2IT80 Discrete Structures

2023-24 Q2

Lecture 8: Graphs II

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Recap

Graph

A graph G is an ordered pair (V, E), where

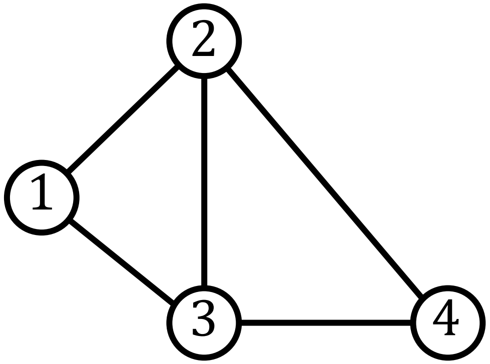
> V is a set of elements, called vertices.
> E a set of 2-element subsets of V, called edges

The degree of a vertex is equal to the number edges it is part of.

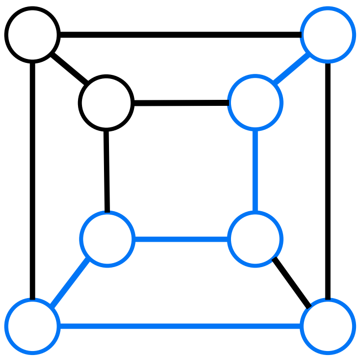
Vertices $v, v' \in V$ are adjacent when $\{v, v'\} \in E$. We say v' is a neighbor of v (and v a neighbor of v').

Example:
$V = \{1, 2, 3, 4\}$
$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}\}$
Degree of vertex 2 is three.

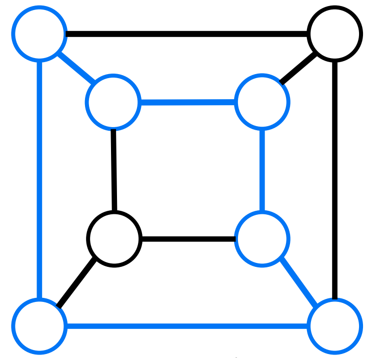
Vertex 2 and vertex 3 are adjacent.



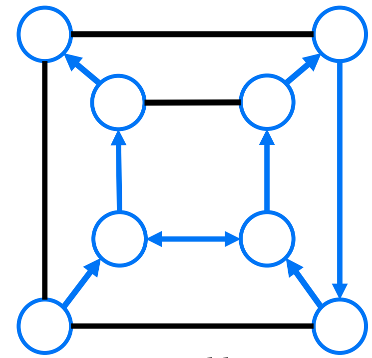
(Induced) subgraphs



Path



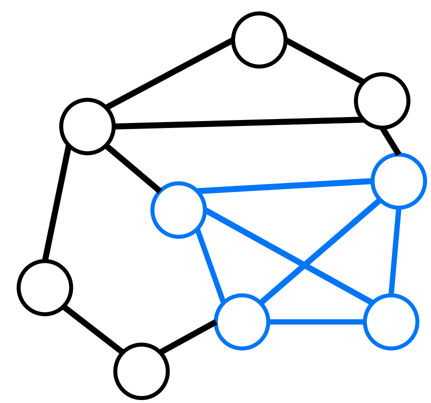
Cycle



Walk

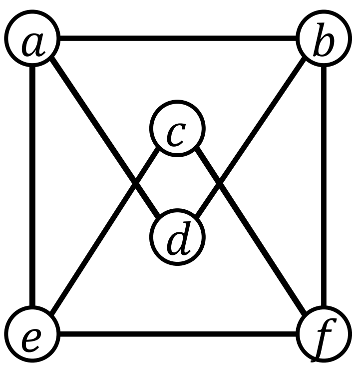
Clique:

Induced subgraph that is the complete graph.



Representations

$G = (V, E)$, where
$V = \{a, b, c, d, e, f\}$,
$E = \{\{a, b\}, \{a, d\}, \{a, e\}, \{b, d\}, \{b, f\}, \{c, e\}, \{c, f\}, \{e, f\}\}$.



$$A_G = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

This lecture

Degree sequences; a not quite representation

Handshake lemma

Graph classes:

- Eulerian
- Hamiltonian
- k-connected

Degree sequences

(1, 1, 1, 2)....

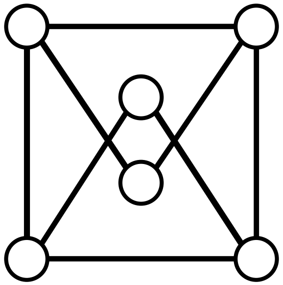
Degree sequence (Graph score)

Let $\deg_G(v)$ be the degree of v in the graph G.

For a graph $G = (V, E)$ with n vertices denoted by $v_1, v_2, \ldots, v_n$, the sequence $(\deg_G(v_1), \deg_G(v_2), \ldots, \deg_G(v_n))$ is called a degree sequence of G.

The order of the degrees does not matter. Generally we write degree sequences in nondecreasing order.

Example:



degree sequence: (2, 2, 3, 3, 3, 3)

Degree sequence: Examples

What is the degree sequence for:

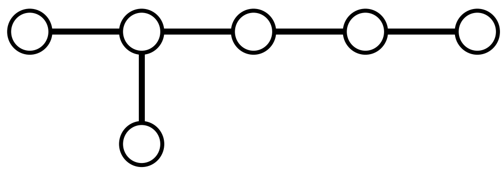
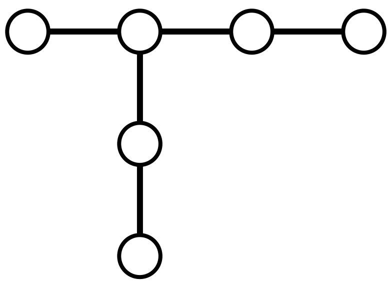
A path-graph of length 5: P_5

A complete bipartite graph: $K_{3,4}$

Degree sequence

Every graph has a degree sequence

But is it unique? What is the graph for (3,2,2,1,1,1)?

No:  

And which tuples of natural numbers correspond to graphs?

Or... How can we tell if a given tuple is a degree sequence?

Existence of sequences

Which of these sequences are degrees sequences of a graph:

$$(1, 1, 1, 2, 4, 4, 4)$$
$$(1, 1, 1, 1, 2, 4, 4)$$
$$(1, 1, 1, 1, 4, 4, 4)$$

What about (1,1,1,2,2,2,3,3,3,4,4,4,5,5,8,10)?

How could we check if a graph exists with this degree sequence?

What do we know of the sum of degrees?

Observation: For each graph $G = (V, E)$ we have $\sum_{v \in V} deg_G(v) = 2|E|$.

Shaking hands

Handshake lemma: For any graph $G = (V, E)$ the number of vertices with odd degree is even.

Shaking hands

Handshake lemma: For any graph $G = (V, E)$ the number of vertices with odd degree is even.

Proof:

We know that $\sum_{v \in \mathrm{V}} deg_G(v) = 2|E|$.

We split the summation on the parity of the vertices.

Let $V_e = \{v \mid v \in V, deg_G(v) \text{ even}\}$
and $V_o = \{v \mid v \in V, deg_G(v) \text{ odd}\}$.
We have $V_e \cup V_o = V$ and $V_e \cap V_o = \emptyset$.

Then $\sum_{v \in \mathrm{V}_e} deg_G(v) + \sum_{v \in \mathrm{V}_o} deg_G(v) = 2|E|$.

We have $2|E|$ is even (multiplication with an even number) and $\sum_{v \in \mathrm{V}_e} deg_G(v)$ is even (sum of even numbers). Then $\sum_{v \in \mathrm{V}_o} deg_G(v)$ must also be even.

This can only be the case if there are an even number of elements (sum of odd numbers). $\square$

Existence of sequences

Which of these sequences are degrees sequences of a graph:

~~(1, 1, 1, 2, 4, 4, 4)~~ By handshaking lemma

(1, 1, 1, 1, 2, 4, 4) ?

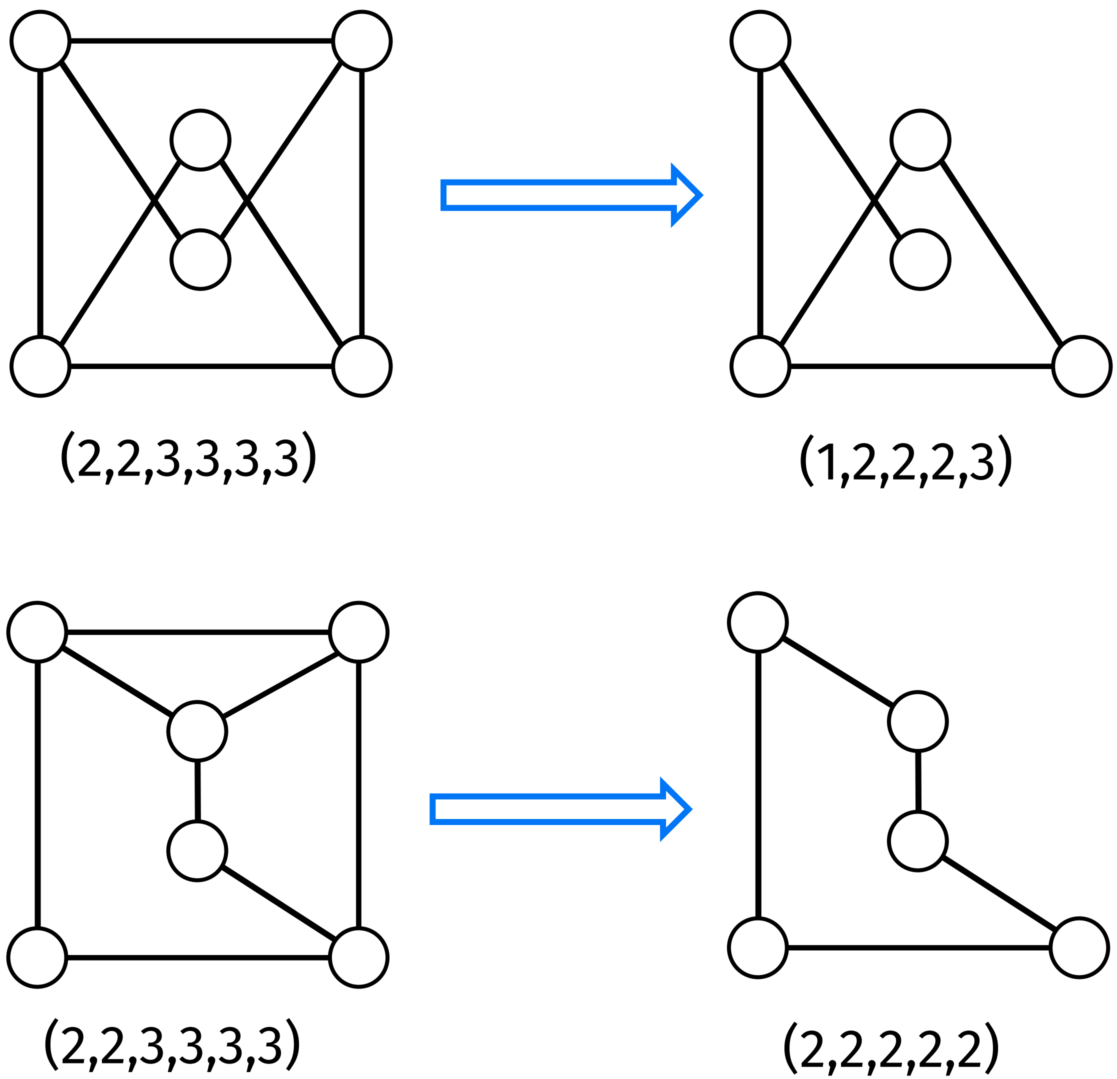
(1, 1, 1, 1, 4, 4, 4)

Sequence that satisfies handshake lemma does not have to be a degree sequence.

(0,0,0,8) is clearly not a degree sequence.

So what next?...

Existence of sequences

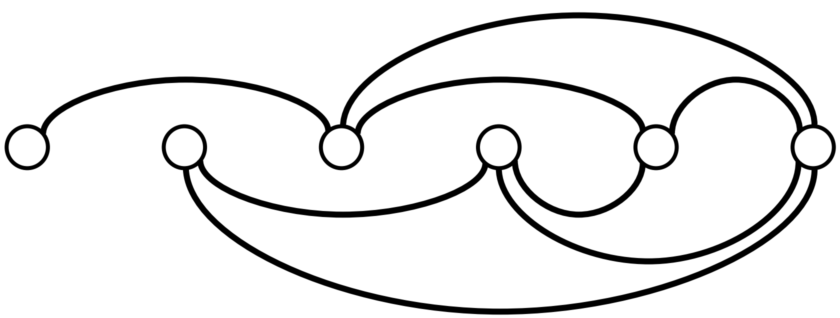


Degree sequences

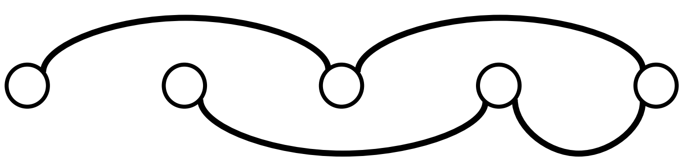
Theorem: Let $D = (d_1, d_2, \ldots, d_n)$ be a sequence of natural numbers, for $n > 1$, where $d_1 \le d_2 \le \cdots \le d_n \le n-1$, and let D' denote the sequence $(d'_1, d'_2 \ldots, d'_{n-1})$, where

$$d'_i = \begin{cases} d_i & \text{for } i < n - d_n \\ d_i - 1 & \text{for } i \ge n - d_n \end{cases}$$

Then D is a degree sequence if and only if D' is a degree sequence.



$(d_1, d_2, \ldots, d_n) = (1, 2, 3, 3, 3, 4)$



$(d'_1, d'_2, \ldots, d'_{n-1}) = (1, 1, 2, 2, 2)$

Degree sequences

Theorem: Let $D = (d_1, d_2, \ldots, d_n)$ be a sequence of natural numbers, for $n > 1$, where $d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$, and let D' denote the sequence $(d_1', d_2' \ldots, d_{n-1}')$, where

$$d_i' = \begin{cases} d_i & \text{for } i < n - d_n \\ d_i - 1 & \text{for } i \geq n - d_n \end{cases}$$

Then D is a degree sequence if and only if D' is a degree sequence and $d_n \leq n - 1$.

Proof (sketch):

To prove: D' is degree sequence $\Rightarrow$ D is degree sequence

To prove: D is degree sequence $\Rightarrow$ D' is degree sequence

Degree sequences

Proof sketch:

D is a degree sequence if and only if D' is a degree sequence

Part 1: D' is a degree sequence $\Rightarrow D$ is a degree sequence

Assume D' is a degree sequence, so there is a graph $G' = (V', E')$,
where $V' = \{v_1, v_2, \dots, v_{n-1}\}$ and $\deg_G(v_i) = d_i'$ for all $i = 1, 2, \dots, n-1$.

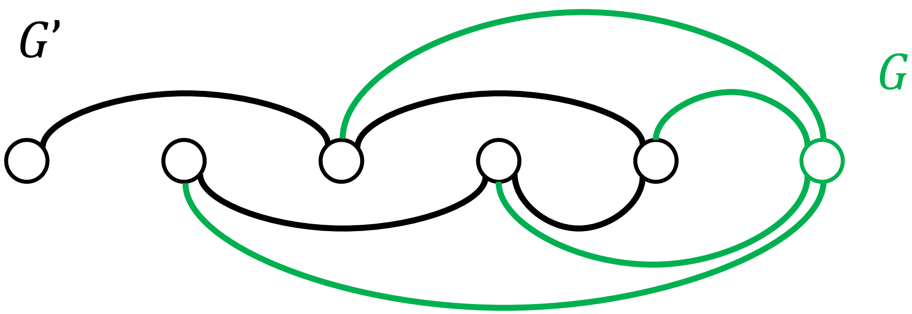
Define a new graph $G = (V, E)$ where $V = V' \cup \{v_n\}$ and

$E = E' \cup \{\{v_i, v_n\}\}$ for $n - d_n \leq i \leq n - 1$.

By construction G has degree sequence D.

D' = (1, 1, 2, 2, 2)

D = (1, 2, 3, 3, 3, 4)



Degree sequences

Proof sketch:

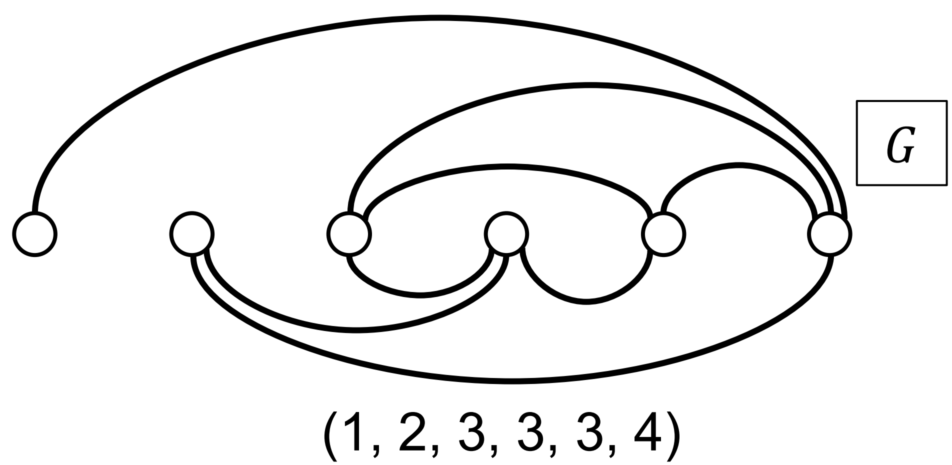
D is a degree sequence if and only if D' is a degree sequence

Part 2: D is a degree sequence $\Rightarrow$ D' is a degree sequence

Assume D is a degree sequence, so there is a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $\deg_G(v_i) = d_i$ for all $i = 1, 2, \ldots, n$.

D' = (1, 1, 2, 2, 2)

D = (1, 2, 3, 3, 3, 4)



Degree sequences

Proof sketch:

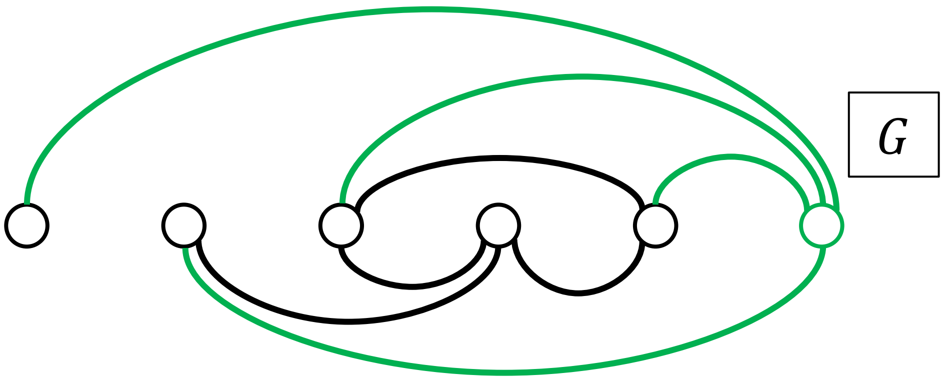
D is a degree sequence if and only if D' is a degree sequence

Part 2: D is a degree sequence ⇒ D' is a degree sequence

Assume D is a degree sequence, so there is a graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ and $\deg_G(v_i) = d_i$ for all $i = 1, 2, \dots, n$.

D' = (1, 1, 2, 2, 2)

D = (1, 2, 3, 3, 3, 4)



(1, 2, 3, 3, 3, 4)

Degree sequences

Proof sketch:

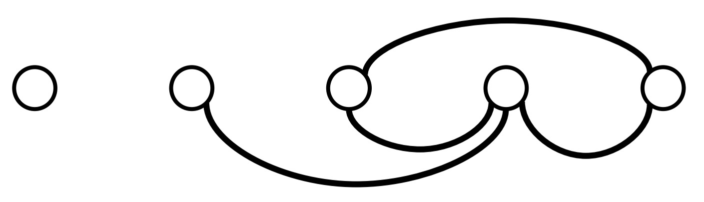
D is a degree sequence if and only if D’ is a degree sequence

Part 2: D is a degree sequence ⇒ D’ is a degree sequence

Assume D is a degree sequence, so there is a graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $\deg_G(v_i) = d_i$ for all $i = 1, 2, \ldots, n$.

D’ = (1, 1, 2, 2, 2)

D = (1, 2, 3, 3, 3, 4)



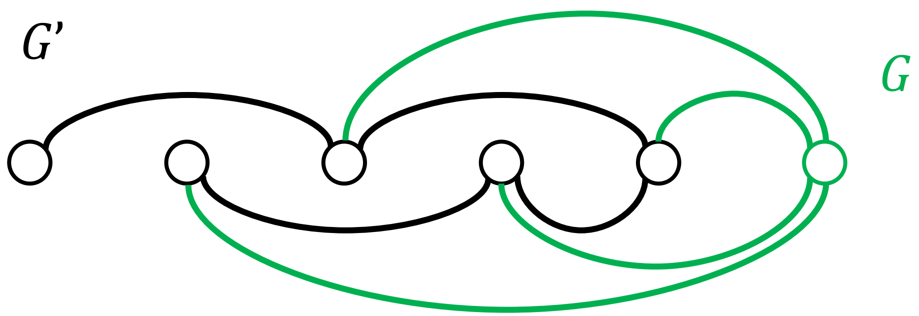
$\boxed{G}$

(0, 1, 2, 2, 3)

Degree sequences

Lemma: Let $D = (d_1, d_2, \dots, d_n)$ be a degree sequence, where $d_1 \leq d_2 \leq \cdots \leq d_n \leq n-1$ and $n > 1$. Then there is a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ and $\deg_G(v_i) = d_i$. Where v_n is adjacent exactly to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

D = (1, 2, 3, 3, 3, 4)



Degree sequences

Lemma: Let $D = (d_1, d_2, \dots, d_n)$ be a degree sequence, where $d_1 \leq d_2 \leq \dots \leq d_n \leq n-1$ and $n > 1$. Then there is a graph $G = (V, E)$ with $V = \{v_1, v_2, \dots, v_n\}$ and $\deg_G(v_i) = d_i$. Where v_n is adjacent exactly to vertices $v_{n-d_n}, v_{n-d_n+1}, \dots, v_{n-1}$.

Proof (sketch):

If $d_n = n-1$, then v_n is connected to all vertices in any graph with D as degrees sequence.

If $d_n < n-1$, for a graph G of degree sequence D, let $j(G)$ be the largest index $j \in \{1,2,\dots,n-1\}$ such that $\{v_j, v_n\} \notin \mathrm{E}(G)$. Let G_0 be a graph with degree sequence D and smallest value of $j(G)$.

We prove $j(G_0) = n - d_n - 1$.

Assume for contradiction that $j(G_0) > n - d_n - 1$.

v_n must be adjacent to a vertex v_i where $i < j$. $\deg(v_i) \leq \deg(v_j)$ and v_i is adjacent to v_n and v_j is not, so there a v_k adjacent to v_j and not v_i.

Create G_0' by removing $\{v_j, v_k\}$ and $\{v_i, v_n\}$ and adding $\{v_j, v_n\}$ and $\{v_i, v_k\}$. G_0' has the same degree sequence but $j(G_0') < j(G_0)$.

Degree sequences

Proof sketch:

D is a degree sequence if and only if D' is a degree sequence

Part 2: D is a degree sequence ⇒ D' is a degree sequence

Assume D is a degree sequence, then from lemma:

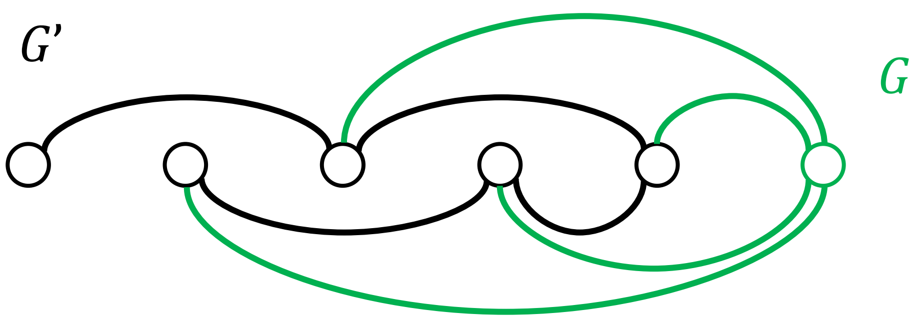
there is a graph $G = (V, E)$ with degree sequence D, where v_n is exactly adjacent to vertices v_i for $n - d_n \leq i \leq n - 1$.

Define a new graph $G' = (V', E')$ where $V' = V \setminus \{v_n\}$ and $E' = E \setminus \{\{v_i, v_n\}: n - d_n \leq i \leq n - 1\}$.

G' has degree sequence D'.

D' = (1, 1, 2, 2, 2)

D = (1, 2, 3, 3, 3, 4)



Degree sequences

So what do we do with this?...

We can easily determine what is a degree sequence!

(1,1,1,1,2,4,4)

(0,0,1,1,1,3)

(0,0,0,0,0)

(1,1,1,1,4,4,4)

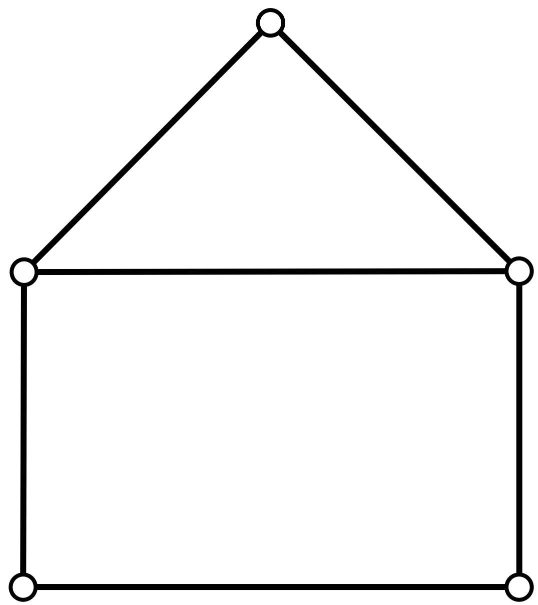
(0,0,1,1,3,3)

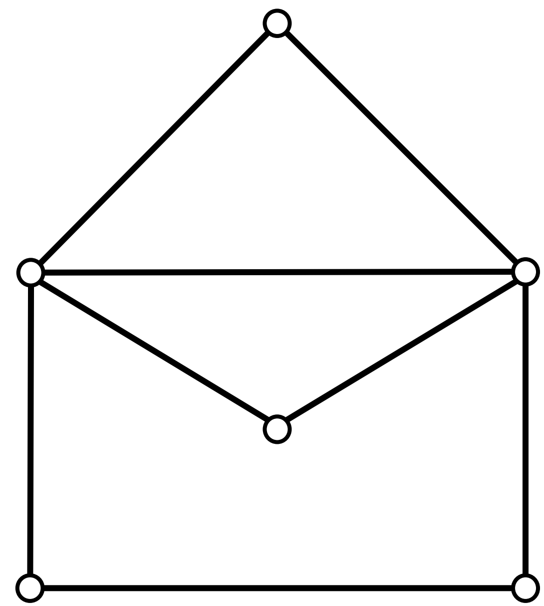
(0,0,0,0,2)

What about (1,1,1,2,2,2,3,3,3,4,4,4,5,5,8,10)?

Special types of paths

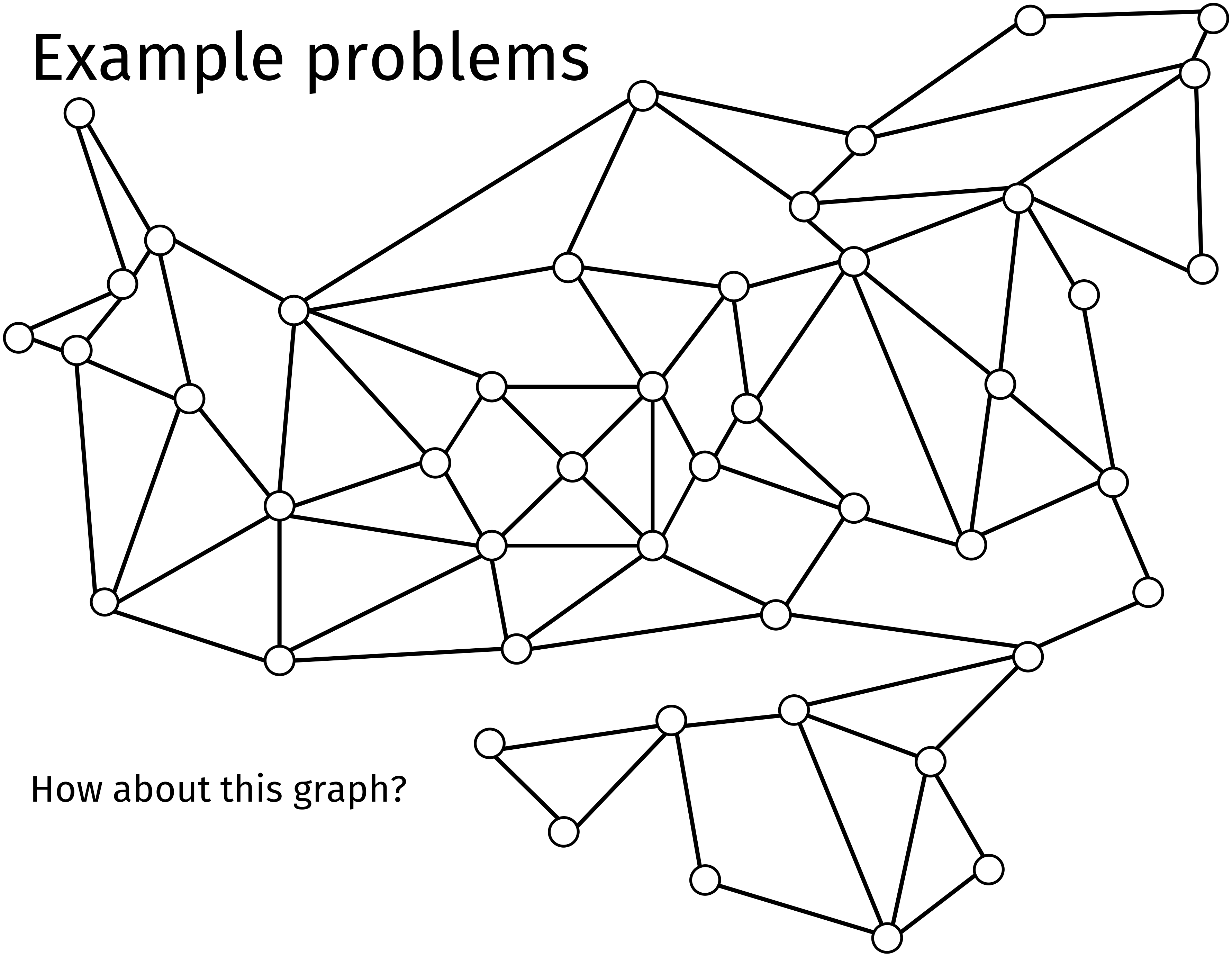
Example problems





Is it possible to draw the given figure without lifting your pen?
What about when you have to end at the same point you started?

Example problems



How about this graph?

Closed Eulerian tour

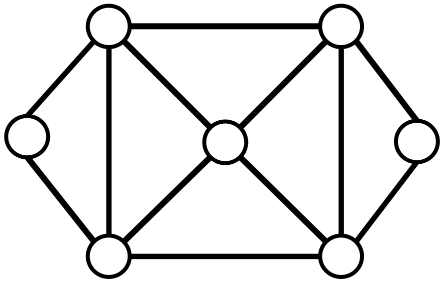
A closed walk containing all the vertices and edges, and each edge exactly once.

Warning: Wikipedia definition does not require this.

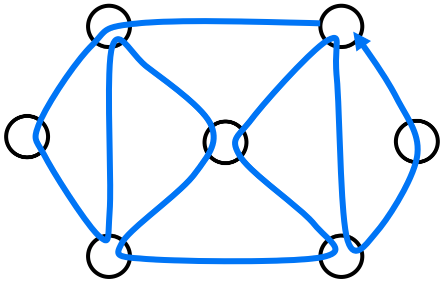
Eulerian Graph

A graph possessing a closed Eulerian tour.

Example:

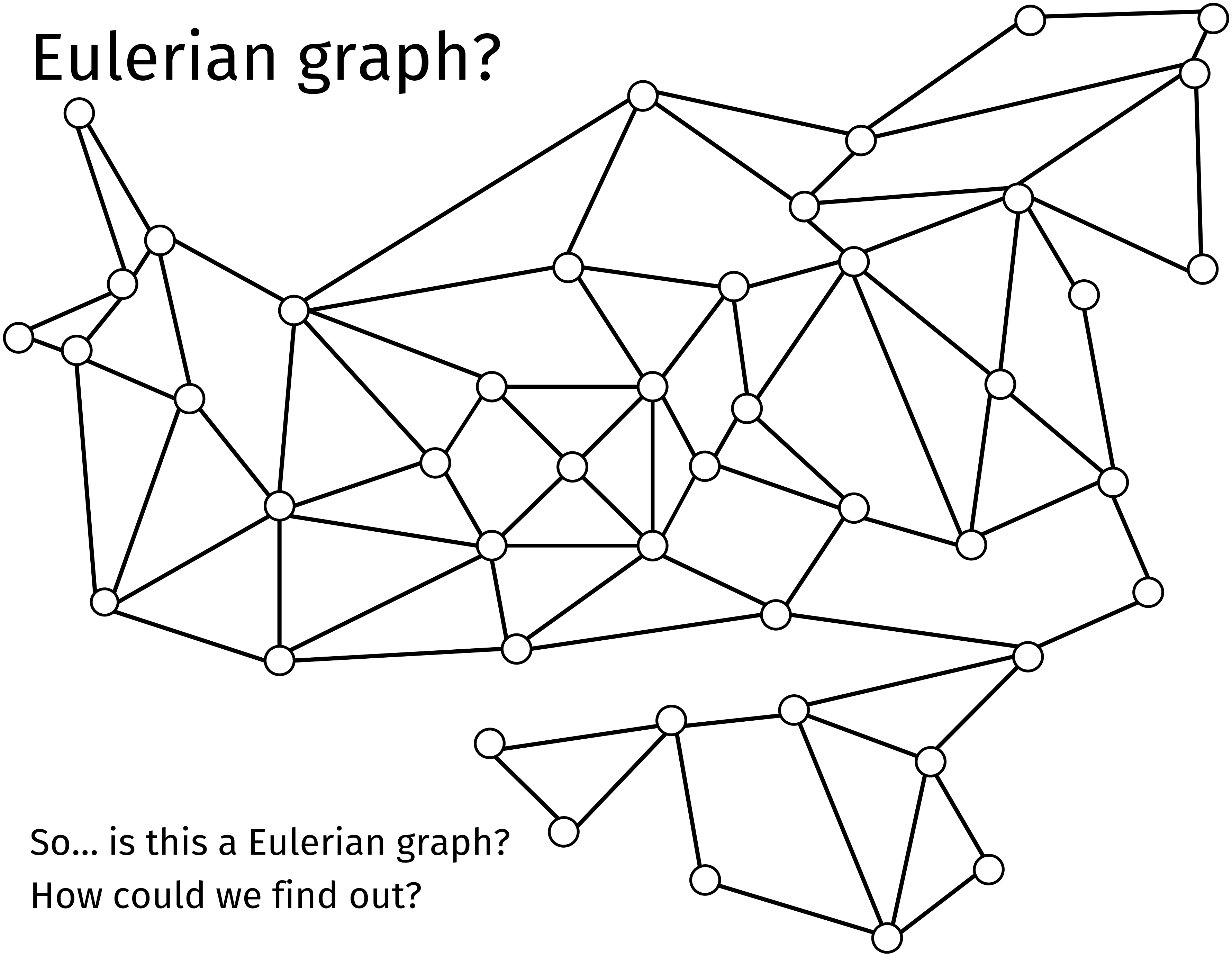


Eulerian graph



Closed Eulerian tour

Eulerian graph?



So... is this a Eulerian graph?
How could we find out?

Eulerian graph

Theorem: A graph $G = (V, E)$ is Eulerian if and only if it is connected and each vertex has an even degree.

Proof (sketch):

G is Eulerian ⇒ even degree + connected

Assume that $G = (V, E)$ is Eulerian. Then there exists an Eulerian tour

The tour proves that G is connected.

Consider a closed Eulerian tour T and an arbitrary vertex $v \in \mathrm{V}$. We classify the edges of v as incoming and outgoing based on T.

But then there must be equally incoming and outgoing edges, so the degree is even.

As we picked v arbitrarily the claim must hold for all vertices.

Eulerian graph

Theorem: A graph $G = (V, E)$ is Eulerian if and only if it is connected and each vertex has an even degree.

Proof (sketch)

Even degree + connected $\Rightarrow$ G is Eulerian

Assume G is connected and every vertex has an even degree. We define a tour as a walk in G in which no edge is repeated.

Let $T = (v_0, e_1, v_2, \dots, e_m, v_m)$ be a tour of maximal length m. We prove that:

1. $v_0 = v_m$
2. $\{e_i\colon i = 1,2, \dots, m\} = E$.

Eulerian graph

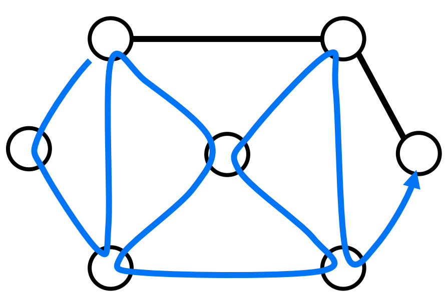
Theorem: A graph $G = (V, E)$ is Eulerian if and only if it is connected and each vertex has an even degree.

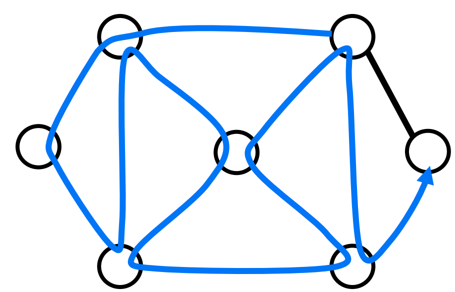
Proof (sketch)

Even degree + connected $\Rightarrow G$ is Eulerian

1. $v_0 = v_m$

Assume $v_0 \neq \mathrm{v}_m$, then v_0 is incident to an odd number of edges in T. We can extend the tour and T is not maximal. Contradiction.





Eulerian graph

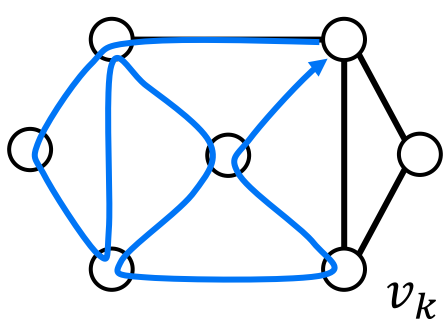
Theorem: A graph $G = (V, E)$ is Eulerian if and only if it is connected and each vertex has an even degree.

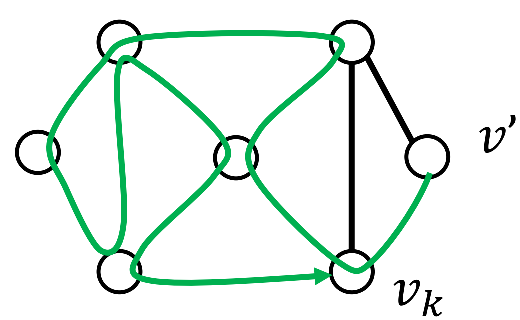
Proof (sketch)

Even degree + connected ⇒ G is Eulerian

1. $v_0 = v_m$
2. $\{e_i\colon i = 1,2, \dots, m\} = E$.

Assume $\{e_i\colon i = 1,2, \dots, m\} \neq E$ and $v_0 = v_m$.
By connectedness there is an edge $e = \{v', v_k\} \notin \mathrm{T}$ such that $v_k \in \mathrm{T}$.
Then the tour $T' = (v', e, v_k, e_{k+1}, \dots, e_m, v_0, e_1, \dots, e_k, v_k)$ has length $m + 1$. Hence T is not maximal. Contradiction.





Eulerian graph

Theorem: A graph $G = (V, E)$ is Eulerian if and only if it is connected and each vertex has an even degree.

Proof (sketch)

G is Eulerian ⇒ even degree + connected

even degree + connected ⇒ G is Eulerian

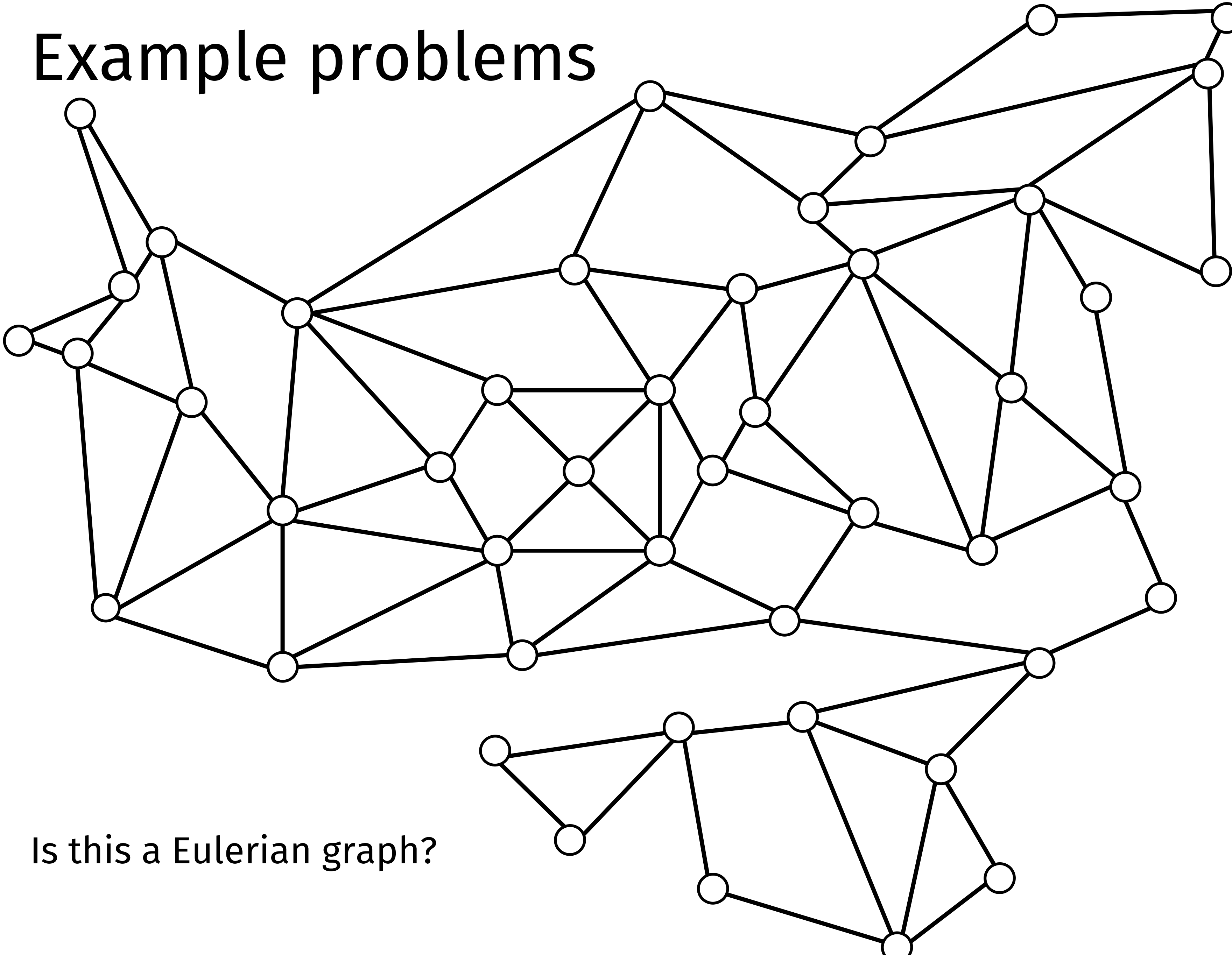
Assume G is connected and every vertex has an even degree. We define a tour as a walk in G in which no edge is repeated.

Let $T = (v_0, e_1, v_2, \dots, e_m, v_m)$ be a tour of maximal length m. We prove that:

1. $v_0 = v_m$
2. $\{e_i\colon i = 1,2, \dots, m\} = E$.

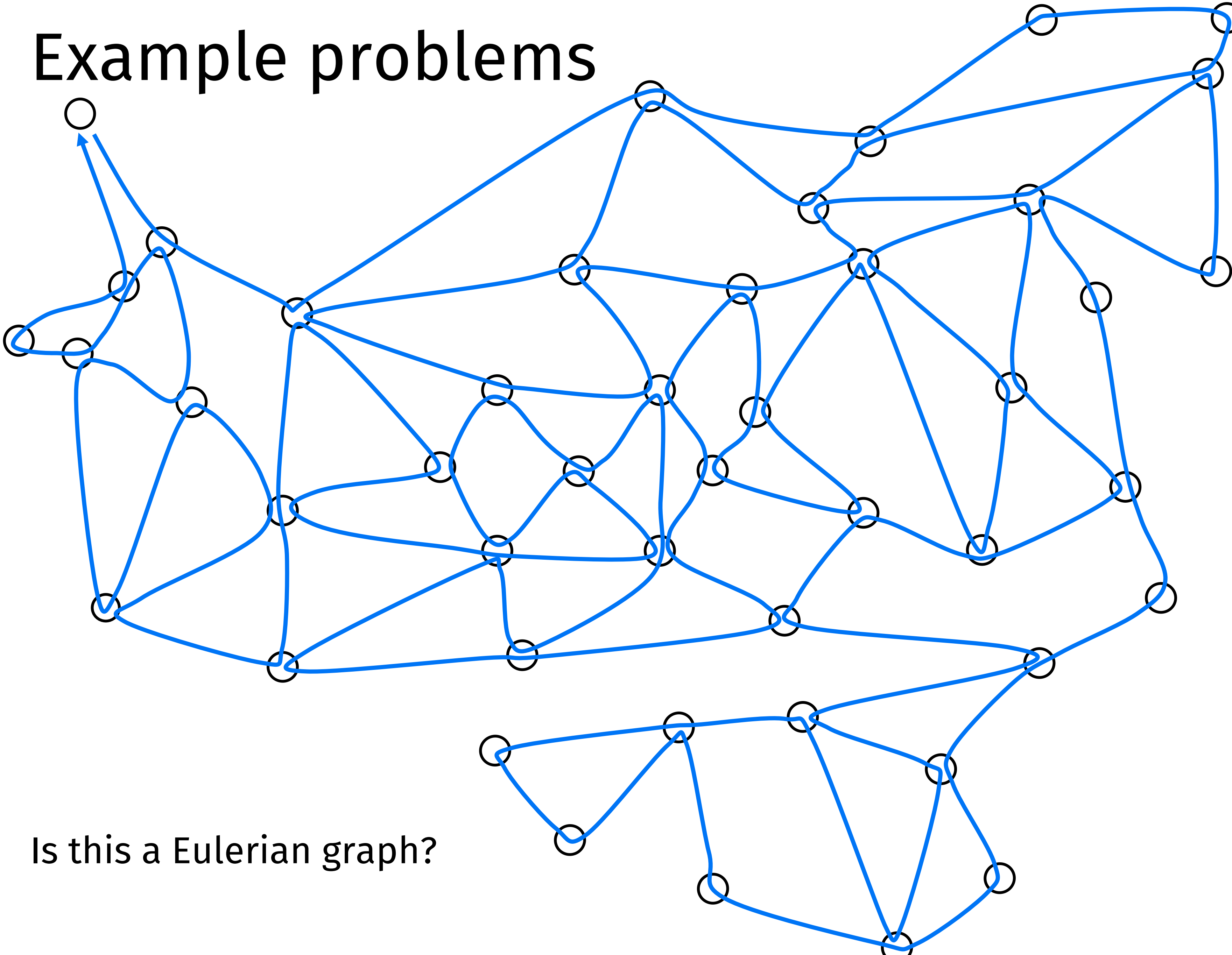
As (1) and (2) are the conditions for a Eulerian tour, G is Eulerian.

Example problems



Is this a Eulerian graph?

Example problems

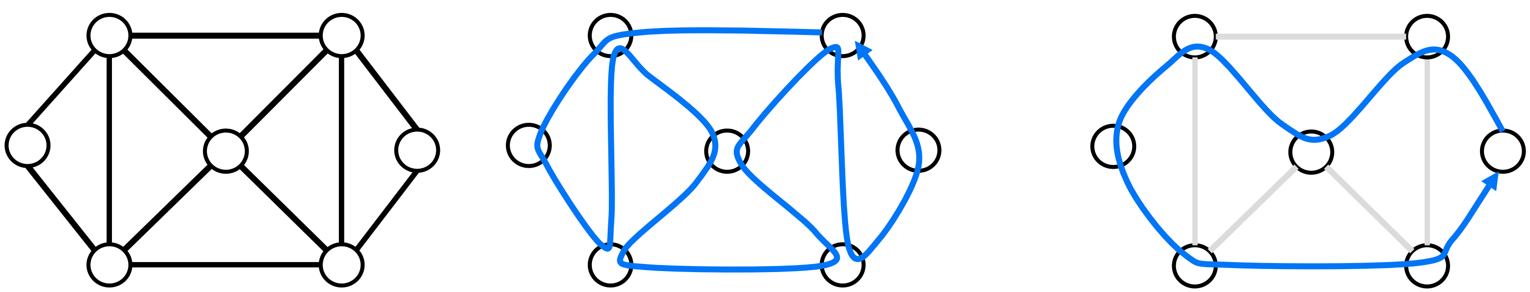


Is this a Eulerian graph?

Hamiltonian cycle

A closed walk containing all the vertices exactly once (except the start/end vertex), but possibly only a subset of the edges.

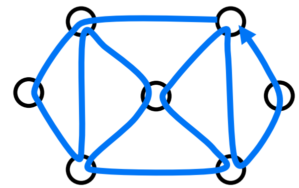
Example:



Eulerian graph Closed Eulerian tour Hamiltonian cycle

Hamiltonian vs. Eulerian

Does $G = (V, E)$ have a Eulerian tour?

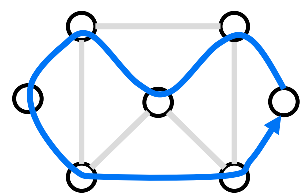


Theorem:

G has a Eulerian tour $\Leftrightarrow$ G is connected, for all $v \in$ V: $\deg(v)$ is even.

Both conditions we can efficiently* check.

Does G=(V,E) have a Hamiltonian cycle?



Theorem:

G has a Hamiltonian cycle $\Leftrightarrow$ something easy to check???

No, this problem is hard. Really, really hard.

NP-hard – we cannot solve it efficiently (even with a computer)*.

*For some definition of efficiently

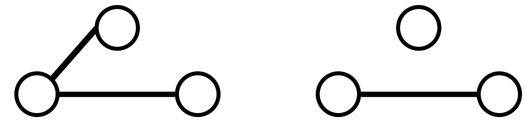
Connections in graphs

Graph operations

Let $G = (V, E)$ be a graph

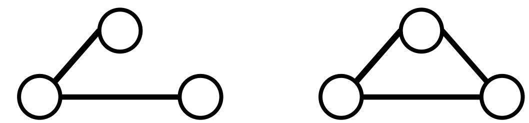
Edge deletion:

$G - e = (V, E \setminus \{e\})$, where $e \in \mathrm{E}$.



Edge insertion:

$G + e = (V, E \cup \{e\})$, where $e \in \binom{V}{2} \setminus \mathrm{E}$.

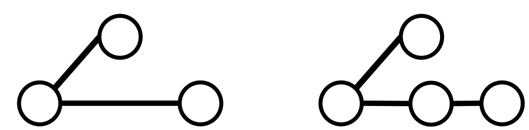


Vertex deletion:

$G - v = (V \setminus \{v\}, \{e \in \mathrm{E} \colon v \notin \mathrm{e}\})$, where $v \in \mathrm{V}$.

Edge subdivision:

$G\%e = (V \cup \{z\}, (E \setminus \{\{x, y\}\}) \cup \{\{x, z\}, \{z, y\}\})$,
where $e = \{x, y\} \in \mathrm{E}$ and $z \notin \mathrm{V}$.

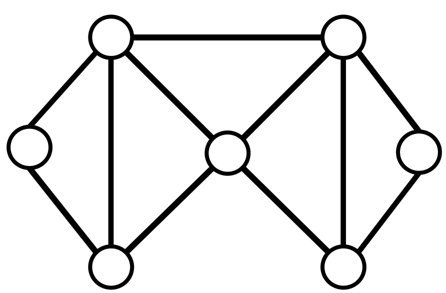


k-vertex-connectivity

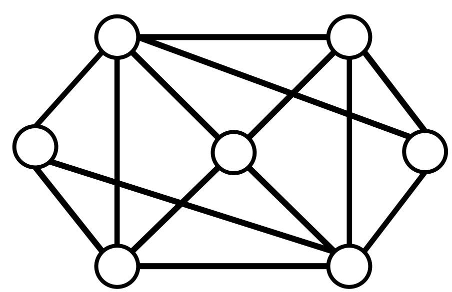
A graph G is called k-vertex-connected if it has at least $k + 1$ vertices and by deleting any $k - 1$ vertices we obtain a connected graph. Often this is abbreviated to k-connected.

For example a graph is 2-connected if it has at least 3 vertices and deleting any 1 vertex does not create a disconnected graph.

Example:



2-connected

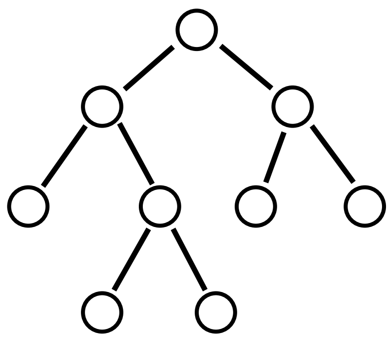


3-connected

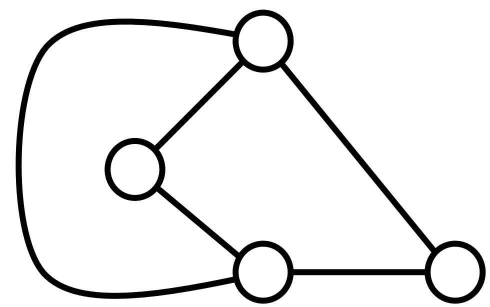
Example

For which k is the graph below k-connected.

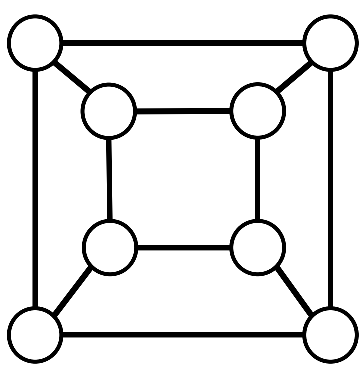
1)



2)



3)

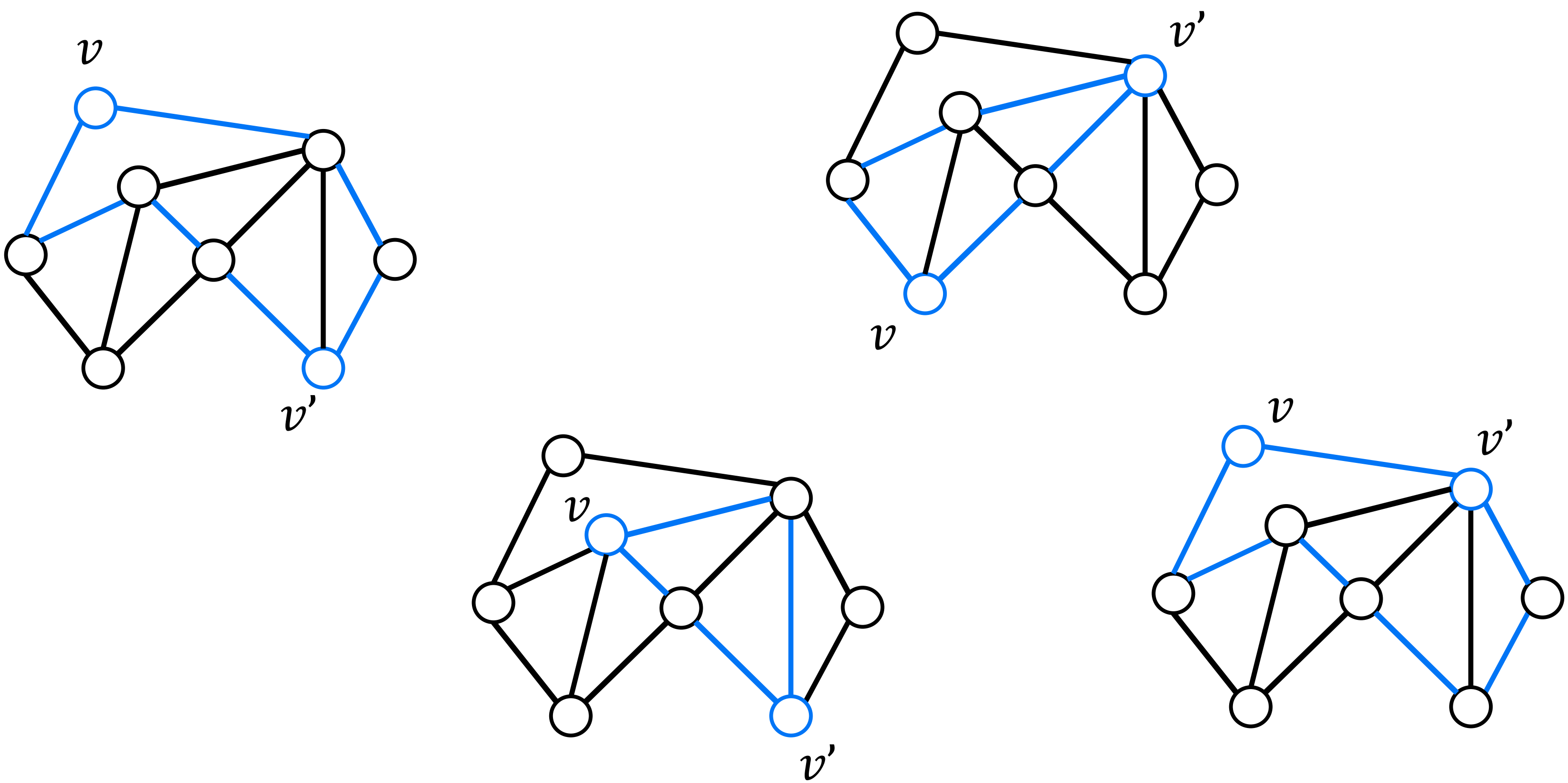


4) The complete graph K_5

5) The complete bipartite graph $K_{3,5}$

2-connected

Theorem: A graph $G = (V, E)$ is 2-connected if and only if there exists, for any two vertices $v, v' \in V$, a cycle in G containing v and v'.



2-connected

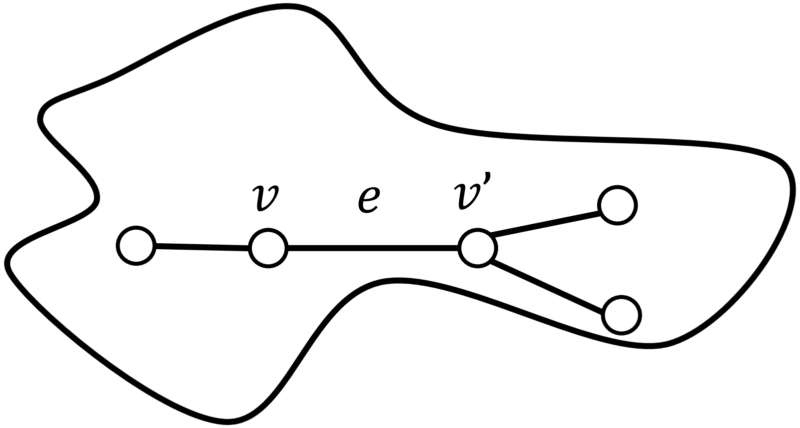
Theorem: A graph $G = (V, E)$ is 2-connected if and only if there exists, for any two vertices $v, v' \in \mathrm{V}$, a cycle in G containing v and v'.

Proof (sketch):

G is 2-connected $\Rightarrow$ every pair on cycle

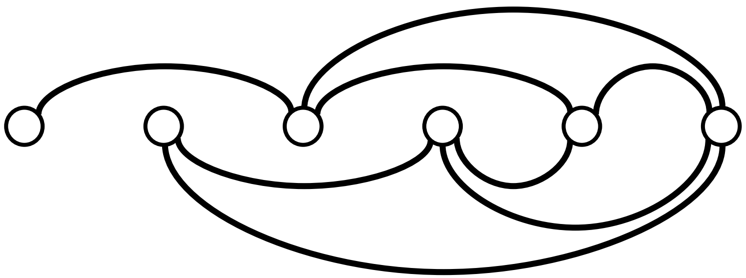
Using induction on $d_G(v, v')$.

See book

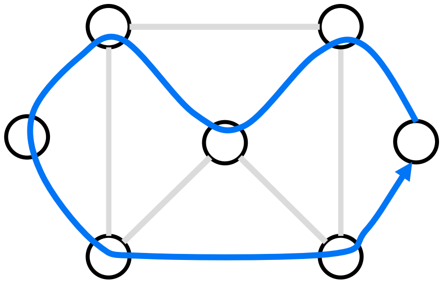


Summary

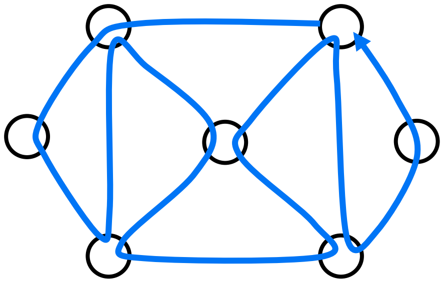
Degree sequences



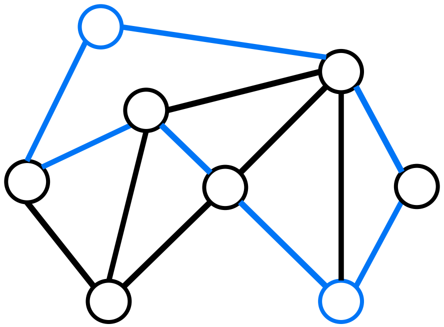
Hamiltonian cycles



Eulerian tour



2-connected graphs



Practical announcements

- Practice set
  - exercise 8,9 (both contain graph proofs)

- Roundtable discussion today (not sure if you can still sign up)