Analysis 1

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Contents

1 Sets, Spaces and Function

1.1 Metric Space

Definition 1.1.1 – distance Let X be a set. A function $d : X \times X \to X$ is called a *distance* on X if it satisfies the following properties:

(i) Positivity: For all $a,b \in X$, it holds that $d(a,b) \geq 0$.

(ii) Non-degeneracy: For all $a,b \in X$, if $d(a,b) = 0$, then $a = b$.

(iii) Symmetry: For all $a,b \in X$, it holds that $d(a,b) = d(b,a)$.

(iv) Triangle inequality: For all $a,b,c \in X$, it holds that $d(a,c) \leq d(a,b) + d(b,c)$.

(v) Reflexivity: For all $a \in X$, it holds that $d(a,a) = 0$.

Usually conditions (ii) and (v) are combined into one condition: For all $a,b \in X, d(a,b) = 0$ if and only if $a = b$.

Definition 1.1.2 – metric space A metric space is a pair $(X, dist)$, where X is a set and $dist$ is a distance function $dist : X \times X \to \mathbb{R}$ on X.

Example 1.1.3 Let $X = \{$Die Hard, Barbie, Oppenheimer$\}$

d	Die Hard	Barbie	Oppenheimer
Die Hard	0	5	2
Barbie	5	0	3
Oppenheimer	2	3	0

Then d is a distance function on X

Definition 1.1.4 – ball in a metric space Let (X,d) be a metric space. Let $c \in X$ and $r \in \mathbb{R}$. The ball of radius r centered at c is the set

$$B(c,r) = \{x \in X | d(c,x) < r\}$$

Example 1.1.5 If $(X,d) = (\mathbb{R}, \mathrm{d}_{\mathbb{R}})$, then $B(1,3) = (-2,4) = \{x \in \mathbb{R} \mid |x-1| < 3\}$

Example 1.1.6 Let $X := \{$Die Hard, Barbie, Oppenheimer$\}$, with distance defined before. Then $B($Barbie, $4) = \{$ Barbie, Oppenheimer $\} = \{x \in X \mid \mathrm{d}(x, \text{Barbie}) < 3\}$.

1.2 Normed Vector Spaces

Definition 1.2.1 – norm Let V be a vector space over $\mathbb{R}$. A norm on V is a function $||\cdot|| : V \to \mathbb{R}$ such that

- Positivity: for all $u,v \in V$ we have $||u|| \geq 0$ and $||u|| = 0$ if and only if $u = 0$.
- Non-degeneracy: for all $u \in V$ if $||u|| = 0$ then $u = 0$.
- Absolute Homogeneity: for all $u \in V$ and for all $\lambda \in \mathbb{R}$ we have $||\lambda u|| = |\lambda| ||u||$.
- Triangle inequality: for all $u,v \in V$ we have $||u+v|| \leq ||u|| + ||v||$.

Example 1.2.2 Let $V = \mathbb{R}^n$. Then $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ is a norm on $\mathbb{R}^n$.

Proposition 1.2.3 – Let $(V, \|\cdot\|)$ be a normed vector space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(u,v) = \|u - v\|$ is a distance on V. And (V,d) is a metric space.

Remark 1.2.4 (Notation for Euclidean distance on $\mathbb{R}^d$ and $\mathbb{R}$). We will usually write $\mathrm{dist}_{\mathbb{R}^d}$ instead of $\mathrm{dist}_{\|\cdot\|_2}$ for the standard (Euclidean) distance on $\mathbb{R}^d$. In particular, if $d \geq 2$, we have

$$\mathrm{dist}_{\mathbb{R}^d}(v,w) = \|v - w\|_2 = \sqrt{\sum_{i=1}^{d} (v_i - w_i)^2}$$

and if $d = 1$ we just have

$$\mathrm{dist}_{\mathbb{R}} = |v - w|$$

And if there is no room for confusion, we will just leave out the subscript altogether and write dist instead of $\mathrm{dist}_{\mathbb{R}^d}$.

1.3 The reverse triangle inequality

Lemma 1.3.1 – Reverse triangle inequality Let $(V, \|\cdot\|)$ be a normed vector space. Then for all $u, v \in V$ we have,

$$|\|v\| - \|w\|| \leq \|v - w\|$$

2 Real Numbers

2.1 What are the real numbers?

Definition 2.1.1 – Real numbers The real numbers are a complete totally ordered field.

2.2 The completeness axiom

Definition 2.2.1 – Upper and Lower bound We say a number $M \in \mathbb{R}$ is an *upper bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \leq M].$$

We say a number $m \in \mathbb{R}$ is a *lower bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \geq M].$$

Given the definition of upper and lower bounds, we define what it means for a set to be bounded from above, bounded from below and just bounded.

Definition 2.2.2 – bounded from above, bounded from below, bounded A set $A \subseteq \mathbb{R}$ is *bounded from above* if there exists an upper bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded from below* if there exists a lower bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded* if it is bounded from above and bounded from below.

Definition 2.2.3 – Least upper bound (supremum) Precisely, M is a *least upper bound* of a subset A if both

1. M is an upper bound of A.
2. For every upper bound $L \in \mathbb{R}$ of A, it holds that $M \leq L$.

Proposition 2.2.4 – Suppose both M and W are a least upper bound of a subset $A \subseteq \mathbb{R}$. Then $M = W$.

Axiom 2.2.5 – Completeness axiom We say that a totally ordered field $\mathbf{R}$ satisfies the *completeness axiom* if every nonempty subset of $\mathbf{R}$ that is bounded from above has a least upper bound.

Lemma 2.2.6 – Every non-empty subset of the real line that is bounded from below has a *largest lower bound*.

Definition 2.2.7 – infimum We usually call the largest lower bound of a non-empty set $A \subseteq \mathbb{R}$ that is bounded from below the *infimum* of A, and we denote it by $\inf A$.

2.3 Alternative characterizations of suprema and infima

Proposition 2.3.1 – alternative characterizationa of supremum Let $A \subseteq \mathbb{R}$ be non-empty and bounded from above. Let $M \in \mathbb{R}$. Then M is the supremum of A if and only if

1. M is an upper bound for A,
2. and

$$\text{for all } \varepsilon > 0, \quad \text{there exists } a \in A, \quad a > M - \varepsilon.$$

Proposition 2.3.2 – alternative characterizationa of infimum Let $A \subseteq \mathbb{R}$ be non-empty and bounded from below. Let $m \in \mathbb{R}$. Then m is the infimum of A if and only if

1. m is a lower bound for A,
2. and

$$\text{for all } \varepsilon > 0, \quad \text{there exists } a \in A, \quad a < m + \varepsilon.$$

These alternative characterizations of the supremum and infimum really provide a standard way to determining the supremum and infimum of subsets of the real line.

2.4 Maxima and minima

Definition 2.4.1 – maximum and minimum Let $A \subseteq \mathbb{R}$ be a subset of the real numbers. We say that $y \in A$ is the *maximum* of A, and write $y = \max A$, if

$$\text{for all } a \in A, \quad a \leq y.$$

We say that $x \in A$ is the *minimum* of A, and write $x = \min A$, if

$$\text{for all } a \in A, \quad a \geq x.$$

Remark 2.4.2. Even if a set $A \subseteq \mathbb{R}$ is non-empty and bounded, it may not have a maximum or minimum. For example, the set $(0,1)$ has no maximum or minimum.

Proposition 2.4.3 – Let A be a subset of $\mathbb{R}$. If A has a maximum, then A is non-empty and bounded from above, and $\sup A = \max A$. If A has a minimum, then A is non-empty and bounded from below, and $\inf A = \min A$.

Proposition 2.4.4 – Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from above. If $\sup A \in A$ then A has a maximum and $\max A = \sup A$.

Proposition 2.4.5 – Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from below. If $\inf A \in A$ then A has a minimum and $\min A = \inf A$.

2.5 The Archimedean property

Proposition 2.5.1 – Archimedeean property For every real number $x \in \mathbb{R}$ there exists a natural number $n \in \mathbb{N}$ such that $x < n$.

Given this proposition, we can define the ceiling function.

Definition 2.5.2 – ceiling function The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is defined as follows. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer $z \in \mathbb{Z}$ such that $x \leq z$.

Proposition 2.5.3 – For every two real numbers $a, b \in \mathbb{R}$ with $a < b$ there exists a $q \in \mathbb{Q}$ with $a < q < b$.

2.6 Computation rules for suprema

In the proposition below, we use the defintions

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for subsets $A, B \subseteq \mathbb{R}$ and a scalar $\lambda \in \mathbb{R}$.

Proposition 2.6.1 – Let A, B, C, D be non-empty subsets of $\mathbb{R}$. Assume that A and B are bounded from above and C and D are bounded from below. Then

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(C + D) = \inf C + \inf D$.
3. For all $\lambda \geq 0$, $\sup(\lambda A) = \lambda \sup A$.
4. For all $\lambda \leq 0$, $\sup(\lambda A) = \lambda \inf A$.
5. $\sup(-C) = -\inf C$.
6. $\inf(-C) = -\sup C$.

2.7 Bernoulli's inequality

Proposition 2.7.1 – Bernoulli's inequality Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

1. If $x \geq -1$, then $(1 + x)^n \geq 1 + nx$.
2. If $x \geq 0$ and $n \geq 2$, then $(1 + x)^n \geq 1 + nx$.

3 Sequences

3.1 Sequence

Definition 3.1.1 – Sequence A sequence is a function for which the domain is $\mathbb{N}$.

$$a : \mathbb{N} \to Y$$

Y can be any set.

Example 3.1.2 Here are some functions that are sequences:

1. $a : \mathbb{N} \to \mathbb{Q}$
2. $b : \mathbb{N} \to (\mathbb{N} \to Y)$
3. $c : \mathbb{N} \to \mathbb{N}$

And some functions that are not sequences:

1. $d : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$
2. $e : \mathbb{Q} \to \mathbb{N}$

3.2 Terminology around sequences

3.2.1 Bounded sequences

Definition 3.2.2 – bouneded sequence Let (X, dist) be a metric space. We say a sequence $a : \mathbb{N} \to X$ is bounded if

$$\begin{aligned}
&\text{there exists } q \in X, \\
&\quad \text{there exists } M > 0, \\
&\quad\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad\quad \text{dist}(a_n, q) \leq M.
\end{aligned}$$

In a normed linear space, we can use a simpler criterion to check whether a sequence is bounded. That is the content of the following proposition.

Proposition 3.2.3 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. The sequence a is bounded if and only if

$$\begin{aligned}
&\text{there exists } M > 0, \\
&\quad \text{for all } n \in \mathbb{N}, \\
&\quad\quad \|a_n\| \leq M.
\end{aligned}$$

3.3 Convergence of sequences

Definition 3.3.1 – Convergence of sequences Let (X, dist) be a metric space. We say that a sequence $a : \mathbb{N} \to X$ converges to a point $p \in X$ if

$$\begin{aligned}
&\text{for all } \varepsilon > 0, \\
&\quad \text{there exists } N \in \mathbb{N}, \\
&\quad\quad \text{for all } n \geq N, \\
&\quad\quad\quad \text{dist}(a_n, p) < \varepsilon.
\end{aligned}$$

We sometimes write

$$\lim_{n\to\infty} a_n = p$$

to express that the sequence (a_n) converges to p.

Definition 3.3.2 – Divergence of sequences Let (X, dist) be a metric space. A sequence $a : \mathbb{N} \to X$ is called *divergent* is it is not convergent.

3.4 Examples and limits of simple sequences

Proposition 3.4.1 – The constant sequence Let (X, dist) be a metric space. Let $p \in X$ and assume that the sequence (a_n) is given by $a_n = p$ for every $n \in \mathbb{N}$. We also say that (a_n) is a constant sequence. Then $\lim_{n\to\infty} = p$.

Example 3.4.2 A standard limit Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence such that $a_n = 1/n$ for $n \geq 1$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to 0.

Proof. Let $\varepsilon > 0$. Choose $N = \lceil 1/\varepsilon \rceil + 1$. Take $n \geq N$. Then

$$\text{dist}_{\mathbb{R}}(a_n, 0) = |a_n - 0| = |1/n| = 1/n \leq 1/N < \varepsilon.$$

□

3.5 Uniqueness of limits

Proposition 3.5.1 – Uniqueness of limits Let (X, dist) be a metric space and let $a : \mathbb{N} \to \mathbb{R}$ be a sequence in X. Assume that $p, q \in X$ and assume that

$$\lim_{n\to\infty} = p \text{ and } \lim_{n\to\infty} a_n = q$$

Then $p = q$.

3.6 More properties of convergent sequences

Proposition 3.6.1 – Let (X, dist) be a metric space and suppose that $a : \mathbb{N} \to X$ is a sequence. Let $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the real-valued sequence

$$n \mapsto \text{dist}(a_n, p)$$

converges to 0 in $\mathbb{R}$.

Proposition 3.6.2 – Convergent sequences are bounded Let (X, dist) be a metric space. Let $a : \mathbb{N} \to X$ be a sequence in X converging to $p \in X$. Then the sequence $a : \mathbb{N} \to X$ is bounded.

Proposition 3.6.3 – Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences. Let $p \in X$ and suppose that $\lim_{n\to\infty} a_n = p$. Then $\lim_{n\to\infty} b_n = p$ if and only if

$$\lim_{n\to\infty} \text{dist}(a_n, b_n) = 0$$

Corollary 3.6.4 – Eventually equal sequences have the same limit Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences such that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n = b_n$$

Then the sequence $a : \mathbb{N} \to X$ converges if and only if the sequence $b : \mathbb{N} \to X$ converges. If the sequences converge, they have the same limit.

3.7 Limit theorems for sequences taking values in a normed vector space

Theorem 3.7.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Assume that the $\lim_{n\to\infty} a_n$ exists and is equal to $p \in V$ and that the $\lim_{n\to\infty} b_n$ exists and is equal to $q \in V$. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $\mu \in \mathbb{R}$. Assume that $\lim_{n\to\infty} \lambda_n = \mu$. Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $p + q$.
2. The $\lim_{n\to\infty}(\lambda_n a_n)$ exists and is equal to μp.

3.8 Index shift

Proposition 3.8.1 – Index shift Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence. Let $k \in \mathbb{N}$ and $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the sequence $(a_{n+k})_n$ (i.e. the sequence $n \mapsto a_{n+k}$) converges to p.

4 Real-valued sequences

4.1 Terminology

Definition 4.1.1 – increasing, decreasing and monotone sequences We say a sequence (a_n) is

1. *increasing* if for every $n \in \mathbb{N}, a_{n+1} \geq a_n$
2. *strictly increasing* if for every $n \in \mathbb{N}, a_{n+1} > a_n$
3. *decreasing* if for every $n \in \mathbb{N}, a_{n+1} \leq a_n$
4. *strictly decreasing* if for every $n \in \mathbb{N}, a_{n+1} < a_n$
5. *monotone* if it is either increasing or decreasing
6. *strictly monotone* if it is either strictly increasing or strictly decreasing

Definition 4.1.2 – upper bound and lower bound for a sequence We say that a number $M \in \mathbb{R}$ is an *upper bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \leq M$$

We say that a number $m \in \mathbb{R}$ is a *lower bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \geq m$$

Definition 4.1.3 – bounded sequence We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that M is an upper bound for a.
We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded below* if there exists an $m \in \mathbb{R}$ such that m is a lower bound for a.

Proposition 4.1.4 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below.

4.2 Monotone, bounded sequences and convergent

Theorem 4.2.1 – Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) convergent and

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad (= \sup\{a_n \mid n \in \mathbb{N}\})$$

Theorem 4.2.2 – Let (a_n) be a decreasing sequence that is bounded from below. Then (a_n) is convergent and

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n \quad (= \inf\{a_n \mid n \in \mathbb{N}\})$$

4.3 Limit theorems

Theorem 4.3.1 – Limit theorems for real-valued sequences Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be two converging sequences, and let $c, d \in \mathbb{R}$ be real numbers such that

$$\lim_{n\to\infty} a_n = c \text{ and } \lim_{n\to\infty} b_n = d.$$

Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $c + d$.
2. The $\lim_{n\to\infty}(a_n b_n)$ exists and is equal to $c \cdot d$.
3. If $d \neq 0$, then $\lim_{n\to\infty}(\frac{a_n}{b_n}$ exists and is equal to $\frac{c}{d}$.
4. For every non-negative integer $m \in \mathbb{N}$, the limit $\lim_{n\to\infty}(a_n)^m$ exists and is equal to c^m.
5. If for every $n \in \mathbb{N}$, the number a_n is non-negative, then for every positive integer $k \in \mathbb{N} \setminus \{0\}$, the limit $\lim_{n\to\infty}(a_n)^{\frac{1}{k}}$ exists and is equal to $c^{\frac{1}{k}}$.

4.4 The squeeze theorem

Theorem 4.4.1 – The squeeze theorem Let $a, b, c : \mathbb{N} \to \mathbb{R}$ be three sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$a_n \leq b_n \leq c_n$$

and assume $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n - L$ for some $L \in \mathbb{R}$. Then $\lim_{n\to\infty} b_n$ exists and is equal to L.

4.5 Divergence to ∞ and $-\infty$

Definition 4.5.1 – We say a sequence $a : \mathbb{N} \to \mathbb{R}$ *diverges to* ∞ and write

$$\lim_{n\to\infty} = \infty$$

if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } N \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N,\\
&\quad\quad\quad a_n > M.
\end{aligned}$$

Similarly, we say a sequence (a_n) *diverges* to $-\infty$ and write

$$\lim_{n\to\infty} a_n = -\infty$$

if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } N \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N,\\
&\quad\quad\quad a_n < M.
\end{aligned}$$

Proposition 4.5.2 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} a_n = \infty.$$

Then the sequence (a_n) is bounded from below.
Similarly, let $b : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} b_n = -\infty.$$

Then the sequence (b_n) is bounded from above.

4.6 Limit theorems for improper limits

Theorem 4.6.1 – **Limit theorems for improper limits** Let $a, b, c, d : \mathbb{N} \to \mathbb{R}$ be four sequences such that

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} c_n = -\infty$$

the sequence (b_n) is bounded from below and the sequence (d_n) is bounded from above. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence bounded below by some $\mu > 0$. Then

i. $\lim_{n\to\infty}(a_n + b_n) = \infty$

ii. $\lim_{n\to\infty}(c_n + d_n) = -\infty$

iii. $\lim_{n\to\infty}(\lambda_n a_n) = \infty$

iv. $\lim_{n\to\infty}(\lambda_n c_n) = -\infty$

Proposition 4.6.2 – Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be two sequences. Then

1. $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}(-a_n) = -\infty$.
2. $\lim_{n\to\infty} b_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

4.7 Standard sequences

4.7.1 Geometric sequence

Proposition 4.7.2 – **Standard limit of of geometric sequence** Let $q \in \mathbb{R}$. The sequence (a_n) defined by $a_n := q^n$ for $n \in \mathbb{N}$

- converges to 0 if $q \in (-1, 1)$
- converges to 1 if $q = 1$
- diverges to ∞ if $q > 1$
- diverges, but not to ∞ or $-\infty$ if $q \leq -1$

4.7.3 The n^{th} root of n

Proposition 4.7.4 – Standard limit of the n^{th} root of n The sequence (a_n) defined by $a_n := \sqrt[n]{n}$ for $n \in \mathbb{N}$ converges to 1.

Corollary 4.7.5 – Let $a > 0$. Then the sequence (b_n) defined by $b_n := \sqrt[n]{a}$ converges to 1.

4.7.6 The number e

First let's define the sequence (a_n) by

$$a_n := \left(1 + \frac{1}{n}\right)^n.$$

We show that (a_n) is increasing and bounded from above by 3. Hence (a_n) converges to some $e \in \mathbb{R}$ by the monotone convergence theorem.

Lemma 4.7.7 – The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is increasing.

Lemma 4.7.8 – The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is bounded from above by 3.

By these two lemmas, the sequence

$$n \mapsto \left(1 + \frac{1}{n}\right)^n$$

converges.

Definition 4.7.9 – (Standard limit of e) We define the number e by

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

4.7.10 Exponentials beat powers

Proposition 4.7.11 – Let $a \in (1, \infty)$ and let $p \in (0, \infty)$. Then

$$\lim_{n \to \infty} \frac{n^p}{a^n} = 0.$$

4.8 Sequences with values in $\mathbb{R}^d$

Proposition 4.8.1 – Consider the metric space $(\mathbb{R}^d, \|\cdot\|_2)$. Let $z \in \mathbb{R}^d$ and let $x : \mathbb{N} \to \mathbb{R}^d$ be a sequence. Denote by y_i the ith componenet of a vector $y \in \mathbb{R}^d$. Then the seequence $(x^{(n)})$ converges to z if and only if for all $i \in \{1, \ldots, d\}$, teh sequence $(x_i^{(n)})$ converges to z_i.

5 Series

5.1 Definition

Definition 5.1.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Let $K \in \mathbb{N}$. We say that a series

$$\sum_{n=K}^{\infty} a_n$$

is *convergent* if the associatedd *sequence of partial sums* $S_k : \mathbb{N} \to V$, i.e. the sequece $(S_K^n)_{n\in\mathbb{N}}$ converges. The term S_K^n is, for $n \in \mathbb{N}$, defined as

$$S_K^n := \sum_{k=K}^{n} a_k$$

If $K = 0$, we usually jusst write S^n or even S_n instead of S_0^n.
If the series $\sum_{n=K}^{\infty} a_n$ is convergent, the *value* of the series is by defintion equal to the limit of the sequence of partial sums, i.e.

$$\sum_{k=K}^{\infty} a_k := \lim_{n\to\infty} S_k^n = \lim_{n\to\infty} \sum_{k=K}^{\infty} a_k$$

5.2 Geometric series

Proposition 5.2.1 – Let $a \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

Proof. We consider

$$\begin{aligned}
(1-a)\sum_{k=0}^{n} a^k &= \sum_{k=0}^{n} a^k - a\sum_{k=0}^{n} a^k \\
&= \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} \\
&= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k \\
&= 1 - a^{n+1}
\end{aligned}$$

□

Proposition 5.2.2 – Geometric series Let $a \in (-1, 1)$. Then the series

$$\sum_{k=0}^{\infty} a^k$$

is convergent and has the value

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

5.3 The harmonic series

Proposition 5.3.1 – Harmonic series The series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

5.4 The hyperharmonic series

Proposition 5.4.1 – Hyperharmonic series Let $p > 1$. Then the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges.

Example 5.4.2 Here is an example of a series taking values in the normed vector space $(\mathbb{R}^2, \|\cdot\|)$:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2}, \left(\frac{1}{2}\right)^k \right)$$

5.5 Only the tail matters for convergence

Lemma 5.5.1 – Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence taking values in V. Let $K, L \in \mathbb{N}$. The series

$$\sum_{n=K}^{\infty} a_n$$

is conovergent is and only if the series

$$\sum_{n=L}^{\infty} a_n$$

is convergent. Moreover, if either the series converges, and $K < L$, then

$$\sum_{n=K}^{\infty} a_n = \sum_{n=K}^{L-1} + \sum_{n=L}^{\infty} a_n.$$

Proposition 5.5.2 – Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and assume that the series

$$\sum_{k=M}^{\infty} a_k$$

is convergent. Then

$$\lim_{m\to\infty} \sum_{k=m}^{\infty} a_k = 0.$$

Proposition 5.5.3 – Index shift for series Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and let $\ell \in \mathbb{N}$. Then the series

$$\sum_{k=M}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=M}^{\infty} a_{k+\ell}$$

converges. Moreoever, if either series converges, then

$$\sum_{k=M}^{\infty} a_{k+\ell} = \sum_{k=M+\ell}^{\infty} a_k.$$

5.6 Divergence test

Proposition 5.6.1 – Let $(V, \|\cdot\|)$ be a normed vector space, and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent to $L \in V$. Then

$$a_n = S_n - S_{n-1}$$

where S_n denote the partial sum $\sum_{k=0}^{n} a_k$. Because S_n and S_{n-1} are both convergent to L, the sequence (a_n) is convergent as well and converges to $L - L = 0$. □

Theorem 5.6.2 – **Divergence test** Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the limit $\lim_{n\to\infty} a_n$ does not exist or is not equal to 0. Then the series

$$\sum_{n=0}^{\infty} a_n$$

is divergent.

5.7 Limit laws for series

Theorem 5.7.1 – **Limit laws for series** Let $(V, \|\cdot\|)$ be a normed vector space and let $a, b : \mathbb{N} \to V$ be sequences in V. Suppose the series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n$$

are convergent. Suppose $\lambda \in \mathbb{R}$. Then

1. The series

$$\sum_{n=0}^{\infty} (a_n + b_n)$$

is converget and converges to

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

2. The series

$$\sum_{n=0}^{\infty} \lambda a_n$$

is convergent and converges to

$$\lambda \sum_{n=0}^{\infty} a_n.$$

6 Series with positive terms

6.1 Comparison test

Theorem 6.1.1 – Comparison test Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences. Assume that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \leq b_n$. Then

1. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Then the series $\sum_{n=1}^{\infty} a_n$ converges as well.
2. Suppose the series $\sum_{n=1}^{\infty} a_n$ diverges. Then the series $\sum_{n=1}^{\infty} b_n$ diverges as well.

Example 6.1.2 Consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2-1}.$$

We first observe that for all $k \geq 2$ we have

$$\frac{k}{k^2-1} \geq \frac{k}{k^2} = \frac{1}{k}.$$

Because the series

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

diverges, the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2-1}$$

diverges as well byt the comparison test.

6.2 Limit comparison test

Theorem 6.2.1 – Limit comparison test Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences.

1. Assume the series $\sum_{k=1}^{\infty} b_k$ converges and assume the limit

   $$\lim_{n\to\infty} \frac{a_n}{b_n}$$

   exists. Then the series $\sum_{k=1}^{\infty} a_k$ converges as well.
2. Assume the series $\sum_{k=1}^{\infty} b_k$ diverges and assume the limit

   $$\lim_{n\to\infty} \frac{a_n}{b_n}$$

   exists and is strictly larger than zero, or that the limit is infinity. Then the series $\sum_{k=1}^{\infty} a_k$ diverges as well.

Example 6.2.2 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1}.$$

We use sequences $a, b : \mathbb{N} \to [0, \infty)$ defined for $k \geq 2$ by

$$a_k = \frac{k}{k^2+1}$$

and

$$b_k = \frac{1}{k}.$$

Then

$$\frac{a_k}{b_k} = \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \frac{1}{1+\frac{1}{k^2}}.$$

By limit laws, we find that the limit of the denominator is 1, i.e.

$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\right) = \lim_{k\to\infty} 1 + \lim_{k\to\infty}\frac{1}{k^2} = 1+0 = 1.$$

Therefore, we may apply the limit law for the quoteient and conclude that

$$lim_{k\to\infty}\frac{a_k}{b_k} = \frac{1}{\lim_{k\to\infty}\left(1+\frac{1}{k^2}\right)} = \frac{1}{1} = 1.$$

The series $\sum_{k=2}^{\infty}\frac{1}{k}$ diverges and therefore it follows from the Limit Comparison Test that the series

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty}\frac{k}{k^2+1}$$

diverges as well.

6.3 Ratio test

Theorem 6.3.1 – Ratio Test Let $a : \mathbb{N} \to [0, \infty)$ be a sequence.

1. if there exists an $N \in \mathbb{N}$ and a $q \in (0, 1)$ such that for all $n \geq N$, it holds that

$$\frac{a_n+1}{a_n} \leq q$$

, then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. if there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it hodls that

$$\frac{a_n+1}{a_n} \geq 1,$$

then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.4 Limit ratio test

Theorem 6.4.1 – Limit Ratio Test Let $a : \mathbb{N} \to (0, \infty)$ be a sequence.

1. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q \in [0, 1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.4.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

6.5 Root test

Theorem 6.5.1 – Root test Let (a_n) be a sequence of non-negative real numbers.

1. If there exists an $N \in \mathbb{N}$ and a $q \in (0, 1)$ such that for all $n \geq N$, it holds that

$$\sqrt[n]{a_n} \leq q,$$

   then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that

$$\sqrt[n]{a_n} \geq 1,$$

   then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.6 Limit root test

Theorem 6.6.1 – Limit Root Test Let (a_n) be a sequence of non-negative real numbers.

1. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q \in [0, 1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.6.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1.$$

7 Series with general terms

7.1 Series with real terms: the Leibniz test

Theorem 7.1.1 – Leibniz test, a.k.a Alternating series test Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences such that for all $k \in \mathbb{N}$, $b_k = (-1)^k a_k$. Assume that there exists a $K \in \mathbb{N}$ such that

1. $a_k \geq 0$ for every $k \geq K$,
2. $a_k \geq a_{k+1}$ for every $k \geq K$,
3. $\lim_{k\to\infty} a_k = 0$.

Then, the series

$$\sum_{k=K}^{\infty} b_k = \sum_{k=K}^{\infty} (-1)^k a_k$$

is convergent. In addition, the following esitmate holds for every $N \geq K$,

$$\left| S_N - \sum_{k=K}^{\infty} b_k \right| \leq a_{N+1}.$$

where for all $n \in \mathbb{N}$, $S_n := \sum_{k=K}^{\infty} b_k$.

Example 7.1.2 We claim that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

We would like to apply the Alternating series test. To do so, we need toe check its conditions. We define the sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_k := \frac{1}{k}$$

for $k \geq 1$ (and $a_0 = a_1 = 1$).

We now check the conditions for the Alternating Series Test.

1. We need to show that $a_k \geq 0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq 0.$$
2. We need to show that $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq \frac{1}{k+1} = a_{k+1}.$$
3. We need to show that
$$\lim_{k\to\infty} a_k = 0$$
. This follow as this is a standard limit.

It follows from the Alternating Series Test that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

7.2 Series charactersization of completeness in normed vector space

Definition 7.2.1 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence of vectors in V. We say the series

$$\sum_{k=0}^{\infty} a_k$$

converges *absolutely* if

$$\sum_{k=0}^{\infty} \|a_k\|$$

converges.

Definition 7.2.2 – Series characterization of completeness We say a normed vector space $(V, \|\cdot\|)$ satisfies the *series characterization of completeness* if every series in V that is absolutely convergent is also convergent.

Proposition 7.2.3 – Every finite-dimensional normed vector space satisfies the series characterization of completeness.

Example 7.2.4 Consider the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}.$$

Since this is not an alternataing series, we cannot apply the Leibniz test.

However, for every k
$in \mathbb{N} \setminus \{0\}$, we have

$$\left| \frac{\sin(k)}{k^2} \right| \leq \frac{1}{k^2}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a standard hyperharmonic seris, of which we know that it converges. By the Cmomparison Test, we conclude that the series

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right|$$

converges as well.

Therefore, the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges absolutely. Since $(\mathbb{R}, |\cdot|)$ is complete, we find that

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges.

Definition 7.2.5 – Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. We say that a series

$$\sum_{k=0}^{\infty} a_k$$

converges *conditionally* if it converges but does not converge absolutely.

7.3 The Cauchy product

Theorem 7.3.1 – Cauchy product Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences. Assume that the series

$$\sum_{k=0}^{\infty} a_k$$

and

$$\sum_{k=0}^{\infty} b_k$$

converge absolutely. Then, the series

$$\sum_{k=0}^{\infty} c_k$$

converges absolutely as well, where

$$c_k := \sum_{\ell=0}^{k} a_\ell b_{k-\ell},$$

and

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

8 Subsequences, lim sup and lim inf

8.1 Index sequences and subsequences

Definition 8.1.1 – Index sequence We say a sequence $n : \mathbb{N} \to \mathbb{N}$ is an *index sequence* if it is strictly increasing.

Example 8.1.2 The sequence $n : \mathbb{N} \to \mathbb{N}$ defined by

$$n_k := 2k$$

is a strictly increasing sequence of natural numbers. In other words, it is an index sequence.

Definition 8.1.3 – Subsequence Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. A sequence $b : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of a if there exists an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that $b = a \circ n$

Just as we often write $(a_n)_{n\in\mathbb{N}}$ for a sequence called a, we often write $(a_{n_k})_{k\in\mathbb{N}}$ for the subsequence $a \circ n$.

8.2 (Sequential) accumulation points

Definition 8.2.1 – (Sequential accumulation points) Let (X, dist) be a metric space. A point $p \in X$ is called an *accumulation point* of a sequence $a : \mathbb{N} \to X$ if there is a subsequence $a \circ n$ of a such that $a \circ n$ converges to p.

8.3 Subsequences of a converging sequence

Proposition 8.3.1 – Let (X, dist) be a metric space. Let (a_n) be a sequence in X converging to $p \in X$. Then every subsequence of (a_n) is convergent to p.

8.4 lim sup

Consider a real-valued sequence (a_n) that is bounded from above and does not diverge to $-\infty$. We can then define a new sequence

$$k \mapsto \sup_{n\geq k} a_n.$$

Note that this sequence is decreasing, because for larger k the supremum is taken over a smaller set.

Lemma 8.4.1 – Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence that is bounded from above and does not diverge to $-\infty$. Then, the sequence $k \mapsto \sup_{n\geq k} a_n$ is bounded from below.

Since the sequence $k \mapsto \sup_{n\geq k} a_n$ is decreasing and bounded from below, it has a limit, and the limit is in fact equal to the infumum of the sequence. This limit is called the lim sup

$$\begin{aligned} \limsup_{n\to\infty} a_n &:= \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n \\ &= \lim_{k\to\infty} \left(\sup_{n\geq k} a_n \right) \end{aligned}$$

For every,

there exists,

Proposition 8.4.2 – **Alternative characterization of** lim sup Let (a_n) be a real-valued sequence. Let $M \in \mathbb{R}$. Then, $M = \limsup_{n\to\infty} a_n$ if and only if

i. For every $\varepsilon > 0$,
 there exists $N \in \mathbb{N}$,
 for all $\ell \geq N$,
 $a_\ell < M + \varepsilon$

ii. For every $\varepsilon > 0$,
 for all $k \in \mathbb{N}$,
 there exists $m \geq k$,
 $a_m > M - \varepsilon$

Theorem 8.4.3 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded from above and does not diverge to $-\infty$. Then $\limsup_{\ell\to\infty} a_\ell$ is a (sequential) accumulation point of a, i.e. there exists a subsequences of a that converges to $\limsup_{\ell\to\infty} a_\ell$.

Corollary 8.4.4 – **Bolzano-Weierstrass** Every bounded, real-valued sequence has a subsequence that converges in $(\mathbb{R}, \text{dist}_{\mathbb{R}})$.

Theorem 8.4.5 – Suppose a sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$. Then

$$\limsup_{\ell\to\infty} a_\ell$$

is the maximum of the set of sequential accumulation points.

8.5 lim inf

Similarly to the lim sup, we can define the lim inf. In some sense,

$$\liminf_{\ell\to\infty} a_\ell = -\limsup_{\ell\to\infty}(-a_\ell)$$

More precisely,

$$\begin{aligned}\liminf_{\ell\to\infty} a_\ell &:= \sup_{\ell\in\mathbb{N}} \inf_{k\geq\ell} a_k \\ &= \lim_{\ell\to\infty} \left(\inf_{k\geq\ell} a_k \right)\end{aligned}$$

Proposition 8.5.1 – **Alternative characterization of** lim inf Let $a : \mathbb{N} \to \mathbb{R}$ and $M \in \mathbb{R}$. Then M equals $\liminf_{\ell\to\infty} a_\ell$ if and only if

1. For every $\varepsilon > 0$,
 there exists $N \in \mathbb{N}$,
 for all $\ell \geq N$,
 $a_\ell > M - \varepsilon$

2. For every $\varepsilon > 0$,
for all $K \in \mathbb{N}$,
there exists $m \geq K$,
$a_m < M + \varepsilon$

Theorem 8.5.2 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is a sequential accumulation point of the sequence a, i.e. there is a subsequence of a that converges to $\liminf_{\ell\to\infty} a_\ell$.

Theorem 8.5.3 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is the minimum of the set of sequential accumulation points.

8.6 Relations between lim, lim sup and lim inf

Proposition 8.6.1 – Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} = L$$

Proposition 8.6.2 – Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences, such that there exists an $N \in \mathbb{N}$ such that for all $\ell \geq N$, $a_\ell \leq b_\ell$. Then

$$\limsup_{\ell\to\infty} a_\ell \leq \limsup_{\ell\to\infty} b_\ell$$

and

$$\liminf_{\ell\to\infty} a_\ell \leq \liminf_{\ell\to\infty} b_\ell.$$

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