

Assignment 6

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1 Exercise 10.7.1

Problem Let $A := \{a, b, \dots, z\}$ be the set of all letters of the alphabet. Let $\alpha : \mathbb{N} \rightarrow A$ be a sequence. Let $v : \mathbb{N} \rightarrow \mathbb{N}$ be an index sequence defined by $v_k = k + 5$ and $\mu : \mathbb{N} \rightarrow \mathbb{N}$ be an index sequence defined by $\mu_k = 3k$. Write first 33 terms of the subsequence $(\alpha_{v_{\mu_k}})_k$

Proof.

$$\begin{aligned}\alpha_{v_{\mu_k}} &= \alpha_{v_{3k}} \\ &= \alpha_{3k+5}\end{aligned}$$

So $(\alpha_{v_{\mu}})_0 = \alpha_5, (\alpha_{v_{\mu}})_1 = \alpha_8, (\alpha_{v_{\mu}})_2 = \alpha_{11}, \dots$. Following the diagram we get the following for the first 33 terms

goodluckinthesecondhalfofanalysis, or with spaces
good luck in the second half of analysis

□

2 Exercise 10.7.3

Problem Let (X, dist) be a metric space and let $a : \mathbb{N} \rightarrow X$ and $b : \mathbb{N} \rightarrow X$ be two sequences, such that $a : \mathbb{N} \rightarrow X$ converges to some $p \in X$. Now consider the following sequence $c : \mathbb{N} \rightarrow X$, defined by

$$c_k := \begin{cases} a_k & \text{if } k \text{ even} \\ b_k & \text{if } k \text{ odd} \end{cases}$$

Show that p is an accumulation point of $c : \mathbb{N} \rightarrow X$.

Proof. Define $n : \mathbb{N} \rightarrow \mathbb{N}$ a index sequence defined by $n_k := 2k$, then the subsequences $(c_{n_k})_k$ is the even terms of c , which is a . Since a converges to p , then p is an accumulation point of c . □

3 Exercise 10.7.5

Problem Let $a : \mathbb{N} \rightarrow \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \rightarrow \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell \rightarrow \infty} a_\ell = \limsup_{\ell \rightarrow \infty} a_\ell = L$$

Proof. We prove both directions.

First assume that $(a_k)_k$ converges to L . Then it holds

$$\begin{aligned} &\text{for all } \epsilon_0 > 0, \\ &\text{there exists } N_0 \in \mathbb{N}, \\ &\text{for all } n \geq N_0, \\ &|a_n - L| < \epsilon. \end{aligned} \tag{1}$$

Need to show $\limsup_{\ell \rightarrow \infty} a_\ell = L$ and $\liminf_{\ell \rightarrow \infty} a_\ell = L$.

By the alternative characterization of \limsup we need to show that

1.

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\text{there exists } N \in \mathbb{N}, \\ &\text{for all } n \geq N, \\ &a_n < L + \epsilon. \end{aligned}$$

2.

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\text{for all } K \in \mathbb{N} \\ &\text{there exists } m \geq K, \\ &a_m > L - \epsilon. \end{aligned}$$

We first show 1.

Let $\epsilon > 0$,

Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$.

Obtain such N_0 ,

Choose $N = N_0$,

Then for all $n \geq N = N_0$, we have

$|a_n - L| < \epsilon$, in particular

$a_n < L + \epsilon$

Now we show 2.

Let $\epsilon > 0$,

Take $K \in \mathbb{N}$,

Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$,

Obtain such N_0 ,

Choose $m = N_0 + K$, then we have

$|a_m - L| < \epsilon_0$, in particular

$a_m > L - \epsilon$

By alternative characterization of \liminf we need to show that

1.

for all $\epsilon > 0$,
there exists $N \in \mathbb{N}$,
for all $n \geq N$,
 $a_n > L - \epsilon$.

2.

for all $\epsilon > 0$,
for all $K \in \mathbb{N}$
there exists $m \geq K$,
 $a_m < L + \epsilon$.

We first show 1.

Let $\epsilon > 0$,

Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$.

Obtain such N_0 ,

Choose $N = N_0$,

Then for all $n \geq N = N_0$, we have

$|a_n - L| < \epsilon_0 = \epsilon$, in particular

$a_n > L - \epsilon$

Now we show 2.

Let $\epsilon > 0$,

Take $K \in \mathbb{N}$,

Choose $\epsilon_0 = \epsilon$ in (1), then there exists $N_0 \in \mathbb{N}$, for all $n > N_0$, $|a_n - L| < \epsilon_0$,

Obtain such N_0 ,

Choose $m = N_0 + K$, then we have

$|a_m - L| < \epsilon_0 = \epsilon$, in particular

$a_m < L + \epsilon$

Now we prove the other direction.

Assume that $\liminf_{\ell \rightarrow \infty} a_\ell = \limsup_{\ell \rightarrow \infty} a_\ell = L$.

We need to show that $(a_k)_k$ converges to L ,

i.e.

for all $\epsilon > 0$,
there exists $N \in \mathbb{N}$,
for all $n > N$,
 $|a_n - L| < \epsilon$

Since $\liminf_{\ell \rightarrow \infty} a_\ell = L$, we have

for all $\epsilon_1 > 0$,
there exists $N_1 \in \mathbb{N}$,
for all $n \geq N_1$,
 $a_n > L - \epsilon_1$ (2)

Since $\limsup_{\ell \rightarrow \infty} a_\ell = L$, we have

for all $\epsilon_2 > 0$,
there exists $N_2 \in \mathbb{N}$,
for all $n \geq N_2$,
 $a_n < L + \epsilon_2$ (3)

Let $\epsilon > 0$,

Choose $\epsilon_1 = \epsilon$ in (2), then there exists $N_1 \in \mathbb{N}$, for all $n \geq N_1$, $a_n > L - \epsilon_1 = L - \epsilon$.

Obtain such N_1 ,

Choose $\epsilon_2 = \epsilon$ in (3), then there exists $N_2 \in \mathbb{N}$, for all $n \geq N_2$, $a_n < L + \epsilon_2 = L + \epsilon$.

Choose $N = \max(N_1, N_2)$, then for all $n \geq N$, we have

$a_n > L - \epsilon$ and $a_n < L + \epsilon$, in particular

$|a_n - L| < \epsilon$

□

4 Exercise 10.7.7

Problem Let $a : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence with at least two sequential accumulation points $p, q \in \mathbb{R}$ with $p \neq q$. Prove that the sequence $a : \mathbb{N} \rightarrow \mathbb{R}$ does not converge.

Proof. Assume (a_n) has two accumulations points $p, q \in \mathbb{R}$ with $p \neq q$.

We argue by contradiction.

Assume (a_n) converges to $L \in \mathbb{R}$.

Since p is a sequential accumulation point we have that

$$\limsup_{k \rightarrow \infty} a_k = p = L$$

Similarly, since q is a sequential accumulation point we have

$$\limsup_{k \rightarrow \infty} a_k = q = L$$

Then $p = q$, which is a contradiction. Therefore (a) does not converge.

□