of the theorem, we first formulate a version in case X is just $\mathbb{R}^d$. Because every closed subset of $\mathbb{R}^d$ is complete, we get the following theorem.

Theorem 21.1.3 (Banach fixed point theorem, $\mathbb{R}^d$ version)**.** Let $D \subset \mathbb{R}^d$ be closed and non-empty. Let $f : D \to D$ be a function, let $\kappa \in [0, 1)$ and assume that for all $x, z \in D$, it holds that

$$\|f(x) - f(z)\|_2 \leq \kappa \, \|x - z\|_2.$$

Then there exists a unique point $p \in D$ such that $f(p) = p$.

Moreover, for all $q \in D$, if we define the sequence $(x^{(n)})_n$ by

$$\begin{aligned} x^{(0)} &:= q \\ x^{(n+1)} &:= f(x^{(n)}) \qquad \text{for } n \in \mathbb{N}. \end{aligned} \tag{21.1.3}$$

Then the sequence $(x^{(n)})_n$ converges to p, and for all $n \in \mathbb{N}$,

$$\|x^{(n)} - p\|_2 \leq \frac{\kappa^n}{1-\kappa} \|x^{(1)} - x^{(0)}\|_2. \tag{21.1.4}$$

Proof of the metric space version of the Banach Fixed Point theorem. We will first show that f has at most one fixed point. To this end, let $z \in X$ be a fixed point of f, i.e. $f(z) = z$, and let $p \in X$ be another fixed point of

f. We need to show that $z = p$. Because z and p are fixed points, we find that

$$\mathsf{dist}_X(p, z) = \mathsf{dist}_X(f(p), f(z)) \leq \kappa \mathsf{dist}_X(p, z)$$

so that

$$(1 - \kappa)\mathsf{dist}_X(p, z) = 0$$

and therefore $\mathsf{dist}_X(p, z) = 0$, from which it indeed follows that $p = z$.

We will now show that for all $q \in D$, the sequence $(x^{(n)})_n$ defined inductively by (21.1.1) converges to a fixed point $p \in D$. From this, of course, it follows immediately that such a fixed point exists.

Let $q \in D$.

Now define the sequence $(x^{(n)})_n$ inductively according to (21.1.1), i.e.

$$\begin{aligned} x^{(0)} &:= q \\ x^{(n+1)} &:= f(x^{(n)}) \qquad \text{for } n \in \mathbb{N}. \end{aligned}$$

Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathsf{dist}_X(x^{(n+2)}, x^{(n+1)}) &= \mathsf{dist}_X(f(x^{(n+1)}), f(x^{(n)})) \\ &\leq \kappa\, \mathsf{dist}_X(x^{(n+1)}, x^{(n)}) \end{aligned} \tag{21.1.5}$$

It follows by induction that for all $n \in \mathbb{N}$ it holds that

$$\mathsf{dist}_X(x^{(n+1)}, x^{(n)}) \leq \kappa^n \mathsf{dist}_X(x^{(1)}, x^{(0)}) \tag{21.1.6}$$

By the triangle inequality, for all $m, n \in \mathbb{N}$ with $m \leq n$,

$$\begin{aligned}\mathsf{dist}_X(x^{(n)}, x^{(m)}) &\leq \mathsf{dist}_X(x^{(n)}, x^{(n-1)}) + \cdots + \mathsf{dist}_X(x^{(m+1)}, x^{(m)})\\ &= \sum_{\ell=m}^{n-1} \mathsf{dist}_X(x^{(\ell+1)}, x^{(\ell)})\\ &\leq \sum_{\ell=m}^{n-1} \kappa^\ell \mathsf{dist}_X(x^{(1)}, x^{(0)})\\ &= \frac{\kappa^m - \kappa^n}{1-\kappa} \mathsf{dist}_X(x^{(1)}, x^{(0)})\\ &\leq \frac{\kappa^m}{1-\kappa} \mathsf{dist}_X(x^{(1)}, x^{(0)}).\end{aligned}$$

Therefore, the sequence $(x^{(n)})$ is a Cauchy sequence. Since D is complete, it follows that $(x^{(n)})$ is convergent, to some $p \in D$ say.

We will now show that $f(p) = p$. For this, we first write inequality (21.1.6) as

$$\mathsf{dist}_X(f(x^{(n)}), x^{(n)}) \leq \kappa^n \mathsf{dist}_X(x^{(1)}, x^{(0)}).$$

Since the sequence $(x^{(n)})$ converges to p, it follows for instance from Proposition 5.6.3 that also the sequence $(f(x^{(n)}))$ converges to p. On the other hand, f is Lipschitz continuous and therefore continuous (see Exercise 13.11.5) so that by the sequence characterization of continuity we know that

$$p = \lim_{n\to\infty} f(x^{(n)}) = f(p).$$

Note that for all $m, n \in \mathbb{N}$, with $m > n$,

$$\begin{aligned}\mathsf{dist}_X(p, x^{(n)}) &\leq \mathsf{dist}_X(p, x^{(m)}) + \mathsf{dist}_X(x^{(m)}, x^{(n)})\\ &\leq \mathsf{dist}_X(p, x^{(m)}) + \frac{\kappa^n}{1-\kappa} \mathsf{dist}_X(x^{(1)}, x^{(0)}).\end{aligned}$$

By taking the limit $m \to \infty$, we find that

$$\mathsf{dist}_X(p, x^{(n)}) \leq \frac{\kappa^n}{1-\kappa} \mathsf{dist}_X(x^{(1)}, x^{(0)}).$$

□

21.2 An example

Example 21.2.1. Consider the function

$$F : [0,1]^2 \to \mathbb{R}^2$$

given by

$$F((x_1, x_2)) := \left(\frac{1}{6}(x_2)^2 + \frac{1}{3}x_1, \frac{1}{\pi} \arctan(x_1) + \frac{1}{2} \right)$$

We would like to show that there is a unique fixed point q of the function F in the set $[0,1]^2$ (i.e. $F(q) = q$). To do this, we would like to apply Banach's fixed point theorem, so we need to check the conditions of the theorem.

First note that $[0,1]^2$ is closed.

We will now show that the range of F is contained in $[0,1]^2 \subset \mathbb{R}^2$.

Let $(x_1, x_2) \in [0,1]^2$. Then

$$0 \leq F_1((x_1, x_2)) = \frac{1}{6}(x_2)^2 + \frac{1}{3}x_1 \leq \frac{1}{6} + \frac{1}{3} < 1$$

and because for all $z \in \mathbb{R}$, $-\pi/2 < \arctan(z) < \pi/2$

$$F_2((x_1, x_2)) = \frac{1}{\pi} \arctan(x_1) + \frac{1}{2} < \frac{1}{\pi}\frac{\pi}{2} + \frac{1}{2} = 1$$

and

$$F_2((x_1, x_2)) = \frac{1}{\pi} \arctan(x_1) + \frac{1}{2} > -\frac{1}{\pi}\frac{\pi}{2} + \frac{1}{2} = 0.$$

So indeed F maps into $[0,1]^2$.

We will now show that F is a contraction. In fact we will show that for all $x, y \in [0,1]^2$,

$$\|F(x) - F(y)\|_2 \leq \kappa \|x - y\|_2$$

with $\kappa = \frac{1}{3}\sqrt{3}$, so indeed κ is strictly smaller than 1.

In proving such an inequality, the Mean-Value Inequality will often play an important role.

$$\begin{aligned}
|F_1((x_1, x_2)) - F_1((y_1, y_2))| &= \left|\frac{1}{6}(x_2)^2 + \frac{1}{3}x_1 - \frac{1}{6}(y_2)^2 - \frac{1}{3}y_1\right| \\
&\leq \left|\frac{1}{6}\left((x_2)^2 - (y_2)^2\right)\right| + \left|\frac{1}{3}(x_1 - y_1)\right| \\
&= \frac{1}{6}|x_2 - y_2||x_2 + y_2| + \frac{1}{3}|x_1 - y_1| \\
&\leq \frac{1}{6}|x_2 - y_2|(|x_2| + |y_2|) + \frac{1}{3}|x_1 - y_1| \\
&\leq \frac{1}{3}|x_2 - y_2| + \frac{1}{3}|x_1 - y_1|
\end{aligned}$$

We now use the inequality that for all $a, b \in \mathbb{R}$,

$$(a + b)^2 \leq 2a^2 + 2b^2,$$

(which follows from the Cauchy-Schwarz inequality).

It follows that

$$|F_1((x_1, x_2)) - F_1((y_1, y_2))|^2 \leq \frac{2}{9}((x_1 - y_1)^2 + (x_2 - y_2)^2)$$

$$|F_2((x_1, x_2)) - F_2((y_1, y_2))| = \left|\frac{1}{\pi}\arctan(x_1) - \frac{1}{\pi}\arctan(y_1)\right|$$

To estimate this, we can use the mean-value inequality. Since for all $t \in \mathbb{R}$,

$$0 \leq \arctan'(t) = \frac{1}{1 + t^2} \leq 1$$

the Mean-Value inequality yields that for all $a, b \in \mathbb{R}$, if $a < b$ then

$$\begin{aligned} |\arctan(a) - \arctan(b)| &\leq \sup_{t \in (a,b)} |\arctan'(t)|(b-a) \\ &\leq 1 \cdot |b-a|. \end{aligned}$$

Because this inequality is symmetric in a and b, we actually don't need the assumption that $a < b$, and we have that for all $a, b \in \mathbb{R}$,

$$|\arctan(a) - \arctan(b)| \leq |b-a|.$$

It follows that

$$\begin{aligned} |F_2((x_1, x_2)) - F_2((y_1, y_2))| &= \left| \frac{1}{\pi} \arctan(x_1) - \frac{1}{\pi} \arctan(y_1) \right| \\ &= \frac{1}{\pi} |\arctan(x_1) - \arctan(y_1)| \\ &\leq \frac{1}{\pi} |x_1 - y_1| \end{aligned}$$

Therefore

$$\begin{aligned} \|F((x_1, x_2)) - F((y_1, y_2))\|_2^2 &= |F_1((x_1, x_2)) - F_1((y_1, y_2))|^2 \\ &\quad + |F_2((x_1, x_2)) - F_2((y_1, y_2))|^2 \\ &\leq \frac{2}{9}((x_1 - y_1)^2 + (x_2 - y_2)^2) + \frac{1}{\pi^2}(x_1 - y_1)^2 \\ &\leq \frac{2}{9}((x_1 - y_1)^2 + (x_2 - y_2)^2) + \frac{1}{9}(x_1 - y_1)^2 \\ &\leq \frac{1}{3}\|(x_1, x_2) - (y_1, y_2)\|_2^2. \end{aligned}$$

21.3 Exercises

Exercise 21.3.1. Consider the function $F : [-1, 1]^2 \to \mathbb{R}^2$ given by

$$F((x_1, x_2)) := \left(\frac{1}{2} \sin(x_2) + \frac{1}{3} x_1 + \frac{1}{6}, \frac{1}{4}(x_1)^3 - \frac{1}{6} \right)$$

Show that the function F has a fixed point.

Chapter 22

Implicit function theorem

22.1 The objective

Before we state the implicit function theorem, it would be good to explain some notation.

We will be considering continuously differentiable functions $f : \Omega \subset \mathbb{R}^{d+m} \to \mathbb{R}^m$. It is good to think of the vector space $\mathbb{R}^{d+m}$ as the vector space $\mathbb{R}^d \oplus \mathbb{R}^m$, i.e. as the vector space of pairs (x, y) of vectors $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^m$.

The implicit function theorem comes to rescue in the following situation: when we want to know that there exists a function g that satisfies for (some) $x \in \mathbb{R}^d$,

$$f(x, g(x)) = 0 \tag{22.1.1}$$

and when we want to know that g has nice properties, i.e. that g itself is continuously differentiable. Rather than giving a functional description of g, the function g is what-is-called *implictly defined* by the equation (22.1.1).

Example 22.1.1. A standard example is when $f : \mathbb{R}^{1+1} \to \mathbb{R}$ is the function $f((x, y)) = x^2 + y^2 - 1$, and we would like to write 'y in terms of x'. More precisely, we would like to find a function g such that

$$f((x, g(x))) = x^2 + (g(x))^2 - 1 = 0.$$

We immediately see two issues here:

- Such a function g cannot be defined for all x (the equation has no solutions if $|x| > 1$),
- and for $|x| < 1$, there are always two possible solutions.

The first issue will be addressed by assumptions in the implicit function theorem: the theorem will need a good starting position, i.e. a point (a, b) such that $f(a, b) = 0$, but it will also need a condition on the derivative of f in the point (a, b). This condition will prohibit us from applying the theorem in the problematic points $(1, 0)$ and $(-1, 0)$ in the example above.

The second issue will be addressed by the conclusions in the implicit function theorem: it only makes statements about points close to (a, b).

22.2 Notation

Before we describe the theorem, we will need to introduce more notation.

Since we will assume the function $f : \Omega \to \mathbb{R}^m$ to be continuously differentiable (with Ω an open subset of $\mathbb{R}^{d+m}$), we will have that in a point $(a, b) \in \Omega$, the derivative $(Df)_{(a,b)}$ exists and is a linear map from $\mathbb{R}^{d+m}$ to $\mathbb{R}^m$. To get a feeling for this, let's see what it looks like in an example.

Example 22.2.1. We could for instance be considering the function $F : \mathbb{R}^{3+2} \to \mathbb{R}^2$ defined by

$$F((x_1, x_2, x_3), (y_1, y_2)) = ((x_1)^2 y_2 + y_1 - 2, \sin(x_2 y_2) + (x_3)^4 - 3)$$

The function F is indeed differentiable and

$$(DF)_{(a,b)}(((h_1, h_2, h_3), (k_1, k_2))) = [DF]_{(a,b)} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ k_1 \\ k_2 \end{pmatrix}$$

where the Jacobian $[DF]_{(a,b)}$ of F in the point $(a, b) \in \mathbb{R}^{3+2}$ is given by

$$\begin{aligned}[DF]_{(a,b)} &= \left(\begin{array}{ccc|cc} \frac{\partial f_1}{\partial x_1}((a,b)) & \frac{\partial f_1}{\partial x_2}((a,b)) & \frac{\partial f_1}{\partial x_3}((a,b)) & \frac{\partial f_1}{\partial y_1}((a,b)) & \frac{\partial f_1}{\partial y_2}((a,b)) \\ \frac{\partial f_2}{\partial x_1}((a,b)) & \frac{\partial f_2}{\partial x_2}((a,b)) & \frac{\partial f_2}{\partial x_3}((a,b)) & \frac{\partial f_2}{\partial y_1}((a,b)) & \frac{\partial f_2}{\partial y_2}((a,b)) \end{array} \right) \\ &= \left(\begin{array}{ccc|cc} 2a_1 b_2 & 0 & 0 & 1 & (a_1)^2 \\ 0 & \cos(a_2 b_2) b_2 & 4(a_3)^3 & 0 & \cos(a_2 b_2) a_2 \end{array} \right)\end{aligned}$$

We will denote by

$$(D_{\mathbf{1}} f)_{(a,b)} : \mathbb{R}^d \to \mathbb{R}^m$$

the restriction of the derivative $(Df)_{(a,b)} : \mathbb{R}^{d+m} \to \mathbb{R}^m$ to the subspace $\mathbb{R}^d \subset \mathbb{R}^{d+m}$. In other words, for all $h \in \mathbb{R}^d$,

$$(D_{\mathbf{1}} f)_{(a,b)}(h) = (Df)_{(a,b)}((h, 0)).$$

Similarly, we will denote by

$$(D_{\mathbf{2}} f)_{(a,b)} : \mathbb{R}^m \to \mathbb{R}^m$$

the restriction of the derivative $(Df)_{(a,b)} : \mathbb{R}^{d+m} \to \mathbb{R}^m$ to the subspace $\mathbb{R}^m \subset \mathbb{R}^{d+m}$. In other words, for all $k \in \mathbb{R}^m$,

$$(D_{\mathbf{2}} f)_{(a,b)}(k) = (Df)_{(a,b)}((0, k)).$$

By linearity of $(Df)_{(a,b)}$, we have the following relationship

$$(Df)_{(a,b)}(h, k) = (D_{\mathbf{1}} f)_{(a,b)}(h) + (D_{\mathbf{2}} f)_{(a,b)}(k).$$

We will denote the matrix representations (with respect to the standard bases) of the maps $(D_{\mathbf{1}} f)_{(a,b)}$ and $(D_{\mathbf{2}} f)_{(a,b)}$ by $[D_{\mathbf{1}} f]_{(a,b)}$ and $[D_{\mathbf{2}} f]_{(a,b)}$ respectively.

Then

$$\begin{aligned}(Df)_{(a,b)}((h,k)) &= (D_{\mathbf{1}} f)_{(a,b)}(h) + (D_{\mathbf{2}} f)_{(a,b)}(k) \\ &= [D_{\mathbf{1}} f]_{(a,b)} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} + [D_{\mathbf{2}} f]_{(a,b)} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\end{aligned}$$

Example 22.2.2. In our previous example, the matrix $[D_{\mathbf{1}}f]_{(a,b)}$ is given by

$$\begin{aligned}
[D_{\mathbf{1}}f]_{(a,b)} &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}((a,b)) & \frac{\partial f_1}{\partial x_2}((a,b)) & \frac{\partial f_1}{\partial x_3}((a,b)) \\ \frac{\partial f_2}{\partial x_1}((a,b)) & \frac{\partial f_2}{\partial x_2}((a,b)) & \frac{\partial f_2}{\partial x_3}((a,b)) \end{pmatrix} \\
&= \begin{pmatrix} 2a_1b_2 & 0 & 0 \\ 0 & \cos(a_2b_2)b_2 & 4(a_3)^3 \end{pmatrix}
\end{aligned}$$

and the matrix $[D_{\mathbf{2}}f]_{(a,b)}$ is given by

$$\begin{aligned}
[D_{\mathbf{2}}f]_{(a,b)} &= \begin{pmatrix} \frac{\partial f_1}{\partial y_1}((a,b)) & \frac{\partial f_1}{\partial y_2}((a,b)) \\ \frac{\partial f_2}{\partial y_1}((a,b)) & \frac{\partial f_2}{\partial y_2}((a,b)) \end{pmatrix} \\
&= \begin{pmatrix} 1 & (a_1)^2 \\ 0 & \cos(a_2b_2)a_2 \end{pmatrix}
\end{aligned}$$

22.3 The implicit function theorem

We are now ready to formulate the implicit function theorem.

Theorem 22.3.1 (Implicit function theorem)**.** Let $\Omega \subset \mathbb{R}^{d+m}$ and let $f : \Omega \to \mathbb{R}^m$ be a function which is continuously differentiable. Let $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^m$ and assume that $(a,b) \in \Omega$ and $f((a,b)) = 0$.

Suppose that $(D_{\mathbf{2}}f)_{(a,b)}$ is invertible (or equivalently, that the matrix $[D_{\mathbf{2}}f]_{(a,b)}$ is non-singular). Then there exists an $r_1 > 0$ and an $r_2 > 0$, and a continuously differentiable function $g : B(a, r_1) \to \mathbb{R}^m$ such that for all $x \in B(a, r_1)$ and all $y \in B(b, r_2)$,

$$f(x,y) = 0 \quad \text{if and only if} \quad y = g(x).$$

Moreover, for all $x \in B(a, r_1)$,

$$(Dg)_x = -(D_{\mathbf{2}}f)^{-1}_{(x,g(x))} \circ (D_{\mathbf{1}}f)_{(x,g(x))}. \tag{22.3.1}$$

The expression for the derivative of g is actually easy to derive when you already know that g is differentiable. Because in that case, we can start from the equality

$$f(x, g(x)) = 0,$$

and use the chain rule to compute that

$$(D_{\mathbf{1}} f)_{(x,g(x))} + (D_{\mathbf{2}} f)_{(x,g(x))} \circ (Dg)_x = 0.$$

We then use that $(D_{\mathbf{2}} f)_{(x,g(x))}$ is non-singular, multiply by its inverse, and conclude the expression (22.3.1).

The proof of this theorem is very technical, but the underlying idea is simple and very beautiful. Given an $x \in \mathbb{R}^d$ close to a, how do we find y such that $f(x, y) = 0$? It won't be possible to find such a y immediately, but let's first see what the guess $y^{(0)} := b$ will give us. We will then make an error, because $f(x, y^{(0)})$ will not be equal to 0 in general. Therefore we will make a new guess, $y^{(1)}$ that aims to correct for the error. We will choose the difference $y^{(1)} - y^{(0)}$ in the way that would give us the exact solution if f were in fact affine:

$$-f(x, y^{(0)}) \approx (D_{\mathbf{2}} f)_{(a,b)}(y^{(1)} - y^{(0)}).$$

Thus, we define

$$y^{(1)} := y^{(0)} - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big(f(x, y^{(0)})\big)$$

where you can see the important assumption that $(D_{\mathbf{2}} f)_{(a,b)}$ is non-singular coming in.

Of course, in general $y^{(1)}$ will also not be the value we are looking for, i.e. in general $f(x, y^{(1)}) \neq 0$, but we can make a new guess $y^{(2)}$, solving for the update $y^{(2)} - y^{(1)}$ again acting as if f were affine:

$$-f(x, y^{(1)}) \approx (D_{\mathbf{2}} f)_{(a,b)}(y^{(2)} - y^{(1)}).$$

Continuing in this fashion, we arrive at the following recursive scheme:

$$y^{(n+1)} := y^{(n)} - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big(f(x, y^{(n)})\big)$$

And that's basically it! We just need to show that this works (that this recursive scheme converges, and that the resulting solutions depend nicely on x). But this is exactly the virtue of the Banach fixed point theorem. In other words, we need to check that we can apply the Banach fixed point theorem, but if we can apply it then we indeed find a y such that

$$y = y - (D_2 f)^{-1}_{(a,b)} f(x,y)$$

from which it follows that indeed $f(x,y) = 0$.

This finishes the intuition for the approach. To show that the involved map is indeed a contraction, we will need to use the Mean-Value Inequality.

Now follows the full proof, but it may be good to skip the proof on first reading.

Proof. Right at the beginning of this proof, we choose two radii r_1 and r_2 according to the following criteria:

i. The Cartesian product $\overline{B}(a, r_1) \times \overline{B}(b, r_2)$ is contained in Ω,

ii. for all $x \in \overline{B}(a, r_1)$ and all $y \in \overline{B}(b, r_2)$ we have

$$\max(\|(D_2 f)^{-1}_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m}, 1) \|\mathsf{Err}^f_{(a,b)}((x,y))\|_2 < \frac{r_2}{3}$$

iii. for all $x \in \overline{B}(a, r_1)$ and all $y \in \overline{B}(b, r_2)$ we have

$$\|(D_2 f)^{-1}_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m} \|(D_1 f)_{(a,b)}\|_{\mathbb{R}^d \to \mathbb{R}^m} \|x - a\|_2 < \frac{r_2}{3}$$

iv. for all $x \in \overline{B}(a, r_1)$ and all $y \in \overline{B}(b, r_2)$ we have

$$\|(D_2 f)^{-1}_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m} \|(D_2 f)_{(x,z)} - (D_2 f)_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m} < \frac{1}{2}.$$

v. For all $x \in \overline{B}(a, r_1)$ and all $y \in \overline{B}(b, r_2)$,

$$\|(D_1 f)_{(x,y)}\|_{\mathbb{R}^d \to \mathbb{R}^m} < 2\|(D_1 f)_{(a,b)}\|_{\mathbb{R}^d \to \mathbb{R}^m}$$

For $x \in \overline{B}(a, r_1)$ we define $F^{(x)} : \overline{B}(b, r_2) \to \mathbb{R}^m$ by

$$F^{(x)}(y) := y - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big(f(x, y)\big).$$

We will want to use the Banach fixed point theorem to the function $F^{(x)}$, and therefore we need to show that $F^{(x)}$ maps every element in $\overline{B}(b, r_2)$ back into $\overline{B}(b, r_2)$, and we need to check that $F^{(x)}$ is a contraction.

We first check that $F^{(x)}$ maps $\overline{B}(b, r_2)$ back to $\overline{B}(b, r_2)$. We show a slightly stronger property, namely that $F^{(x)}$ maps $\overline{B}(b, r_2)$ to $B(b, 2r_2/3)$. Indeed, by the criteria above, if $x \in \overline{B}(a, r_1)$ and $y \in \overline{B}(b, r_2)$ then

$$\begin{aligned} \|F^{(x)}(y) - b\|_2 &= \|y - b - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big(f(x, y)\big)\|_2 \\ &= \|y - b - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big(f(a, b) + (D_{\mathbf{1}} f)_{(a,b)}(x - a) \\ &\qquad + (D_{\mathbf{2}} f)_{(a,b)}(y - b) + \mathsf{Err}^f_{(a,b)}((x, y))\big)\|_2 \\ &= \| - (D_{\mathbf{2}} f)^{-1}_{(a,b)}\big((D_{\mathbf{1}} f)(x - a) + \mathsf{Err}^f_{(a,b)}((x, y))\big)\|_2 \\ &< \frac{2r_2}{3}. \end{aligned} \tag{22.3.2}$$

where we used the assumption that $f(a, b) = 0$.

Next, we check that $F^{(x)} : \overline{B}(b, r_2) \to \overline{B}(b, r_2)$ is a contraction.

We note that $F^{(x)}$ is continuously differentiable and

$$(DF^{(x)})_z = I - (D_{\mathbf{2}} f)^{-1}_{(a,b)} \circ (D_{\mathbf{2}} f)_{(x,z)}.$$

We compute

$$\begin{aligned}
&\|(DF^{(x)})_z\|_{\mathbb{R}^m\to\mathbb{R}^m}\\
&= \|I - (D_2 f)^{-1}_{(a,b)} \circ (D_2 f)_{(x,z)}\|_{\mathbb{R}^m\to\mathbb{R}^m}\\
&= \|I - (D_2 f)^{-1}_{(a,b)} \circ ((D_2 f)_{(a,b)} + (D_2 f)_{(x,z)} - (D_2 f)_{(a,b)})\|_{\mathbb{R}^m\to\mathbb{R}^m}\\
&= \|(D_2 f)^{-1}_{(a,b)} \circ ((D_2 f)_{(x,z)} - (D_2 f)_{(a,b)})\|_{\mathbb{R}^m\to\mathbb{R}^m}\\
&\leq \|(D_2 f)^{-1}_{(a,b)}\|_{\mathbb{R}^m\to\mathbb{R}^m}\|((D_2 f)_{(x,z)} - (D_2 f)_{(a,b)})\|_{\mathbb{R}^m\to\mathbb{R}^m}\\
&< \frac{1}{2}
\end{aligned}$$

where we used criterion (iv) on the choice of r_1 and r_2.

It follows by the Mean-Value Inequality that for every $x \in \overline{B}(a, r_1)$, the function $F^{(x)} : \overline{B}(b, r_2) \to \overline{B}(b, r_2)$ is a $(1/2)$-contraction.

By the Banach fixed point theorem, the function $F^{(x)}$ has a unique fixed point. We define by $g : \overline{B}(a, r_1) \to \mathbb{R}^m$ the function that assigns to $x \in \overline{B}(r_1)$ the fixed point of $F^{(x)}$. Then for every $x \in \overline{B}(a, r_1)$,

$$f(x, g(x)) = 0.$$

We will now show that the function $g : B(a, r_1) \to \mathbb{R}^m$ is differentiable. Unfortunately and fortunately we need to do this in two steps. First we will show that g is Lipschitz, and only then will we be able show that g is differentiable.

proof that the function $g : \overline{B}(a, r_1) \to \overline{B}(b, r_2)$ is Lipschitz. To derive that g is Lipschitz continuous, we take $u \in \overline{B}(a, r_1)$ and we are going to just use $g(u)$ as an initial condition for the fixed-point iteration of $F^{(x)}$.

We would like to see how large is the difference

$$F^{(x)}(g(u)) - g(u).$$

Therefore we compute

$$\begin{aligned} F^{(x)}(g(u)) - g(u) &= -(D_{\mathbf{2}}f)^{-1}_{(a,b)}(f(x,g(u))) \\ &= -(D_{\mathbf{2}}f)^{-1}_{(a,b)}\big(f(x,g(u)) - f(u,g(u))\big) \end{aligned}$$

where we used that $f(u,g(u)) = 0$.

Define

$$M := 2\|(D_{\mathbf{2}}f)^{-1}_{(a,b)}\|_{\mathbb{R}^m\to\mathbb{R}^m} \sup_{(\sigma,\tau)\in\overline{B}(a,r_1)\times\overline{B}(b,r_2)} \|(D_{\mathbf{1}}f)_{(\sigma,\tau)}\|_{\mathbb{R}^m\to\mathbb{R}^m}$$

which exists for instance by criterion v above.

It follows by the Mean-Value Inequality that

$$\|F^{(x)}(g(u)) - g(u)\|_{\mathbb{R}^m} \le \frac{1}{2}M\|x-u\|_{\mathbb{R}^d}$$

By the estimate from the Banach fixed-point theorem, also

$$\|g(x) - g(u)\|_{\mathbb{R}^m} \le M\|x-u\|_{\mathbb{R}^d}.$$

so that g is indeed M-Lipschitz.

proof that the function $g : B(b,r_2) \to B(b,r_2)$ is differentiable. Let $u \in B(a,r_1)$ and define $v := g(u)$. Note that in fact, $v \in B(b, 2r_2/3)$ by the estimate in (22.3.2).

We expect that the derivative of g in the point u would be

$$(Dg)_u = -(D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}$$

(as this expression can be derived from the chain rule assuming that g is indeed differentiable).

Therefore, we consider the error function

$$\mathsf{Err}^g_u(x) := g(x) - g(u) + (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)$$

and we need to show that

$$\lim_{x \to u} \frac{\|\mathsf{Err}_u^g(x)\|_{\mathbb{R}^m}}{\|x-u\|_{\mathbb{R}^d}} = 0.$$

We are now going to use

$$g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)$$

as the initial condition in the fixed-point iteration at x. Note that this is possible for x close enough to u, since then this point is in $\overline{B}(b, r_2)$. We would like to see how large is the difference

$$\begin{aligned} F^{(x)}&\Big(\big(g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)\big)\Big) \\ &- \big(g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)\big) \end{aligned}$$

as by the estimate from the Banach fixed-point theorem, this would immediately give us control over

$$g(x) - \big(g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)\big)$$

as well.

Therefore, we compute

$$\begin{aligned} &F^{(x)}\Big(\big(g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)\big)\Big) \\ &\quad - \big(g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)\big) \\ &= -(D_2 f)^{-1}_{(a,b)}\big(f(x, g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u))\big) \\ &= -(D_2 f)^{-1}_{(a,b)}\big(f(x, g(u) - (D_2 f)^{-1}_{(u,v)} \circ (D_1 f)_{(u,v)}(x-u)) \\ &\qquad - f(u, g(u))\big) \end{aligned}$$

where we used in the last line that $f(u, g(u)) = 0$. Because f is differentiable, we find

$$\begin{aligned}
&F^{(x)}\big(\big(g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)\big)\big) \\
&\quad - \big(g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)\big) \\
&= -(D_{\mathbf{2}}f)^{-1}_{(a,b)}\big((D_{\mathbf{1}}f)_{(u,v)}(x-u) \\
&\qquad + (D_{\mathbf{2}}f)_{(u,v)}(-(D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)) \\
&\qquad + \mathsf{Err}^{f}_{(u,v)}((x, g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)))\big) \\
&= -(D_{\mathbf{2}}f)^{-1}_{(a,b)}\big(\mathsf{Err}^{f}_{(u,v)}((x, g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)))\big)
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|F^{(x)}\big(\big(g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)\big)\big) \\
&\quad - \big(g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)\big)\|_{\mathbb{R}^m} \\
&\leq \|(D_{\mathbf{2}}f)^{-1}_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m} \\
&\quad \times \|\mathsf{Err}^{f}_{(u,v)}((x, g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)))\|_{\mathbb{R}^m}
\end{aligned}$$

By the estimate in the Banach fixed-point theorem, we find

$$\begin{aligned}
&\|\mathsf{Err}^{g}_{a}(x)\|_{\mathbb{R}^m} \\
&= \|g(x) - (g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u))\| \\
&\leq 2\|(D_{\mathbf{2}}f)^{-1}_{(a,b)}\|_{\mathbb{R}^m \to \mathbb{R}^m} \\
&\quad \times \|\mathsf{Err}^{f}_{(u,v)}((x, g(u) - (D_{\mathbf{2}}f)^{-1}_{(u,v)} \circ (D_{\mathbf{1}}f)_{(u,v)}(x-u)))\|_{\mathbb{R}^m}
\end{aligned}$$

so that, using the Lipschitz continuity of g, it follows by the squeeze theorem that indeed

$$\lim_{x \to a} \frac{\|\mathsf{Err}^{g}_{a}(x)\|_{\mathbb{R}^m}}{\|x-a\|_{\mathbb{R}^m}} = 0.$$

□

22.4 The inverse function theorem

Theorem 22.4.1 (Inverse function theorem)**.** Let $\Omega \subset \mathbb{R}^m$ be open and let $h : \Omega \to \mathbb{R}^m$ be a function which is continuously differentiable. Suppose $b \in \Omega$ and suppose that $(Dh)_b$ is non-singular.

Then there exists an $r_1 > 0$ and an $r_2 > 0$, and a continuously differentiable function $g : B(h(b), r_1) \to \mathbb{R}^m$ such that for all $x \in B(h(b), r_1)$ and all $y \in B(b, r_2)$,

$$x = h(y) \qquad \text{if and only if} \qquad y = g(x).$$

Moreover, for all $x \in B(h(b), r_1)$,

$$(Dg)_x = (Dh)^{-1}_{g(x)}.$$

In particular, for $r_3 > 0$ small enough, the function h restricted to $B(b, r_3)$ mapping to $h(B(b, r_3))$ is invertible with continuously differentiable inverse g.

Proof. The proof of the theorem follows from applying the Implicit Function Theorem to the function $F : \mathbb{R}^m \times \Omega \to \mathbb{R}^m$ given by

$$F(x, y) := x - h(y)$$

□

The inverse function theorem is useful to conclude for instance that for $k \in \mathbb{N}$, the function $x \mapsto x^{1/k}$ is differentiable on the domain $(0, \infty)$.

The implicit function theorem would also allow us to conclude that the function $\ln : (0, \infty) \to \mathbb{R}$ is differentiable on its domain, given that the exponentional function $\exp : \mathbb{R} \to \mathbb{R}$ is differentiable. But truly, we still haven't given a proper definition of the exponential function. The next chapters will allow us to provide such a definition.

22.5 Exercises

Exercise 22.5.1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x, y) = x^2 - y^3 + 3y$$

a. Specify precisely the points (a, b) on the curve

$$\Gamma := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 1\}$$

such that there **does not** exist $r_1, r_2 > 0$ and a continuously differentiable function $g : B(a, r_1) \to \mathbb{R}$ such that for all $x \in B(a, r_1)$ and $y \in B(b, r_2)$

$$F(x, y) = 1 \qquad \text{if and only if} \qquad y = g(x).$$

b. For a continuously differentiable function $g : B(a, r_1) \to \mathbb{R}$ such that

$$F(x, g(x)) = 1$$

compute $g'(x)$ for every $x \in B(a, r_1)$ in terms of x and $g(x)$.

Exercise 22.5.2. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, and consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, t) := (x + \rho(x)t, t)$$

Show that there exist radii $r_1 > 0$ and $r_2 > 0$ and a function $G : B(0, r_2) \to B(0, r_1)$ such that for all (y, s) in $B(0, r_2)$ and (x, t) in $B(0, r_1)$

$$(y, s) = F((x, t)) \text{ if and only if } (x, t) = G((y, s)).$$

and for every (y, s) give an expression for

$$[DG]_{(y,s)}$$

Exercise 22.5.3. Consider the function $F : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x_1, x_2, x_3, y_1, y_2) = ((x_2)^2 y_1 + y_2 - 3, \cos(x_3 y_2) + y_2 + (x_1)^2)$$

Prove that there exist $r_1, r_2 > 0$ and a continuously differentiable function $g : B((1, 3, 2), r_1) \to \mathbb{R}^2$ such that for all $x \in B((1, 3, 2), r_1)$ and all $y \in B((1, 0), r_2)$,

$$F(x_1, x_2, x_3, y_1, y_2) = (6, 2) \quad \text{if and only if} \quad y = g(x).$$

Moreover, compute the Jacobian $[Dg]_{(1,3,2)}$.

Chapter 23

Function sequences

23.1 Pointwise convergence

Definition 23.1.1. Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$. We say that a sequence of functions $f : \mathbb{N} \to (D \to Y)$ converges *pointwise* to a function $f^* : D \to Y$ if

$$\begin{aligned}&\text{for all } x \in D,\\ &\quad \lim_{n\to\infty} f_n(x) = f^*(x).\end{aligned}$$

Example 23.1.2. We consider a case in which $(X, \mathsf{dist}_X) = (\mathbb{R}, \mathsf{dist}_{\mathbb{R}})$ and D is the interval $[0,1] \subset \mathbb{R}$. We consider the sequence of functions $f : \mathbb{N} \to ([0,1] \to \mathbb{R})$ defined by

$$f_n(x) = x^n.$$

Then the sequence (f_n) converges pointwise to the funtion $f^* : [0,1] \to \mathbb{R}$ defined by

$$f^*(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1. \end{cases}$$

To show this, let $x \in [0,1]$. Then we consider two cases. In case $x \in$

$[0, 1)$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 = f^*(x)$$

as this is a standard limit. In case $x = 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 1^n = 1 = f^*(x).$$

23.2 Uniform convergence

Definition 23.2.1. Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subset X$. We say that a sequence of functions $f : \mathbb{N} \to (D \to Y)$ converges *uniformly* to a function $f^* : D \to Y$ if

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } N \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N,\\
&\quad\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad\quad \mathsf{dist}_Y(f_n(x), f^*(x)) < \epsilon.
\end{aligned}$$

Proposition 23.2.2. Let (X, dist_X) and (Y, dist_Y) be two metric spaces, let $D \subset X$, and assume that a sequence of functions $f : \mathbb{N} \to (D \to Y)$ converges uniformly to a function $f^* : D \to Y$. Then (f_n) converges to f^* pointwise on D.

Proof. We need to show that (f_n) converges pointwise to f^*. That means that we need to show that

$$\begin{aligned}
&\text{for all } x \in D,\\
&\quad \lim_{n\to\infty} f_n(x) = f^*(x).
\end{aligned}$$

Let $x \in D$. We need to show that for every $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\mathsf{dist}_Y(f_n(x), f^*(x)) < \epsilon.$$

Let $\epsilon > 0$. Since (f_n) converges to f^* uniformly, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $p \in D$,

$$\mathsf{dist}_Y(f_n(p), f^*(p)) < \epsilon.$$

Choose $N_1 := N$. Let $n \geq N_1$. Then, since $n \geq N$, also

$$\mathsf{dist}_Y(f_n(x), f^*(x)) < \epsilon.$$

□

The previous proposition has a simple consequence that is very useful in practice. Suppose for instance that you need to check whether a sequence of functions converges uniformly, and you already know that the sequence converges *pointwise* to a function f^*. Then you only need to check whether the sequence of functions converges uniformly to f^*.

Corollary 23.2.3. Suppose a sequence of functions $f : \mathbb{N} \to (D \to Y)$ converges *pointwise* to a function $f^* : D \to Y$. Then (f_n) converges uniformly on D if and only if (f_n) converges uniformly to f^* on D.

23.3 Preservation of continuity under uniform convergence

Theorem 23.3.1. Let (f_n) be a sequence of continuous functions from a domain D in the metric space (X, dist_X) to the metric space (Y, dist_Y) that converges uniformly to a function $g : D \to Y$. Then the function g is also continuous on D.

Proof. We need to show that $g : D \to Y$ is continuous, so we need to show that for all $a \in D$, the function g is continuous in a.

Let $a \in D$. We need to show that

for all $\epsilon > 0$,
there exists $\delta > 0$,
for all $x \in D$,
if $0 < \mathsf{dist}_X(x, a) < \delta$
then $\mathsf{dist}_Y(g(x), g(a)) < \epsilon$.

Let $\epsilon > 0$. Since the function sequence (f_n) converges to g uniformly, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, and all $x \in D$

$$\mathsf{dist}_Y(f_n(x), g(x)) < \frac{\epsilon}{3}.$$

Choose such an $N \in \mathbb{N}$. Because the function f_N is continuous, there exists a $\delta_0 > 0$ such that for all $x \in D$, if $0 < \mathsf{dist}_X(x, a) < \delta_0$, then

$$\mathsf{dist}_Y(f_N(x), f_N(a)) < \frac{\epsilon}{3}.$$

Choose $\delta := \delta_0$.

Let $x \in D$. Assume that $0 < \mathsf{dist}_X(x, a) < \delta$. Then by the triangle inequality

$$\begin{aligned}\mathsf{dist}_Y(g(x), g(a)) &\leq \mathsf{dist}_Y(g(x), f_N(x)) + \mathsf{dist}_Y(f_N(x), f_N(a)) \\ &\qquad + \mathsf{dist}(f_N(a), g(a)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

□

The previous theorem is sometimes very useful to rule out that a sequence of functions converges *uniformly*. If the functions in the sequence are all continuous, but the pointwise limit is not continuous, then the sequence does not converge uniformly.

Example 23.3.2. Consider the sequence of functions (f_n) from $[0,1]$ to $\mathbb{R}$ defined by

$$f_n(x) = x^n.$$

We have seen that the pointwise limit is $g : [0,1] \to \mathbb{R}$ given by

$$g(x) := \begin{cases} 0, & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1. \end{cases}$$

Because the function g is not continuous, but for every $n \in \mathbb{N}$ the function f_n *is* continuous (as it is a polynomial), we conclude that the sequence (f_n) does not converge to g uniformly.

23.4 Differentiability theorem

Theorem 23.4.1. Let (f_n) be a sequence of functions from an open domain Ω in a vector space V to $\mathbb{R}$ and suppose the sequence converges pointwise to a function $g : \Omega \to \mathbb{R}$. Suppose moreover that the functions f_n are continuously differentiable on Ω and suppose the sequence of functions $Df_n : \Omega \to \mathsf{Lin}(V, \mathbb{R})$ converges uniformly to a function $\Delta : \Omega \to \mathsf{Lin}(V, \mathbb{R})$. Then the function g is differentiable on Ω as well and

$$Dg = \Delta.$$

Proof. To show that $g : \Omega \to \mathbb{R}$ is differentiable on Ω, we need to show that for all $a \in \Omega$, the function g is differentiable in a.

Let $a \in \Omega$.

Define the error function

$$\mathsf{Err}_a^g(x) := g(x) - g(a) - \Delta_a(x - a).$$

We need to show that

$$\lim_{x \to a} \frac{\left|\mathsf{Err}_a^g(x)\right|}{\|x - a\|_V} = 0.$$

Let $\epsilon > 0$.

Because the functions Df_n converge to Δ uniformly, the function Δ is continuous by Theorem 23.3.1. Therefore, there exists a $\delta_0 > 0$, such that for all $z \in \Omega$, if $0 < \|z - a\|_V < \delta_0$ then

$$\|\Delta_z - \Delta_a\|_{V \to \mathbb{R}} < \frac{\epsilon}{3}.$$

Choose such a δ_0.

Choose $\delta := \delta_0$.

Let $x \in \Omega$ and assume that $0 < \|x - a\|_V < \delta$.

Moreover, since the functions Df_n converge to Δ uniformly on Ω, there exists an $N_0 \in \mathbb{N}$ such that for all $z \in \Omega$ and all $n \geq N_0$,

$$\|(Df_n)_z - \Delta_z\|_{V \to \mathbb{R}} < \frac{\epsilon}{3}.$$

Now because the function sequence (f_n) converges to g pointwise, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$|f_n(x) - g(x)| < \frac{\epsilon}{6}\|x - a\|_V.$$

Similarly, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$,

$$|f_n(a) - g(a)| < \frac{\epsilon}{6}\|x - a\|_V.$$

Choose $N := \max(N_0, N_1, N_2)$. Then

$$\begin{aligned}|\mathsf{Err}_a^g(x)| &= |g(x) - g(a) - \Delta_a(x-a)| \\ &\leq |f_N(x) - f_N(a) - \Delta_a(x-a)| + |f_N(x) - g(x)| + |f_N(a) - g(a)| \\ &< |f_N(x) - f_N(a) - \Delta_a(x-a)| + \frac{\epsilon}{3}\|x-a\|_V.\end{aligned}$$

By the Mean-Value Theorem, there exists a point y on the line segment from a to x such that

$$f_N(x) - f_N(a) = (Df_N)_y(x-a).$$

Therefore

$$\begin{aligned}|\mathsf{Err}_a^g(x)| &< |f_N(x) - f_N(a) - \Delta_a(x-a)| + \frac{\epsilon}{3}\|x-a\|_V \\ &= |(Df_N)_y(x-a) - \Delta_a(x-a)| + \frac{\epsilon}{3}\|x-a\|_V \\ &= |((Df_N)_y - \Delta_a)(x-a)| + \frac{\epsilon}{3}\|x-a\|_V \\ &\leq \|(Df_N)_y - \Delta_a\|_{V\to\mathbb{R}}\|x-a\|_V + \frac{\epsilon}{3}\|x-a\|_V \\ &\leq (\|(Df_N)_y - \Delta_y\|_{V\to\mathbb{R}} + \|\Delta_y - \Delta_a\|_{V\to\mathbb{R}})\|x-a\|_V + \frac{\epsilon}{3}\|x-a\|_V \\ &< \epsilon\|x-a\|_V\end{aligned}$$

which is what we needed to show. □

23.5 The normed vector space of bounded functions

Definition 23.5.1. Let D be a set. The normed vector space $(\mathcal{B}(D), \|\cdot\|_\infty)$ is defined as the vector space of *bounded* functions from D to $\mathbb{R}$ with norm $\|\cdot\|_\infty$ given by

$$\|f\|_\infty = \sup_{x\in D}|f(x)|.$$

The vector space $\mathcal{B}(D)$ is infinite-dimensional if D has infinitely many elements.

Proposition 23.5.2. Let (f_n) be a sequence of functions from D to $\mathbb{R}$ and let f be a function. Then the sequence (f_n) converges uniformly to f if and only if there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, the function $(f_n - f)$ is bounded, and such that the sequence $n \mapsto (f_{N+n} - f)$ converges to 0 in $\mathcal{B}(D)$.

Proposition 23.5.3. Let (f_n) be a sequence of functions from a domain D in the metric space (X, dist_X) to the metric space (Y, dist_Y) and let $g : D \to Y$ be a function. Then (f_n) converges to g uniformly if and only if there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, the function h_n given by

$$h_n(x) := \mathsf{dist}_Y(f_n(x), g(x))$$

is bounded and the sequence $n \mapsto h_{N+n}$ converges to 0 in $\mathcal{B}(D)$.

23.6 Exercises

Exercise 23.6.1. Determine whether the following functions converge pointwise and whether they converge uniformly on the indicated domain. If they converge pointwise, give the pointwise limit.

a. $a_n(x) = nx \exp(-nx)$ on the domain $\mathbb{R}$.

b. $b_n(x) = \sin(nx)$ on the domain $\mathbb{R}$.

c. $c_n(x) = x^n \cos(nx)$ on the domain $[0, 1)$.

d. $d_n(x) = \tan(x/n)$ on the domain $[-1, 1]$.

e. $e_n(x) = \exp(x - n)$ on the domain $(-\infty, 5]$.

f. $f_n(x) = \arctan(n(x - 2))$ on the domain $\mathbb{R}$.

Chapter 24

Function series

24.1 Definitions

Let (X, dist_X) be a metric space. Let (f_n) be a sequence of functions from $\Omega \subset X$ to $\mathbb{R}$.

We say that the function series

$$\sum_{k=0}^{\infty} f_k$$

converges *pointwise* to a function $s : \Omega \to \mathbb{R}$ if the function sequence of partial sums

$$S_n(x) := \sum_{k=0}^{n} f_k(x)$$

converges *pointwise* to the function $s : \Omega \to \mathbb{R}$.

We say that the function series $\sum_{k=0}^{\infty} f_k$ converges to s *uniformly* if the sequence of partial sums converges to the function s *uniformly*.

24.2 The Weierstrass M-test

Theorem 24.2.1 (Weierstrass M-test)**.** Let (f_n) be a sequence of functions from Ω to $\mathbb{R}$, and suppose there exists a sequence of real numbers (M_n) such that for all $n \in \mathbb{N}$, all $x \in \Omega$ it holds that

$$|f_n(x)| \leq M_n$$

and suppose the series

$$\sum_{k=0}^{\infty} M_k$$

converges.

Then the function series

$$\sum_{k=0}^{\infty} f_k$$

converges absolutely and uniformly on Ω.

Proof. We first show that the function series

$$\sum_{k=0}^{\infty} f_k$$

converges absolutely. For that we need to show that for every $x \in \Omega$, the series

$$\sum_{k=0}^{\infty} f_k(x)$$

converges absolutely.

Let $x \in \Omega$. By assumption, for every $k \in \mathbb{N}$, we have $|f_k(x)| \leq M_k$. Since the series

$$\sum_{k=0}^{\infty} M_k$$

converges, it follows by the comparison test that also the series

$$\sum_{k=0}^{\infty} |f_k(x)|$$

converges. Hence, the series

$$\sum_{k=0}^{\infty} f_k(x)$$

converges absolutely.

In particular, the series

$$\sum_{k=0}^{\infty} f_k$$

converges pointwise to a function $s : \Omega \to \mathbb{R}$.

We will now show that the series

$$\sum_{k=0}^{\infty} f_k$$

converges to s uniformly. Since the series

$$\sum_{k=0}^{\infty} M_k$$

converges, we know that

$$\lim_{\ell \to \infty} \sum_{k=\ell+1}^{\infty} M_k = 0$$

Let $x \in \Omega$. Then

$$\begin{aligned} |\sum_{k=0}^{\ell} f_k(x) - s(x)| &= \left| \sum_{k=\ell+1}^{\infty} f_k(x) \right| \\ &\leq \sum_{k=\ell+1}^{\infty} |f_k(x)| \\ &\leq \sum_{k=\ell+1}^{\infty} M_k \end{aligned}$$

from which it follows that

$$0 \leq \| \sum_{k=0}^{\ell} f_k - s\|_\infty \leq \sum_{k=\ell+1}^{\infty} M_k$$

and from the squeeze theorem we get that

$$\lim_{\ell\to\infty} \| \sum_{k=0}^{\ell} f_k - s\|_\infty = 0.$$

In other words, the series

$$\sum_{k=0}^{\infty} f_k$$

converges to s uniformly. □

The statement of the above theorem is practically equivalent to the statement that the normed vector space of bounded functions on Ω, denoted by $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$, is complete.

Example 24.2.2. Consider the function series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

We claim that for every $r > 0$, this series converges *uniformly* on the interval $[-r, r]$.

Proof. Let $r > 0$. To verify the claim, we will apply the Weierstrass M-test. The functions f_k appearing in the theorem correspond to the functions

$$f_k(x) = \frac{x^k}{k!}.$$

We need to check all conditions in the Weierstrass M-test, so we need to define a sequence (M_k) and verify for that choice that for all $x \in [-r, r]$ and all $k \in \mathbb{N}$,

$$|f_k(x)| \leq M_k,$$

and in addition we would need to verify that the series

$$\sum_{k=0}^{\infty} M_k$$

converges.

First we note that for all $k \in \mathbb{N}$ and for all $x \in [-r, r]$,

$$|f_k(x)| = \frac{|x|^k}{k!} \leq \frac{r^k}{k!}.$$

Therefore, we choose for (M_k) the sequence

$$M_k := \frac{r^k}{k!}.$$

We then indeed find that for all $k \in \mathbb{N}$ and for all $x \in [-r, r]$,

$$|f_k(x)| \leq M_k.$$

We can verify by the ratio test that the series

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \frac{r^k}{k!}$$

converges. It follows from the Weierstrass M-test that the function series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges uniformly on the interval $[-r, r]$. □

24.3 Conditions for differentiation of function series

The following theorem is a direct consequence of Theorem 23.4.1 on the preservation of differentiability.

Theorem 24.3.1. Let (f_n) be a sequence of functions from $\Omega \subset \mathbb{R}$ to $\mathbb{R}$. Suppose

i. for every $n \in \mathbb{N}$, the function f_n is continuously differentiable on Ω.

ii. the function series
$$\sum_{k=0}^{\infty} f_k$$
converges pointwise to a function $g : \Omega \to \mathbb{R}$.

iii. the function series
$$\sum_{k=0}^{\infty} f_k'$$
converges uniformly.

Then the function g is differentiable on Ω as well and for all $x \in \Omega$,

$$g'(x) = \sum_{k=0}^{\infty} f_k'(x)$$

24.4 Exercises

Exercise 24.4.1. Let (f_k) be a sequence of continuously differentiable, bounded functions $\mathbb{R} \to \mathbb{R}$. Suppose that for all $k \in \mathbb{N}$,

$$\|f_k\|_\infty \leq 1$$

and

$$\|f_k'\|_\infty \leq 5$$

a. Show that the function series
$$\sum_{k=0}^{\infty} 2^{-k} f_k$$
converges pointwise to some function $s : \mathbb{R} \to \mathbb{R}$.

b. Show that the function $s : \mathbb{R} \to \mathbb{R}$ is differentiable.

Exercise 24.4.2. Let D be a subset of $\mathbb{R}$, and let (f_k) be a sequence of bounded functions from D to $\mathbb{R}$, i.e. (f_k) is a sequence in the normed vector space $(\mathcal{B}(D), \|\cdot\|_\infty)$. Assume that the series

$$\sum_{k=0}^{\infty} f_k$$

converges uniformly. Show that

$$\lim_{k\to\infty} \|f_k\|_\infty = 0.$$

Exercise 24.4.3. Consider the sequence of functions (f_k) from $(0, \infty)$ to $\mathbb{R}$ given by

$$f_k(x) = k^2 x^2 \exp(-k^2 x)$$

a. Show that the series

$$\sum_{k=0}^{\infty} f_k$$

converges uniformly to some function $s : (0, \infty) \to \mathbb{R}$.

Hint: Note that f_k can be written as

$$f_k(x) = \frac{1}{k^2} g(k^2 x)$$

for some function $g : (0, \infty) \to (0, \infty)$. What is g?

b. Show that the function $s : (0, \infty) \to \mathbb{R}$ is continuous.

c. Show that the series

$$\sum_{k=0}^{\infty} f_k'$$

does *not* converge uniformly on $(0, \infty)$.

d. Show that for every $a \in (0, \infty)$, the series

$$\sum_{k=0}^{\infty} f_k'$$

does converge uniformly on the interval (a, ∞).

Hint: Note that there exists a $z \in \mathbb{R}$ such that for $b \geq z$,

$$b^2 \exp(-b) \leq \exp(-b/2)$$

Therefore, for k large enough, it holds for all $x \in (a, \infty)$ that

$$(k^2 x)^2 \exp(-k^2 x) < \exp(-k^2 x/2)$$

e. Show that the function $s : (0, \infty) \to \mathbb{R}$ is differentiable on $(0, \infty)$.

Chapter 25

Power series

25.1 Definition

Definition 25.1.1. A *power series* at a point $c \in \mathbb{R}$ is a function series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

where $a : \mathbb{N} \to \mathbb{R}$ is a real-valued sequence.

25.2 Convergence of power series

Lemma 25.2.1. Suppose a power series

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

converges at a point $z \in \mathbb{R}$. Let $\delta > 0$ be such that $\delta < |z-c|$. Then the power series converges absolutely and uniformly on the interval $[c-\delta, c+\delta]$.

Proof. We will apply the Weierstrass M-test.

Since the series
$$\sum_{k=0}^{\infty} a_k (z-c)^k$$
converges, the sequence $k \mapsto a_k(z-c)^k$ converges (to zero) and therefore it is bounded. In other words, there exists a $C > 0$ such that for all $k \in \mathbb{N}$,
$$|a_k(z-c)^k| \leq C.$$

Now note that for all $k \in \mathbb{N}$, and all $x \in [c-\delta, c+\delta]$,
$$|a_k(x-c)^k| = |a_k| \left| \frac{x-c}{z-c} \right|^k |z-c|^k \leq C \left(\frac{\delta}{|z-c|} \right)^k =: M_k. \quad (25.2.1)$$

Now note that since $\delta < |z-c|$, it follows that $\delta/|z-c| < 1$ so that
$$\sum_{k=0}^{\infty} \left(\frac{\delta}{|z-c|} \right)^k$$
is a standard convergent geometric series. By limit laws for series, the series
$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} C \left(\frac{\delta}{|z-c|} \right)^k$$
is convergent as well.

By the Weierstrass M-test, the function series
$$\sum_{k=0}^{\infty} a_k (x-c)^k$$
converges absolutely and uniformly on the interval $[c-\delta, c+\delta]$. $\square$

Corollary 25.2.2. For every power series
$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

around a point $c \in \mathbb{R}$ exactly one of the following occurs:

i. The series converges for $x = c$ and diverges for $x \neq c$. In this case we say that the radius of convergence of the power series is 0.

ii. There exists an $R > 0$ such that for all $x \in (c - R, c + R)$ the power series converges and for all $x \in \mathbb{R} \backslash [c - R, c + R]$ the series diverges. In this case we say the radius of convergence equals R.

iii. The series converges for all $x \in \mathbb{R}$. In this case we say the radius of convergence is ∞.

The following proposition gives a way to determine the radius of convergence of a power series.

Proposition 25.2.3. Let
$$\sum_{k=0}^{\infty} a_k (x - c)^k$$
be a power series around c and define the (extended real) number
$$L := \limsup_{k \to \infty} \sqrt[k]{|a_k|}$$
Then

i. if $L = \infty$, then the radius of convergence of the power series is 0,

ii. if $L \in (0, \infty)$, then the radius of convergence of the power series is $1/L$,

iii. if $L = 0$, then the radius of convergence of the power series is ∞.

For the proof of the proposition, let us first give an alternative version of the root test, Theorem 8.4.1.

Theorem 25.2.4 (Root test, lim sup version)**.** Let (b_k) be a sequence of nonnegative real numbers.

i. If

$$\limsup_{k\to\infty} \sqrt[k]{b_k} < 1,$$

then the series $\sum_k b_k$ converges.

ii. If

$$\limsup_{k\to\infty} \sqrt[k]{b_k} > 1$$

then the series $\sum_k b_k$ diverges.

Proof. First suppose that

$$\limsup_{k\to\infty} \sqrt[k]{b_k} < 1.$$

Denote by $M := \limsup_{k\to\infty} \sqrt[k]{b_k}$. Then by the alternative characterization of the lim sup, there exists an $N \in \mathbb{N}$ such that for all $k \geq N$,

$$\sqrt[k]{b_k} < M + \frac{1-M}{2} = \frac{1+M}{2} < 1.$$

It follows by Theorem 8.4.1 that the series $\sum_k b_k$ converges.

Now suppose that

$$\limsup_{k\to\infty} \sqrt[k]{b_k} = \infty.$$

Then the sequence $(\sqrt[k]{b_k})$ is not bounded from above, and as a consequence the sequence (b_k) is not bounded from above. Therefore, the series $\sum_k b_k$ diverges.

If

$$M := \limsup_{k\to\infty} \sqrt[k]{b_k} \in (1, \infty)$$

then by the alternative characterization of the lim sup, for every $K \in \mathbb{N}$, there exists an $\ell \geq K$ such that

$$\sqrt[\ell]{b_\ell} > 1.$$

For such ℓ, also $b_\ell > 1$. It follows that the sequence (b_k) does not converge to zero, and therefore the series $\sum_k b_k$ diverges. □

With this version of the root test, we can now prove Proposition 25.2.3.

Proof of Proposition 25.2.3. We would like to apply the root test. We therefore consider

$$\limsup_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} = \limsup_{k\to\infty} \left(\sqrt[k]{|a_k|}|x-c|\right)$$

It is clear that the power series always converges for $x = c$.

If $L = \infty$, and $x \neq c$, then

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k|}|x-c|\right) = \infty$$

and in particular the terms do not converge to zero. Therefore, if $L = \infty$, the power series only converges for $x = c$.

If $L \in (0, \infty)$, the root test implies that the series converges if $|x - c| < 1/L$, then

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k|}|x-c|\right) < 1$$

so that by the root test it follows that the series converges. On the other hand,if $L \in (0, \infty)$ and if $|x - c| > 1/L$, then

$$\limsup_{k\to\infty} \left(\sqrt[k]{|a_k|}|x-c|\right) > 1$$

and in particular the terms do not converge to zero. Therefore, the power series diverges.

If $L = 0$, it follows from the root test that the series converges for all $x \in \mathbb{R}$. □

25.3 Standard functions defined as power series

Proposition 25.3.1. The power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

has radius of convergence $R = \infty$.

Definition 25.3.2. The function $\exp : \mathbb{R} \to \mathbb{R}$ is defined as the power series

$$\exp(x) := \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

It has radius of convergence $R = \infty$.

Definition 25.3.3. The function $\sin : \mathbb{R} \to \mathbb{R}$ is defined as the power series

$$\sin(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

It has radius of convergence $R = \infty$.

Definition 25.3.4. The function $\cos : \mathbb{R} \to \mathbb{R}$ is defined as the power series

$$\cos(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}.$$

It has radius of convergence $R = \infty$.

25.4 Operations with power series

Proposition 25.4.1 (Sums of power series)**.** Let

$$\sum_{k=0}^{\infty} a_k (x-c)^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k (x-c)^k$$

be two power series around c, with radii of convergence R_1 and R_2 respectively. The sum of these functions is the power series

$$\sum_{k=0}^{\infty} (a_k + b_k)(x-c)^k$$

and the radius of convergence R for this new power series satisfies

$$R \geq \min(R_1, R_2).$$

Proposition 25.4.2 (Products of power series)**.** Let

$$\sum_{k=0}^{\infty} a_k (x-z)^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} b_k (x-z)^k$$

be two power series around a point $z \in \mathbb{R}$ with radii of convergence R_1 and R_2 respectively. Then the product of the two power series is again a power series

$$\sum_{k=0}^{\infty} c_k (x-z)^k$$

where

$$c_k := \sum_{\ell=0}^{k} a_\ell b_{k-\ell}.$$

The radius of convergence R of this new power series satisfies

$$R \geq \min(R_1, R_2).$$

Proof. It suffices to show that for all $x \in \mathbb{R}$ such that $|x - z| < \min(R_1, R_2)$, the power series

$$\sum_{k=0}^{\infty} c_k (x-z)^k$$

converges and

$$\sum_{k=0}^{\infty} c_k (x-z)^k = \left(\sum_{k=0}^{\infty} a_k (x-z)^k \right) \left(\sum_{k=0}^{\infty} b_k (x-z)^k \right)$$

Let therefore $x \in \mathbb{R}$ be such that $|x - z| < \min(R_1, R_2)$. We can now choose $A_k := a_k(x-z)^k$ and $B_k := b_k(x-z)^k$ in Theorem 9.3.1. It follows from the theorem that with

$$\begin{aligned} C_k &:= \sum_{\ell=0}^{k} A_\ell B_{k-\ell} = \sum_{\ell=0}^{k} a_\ell b_{k-\ell} (x-z)^\ell (x-z)^{k-\ell} \\ &= c_k (x-z)^k \end{aligned}$$

we have that the series

$$\sum_{k=0}^{\infty} C_k = \sum_{k=0}^{\infty} c_k (x-z)^k$$

converges and indeed

$$\sum_{k=0}^{\infty} c_k (x-z)^k = \left(\sum_{k=0}^{\infty} a_k (x-z)^k \right) \left(\sum_{k=0}^{\infty} b_k (x-z)^k \right).$$

□

25.5 Differentiation of power series

The following proposition follows directly from the differentiability theorem.

Proposition 25.5.1. Let
$$\sum_{k=0}^{\infty} a_k (x-c)^k$$
be a power series with radius of convergence R. Then, the power series is differentiable on the interval $(c-R, c+R)$ and on this interval, its derivative equals the power series
$$\sum_{\ell=0}^{\infty} (\ell+1) a_{\ell+1} (x-c)^{\ell}$$
which has the same radius of convergence.

Corollary 25.5.2. Let
$$\sum_{k=0}^{\infty} a_k (x-c)^k$$
be a power series with radius of convergence R. Then the power series is infinitely many times differentiable on the interval $(c-R, c+R)$, and for every $\ell \in \mathbb{N}$, the ℓth derivative has the same radius of convergence.

Theorem 25.5.3 (Identification of coefficients)**.** Let $R > 0$ and let $f : (c-R, c+R) \to \mathbb{R}$ be given by a power series
$$f(x) := \sum_{k=0}^{\infty} a_k (x-c)^k.$$
Then for all $k \in \mathbb{N}$,
$$a_k = \frac{f^{(k)}(c)}{k!}.$$

Theorem 25.5.4 (Identity theorem for power series)**.** Let $R > 0$ and let $f, g : (c-R, c+R) \to \mathbb{R}$ be given by power series
$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k \qquad \text{and} \qquad g(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$$

and assume for all $x \in (c - R, c + R)$,

$$f(x) = g(x).$$

Then for all $k \in \mathbb{N}$,

$$a_k = b_k.$$

25.6 Taylor series

Definition 25.6.1. Let $f : \Omega \to \mathbb{R}$ be a function that is infinitely many times differentiable on some open set Ω. Assume $c \in \Omega$.

Then the function series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

is called the *Taylor series* of f around c.

25.7 Exercises

Exercise 25.7.1. Give the Taylor series of the function

$$f(x) := \frac{1}{3x^2}$$

around the point $c = 1$ and give its radius of convergence.

Exercise 25.7.2. Prove Proposition 25.4.1.

Exercise 25.7.3. Now that the definitions of cos and sin are provided:

i. Prove using power series, (i.e. without using Proposition 16.3.1) that for all $x \in \mathbb{R}$,

$$\cos'(x) = -\sin(x)$$

and

$$\sin'(x) = \cos(x).$$

ii. Show that for all $x \in \mathbb{R}$,

$$\sin^2(x) + \cos^2(x) = 1.$$

Exercise 25.7.4. Determine, as a power series around 0, the solution to the differential equation

$$x^2 f''(x) - 2x f'(x) + (2 - x^2) f(x) = 0$$

with

$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0.$$

and determine the radius of convergence of the power series. Hint, use the Ansatz

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

and use the theorems about the sums of power series, products of power series and the identity theorem to determine the coefficients a_k.

Chapter 26

Riemann integration in one dimension

In this chapter, we will introduce a method to integrate functions. We will define the Riemann integral, but we cannot define the Riemann integral of all functions.

The main messages for this chapter are:

- Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.
- The fundamental theorem of calculus, mainly the part that when $F : [a, b] \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ satisfy $F' = f$ and f is bounded and Riemann integrable, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

The last statement gives a way to compute integrals in practice.

26.1 Riemann integrable functions and the Riemann integral

Definition 26.1.1. A *partition* P of an interval $[a, b]$ (with n intervals) is a subset $\{x_0, x_1, \dots, x_n\} \subset [a, b]$ such that $a = x_0 < x_1 < \dots < x_n = b$.

Definition 26.1.2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a, b]$. Then the upper sum of f with respect to P is defined as

$$U(P, f) := \sum_{k=1}^{n} M_k \Delta x_k$$

where $\Delta x_k := (x_k - x_{k-1})$ and

$$M_k := \sup_{x \in [x_{k-1}, x_k]} f(x)$$

Similarly, we define the lower sum of f with respect to P as

$$L(P, f) := \sum_{k=1}^{n} m_k \Delta x_k$$

where

$$m_k := \inf_{x \in [x_{k-1}, x_k]} f(x).$$

Definition 26.1.3. Let $\tilde{P}$ be a partition of $[a, b] \subset \mathbb{R}$. A partition P is called a *refinement* of $\tilde{P}$ if $\tilde{P} \subset P$.

If $\tilde{P}$ and $\tilde{Q}$ are two partitions of $[a, b]$. Then a partition P is called a *common refinement* of $\tilde{P}$ and $\tilde{Q}$ if P is both a refinement of $\tilde{P}$ and a refinement of $\tilde{Q}$.

Note that two partitions $\tilde{P}$ and $\tilde{Q}$ always have a common refinement P: For P one could just take $\tilde{P} \cup \tilde{Q}$.

Proposition 26.1.4. For every bounded $f : [a, b] \to \mathbb{R}$ and every partition P of $[a, b]$, we have

$$L(P, f) \leq U(P, f).$$

Proof. Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then

$$L(P, f) = \sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k = U(P, f)$$

because

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) \leq \sup_{x \in [x_{k-1}, x_k]} f(x) = M_k.$$

□

Definition 26.1.5. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We define the upper Darboux integral of f as

$$\overline{\int_a^b} f dx := \inf\{U(P, f) \mid P \text{ partition of } [a, b]\}$$

and the lower Darboux integral of f as

$$\underline{\int_a^b} f dx := \sup\{L(P, f) \mid P \text{ partition of } [a, b]\}$$

Proposition 26.1.6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then

$$\underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx.$$

We are most interested in those functions for which the upper Darboux integral agrees with the lower Darboux integral. We will call those functions *Riemann integrable*.

Definition 26.1.7 (Riemann integrability and the Riemann integral)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. We say f is *Riemann integrable* if

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx.$$

In this case we say that the *Riemann integral* of f equals this common value, i.e.

$$\int_a^b f dx := \overline{\int_a^b} f dx = \underline{\int_a^b} f dx.$$

Proposition 26.1.8 (Alternative characterization of Riemann integrability)**.** Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if

for all $\epsilon > 0$,
  there exists a partition P of $[a, b]$,
    $U(P, f) - L(P, f) < \epsilon$.

Proof. First suppose that f is Riemann integrable. We need to show that

for all $\epsilon > 0$,
  there exists a partition P of $[a, b]$,
    $U(P, f) - L(P, f) < \epsilon$.

Let $\epsilon > 0$. There exists a partition $\tilde{P}$ of $[a, b]$ such that

$$U(\tilde{P}, f) < \int_a^b f(x)dx + \epsilon/2.$$

Moreover, there exists a partition $\tilde{Q}$ of $[a, b]$ such that

$$L(\tilde{Q}, f) > \int_a^b f(x)dx - \epsilon/2.$$

Define the partition P of $[a,b]$ as $P := \tilde{P} \cup \tilde{Q}$. Then P is a common refinement of $\tilde{P}$ and $\tilde{Q}$, i.e. it is both a refinement of $\tilde{P}$ and it is a refinement of $\tilde{Q}$. Therefore,

$$\begin{aligned}\underline{\int_a^b} f(x)dx - \epsilon/2 &< L(\tilde{Q}, f)\\ &\leq L(P,f) \leq U(P,f) \leq U(\tilde{P}, f)\\ &< \overline{\int_a^b} f(x)dx + \epsilon/2\end{aligned}$$

It follows that

$$U(P,f) - L(P,f) < \epsilon.$$

Now suppose that

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists a partition } P \text{ of } [a,b],\\ &\qquad U(P,f) - L(P,f) < \epsilon.\end{aligned}$$

We need to show that f is Riemann integrable. Suppose not, then we may set

$$\epsilon := \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx > 0.$$

Then by assumption there exists a partition P such that

$$U(P,f) - L(P,f) < \epsilon.$$

But then

$$\epsilon = \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < U(P,f) - L(P,f) < \epsilon.$$

□

Definition 26.1.9. We denote the set of bounded, Riemann-integrable functions $f : [a,b] \to \mathbb{R}$ by $\mathcal{R}[a,b]$.

26.2 Sums, products of Riemann integrable functions

Proposition 26.2.1 ($\mathcal{R}[a,b]$ is a vector space)**.** Let $f, g : [a,b] \to \mathbb{R}$ be two bounded and Riemann-integrable functions. Then

i. The function $f+g$ is Riemann integrable and

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

ii. For every $\lambda \in \mathbb{R}$, the function λf is Riemann integrable and

$$\int_a^b (\lambda f(x))dx = \lambda \int_a^b f(x)dx.$$

Definition 26.2.2. If $f : [a,b] \to \mathbb{R}$ is bounded and Riemann integrable on $[a,b]$, then we define

$$\int_b^a f(x)dx := -\int_a^b f(x)dx.$$

Proposition 26.2.3 (Further properties of the Riemann integral)**.** Let $f, g : [a,b] \to \mathbb{R}$ be two bounded and Riemann integrable functions.

i. We have
$$\int_a^b 1dx = b-a$$

ii. Monotonicity: if for all $x \in [a,b]$ it holds that $f(x) \leq g(x)$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

iii. Restriction: If $z \in (a,b)$, then f is Riemann integrable on $[a,z]$

and on $[z, b]$ and

$$\int_a^b f(x)dx = \int_a^z f(x)dx + \int_z^b f(x)dx.$$

iv. Triangle inequality: The function $|f|$ is Riemann integrable on $[a, b]$ and we have the following version of the triangle inequality

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

v. The function fg is Riemann integrable on $[a, b]$.

26.3 Continuous functions are Riemann integrable

Proposition 26.3.1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof. We will use the alternative characterization of Riemann integrability, namely Proposition 26.1.8. We therefore need to show that

for all $\epsilon > 0$,
there exists a partition P of $[a, b]$,
$U(P, f) - L(P, f) < \epsilon$.

Let $\epsilon > 0$.

Because the closed interval $[a, b]$ is compact, the function f is uniformly continuous. Therefore, there exists a $\delta > 0$ such that for all $x, y \in [a, b]$, if $0 < |x - y| < \delta$ then

$$|f(x) - f(y)| < \frac{\epsilon}{2(b - a)}.$$

Now choose an $N \in \mathbb{N}$ such that $N > 1/\delta$. We define the partition

$$P := (x_0, x_1, \dots, x_{N+1})$$

by

$$x_i := a + i\frac{b-a}{N}.$$

Then

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{k=1}^{N} M_k \Delta x_k - \sum_{k=1}^{N} m_k \Delta x_k \\ &= \sum_{k=1}^{N} (M_k - m_k) \Delta x_k \end{aligned}$$

with

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$$

and

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x).$$

Therefore, for all $k \in \{1, \dots, N\}$,

$$0 \leq M_k - m_k \leq \frac{\epsilon}{2(b-a)}.$$

We find that

$$\begin{aligned} |U(P, f) - L(P, f)| &\leq \sum_{k=1}^{N} |M_k - m_k| \Delta x_k \\ &\leq \frac{\epsilon}{2(b-a)} \sum_{k=1}^{N} \Delta x_k \\ &= \frac{\epsilon}{2(b-a)} (b-a) \\ &< \epsilon. \end{aligned}$$

□

26.4 Fundamental theorem of calculus

Theorem 26.4.1 (Fundamental theorem of calculus)**.**

i. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the function

$$F : [a, b] \to \mathbb{R}$$

given by

$$F(x) := \int_a^x f(s)ds$$

is differentiable on (a, b) and for all $x \in (a, b)$

$$F'(x) = f(x).$$

ii. Let $F : [a, b] \to \mathbb{R}$ be an anti-derivative of a function $f : [a, b] \to \mathbb{R}$, i.e. for all $x \in (a, b)$,

$$F'(x) = f(x)$$

and suppose that f is bounded and Riemann integrable on $[a, b]$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. We will first show part (i).

We will show that the function F is differentiable on (a, b) and satisfies for all $c \in (a, b)$

$$F'(c) = f(c).$$

To do so, we define $\mathsf{Err}_c : (a, b) \to \mathbb{R}$ by

$$\mathsf{Err}_c(x) := F(x) - (F(c) + f(c)(x - c))$$

and we need to show that

$$\lim_{x \to c} \frac{\mathsf{Err}_c(x)}{|x - c|} = 0.$$

Let $\epsilon > 0$. Because f is continuous in c, there exists a $\delta > 0$ such that for all $x \in (a, b)$, it holds that if $0 < |x - c| < \delta$ then

$$|f(x) - f(c)| < \frac{\epsilon}{2}.$$

Choose such a δ.

Let $x \in (a, b)$ and assume $0 < |x - c| < \delta$. We have that

$$\begin{aligned}
|\mathsf{Err}_c(x)| &= |F(x) - F(c) - f(c)(x - c)| \\
&= \left| \int_c^x f(s) ds - f(c)(x - c) \right| \\
&= \left| \int_c^x (f(s) - f(c) dx \right| \\
&\leq \int_c^x |f(s) - f(c)| dx \\
&\leq \frac{\epsilon}{2} |x - c|
\end{aligned}$$

Therefore

$$\frac{|\mathsf{Err}_c(x)|}{|x - c|} < \epsilon.$$

We will now show part (ii) of the theorem.

Let $F : [a, b] \to \mathbb{R}$ be an anti-derivative of a function $f : [a, b] \to \mathbb{R}$ and assume that f is bounded and Riemann integrable.

We need to show that

$$\int_a^b f(x) dx = F(b) - F(a).$$

Let $P := \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. We can write $F(b) - F(a)$ as a telescoping sum

$$F(b) - F(a) = \sum_{k=1}^{n} \big(F(x_k) - F(x_{k-1}) \big)$$

and then use the Mean-Value Theorem to conclude that there are points $c_k \in (x_{k-1}, x_k)$ such that

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{n} \big(F(x_k) - F(x_{k-1})\big) \\ &= \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \end{aligned}$$

We now use the definition of

$$M_k := \sup_{x \in [x_{k-1}, x_k]} f(x)$$

to estimate

$$\begin{aligned} F(b) - F(a) &= \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} M_k \Delta x_k \\ &= U(P, f). \end{aligned}$$

It follows that

$$F(b) - F(a) \leq \overline{\int_a^b} f(x) dx$$

Similarly, we can prove that

$$\underline{\int_a^b} f(x) dx \leq F(b) - F(a)$$

Therefore,

$$\underline{\int_a^b} f(x) dx \leq F(b) - F(a) \leq \overline{\int_a^b} f(x) dx$$

Because f is integrable the left- and right-hand side of this inequality have the same value so that we can conclude that

$$F(b) - F(a) = \int_a^b f(x)dx.$$

□

26.5 Exercises

Exercise 26.5.1. Let a, b and c be three real numbers, with $a < b < c$ and let $g : [a, b] \to \mathbb{R}$ and $h : [b, c] \to \mathbb{R}$ be two bounded and Riemann-integrable functions. Let $f : [a, c] \to \mathbb{R}$ be such that for all $x \in (a, b)$

$$f(x) = g(x)$$

and for all $x \in (b, c)$,

$$f(x) = h(x).$$

Show that f is Riemann integrable. (Note that the values of f in the points a, b and c are not specified.)

Exercise 26.5.2. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2 + x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ -3 & \text{if } x \in \mathbb{Q}. \end{cases}$$

i. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0, 1]$. Show that

$$U(P, f) - L(P, f) \geq 5.$$

ii. Show that f is not Riemann integrable.

Exercise 26.5.3. Compute the following integral, carefully quoting the theorems that you use in your computation

$$\int_{-1}^{\sqrt{3}} \frac{1}{1 + x^2} dx.$$

Exercise 26.5.4. Let a, b and c be three real numbers, with $a < b < c$. Assume $f : [a, c] \to \mathbb{R}$ is bounded, and assume f is Riemann integrable on $[a, b]$ and Riemann integrable on $[b, c]$.

Prove that f is Riemann integrable on $[a, c]$.

Exercise 26.5.5. Suppose $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are bounded. Assume there exists an $n \in \mathbb{N}$ and that there are points $y_1, \ldots, y_n$ in $[a, b]$ such that for all $x \in [a, b] \setminus \{y_1, \ldots, y_n\}$

$$f(x) = g(x).$$

Hint: First show that for all $y \in [a, b]$, the function

$$\mathbf{1}_y : [a, b] \to \mathbb{R}$$

defined as

$$\mathbf{1}_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

is Riemann integrable.

Chapter 27

Riemann integration in multiple dimensions

In this chapter we define Riemann integration for functions defined on $\mathbb{R}^d$.

27.1 Partitions in multiple dimensions

Definition 27.1.1. By a *closed rectangle* in $\mathbb{R}^d$ we mean a set R of the form
$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

Definition 27.1.2 (Partition of a rectangle)**.** Let

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

be a closed rectangle. By a partition Q of R we mean a Cartesian product
$$Q = P^1 \times \cdots \times P^d$$
where for $i \in \{1, \ldots, d\}$, the partition $P^i = \{x^i_1, \ldots, x^i_{n_i}\}$ is a partition of $[a_i, b_i]$.

27.2 Riemann integral on rectangles in $\mathbb{R}^d$

Riemann integrability of functions defined on a rectangle $R \subset \mathbb{R}^d$ is defined completely analogously to the Riemann integral for functions defined on an interval $[a, b] \subset \mathbb{R}$.

Definition 27.2.1. Let $R \subset \mathbb{R}^d$ be a closed rectangle,

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

let $f : R \to \mathbb{R}$ be a bounded function and let

$$Q = P^1 \times \cdots \times P^d$$

be a partition of R, where for every $j \in \{1, \cdots, d\}$,

$$P^j = \{x_0^j, x_1^j, \ldots, x_{n_j}^j\}$$

is a partition of $[a_j, b_j]$.
Then the *upper sum* of f with respect to Q is defined as

$$U(Q, f) := \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} M_{k_1,\cdots,k_d} \Delta x_{k_1}^1 \cdots \Delta x_{k_d}^d$$

where $\Delta x_k^j := (x_k^j - x_{k-1}^j)$ and

$$M_{k_1,\ldots,k_d} := \sup_{x \in [x_{k_1-1}^1, x_{k_1}^1] \times \cdots \times [x_{k_d-1}^d, x_{k_d}^d]} f(x)$$

Similarly, we define the *lower sum* of f with respect to Q as

$$L(Q, f) := \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} m_{k_1,\cdots,k_d} \Delta x_{k_1}^1 \cdots \Delta x_{k_d}^d$$

where
$$m_{k_1,\ldots,k_d} := \inf_{x \in [x^1_{k_1-1}, x^1_{k_1}] \times \cdots \times [x^d_{k_d-1}, x^d_{k_d}]} f(x).$$

Definition 27.2.2. Let $R \subset \mathbb{R}^d$ be a closed rectangle, and let $f : R \to \mathbb{R}$ be a bounded function. We define the upper Darboux integral of f as

$$\overline{\int_R} f dx := \inf\{U(P, f) \mid P \text{ partition of } R\}$$

and the lower Darboux integral of f as

$$\underline{\int_R} f dx := \sup\{L(P, f) \mid P \text{ partition of } R\}$$

Definition 27.2.3. Let $R \subset \mathbb{R}^d$ be a closed rectangle and let $f : R \to \mathbb{R}$ be a bounded function. We say f is *Riemann integrable* if

$$\overline{\int_R} f dx = \underline{\int_R} f dx.$$

In this case we say that the *Riemann integral* of f equals this common value, i.e.
$$\int_R f dx := \overline{\int_R} f dx = \underline{\int_R} f dx.$$

Proposition 27.2.4 (Alternative characterization of Riemann integrability)**.** Let $R \subset \mathbb{R}^d$ be a closed rectangle and let $f : R \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if

$$\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists a partition } P \text{ of } R, \\
&\quad\quad U(P, f) - L(P, f) < \epsilon.
\end{aligned}$$

27.3 Properties of the multi-dimensional Riemann integral

Just as in the one-dimensional case, the set of all Riemann integrable functions on a rectangle R forms a vector space, and the integral is linear.

Proposition 27.3.1. Let R be a closed rectangle and let $f, g : R \to \mathbb{R}$ be bounded and Riemann integrable on R. Then

i. the function $f + g$ is Riemann integrable on R and

$$\int_R (f(x) + g(x))dx = \int_R f(x)dx + \int_R g(x)dx$$

ii. for all $\lambda \in \mathbb{R}$, the function λf is Riemann integrable on R and

$$\int_R \lambda f(x)dx = \lambda \int_R f(x)dx.$$

Proposition 27.3.2. Let $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ be a closed rectangle in $\mathbb{R}^d$ and let $f, g : R \to \mathbb{R}$ be bounded and Riemann integrable on R. Then

i. (volume)

$$\int_R 1dx = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d) =: \mathrm{Vol}(R)$$

ii. (monotonicity) if for all $x \in R$, $f(x) \leq g(x)$ then

$$\int_R f(x)dx \leq \int_R g(x)dx$$

iii. (triangle inequality) the function $|f|$ is Riemann integrable on R and

$$\left| \int_R f(x)dx \right| \leq \int_R |f(x)|dx.$$

iv. (additivity of domain) if Q is a closed rectangle contained in R, then f is integrable on Q. Moreover, if $Q_1, \ldots, Q_N$ are finitely many closed rectangles,

- if their interiors are disjoint, i.e. $\text{int } Q_i \cap \text{int } Q_j = \emptyset$ if $i \neq j$ and
- if the union of Q_i's equals R, i.e.

$$\bigcup_{i=1}^{N} Q_i = R,$$

then

$$\int_R f(x)dx = \sum_{i=1}^{N} \int_{Q_i} f(x)dx.$$

27.4 Continuous functions are Riemann integrable

Just as with integration in one dimension, *if* the function f is continuous on R, *then* it is Riemann integrable on R.

Theorem 27.4.1. Let $R \subset \mathbb{R}^d$ be a closed rectangle, and let $f : R \to \mathbb{R}$ be a continuous function. Then f is bounded and Riemann integrable on R.

The proof is practically identical to the proof in the one-dimensional case.

27.5 Fubini's theorem

To effectively compute the value of integrals, we need some more tools. In fact, if possible, we would like to use the fundamental theorem of calculus in the computation. That theorem is however a statement about integrals in one dimension. We therefore need a way to use one-dimensional integrals in the computation of multi-dimensional integrals. The following theorem, called Fubini's theorem, provides such a way.

Theorem 27.5.1 (Fubini)**.** Let $R = A \times B$ be a rectangle in $\mathbb{R}^{d+m}$. Let $f : R \to \mathbb{R}$ be bounded and Riemann integrable on R, and suppose for every $x \in \mathbb{R}^d$ the function $h_x : \mathbb{R}^m \to \mathbb{R}$ defined by

$$h_x(y) := f(x, y)$$

is Riemann integrable. Then the function $F : \mathbb{R}^d \to \mathbb{R}$ given by

$$F(x) := \int_B f(x, y) dy$$

is Riemann integrable and

$$\int_R f(z) dz = \int_A \left(\int_B f(x, y) dy \right) dx$$

27.6 The (topological) boundary of a set

The topological boundary of a set in a metric space is defined as those points that are neither in the interior of the set, nor in the interior of the complement of the set. For the subsets of $(\mathbb{R}^d, \|\cdot\|_2)$, this comes down to the following definition.

Definition 27.6.1 (Topological boundary)**.** Let E be a subset of the normed vector space $(\mathbb{R}^d, \|\cdot\|)$. The boundary of E is defined as

$$\partial E := \mathbb{R}^d \setminus \big((\text{int } E) \cup (\text{int}(\mathbb{R}^d \setminus E)) \big).$$

27.7 Jordan content

Definition 27.7.1 (Volume of a rectangle)**.** Let

$$R = [a_1, b_1] \times \cdots [a_d, b_d] \subset \mathbb{R}^d$$

be a rectangle. Then the volume of R is defined as

$$\mathrm{Vol}(R) := (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d).$$

Definition 27.7.2. We say that a closed rectangle $R \subset \mathbb{R}^d$ is a *cube* if all sides have the same length, i.e. if

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d]$$

then for all $i, j \in \{1, \dots, d\}$,

$$b_i - a_i = b_j - a_j.$$

Definition 27.7.3. We say that a subset $S \subset \mathbb{R}^d$ has *Jordan content zero* if

for all $\epsilon > 0$,
there exists $N \in \mathbb{N}$,
there exist rectangles $R_1, \dots, R_N$,

$$S \subset \bigcup_{i=1}^{N} R_i \qquad \text{and} \qquad \sum_{i=1}^{N} \mathrm{Vol}\, R_i < \epsilon.$$

Lemma 27.7.4. Suppose a set $S \subset \mathbb{R}^d$ has Jordan content zero. Then for all $\epsilon > 0$, there exists an $M \in \mathbb{N}$ and cubes $Q_1, \dots, Q_M$, such that

$$S \subset \bigcup_{i=1}^{M} Q_i \qquad \text{and} \qquad \sum_{i=1}^{M} \mathrm{Vol}\, Q_i < \epsilon.$$

Proposition 27.7.5. Let $S \subset \mathbb{R}^d$ be a subset with Jordan content zero and let $F : S \to \mathbb{R}^d$ be Lipschitz. Then $F(S)$ has Jordan content zero.

Proposition 27.7.6. Let E be a bounded subset on $\mathbb{R}^d$, and let $F : E \to \mathbb{R}^{d+m}$ be Lipschitz where $m \geq 1$. Then $F(E)$, as a subset of $\mathbb{R}^{d+m}$, has Jordan content zero.

27.8 Integration over general domains

Definition 27.8.1 (Integration over bounded subsets)**.** Let E be a bounded subset of $\mathbb{R}^d$. We say that a function $f : E \to \mathbb{R}$ is integrable on E if, with R some rectangle in $\mathbb{R}^d$ containing E, the function $f_E : R \to \mathbb{R}$ defined by

$$f_E(x) := \begin{cases} f(x) & \text{if } x \in E, \\ 0 & \text{if } x \notin E \end{cases}$$

is integrable on R. Moreover, we define

$$\int_E f(x)dx := \int_R f_E(x)dx$$

The above definition is actually independent of the choice of rectangle R that contains E.

Definition 27.8.2. Let $E \subset \mathbb{R}^d$. We say that E is a *Jordan set* if the topological boundary ∂E of E has Jordan content zero.

Proposition 27.8.3. Let $R \subset \mathbb{R}^d$ be a closed rectangle and assume that $E \subset R$ is a bounded subset of $\mathbb{R}^d$ and assume E is a Jordan set. Let f be a bounded and Riemann integrable function on R. Then f is integrable on E.

27.9 The volume of bounded sets

Definition 27.9.1 (Characteristic function of a set)**.** Let $E \subset \mathbb{R}^d$. The characteristic function of E is the function $\mathbf{1}_E : \mathbb{R}^d \to \mathbb{R}$ given by

$$\mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Definition 27.9.2. Let E be a bounded set such that the characteristic function $\mathbf{1}_E : \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable. Then the volume of E is defined as

$$\mathrm{Vol}(E) := \int_E 1 dx.$$

27.10 Exercises

Exercise 27.10.1. Let $S_1, \ldots, S_m \subset \mathbb{R}^d$ have Jordan content zero. Show that the union

$$\bigcup_{i=1}^m S_i$$

also has Jordan content zero.

Exercise 27.10.2. Show that the unit ball $B(0,1)$ in $\mathbb{R}^2$ is a Jordan set. **Hint:** Make a parametrization of the boundary of the unit ball

Exercise 27.10.3. Let $E \subset \mathbb{R}^2$ be the subset of $[0,1]^2$ above the curve $y = x^4$, and to the right of the curve $x = \sin(y\pi)^3$. Show that E is a Jordan set.

Exercise 27.10.4. Let $E \subset \mathbb{R}^2$ be the subset of $[0,1]^2$ above the curve $y = x^4$, and to the right of the curve $y = \sqrt[3]{x}$. Show that E is a Jordan set.

Chapter 28

Change-of-variables Theorem

28.1 The Change-of-variables Theorem

Theorem 28.1.1 (Change-of-variables theorem)**.** Let $\Omega \subset \mathbb{R}^d$ be open. Let $E \subset \Omega$ be a Jordan set such that also its closure $\bar{E} \subset \Omega$. Let $\Phi : \Omega \to \mathbb{R}^d$ be continuously differentiable and injective. Assume that also the inverse function Φ^{-1} is differentiable.

Assume that $f : \Phi(E) \to \mathbb{R}$ is integrable on $\Phi(E)$ and assume that $f \circ \Phi$ is integrable on E.

Then

$$\int_{\Phi(E)} f(x)dx = \int_E f(\Phi(y))\big|\det([D\Phi]_y)\big|dy.$$

We will review in this chapter a few standard transformations, and therefore standard applications of the change-of-variables theorem. These involve transformations from and to polar, cylindrical and spherical coordinates. It is important to know these transformations, and the corresponding determinants of the Jacobians, by heart.

28.2 Polar coordinates

The transformation is given by the function

$$\Phi_{\mathsf{pol}} : (0, \infty) \times (0, 2\pi) \to \mathbb{R}^2$$

defined by

$$\Phi_{\mathsf{pol}}(r, \phi) := (r \cos \phi, r \sin \phi)$$

Here,

$$\det [D\Phi_{\mathsf{pol}}]_{(r,\phi)} = r$$

In many situations in which one would like to change the polar coordinates to compute an integral, the Change-of-variables Theorem does not directly apply. The transformation from polar to Cartesian coordinates is so nice, however, that one can obtain a statement that can be applied more conveniently.

A subset $E \subset (0, \infty) \times (0, 2\pi)$ is a Jordan set if and only if $\Phi_{\mathsf{pol}}(E)$ is a Jordan set. Moreover, if one of these holds, a function $f : \mathbb{R}^2 \to \mathbb{R}$ is Riemann integrable on $\Phi_{\mathsf{pol}}(E)$ if and only if $f \circ \Phi_{\mathsf{pol}}$ is Riemann integrable on E and

$$\int_{\Phi_{\mathsf{pol}}(E)} f(x)dx = \int_E f(\Phi_{\mathsf{pol}}(r, \phi))rdrd\phi$$

28.3 Cylindrical coordinates

The transformation is given by the function

$$\Phi_{\mathsf{cyl}} : (0, \infty) \times (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$$

defined by

$$\Phi_{\mathsf{cyl}}(r, \phi, z) := (r \cos \phi, r \sin \phi, z)$$

Here,

$$\det [D\Phi_{\mathsf{cyl}}]_{(r,\phi,z)} = r$$

A subset $E \subset (0, \infty) \times (0, 2\pi) \times \mathbb{R}$ is a Jordan set if and only if $\Phi_{\mathsf{cyl}}(E)$ is a Jordan set. Moreover, if one of these holds, a function $f : \mathbb{R}^3 \to \mathbb{R}$ is

Riemann integrable on $\Phi_{\mathsf{cyl}}(E)$ if and only if $f \circ \Phi_{\mathsf{cyl}}$ is Riemann integrable on E and

$$\int_{\Phi_{\mathsf{cyl}}(E)} f(x)dx = \int_E f(\Phi_{\mathsf{cyl}}(r,\phi,z))rdrd\phi dz$$

28.4 Spherical coordinates

The transformation is given by the function

$$\Phi_{\mathsf{sph}} : (0,\infty) \times (0,2\pi) \times (0,\pi) \to \mathbb{R}^3$$

given by

$$\Phi_{\mathsf{sph}}(\rho,\phi,\theta) := (\rho\cos\phi\sin\theta, \rho\sin\phi\sin\theta, \rho\cos\theta)$$

Here,

$$\det\left[D\Phi_{\mathsf{sph}}\right]_{(\rho,\phi,\theta)} = \rho^2\sin\theta.$$

A subset $E \subset (0,\infty) \times (0,2\pi) \times (0,\pi)$ is a Jordan set if and only if $\Phi_{\mathsf{sph}}(E)$ is a Jordan set. Moreover, if one of these holds, a function $f : \mathbb{R}^3 \to \mathbb{R}$ is Riemann integrable on $\Phi_{\mathsf{sph}}(E)$ if and only if $f \circ \Phi_{\mathsf{sph}}(E)$ is Riemann integrable on E and

$$\int_{\Phi_{\mathsf{sph}}(E)} f(x)dx = \int_E f(\Phi_{\mathsf{sph}}(r,\phi,\theta))r^2\sin\theta drd\phi d\theta.$$

28.5 Exercises

Exercise 28.5.1. Determine

$$\int_0^{\pi/2}\int_0^y \frac{\sin(y)}{\sqrt{4-\sin^2(x)}}dxdy$$

Exercise 28.5.2. Let K be the following subset of $\mathbb{R}^3$

$$K := \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le 1,\ x^2+y^2 < z\}$$

Determine

$$\int_K \exp(-(x^2+y^2))dxdydz.$$

Exercise 28.5.3. Let K be the subset of those points in $\mathbb{R}^3$ that are inside the ball around the origin of radius 4 but outside the cylinder around the z-axis of radius 1, i.e.

$$K := B(0,4) \setminus \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 < 1\}$$

Determine the volume of K.

Exercise 28.5.4. The center of mass of a Jordan set $K \subset \mathbb{R}^d$ is defined as

$$\text{c.m.}(K) := \frac{1}{\text{Vol}(K)} \left(\int_K x_1 dx, \int_K x_2 dx, \cdots, \int_K x_d dx \right)$$

Let $E \subset \mathbb{R}^d$ be a Jordan set. Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be linear and invertible. Show that the center of mass of $\Phi(E)$ is equal to

$$\text{c.m.}(\Phi(E)) = \Phi(\text{c.m.}(E)).$$

Exercise 28.5.5.

i. Let $e_1, \ldots, e_d$ be the standard basis in $\mathbb{R}^d$. Consider the simplex

$$\mathcal{S} := \{x \in \mathbb{R}^d \mid x_1 + \cdots + x_d \leq 1 \text{ and for all } i \in \{1, \ldots, d\}, 0 \leq x_i \leq 1, \}$$

Compute the volume of $\mathcal{S}$.

ii. Now let $v_1, \ldots, v_d$ be a basis in $\mathbb{R}^d$. Let M denote the $d \times d$ matrix for which the column vectors are the vectors $v_1, \ldots, v_d$. Let the map $\iota : \mathbb{R}^d \to \mathbb{R}^d$ be given by

$$\iota(x) = x_1 v_1 + \cdots + x_d v_d.$$

Show that

$$\text{Vol}(\iota(\mathcal{S})) = \frac{1}{d!} |\det M|.$$

Appendix A

Best practices

i. Always start by writing down what is **given** and what you need **to show**.

ii. To directly prove a statement

> for all $a \in A$,
> $(\ldots)$

you first introduce $a \in A$ by writing

> Let $a \in A$.

and then you continue to prove $(\ldots)$.

iii. To directly prove a statement

> there exists $a \in A$,
> $(\ldots)$

you make a choice for a and write

> Choose $a := \dots$

and then you continue to prove $(\dots)$.

iv. If you need to show a statement of the form

> if A then B

start with writing

> Assume A.

and continue showing B.

v. At the end of a proof in analysis, you often need to show an (in)equality. You can do so by chaining several (in)equalities together, of course making sure that they all hold.

vi. (Backwards reasoning.) Sometimes you need to show a statement B, but it would follow directly from another statement A. In that case you could write

> It suffices to show A.

and continue showing A.

vii. (Forward reasoning.) Perhaps more often than *backwards reasoning* you would like to apply *forward reasoning*. You usually do so in the middle of a proof, stating a new fact that can be derived from your earlier conclusions.

> It holds that A.

If you want to use a theorem (or lemma, proposition etc.), first explicitly check all the assumptions, then afterwards you can use the conclusion of the theorem. A template is:

> ... *check conditions of theorem* ...
> Therefore, by Theorem *(insert reference to theorem)*
> it holds that (*theorem conclusion*).

viii. If you know that a for-all statement such as

$$\begin{aligned}&\text{for all } a \in A,\\ &\quad (\dots)\end{aligned} \tag{A.0.1}$$

holds, you can **use** it as follows

> Choose $a := \dots$ in $(A.0.1)$. Then $(\dots)$.

ix. If you know that a there-exists statement such as

$$\begin{aligned}&\text{there exists } a \in A,\\ &\quad (\dots)\end{aligned} \tag{A.0.2}$$

holds, you can **use** it as follows

> Obtain an $a \in A$ according to $(A.0.2)$

or as

> Obtain an $a \in A$ such that $(\dots)$ according to $(A.0.2)$

or just

> Obtain such an $a \in A$.

x. To prove a statement

> $(\dots)$

by contradiction, you can use the following template:

> We argue by contradiction. Suppose $\neg(\dots)$.
>
> $\dots$ *derivation that leads to a contradiction* $\dots$
>
> Contradiction. We conclude that $(\dots)$ holds.

xi. Sometimes you need to make a case distinction: you might for instance want to argue differently if a real number is strictly negative or positive. A template for a case distinction is as follows.

> Case A.
>
> $\dots$*proof in Case A*$\dots$
>
> Case B.
>
> $\dots$*proof in Case B*$\dots$
>
> Case C.
>
> $\dots$*proof in Case C*$\dots$
>
> *etc*$\dots$

Make sure to really cover all possible cases.

xii. You can use natural induction to show a statement

> for all $n \in \mathbb{N}$,
>
> $P(n)$

where $P(n)$ is a statement depending on n. The template is as follows:

> We use induction on $n \in \mathbb{N}$.
> We first show the base case, i.e. that $P(0)$ holds.
> ... *insert here a proof of* $P(0)$ ...
> We now show the induction step.
> Let $k \in \mathbb{N}$ and assume that $P(k)$ holds.
> We need to show that $P(k+1)$ holds.
> ... *insert here a proof of* $P(k+1)$ ...

xiii. Especially when constructing subsequences, we will often need *inductive definitions* of sequences. If X is a set, we might want to inductively define a sequence $f : \mathbb{N} \to X$. We can use the following template for this.

> We will inductively define a sequence $f : \mathbb{N} \to X$.
> ... *possible auxiliary derivations* ...
> Define $f(0) := \ldots$
>
> Let $k \in \mathbb{N}$ and assume $f(0), \ldots, f(k)$ are defined .
> ... *possible auxiliary derivations* ...
> Define $f(k+1) := \ldots$

xiv. Make sure that every variable that you are using is defined. In particular:

- After writing a sentence:

  > for all $\epsilon > 0, \quad \ldots$

  the variable ϵ is not defined, and you cannot refer to it.
- After writing the sentence

  > there exists $N \in \mathbb{N}, \quad \ldots$

the variable N is not defined, and you cannot refer to it. To use it in the rest of a proof, you can follow up with

> Choose such an $N \in \mathbb{N}$.

See also item (ix) of this best-practices list.

xv. Beware that $(A \implies B)$ means "if A then B". Importantly, it **does not mean** "A holds therefore B holds". Because this goes wrong very often, I recommend to use

> if ... then ...

in your proof rather than implication symbols $\Rightarrow$ and $\Leftrightarrow$.

xvi. If the statement that you need to show is an "if and only if" statement, show the "if" and "only if" statements separately.

xvii. Indicate whether a statement that you write down is a statement you want to show, or whether it is a statement that you assume, or whether it is a consequence of your earlier derivations.

xviii. Care about your presentation of the proof.

xix. At several times, remind the reader (and yourself) of what you need to show at that stage.

xx. If you hand-write your proof, make sure that you use your best handwriting.

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