Linear Algebra Exercises

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1 Summary

1.1 Vector/Parametric descriptions of lines and planes

1.1.1 Lines

$$l : \underline{x} = \underline{v} + \lambda \underline{u}$$

where $\underline{v}$ lands on the line and is called the *position vector*, and $\underline{u}$ is vector that is in the "direction" of the line, called the direction vector.

1.1.2 Plane

$$V : \underline{x} = \underline{u} + \lambda \underline{v} + \mu \underline{w}$$

where $\underline{u}$ is a vector that lands on the plane, and $\underline{v}$ and $\underline{w}$ are two linearly independent vectors in the plane.

1.2 Determining parametric description from equations

Example 1.1 Suppose $2x_1 - x_2 + 3x_3 = 4$ is the equation of the plane V. To determine a parametric description we proceed as follows. If you assign any value, say λ to x_2, and any value, say μ to x_3, then x_1 is determined: $x_1 = 2 + \lambda/2 - 3\mu/2$. So

$$x_1 = 2 + \lambda/2 - 3\mu/2, x_2 = \lambda, x_3 = \mu$$

In vector notation:

$$(x_1, x_2, x_3) = (2 + \lambda/2 - 3\mu/2, \lambda, \mu) = (2, 0, 0) + \lambda(1/2, 1, 0) + \mu(-3/2, 0, 1).$$

Then the vector parametric description is

$$V : \underline{x} = (2, 0, 0) + \lambda(\frac{1}{2}, 1, 0) + \mu(-\frac{3}{2}, 0, 1).$$

To avoid fraction, you could also take

$$V : \underline{x} = (2, 0, 0) + \rho(1, 2, 0) + \sigma(-3, 0, 2).$$

Example 1.2 To find an equation of the plane V with vector parametric equation $\underline{x} = (2,0,0) + \lambda(1,1,0) + \mu(0,2,1)$. We can find two vectors in the plane, say $\underline{v}$ and $\underline{w}$, and then use the cross product to find the normal vector to the plane. Then we can use the normal vector to find the equation of the plane.

For example, we can take $\underline{v} = (1,1,0)$ and $\underline{w} = (0,2,1)$. Then the normal vector to the plane is $\underline{n} = \underline{v} \times \underline{w} = (1,1,0) \times (0,2,1) = (1,-1,2)$. Then the equation of the plane is

$$V : \underline{n} \cdot (\underline{x} - (2,0,0)) = 0$$

$$\begin{aligned} (1,-1,2) \cdot (x_1 - 2, x_2, x_3) &= 0 \\ x_1 - 2 - x_2 + 2x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 2 \end{aligned}$$

Definition 1.1 – Cross product Let $\underline{u} = (u_1, u_2, u_3)$ and $\underline{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{R}^3$. The cross product of $\underline{u}$ and $\underline{v}$ is the vector

$$\underline{u} \times \underline{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

The cross product is orthogonal to both $\underline{u}$ and $\underline{v}$.