Practice Midterm

Jim Portegies

Exercise 1

Consider the set

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 - \tfrac{1}{10} < y < x^2 + \tfrac{1}{10}\}$$

Suppose that $f : \Omega \to W$ is differentiable on Ω and that for every $a \in \Omega$,

$$\|(Df)_a\|_{\mathbb{R}^2 \to W} \leq 2.$$

and suppose $\|f((-1, 1))\|_W = 3$.
Show that

$$\|f((2, 4))\|_W \leq 30.$$

Exercise 2

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f((x_1, x_2)) = \begin{cases} (x_1)^2 (x_2)^2, & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

a. Prove that f is differentiable on $\mathbb{R}$.

The previous version of this exercise said "twice differentiable", but that was a typo and is not true. In any case, the solution of this exercise is a bit too much work for a midterm.

b. Give the first order Taylor order polynomial of f in $0 \in \mathbb{R}^2$.

The previous version of this exercise said "second order Taylor polynomial".

Exercise 3

Determine whether the following limits exist, and if so, determine their value.

a.

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x)+\frac{1}{2}x^2+\sin(y)^4}{x^4+y^4}$$

b.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)\cos(y)-x}{x^4+y^4}$$

Exercise 4

Does there exist a three times differentiable function $f : \mathbb{R}^4 \to \mathbb{R}$ such that for all $u \in \mathbb{R}^4$,

$$(D^3 f)_0(u,u,u) = u_1 + 4u_4$$

Either give an example of such a function or show why such a function does not exist.