Discrete Structures

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1 Sets

1.1 Functions

Proposition 1.1.1 Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

(i) If f, g are injective, then $g \circ f$ is also injective.

(ii) If f, g are surjective, then $g \circ f$ is also surjective.

(iii) If f, g are bijective, then $g \circ f$ is also bijective.

(iv) For any function $f : X \to Y$, there exist a set Z, an injective function $h : Z \to Y$, and a surjective function $g : X \to Z$, such that $f = h \circ g$

1.2 Relations

Proposition 1.2.1 Let R be a relation on a set X. Then

1. R is reflexive if and only if $\Delta_X \subseteq R$.
2. R is symmetric if and only if $R = R^{-1}$.
3. R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta_X$.
4. R is transitive if and only if $R \circ R \subseteq R$.

Where $\Delta_X = \{(x, x) \mid x \in X\}$.

1.3 Orders

Proposition 1.3.1 Let R be an order on a set X, Then $R \cup R^{-1} = X \times X$

Definition 1.3.2 (Immediate predecessor) Let $(X, \sqsubseteq)$ be an ordered set. We say that an element $x \in X$ is an *immediate predecessor* of an element $y \in X$ if

- $x \sqsubseteq y$, and
- there is no $t \in X$ such that $x \sqsubseteq y \sqsubseteq y$.

Proposition 1.3.3 Let $(X, \sqsubseteq)$ be a finite-ordered set and let $\lhd$ be the corresponding immediate predecessor relation. Then for any two elements $x, y \in X, x \sqsubseteq y$ holds if and only if there exists elements $x_1, \ldots, x_k \in X$ such that $x \lhd x_1 \lhd \cdots \lhd x_k \lhd y$ (possibly $k = 0$, i.e. we may also have $x \lhd y$).

Theorem 1.3.4 (Linear extension) Let $(X, \sqsubseteq)$ be a finite-poset. Then there exists a linear ordering $\leq$ on X such that $x \sqsubseteq y \implies x \leq y$. The order $\leq$ is called a *linear extension* of $\sqsubseteq$.

Definition 1.3.5 **(Minimal/Maximal element)** Let $(X, \sqsubseteq)$ be an ordered set. An element $a \in X$ is called a *minimal element* if there is no $x \in X$ such that $x \sqsubseteq a$ and $x \neq a$. An element $b \in X$ is called a *maximal element* if there is no $y \in X$ such that $b \sqsubseteq y$ and $b \neq y$.

Theorem 1.3.6 Every finite partially ordered set has at least one minimal element.

Definition 1.3.7 **(Smallet/Largest element)** Let $(X, \sqsubseteq)$ be an ordered set. An element $a \in X$ is called a *smallest element* if $\forall x \in X[a \sqsubseteq x]$. An element $b \in X$ is called a *largest element* if $\forall y \in X[y \sqsubseteq b]$.

Definition 1.3.8 **(Embedding)** Let $(X, \sqsubseteq)$ and $(X', \sqsubseteq')$ be two ordered sets. A mapping $f : X \to X'$ is called an *embedding* of $(X, \sqsubseteq)$ into $(X', \sqsubseteq')$ if the following conditions hold:

(i) f is injective

(ii) $\forall x, y \in X[x \sqsubseteq y \iff f(x) \sqsubseteq' f(y)]$

Definition 1.3.9 **(Isomorhpism)** A surjective embedding is called an *isomorphism*.

Theorem 1.3.10 For every ordered set $(X, \sqsubseteq)$ there exists an embedding into the ordered set $(\mathcal{P}(X), \subseteq)$

Definition 1.3.11 A set $A \subseteq X$ is called *independent* in $(X, \sqsubseteq)$ if we never have $x \sqsubseteq y$ for two distinct elements $x, y \in A$. It is also referred to as an *antichain*.

Remark 1.3.12. The set of all minimal elements in $(X, \sqsubseteq)$ is independeent.

Definition 1.3.13 A set $A \subseteq X$ is called a *chain* in $(X, \sqsubseteq)$ if for any two elements $x, y \in A$ we have $x \sqsubseteq y$ or $y \sqsubseteq x$.

Theorem 1.3.14 Let $(X, \sqsubseteq)$ be a poset and α be the maximum size of an independent set in X and ω be the maximum size of a chain in X. Then $\alpha \cdot \omega \geq |X|$.

Theorem 1.3.15 **(Erdös - Szekeres)** Let $n \in \mathbb{N}$. Then every sequence of $n^2 + 1$ distinct real numbers contains a monotone subsequence of length $n + 1$.

2 Counting

2.1 functions

Proposition 2.1.1 Let $|N| = n, |M| = m$. Then number of all possible mappings $f : N \to M$ is m^n.

Proposition 2.1.2 An n-element set X has exactly 2^n subsets.

Proposition 2.1.3 Let $n \geq 1$. Each n-element set has exactly 2^{n-1} subsets of an odd size and exactly 2^{n-1} subsets of an even size.

Proposition 2.1.4 For given numbers $n, m \geq 0$, there exists exactly

$$m(m-1)\dots(m-n+1) = \prod_{i=0}^{n-1}(m-i)$$

injective mappings $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$.

2.2 Permutations

Definition 2.2.1 (Permutation) $|\mathrm{Sym}_n| = n!$

2.3 Binomial coefficients

Proposition 2.3.1 For any finite set X, the number of all k-element subsets equals $\binom{|X|}{k}$.

Proposition 2.3.2 (Balls and bins) $\binom{m+r-1}{r-1}$

2.4 Include-exclude principle

3 Graphs

3.1 Graphs

Definition 3.1.1 (Isomorhpism) Two graphs $G = (V, E)$ and $G' = (V', E')$ are called *isomorphic* if there exists a bijection $f : V \to V'$ such that $\forall x, y \in V, x \neq y[\{x, y\} \in E \iff \{f(x), f(y)\} \in E']$ Such a bijection f is called an *isomorphism* from G to G'.

Theorem 3.1.2 (Counting graphs) Let $V = \{1, 2, \ldots, n\}$. There are $2^{\binom{n}{2}}$ possible graphs.

3.2 Subgraphs

Definition 3.2.1 (Subgraph) A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Definition 3.2.2 (Induced subgraph) A graph $G' = (V', E')$ is an *induced subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{\{x, y\} \in E \mid x, y \in V'\}$.

3.3 Common graphs

Complete graphs

Definition 3.3.1 (Complete graph) A graph $G = (V, E)$ is a *complete graph* if $\forall x, y \in V, x \neq y[\{x, y\} \in E]$.

Proposition 3.3.2 Let $V = \{1, 2, \ldots, n\}$ and K_n be the complete graph on V. Then K_n has $\binom{n}{2}$ edges.

Star graphs

Definition 3.3.3 (Star graph) A graph $G = (V, E)$ is a *star graph* if $V = \{u\} \cup \{v_1, \ldots, v_n\}$ and $E = \{\{u, v_j\} \mid j = 1, 2, \ldots, n\}$.

Proposition 3.3.4 Let $V = \{1, 2, \ldots, n\}$ and S_n be the star graph on V. Then S_n has $n - 1$ edges.

Complete bipartite graphs

Definition 3.3.5 (Complete bipartite graph) A graph $G = (V, E)$ is a *complete bipartite graph* if $V = V_1 \cup V_2$ and $E = \{\{x, y\} \mid x \in V_1, y \in V_2\}$.

Proposition 3.3.6 Let $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{n+1, n+2, \ldots, n+m\}$ and $K_{n,m}$ be the complete bipartite graph on $V_1 \cup V_2$. Then $K_{n,m}$ has $n \cdot m$ edges.

Paths

Definition 3.3.7 **(Path)** A graph $G = (V, E)$ is a *path* if $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \dots, n-1\}$.

Proposition 3.3.8 Let $V = \{1, 2, \dots, n\}$ and P_n be the path on V. Then P_n has $n - 1$ edges.

Cycles

Definition 3.3.9 **(Cycle)** A graph $G = (V, E)$ is a *cycle* if $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} \mid i = 1, 2, \dots, n-1\} \cup \{\{v_n, v_1\}\}$.

Proposition 3.3.10 Let $V = \{1, 2, \dots, n\}$ and C_n be the cycle on V. Then C_n has n edges.

3.4 Walks

Definition 3.4.1 **(Walk)** A *walk* in a graph $G = (V, E)$ is a sequence of vertices $v_1, v_2, \dots, v_n$ such that $\forall i \in \{1, 2, \dots, n-1\}[\{v_i, v_{i+1}\} \in E]$.
Another definition is that a walk is a sequence of edges $e_1, e_2, \dots, e_n$ such that $\forall i \in \{1, 2, \dots, n-1\}[e_i = \{v_i, v_{i+1}\}]$.

3.5 Connected and Components

Definition 3.5.1 **(Connected)** A graph $G = (V, E)$ is *connected* if $\forall x, y \in V$ [$\exists$ a walk from x to y].

Definition 3.5.2 **(Component)** The components of a graph G are the equivalence classes defined by the relation on the set $V(G)$, where x y $\iff$ $\exists$a walk from x to y in G

Theorem 3.5.3 Any graph $G = (V, E)$ where each vertex $v \in V$ has $\deg_G(v) \geq \frac{n-1}{2}$ is connected, where $n = |V|$.

3.6 Graph distance

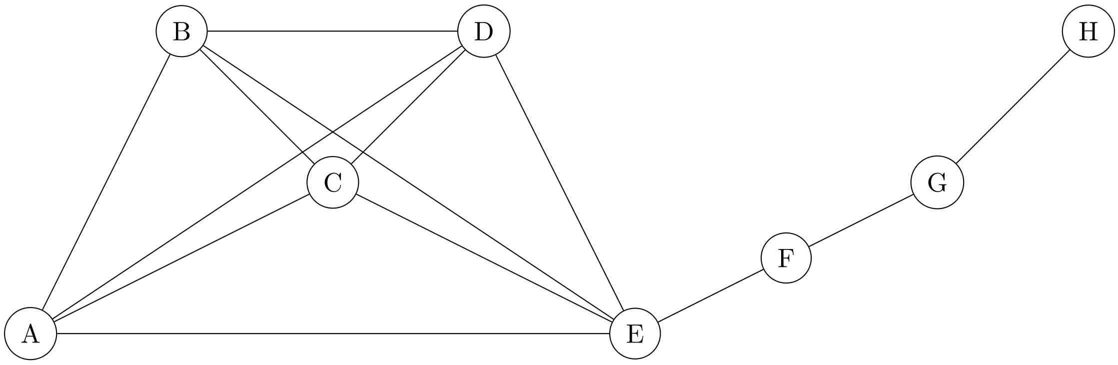
Definition 3.6.1 **(Distance)** The *distance* between two vertices x and y in a graph G is the length of the shortest walk from x to y in G.

Theorem 3.6.2 Let $G = (V, E)$ with vertex set $V = \{v_1, \dots, v_n\}$ be a graph and let A be its adjacency matrix. Let $a^k_{i,j}$ denote the element of A^k at position (i, j). Then $a^k_{i,j}$ is the number of walks of length k from v_i to v_j.

3.7 Degree sequence

Definition 3.7.1 **(Degree sequence)** The *degree sequence* of a graph $G = (V, E)$ is the sequence of degrees of the vertices of G.
The order doesn't matter. Generally, we sort the sequence in non-decreasing order.

Example 3.7.2 The degree sequence of the graph G is $(1, 2, 2, 4, 4, 4, 4, 4)$.



Example 3.7.3 A path of length n has degree sequence $(2, 1, 1, \dots, 1, 2)$.

Example 3.7.4 A cycle of length n has degree sequence $(2, 2, \dots, 2)$.

Example 3.7.5 A complete graph of n vertices has degree sequence $(n-1, n-1, \dots, n-1)$.

Example 3.7.6 A complete bipartite graph $K_{n,m}$ has degree sequence $(m, m, \dots, m, n, n, \dots, n)$.

Remark 3.7.7. A sequence of non-negative integers is a degree sequence of some graph if and only if the sum of the integers is even.

Lemma 3.7.8 Hankshake lemma For any graph $G = (V, E)$ the number of vertices of odd degree is even.

Theorem 3.7.9 Let $D = (d_1, \dots, d_n)$ be a sequence of natural numbers, for $n > 1$, where $d_1 \leq d_2 \leq \dots \leq d_n \leq n_1$ and let D' denote the sequence $(d'_1, \dots, d'_{n-1})$ where

$$d'_i = \begin{cases} d_i & \text{for } i < n - d_n \\ d_i - 1 & \text{for } i \geq n - d_n \end{cases}$$

Then D is a degree sequence if and only if D' is a degree sequence.

3.8 Eulerian graphs

Definition 3.8.1 **(Closed Eulerian tour)** A closed walk containing all the vertices and edges, and each edge exactly once, is called a *closed Eulerian tour*.

Definition 3.8.2 **(Eulerian graph)** A graph $G = (V, E)$ is *Eulerian* if it has a closed Eulerian tour.

Theorem 3.8.3 A graph $G = (V, E)$ is Eulerian if and only if G is connected and each vertex has even degree.

3.9 Hamiltonian cycle

Definition 3.9.1 **(Hamiltonian cycle)** A *Hamiltonian cycle* is a cycle that contains all the vertices of a graph.

3.10 Graph operations

Definition 3.10.1 **(Graph operations)** Let $G = (V, E)$ be a graph

1. Edge deletion: $G - e = (V, E \setminus \{e\})$, where $e \in E$.
2. Edge insertion: $G + e = (V, E \cup \{e\})$, where $e \in \binom{V}{2} \setminus E$
3. Vertex deletion: $G - v = (V - \{v\}, \{e \in E \mid v \in e\})$, where $v \in V$.

4. Edge subdivision: $G\%e = (V \cup \{z\}, (E \setminus \{\{x,y\}\}) \cup \{\{x,z\},\{z,y\}\})$ where $e = \{x,y\} \in E$ and $z \notin V$.

3.11 K-vertex-connectivity

Definition 3.11.1 A graph G is called *k-vertex-connected* if $|V(G)| \geq k+1$ and $G - v$ is connected for every $v \in V(G)$. Often we say G is k-connected.

Example 3.11.2 K_n is $(n-1)$-connected.

Theorem 3.11.3 A graph $G = (V, E)$ is 2-connected if and only if for any two vertices $v, w \in V$, there exists a cycle containing v and w.

4 Trees

4.1 Definition

Definition 4.1.1 **(Tree)** A *tree* is a connected graph with no cycles.

Theorem 4.1.2 For a non-empty graph $G = (V, E)$, the following are equivalent:

1. The graph G is a tree.
2. For any two distinct vertices $u, v \in V$, there is a unique path from u to v. (unique paths)
3. The graph G is connected and $\forall e \in E, G - e$ is disconnected. (minimal connected graph)
4. The graph G is acyclic and $\forall e \in \binom{V}{2} \setminus E, G + e$ contains a cycle. (maximal acyclic graph)
5. G is connected and $|V| = |E| + 1$. (Euler's formula)

4.2 Induction on trees

Lemma 4.2.1 **(end-vertex)** Every tree with at least two vertices has at least two leaves.

Lemma 4.2.2 **(tree-growing)** Let G be a graph and v be a leaf in G. Then $G - v$ is a tree.

4.3 Rooted trees

Definition 4.3.1 **(Rooted tree)** A *rooted tree* is a pair (T, r) where T is a tree and $r \in V(T)$ is a distinguished vertex of T called *the root.*
A node u in a rooted tree T may have a:

1. parent: the unique vertex $v \in V(T)$ such that $\{u, v\} \in E(T)$ and v lies on the unique path from u to r,
2. ancestor: a vertex $v \in V(T)$ such that v lies on the unique path from u to r,
3. child: a vertex $v \in V(T)$ where u is the parent of v,
4. descendant: a vertex $v \in V(T)$ where u is an ancestor of v,

4.4 Subtree

Definition 4.4.1 **(Subtree)** The *subtree rooted at* $v \in V(T)$ in a rooted tree is the induced subgraph defined by all vertices that are descendants of v, rooted at v.

4.5 Binary trees

Definition 4.5.1 **(Binary tree)** A *binary tree* is a rooted tree where each node has at most two children.

Definition 4.5.2 **(Strict binary tree)** A *strict binary tree* is a rooted tree where each node has exactly zero or two children.

Lemma 4.5.3 A strict binary tree with n vertices has $\frac{n-1}{2}$ internal vertices.

4.6 Ear decomposition

Lemma 4.6.1 Let $G = (V, E)$ be a 2-connected graph, then

1. $G\%e$ is 2-connected graph, where $e \in E$
2. $G + e$ is a 2 connected graph, where $e \in \binom{V}{2} \setminus E$

Proposition 4.6.2 Any 2-connected graph $G = (V, E)$ can be connected from K_3 by a sequence of edges subdivisions and edge additions.

Definition 4.6.3 **(Ear decomposition)** An *ear decomposition* of a graph $G = (V, E)$ is a sequence of subgraphs $G_0, G_1, \ldots, G_k$ of G such that

1. G_0 is a cycle,
2. $G_k = G$,
3. $G_i = G_{i-1}\%e_i$ or $G_i = G_{i-1} + e_i$ for $i = 1, 2, \ldots, k$.

Theorem 4.6.4 Any 2-connected graph G has an ear decomposition.

5 Directed Graphs

5.1 Definition

Definition 5.1.1 **(Directeed graph)** A directed graph G is an order pari (V, E), where V is some set of elements and $E \subseteq V \times V$.
A *directed edge* $e = (u, v)$, called an edge from u to v, has *head* v and *tail* u.
The *indegree* $\deg_G^+(v)$ of a vertex v is the number of edges having v as head. The *outdegree* $\deg_G^-(v)$ is the number of edges having v as tail.

5.2 Connectedness

Definition 5.2.1 **(Symmetrization)** The *symmetrization* of a directed graph $G = (V, E)$ is the undirected graph $\mathrm{Sym}(G) = (V, \overline{E})$ where $\overline{E} = \{\{u, v\} \mid (u, v) \in E \lor (v, u) \in E\}$

Definition 5.2.2 **(Weakly connected)** A directed graph G is called *weakly connected* if its symmetrization $\mathrm{Sym}(G)$ is connected.

Definition 5.2.3 (Strongly connected) A directed graph G is called *strongly connected* if for every two vertices $u, v \in V$ there is a directed path from u to v and a directed path from v to u.

Definition 5.2.4 (Weakly conneccted components) *Weakly connected components* of a directed graph G are the equivalence classes defined by the relation on the set $V(G)$, where $x \quad y \iff \exists$a walk from x to y in $\mathrm{Sym}(G)$

Definition 5.2.5 (Strongly connected components) *Strongly connected components* of a directed graph G are the equivalence classes defined by the relation on the set $V(G)$, where $x \quad y \iff \exists$a directed walk from x to y and from y to x in G

5.3 Eulerian directed graphs

Definition 5.3.1 (Eulerian directed tour) A closed directed walk containing all the vertices and edges, and each edge exactly once is an *Eulerian directed tour.*

Definition 5.3.2 (Eulerian directed graph) A directed graph G is *Eulerian* if it has an Eulerian directed tour.

Theorem 5.3.3 A directed graph is Eulerian if and only if its symmetrization is connected and $\deg^+(v) = \deg^-(v)$ for all $v \in V$.

5.4 De Bruijn Graphs

Lemma 5.4.1 Every vertex v in a De Bruijn graph has $\deg^+(v) = \deg^-(v)$.

Lemma 5.4.2 For any De Bruijn graph G, $\mathrm{Sym}(G)$ is connected.

5.5 Directed acyclic graphs

Definition 5.5.1 (Directed acyclic graph) A *directed acyclic graph* is a directed graph with no directed cycles.

Definition 5.5.2 (Source) A *source* in a directed graph G is a vertex v such that $\deg^+(v) = 0$.

Definition 5.5.3 (Sink) A *sink* in a directed graph G is a vertex v such that $\deg^-(v) = 0$.

Theorem 5.5.4 Every (finite) DAG $G = (V, E)$ has at least one sink.

6 Planar Graphs

6.1 Definitions

Definition 6.1.1 (Planar graph) A *planar graph* is a graph that can be drawn in the plane without any edges crossing.

Definition 6.1.2 (Arc) An *arc* is an injective continuous function $\gamma : [0, 1] \to \mathbb{R}^2$.

Definition 6.1.3 (Drawing) A *drawing* of a graph $G = (V, E)$ is an assignment:

- to every vertex $v \in V$ a point $b(v)$ of the plane
- to every edge $e = \{u, v\} \in E$, assign an arc $a(e)$ in the plane with endpoints $b(u)$ and $b(v)$

such that

- the mapping b is injective
- no point $b(v)$ lies on any of the arcs $a(e)$ unless it is an endpoint of that arc

Definition 6.1.4 (Topological graph) A *topological graph* is a graph together with a drawing.

6.2 Faces

Definition 6.2.1 (Face) A *face* of a drawing of a graph G is a maximal connected subset of the plane whose boundary consists of arcs $a(e)$ for edges $e \in E$.

Definition 6.2.2 (Jordan curve) A *Jordan curve* is an arc whose endpoints coincide.

Theorem 6.2.3 Any Jordan curve k divides the plane into exactly two connected parts, the "interior" and the "exterior" of k, and k is the boundary of both the interior and exterior.

6.3 Planar graphs

Proposition 6.3.1 K_1, K_2, K_3, K_4 are planar.
K_5 is not planar.

Proposition 6.3.2 $K_{3,3}$ is not planar

Theorem 6.3.3 (Kuratowski's theorem) A graph G is planar if and only if it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or to a subdivision of K_5

6.4 Properties of planar graphs

Theorem 6.4.1 (Euler's formula) Let $G = (V, E)$ be a connected planar graph and let f be the number of faces of some planar drawing of G. Then we have

$$|V| - |E| + f = 2$$

Theorem 6.4.2 Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then

$$|E| \leq 3|V| - 6.$$

Corollary 6.4.3 Every planar graph contains a vertex of degree at most 5.

6.5 Coloring maps

Definition 6.5.1 A mapping $c : V \to \{1, 2, \ldots, k\}$ is called a *coloring* of a graph $G = (V, E)$ if $c(u) \neq c(v)$ for every edge $\{u, v\} \in E$.

Definition 6.5.2 (Chromatic number) The *chromatic number*, denoted by $\chi(G)$, of a graph G is the smallest k such that G has a coloring $c : V \to \{1, 2, \ldots, k\}$.

Example 6.5.3 $\chi(K_n) = n$.
Example 6.5.4 $\chi(K_{n,m}) = 2$.
Example 6.5.5 $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}.$
Example 6.5.6 $\chi(P_n) = 2$.
Example 6.5.7 $\chi(T_n) = 2$.

Theorem 6.5.8 Any planar graph satisfies $\chi(G) \leq 4$.

7 Double Counting

7.1 Double Counting

Theorem 7.1.1 If $G = (V, E)$ is a triangle-free graph with n vertices, then G has at most $\frac{n^2}{4}$ edges.

Theorem 7.1.2 If $G = (V, E)$ is a n-vertex graph without a $K_{2,2}$ subgraph, then G has at most $\frac{1}{2}(n^{\frac{3}{2}} + n)$ edges