2IT80 Discrete Structures

2023-24 Q2

Lecture 12: Double Counting

EINDHOVEN UNIVERSITY OF TECHNOLOGY

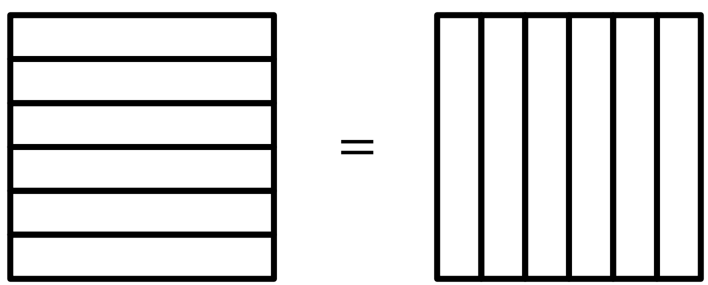
Double Counting as a Technique

Count the same thing in two different ways.

This yields two different formulas, but since they count the same things, they are equal.

For example, if A is a n by m matrix then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$$



Main difficulty: what exactly should be double counted??

Example: Density of planar graphs

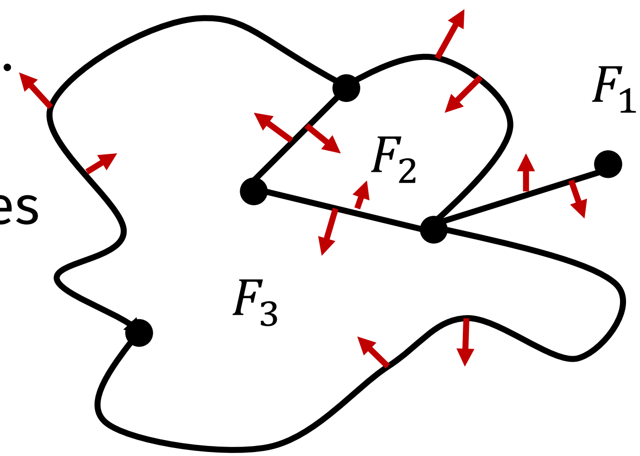
Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then $|E| \leq 3|V| - 6$.

Proof: W.l.o.g. assume G is connected.

Consider a planar drawing of G with k faces.

Denote the faces by $F_1, \dots, F_k$.

Denote by f_i the number of edge adjacencies that face F_i has.



We have $\sum_{i=1}^{k} f_i = 2 \cdot |E|$.

Observe that $f_i \geq 3$ since we do not allow parallel edges.

Therefore $3k \leq 2|E|$.

Euler's formula: $|V| - |E| + k = 2$.

Recap: Orderings

Ordering relation

An ordering relation on a set X is a reflexive, antisymmetric, transitive relation on X.

A partially ordered set (poset) is a pair (X, R) where X is a set and R is an ordering relation on X.

We often use notation $\leq$ and $\preccurlyeq$.

A relation R is a linear or total order if for every two elements x, y we have xRy or yRx.

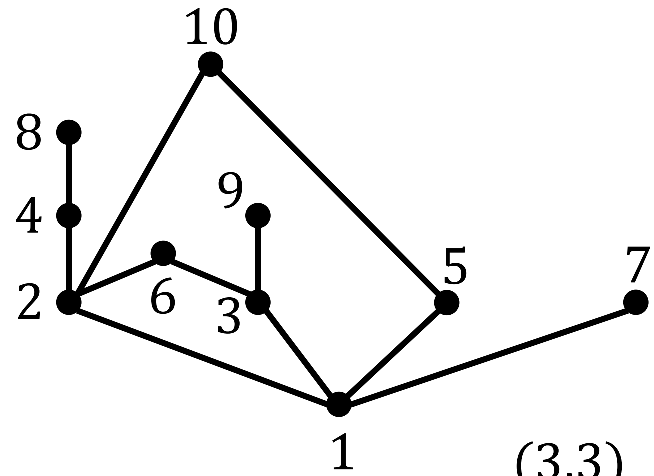
Reflexive, if xRx for all $x \in X$

Antisymmetric if, xRy and yRx implies $x = y$

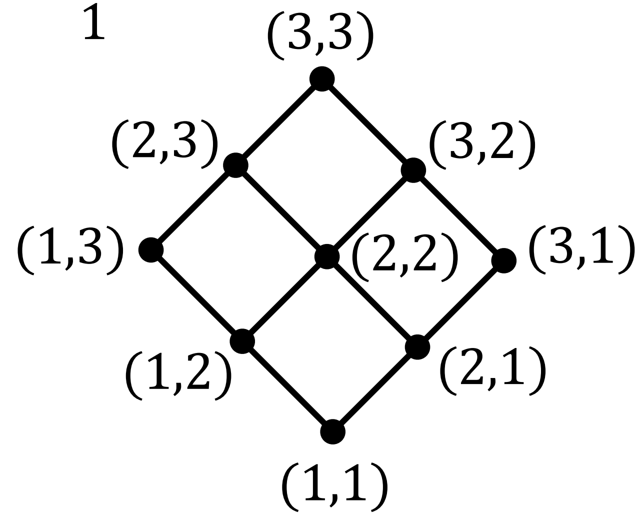
Transitive, if xRy and yRz implies xRz for all $x, y, z \in X$

Hasse diagrams

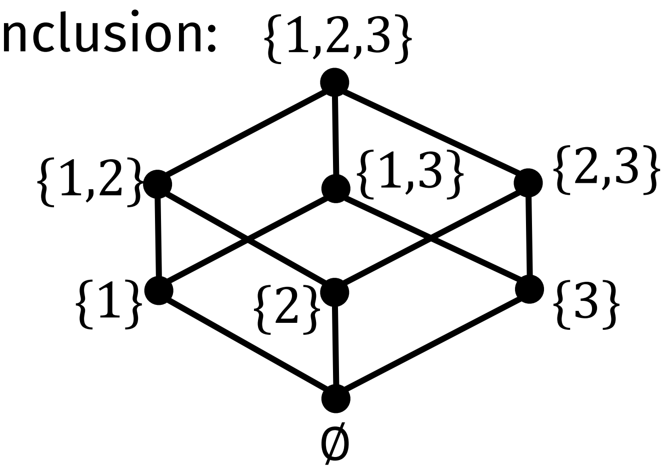
Divisibility relation: $(\{1,2,\ldots,10\},|)$:



Set $\{1,2,3\} \times \{1,2,3\}$ where $(a_1, b_1) \preccurlyeq (a_2, b_2)$
if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$:



Subsets of $\{1,2,3\}$ ordered by inclusion:



Counting sets

Independent sets

Given a set N of n elements, how many subsets are there?

Independent: Two different sets A, B are independent we do not have $A \subseteq B$ or $B \subseteq A$.

Example:

$\{1,2,3,4\}$ and $\{1,2,5,6\}$ are independent

$\{1,2,3,4\}$ and $\{1,3\}$ are not

Independent system of subsets: Set of subsets that are pairwise independent.

Example

$N = \{1,2,3,4\}$

$\{\{1,2\}, \{2,3\}, \{2,4\}, \{1,4\}\}$ is an independent system of subsets of N

How large can an independent system of subsets be?

Independent sets

- What is a largest system of independent subsets of
  - $\{1,2\}$
  - $\{1,2,3\}$
  - $\{1,2,3,4\}$

- Which sets cannot be included in a large system of independent subsets?

Sperner's Theorem

Theorem: Any independent system of subsets of an n-element set contains at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets.

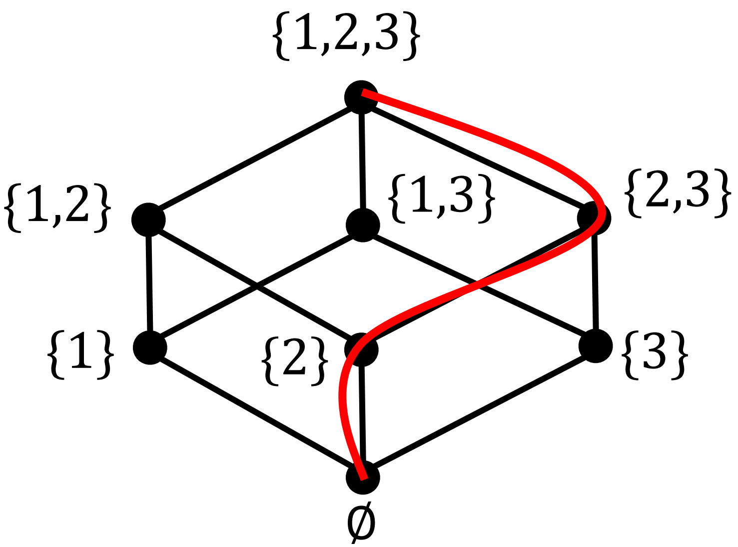
Proof: By double counting with partially ordered sets.

Let $X = \{x_1, \dots, x_n\}$

Consider the poset $(2^X, \subseteq)$

Maximal chain: a sequence of subsets $A_0, \dots, A_n$ with $A_0 = \emptyset$ and $A_{i+1} = A_i \cup \{x_j\}$, where $x_j \in X$ is not in A_i.

For example: $\emptyset \subseteq \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\}$



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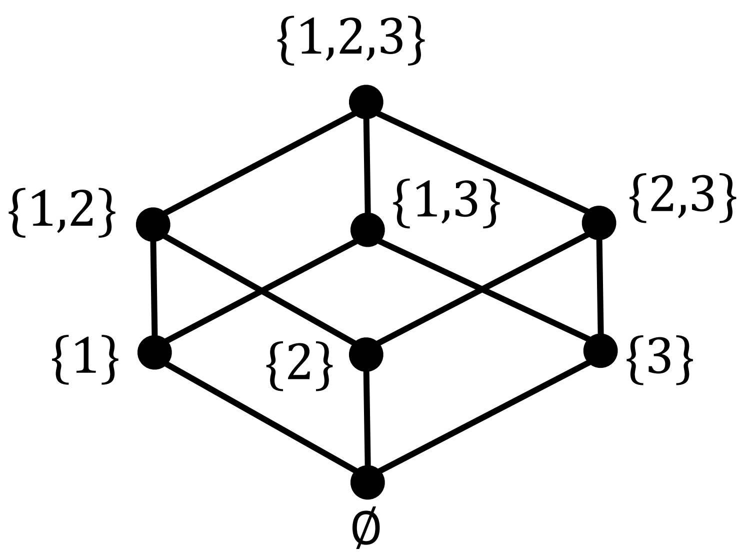

Let I a set of m independent subsets of X

How large can I be?

Count pairs (S, R) where $S \in I$, R a maximal chain, and S is in R.

Sperner's Theorem

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Proof (continued): Let $X = \{x_1, \dots, x_n\}$

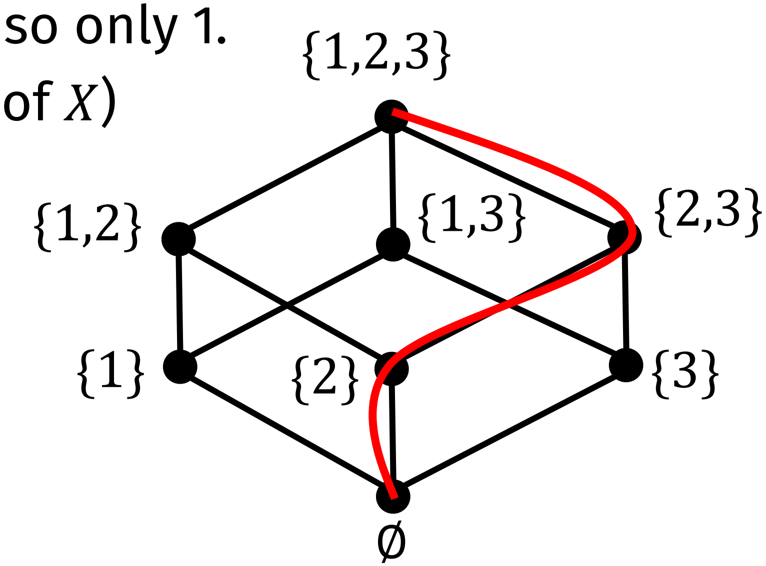
Let I a set of m independent subsets of X

Count pairs (S, R) where $S \in I$, R a maximal chain, and S is in R.

How many pairs can a maximal chain R appear in?

Two sets in a chain are not independent, so only 1.

$\#pairs \leq n!$ (each chain is a permutation of X)



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Proof (continued): Let $X = \{x_1, \dots, x_n\}$

Let I a set of m independent subsets of X

Count pairs (S, R) where $S \in I$, R a maximal chain, and S is in R.

$\#pairs \leq n!$

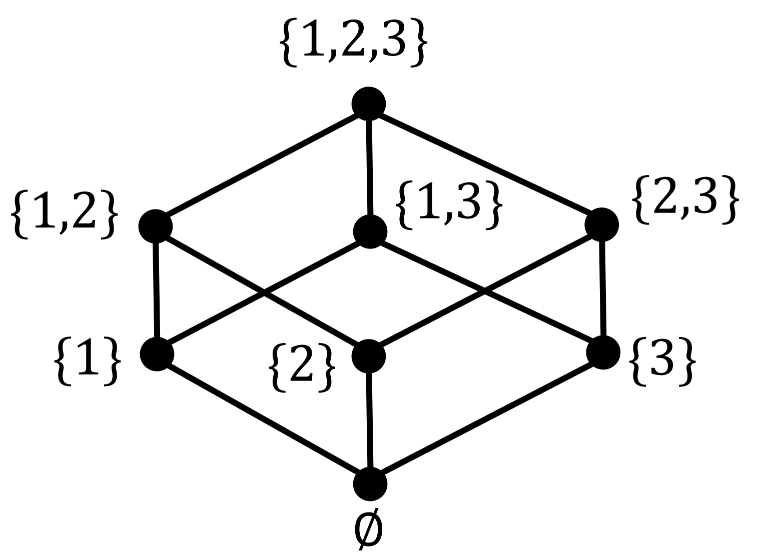
How many pairs (or maximal chains) can a subset S appear in?

If $|S| = k$ then $A_k = S$, $A_i \subseteq A_k$ for $i < k$,

$A_j \supseteq A_k$ for $j > k$

So any R that represents a permutation where the first k elements are exactly those of S. So S appears $k!\,(n-k)!$ times

$$\#pairs = \sum_{S \in I} |S|!\,(n - |S|)!$$



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Proof (continued): Let $X = \{x_1, \dots, x_n\}$

Count pairs (S, R) where $S \in I$, R a maximal chain, and S is in R.

Count pairs (S, R) where $S \in I$ and R a maximal chain!

$\#pairs \leq n!$

$$\#pairs = \sum_{S \in I} |S|!\,(n - |S|)!$$

$$\sum_{S \in I} |S|!\,(n - |S|)! \leq n!$$

$$\sum_{S \in I} \frac{|S|!\,(n - |S|)!}{n!} \leq 1$$

$$\sum_{S \in I} \frac{1}{\binom{n}{|S|}} \leq 1$$

$$\sum_{S \in I} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{S \in I} \frac{1}{\binom{n}{|S|}} \leq 1$$

$$|I| \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq 1$$

$$|I| \leq \binom{n}{\lfloor n/2 \rfloor}$$

A game: HEX

HEX

Game on a graph as follows:

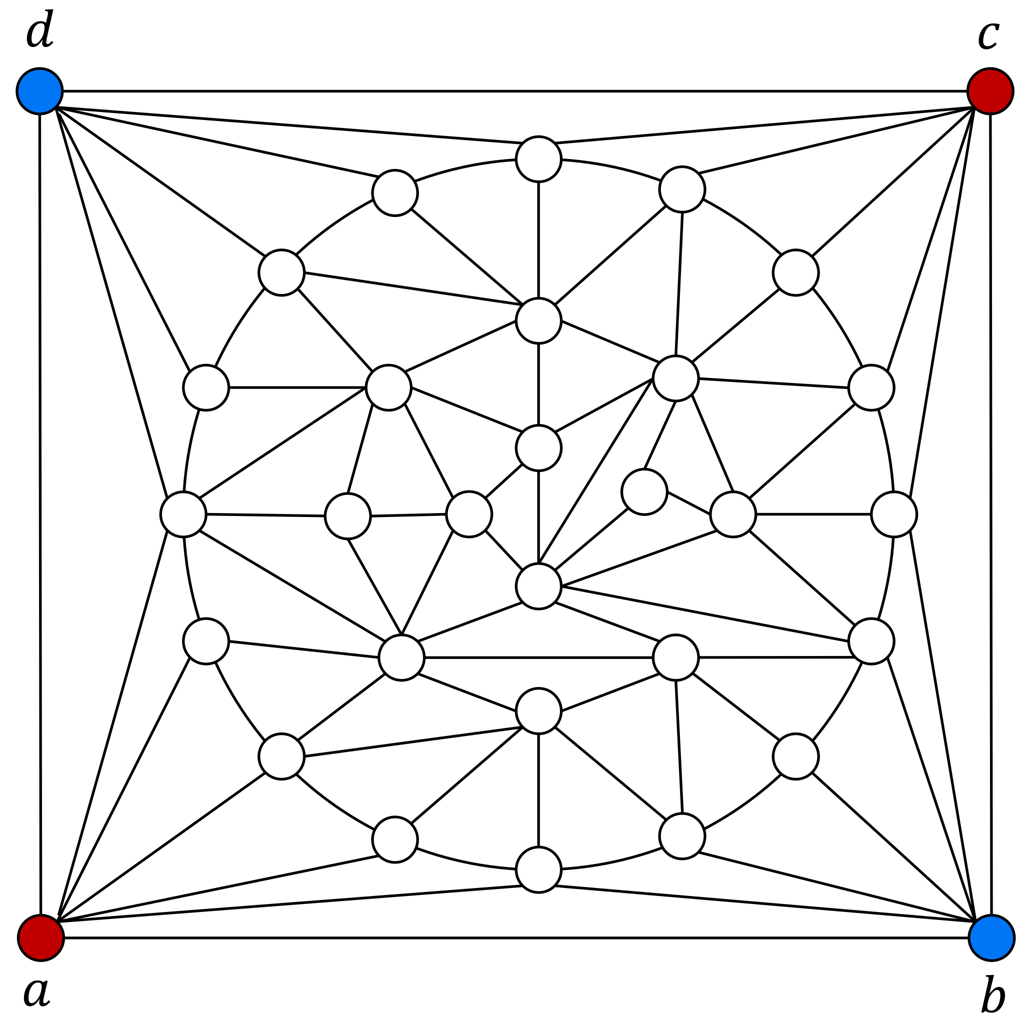
- Plane
- Outer face is a square
- Inner faces are triangles

Two players move alternately:

- Alice marks nodes ○ ⇨ ●
- Betty marks nodes ○ ⇨ ●

Winning conditions:

- Alice wins if she marks all nodes on a path from a to c
- Betty wins if she marks all nodes on a path from b to d



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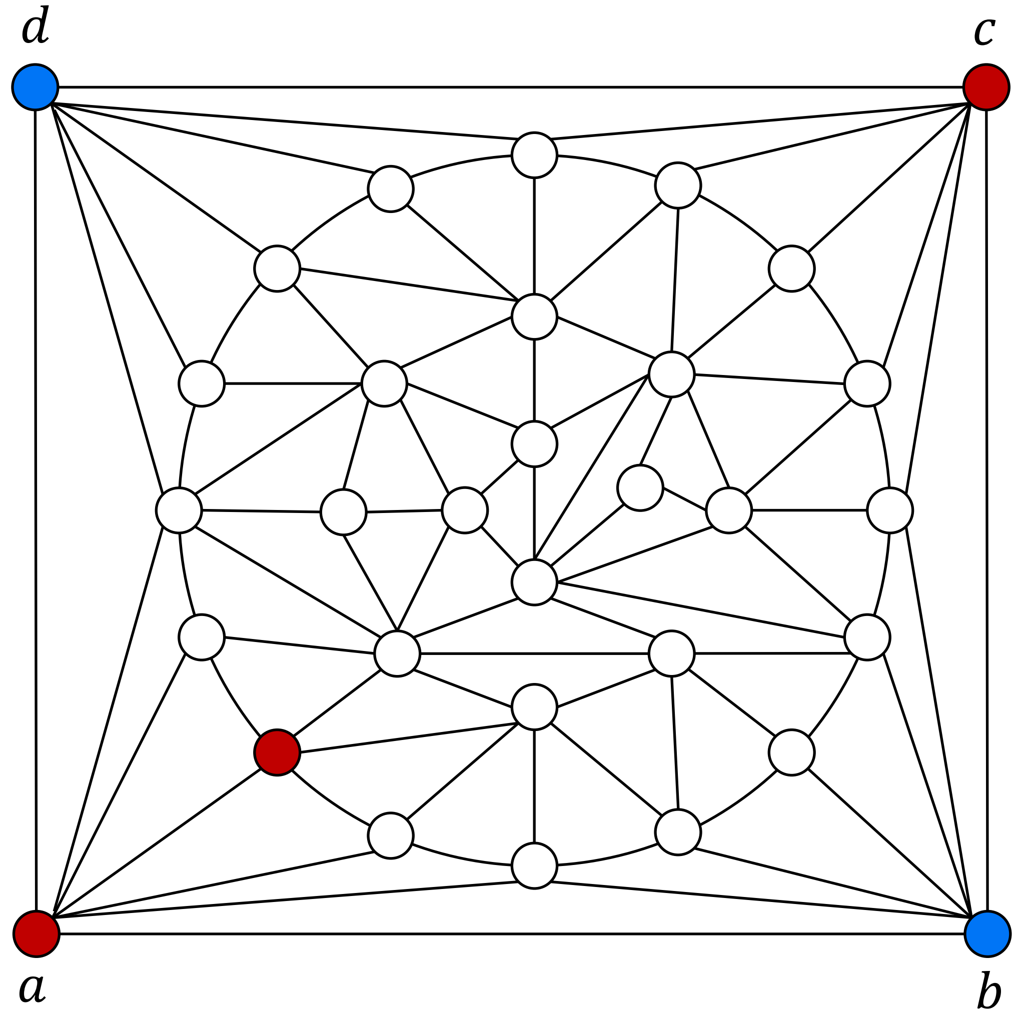

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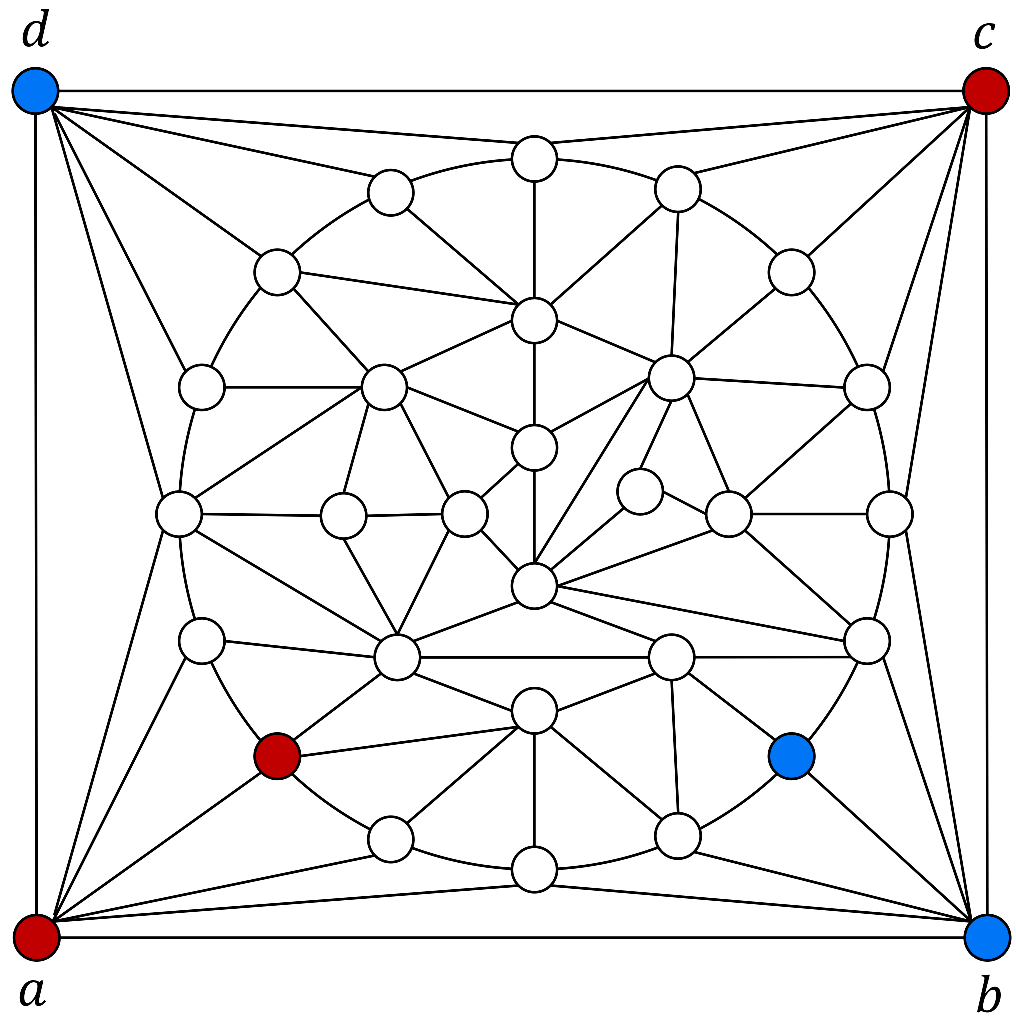

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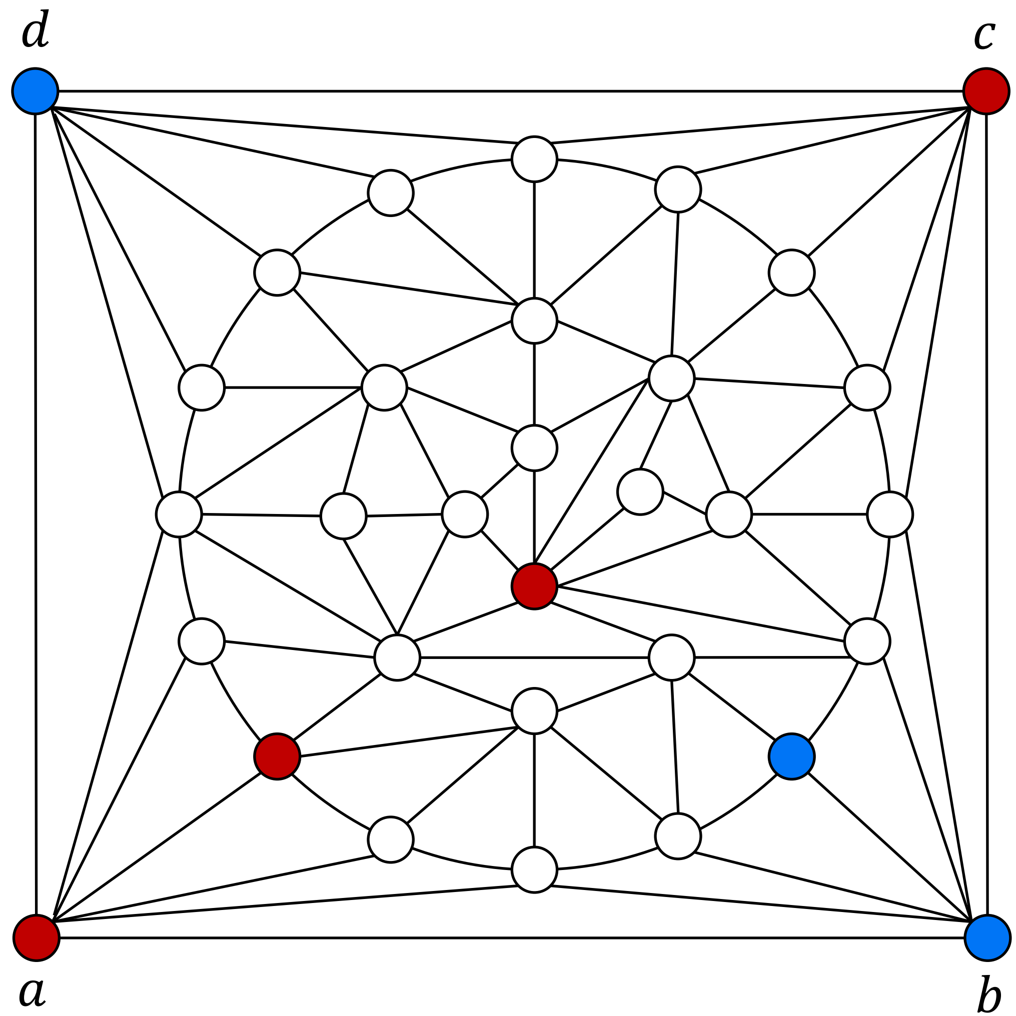

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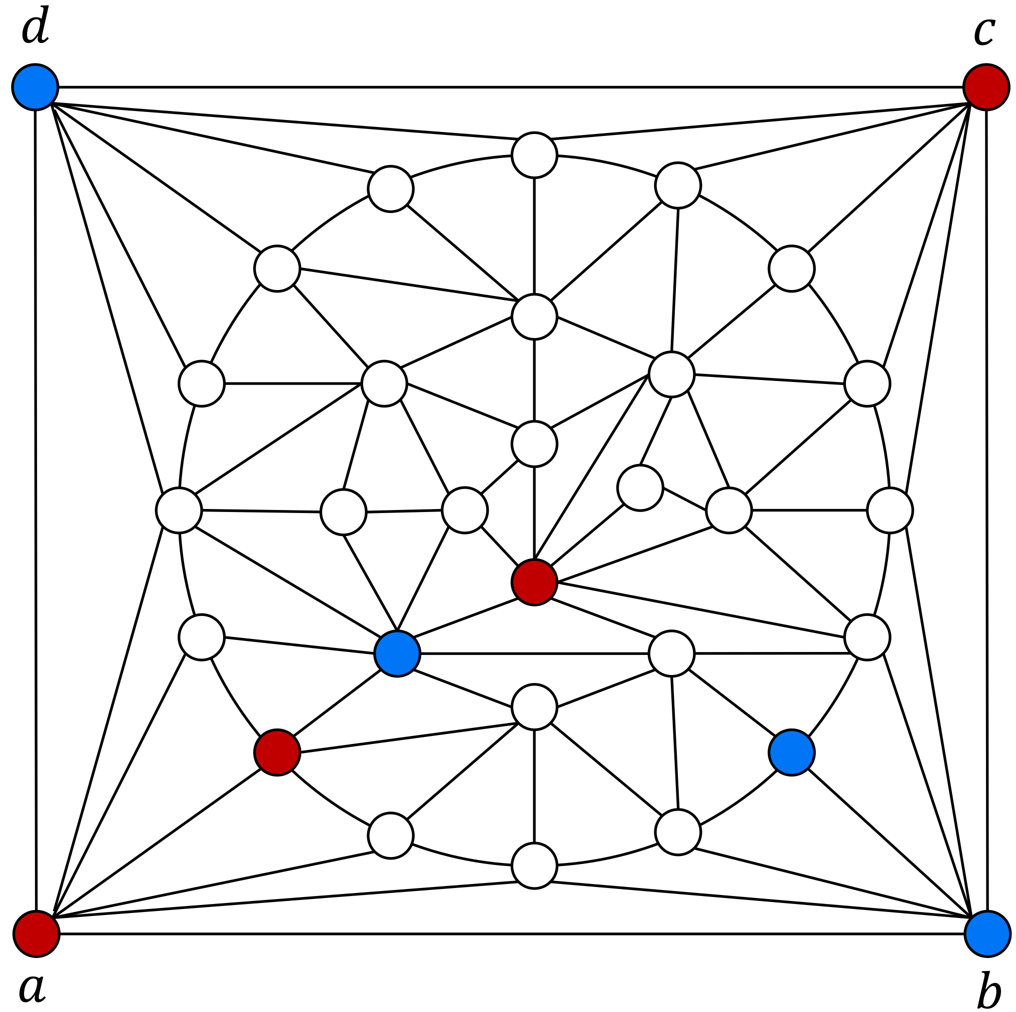

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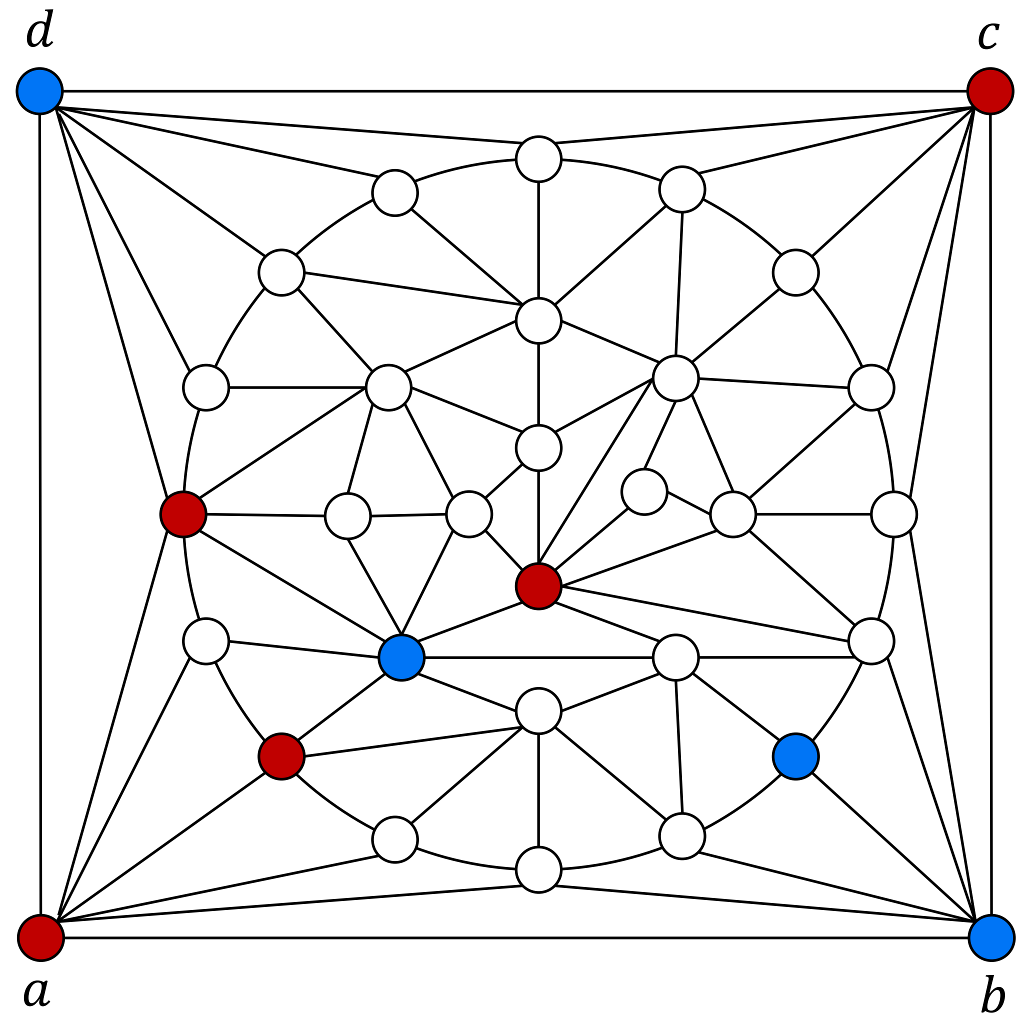

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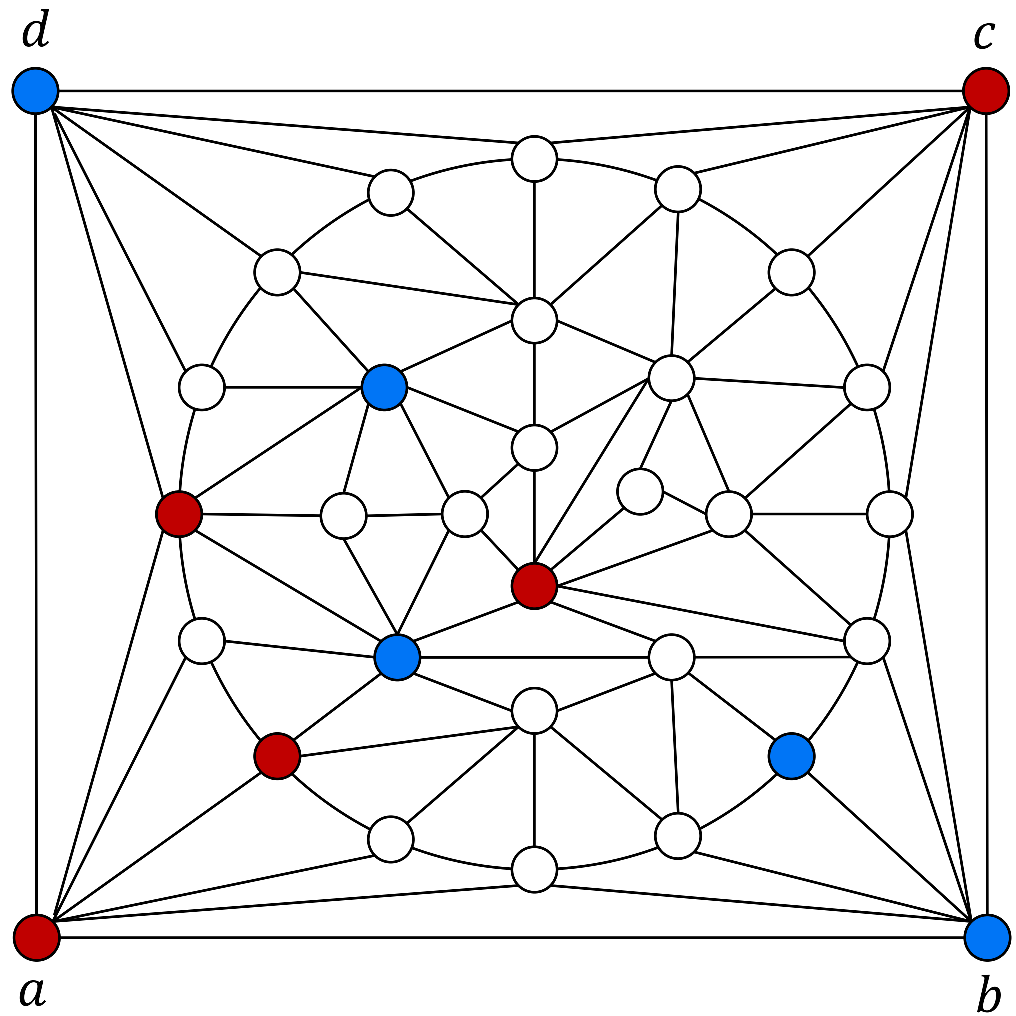

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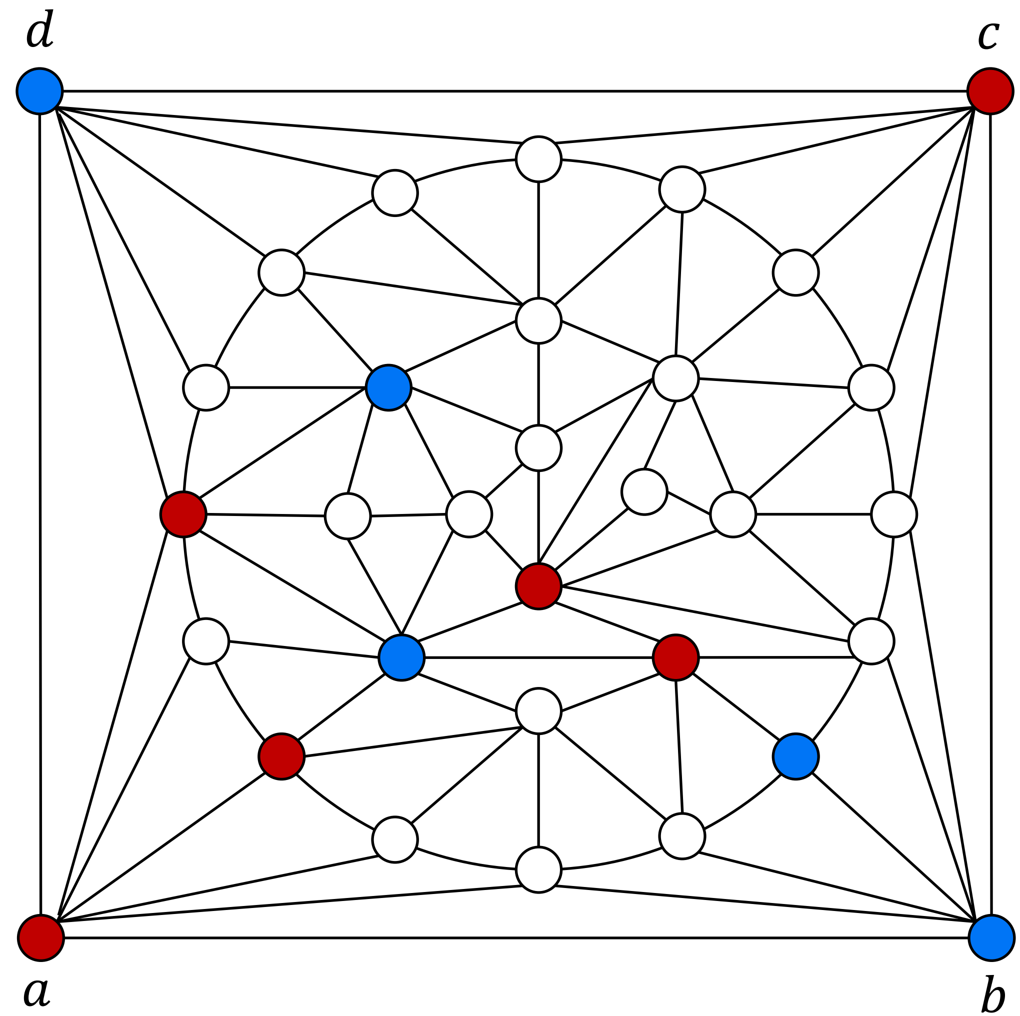

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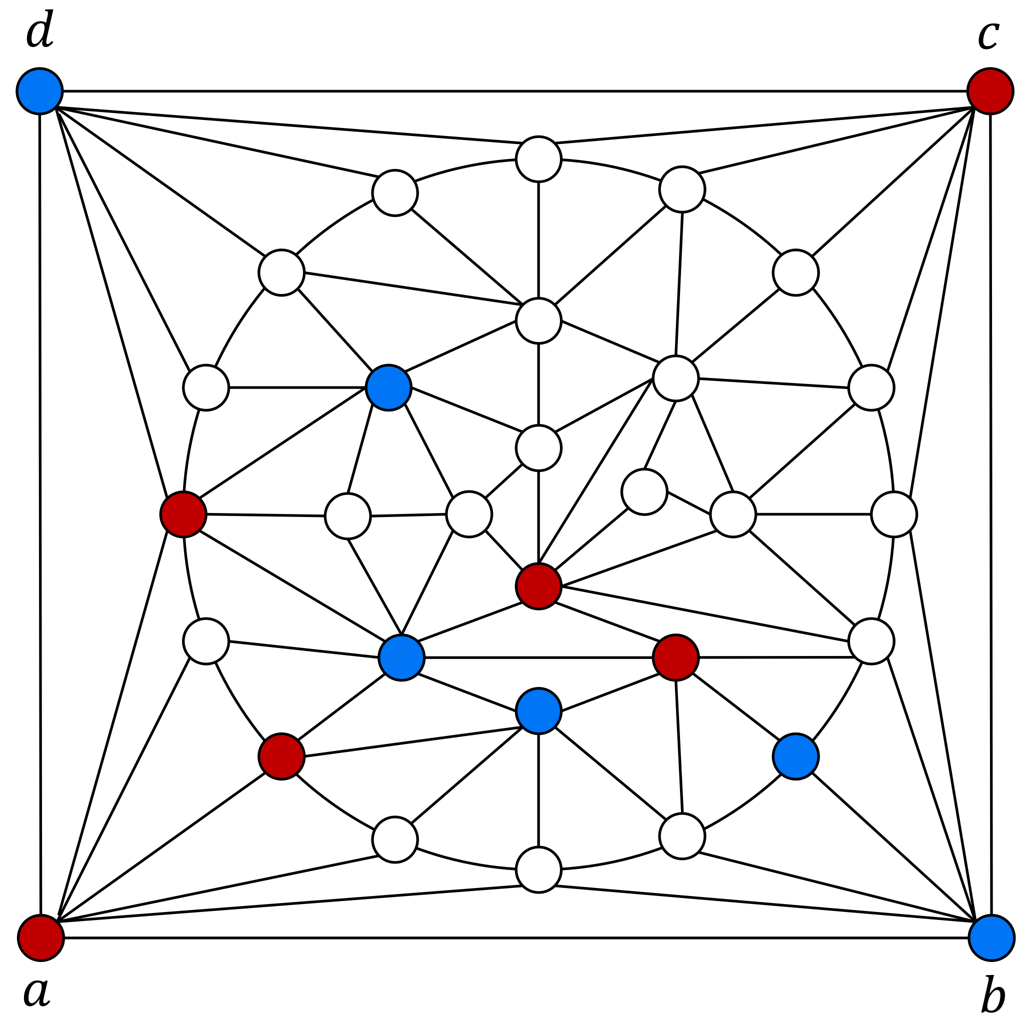

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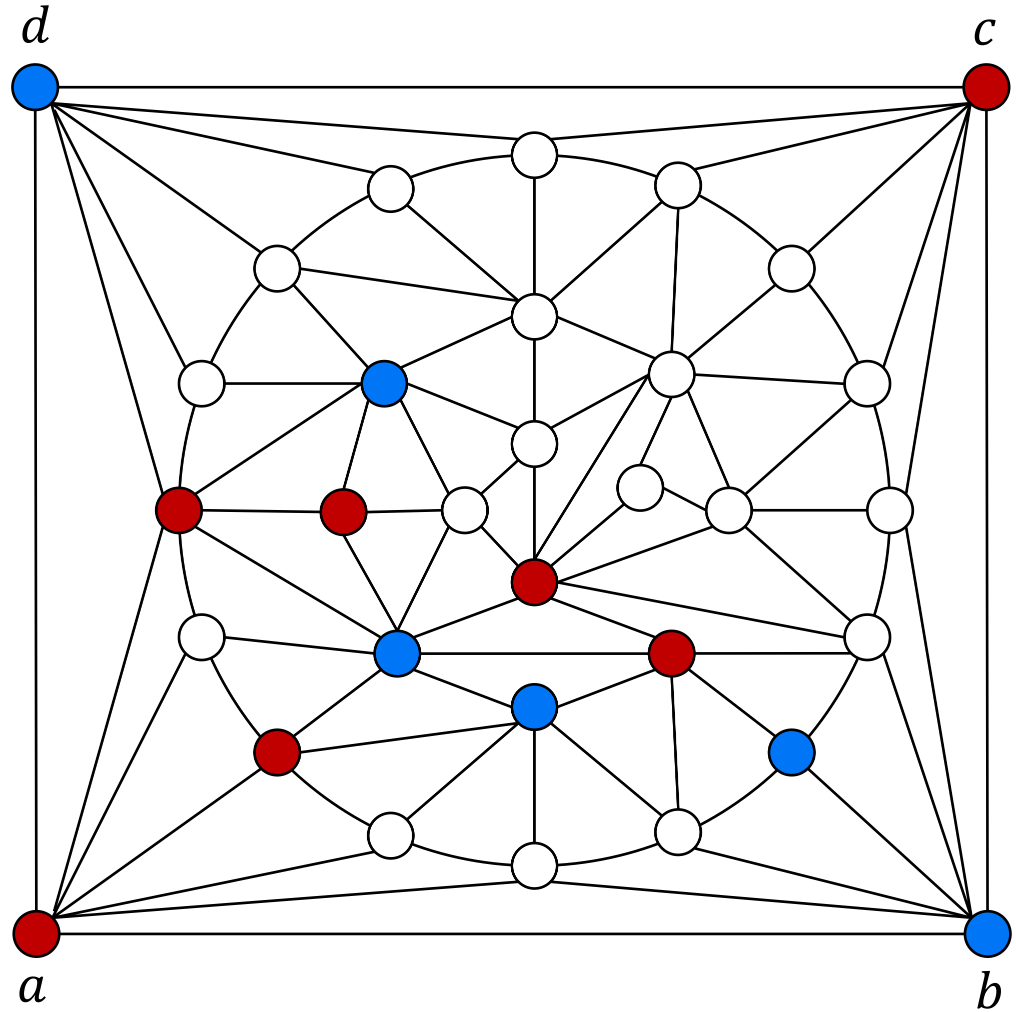

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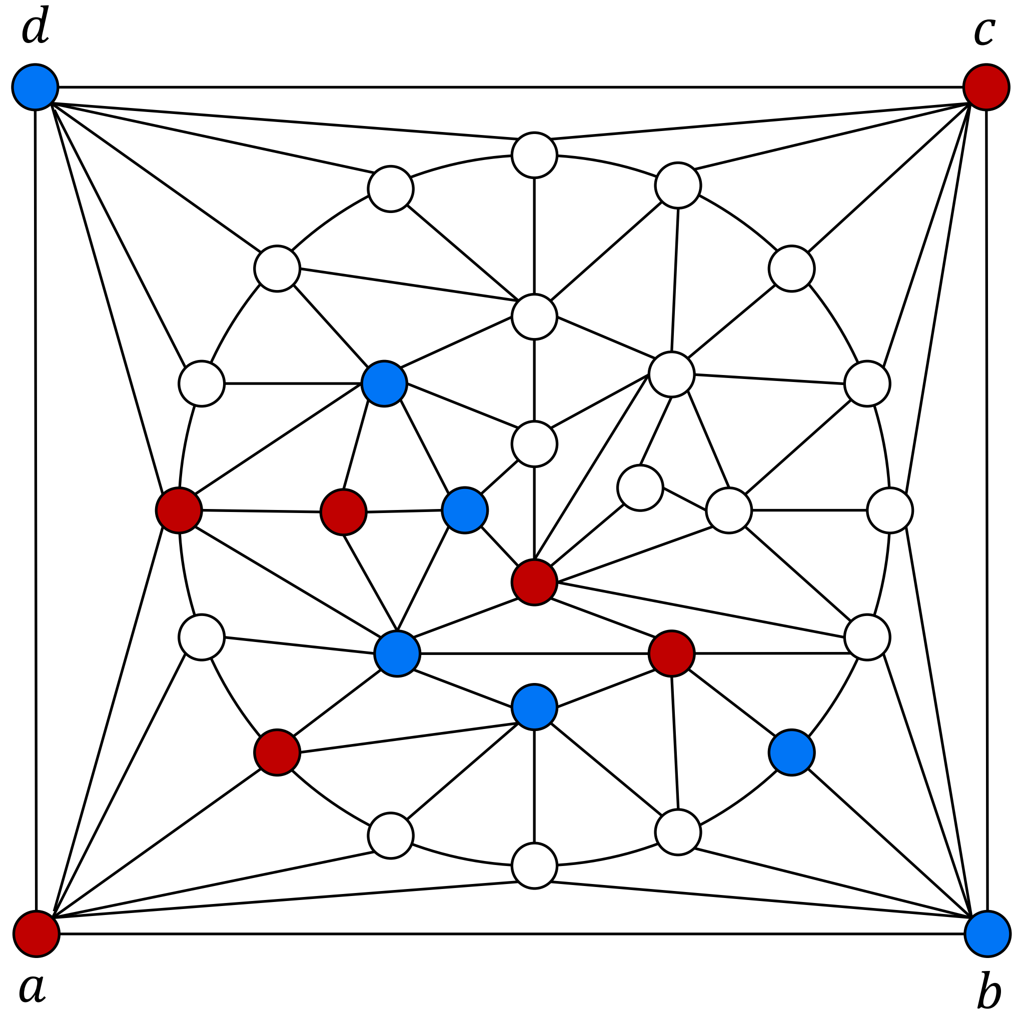

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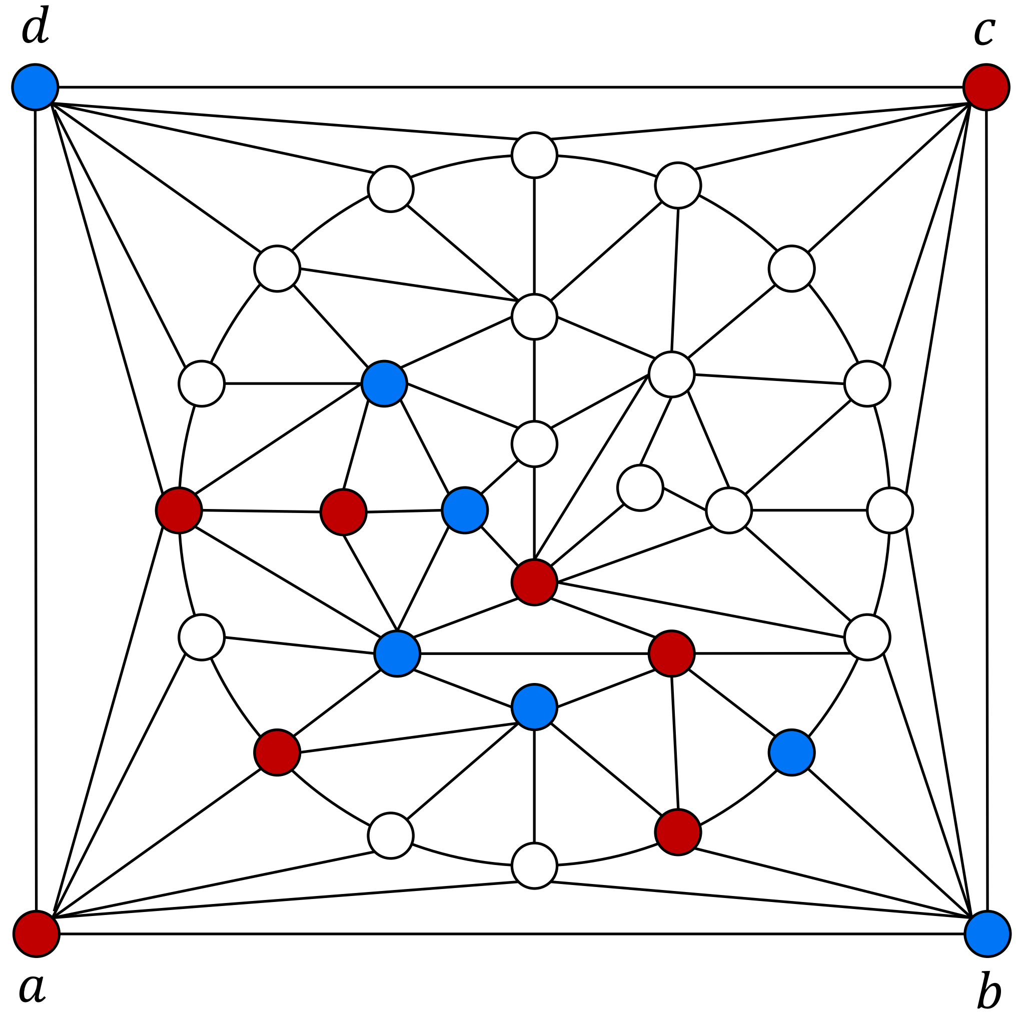

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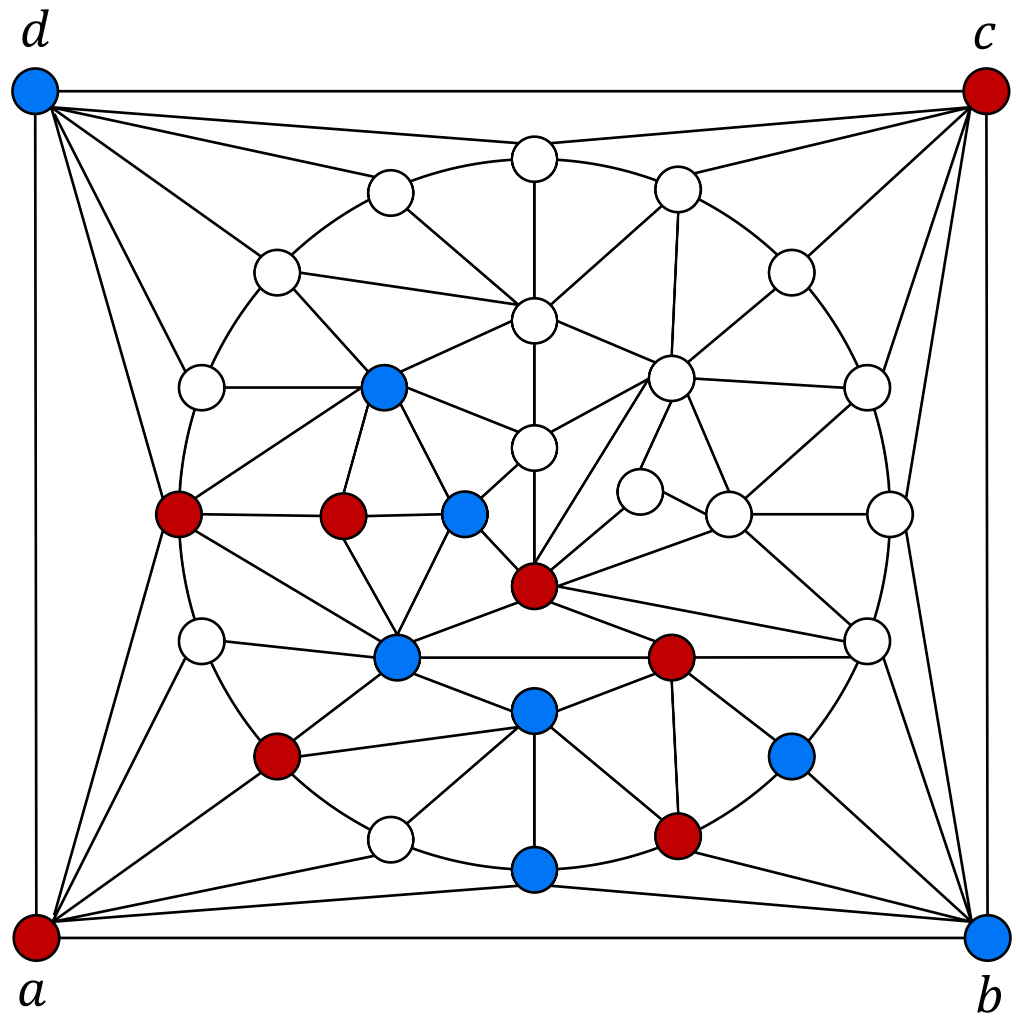

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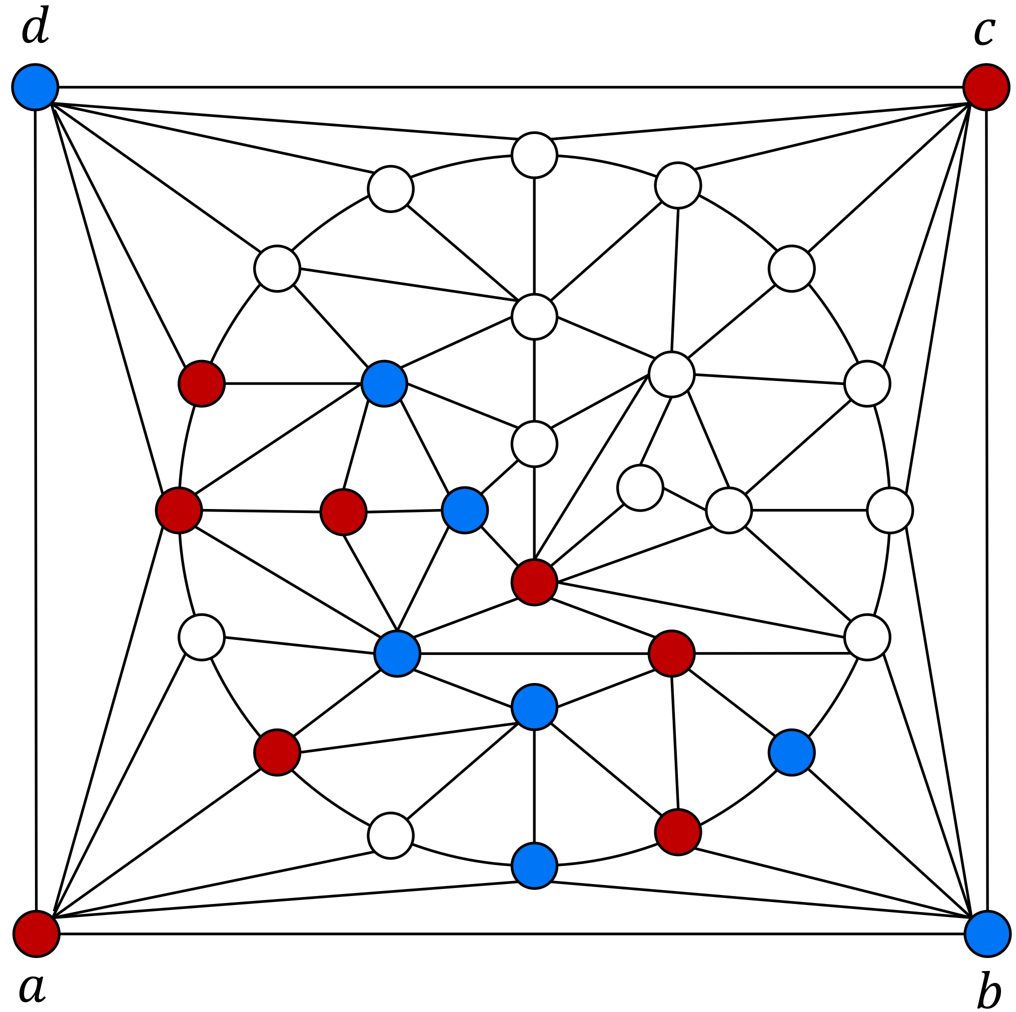

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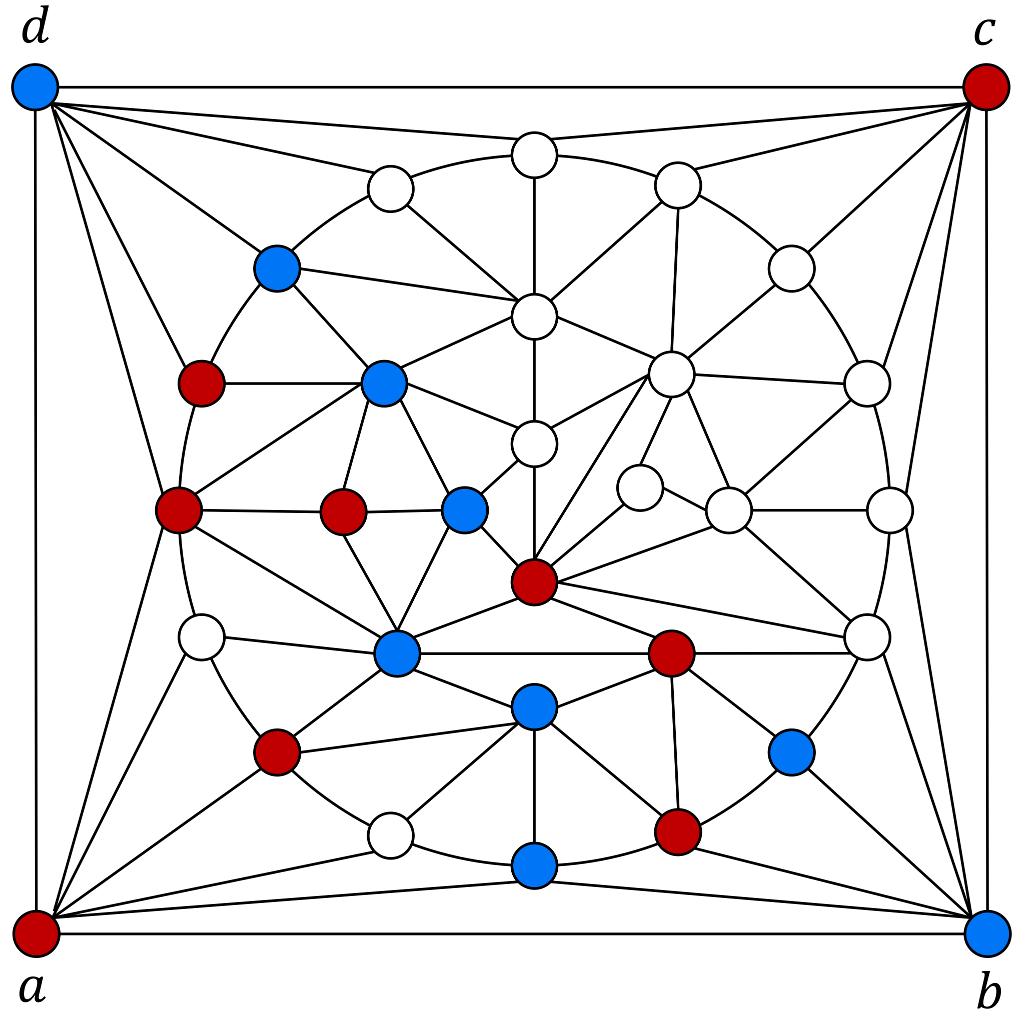

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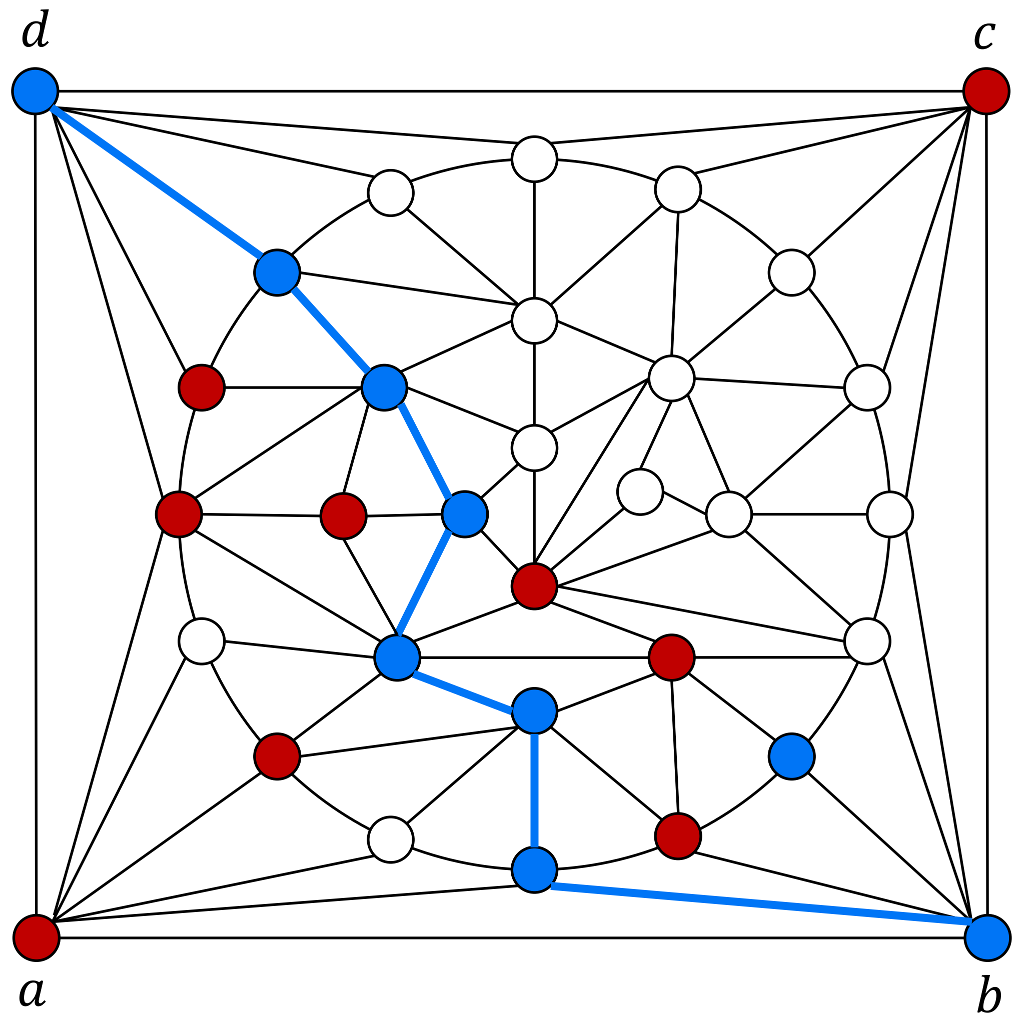

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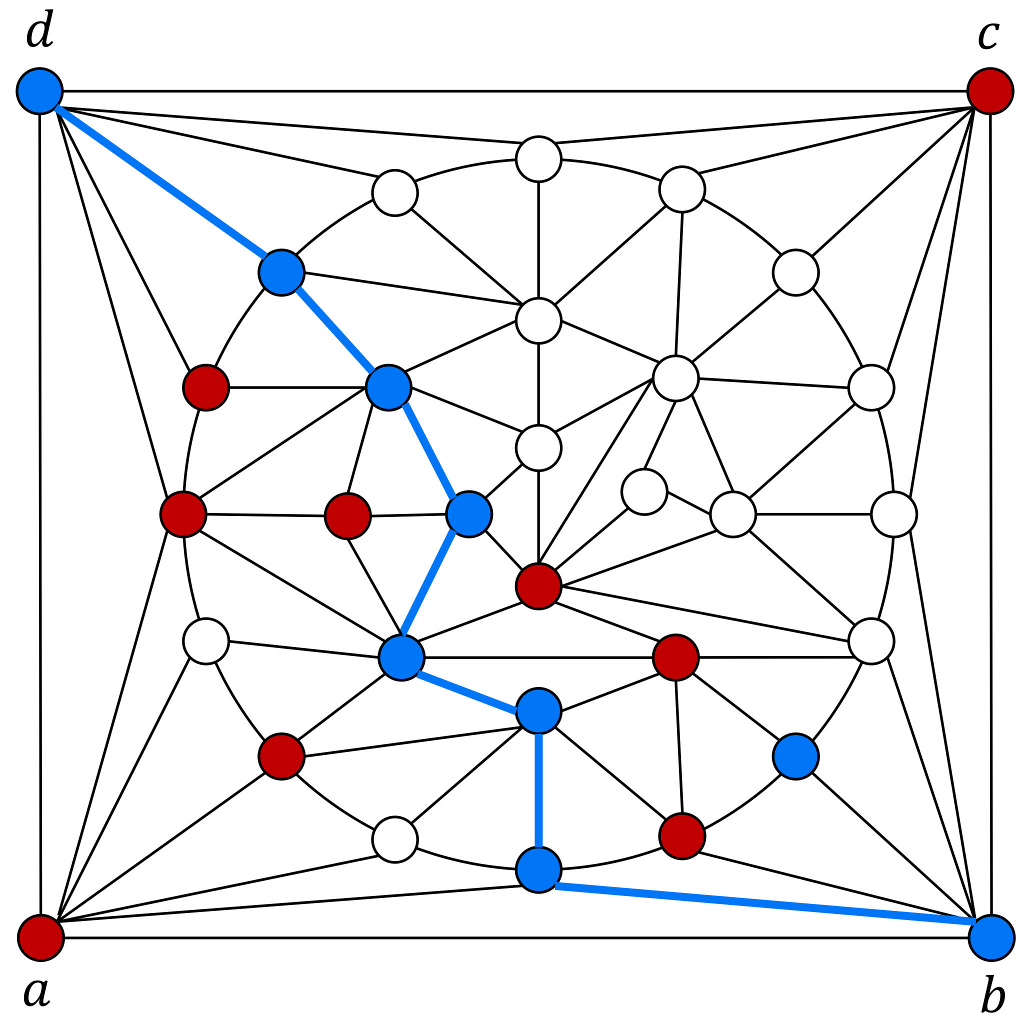

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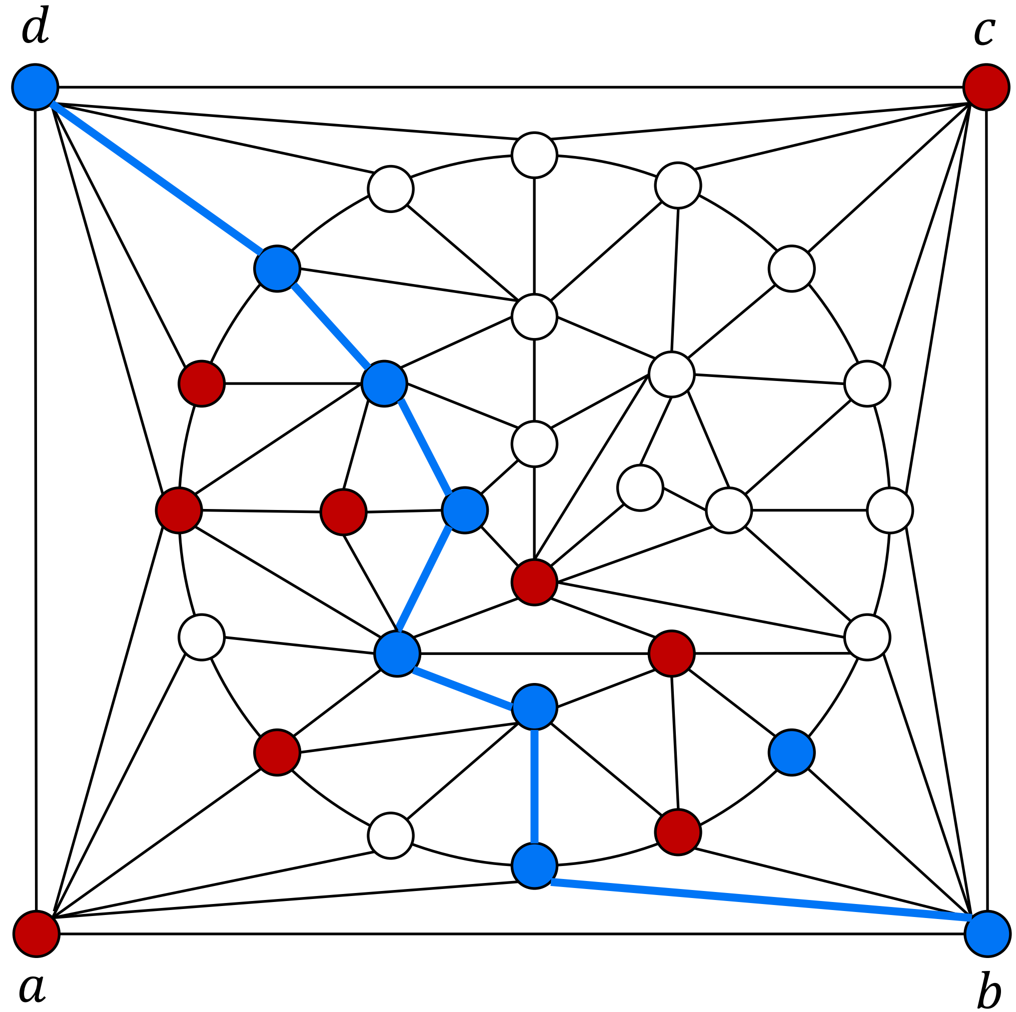
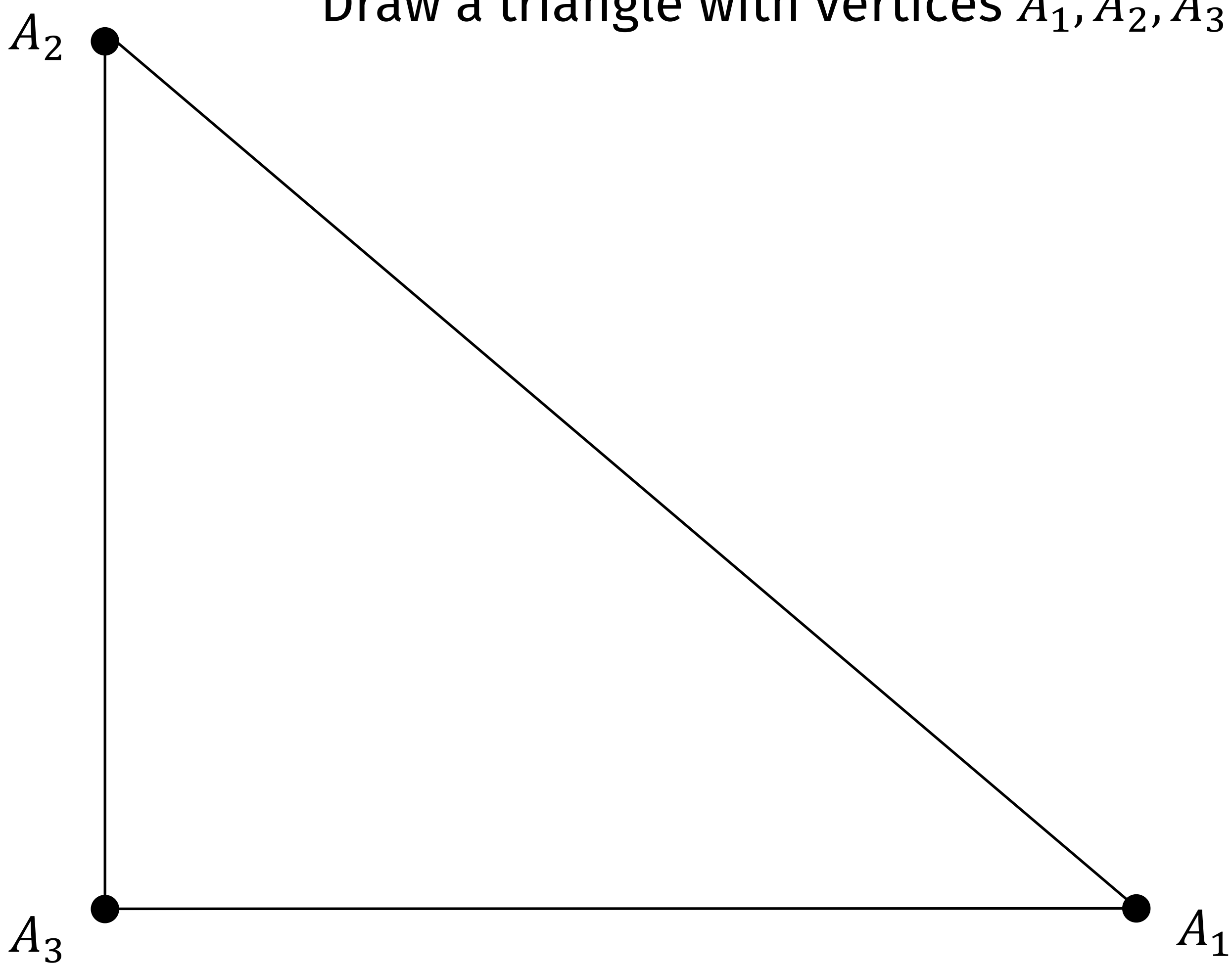


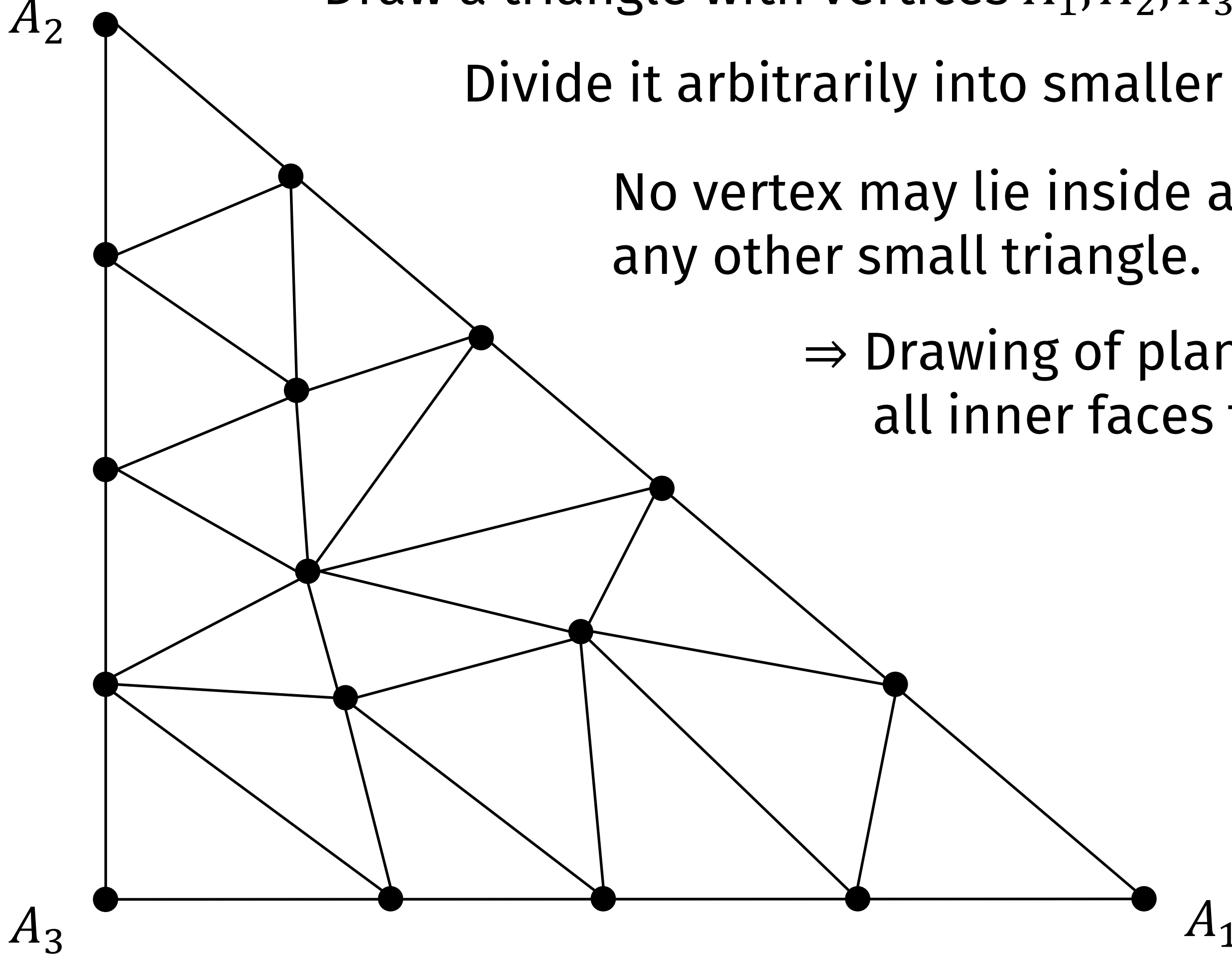
Proposition: On a board of the given type (the outer face is a square, all inner faces are triangles), a draw is impossible.

Sperner's Lemma

Draw a triangle with vertices A_1, A_2, A_3 in the plane.



Sperner's Lemma



Draw a triangle with vertices A_1, A_2, A_3 in the plane.

Divide it arbitrarily into smaller triangles.

No vertex may lie inside an edge of any other small triangle.

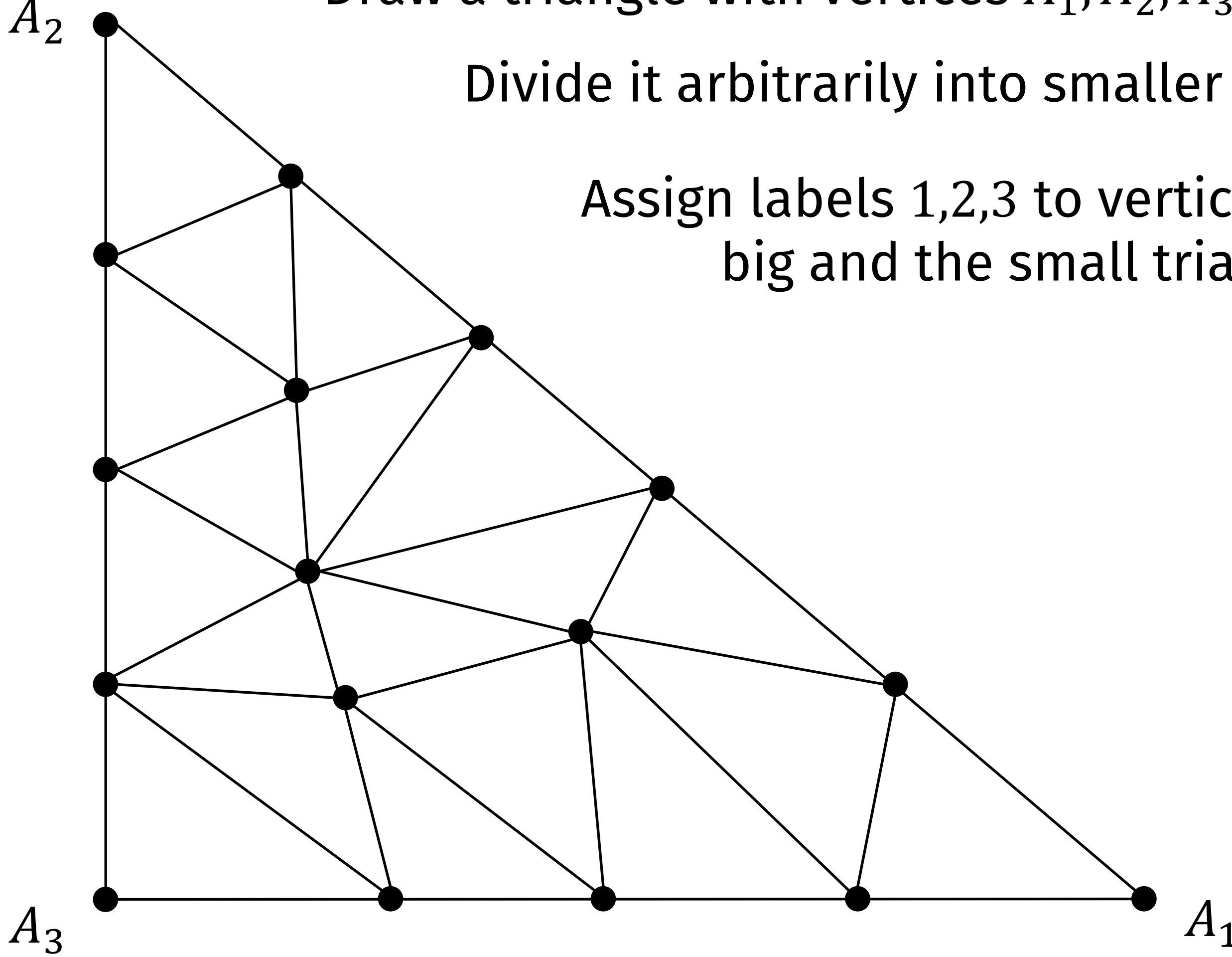
$\Rightarrow$ Drawing of planar graph; all inner faces triangles.

Sperner's Lemma

Draw a triangle with vertices A_1, A_2, A_3 in the plane.

Divide it arbitrarily into smaller triangles.

Assign labels $1,2,3$ to vertices of the big and the small triangles:



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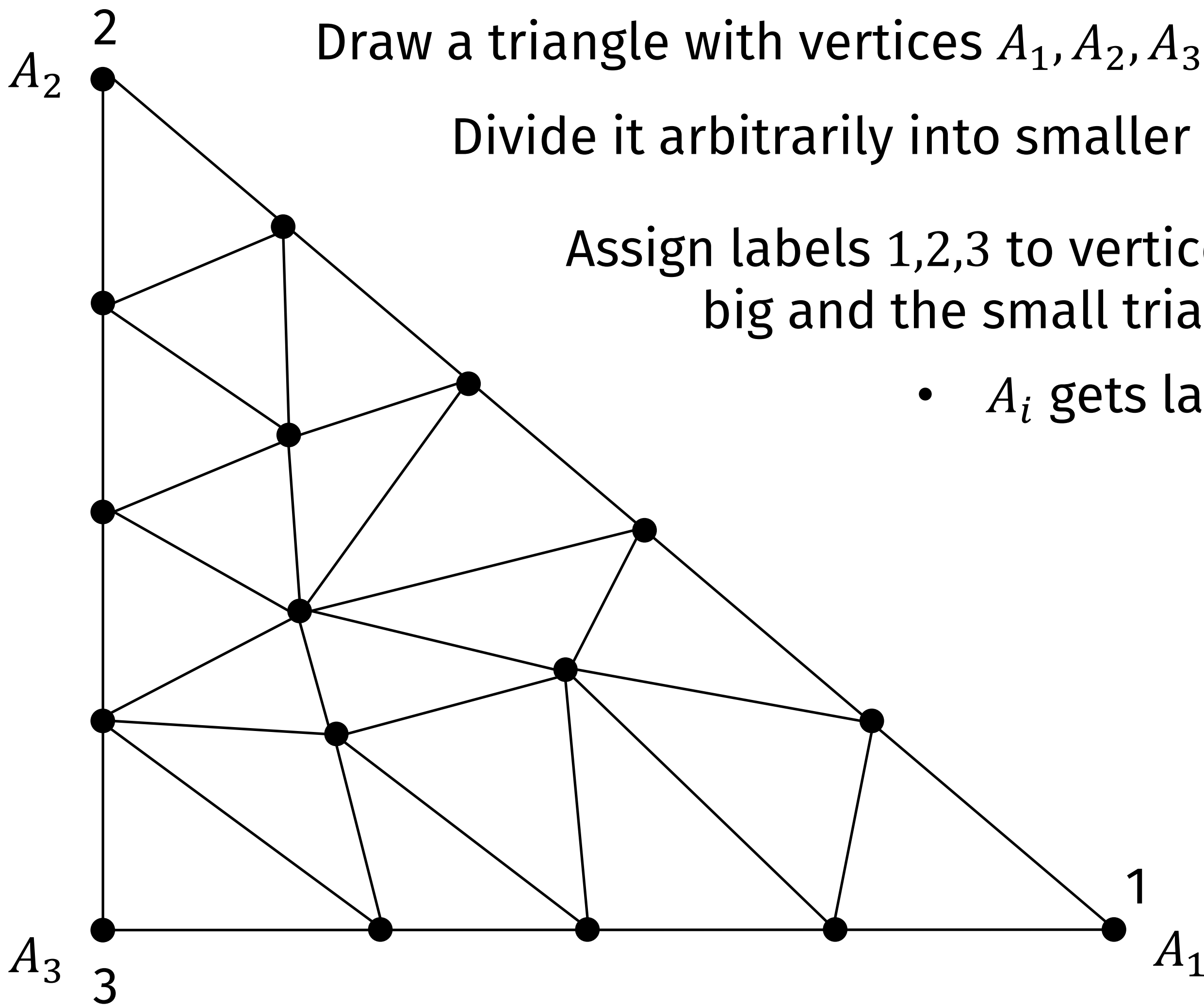


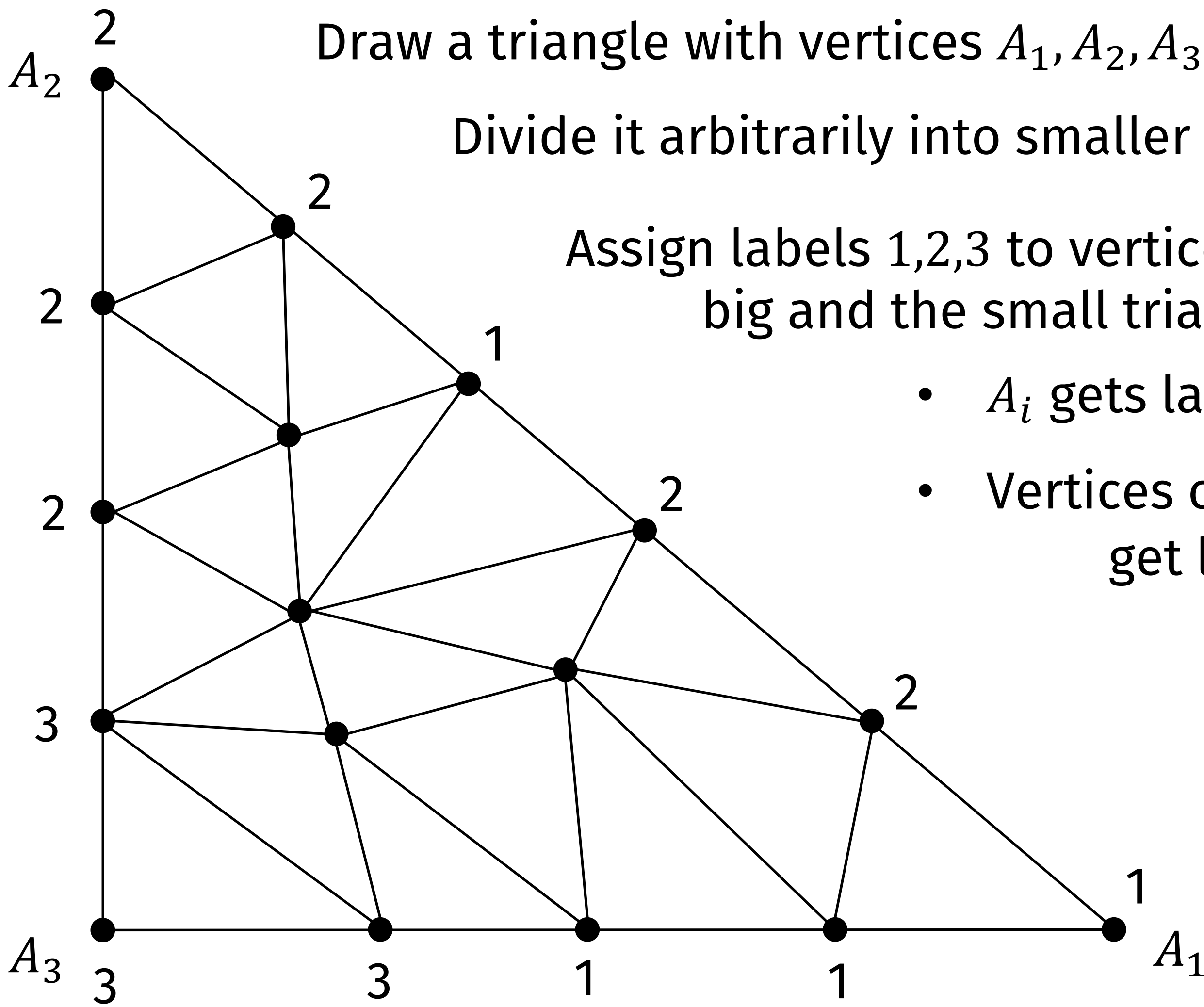
Draw a triangle with vertices A_1, A_2, A_3 in the plane.

Divide it arbitrarily into smaller triangles.

Assign labels $1,2,3$ to vertices of the big and the small triangles:

- A_i gets label $i \in \{1,2,3\}$

Sperner's Lemma



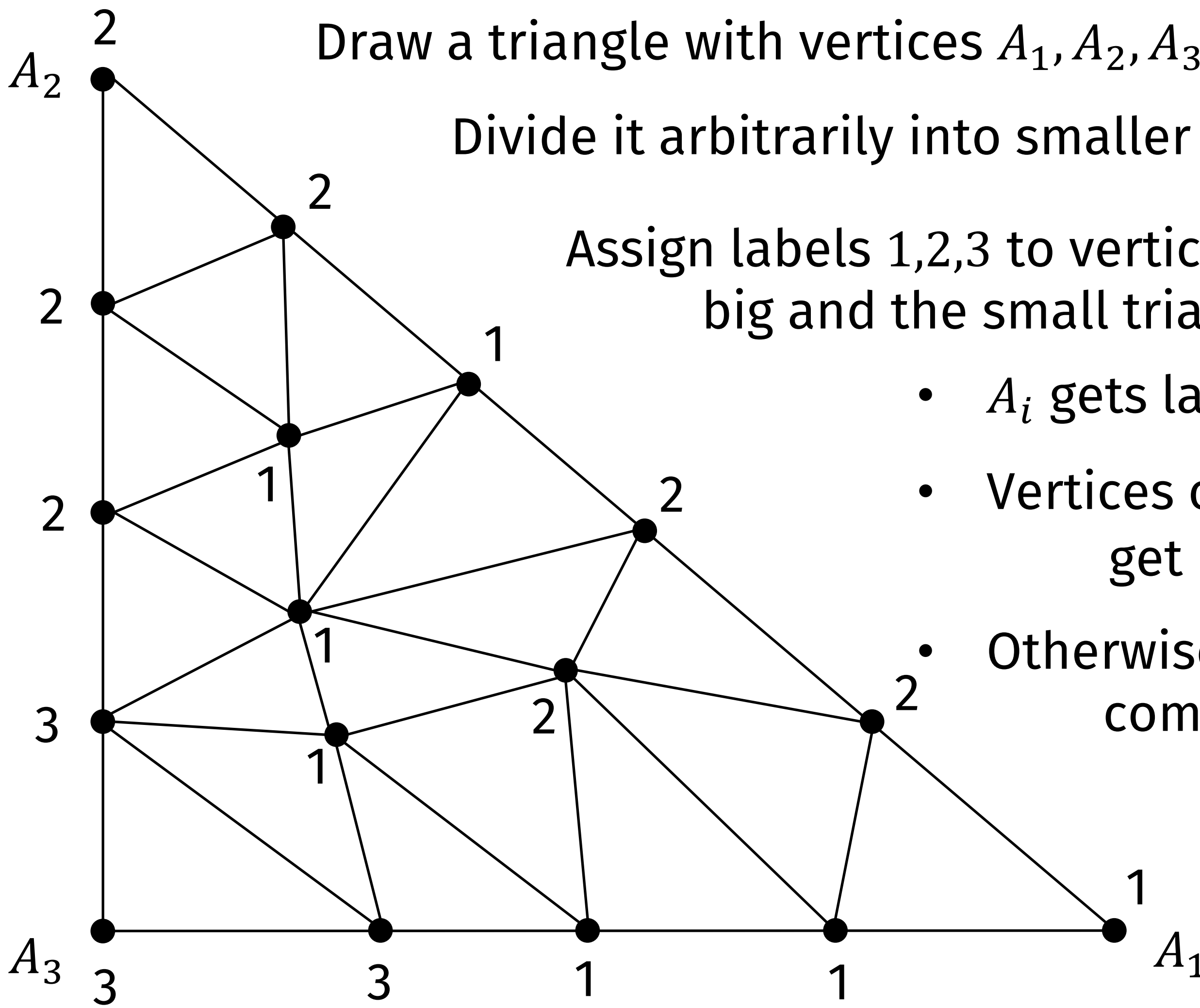
Draw a triangle with vertices A_1, A_2, A_3 in the plane.

Divide it arbitrarily into smaller triangles.

Assign labels $1,2,3$ to vertices of the big and the small triangles:

- A_i gets label i, $i = 1,2,3$.
- Vertices on the edge A_iA_j get label i or j

Sperner's Lemma



Draw a triangle with vertices A_1, A_2, A_3 in the plane.

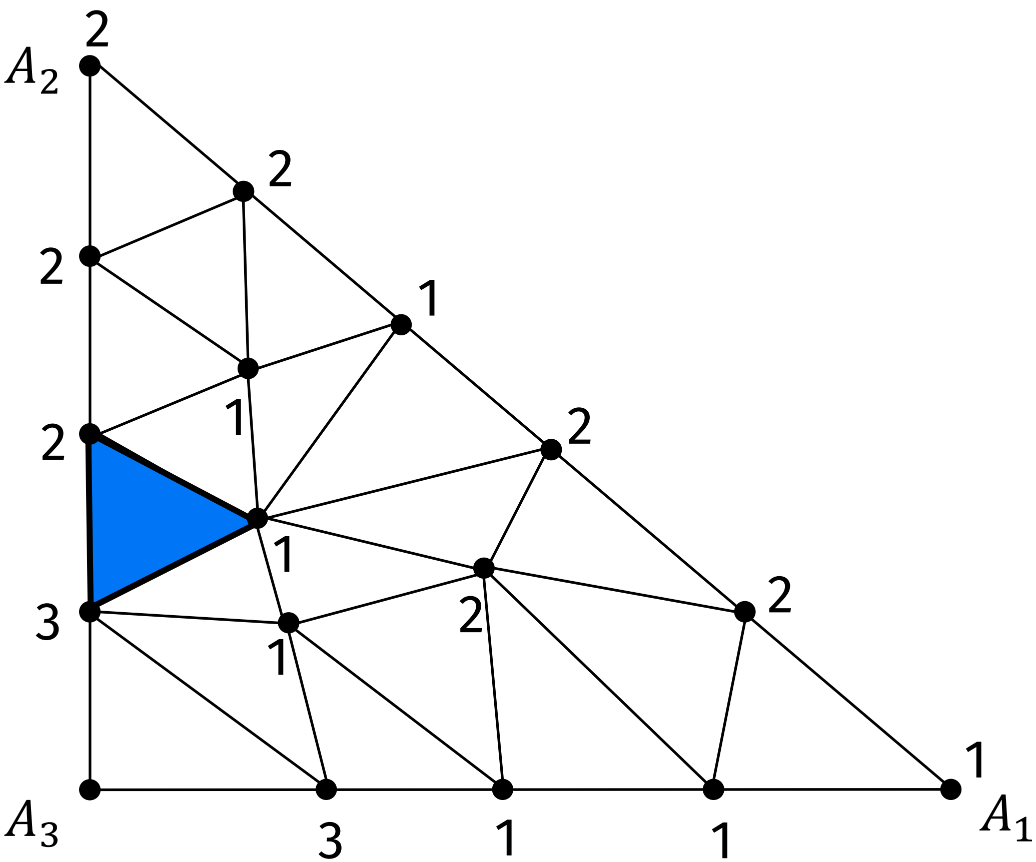
Divide it arbitrarily into smaller triangles.

Assign labels $1,2,3$ to vertices of the big and the small triangles:

- A_i gets label i, $i = 1,2,3$.
- Vertices on the edge A_iA_j get label i or j
- Otherwise assignment is completely arbitrary.

Sperner's Lemma

Proposition: In the situation described before, a small triangle always exists whose vertices are assigned all the three labels $1,2,3$.



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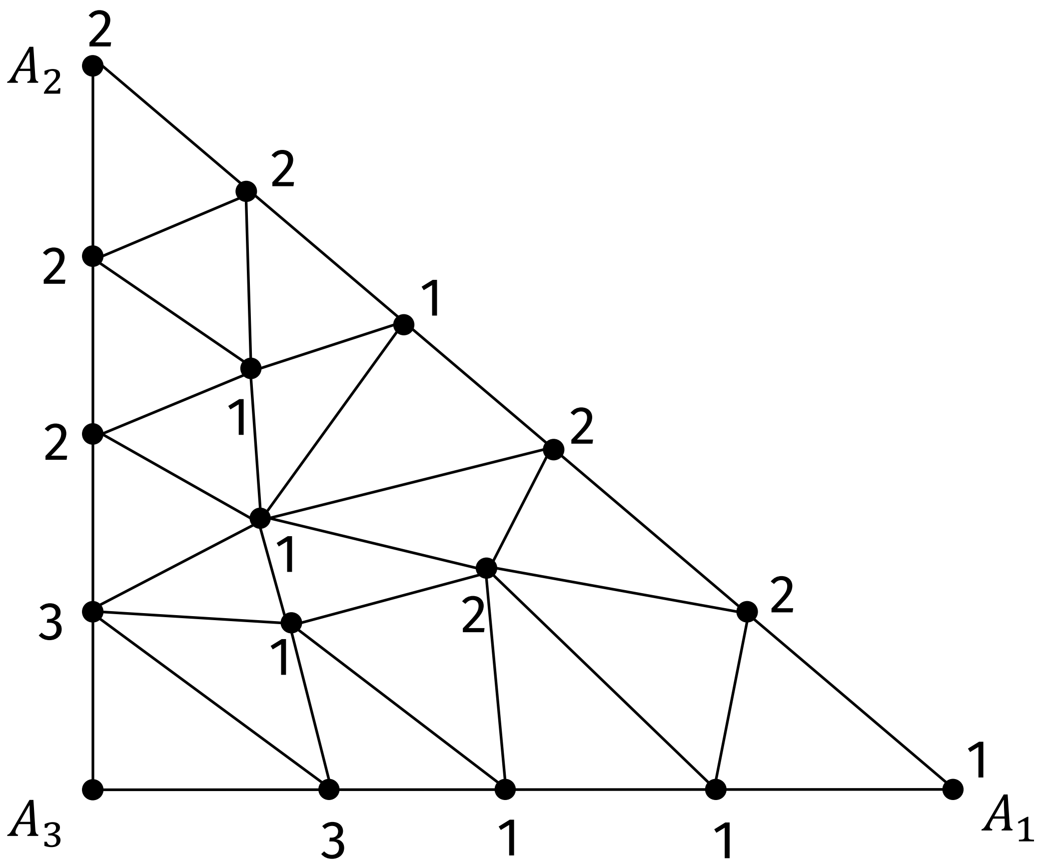
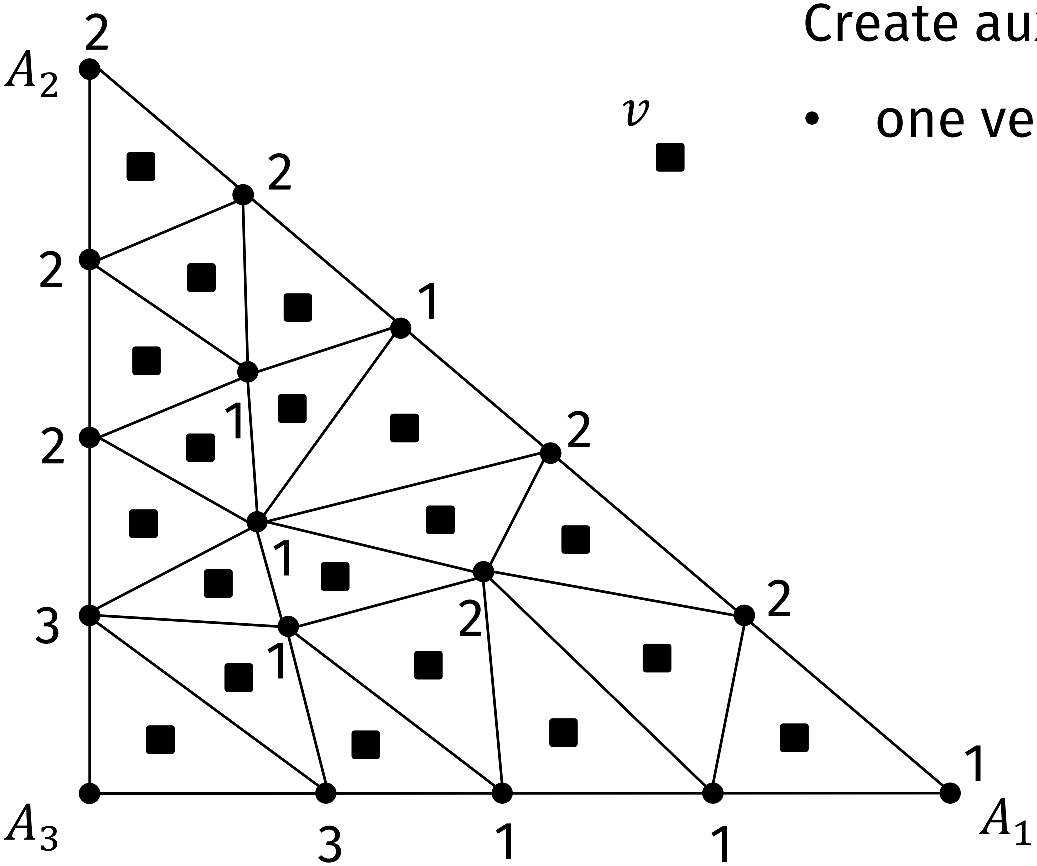


Sperner's Lemma

Proposition: In the situation described before, a small triangle always exists whose vertices are assigned all the three labels $1,2,3$.



Create auxiliary graph G:

- one vertex per face of the triangulation

Sperner's Lemma

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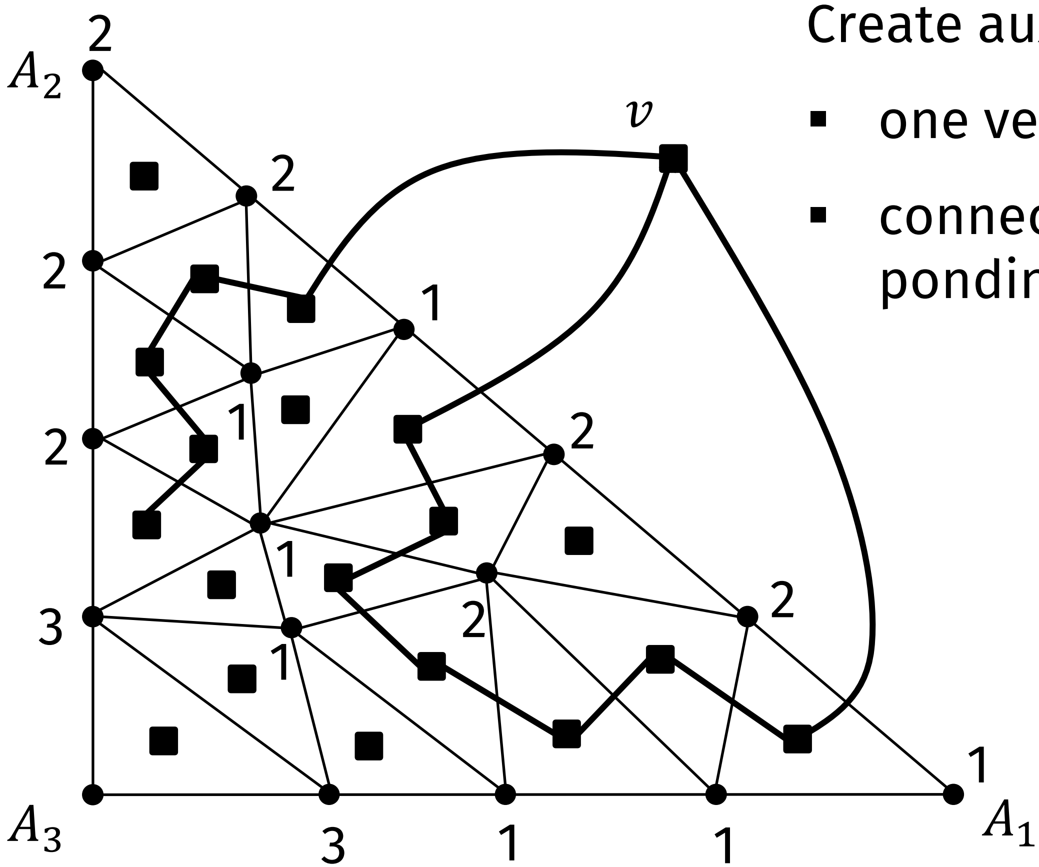
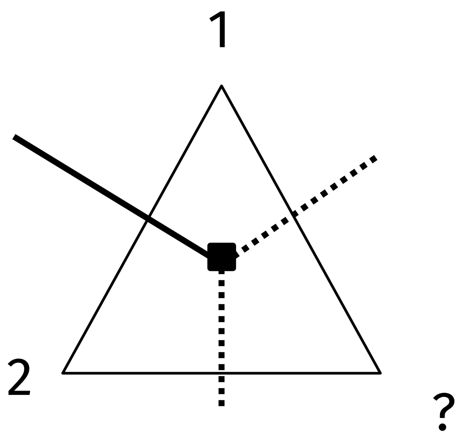


Create auxiliary graph G:

- one vertex per face of the triangulation
- connect two vertices if the corresponding faces share a $\{1,2\}$-edge

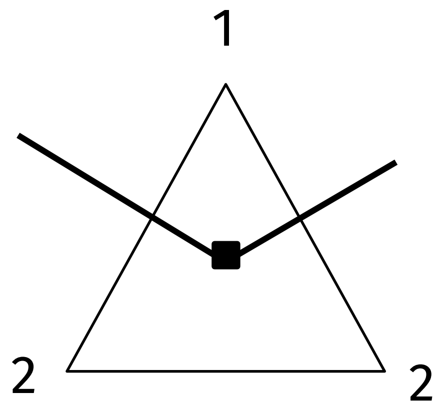
Degrees of Triangles

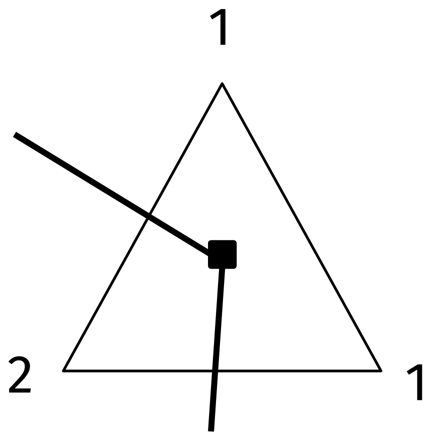
A small triangle is connected to some of its neighbors in G if and only if one of its vertices is labelled by 1 and another one by 2.

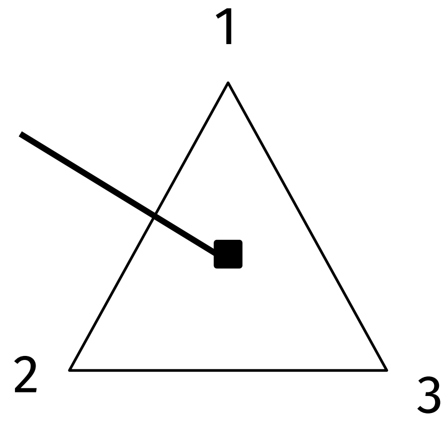


If the remaining vertex has label 3, then the considered triangle is adjacent to exactly one neighbour.

This is the only case where the degree of a small triangle in the graph G is odd.



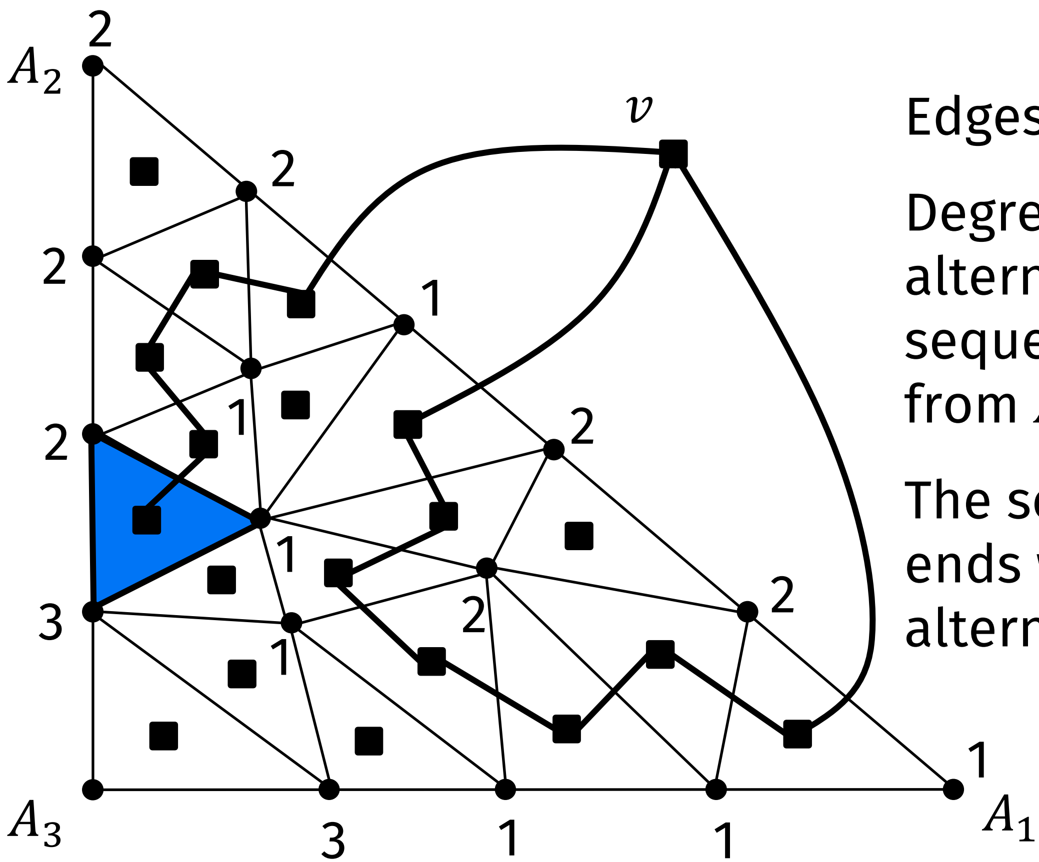




Finishing the proof

Know: Inner face of odd degree corresponds to desired face.

Show: v has odd degree, then existence of odd inner face follows from the handshake lemma.



Edges of v only cross the side A_1A_2.

Degree of v is the number of alternations between 1 and 2 of the sequence of labels when moving from A_1 to A_2.

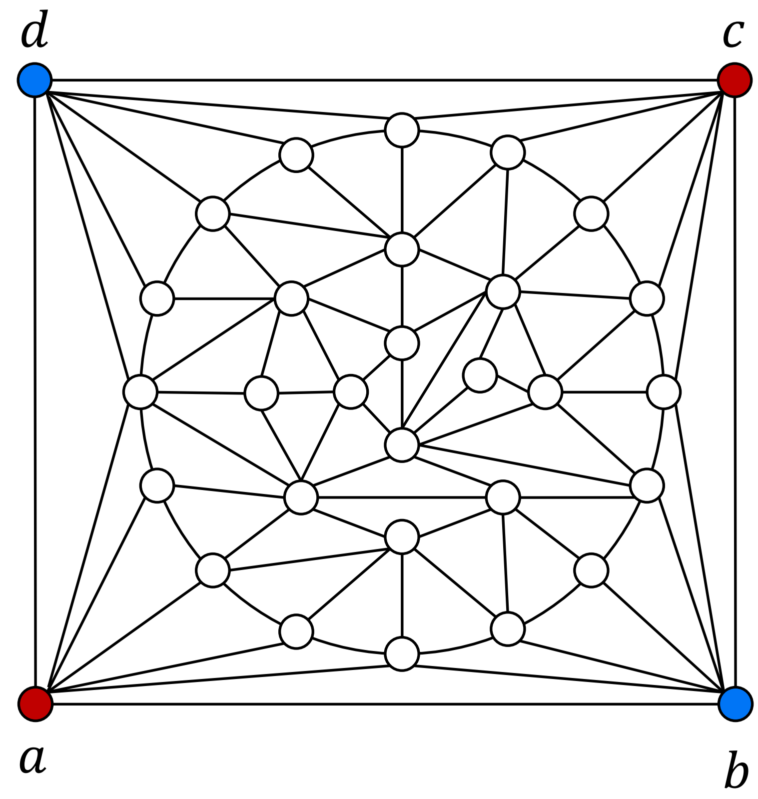
The sequence begins with 1 and ends with 2, so the number of alternations must be odd.

Hence v has odd degree.

No draws!

Proposition: On a board of the given type (the outer face is a square, all inner faces are triangles), a draw is impossible.

Proof: Assume for contradiction that a draw occurred.



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Proposition: On a board of the given type (the outer face is a square, all inner faces are triangles), a draw is impossible.

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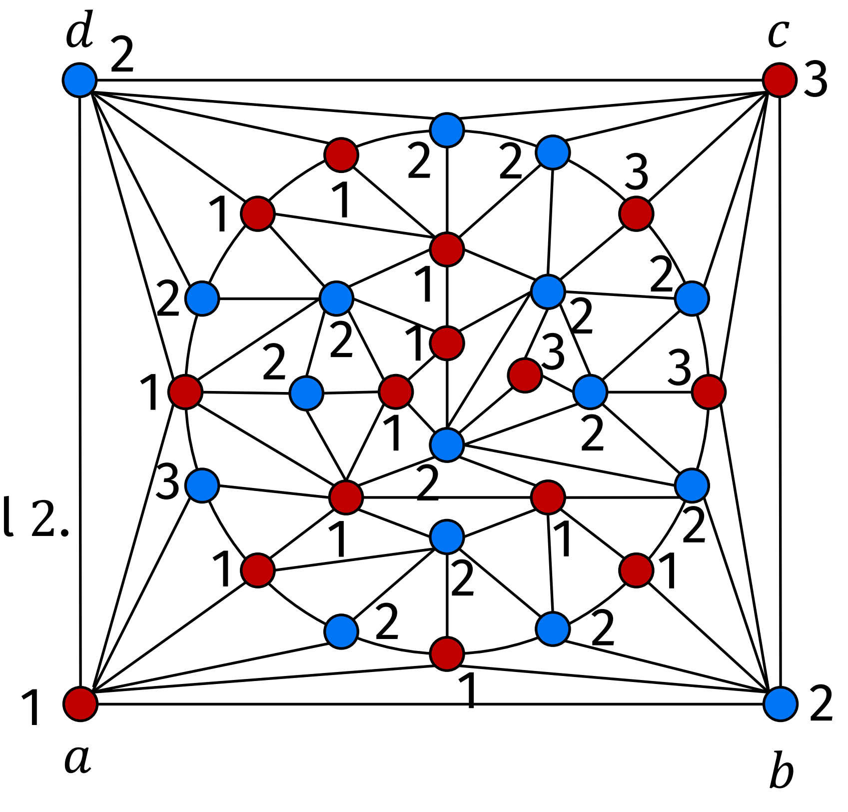
- A: set of nodes marked by Alice ()
- B: set of nodes marked by Betty ()

Assign labels $1,2,3$ as follows:

- Node in A gets label 1 if it can be connected to a by a path with all vertices belonging to A.
- Similarly, nodes in B connected to b by a path entirely lying in B get label 2.
- Remaining nodes get label 3.



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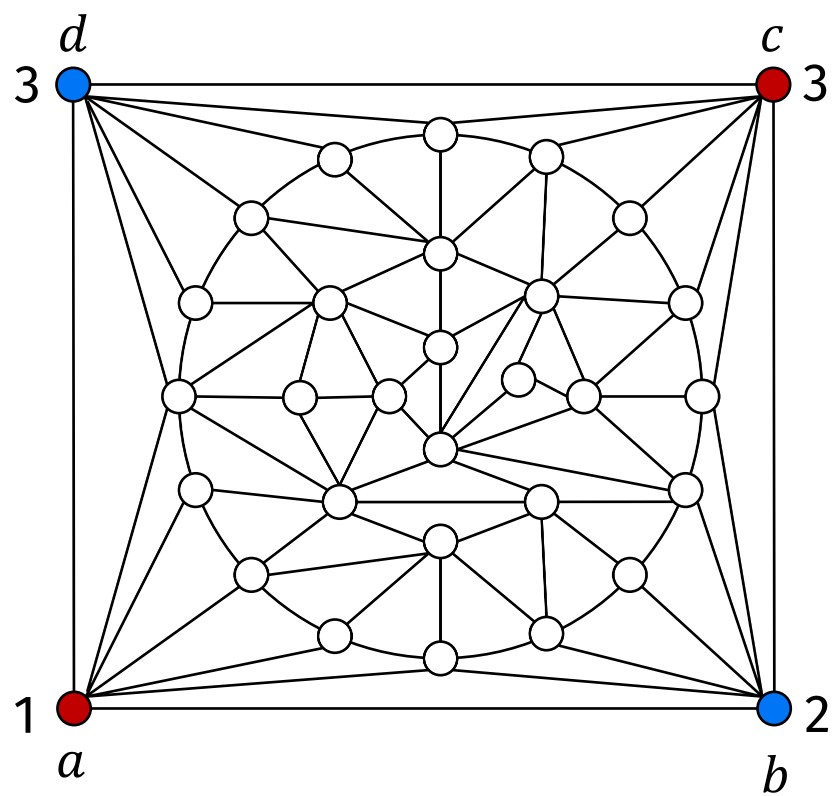
Proof (continued):

Assume for a contradiction that c, d receive label 3.



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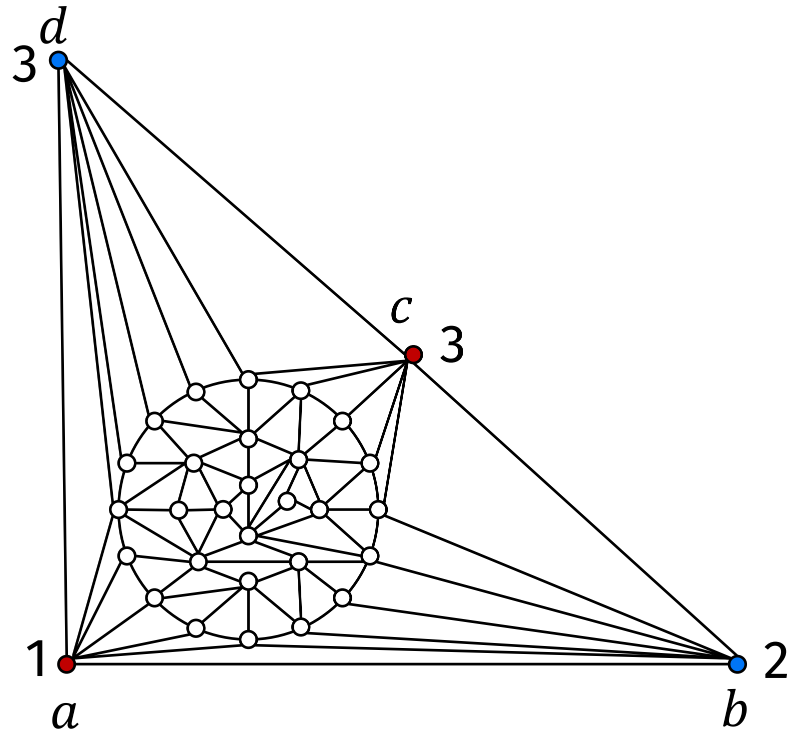

No draws!

Proposition: On a board of the given type (the outer face is a square, all inner faces are triangles), a draw is impossible.

Proof (continued):

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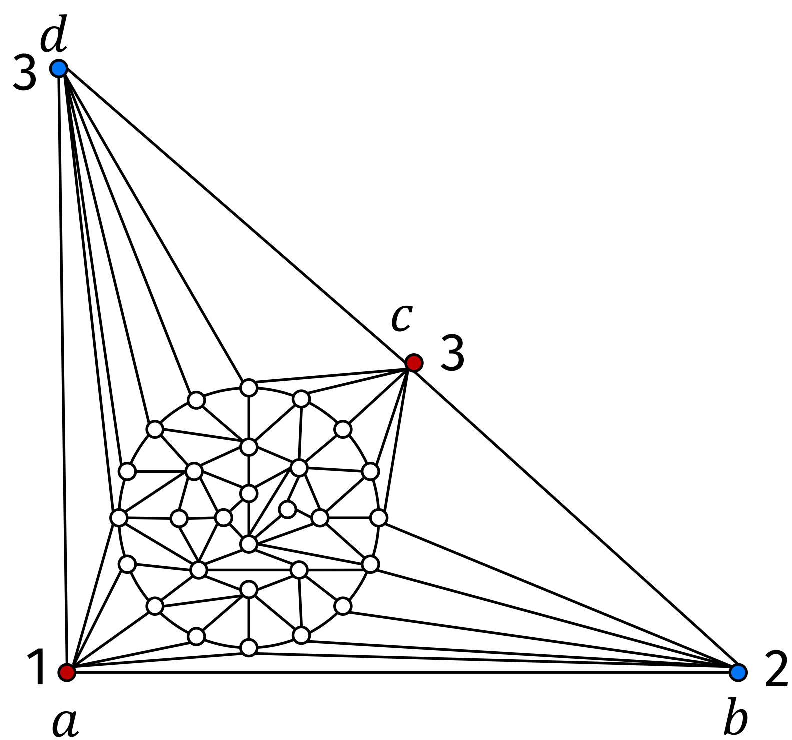
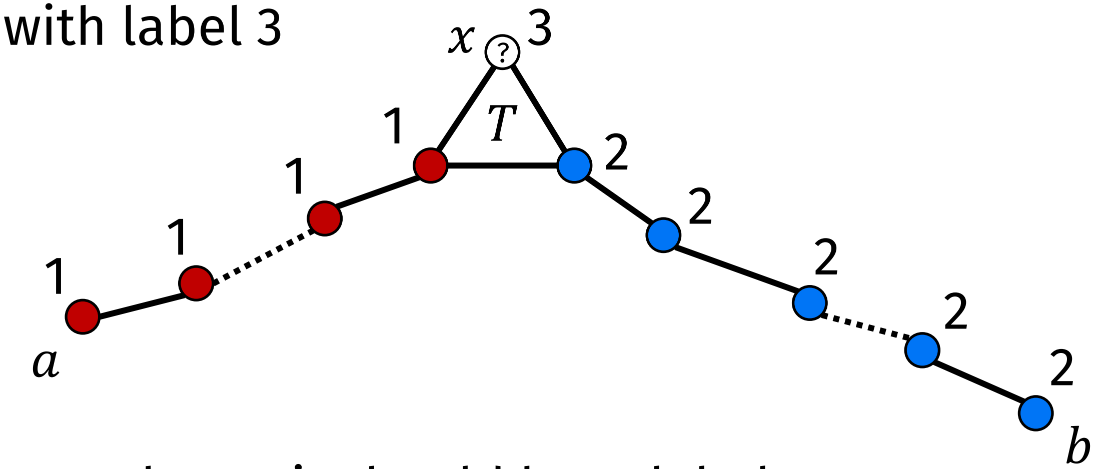
Sperner's Lemma ⇒ there is a triangle T with all three labels.



Finishing the proof.

- x: node of T with label 3



- If x belongs to A, then x it should have label 1, so $x \notin A$
- If x belongs to B, then x it should have label 2, so $x \notin B$

- So x must be uncolored, but then the game has not ended.
  A contradiction, to our assumption that c, d have label 3.

The real HEX

How does this relate to the game discussed above?
→ see Practice!

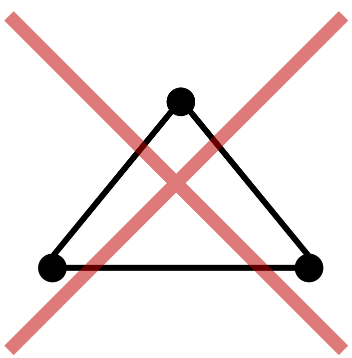
Two Results in Extremal Graph Theory

Extremal Graph Theory

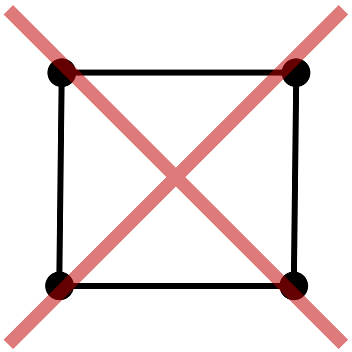
Extremal Graph Theory asks (and answers) questions of the following type:

How many edges can a graph have if it does not contain the following substructure?

Examples:



No (non-induced) subgraph isomorphic to K_3.



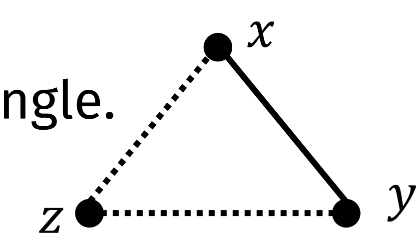
No (non-induced) subgraph isomorphic to $K_{2,2}$.

Triangle-free graphs

Proposition: if $G = (V, E)$ is a triangle-free graph with n vertices, then G has at most $\frac{n^2}{4}$ edges.

Proof: We double count the degrees on the endpoints of edges.

- For each edge $\{x, y\}$ it is $\deg(x) + \deg(y) \leq n$.
  Otherwise: find z adjacent to both x, y giving a triangle.



$$\sum_{v \in V} \deg(v)^2 = \sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) \leq |E|n$$

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|$$

???

One more ingredient

The Cauchy-Schwarz inequality:

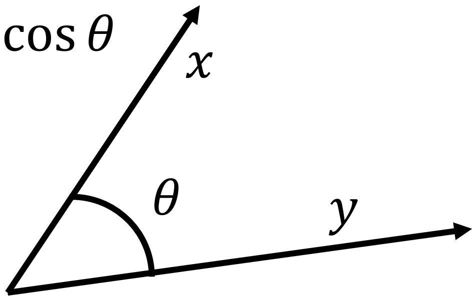
Proposition: For arbitrary real numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ we have

$$\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}$$

No proof, but motivation:

scalar product of vectors $x, y \le$ product of lengths of x, y

$$\sum_{i=1}^{n} x_i y_i = (x_1, \dots x_n) \cdot (y_1, \dots, y_n) = ||x|| \, ||y|| \, \cos\theta$$



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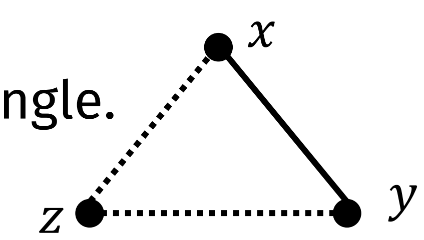

$$\sum_{v \in V} \deg(v)^2 = \sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) \leq |E|n$$

$$\sum_{v \in V} \deg(v) \cdot 1 = 2 \cdot |E|$$

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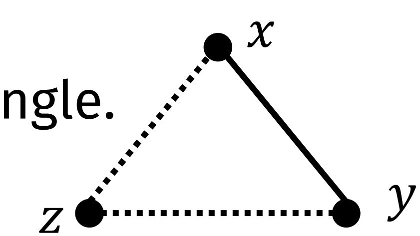


$$\sum_{v \in V} \deg(v)^2 = \sum_{\{x,y\} \in E} (\deg(x) + \deg(y)) \leq |E| n$$

Cauchy-Schwarz:

$$\sqrt{\sum_{v \in V} \deg(v)^2} \cdot \sqrt{\sum_{v \in V} 1^2} \geq \sum_{v \in V} \deg(v) \cdot 1 = 2 \cdot |E|$$

$$\Rightarrow \sum_{v \in V} \deg(v)^2 \cdot n \geq 4\,|E|^2$$

Triangle-free graphs

Proposition: if $G = (V, E)$ is a triangle-free graph with n vertices, then G has at most $\frac{n^2}{4}$ edges.

Proof (continued):

$$\sum_{v \in V} \deg(v)^2 \leq n \cdot |E|$$

$$\sum_{v \in V} \deg(v)^2 \cdot n \geq 4\,|E|^2$$

$$\Rightarrow 4\,|E|^2 \leq \sum_{v \in V} \deg(v)^2 \cdot n \leq n^2 |E| \Leftrightarrow |E| \leq \frac{n^2}{4}$$

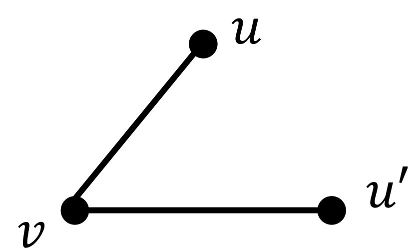
Four-cycle-free graphs

Proposition: if $G = (V, E)$ is an n-vertex graph without $K_{2,2}$ as a (non-induced) subgraph, then G has at most $\frac{1}{2}\left(n^{3/2} + n\right)$edges.

Proof (sketch): We double count the set M of pairs $(\{u, u'\}, \mathrm{v})$, where $v \in V$ and $\{u, u'\} \in \binom{V}{2}$ and v has an edge to both u and u'.

- For a fixed pair $\{u, u'\}$ at most one vertex v may be joined to both: otherwise we get $K_{2,2}$.

$$\Rightarrow |M| \leq \binom{n}{2}$$



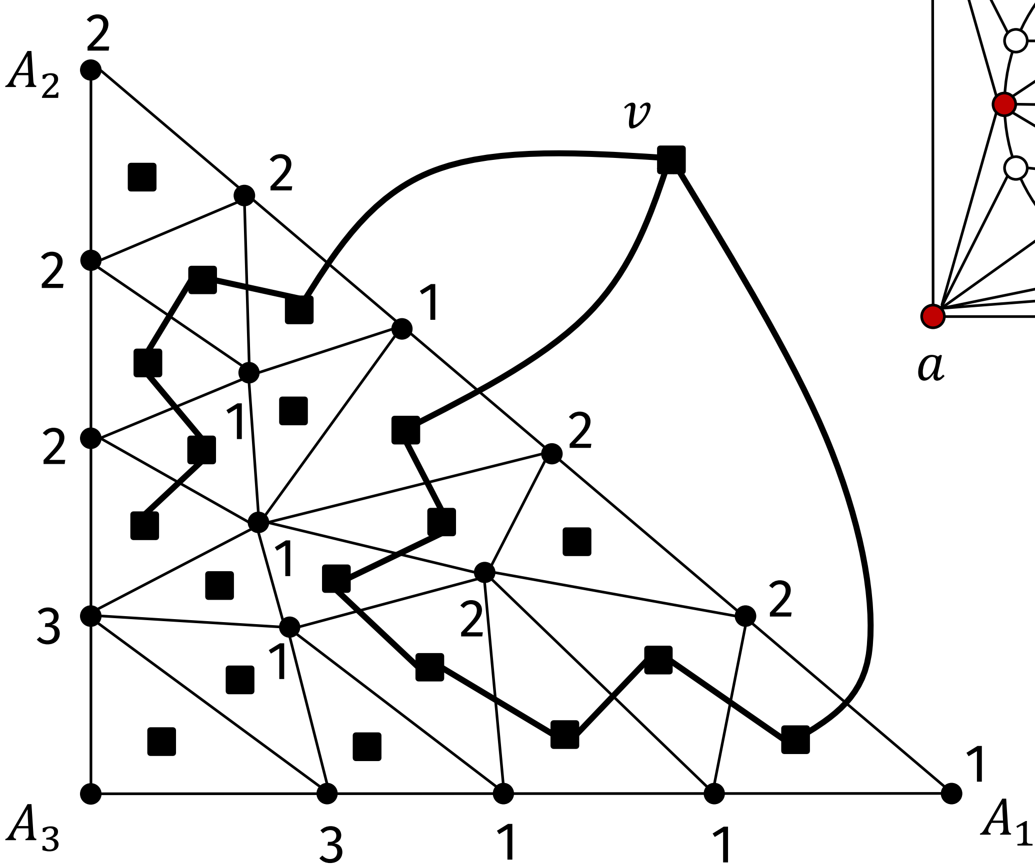
- A fixed vertex v contributes exactly $\binom{\deg(v)}{2}$ pairs.

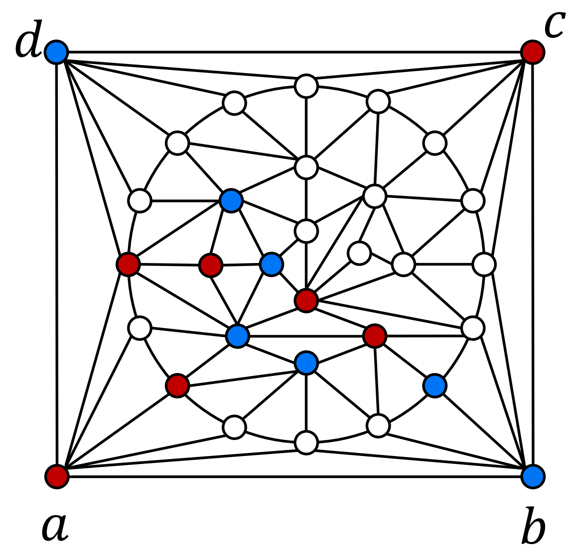
$$|M| = \sum_{v \in V} \binom{\deg(v)}{2}$$

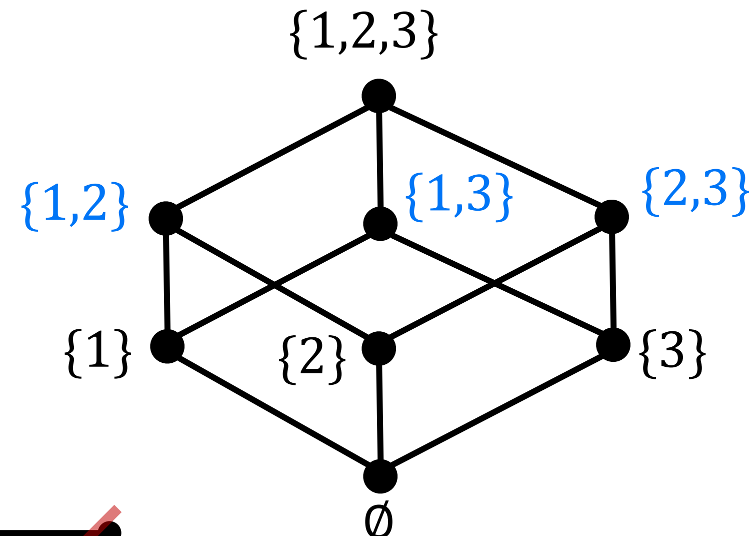
$$\Rightarrow \sum_{v \in V} \binom{\deg(v)}{2} \leq \binom{n}{2}$$

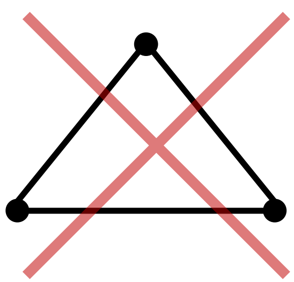
Then result follows from Cauchy-Schwarz (see book).

Summary

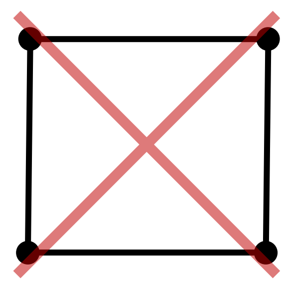








$\Rightarrow \ \leq \frac{1}{4}n^2$ edges



$\Rightarrow \ \leq \frac{1}{2}(n^{\frac{3}{2}} + n)$ edges

Organizational

- Exam prep lecture
  - Short summary of course topics
  - What to expect for the exam
  - Questions
- Discussion group
  - Practice exams will be available soon

Timeline

- Test on Monday 15th
- graded before Friday 19^{th}
- discussions until Sunday night
- final grades on Monday
- Will be deregistered from exam if you did not make the 5.5 threshold after that