Department of Mathematics and Computer Science
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Lines and Ratios in a Triangle

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1 Problem Description

In $\triangle ABC$ (the points A, B, C are non co-linear), P is the midpoint of the segment BC and R is the point on the line AB such that A is the midpoint of the segment BR. Use vectors to determine the point of intersection Q of lines PR and AC, and show that $AQ : QC = 1 : 2$.

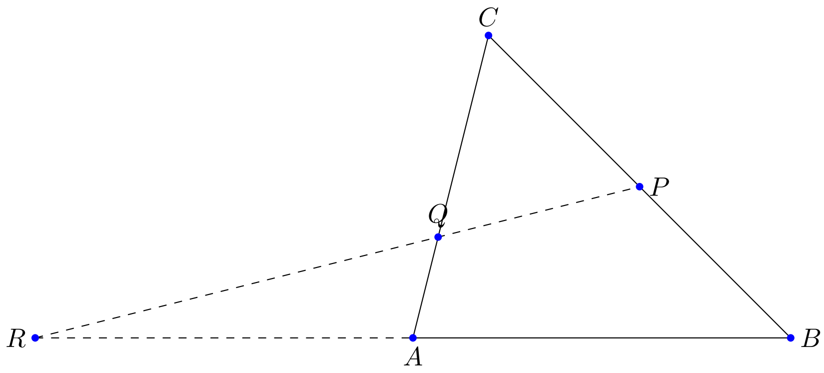

Figure 1: Example Construction

2 Solution

In solving this problem, we will mainly use vector techniques. In geometric exercises that involve vectors, we first need to choose where to put the origin if it is not explicitly specified. Without loss of generality, we can choose the point A as our origin and assign the vectors $\underline{b}, \underline{c}, \underline{p}, \underline{q}, \underline{r}$ to the points B, C, P, Q, R, respectively.

The way we solve the exercise is first we find vector representations of the lines RP and AC, then by finding their point of intersection we find Q.

Since A is the origin and the midpoint of the segment BR, we get $\underline{r} = -\underline{b}$. Additionally, since P is the midpoint of the segment BC, we get:

$$\underline{p} = \underline{b} + \frac{1}{2}\left(\underline{c} - \underline{b}\right) = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}.$$

The line RP can be defined as $RP : \underline{x} = \underline{r} + \lambda\left(\underline{p} - \underline{r}\right)$, where $\lambda \in \mathbb{R}$ [1], using our above defined substitutions, we thus get:

$$RP : \underline{x} = -\underline{b} + \lambda\left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}\right) \tag{1}$$

And the parametric equation for AC is

$$AC : \underline{x} = \mu\underline{c},\ \text{where } \mu \in \mathbb{R} \tag{2}$$

since AC passes through the origin (A) and has direction $\underline{c}$. Intersecting eq. (1) and eq. (2) we get:

$$-\underline{b} + \lambda \left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c} \right) = \mu\underline{c}$$

$$-\underline{b} + \lambda \left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c} \right) - \mu\underline{c} = \underline{0}$$

$$\left(-1 + \frac{3}{2}\lambda \right) \underline{b} + \left(\frac{1}{2}\lambda - \mu \right) \underline{c} = \underline{0}$$

The points B and C are not co-linear, therefore, $\underline{b}$ and $\underline{c}$ are linearly independent. Thus from [2] we get:

$$\begin{cases} -1 + \frac{3}{2}\lambda = 0 \\ \frac{1}{2}\lambda - \mu = 0 \end{cases} \iff \begin{cases} \frac{3}{2}\lambda = 1 \\ \mu = \frac{1}{2}\lambda \end{cases} \iff \begin{cases} \lambda = \frac{2}{3} \\ \mu = \frac{1}{3} \end{cases}$$

Since $\underline{q} \in AC$, we can write

$$\underline{q} = \mu\underline{c} = \frac{1}{3}\underline{c}.$$

From this, we get:

$$AQ = \|\frac{1}{3}\underline{c}\| = \frac{1}{3}\|\underline{c}\|$$

and

$$QC = AC - AQ = \|\underline{c}\| - \frac{1}{3}\|\underline{c}\| = \frac{2}{3}\|\underline{c}\|.$$

Having calculated the lengths of AQ and QC we see that

$$\frac{AQ}{QC} = \frac{\frac{1}{3}\|\underline{c}\|}{\frac{2}{3}\|\underline{c}\|}$$

$$\frac{AQ}{QC} = \frac{1}{2}$$

$$\therefore AQ : QC = 1 : 2$$

3 Conclusion

We were able able to prove a geometric property using vector arithmetic and, indeed, we found that the intersection Q of lines RP and AC divides the segment AC into two segments AQ and QC, where QC is twice the length of AQ (i.e. $AQ : QC = 1 : 2$).

Certainly, there are also other ways in which we could have solved this exercise. For instance, instead of choosing A as origin, we could have chosen any other given point or even an arbitrary point on the plane. This, however, may lead to more complicated expressions for the lines RP and AC, thus making it more prone to computational errors.

Additionally, it may also be possible to solve the given problem using only geometric properties of triangles, however, its investigation is beyond the scope of this article and this course.

4 Roles of Group Members

- Jiaqi Wang - document organization, visual organization and final editing
- Mil Majerus - solution writing
- Jean Nguyen - proof reading
- Long Pham - proof reading, visual organization and final editing

References

[1] Hans Sterk. *Linear Algebra 1*, chapter 1.2.1. Technische Universiteit Eindhoven, 2023-2024.

[2] Hans Sterk. *Linear Algebra 1*, chapter 3.2.10. Tecnhische Universiteit Eindhoven, 2023-2024.