Assignment 6

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1 Exercise 10.7.1

Problem 1.1 Let $A := \{a, b, \ldots, z\}$ be the set of all letters of the alphabet. Let $\alpha : \mathbb{N} \to A$ be a sequence. Let $v : \mathbb{N} \to \mathbb{N}$ be an index sequence defined by $v_k = k + 5$ and $\mu : \mathbb{N} \to \mathbb{N}$ be an index sequence defined by $\mu_k = 3k$. Write first 33 terms of the subsequence $(\alpha_{v_{\mu_k}})_k$

Proof.

$$\begin{aligned} \alpha_{v_{\mu_k}} &= \alpha_{v_{3k}} \\ &= \alpha_{3k+5} \end{aligned}$$

So $(\alpha_{v_\mu})_0 = \alpha_5, (\alpha_{v_\mu})_1 = \alpha_8, (\alpha_{v_\mu})_2 = \alpha_{11}, \ldots$. Following the diagram we get the following for the first 33 terms

goodluckinthesecondhalfofanalysis, or with spaces
good luck in the second half of analysis

□

2 Exercise 10.7.3

Problem 2.1 Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences, such that $a : \mathbb{N} \to X$ converges to some $p \in X$.

Now consider the following sequence $c : \mathbb{N} \to X$, defined by

$$c_k := \begin{cases} a_k & \text{if } k \text{ even} \\ b_k & \text{if } k \text{ odd} \end{cases}$$

Show that p is an accumulation point of $c : \mathbb{N} \to X$.

Proof. Define $n : \mathbb{N} \to \mathbb{N}$ a index sequence defined by $n_k := 2k$, then the subsequences $(c_{n_k})_k$ is the even terms of c, which is a. Since a converges to p, then p is an accumulation point of c. □

3 Exercise 10.7.5

Problem 3.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} a_\ell = L$$

Proof. We prove both directions.

First assume that $(a_k)_k$ converges to L. Then it holds

$$\begin{aligned}&\text{for all } \epsilon_0 > 0,\\ &\quad \text{there exists } N_0 \in \mathbb{N},\\ &\qquad \text{for all } n \geq N_0,\\ &\qquad\quad |a_n - L| < \epsilon.\end{aligned} \tag{1}$$

Need to show $\limsup_{\ell\to\infty} a_\ell = L$ and $\liminf_{\ell\to\infty} a_\ell = L$.
By the alternative characterization of lim sup we need to show that
1.

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\qquad \text{for all } n \geq N,\\ &\qquad\quad a_n < L + \epsilon.\end{aligned}$$

2.

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{for all } K \in \mathbb{N}\\ &\qquad \text{there exists } m \geq K,\\ &\qquad\quad a_m > L - \epsilon.\end{aligned}$$

We first show 1.
Let $\epsilon > 0$,
Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$.
Obtain such N_0,
Choose $N = N_0$,
Then for all $n \geq N = N_0$, we have
$|a_n - L| < \epsilon$, in particular
$a_n < L + \epsilon$

Now we show 2.
Let $\epsilon > 0$,
Take $K \in N$,
Choose $\epsilon_0 =$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$,
Obtain such N_0,
Choose $m = N_0 + K$, then we have
$|a_m - L| < \epsilon_0$, in particular
$a_m > L - \epsilon$

By alternative characterization of lim inf we need to show that
1.

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\qquad \text{for all } n \geq N,\\ &\qquad\quad a_n > L - \epsilon.\end{aligned}$$

2.

$$
\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{for all } K \in \mathbb{N} \\
&\qquad \text{there exists } m \geq K, \\
&\qquad\quad a_m < L + \epsilon.
\end{aligned}
$$

We first show 1.
Let $\epsilon > 0$,
Choose $\epsilon_o = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$.
Obtain such N_0,
Choose $N = N_0$,
Then for all $n \geq N = N_0$, we have
$|a_n - L| < \epsilon_0 = \epsilon$, in particular
$a_n > L - \epsilon$

Now we show 2.
Let $\epsilon > 0$,
Take $K \in \mathbb{N}$,
Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$,
Obtain such N_0,
Choose $m = N_0 + K$, then we have
$|a_m - L| < \epsilon_0 = \epsilon$, in particular
$a_m < L + \epsilon$

Now we prove the other direction.
Assume that $\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} a_\ell = L$.
We need to show that $(a_k)_k$ converges to L,
i.e.

$$
\begin{aligned}
&\text{for all } \epsilon > 0, \\
&\quad \text{there exists } N \in \mathbb{N}, \\
&\qquad \text{for all } n > N, \\
&\qquad\quad |a_n - L| < \epsilon
\end{aligned}
$$

Since $\liminf_{\ell\to\infty} a_\ell = L$, we have

$$
\begin{aligned}
&\text{for all } \epsilon_1 > 0, \\
&\quad \text{there exists } N_1 \in \mathbb{N}, \\
&\qquad \text{for all } n \geq N_1, \\
&\qquad\quad a_n > L - \epsilon_1
\end{aligned} \tag{2}
$$

Since $\limsup_{\ell\to\infty} a_\ell = L$, we have

$$
\begin{aligned}
&\text{for all } \epsilon_2 > 0, \\
&\quad \text{there exists } N_2 \in \mathbb{N}, \\
&\qquad \text{for all } n \geq N_2, \\
&\qquad\quad a_n < L + \epsilon_2
\end{aligned} \tag{3}
$$

Let $\epsilon > 0$,
Choose $\epsilon_1 = \epsilon$ in (2), then there exists $N_1 \in \mathbb{N}$, for all $n \geq N_1$, $a_n > L - \epsilon_1 = L - \epsilon$.
Obtain such N_1,
Choose $\epsilon_2 = \epsilon$ in (3), then there exists $N_2 \in \mathbb{N}$, for all $n \geq N_2$, $a_n < L + \epsilon_2 = L + \epsilon$.
Choose $N = \max(N_1, N_2)$, then for all $n \geq N$, we have
$a_n > L - \epsilon$ and $a_n < L + \epsilon$, in particular
$|a_n - L| < \epsilon$

$\square$

4 Exercise 10.7.7

Problem 4.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence with at least two sequential accumulation points $p, q \in \mathbb{R}$ with $p \neq q$. Prove that the sequence $a : \mathbb{N} \to \mathbb{R}$ does not converge.

Proof. Assume (a_n) has two accumulations points $p, q \in \mathbb{R}$ with $p \neq q$.

We argue by contradiction.
Assume (a_n) converges to $L \in \mathbb{R}$.
Since p is a sequential accumulation point we have that

$$\limsup_{k\to\infty} a_k = p = L$$

Similarly, since q is a sequential accumulation point we have

$$\limsup_{k\to\infty} a_k = q = L$$

Then $p = q$, which is a contradiction. Therefore (a) does not converge. $\square$