1 Exercise 15.12.1

Problem Let $A : V \to W$ be a linear map from a finite-dimensional normed vector space $(V, \|\cdot\|_V)$ to a normed vector space $(W, \|\cdot\|_W)$. Show that A is differentiable on V.

Proof. We need to show that for all $a \in V$ there exists a linear map $L_a : V \to W$ such that if we define

$$\mathrm{Err}(x) := A(x) - A(a) - L_a(x - a).$$

Then

$$\lim_{x \to a} \frac{\|\mathrm{Err}(x)\|_W}{\|x - a\|_V} = 0$$

Let $a \in V$.
Define $L_a : V \to W$ by $L_a = A$.
Then

$$\mathrm{Err}(x) = A(x) - A(a) - A(x - a) = A(x) - A(a) - A(x) + A(a) = 0$$

Then it holds that

$$\lim_{x \to a} \frac{\|\mathrm{Err}(x)\|_W}{\|x - a\|_V} = 0$$

□

2 Exercise 15.12.3

Problem The function $\ln : (0, \infty) \to \mathbb{R}$ is the unique, differentiable function such that $\ln(1) = 0$ and $\ln'(x) = 1/x$. Show that for all $x \in (-1, \infty)$ it holds that

$$\ln(1 + x) \leq x$$

with equality if and only if $x = 0$.

Proof. Define $f(x) = \ln(1 + x) - x$.
Then $f'(x) = \frac{1}{1+x} - 1$
It holds that for all $x \geq 0, f'(x) \leq 0$
So

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \leq 0$$
$$\lim_{x \to a} \frac{\ln(1 + x) - x - \ln(1 + a) + a}{x - a} \leq 0$$

□

3 Exercise 15.12.4

Problem Let $\Omega \subseteq \mathbb{R}$ be an open subset of $\mathbb{R}$ and consider a function $f : \Omega \to W$ interpreted as a function from the subset Ω of the normed vector space $(\mathbb{R}, |\cdot|)$ to the normed vector space $(W, \|\cdot\|_W)$. Then f is differentiable in $a \in \Omega$ if and only if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Moreover, if this limit exists, we call it $f'(a)$, and then for all $h \in \mathbb{R}$

$$f'(a) \cdot h = (Df)_a(h).$$

Proof. Let $a \in \Omega$ We prove the forward direction.
Assume f is differentiable in a, then
there exists $L : V \to W$ such that for all $x \in \Omega$ □

4 Exercise 15.12.5

Problem Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two two-dimensional vector spaces with bases v_1, v_2 and w_1, w_2 respectively. Assume that a function $f : V \to W$ is differentiable in 0 with

$$(Df)_0(v_1 + v_2) = w_1$$

and

$$(Df)_0(v_1 - 2v_2) = w_1 - w_2.$$

Give the matrix representation of the linear map $(Df)_0 : V \to W$ with respect to the bases v_1, v_2 and w_1, w_2.

Proof. □

5 Exercise 16.4.4

Problem 1. Consider the function $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) := (\cos(t), \sin(t), \arctan(t).$$

Show that f is differentiable and give an expression for the function $f' : \mathbb{R} \to \mathbb{R}^3$ and for the derivative $(Df) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, \mathbb{R}^3)$.

2. Let w_1, w_2 be two vectors in a finite-dimensional normed vector space $(W, \|\cdot\|_W)$. Consider the function $g : \mathbb{R} \to W$ given by

$$g(t) = \cosh(t) w_1 + \sinh(t) w_2$$

Show that g is differentiable and give an expression for the function $g' : \mathbb{R} \to W$ and for the derivative $(Dg) : \mathbb{R} \to \mathrm{Lin}(\mathbb{R}, W)$.

Proof. □