2MBA60 Analysis 2, Group 4-4

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18.4.1

Corollary 18.2.1 (Mean-value inequality). Let $f : \Omega \to W$ be differentiable on an open domain $\Omega \subset V$. Then, for all $a, b \in \Omega$, if for every $\tau \in (0, 1)$, also

$$(1-\tau)a + \tau b \in \Omega$$

then

$$\|f(b) - f(a)\|_W \leq \sup_{\tau \in (0,1)} \left\|(Df)_{(1-\tau)a+\tau b}\right\|_{V \to W} \|b - a\|_V .$$

Prove Corollary 18.2.1.

Let $f : \Omega \to W$ be differentiable on an open domain. $\Omega \subset V$. Take $a, b \in \Omega$
Assume that $\gamma : [0, 1] \to V, \gamma(\tau) = (1-\tau)a + \tau b, \forall \tau \in [0, 1], \gamma(\tau) \in \Omega$.
Then γ is an affine function, thus continuous in $[0, 1]$ and differentiable on (0, 1), with $\gamma'(\tau) = b-a$.
Define $g : [0, 1] \to W, g := f \circ \gamma$.
Since f is differentiable on Ω, which is where γ maps to, it also is continuous on Ω and thus g is a composition of continuous functions f, γ and continuous itself.
By the chain rule, since f, γ are differentiable, g is differentiable on $(0, 1)$.
$\|f(b) - f(a)\|_W = \|f(\gamma(1)) - f(\gamma(0))\|_W = \|g(1) - g(0)\|_W$.
Since g is continuous on [a, b] and differentiable on (a, b) by the mean value inequality it holds that $\|g(1) - g(0)\|_W \leq \sup_{\tau \in (0,1)} \|g'(\tau)\|_W (1 - 0)$.
By the chain rule it holds that

$$\begin{aligned}
g'(\tau) &= (Dg)_\tau(1) \\
&= (D(f \circ \gamma))_\tau(1) \\
&= (Df)_{\gamma(\tau)} \circ (D\gamma)_\tau(1) \\
&= (Df)_{\gamma(\tau)} \circ (\gamma'(\tau)) \\
&= (Df)_{\gamma(\tau)}(b - a)
\end{aligned}$$

We thus have

$$\begin{aligned}
\|f(b) - f(a)\|_W = \|g(1) - g(0)\|_W &\leq \sup_{\tau \in (0,1)} \left\|(Df)_{\gamma(\tau)}(b - a)\right\|_W \\
&\leq \sup_{\tau \in (0,1)} \left\|(Df)_{\gamma(\tau)}\right\|_{V \to W} \|b - a\|_V \\
&= \sup_{\tau \in (0,1)} \left\|(Df)_{(1-\tau)a+\tau b}\right\|_{V \to W} \|b - a\|_V
\end{aligned}$$

18.4.2

Lemma 18.2.2. Suppose $f : \Omega \to W$ is differentiable on Ω and suppose its derivative function $Df : \Omega \to \text{Lin}(V, W)$ is bounded. Let $a \in \Omega$ and assume $r > 0$ is such that $B(a, r) \subset \Omega$. Then for all $x \in B(a, r)$,

$$\left\|\text{Err}_a^f(x)\right\|_W \leq \sup_{z \in B(a,r)} \|(Df)_z - (Df)_a\|_{V \to W} \|x - a\|_V$$

Give a proof of Lemma 18.2.2.

Assume that $f : \Omega \to W$ is differentiable on Ω, $Df : \Omega \to \text{Lin}(V, W)$ is bounded. Assume that $a \in \Omega$ and assume that there exists $r > 0, B(a, r) \in \Omega$.
Obtain such a r.
Let $x \in B(a, r)$, then it holds that $x \in \Omega$ and for every $\tau \in (0, 1), (1 - \tau)a + \tau x \in B(a, r) \subset \Omega$.
Define $g : \Omega \to W, g(x) = \text{ Err}_a^f(x) = f(x) - f(a) - (Df)_a(x - a)$.
By the chain rule it holds that g is differentiable on Ω.
Since $a, x \in \Omega$ and for every $\tau \in (0, 1), (1 - \tau)a + \tau x \in \Omega$ we can apply the mean value inequality to obtain the following statement:

$$\|g(x) - g(a)\|_W \leq \sup_{\tau \in (0,1)} \left\|(Dg)_{(1-\tau)a+\tau x}\right\|_{V \to W} \|x - a\|_W$$

It holds that $(Dg)_x = (Df)_x - (Df)_a$ and $g(a) = f(a) - f(a) + (Df)_a(a - a) = 0$.

$$\|g(x)\|_W \leq \sup_{\tau \in (0,1)} \left\|(Df)_{(1-\tau)a+\tau x} - (Df)_a\right\|_{V \to W} \|x - a\|_V$$

Since for all $\tau \in (0, 1)$ it holds that $(1 - \tau)a + \tau x \in B(a, r)$ it holds that
$\sup\limits_{\tau \in (0,1)} \left\|(Df)_{(1-\tau)a+\tau x} - (Df)_a\right\|_{V \to W} \leq \sup\limits_{z \in B(a,r)} \|(Df)_z - (Df)_a\|_{V \to W}$.
We conclude that

$$\left\|\text{Err}_a^f(x)\right\|_W = \|g(x)\|_W \leq \sup_{z \in B(a,r)} \|(Df)_z - (Df)_a\|_{V \to W} \|x - a\|_V$$

18.4.3

Define the subset $\Omega \subset \mathbb{R}^2$ as follows

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid \|(x_1, x_2)\|_2 > 1.9\}.$$

Let $f : \Omega \to W$ be a differentiable function and assume that for all $a \in \Omega$,

$$\|(Df)_a\|_{\mathbb{R}^2 \to W} \leq 5. \tag{1}$$

Prove that

$$\|f((2, 0)) - f((-2, 0))\|_W \leq 10\pi.$$

It holds that $a = (-2, 0), b = (2, 0) \in \Omega$.
Define $\gamma : \mathbb{R} \to \mathbb{R}^2$ as

$$(2 \cos(\pi - \tau), 2 \sin(\tau)) \in \Omega.$$

Define $g : [0, \pi] \to W$ by $g := f \circ \gamma$.
The component functions $\gamma_1 : \mathbb{R} \to \mathbb{R}$ and $\gamma_2 : \mathbb{R} \to \mathbb{R}$ are given by

$$\gamma_1(\tau) = 2 \cos(\pi - \tau),\ \gamma_2(\tau) = 2 \sin(\tau)$$

By Corollary 15.8.2 it holds that if the component functions γ_1 and γ_2 are differentiable, γ is also

differentiable.
Since γ_1 and γ_2 are both the product of a constant and a standard differentiable function, by the product rule it holds that γ_1 and γ_2 are both differentiable.
So it also holds that γ is differentiable.
By the chain rule, since f, γ are differentiable, g is differentiable on $(0, \pi)$.
Since f is differentiable on Ω, it is also continuous on Ω.
It holds that γ_1 and γ_2 are both continuous functions.
Since the component functions of γ are both continuous it holds that γ is also continuous.
Since f is continuous on Ω and γ is continuous, g is a composition of continuous functions f, γ and is continuous itself, and thus continous on $[0, \pi]$.
It holds that $\|f(b) - f(a)\|_W = \|f(\gamma(\pi)) - f(\gamma(0))\|_W = \|g(\pi) - g(0)\|_W$.
Since g is continuous on $[a, b]$ and differentiable on (a, b), by the mean value inequality it holds that $\|g(\pi) - g(0)\|_W \leq \sup_{\tau \in (0,\pi)} \|g'(\tau)\|_W (\pi - 0)$.
By the chain rule it holds that

$$g'(\tau) = (Df)_{\gamma(\tau)} \circ (\gamma'(\tau)) = (Df)_{\gamma(\tau)}(2\sin(\pi - \tau), 2\cos(\tau)). \tag{2}$$

Since $(2\cos(\pi - \tau), 2\sin(\tau)) \in \Omega$ is a circle with radius 2, it holds that

$$\|(2\cos(\pi - \tau), 2\sin(\tau))\|_2 = 2 \quad \forall \tau \in \mathbb{R}. \tag{3}$$

Now combining everything gives us

$$\begin{aligned}
\|f(b) - f(a)\|_W = \|g(\pi) - g(0)\|_W &\leq \sup_{\tau \in (0,\pi)} \|g'(\tau)\|_W (\pi - 0) \\
(2) \quad &= \sup_{\tau \in (0,\pi)} \left\|(Df)_{\gamma(\tau)}(2\sin(\pi - \tau), 2\cos(\tau))\right\|_W \pi \\
&\leq \sup_{\tau \in (0,\pi)} \left\|(Df)_{\gamma(\tau)}\right\|_{\mathbb{R}^2 \to W} \|(2\sin(\pi - \tau), 2\cos(\tau))\|_2 \pi \\
(3) \quad &= \sup_{\tau \in (0,\pi)} \left\|(Df)_{(2\cos(\pi-\tau), 2\sin(\tau))}\right\|_{\mathbb{R}^2 \to W} 2\pi \\
(1) \quad &\leq 5 \cdot 2\pi = 10\pi.
\end{aligned}$$

Since we have $a = (-2, 0), b = (2, 0) \in \Omega$, we conclude that

$$\|f((2, 0)) - f((-2, 0))\|_W \leq 10\pi.$$

18.4.4

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f((x_1, x_2)) = \begin{cases} \frac{(x_1)^2 (x_2)^7}{(x_1)^2 + (x_2)^2} & \text{if } (x_1, x_2) \neq (0, 0) \\ 0 & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

a Show that f is differentiable on $\mathbb{R}^2$ by showing that the partial derivatives exist and are continuous.

b for $a \in \mathbb{R}^2$, compute $\nabla f(a)$.

1.

Let $x \in \mathbb{R}^2 \setminus \{0\}$, then $f'(x)$ exists since $f(x) = \frac{(x_1)^2(x_2)^7}{(x_1)^2+(x_1)^2}$ and thus a rational function of polynomials. Thus

$$\begin{aligned}\frac{\partial f}{\partial x_2}(x) &= \frac{7(x_1)^2(x_2)^6((x_1)^2+(x_2)^2) - 2(x_1)^2(x_2)^8}{((x_1)^2+(x_2)^2)^2} \\ &= \frac{5(x_1)^2(x_2)^8 + 7(x_1)^4(x_2)^6}{((x_1)^2+(x_2)^2)^2}\end{aligned}$$

and

$$\frac{\partial f}{\partial x_1}(x) = \frac{2(x_1)(x_2)^7((x_1)^2+(x_2)^2) - 2(x_1)^3(x_2)^7}{((x_1)^2+(x_2)^2)} = \frac{2(x_1)(x_2)^9}{((x_1)^2+(x_2)^2)^2}$$

Both of the partial derivatives above are continuous in their domain of definition ($\mathbb{R}^2 \setminus \{0\}$) since they are rationals of polynomials.
Then by the definition of a partial derivative we have

$$\frac{\partial f}{\partial x_1}(0) = \lim_{h\to 0} \frac{f(0+he_1) - f(0)}{h} = \lim_{h\to 0} \frac{f(he_1)}{h} = \frac{\frac{(h)^2(0)^7}{h^2+0^2}}{h} = 0$$

and

$$\frac{\partial f}{\partial x_2}(0) = \lim_{h\to 0} \frac{f(0+he_2) - f(0)}{h} = \lim_{h\to 0} \frac{f(he_2)}{h} = \lim_{h\to 0} \frac{\frac{(0)^2(h)^7}{0^2+h^2}}{h} = 0$$

Hence to show the partial derivatives are continuous in $\mathbb{R}^2$ we need to show that

$$\lim_{x\to 0} \frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_1}(0)$$

and

$$\lim_{x\to 0} \frac{\partial f}{\partial x_2}(x) = \frac{\partial f}{\partial x_2}(0)$$

.
We need to show that $\lim_{x\to 0} \frac{\partial f}{\partial x_1}(x) = 0$ and $\lim_{x\to 0} \frac{\partial f}{\partial x_2}(x) = 0$.
We need to show that $\lim_{x\to 0} \frac{2(x_1)(x_2)^9}{((x_1)^2+(x_2)^2)^2} = 0$.
By the sequence characterization of limits it suffices to show that for all sequences $z : \mathbb{N} \to \mathbb{R}^2 \setminus (0,0)$ converging to 0, $\lim\limits_{n\to\infty} \frac{\partial f}{\partial x_1}(z^{(n)}) = 0$.
Let $z : \mathbb{N} \to \mathbb{R}^2 \setminus (0,0)$ be a sequence converging to 0.
Note that for all $n \in \mathbb{N}$, $|z_i^{(n)}| \le \left\|z^{(n)}\right\|_2$ for all $i \in \mathbb{N}$.
It also holds that for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \quad & |\tfrac{\partial f}{\partial x_1}(z^{(n)})| & \le \frac{2\left\|z^{(n)}\right\|_2 \left\|z^{(n)}\right\|_2^9}{((z_1^{(n)})^2+(z_2^{(n)})^2)^2} \\ \frac{2(z_1^{(n)})^2(z_2^{(n)})^9}{((z_1^{(n)})^2+(z_2^{(n)})^2)^2} \le \quad & \tfrac{2\left\|z^{(n)}\right\|_2^{10}}{((z_1^{(n)})^2+(z_2^{(n)})^2)^2} & \le \frac{2\left\|z^{(n)}\right\|_2^{10}}{\left\|z^{(n)}\right\|_2^4} = 2\left\|z^{(n)}\right\|_2^6 \end{aligned}$$

By the squeeze theorem it holds that

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{2(z_1^{(n)})^2(z_2^{(n)})^9}{((z_1^{(n)})^2+(z_2^{(n)})^2)^2} \le \lim_{n\to\infty} 2 \cdot \left\|z^{(n)}\right\|_2^6 = 0$$

We conclude that $\frac{\partial f}{\partial x_1}(x)$ is continuous.
We need to show that $\lim\limits_{x\to 0} \frac{\partial f}{\partial x_2}(x) = 0$.

By the sequence characterization of limits it suffices to show that for all $z : \mathbb{N} \to \mathbb{R}^2 \setminus (0,0)$ converging to 0, $\lim_{n\to\infty} \frac{\partial f}{\partial x_2}(z^{(n)}) = 0$.
Let $z : \mathbb{N} \to \mathbb{R}^2 \setminus (0,0)$ be a sequence converging to 0.
It holds that for all $n \in \mathbb{N}, i \in 1, 2, |z_i^{(n)}| \leq \left\|z^{(n)}\right\|_2$.
Note that for all $n \in \mathbb{N}$,

$$\begin{aligned}0 \leq |\frac{\partial f}{\partial x_2}(z^{(n)})| &\leq \frac{5\left\|z^{(n)}\right\|_2^2\left\|z^{(n)}\right\|_2^8 + 7\left\|z^{(n)}\right\|_2^4\left\|z^{(n)}\right\|_2^6}{((z_1^{(n)})^2 + (z_2^{(n)})^2)^2} \\ &= \frac{12\left\|z^{(n)}\right\|_2^{10}}{\left\|z^{(n)}\right\|_2^4} = 12\left\|z^{(n)}\right\|_2^6\end{aligned}$$

By the squeeze theorem it holds that

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{5(z_1^{(n)})^2(z_2^{(n)})^8 + 7(z_1^{(n)})^4(z_2^{(n)})^6}{((z_1^{(n)})^2 + (z_2^{(n)})^2)^2} \leq \lim_{n\to\infty} 12 \cdot \left\|z^{(n)}\right\|_2^6 = 0$$

We conclude that $\frac{\partial f}{\partial x_2}(x)$ is continuous.
Since the partial derivatives exists and are continuous, we conclude that f is differentiable on $\mathbb{R}^2$.

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From exercise a, we compute

$$\nabla f(a) = \begin{pmatrix} \frac{2(a_1)(a_2)^9}{((a_1)^2+(a_2)^2)^2} \\ \frac{5(a_1)^2(a_2)^8+7(a_1)^4(a_2)^6}{((a_1)^2+(a_2)^2)^2} \end{pmatrix}$$