2IT80 Discrete Structures

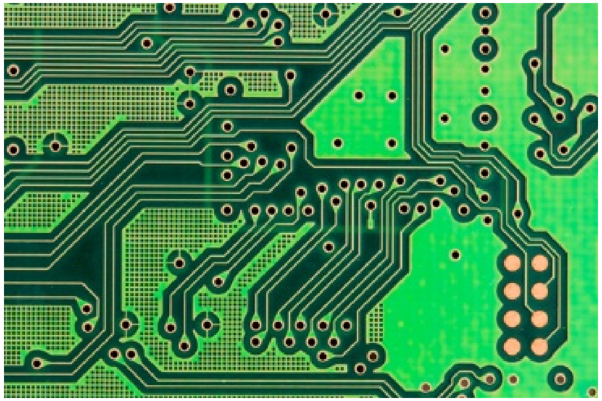
2023-24 Q2

Lecture 7: Graphs I

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Graphs

Many problems can be captured by some set of points and the connections between them. This implicit structure may help us solve these problems.



Graph

A graph G is an ordered pair (V, E), where

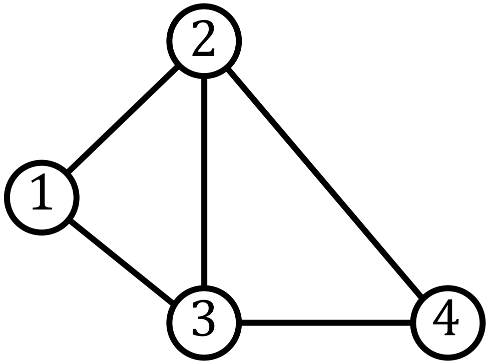
- V is a set of elements, called vertices.
- E a set of 2-element subsets of V, called edges

The degree of a vertex is equal to the number edges it is part of.

Vertices $v, v' \in V$ are adjacent when $\{v, v'\} \in E$. We say v' is a neighbor of v (and v a neighbor of v').

Example:
$V = \{1, 2, 3, 4\}$
$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$
Degree of vertex 2 is three.

Vertex 2 and vertex 3 are adjacent.



Examples

(Of things that are often modeled as a graph.)

- Social network:
  Vertices: profiles
  Edges: $\{a, b\}$ if profiles a and b are friends

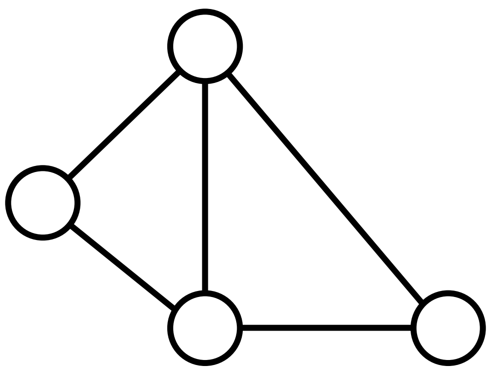
- The internet:
  Vertices: Computers, phones, routers, servers, etc...
  Edges: $\{a, b\}$ if a has a direct connection to b

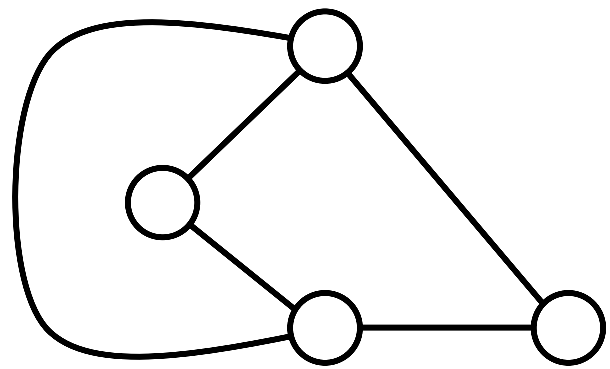
- Software architecture
  Vertices: Components
  Edges: $\{a, b\}$ if a interfaces with b

In this course: mainly consider the mathematical construct.

Drawing graphs

Here we use ○ for a vertex and connected vertices of edges by drawing lines (or arcs).



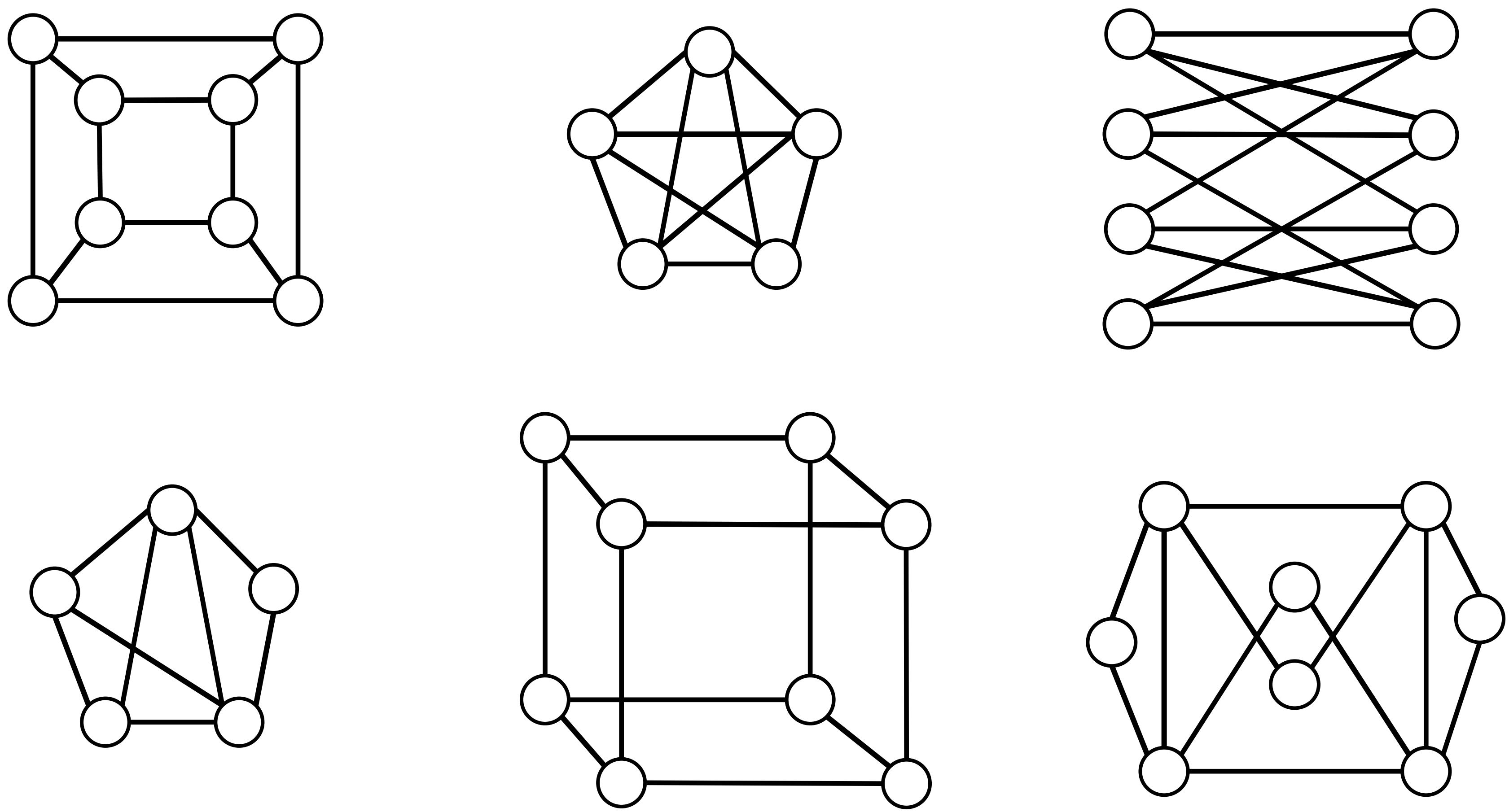


More formal definition of a drawing when we talk planar graphs.

Graph drawings

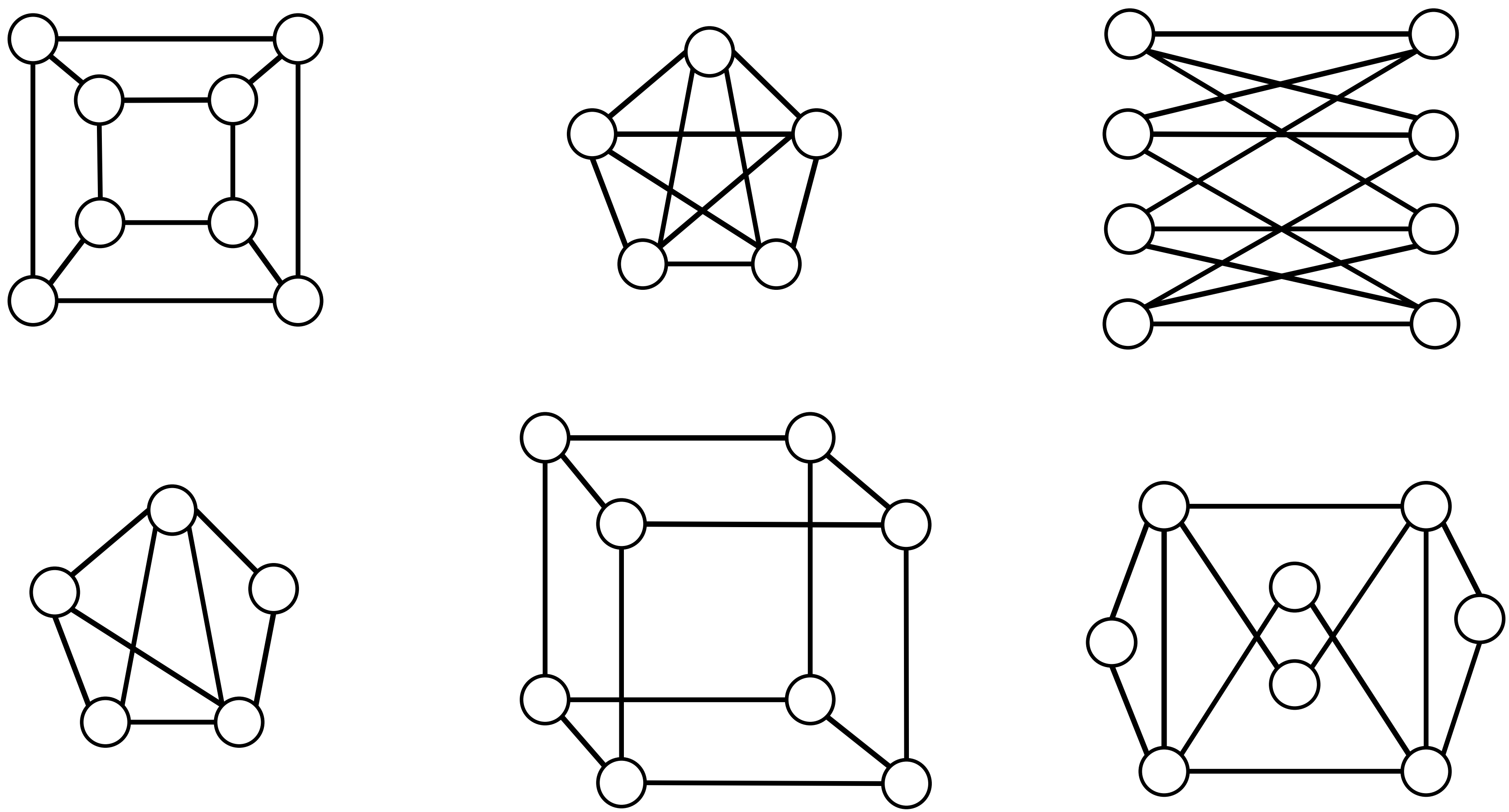
A graph defines only connectivity, but we can draw it.

Planar graphs *can* be drawn in the plane without edge crossings.



Graph drawings

Differently drawn graphs, may actually have the same structure.

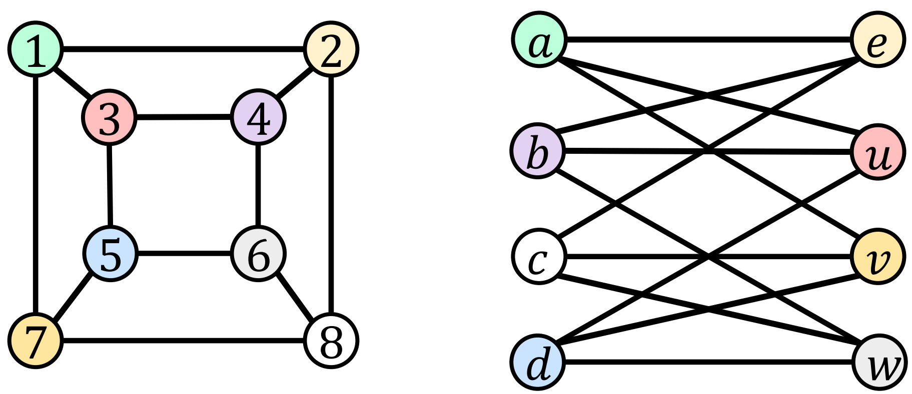


Graph isomorphism

Two graphs $G = (V, E)$ and $G' = (V', E')$ are called isomorphic if a bijection $f : V \to V'$ exists such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$ holds for all $x, y \in V$, $x \neq y$.

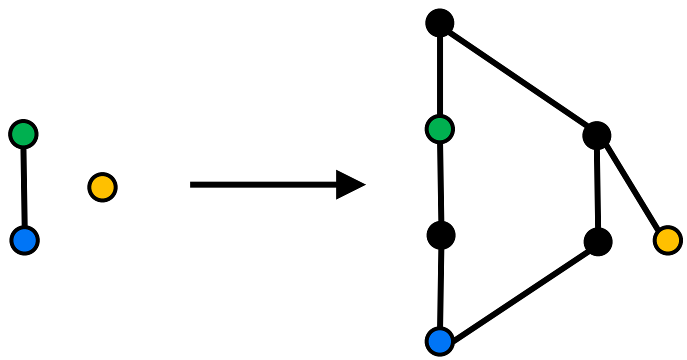
Such an f is called an isomorphism of the graphs G and G'. We indicate that two graphs G and G' are isomorphic by $G \cong G'$.

Example:
$f : 1 \mapsto a; 2 \mapsto e; 3 \mapsto u; 4 \mapsto b; 5 \mapsto d; 6 \mapsto w; 7 \mapsto v; 8 \mapsto c$



Reminder: Embedding

Let $(X, \preccurlyeq)$ and $(X', \preccurlyeq')$ be ordered sets. A mapping $f : X \to X'$ is called an embedding of $(X, \preccurlyeq)$ into $(X', \preccurlyeq')$ if the following conditions hold:
(i) f is injective;
(ii) $f(x) \preccurlyeq' f(y)$ if and only if $x \preccurlyeq y$.

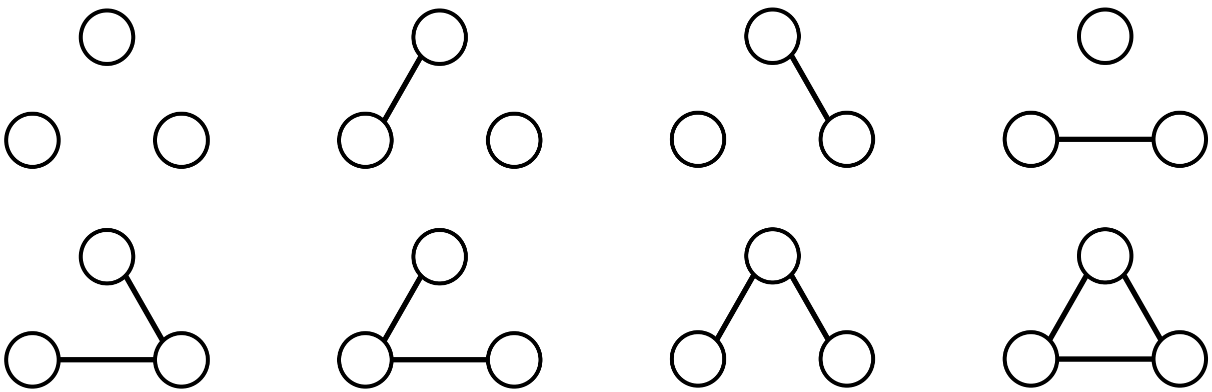


If an embedding is surjective, then it is an isomorphism.

Counting graphs

Let $V = \{1, 2, \ldots, n\}$. How many graphs can we make?

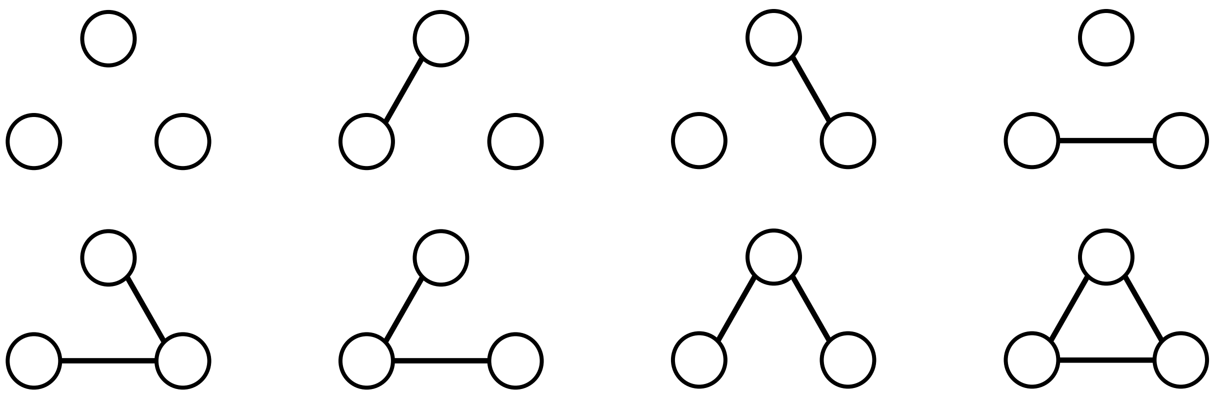
This depends only on the choice of edges. There are $\binom{n}{2}$ possible edges we can choose. Thus we get $2^{\binom{n}{2}}$ possible graphs.



Counting non-isomorphic graphs

Let $V = \{1, 2, \dots, n\}$. How many graphs can we make?

This depends only on the choice of edges. There are $\binom{n}{2}$ possible edges we can choose. Thus we get $2^{\binom{n}{2}}$ possible graphs.



How many pairwise non-isomorphic graphs can we make?

Non-isomorphic graphs

Question: How many classes of the equivalence relation $\cong$ on the set of all graphs with vertex set $V = \{1, 2,...,n\}$ exist?

Proof sketch:

Upperbound:

At most $2^{\binom{n}{2}}$ graphs, implies at most $2^{\binom{n}{2}}$ equivalence classes.

Lowerbound:

If two graphs are isomorphic then there exists a bijection between their vertices.

The total number of possible bijections is $n!$, so each equivalence class has at most $n!$ graphs.

There are at least $\frac{2^{\binom{n}{2}}}{n!}$ equivalence classes. We can show that this function does not grow much slower than $2^{\binom{n}{2}}$.

Non-isomorphic graphs

More formal derivation for the lower bound

Define E^* as the set of equivalence classes for the set G^* of all graphs on V under the isomorphism relation $\cong$.
Then each $E \in E^*$ is an equivalence class that contains graphs that are all isomorphic to each other.

Consider an $E \in E^*$ and a graph $G \in E$. For each graph $G' \in E$ there is a bijection from $V(G)$ to $V(G')$ since G and G' are isomorphic.
There are at most $n!$ bijections between sets of n elements, so there are at most $n!$ graphs in any equivalence class E.

The equivalence classes partition the set G^* of graphs.
So $|G| = \sum_{E \in E^*} |E| \leq \sum_{E \in E^*} n! = |E^*| * n!$

So $|G| \leq |E^*| * n! \Rightarrow |E^*| \geq |G^*| / n! = 2^{\binom{n}{2}} / n!$

Subgraphs

(Induced) subgraph

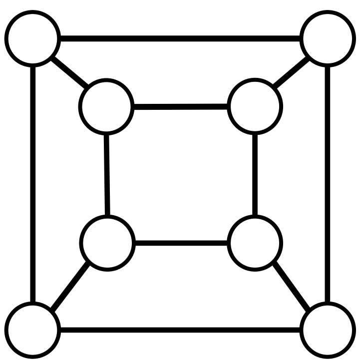
Let G and G' be graphs.
G is a subgraph of G' if

$$V(G) \subseteq V(G') \text{ and } E(G) \subseteq E(G') \cap \binom{V(G)}{2}.$$

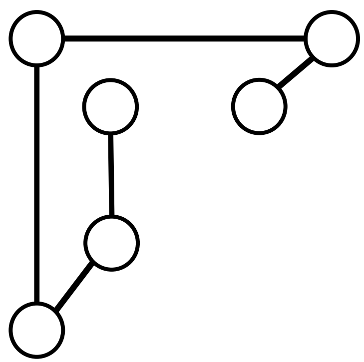
G is an induced subgraph of G' if

$$V(G) \subseteq V(G') \text{ and } E(G) = E(G') \cap \binom{V(G)}{2}.$$

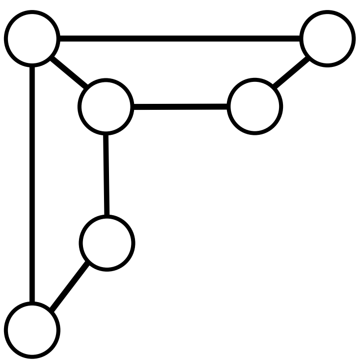
Example:



Graph G



Subgraph of G



Induced subgraph of G

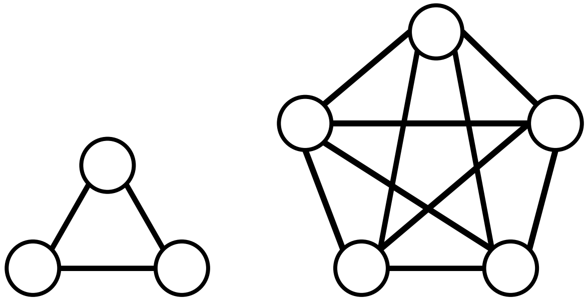
Common graph

Complete graph K_n

$V = \{1,2,\dots,n\}$

$E = \binom{V}{2}$

$|E| = \frac{n\cdot(n-1)}{2}$



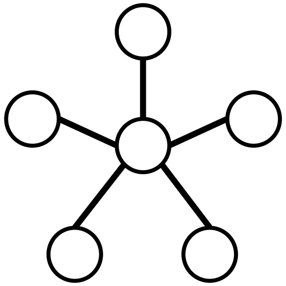
K_3 K_5

Star graph S_n $(K_{1,n})$

$V = \{u_1\} \cup \{v_1, v_2, \dots, v_n\}$

$E = \left\{\{u_1, v_j\}: j = 1,2,\dots,n\right\}$

$|E| = n$



S_5

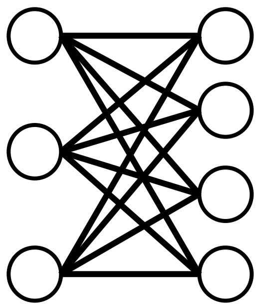
Common graphs

Complete bipartite graph $K_{n,m}$

$V = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_m\}$

$E = \left\{\{u_i, v_j\}: i = 1, 2, \dots, n; j = 1,2, \dots, m\right\}$

$|E| = n \cdot m$



$K_{3,4}$

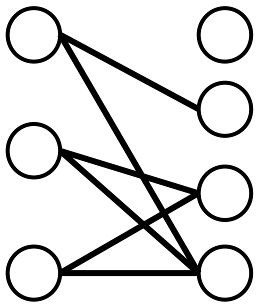
Bipartite graph (family of graphs)

$V = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_m\}$

$E^* = \left\{\{u_i, v_j\}: i = 1, 2, \dots, n; j = 1,2, \dots, m\right\}$

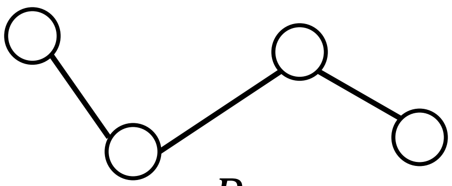
$E \subseteq E^*$

$|E| \leq n \cdot m$



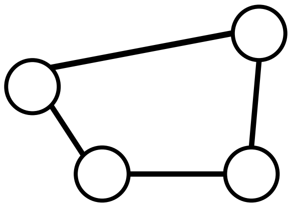
Common graphs

Path P_n
$V = \{0,1,2,\dots,n\}$
$E = \{\{i, i+1\}: i = 0,1,2,\dots,n-1\}$
$|E| = n$



P_3

Cycle C_n
$V = \{1,2,\dots,n\}$
$E = \{\{i, i+1\}: i = 1,2,\dots,n-1\} \cup \{\{1,n\}\}$
$|E| = n$



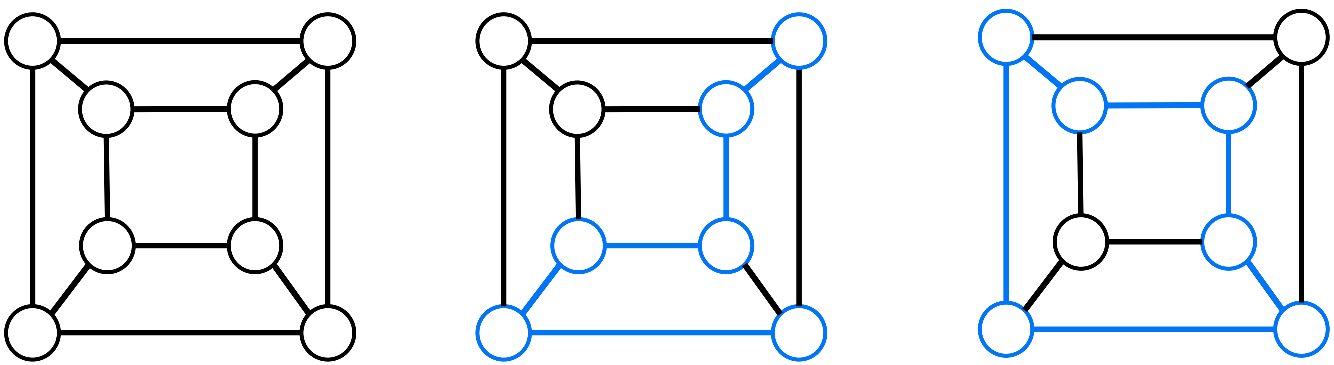
C_4

Paths and cycles

A path in a graph G is a subgraph of G that is isomorphic to P_t, for some nonnegative integer t.

A cycle in a graph G is a subgraph of G that is isomorphic to C_t, for any integer $t \geq 3$. (Also known as a circuit.)

Example:



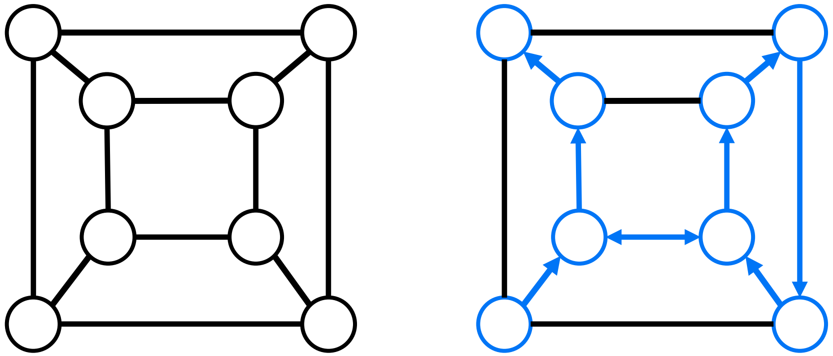
Graph G Path in G Cycle in G

Walks

A walk is an interleaved sequence of vertices and edges $(v_0, e_1, v_1, e_2, \dots, e_t, v_t)$ where $e_i = \{v_{i-1}, v_i\} \in \mathrm{E}(G)$ for all $i = 1, \dots, t$.

Intuition: A walk is a path that is allowed to visit the same vertices and edges again.

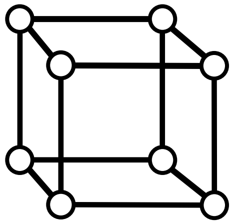
Example:



Graph G Walk in G

Quiz

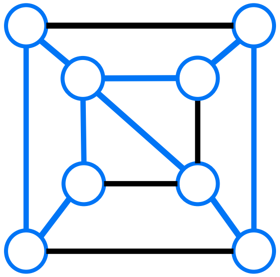
The graph drawn on the right is a planar graph.



Yes, because we *can* draw it without intersecting edges.

Graph G is an induced subgraph of G' if $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$.

No, an induced subgraph has all the edges between the selected subset of the vertices. That is, $E(G) = E(G') \cap \binom{V(G)}{2}$.



The blue edges and vertices form a cycle.

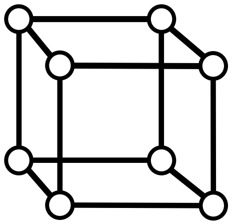
No, a cycle does not pass through the same vertex multiple times.

An interleaved sequence of vertices and edges $(v_0, e_1, v_1, e_2, \dots, e_t, v_t)$, where $e_i = \{v_{i-1}, v_i\} \in \mathrm{E}(G)$ for all $i = 1, \dots, t$.

a) ...is a walk.   b) ... is a cycle.   c) ... is a mess.

Quiz

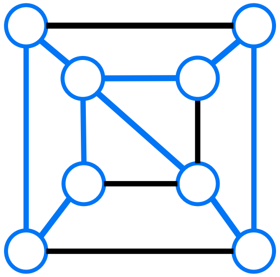
The graph drawn on the right is a planar graph.



Yes, because we *can* draw it without intersecting edges.

Graph G is an induced subgraph of G' if $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$.

No, an induced subgraph has all the edges between the selected subset of the vertices. That is, $E(G) = E(G') \cap \binom{V(G)}{2}$.



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No, a cycle does not pass through the same vertex multiple times.

An interleaved sequence of vertices and edges $(v_0, e_1, v_1, e_2, \dots, e_t, v_t)$, where $e_i = \{v_{i-1}, v_i\} \in \mathrm{E}(G)$ for all $i = 1, \dots, t$.

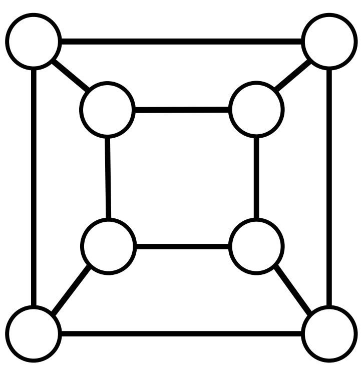
a) ...is a walk. b) ... is a cycle. c) ... is a mess.

Connected and components

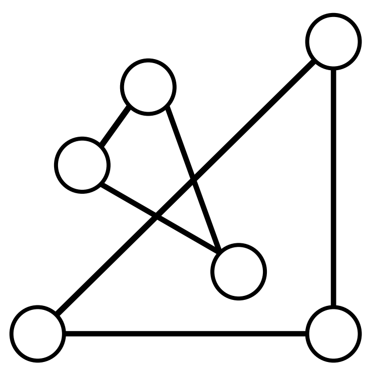
A graph is connected if for any two vertices $v, w \in V(G)$, G contains a walk from v to w.

The components of a graph G are the equivalence classes defined by the relation $\sim$ on the set $V(G)$, where $x{\sim}y$ if and only if there exists a walk from x to y in G.

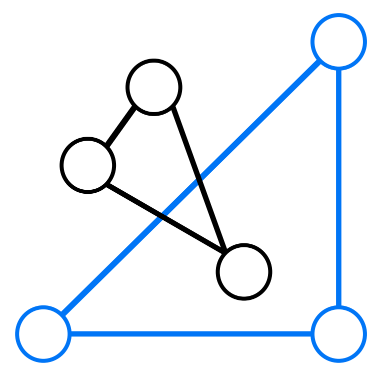
Example:



Connected



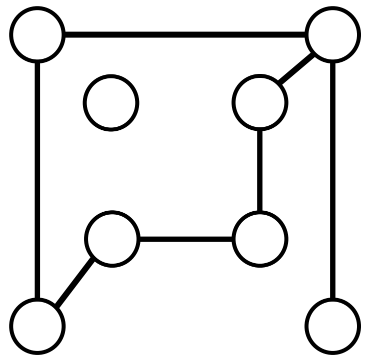
Not connected



Component

Examples

How many components in the following graphs?



Examples

How many components in the following graphs?

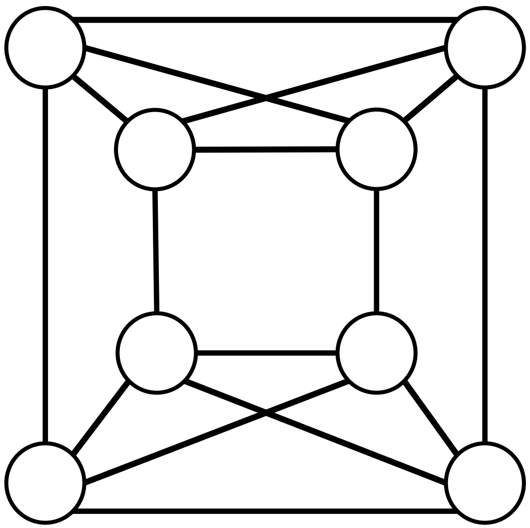
$G = (V, E)$

V : integers from 2 to 10

E : $\{a, b\}$ if $a \mid b$ or $b \mid a$

Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in V$ has $deg_G(v) \geq {}^{(n-1)}/_2$ is connected, where $n = |V|$.

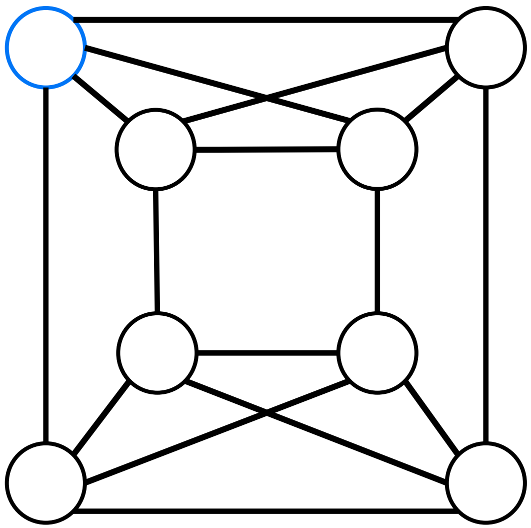


A graph is connected if for any two vertices $v, w \in V(G)$, G contains a walk for v to w.

Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in V$ has $deg_G(v) \geq {}^{(n-1)}/_2$ is connected, where $n = |V|$.

$v = w$



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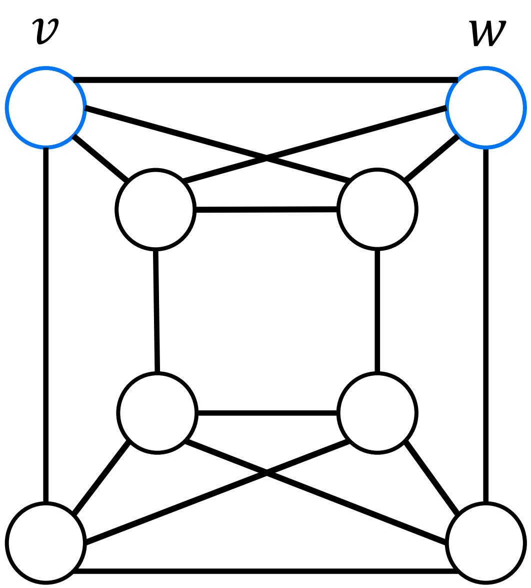

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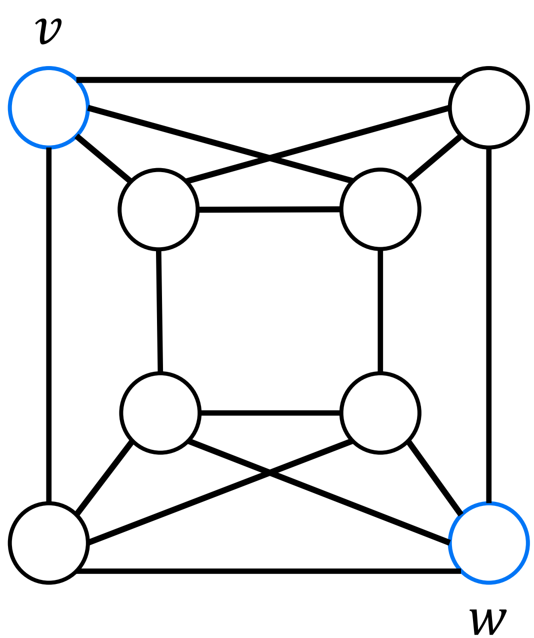
Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in \mathrm{V}$ has $deg_G(v) \geq {}^{(n-1)}/_2$ is connected, where $n = |V|$.



A graph is connected if for any two vertices $v, w \in V(G)$, G contains a walk for v to w.

Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in \mathrm{V}$ has $deg_G(v) \geq {}^{(n-1)}/_{2}$ is connected, where $n = |V|$.

Proof: Graph G is connected if for any two vertices $v, w \in V$, G contains a walk from v to w. Take two arbitrary vertices $v, w \in \mathrm{V}$.

Now a case distinction based on v, w

Case 1: $v = w$

Case 2: there is an edge $\{v, w\}$

Case 3: “Otherwise” $\rightarrow$ $v \neq w$ and there is no edge $\{v, w\}$

Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in \mathrm{V}$ has $deg_G(v) \geq {}^{(n-1)}/_{2}$ is connected, where $n = |V|$.

Proof: Graph G is connected if for any two vertices $v, w \in V$, G contains a walk from v to w. Take two arbitrary vertices $v, w \in \mathrm{V}$.

Now a case distinction based on v, w

Case 1: $v = w$

Then there a walk (v) from v to w

Case 2: there is an edge $\{v, w\}$

Then there is a walk $(v, \{v, w\}, w)$ from v to w

Case 3: "Otherwise" $\rightarrow$ $v \neq w$ and there is no edge $\{v, w\}$

Little more work

Connected

Theorem: Any graph $G = (V, E)$ where each vertex $v \in \mathrm{V}$ has $deg_G(v) \geq {}^{(n-1)}/_{2}$ is connected, where $n = |V|$.

Proof: Graph G is connected if for any two vertices $v, w \in V$, G contains a walk from v to w. Take two arbitrary vertices $v, w \in \mathrm{V}$.

Case 3: $v \neq \mathrm{w}$ and $\{v, w\} \notin E$.

Let A be the set of neighbors of v, and B be the set of neighbors of w.

We have $|A \cup \mathrm{B}| = |A| + |B| - |A \cap \mathrm{B}|$ (inclusion-exclusion).

$|A \cap \mathrm{B}| = |A| + |B| - |A \cup \mathrm{B}|$ (rewriting)

$|A \cap \mathrm{B}| \geq (n-1) - |A \cup \mathrm{B}|$ (by degree constraint)

$|A \cap \mathrm{B}| \geq (n-1) - (n-2)$ (since $A \cup B$ does not contain v,w)

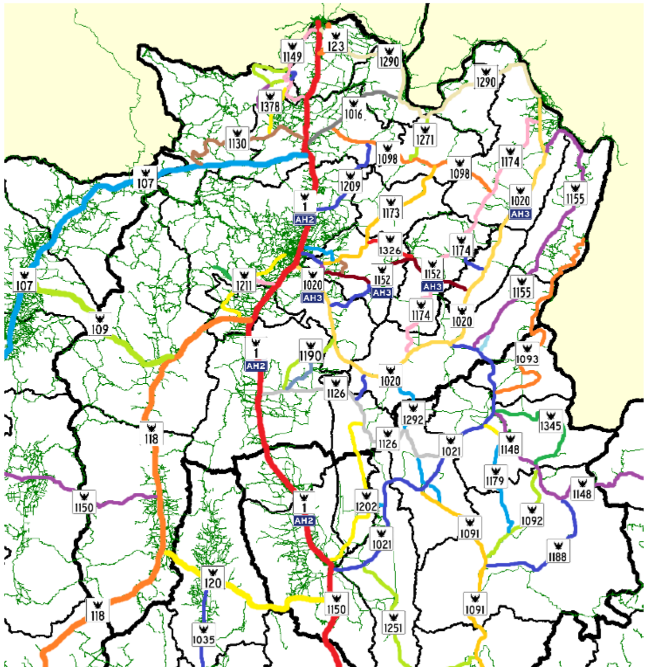
$|A \cap \mathrm{B}| \geq 1.$

So there is at least one common neighbor and hence there must be a path from v to w.

As we picked v and w arbitrarily this must hold for any two vertices. Hence G is connected.

Graph distance

Distance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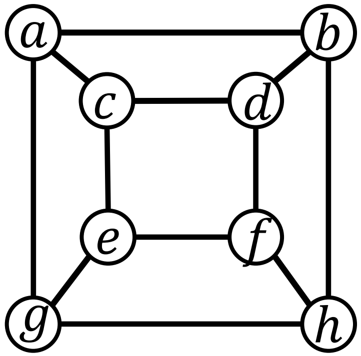
1148
1179
1148
1150
1
AH2
1202
1091
1092
1021
1188
120
1150
1091
118
1035
1251

Graph distance

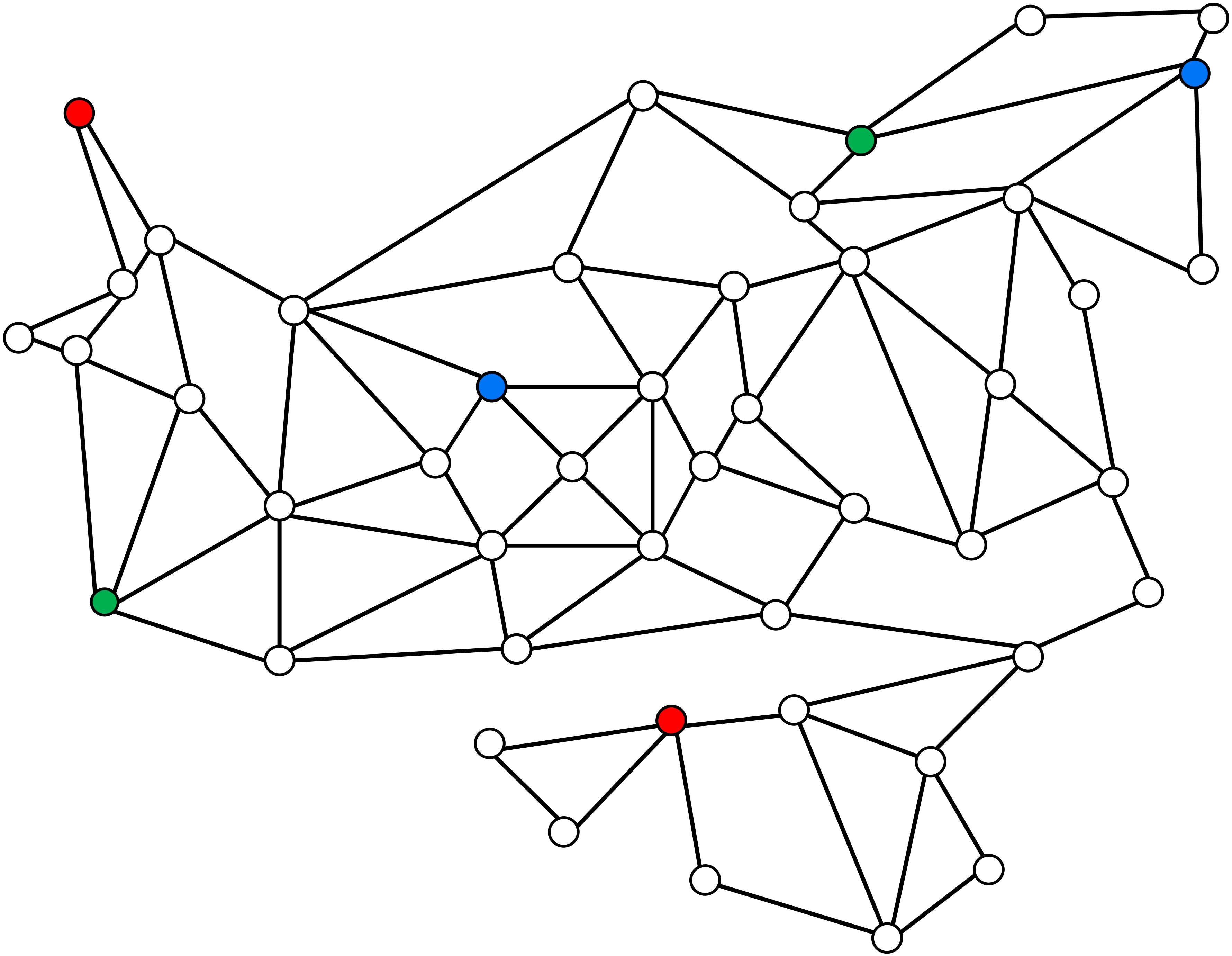
$d_G: V \times V \to \mathbb{R}$ is the distance function of graph G.

Distance between two vertices $v, v' \in V(G)$, denoted by $d_G(v, v')$, is the minimum number of edges on a path from v to v'.
(*Intuition*: The length of the shortest path from v to v'.)

Example:



$$d_G(a,a) = 0$$
$$d_G(b,h) = 1$$
$$d_G(a,h) = 2$$
$$d_G(d,e) = 2$$
$$d_G(b,e) = 3$$

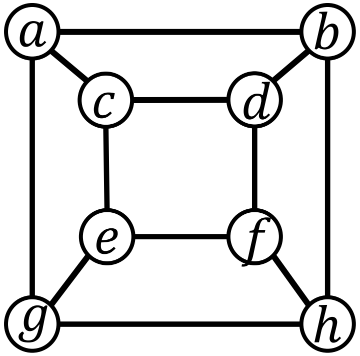


Graph distance properties

$d_G \colon V \times V \to \mathbb{R}$ is the distance function of graph G.

Distance between two vertices $v, v' \in V(G)$, denoted by $d_G(v, v')$, is the minimum number of edges on a path from v to v'.
(*Intuition*: The length of the shortest path from v to v'.)

1. $d_G(v, v') \geq 0$, and $d_G(v, v') = 0$ if and only if $v = v'$;



Graph distance properties

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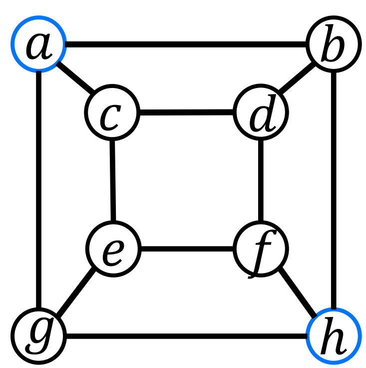
2. $d(v, v') = d(v', v)$ for any pair of vertices v, v' (symmetry);



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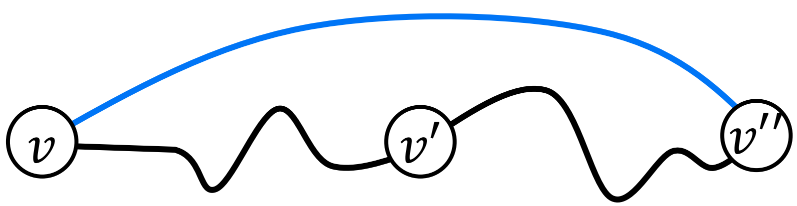
3. $d(v, v'') \leq d(v, v') + d(v', v'')$ for any three vertices $v, v', v'' \in V(G)$ (triangle inequality);



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Metric

4. $d_G(v, v'')$ is a nonnegative integer for any two vertices v, v';
5. If $d_G(v, v'') > 1$ then there exists a vertex $v', v \neq v' \neq v''$, such that $d_G(v, v') + d_G(v', v'') = d_G(v, v'')$.

Metrics

$d\colon V \times V \to \mathbb{R}$ is a metric on the set V if and only if for any $v, v' \in$ V

Properties:

1. $d(v, v') \geq 0$, and $d(v, v') = 0$ if and only if $v = v'$;
2. $d(v, v') = d(v', v)$ for any pair of vertices v, v' (symmetry);
3. $d(v, v'') \leq d(v, v') + d(v', v'')$ for any three vertices $v, v', v'' \in V(G)$ (triangle inequality);

The metric d together with V is called a metric space.

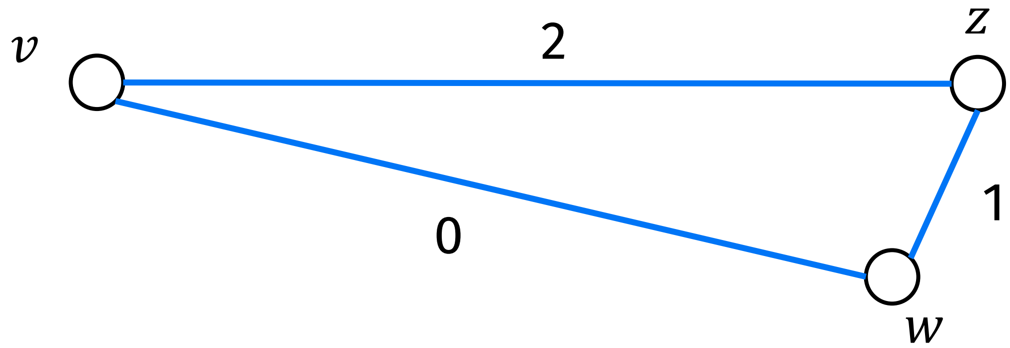
Example metrics:

- Graph distance
- Euclidean distance

Quiz

Metric or not a metric?

$d(v, w) = 0$, $d(v, z) = 2$, and $d(w, z) = 1$

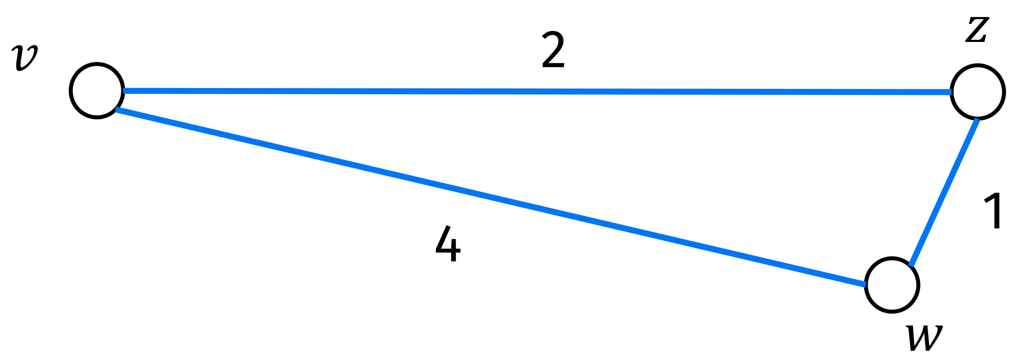


Quiz

Metric or not a metric?

$d(v, w) = 0$, $d(v, z) = 2$, and $d(w, z) = 1$ can't be (prop. 1)

$d(v, w) = 4$, $d(v, z) = 2$, and $d(w, z) = 1$



Quiz

Metric or not a metric?

$d(v,w) = 0$, $d(v,z) = 2$, and $d(w,z) = 1$ — can't be (prop. 1)

$d(v,w) = 4$, $d(v,z) = 2$, and $d(w,z) = 1$ — can't be (prop. 3)

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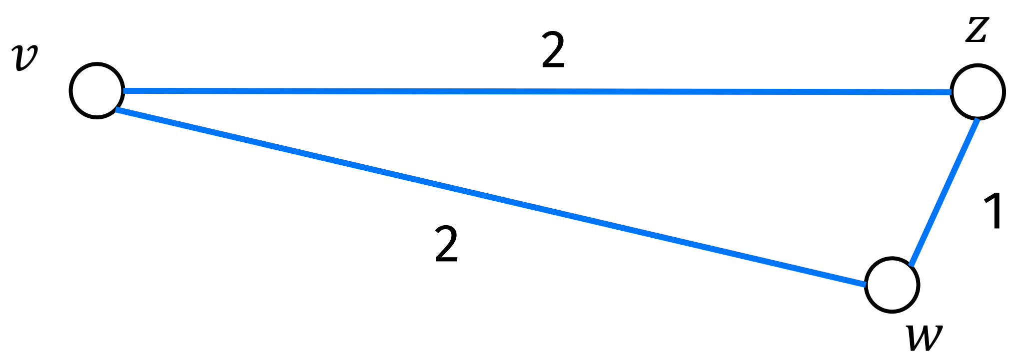
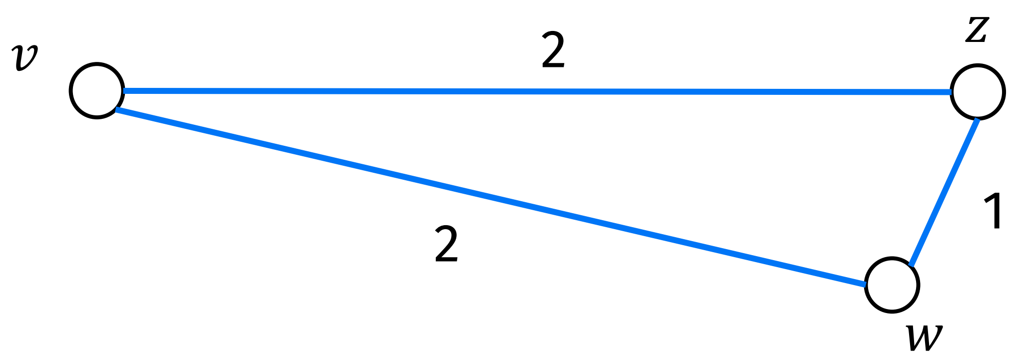


Quiz

Metric or not a metric?

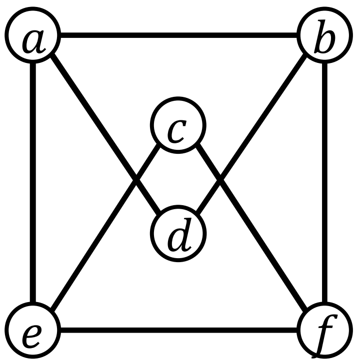
$d(v,w) = 0$, $d(v,z) = 2$, and $d(w,z) = 1$	can't be (prop. 1)
$d(v,w) = 4$, $d(v,z) = 2$, and $d(w,z) = 1$	can't be (prop. 3)
$d(v,w) = 2$, $d(v,z) = 2$, and $d(w,z) = 1$	metric



Representing graphs

Representations

We have seen several ways to represent graphs.



$G = (V, E)$, where
$V = \{a, b, c, d, e, f\}$,
$E = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, d\}, \{b, f\}, \{c, e\}, \{c, f\}, \{e, f\}\}$.

What other options could there be?

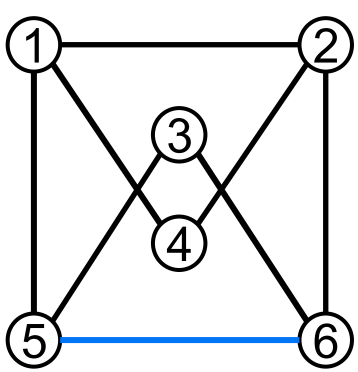
Adjacency matrix

Let $G = (V, E)$ be a graph with n vertices. Denote the vertices by $v_1, v_2, \ldots, v_n$ (in some arbitrary order). The adjacency matrix of G is an $n \times n$ matrix A_G defined by the following rule:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Here a_{ij} is the element on the i-th row and j-th column of A_G.

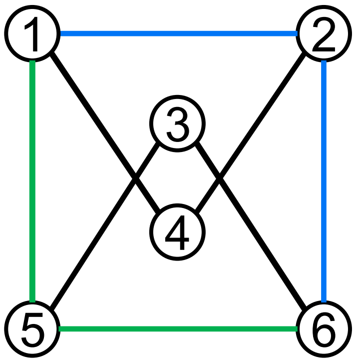
Example:



$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Distance

Theorem: Let $G = (V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a_{i,j}^{(k)}$ denote the element of A^k at position (i, j). Then $a_{i,j}^{(k)}$ is the number of walks of length exactly k from vertex v_i to vertex v_j in graph G.



$$A^2 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

$a_{6,1}^{(2)}$

Distance

Theorem: Let $G = (V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a_{i,j}^{(k)}$ denote the element of A^k at position (i, j). Then $a_{i,j}^{(k)}$ is the number of walks of length exactly k from vertex v_i to vertex v_j in graph G.

$$A^3 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix}$$

$a_{6,2}^{(3)}$

$$a_{6,2}^{(3)} = 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 + 3 \cdot 1 = 6$$

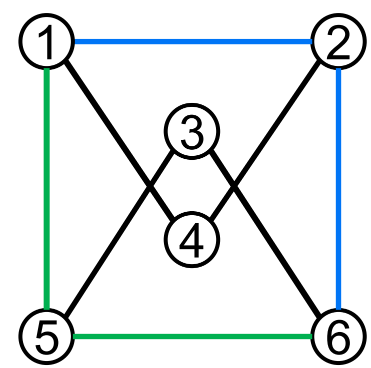
Distance

Theorem: Let $G = (V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a_{i,j}^{(k)}$ denote the element of A^k at position (i, j). Then $a_{i,j}^{(k)}$ is the number of walks of length exactly k from vertex v_i to vertex v_j in graph G.

$\{1,2\} \in E$

$$A^3 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$a_{6,1}^{(2)}$



Distance

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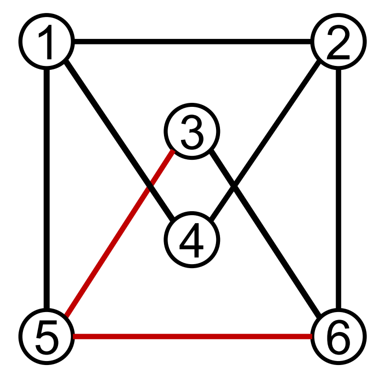
$\{3{,}2\} \in E$

$$A^3 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

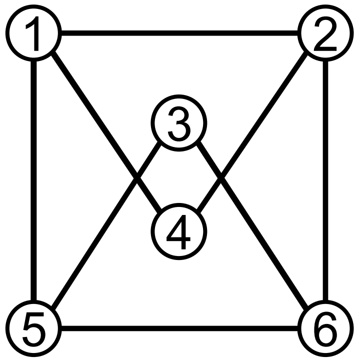
$a_{6,3}^{(2)}$



Distance

Theorem: Let $G = (V, E)$ with vertex set $V = \{v_1, v_2, \dots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a_{i,j}^{(k)}$ denote the element of A^k at position (i, j). Then $a_{i,j}^{(k)}$ is the number of walks of length exactly k from vertex v_i to vertex v_j in graph G.

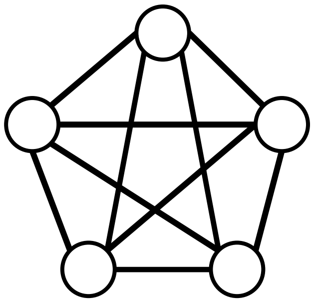
$$A^3 = \begin{pmatrix} 3 & 1 & 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 3 & 1 \\ 2 & 0 & 1 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$



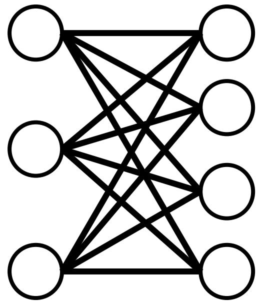
$$a_{6,2}^{(3)} = 2 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 + 3 \cdot 1 = 6$$

Summary

Graph classes



K_5



$K_{3,4}$

Isomorphism

Two graphs $G = (V, E)$ and $G' = (V', E')$ are called isomorphic if a bijection $f : V -> V'$ exists such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$ holds for all $x, y \in V$, $x \neq y$.

Adjacency matrices

$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$a_{6,2}$

Organizational

- Practice set:
  - Ex. 1,4: Practice definitions
  - Ex. 2,3,5,6: Proofs
  - Do exercise 4 before study group (practice definitions)
  - Try exercise 2,6 (these are somewhat harder)

- Round table discussion:
  - Opportunity to discuss what you do and do not like about course
  - Register at 2IC01 on Canvas --------->