Analysis 1

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1 Sets, Spaces and Function

1.1 Metric Space

Definition 1.1.1 **(distance)** Let X be a set. A function $d : X \times X \to X$ is called a *distance* on X if it satisfies the following properties:

(i) Positivity: For all $a, b \in X$, it holds that $d(a, b) \geq 0$.

(ii) Non-degeneracy: For all $a, b \in X$, if $d(a, b) = 0$, then $a = b$.

(iii) Symmetry: For all $a, b \in X$, it holds that $d(a, b) = d(b, a)$.

(iv) Triangle inequality: For all $a, b, c \in X$, it holds that $d(a, c) \leq d(a, b) + d(b, c)$.

(v) Reflexivity: For all $a \in X$, it holds that $d(a, a) = 0$.

Usually conditions (ii) and (v) are combined into one condition: For all $a, b \in X, d(a, b) = 0$ if and only if $a = b$.

Definition 1.1.2 **(metric space)** A metric space is a pair $(X, dist)$, where X is a set and $dist$ is a distance function $dist : X \times X \to \mathbb{R}$ on X.

Example 1.1.3 Let $X = \{$Die Hard, Barbie, Oppenheimer$\}$

d	Die Hard	Barbie	Oppenheimer
Die Hard	0	5	2
Barbie	5	0	3
Oppenheimer	2	3	0

Then d is a distance function on X

Definition 1.1.4 **(ball in a metric space)** Let (X, d) be a metric space. Let $c \in X$ and $r \in \mathbb{R}$. The ball of radius r centered at c is the set

$$B(c, r) = \{x \in X | d(c, x) < r\}$$

Example 1.1.5 If $(X, d) = (\mathbb{R}, d_{\mathbb{R}})$, then $B(1, 3) = (-2, 4) = \{x \in \mathbb{R} \mid |x - 1| < 3\}$

Example 1.1.6 Let $X := \{$Die Hard, Barbie, Oppenheimer$\}$, with distance defined before. Then $B($Barbie, $4) = \{$ Barbie, Oppenheimer $\} = \{x \in X \mid d(x, Barbie) < 3\}$.

1.2 Normed Vector Spaces

Definition 1.2.1 **(norm)** Let V be a vector space over $\mathbb{R}$. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

- Positivity: for all $u, v \in V$ we have $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$.
- Non-degeneracy: for all $u \in V$ if $\|u\| = 0$ then $u = 0$.
- Absolute Homogeneity: for all $u \in V$ and for all $\lambda \in \mathbb{R}$ we have $\|\lambda u\| = |\lambda|\|u\|$.
- Triangle inequality: for all $u, v \in V$ we have $\|u + v\| \leq \|u\| + \|v\|$.

Example 1.2.2 Let $V = \mathbb{R}^n$. Then $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$ is a norm on $\mathbb{R}^n$.

Proposition 1.2.3 Let $(V, \|\cdot\|)$ be a normed vector space. Then the function $d : V \times V \to \mathbb{R}$ defined by $d(u, v) = \|u - v\|$ is a distance on V. And (V, d) is a metric space.

Remark 1.2.4 (Notation for Euclidean distance on $\mathbb{R}^d$ and $\mathbb{R}$). We will usually write $\mathrm{dist}_{\mathbb{R}^d}$ instead of $\mathrm{dist}_{\|\cdot\|_2}$ for the standard (Euclidean) distance on $\mathbb{R}^d$. In particular, if $d \geq 2$, we have

$$\mathrm{dist}_{\mathbb{R}^d}(v,w) = \|v-w\|_2 = \sqrt{\sum_{i=1}^{d}(v_i - w_i)^2}$$

and if $d = 1$ we just have

$$\mathrm{dist}_{\mathbb{R}} = |v-w|$$

And if there is no room for confusion, we will just leave out the subscript altogether and write dist instead of $\mathrm{dist}_{\mathbb{R}^d}$.

1.3 The reverse triangle inequality

Lemma 1.3.1 **(Reverse triangle inequality)** Let $(V, \|\cdot\|)$ be a normed vector space. Then for all $v, w \in V$ we have,

$$|\|v\| - \|w\|| \leq \|v - w\|$$

2 Real Numbers

2.1 What are the real numbers?

Definition 2.1.1 **(Real numbers)** The real numbers are a complete totally ordered field.

2.2 The completeness axiom

Definition 2.2.1 **(Upper and Lower bound)** We say a number $M \in \mathbb{R}$ is an *upper bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \leq M].$$

We say a number $m \in \mathbb{R}$ is a *lower bound* for a set $A \subseteq \mathbb{R}$ if

$$\forall a \in A[a \geq m].$$

Given the definition of upper and lower bounds, we define what it means for a set to be bounded from above, bounded from below and just bounded.

Definition 2.2.2 **(bounded from above, bounded from below, bounded)** A set $A \subseteq \mathbb{R}$ is *bounded from above* if there exists an upper bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded from below* if there exists a lower bound for A.
A set $A \subseteq \mathbb{R}$ is *bounded* if it is bounded from above and bounded from below.

Definition 2.2.3 **(Least upper bound (supremum))** Precisely, M is a *least upper bound* of a subset A if both

1. M is an upper bound of A.
2. For every upper bound $L \in \mathbb{R}$ of A, it holds that $M \leq L$.

Proposition 2.2.4 Suppose both M and W are a least upper bound of a subset $A \subseteq \mathbb{R}$. Then $M = W$.

Axiom 2.2.5 **(Completeness axiom)** We say that a totally ordered field $\mathbf{R}$ satisfies the *completeness axiom* if every nonempty subset of $\mathbf{R}$ that is bounded from above has a least upper bound.

Lemma 2.2.6 Every non-empty subset of the real line that is bounded from below has a *largest lower bound.*

Definition 2.2.7 **(infimum)** We usually call the largest lower bound of a non-empty set $A \subseteq \mathbb{R}$ that is bounded from below the *infimum* of A, and we denote it by $\inf A$.

2.3 Alternative characterizations of suprema and infima

Proposition 2.3.1 **(alternative characterizationa of supremum)** Let $A \subseteq \mathbb{R}$ be non-empty and bounded from above. Let $M \in \mathbb{R}$. Then M is the supremum of A if and only if

1. M is an upper bound for A,
2. and

$$\begin{aligned} &\text{for all } \varepsilon > 0, \\ &\quad \text{there exists } a \in A, \\ &\quad\quad a > M - \varepsilon. \end{aligned}$$

Proposition 2.3.2 **(alternative characterizationa of infimum)** Let $A \subseteq \mathbb{R}$ be non-empty and bounded from below. Let $m \in \mathbb{R}$. Then m is the infimum of A if and only if

1. m is a lower bound for A,
2. and

$$\begin{aligned}&\text{for all } \varepsilon > 0,\\ &\quad\text{there exists } a \in A,\\ &\qquad a < m + \varepsilon.\end{aligned}$$

These alternative characterizations of the supremum and infimum really provide a standard way to determining the supremum and infimum of subsets of the real line.

2.4 Maxima and minima

Definition 2.4.1 **(maximum and minimum)** Let $A \subseteq \mathbb{R}$ be a subset of the real numbers. We say that $y \in A$ is the *maximum* of A, and write $y = \max A$, if

$$\begin{aligned}&\text{for all } a \in A,\\ &\quad a \leq y.\end{aligned}$$

We say that $x \in A$ is the *minimum* of A, and write $x = \min A$, if

$$\begin{aligned}&\text{for all } a \in A,\\ &\quad a \geq x.\end{aligned}$$

Remark 2.4.2. Even if a set $A \subseteq \mathbb{R}$ is non-empty and bounded, it may not have a maximum or minimum. For example, the set $(0, 1)$ has no maximum or minimum.

Proposition 2.4.3 Let A be a subset of $\mathbb{R}$. If A has a maximum, then A is non-empty and bounded from above, and $\sup A = \max A$. If A has a minimum, then A is non-empty and bounded from below, and $\inf A = \min A$.

Proposition 2.4.4 Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from above. If $\sup A \in A$ then A has a maximum and $\max A = \sup A$.

Proposition 2.4.5 Let A be a subset of $\mathbb{R}$. Assume that A is non-empty and bounded from below. If $\inf A \in A$ then A has a minimum and $\min A = \inf A$.

2.5 The Archimedean property

Proposition 2.5.1 **(Archimedeean property)** For every real number $x \in \mathbb{R}$ there exists a natural number $n \in \mathbb{N}$ such that $x < n$.

Given this proposition, we can define the ceiling function.

Definition 2.5.2 **(ceiling function)** The *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \to \mathbb{Z}$ is defined as follows. For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer $z \in \mathbb{Z}$ such that $x \leq z$.

Proposition 2.5.3 For every two real numbers $a, b \in \mathbb{R}$ with $a < b$ there exists a $q \in \mathbb{Q}$ with $a < q < b$.

2.6 Computation rules for suprema

In the proposition below, we use the definitions

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for subsets $A, B \subseteq \mathbb{R}$ and a scalar $\lambda \in \mathbb{R}$.

Proposition 2.6.1 Let A, B, C, D be non-empty subsets of $\mathbb{R}$. Assume that A and B are bounded from above and C and D are bounded from below. Then

1. $\sup(A + B) = \sup A + \sup B$.
2. $\inf(C + D) = \inf C + \inf D$.
3. For all $\lambda \geq 0$, $\sup(\lambda A) = \lambda \sup A$.
4. For all $\lambda \leq 0$, $\sup(\lambda A) = \lambda \inf A$.
5. $\sup(-C) = -\inf C$.
6. $\inf(-C) = -\sup C$.

2.7 Bernoulli's inequality

Proposition 2.7.1 **(Bernoulli's inequality)** Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

1. If $x \geq -1$, then $(1 + x)^n \geq 1 + nx$.
2. If $x \geq 0$ and $n \geq 2$, then $(1 + x)^n \geq 1 + nx$.

3 Sequences

3.1 Sequence

Definition 3.1.1 **(Sequence)** A sequence is a function for which the domain is $\mathbb{N}$.

$$a : \mathbb{N} \to Y$$

Y can be any set.

Example 3.1.2 Here are some functions that are sequences:

1. $a : \mathbb{N} \to \mathbb{Q}$
2. $b : \mathbb{N} \to (\mathbb{N} \to Y)$
3. $c : \mathbb{N} \to \mathbb{N}$

And some functions that are not sequences:

1. $d : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$
2. $e : \mathbb{Q} \to \mathbb{N}$

3.2 Terminology around sequences

3.2.1 Bounded sequences

Definition 3.2.2 **(bouneded sequence)** Let (X, dist) be a metric space. We say a sequence $a : \mathbb{N} \to X$ is bounded if

$$\text{there exists } q \in X,$$

$$\text{there exists } M > 0,$$

$$\text{for all } n \in \mathbb{N},$$

$$\mathrm{dist}(a_n, q) \leq M.$$

In a normed linear space, we can use a simpler criterion to check whether a sequence is bounded. That is the content of the following proposition.

Proposition 3.2.3 Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. The sequence a is bounded if and only if

$$\text{there exists } M > 0,$$

$$\text{for all } n \in \mathbb{N},$$

$$\|a_n\| \leq M.$$

3.3 Convergence of sequences

Definition 3.3.1 **(Convergence of sequences)** Let (X, dist) be a metric space. We say that a sequence $a : \mathbb{N} \to X$ converges to a point $p \in X$ if for all $\epsilon > 0$,

there exists $N \in \mathbb{N}$,

for all $n \geq N$,

$\operatorname{dist}(a_n, p) < \epsilon$.

We sometimes write

$$\lim_{n \to \infty} a_n = p$$

to express that the sequence (a_n) converges to p.

Definition 3.3.2 (Divergence of sequences) Let (X, dist) be a metric space. A sequence $a : \mathbb{N} \to X$ is called *divergent* is it is not convergent.

3.4 Examples and limits of simple sequences

Proposition 3.4.1 (The constant sequence) Let (X, dist) be a metric space. Let $p \in X$ and assume that the sequence (a_n) is given by $a_n = p$ for every $n \in \mathbb{N}$. We also say that (a_n) is a constant sequence. Then $\lim_{n \to \infty} = p$.

Example 3.4.2 A standard limit Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence such that $a_n = 1/n$ for $n \geq 1$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to 0.

Proof. Let $\epsilon > 0$. Choose $N = \lceil 1/\epsilon \rceil + 1$. Take $n \geq N$. Then

$$\operatorname{dist}_{\mathbb{R}}(a_n, 0) = |a_n - 0| = |1/n| = 1/n \leq 1/N < \epsilon.$$

□

3.5 Uniqueness of limits

Proposition 3.5.1 (Uniqueness of limits) Let (X, dist) be a metric space and let $a : \mathbb{N} \to \mathbb{R}$ be a sequence in X. Assume that $p, q \in X$ and assume that

$$\lim_{n \to \infty} = p \text{ and } \lim_{n \to \infty} a_n = q$$

Then $p = q$.

3.6 More properties of convergent sequences

Proposition 3.6.1 Let (X, dist) be a metric space and suppose that $a : \mathbb{N} \to X$ is a sequence. Let $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the real-valued sequence

$$n \mapsto \operatorname{dist}(a_n, p)$$

converges to 0 in $\mathbb{R}$.

Proposition 3.6.2 (Convergent sequences are bounded) Let (X, dist) be a metric space. Let $a : \mathbb{N} \to X$ be a sequence in X converging to $p \in X$. Then the sequence $a : \mathbb{N} \to X$ is bounded.

Proposition 3.6.3 Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences. Let $p \in X$ and suppose that $\lim_{n \to \infty} a_n = p$. Then $\lim_{n \to \infty} b_n = p$ if and only if

$$\lim_{n \to \infty} \operatorname{dist}(a_n, b_n) = 0$$

Corollary 3.6.4 (Eventually equal sequences have the same limit) Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ and $b : \mathbb{N} \to X$ be two sequences such that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_n = b_n$$

Then the sequence $a : \mathbb{N} \to X$ converges if and only if the sequence $b : \mathbb{N} \to X$ converges. If the sequences converge, they have the same limit.

3.7 Limit theorems for sequences taking values in a normed vector space

Theorem 3.7.1 Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ and $b : \mathbb{N} \to V$ be two sequences. Assume that the $\lim_{n\to\infty} a_n$ exists and is equal to $p \in V$ and that the $\lim_{n\to\infty} b_n$ exists and is equal to $q \in V$. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Let $\mu \in \mathbb{R}$. Assume that $\lim_{n\to\infty} \lambda_n = \mu$. Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $p + q$.
2. The $\lim_{n\to\infty}(\lambda_n a_n)$ exists and is equal to μp.

3.8 Index shift

Proposition 3.8.1 (Index shift) Let (X, dist) be a metric space and let $a : \mathbb{N} \to X$ be a sequence. Let $k \in \mathbb{N}$ and $p \in X$. Then the sequence $a : \mathbb{N} \to X$ converges to p if and only if the sequence $(a_{n+k})_n$ (i.e. the sequence $n \mapsto a_{n+k}$) converges to p.

4 Real-valued sequences

4.1 Terminology

Definition 4.1.1 **(increasing, decreasing and monotone sequences)** We say a sequence (a_n) is

1. *increasing* if for every $n \in \mathbb{N}, a_{n+1} \geq a_n$
2. *strictly increasing* if for every $n \in \mathbb{N}, a_{n+1} > a_n$
3. *decreasing* if for every $n \in \mathbb{N}, a_{n+1} \leq a_n$
4. *strictly decreasing* if for every $n \in \mathbb{N}, a_{n+1} < a_n$
5. *monotone* if it is either increasing or decreasing
6. *strictly monotone* if it is either strictly increasing or strictly decreasing

Definition 4.1.2 **(upper bound and lower bound for a sequence)** We say that a number $M \in \mathbb{R}$ is an *upper bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \leq M$$

We say that a number $m \in \mathbb{R}$ is a *lower bound* for a sequence $a : \mathbb{N} \to \mathbb{R}$ if

$$\text{for all } n \in \mathbb{N}$$

$$a_n \geq m$$

Definition 4.1.3 **(bounded sequence)** We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded above* if there exists an $M \in \mathbb{R}$ such that M is an upper bound for a.
We say that a sequence $a : \mathbb{N} \to \mathbb{R}$ is *bounded below* if there exists an $m \in \mathbb{R}$ such that m is a lower bound for a.

Proposition 4.1.4 Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below.

4.2 Monotone, bounded sequences and convergent

Theorem 4.2.1 Let (a_n) be an increasing sequence that is bounded from above. Then (a_n) convergent and

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n \quad (= \sup\{a_n \mid n \in \mathbb{N}\})$$

Theorem 4.2.2 Let (a_n) be a decreasing sequence that is bounded from below. Then (a_n) is convergent and

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n \quad (= \inf\{a_n \mid n \in \mathbb{N}\})$$

4.3 Limit theorems

Theorem 4.3.1 **(Limit theorems for real-valued sequences)** Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to \mathbb{R}$ be two converging sequences, and let $c, d \in \mathbb{R}$ be real numbers such that

$$\lim_{n\to\infty} a_n = c \text{ and } \lim_{n\to\infty} b_n = d.$$

Then

1. The $\lim_{n\to\infty}(a_n + b_n)$ exists and is equal to $c + d$.
2. The $\lim_{n\to\infty}(a_n b_n)$ exists and is equal to $c \cdot d$.
3. If $d \neq 0$, then $\lim_{n\to\infty}(\frac{a_n}{b_n}$ exists and is equal to $\frac{c}{d}$.
4. For every non-negative integer $m \in \mathbb{N}$, the limit $\lim_{n\to\infty}(a_n)^m$ exists and is equal to c^m.
5. If for every $n \in \mathbb{N}$, the number a_n is non-negative, then for every positive integer $k \in \mathbb{N} \setminus \{0\}$, the limit $\lim_{n\to\infty}(a_n)^{\frac{1}{k}}$ exists and is equal to $c^{\frac{1}{k}}$.

4.4 The squeeze theorem

Theorem 4.4.1 **(The squeeze theorem)** Let $a, b, c : \mathbb{N} \to \mathbb{R}$ be three sequences. Suppose that there exists an $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$a_n \leq b_n \leq c_n$$

and assume $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ for some $L \in \mathbb{R}$. Then $\lim_{n\to\infty} b_n$ exists and is equal to L.

4.5 Divergence to ∞ and $-\infty$

Definition 4.5.1 We say a sequence $a : \mathbb{N} \to \mathbb{R}$ *diverges to* ∞ and write

$$\lim_{n\to\infty} = \infty$$

if

$$\text{for all } M \in \mathbb{R},$$

$$\text{there exists } N \in \mathbb{N},$$

$$\text{for all } n \geq N,$$

$$a_n > M.$$

Similarly, we say a sequence (a_n) *diverges* to $-\infty$ and write

$$\lim_{n\to\infty} a_n = -\infty$$

if

$$\text{for all } M \in \mathbb{R},$$

$$\text{there exists } N \in \mathbb{N},$$

$$\text{for all } n \geq N,$$

$$a_n < M.$$

Proposition 4.5.2 Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} a_n = \infty.$$

Then the sequence (a_n) is bounded from below.

Similarly, let $b : \mathbb{N} \to \mathbb{R}$ be a sequence such that

$$\lim_{n\to\infty} b_n = -\infty.$$

Then the sequence (b_n) is bounded from above.

4.6 Limit theorems for improper limits

Theorem 4.6.1 **(Limit theorems for improper limits)** Let $a, b, c, d : \mathbb{N} \to \mathbb{R}$ be four sequences such that

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} c_n = -\infty$$

the sequence (b_n) is bounded from below and the sequence (d_n) is bounded from above. Let $\lambda : \mathbb{N} \to \mathbb{R}$ be a sequence bounded below by some $\mu > 0$. Then

i. $\lim_{n\to\infty}(a_n + b_n) = \infty$

ii. $\lim_{n\to\infty}(c_n + d_n) = -\infty$

iii. $\lim_{n\to\infty}(\lambda_n a_n) = \infty$

iv. $\lim_{n\to\infty}(\lambda_n c_n) = -\infty$

Proposition 4.6.2 Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be two sequences. Then

1. $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty}(-a_n) = -\infty$.
2. $\lim_{n\to\infty} b_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

4.7 Standard sequences

4.7.1 Geometric sequence

Proposition 4.7.2 **(Standard limit of of geometric sequence)** Let $q \in \mathbb{R}$. The sequence (a_n) defined by $a_n := q^n$ for $n \in \mathbb{N}$

- converges to 0 if $q \in (-1, 1)$
- converges to 1 if $q = 1$
- diverges to ∞ if $q > 1$
- diverges, but not to ∞ or $-\infty$ if $q \leq -1$

4.7.3 The n^{th} root of n

Proposition 4.7.4 **(Standard limit of the n^{th} root of n)** The sequence (a_n) defined by $a_n := \sqrt[n]{n}$ for $n \in \mathbb{N}$ converges to 1.

Corollary 4.7.5 Let $a > 0$. Then the sequence (b_n) defined by $b_n := \sqrt[n]{a}$ converges to 1.

4.7.6 The number e

First let's define the sequence (a_n) by

$$a_n := \left(1 + \frac{1}{n}\right)^n.$$

We show that (a_n) is increasing and bounded from above by 3. Hence (a_n) converges to some $e \in \mathbb{R}$ by the monotone convergence theorem.

Lemma 4.7.7 The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is increasing.

Lemma 4.7.8 The sequence (a_n) defined by $a_n := \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N} \setminus \{0\}$ and $a_0 = 1$ is bounded from above by 3.

By these two lemmas, the sequence

$$n \mapsto \left(1 + \frac{1}{n}\right)^n$$

converges.

Definition 4.7.9 ((Standard limit of e)) We define the number e by

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

4.7.10 Exponentials beat powers

Proposition 4.7.11 Let $a \in (1, \infty)$ and let $p \in (0, \infty)$. Then

$$\lim_{n \to \infty} \frac{n^p}{a^n} = 0.$$

4.8 Sequences with values in $\mathbb{R}^d$

Proposition 4.8.1 Consider the metric space $(\mathbb{R}^d, \|\cdot\|_2)$. Let $z \in \mathbb{R}^d$ and let $x : \mathbb{N} \to \mathbb{R}^d$ be a sequence. Denote by y_i the ith component of a vector $y \in \mathbb{R}^d$. Then the sequence $(x^{(n)})$ converges to z if and only if for all $i \in \{1, \dots, d\}$, the sequence $(x_i^{(n)})$ converges to z_i.

5 Series

5.1 Definition

Definition 5.1.1 Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Let $K \in \mathbb{N}$. We say that a series

$$\sum_{n=K}^{\infty} a_n$$

is *convergent* if the associatedd *sequence of partial sums* $S_k : \mathbb{N} \to V$, i.e. the sequence $(S_K^n)_{n\in\mathbb{N}}$ converges. The term S_K^n is, for $n \in \mathbb{N}$, defined as

$$S_K^n := \sum_{k=K}^{n} a_k$$

If $K = 0$, we usually jusst write S^n or even S_n instead of S_0^n.
If the series $\sum_{n=K}^{\infty} a_n$ is convergent, the *value* of the series is by definition equal to the limit of the sequence of partial sums, i.e.

$$\sum_{k=K}^{\infty} a_k := \lim_{n\to\infty} S_k^n = \lim_{n\to\infty} \sum_{k=K}^{n} a_k$$

5.2 Geometric series

Proposition 5.2.1 Let $a \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}.$$

Proof. We consider

$$\begin{aligned}
(1-a)\sum_{k=0}^{n} a^k &= \sum_{k=0}^{n} a^k - a\sum_{k=0}^{n} a^k \\
&= \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} \\
&= \sum_{k=0}^{n} a^k - \sum_{k=1}^{n+1} a^k \\
&= 1 - a^{n+1}
\end{aligned}$$

□

Proposition 5.2.2 (Geometric series) Let $a \in (-1, 1)$. Then the series

$$\sum_{k=0}^{\infty} a^k$$

is convergent and has the value

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

5.3 The harmonic series

Proposition 5.3.1 **(Harmonic series)** The series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges.

5.4 The hyperharmonic series

Proposition 5.4.1 **(Hyperharmonic series)** Let $p > 1$. Then the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges.

Example 5.4.2 Here is an example of a series taking values in the normed vector space $(\mathbb{R}^2, \|\cdot\|)$:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^2}, \left(\frac{1}{2}\right)^k \right)$$

5.5 Only the tail matters for convergence

Lemma 5.5.1 Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence taking values in V. Let $K, L \in \mathbb{N}$. The series

$$\sum_{n=K}^{\infty} a_n$$

is conovergent is and only if the series

$$\sum_{n=L}^{\infty} a_n$$

is convergent. Moreover, if either the series converges, and $K < L$, then

$$\sum_{n=K}^{\infty} a_n = \sum_{n=K}^{L-1} + \sum_{n=L}^{\infty} a_n.$$

Proposition 5.5.2 Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and assume that the series

$$\sum_{k=M}^{\infty} a_k$$

is convergent. Then

$$\lim_{m \to \infty} \sum_{k=m}^{\infty} a_k = 0.$$

Proposition 5.5.3 **(Index shift for series)** Let $a : \mathbb{N} \to V$ be a sequence, let $M \in \mathbb{N}$ and let $\ell \in \mathbb{N}$. Then the series

$$\sum_{k=M}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=M}^{\infty} a_{k+\ell}$$

converges. Moreoever, if either series converges, then

$$\sum_{k=M}^{\infty} a_{k+\ell} = \sum_{k=M+\ell}^{\infty} a_k.$$

5.6 Divergence test

Proposition 5.6.1 Let $(V, \|\cdot\|)$ be a normed vector space, and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Suppose the series $\sum_{n=0}^{\infty} a_n$ is convergent to $L \in V$. Then

$$a_n = S_n - S_{n-1}$$

where S_n denote the partial sum $\sum_{k=0}^{n} a_k$. Because S_n and S_{n-1} are both convergent to L, the sequence (a_n) is convergent as well and converges to $L - L = 0$. □

Theorem 5.6.2 (Divergence test) Let $(V, \|\cdot\|)$ be a normed vector space and let $a : \mathbb{N} \to V$ be a sequence in V. Suppose the limit $\lim_{n\to\infty} a_n$ does not exist or is not equal to 0. Then the series

$$\sum_{n=0}^{\infty} a_n$$

is divergent.

5.7 Limit laws for series

Theorem 5.7.1 (Limit laws for series) Let $(V, \|\cdot\|)$ be a normed vector space and let $a, b : \mathbb{N} \to V$ be sequences in V. Suppose the series

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n$$

are convergent. Suppose $\lambda \in \mathbb{R}$. Then

1. The series
$$\sum_{n=0}^{\infty} (a_n + b_n)$$
is converget and converges to
$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n.$$

2. The series
$$\sum_{n=0}^{\infty} \lambda a_n$$
is convergent and converges to
$$\lambda \sum_{n=0}^{\infty} a_n.$$

6 Series with positive terms

6.1 Comparison test

Theorem 6.1.1 (Comparison test) Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences. Assume that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n \leq b_n$. Then

1. Suppose the series $\sum_{n=1}^{\infty} b_n$ converges. Then the series $\sum_{n=1}^{\infty} a_n$ converges as well.
2. Suppose the series $\sum_{n=1}^{\infty} a_n$ diverges. Then the series $\sum_{n=1}^{\infty} b_n$ diverges as well.

Example 6.1.2 Consider the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}.$$

We first observe that for all $k \geq 2$ we have

$$\frac{k}{k^2 - 1} \geq \frac{k}{k^2} = \frac{1}{k}.$$

Because the series

$$\sum_{k=2}^{\infty} \frac{1}{k}$$

diverges, the series

$$\sum_{k=2}^{\infty} \frac{k}{k^2 - 1}$$

diverges as well by the comparison test.

6.2 Limit comparison test

Theorem 6.2.1 (Limit comparison test) Let $a, b : \mathbb{N} \to [0, \infty)$ be two sequences.

1. Assume the series $\sum_{k=1}^{\infty} b_k$ converges and assume the limit

   $$\lim_{n \to \infty} \frac{a_n}{b_n}$$

   exists. Then the series $\sum_{k=1}^{\infty} a_k$ converges as well.
2. Assume the series $\sum_{k=1}^{\infty} b_k$ diverges and assume the limit

   $$\lim_{n \to \infty} \frac{a_n}{b_n}$$

   exists and is strictly larger than zero, or that the limit is infinity. Then the series $\sum_{k=1}^{\infty} a_k$ diverges as well.

Example 6.2.2 Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 1}.$$

We use sequences $a, b : \mathbb{N} \to [0, \infty)$ defined for $k \geq 2$ by

$$a_k = \frac{k}{k^2 + 1}$$

and
$$b_k = \frac{1}{k}.$$
Then
$$\frac{a_k}{b_k} = \frac{\frac{k}{k^2+1}}{\frac{1}{k}} = \frac{1}{1+\frac{1}{k^2}}.$$
By limit laws, we find that the limit of the denominator is 1, i.e.
$$\lim_{k\to\infty}\left(1+\frac{1}{k^2}\right) = \lim_{k\to\infty} 1 + \lim_{k\to\infty}\frac{1}{k^2} = 1+0 = 1.$$
Therefore, we may apply the limit law for the quotient and conclude that
$$lim_{k\to\infty}\frac{a_k}{b_k} = \frac{1}{\lim_{k\to\infty}(1+\frac{1}{k^2})} = \frac{1}{1} = 1.$$
The series $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k}$ diverges, and therefore it follows from the Limit Comparison Test that the series
$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty}\frac{k}{k^2+1}$$
diverges as well.

6.3 Ratio test

Theorem 6.3.1 **(Ratio Test)** Let $a : \mathbb{N} \to [0,\infty)$ be a sequence.

1. If there exists an $N \in \mathbb{N}$ and a $q \in (0,1)$ such that for all $n \geq N$, it holds that
$$\frac{a_{n+1}}{a_n} \leq q,$$
then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that
$$\frac{a_{n+1}}{a_n} \geq 1,$$
then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.4 Limit ratio test

Theorem 6.4.1 **(Limit Ratio Test)** Let $a : \mathbb{N} \to (0,\infty)$ be a sequence.

1. If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = q$ with $q \in [0,1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.4.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1.$$

6.5 Root test

Theorem 6.5.1 (Root test) Let (a_n) be a sequence of non-negative real numbers.

1. If there exists an $N \in \mathbb{N}$ and a $q \in (0,1)$ such that for all $n \geq N$, it holds that
$$\sqrt[n]{a_n} \leq q,$$
then the series $\sum_{k=1}^{\infty} a_k$ converges.

2. If there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, it holds that
$$\sqrt[n]{a_n} \geq 1,$$
then the series $\sum_{k=1}^{\infty} a_k$ diverges.

6.6 Limit root test

Theorem 6.6.1 (Limit Root Test) Let (a_n) be a sequence of non-negative real numbers.

1. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q \in [0,1)$, then the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\lim_{n\to\infty} \sqrt[n]{a_n} = q$ with $q > 1$, or if $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Remark 6.6.2. We cannot conclude anything about the convergence of a series $\sum_k a_k$ when
$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1.$$

7 Series with general terms

7.1 Series with real terms: the Leibniz test

Theorem 7.1.1 **(Leibniz test, a.k.a Alternating series test)** Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences such that for all $k \in \mathbb{N}$, $b_k = (-1)^k a_k$. Assume that there exists a $K \in \mathbb{N}$ such that

1. $a_k \geq 0$ for every $k \geq K$,
2. $a_k \geq a_{k+1}$ for every $k \geq K$,
3. $\lim_{k\to\infty} a_k = 0$.

Then, the series

$$\sum_{k=K}^{\infty} b_k = \sum_{k=K}^{\infty} (-1)^k a_k$$

is convergent. In addition, the following esitmate holds for every $N \geq K$,

$$\left| S_N - \sum_{k=K}^{\infty} b_k \right| \leq a_{N+1}.$$

where for all $n \in \mathbb{N}$, $S_n := \sum_{k=K}^{\infty} b_k$.

Example 7.1.2 We claim that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

We would like to apply the Alternating series test. To do so, we need toe check its conditions.

We define the sequence $a : \mathbb{N} \to \mathbb{R}$ by

$$a_k := \frac{1}{k}$$

for $k \geq 1$ (and $a_0 = a_1 = 1$).

We now check the conditions for the Alternating Series Test.

1. We need to show that $a_k \geq 0$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq 0.$$
2. We need to show that $a_k \geq a_{k+1}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Then,
$$a_k = \frac{1}{k} \geq \frac{1}{k+1} = a_{k+1}.$$
3. We need to show that
$$\lim_{k\to\infty} a_k = 0$$
. This follow as this is a standard limit.

It follows from the Alternating Series Test that the series

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$$

converges.

7.2 Series charactersization of completeness in normed vector space

Definition 7.2.1 Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence of vectors in V. We say the series

$$\sum_{k=0}^{\infty} a_k$$

converges *absolutely* if

$$\sum_{k=0}^{\infty} \|a_k\|$$

converges.

Definition 7.2.2 (Series characterization of completeness) We say a normed vector space $(V, \|\cdot\|)$ satisfies the *series characterization of completeness* if every series in V that is absolutely convergent is also convergent.

Proposition 7.2.3 Every finite-dimensional normed vector space satisfies the series characterization of completeness.

Example 7.2.4 Consider the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}.$$

Since this is not an alternataing series, we cannot apply the Leibniz test.

However, for every k
$in\mathbb{N} \setminus \{0\}$, we have

$$\left| \frac{\sin(k)}{k^2} \right| \leq \frac{1}{k^2}.$$

The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is a standard hyperharmonic seris, of which we know that it converges. By the Cmomparison Test, we conclude that the series

$$\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right|$$

converges as well.

Therefore, the series

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges absolutely. Since $(\mathbb{R}, |\cdot|)$ is complete, we find that

$$\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$$

converges.

Definition 7.2.5 Let $(V, \|\cdot\|)$ be a normed vector space. Let $a : \mathbb{N} \to V$ be a sequence. We say that a series

$$\sum_{k=0}^{\infty} a_k$$

converges *conditionally* if it converges but does not converge absolutely.

7.3 The Cauchy product

Theorem 7.3.1 (Cauchy product) Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences. Assume that the series

$$\sum_{k=0}^{\infty} a_k$$

and

$$\sum_{k=0}^{\infty} b_k$$

converge absolutely. Then, the series

$$\sum_{k=0}^{\infty} c_k$$

converges absolutely as well, where

$$c_k := \sum_{\ell=0}^{k} a_\ell b_{k-\ell},$$

and

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

8 Subsequences, lim sup and lim inf

8.1 Index sequences and subsequences

Definition 8.1.1 (Index sequence) We say a sequence $n : \mathbb{N} \to \mathbb{N}$ is an *index sequence* if it is strictly increasing.

Example 8.1.2 The sequence $n : \mathbb{N} \to \mathbb{N}$ defined by

$$n_k := 2k$$

is a strictly increasing sequence of natural numbers. In other words, it is an index sequence.

Definition 8.1.3 (Subsequence) Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. A sequence $b : \mathbb{N} \to \mathbb{R}$ is called a *subsequence* of a if there exists an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that $b = a \circ n$

Just as we often write $(a_n)_{n \in \mathbb{N}}$ for a sequence called a, we often write $(a_{n_k})_{k \in \mathbb{N}}$ for the subsequence $a \circ n$.

8.2 (Sequential) accumulation points

Definition 8.2.1 ((Sequential) accumulation points) Let (X, dist) be a metric space. A point $p \in X$ is called an *accumulation point* of a sequence $a : \mathbb{N} \to X$ if there is a subsequence $a \circ n$ of a such that $a \circ n$ converges to p.

8.3 Subsequences of a converging sequence

Proposition 8.3.1 Let (X, dist) be a metric space. Let (a_n) be a sequence in X converging to $p \in X$. Then every subsequence of (a_n) is convergent to p.

8.4 lim sup

Consider a real-valued sequence (a_n) that is bounded from above and does not diverge to $-\infty$. We can then define a new sequence

$$k \mapsto \sup_{n \geq k} a_n.$$

Note that this sequence is decreasing, because for larger k the supremum is taken over a smaller set.

Lemma 8.4.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence that is bounded from above and does not diverge to $-\infty$. Then, the sequence $k \mapsto \sup_{n \geq k} a_n$ is bounded from below.

Since the sequence $k \mapsto \sup_{n \geq k} a_n$ is decreasing and bounded from below, it has a limit, and the limit is in fact equal to the infumum of the sequence. This limit is called the lim sup

$$\begin{aligned} \limsup_{n \to \infty} a_n &:= \inf_{k \in \mathbb{N}} \sup_{n \geq k} a_n \\ &= \lim_{k \to \infty} \left(\sup_{n \geq k} a_n \right) \end{aligned}$$

Proposition 8.4.2 (Alternative characterization of lim sup) Let (a_n) be a real-valued sequence. Let $M \in \mathbb{R}$. Then, $M = \limsup_{n \to \infty} a_n$ if and only if

i.
$$\begin{aligned} &\text{For every } \epsilon > 0, \\ &\quad \text{there exists } N \in \mathbb{N}, \\ &\quad\quad \text{for all } \ell \geq N, \\ &\quad\quad\quad a_\ell < M + \epsilon \end{aligned}$$

ii.
$$\begin{aligned}&\text{For every } \epsilon > 0,\\ &\quad \text{for all } k \in \mathbb{N},\\ &\quad\quad \text{there exists } m \geq k,\\ &\quad\quad\quad a_m > M - \epsilon\end{aligned}$$

Theorem 8.4.3 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded from above and does not diverge to $-\infty$. Then $\limsup_{\ell\to\infty} a_\ell$ is a (sequential) accumulation point of a, i.e. there exists a subsequences of a that converges to $\limsup_{\ell\to\infty} a_\ell$.

Corollary 8.4.4 **(Bolzano-Weierstrass)** Every bounded, real-valued sequence has a subsequence that converges in $(\mathbb{R}, \mathrm{dist}_{\mathbb{R}})$.

Theorem 8.4.5 Suppose a sequence $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and does not diverge to $-\infty$. Then

$$\limsup_{\ell\to\infty} a_\ell$$

is the maximum of the set of sequential accumulation points.

8.5 lim inf

Similarly to the lim sup, we can define the lim inf. In some sense,

$$\liminf_{\ell\to\infty} a_\ell = -\limsup_{\ell\to\infty}(-a_\ell)$$

More precisely,

$$\begin{aligned}\liminf_{\ell\to\infty} a_\ell &:= \sup_{\ell\in\mathbb{N}} \inf_{k\geq\ell} a_k\\ &= \lim_{\ell\to\infty}\left(\inf_{k\geq\ell} a_k\right)\end{aligned}$$

Proposition 8.5.1 **(Alternative characterization of** $\liminf$**)** Let $a : \mathbb{N} \to \mathbb{R}$ and $M \in \mathbb{R}$. Then M equals $\liminf_{\ell\to\infty} a_\ell$ if and only if

1.
$$\begin{aligned}&\text{For every } \epsilon > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } \ell \geq N,\\ &\quad\quad\quad a_\ell > M - \epsilon\end{aligned}$$

2.
$$\begin{aligned}&\text{For every } \epsilon > 0,\\ &\quad \text{for all } K \in \mathbb{N},\\ &\quad\quad \text{there exists } m \geq K,\\ &\quad\quad\quad a_m < M + \epsilon\end{aligned}$$

Theorem 8.5.2 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is a sequential accumulation point of the sequence a, i.e. there is a subsequence of a that converges to $\liminf_{\ell\to\infty} a_\ell$.

Theorem 8.5.3 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence that is bounded below and does not diverge to ∞. Then $\liminf_{\ell\to\infty} a_\ell$ is the minimum of the set of sequential accumulation points.

8.6 Relations between $\lim$, $\limsup$ and $\liminf$

Proposition 8.6.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence and let $L \in \mathbb{R}$. Then $a : \mathbb{N} \to \mathbb{R}$ converges to L if and only if

$$\liminf_{\ell\to\infty} a_\ell = \limsup_{\ell\to\infty} = L$$

Proposition 8.6.2 Let $a, b : \mathbb{N} \to \mathbb{R}$ be two real-valued sequences, such that there exists an $N \in \mathbb{N}$ such that for all $\ell \geq N$, $a_\ell \leq b_\ell$. Then

$$\limsup_{\ell\to\infty} a_\ell \leq \limsup_{\ell\to\infty} b_\ell$$

and

$$\liminf_{\ell\to\infty} a_\ell \leq \liminf_{\ell\to\infty} b_\ell.$$

9 Point-set topology of metric spaces

Here we introduce three properties for subsets of a metric space: *closedness*, *completeness*, and *compactness*. For those three properties we known that every compact set is complete, and every complete set is closed. However, not every closed set is complete, and not every complete set is compact.

9.1 Open sets

Definition 9.1.1 **(Open set)** Let (X, dist) be a metric space. We say that a subset $O \subseteq X$ is *open* if every $x \in O$ is an interior point of O.

Now we need to say what it means to be an interior point.

Definition 9.1.2 **(Interior point)** Let (X, dist) be a metric space and let A be subset of X. A point $a \in A$ is called an *interior point* of A if

$$\begin{gathered}\text{there exists } r > 0\\ B(a, r) \subseteq A\end{gathered}$$

where $B(a, r)$ is an (open) ball around point a with radius r (definition 1.1.4).

Proposition 9.1.3 Let (X, dist) be a metric space. The ball

$$B(p, r) := \{x \in X | \text{dist}(x, p) < r\}$$

is indeed open.

Proposition 9.1.4 **('Open' intervals are open)** Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals $(a, b), (-\infty, b), (a, \infty)$ are all open subsets of $\mathbb{R}$.

Proposition 9.1.5 Let (X, dist) be a metric space. Then both the empty set $\emptyset$ and the set X itself (both of these are subsets of X) are open.

Proof. We first show that the empty set is open. We argue by contradiction. Suppose there exists a point $x \in \emptyset$ such that x is not an interior point of X. Then we have a contradiction, because the empty set has no elements.

We will now show that X is open. Let $x \in X$. We will show that x is an interior point, i.e. we will show that there exists an $r > 0$ such that $B(x, r) \subseteq X$.
Choose $r := 1$. Then $B(x, r) = B(x, 1) \subseteq X$. □

The set of all interior points of a subset $A \subseteq X$ is called the *interior* of the set A.

Definition 9.1.6 **(The interior of a set)** Let (X, dist) be a metric space and let $A \subseteq X$ be a subset of X. Then the *interior* of the set A, denoted by $\text{int } A$ is the set of all interior points of A, i.e $\text{int } A$ is defined as

$$\text{int } A := \{x \in A \mid x \text{ is an interior point of } A\}.$$

Example 9.1.7 The interior of the interval $[2, 5)$ (viewed as subset of $(\mathbb{R}, |\cdot|)$) is the interval $(2, 5)$.

The interior of a set is always open.

Proposition 9.1.8 Let (X, dist) be a metric space and let $A \subseteq X$. Then $\text{int } A$ is open.

The union of open sets is always open

Unions of open sets are always open. You may recall that if $\mathcal{I}$ is some set and if for every $\alpha \in \mathcal{I}$ we have a subset $A_\alpha \subseteq X$, then the union

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha$$

is defined as

$$\bigcup_{\alpha \in \mathcal{I}} A_\alpha := \{x \in X \mid \text{ there exists } \alpha \in \mathcal{I} \text{ such that } x \in A_\alpha\}$$

Proposition 9.1.9 Let (X, dist) be a metric space, let $\mathcal{I}$ be some set and assume that for every $\alpha \in \mathcal{I}$, we have a subset $O_\alpha \subseteq X$. Suppose that for all $\alpha \in \mathcal{I}$ the set O_α is open. Then also the union

$$\bigcup_{\alpha \in \mathcal{I}} O_\alpha$$

is open.

Example 9.1.10 We already know that for every $n \in \mathbb{N}$, the interval $(2n, 2n+1)$ is an open subset of $(\mathbb{R}, |\cdot|)$. Therefore, (choosing $\mathcal{I} = \mathbb{N}$ and $O_\alpha = (2\alpha, 2\alpha + 1)$ in the previous proposition,) we also know that the set

$$\bigcup_{n \in \mathbb{N}} (2n, 2n+1)$$

is an open set of $(\mathbb{R}, |\cdot|)$ as well.

Finite intersections of open sets are open

Proposition 9.1.11 Let (X, dist) be a metric space and let $O_1, \dots, O_N$ be open subsets of X. Then the intersection

$$O_1 \cap \cdots \cap O_N$$

is also open.

Cartesian products of open sets

Proposition 9.1.12 Let $O_1, \dots, O_d$ be open subsets of $\mathbb{R}$. Then

$$O_1 \times \cdots \times O_d (= \{(o_1, \dots, o_d) \mid o_i \in O_i\})$$

is an open subset of $(\mathbb{R}^d, \|\cdot\|_2)$.

9.2 Closed sets

Definition 9.2.1 Let (X, dist) be a metric space. We say that a subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open.

Proposition 9.2.2 Let (X, dist) be a metric space. Then both the empty set $\emptyset$ and the set X itself (both of these are subsets of X) are closed.

Warning If you want to show that a set is closed *it is not enough* to show that the set is not open.

Proposition 9.2.3 (Sequence characterization of closedness) A set $C \subseteq X$ is closed if and only if for every sequence (c_n) in C converging to some $x \in X$, it holds that $x \in C$.

Example 9.2.4 Consider the subset A of the metric space $(\mathbb{R}^2, \|\cdot\|)$ defined by

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq (x_2)^2\}$$

Proof. By the sequence characterization of closedness, it suffices to show that for all sequences $y : \mathbb{N} \to A$, if the sequence y converges to some point $z \in \mathbb{R}^2$, then actually $z \in A$.
Let $y : \mathbb{N} \to A$ be a sequence in A.
Assume that the sequence (y) converges to some point $z \in \mathbb{R}^2$.
We need to show that $z \in A$.
Since y converges to z, we know that the components sequences y_1 and y_2 of y converge to the components z_1 and z_2 of z, namely

$$\lim_{n\to\infty} y_1^{(n)} = z_1 \quad \text{and} \quad \lim_{n\to\infty} y_2^{(n)} = z_2.$$

By limit theorems, we know that

$$\lim_{n\to\infty} \left(y_2^{(n)}\right)^2 = (z_2)^2.$$

Since for all $n \in \mathbb{N}, y^{(n)} \in A$, we also know that for all $n \in \mathbb{N}, y_1(n) \leq (y_2(n))^2$. Therefore,

$$z_1 = \lim_{n\to\infty} y_1^{(n)} \leq \lim_{n\to\infty} \left(y_2^{(n)}\right)^2 = (z_2)^2.$$

We conclude that indeed $z \in A$. □

Proposition 9.2.5 Let $a, b \in \mathbb{R}$ with $a < b$. Then the intervals $[a, b], (-\infty, b]$ and $[a, \infty)$ are all closed.

We now provide a few ways to create new closed sets out of sets about which you already know that they are closed.

Intersections of closed sets are always closed

Let (X, dist) be a metric space. If $\mathcal{I}$ is a set, and for every $\alpha \in \mathcal{I}$, we have a subset A_α of X, then the intersection

$$\bigcap_{\alpha\in\mathcal{I}} A_\alpha$$

is defined as

$$\bigcap_{\alpha\in\mathcal{I}} A_\alpha := \{x \in X \mid \text{ for all } \alpha \in \mathcal{I}, x \in A_\alpha\}.$$

Proposition 9.2.6 Let (X, dist) be a metric space. Let $\mathcal{I}$ be a set and suppose for every $\alpha \in \mathcal{I}$ we have a subset $C_\alpha \subseteq X$. Assume that for every $\alpha \in \mathcal{I}$ the set C_α is closed. Then the intersection

$$\bigcap_{\alpha\in\mathcal{I}} C_\alpha$$

is closed as well.

Finite unions of closed sets are closed

Proposition 9.2.7 Let (X, dist) be a metric space. Let $C_1, \ldots, C_N$ be closed subsets of X. Then the finite union

$$C_1 \cup \cdots \cup C_N$$

is also closed.

Products of closed sets

Proposition 9.2.8 Let $C_1, \ldots, C_d$ be closed subsets of $\mathbb{R}$. Then the Cartesian product

$$C_1 \times \cdots \times C_d (= \{(c_1, \ldots, c_d) \mid c_i \in C_i\})$$

is a closed subset of $(\mathbb{R}^d, |\cdot|)$

The topological boundary of a set

Definition 9.2.9 (The topological boundary) Let (X, dist) be a metric space and let $A \subseteq X$. The *topological boundary* of a set A is denoted by ∂A and defined as

$$\partial A := X \setminus ((\mathrm{int} A) \cup (\mathrm{int}(X \setminus A)))$$

Example 9.2.10 The topological boundary of the interval $[2, 5)$ is the set $\{2, 5\}$ that consists of exactly the points 2 and 5.

9.3 Cauchy sequences

Definition 9.3.1 (Cauchy sequence) Let (X, dist) be a metric space. We say that a sequence $a : \mathbb{N} \to X$ is a Cauchy sequence if

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } N \in \mathbb{N}, \\ &\quad\quad \text{for all } m, n \geq N, \\ &\quad\quad\quad \mathrm{dist}(a_m, a_n) < \epsilon \end{aligned}$$

Proposition 9.3.2 Every Cauchy sequence is bounded

Proposition 9.3.3 Let $a : \mathbb{N} \to X$ be a Cauchy sequence and assume that a has a subsequence converging to $p \in X$. Then the sequence a itself converge to p.

Proposition 9.3.4 Let (X, dist) be a metric space. Let (x_n) be a converging sequence in X. Then (x_n) is a Cauchy sequence.

9.4 Completeness

Definition 9.4.1 Let (X, dist) be a metric space. We say that a subset $A \subseteq X$ is *complete* (in (X, dist)) if every Cauchy sequence in A is convergent, with limit in A.
We also say the metric space (X, dist) itself is complete if X is a complete subset of X in (X, dist).

Theorem 9.4.2 The metric space $(\mathbb{R}, \mathrm{dist}_{\mathbb{R}})$ is complete.

Proof. Let $a : \mathbb{N} \to \mathbb{R}$ be a Cauchy sequence. Because a is a Cauchy sequence, it is in particular bounded. as a consequence, by theorem 8.4.3, there is a subsequence $a \circ n$ such that $a \circ n$ converges to

$$\limsup_{k \to \infty} a_k$$

Finally, we know from proposition 9.3.3 that if a subsequence of a Cauchy sequence converges, that then the whole sequence converges. Therefore, the sequence $a : \mathbb{N} \to \mathbb{R}$ converges. □

Proposition 9.4.3 The metric space $(\mathbb{R}^d, \mathrm{dist}_{\|\cdot\|_2})$ is complete, where $\|\cdot\|_2$ is the Euclidean norm.

Proposition 9.4.4 Let (X, dist) be a metric space. Suppose $A \subseteq X$ is complete. Then A is closed

Proposition 9.4.5 Let (X, dist) be a metric space and let $C \subseteq X$ be a complete subset. Let $A \subseteq C$ be a subset of C. Then, A is complete if and only if A is closed.

Series characterization of completeness in normed vector spaces

Theorem 9.4.6 Let $(V, \|\cdot\|)$ be a normed vector space. Then $(V, \|\cdot\|)$ is complete if and only if every absolutely converging series is convergent.

Corollary 9.4.7 Let $a : \mathbb{N} \to \mathbb{R}$ be a real-valued sequence. Suppose the series

$$\sum_{n=0}^{\infty} a_n$$

converges absolutely, i.e. the series

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Then also the series

$$\sum_{n=0}^{\infty} a_n$$

converges.

Example 9.4.8 The series

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k^2}$$

converges, because it converges absolutely.

10 Compactness

10.1 Definition of (sequential) compactness

Definition 10.1.1 **((Sequential) compactness)** Let (X, dist) be a metric space. We say a subset $K \subseteq X$ is *(sequentially) compact* if every sequence $x : \mathbb{N} \to K$ has a converging subsequence $x \circ n$, converging to a point $z \in K$.

10.2 Boundedness and total boundedness

Definition 10.2.1 **(Bounded sets)** Let (X, dist) be a metric space. We say that a subset $A \subseteq X$ is *bounded* if

$$\begin{aligned}&\text{there exists } q \in X,\\ &\quad \text{there exists } M > 0,\\ &\qquad \text{for all } p \in A,\\ &\qquad\quad \mathrm{dist}(p, q) \leq M.\end{aligned}$$

Just as with the concept of boundedness for sequences, in normed vector spaces boundedness has a somewhat easier alternative characterization.

Proposition 10.2.2 Let $(V, \|\cdot\|)$ be a normed linear space. A subset $A \subseteq V$ is bounded if and only if

$$\begin{aligned}&\text{there exists } M > 0,\\ &\quad \text{for all } v \in A,\\ &\qquad \|v\| \leq M.\end{aligned}$$

Definition 10.2.3 **(Totally bounded sets)** Let (X, dist) be a metric space. We say that a subset $A \subseteq X$ is *totally bounded* if

$$\begin{aligned}&\text{for all } r > 0,\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\qquad \text{there exists } p_1, \dots, p_N \in X,\\ &\qquad\quad A \subseteq \textstyle\bigcup_{i=1}^N B(p_i, r).\end{aligned}$$

In the next proposition we will say that "total boundedness" is a stronger property than just "boundedness".

Proposition 10.2.4 Let (X, dist) be a metric space and let A be a subset of X. If A is totally bounded, it is bounded.

In the special case of the normed vector space $(\mathbb{R}^d, \|\cdot\|_2)$, however, a subset is totally bounded if and only if it is bounded.

Proposition 10.2.5 Consider now the normed vector space $(\mathbb{R}^d, \|\cdot\|_2)$. A subset $A \subseteq \mathbb{R}^d$ is bounded in $(\mathbb{R}^d, \|\cdot\|_2)$ if and only if it is totally bounded.

10.3 Alternative characterization of compactness

Theorem 10.3.1 A subset $K \subseteq X$ is compact if and only if it is complete and totally bounded.

In the special case of $(\mathbb{R}^d, \|\cdot\|)$ we have an easier alternative characterization of compactness.

Theorem 10.3.2 **(Heine-Borel Theorem)** A subset of $(\mathbb{R}^d, \|\cdot\|_2)$ is compact if and only if it is closed and bounded.

11 Limits and continuity

We will consider functions $f : D \to Y$ mappings from a subset $D \subseteq X$ of a metric space (X, dist_X) to a metric space (Y, dist_Y). These are quite some actors: an input metric space (X, dist_X), a subset D of the metric space, and an output metric space (Y, dist_Y). And the concept of *limits* and *continuity* depend on all these actors.

On the coarsest level, if $p \in X$ and $q \in Y$, then the statement that

$$\lim_{x \to p} f(x) = q$$

will mean that if the distance between x and p is small, but not zero, the distance between $f(x)$ and q will be small.

11.1 Accumulation points

To get a useful concept of a limit in a point $p \in X$, the point p needs to be an *accumulation point* of the domain D of the function.

Definition 11.1.1 **(Accumulation points)** Let (X, dist_X) be a metric space and let $D \subseteq X$ be a subset of X. We say a point $p \in X$ is an *accumulation point* of the set D if

for all $\epsilon > 0$,
  there exists $x \in D$.
    $0 < \mathrm{dist}_X(x, p) < \epsilon$

We denote the set of accumulation points of a set D by D'.

Note that accumulation points of a set D do not have to lie in the set D themselves. If a point odes lie in D, but is not an accumulation point, then we call it an *isolated point* of D.

Definition 11.1.2 **(Isolated points)** Let (X, dist) be a metric space and let $D \subseteq X$ be a subset of X. We say a point $a \in D$ is an *isolated point* if it is not an accumulation point, i.e. if $a \in D \setminus D'$.

11.2 Limit in an accumulation point

We can now define limits in accumulation points of D.

Definition 11.2.1 **(Limit in an accumulation point)** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subseteq X$ be a subset of X. Let $f : D \to Y$ be a function and let $q \in Y$ be a point in Y. Let $a \in D'$ be an accumulation point of D. Then we say f converges to q as x goes to a, and write

$$\lim_{x \to a} f(x) = q$$

if

for all $\epsilon > 0$,
  there exists $\delta > 0$,
    for all $x \in D$,
      if $0 < \mathrm{dist}_X(x, a) < \delta$, then $\mathrm{dist}_Y(f(x), q) < \epsilon$.

11.3 Uniqueness of limits

Proposition 11.3.1 Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subseteq X$ be a subset of X. Let $f : D \to Y$ be a function on D. Let $a \in D'$ and assume

$$\lim x \to a f(x) = p \quad \text{and} \quad \lim_{x \to a} f(x) = q$$

for points $p, q \in Y$. Then $p = q$.

11.4 Sequential characterization of limits

Theorem 11.4.1 **(Sequence characterization of limits)** Let (X, dist_X) and (Y, dist_Y) be two metric spaces. Let $D \subseteq X$. Let $f : D \to Y$ and let $a \in D'$. Let $q \in Y$. Then

$$\lim_{x \to a} f(x) = q$$

if and only if

$$\begin{aligned}&\text{for all sequences } (x^n) \text{ in } D \setminus \{a\} \text{ converging to } a,\\ &\quad \lim_{n\to\infty} f(x^n) = q\end{aligned}$$

11.5 Limit laws

Theorem 11.5.1 Let (X, dist_X) be a metric space and let $(V, \|\cdot\|)$ be a normed vector space. Let $D \subseteq X$ and let $f : D \to V$ and $g : D \to V$ be two functions. Let $a \in D'$. Moreover, assume that the limit $lim_{n \to a} f(x)$ exists and equals $p \in V$ and that limii $\lim_{n \to a} g(x)$ exists and equals $q \in V$. Let $\lambda \in \mathbb{R}$. Then

1. The limit $\lim_{x\to a}(f(x) + g(x))$ exists and equals $p + q$.
2. The limit $\lim_{x\to a}(\lambda f(x))$ exists and equals λp.

11.6 Continuity

Definition 11.6.1 **(Continuity in a point)** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subseteq X$ be a subset of X. We say a function $f : D \to Y$ is *continuous* in a point $a \in D \cap D'$ if

$$\lim_{x \to a} f(x) = f(a).$$

If $a \in D$ is an isolated point, i.e. if $a \in D \setminus D'$, then we also say that f is continuous in a.

We say a function is continuous if it is continuous in every point in its domain.

Definition 11.6.2 **(Continuity on the domain)** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subseteq X$ be a subset of X. We say a function $f : D \to Y$ is *continuous on* D if f is continuous in a for every $a \in D$.

Sometimes it si a bit cumbersome to make the distinction between isolated points and accumulation points. The following alternative characterization of continuity in a point circumvecnts this issue.

Proposition 11.6.3 **(Alternative $\epsilon - \delta$ characterization of continuity in a point)** Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $D \subseteq X$ be a subset of X. Let $a \in D$. Then the function f is continuous in a if and only if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } \delta > 0,\\ &\qquad \text{for all } x \in D,\\ &\qquad\quad \text{if } 0 < \text{dist}_X(x, a) < \delta, \text{ then } \text{dist}_Y(f(x), f(a)) < \epsilon.\end{aligned}$$

11.7 Sequence characterization of continuity

As with many concepts in analysis, continuity is conveniently probed with sequences.

Theorem 11.7.1 **(Sequence characterization of continuity)** Let (X, dist_X) and (Y, dist_Y) be metric spaces. Let $D \subseteq X$ and let $f : D \to Y$ be function. Let $a \in D$. The function f is continuous in a if and only if

$$\begin{aligned}&\text{for all sequences } (x^n) \text{ in } D \text{ converging to } a,\\ &\quad \lim_{n\to\infty} f(x^n) = f(a).\end{aligned}$$

11.8 Rules for continuous functions

The following proposition implies that the composition of two continuous functions is also continuous.

Proposition 11.8.1 Let (X, dist_X), (Y, dist_Y) and $Z, \mathrm{dist}_Z)$ be metric spaces, let $D \subseteq X$ and $E \subseteq Y$. Let $f : D \to Y$ and $g : E \to Z$ be two functions, and assume that $f(D) \subseteq E$. Let $a \in D$. If f is continuous in a and g is continuous in $f(a)$, then $g \circ f$ is continuous in a.

11.9 Images of compact sets under continuous functions are compact

Proposition 11.9.1 Let (X, dist_X) and (Y, dist_Y) be two metric spaces and let $K \subseteq X$ be a a compact subset of X. Let $f : K \to Y$ be continuous on K. Then $f(K)$ is a compact subset of Y.

11.10 Uniform continuity

Definition 11.10.1 Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subseteq X$ be a non-empty subset. We say that $f : D \to Y$ is *uniformly continuous* on D if

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } \delta > 0,\\ &\quad\quad \text{for all } p, q \in D,\\ &\quad\quad\quad 0 < \mathrm{dist}_X(p, q) < \delta \implies \mathrm{dist}_Y(f(p), f(q)) < \epsilon.\end{aligned}$$

The following proposition shows that *uniform continuity* is a stronger property that continuity.

Proposition 11.10.2 Let (X, dist_X) and (Y, dist_Y) be metric spaces and let $D \subseteq X$ be a non-empty subset. Let $f : D \to Y$ be uniformly continuous on D. Then f is continuous on D.

Although uniform continuity is a stronger property than continuity, it is not as strong as continuity on compact sets.

Theorem 11.10.3 Let (X, dist_X) and (Y, dist_Y) be metric spaces, let $K \subseteq X$ be compact and let $f : K \to Y$ be continuous on K. Then f is uniformly continuous on K.

12 Real-valued functions

12.1 More limit laws

Theorem 12.1.1 **(Limit laws for real-valued functinos)** Let (X, dist) be a metric space, let D be a subset of X and assume that $a \in D'$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two real-valued functions and assume that $\lim_{x\to a} f(x)$ exists and equals $M \in \mathbb{R}$ and that $\lim_{x\to a} g(x)$ exists and equals $L \in \mathbb{R}$. Then

1. For every $m \in \mathbb{N}$, the limit $\lim_{x\to a}(f(x))^m$ exists and equals M^m.
2. The limit $\lim_{x\to a}(f(x)g(x))$ exists and equals ML.
3. If $L \neq 0$, then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and equals $\frac{M}{L}$.
4. If for all $x \in D$, $f(x) \geq 0$, then for every $k \in \mathbb{N} \setminus \{0\}$,

$$\lim_{x\to a} \sqrt[k]{f(x)} = \sqrt[k]{M}$$

12.2 Building new continuous functions

The following theorem translates the limit laws from the previous section into statements about continuity.

Theorem 12.2.1 Let (X, dist) be a metric space, let D be a subset of X and assume $a \in D$. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two real-valued functions that are continuous in a. Then

1. For every $m \in \mathbb{N}$, the function f^m is continuous in a.
2. The function $f + g$ is continuous in a.
3. The function fg is continuous in a.
4. If $g(a) \neq 0$, then the function $\frac{f}{g}$ is continuous in a.
5. If for all $x \in D$, $f(x) \geq 0$, then for every $k \in \mathbb{N} \setminus \{0\}$, the function $\sqrt[k]{f}$ is continuous in a.

12.3 Continuity of standard functions

Proposition 12.3.1 **(Polynomials are continuous)** Every (possibly multivariate) polynomial is continuous as a function from $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$.

Proposition 12.3.2 **(Rational functions are continuous)** Every (possibly multivariate) rational function is continuous as a function from $(\mathbb{R}^d, \|\cdot\|_2)$ to $(\mathbb{R}, |\cdot|)$.

Proposition 12.3.3 **(Continuity of some standard functions)** The functions

$$\begin{aligned}
&\exp : \mathbb{R} \to \mathbb{R} \qquad && \ln : (0, \infty) \to \mathbb{R} \\
&\sin : \mathbb{R} \to \mathbb{R} && \arcsin : [-1, 1] \to \mathbb{R} \\
&\cos : \mathbb{R} \to \mathbb{R} && \arccos : [-1, 1] \to \mathbb{R} \\
&\tan : (-\pi/2, \pi/2) \to \mathbb{R} && \arctan : \mathbb{R} \to \mathbb{R}
\end{aligned}$$

are all continuous.

12.4 Limits from the left and from the right

Definition 12.4.1 (Limit from the left) Let (Y, dist_Y) be a metric space, and let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. Let $f : D \to Y$ be a function. Let $a \in \mathbb{R}$ be such that $a \in ((-\infty, a) \cap D)'$, i.e. such that a is an accumulation point in the set $(-\infty, a) \cap D$ in the metric space $(\mathbb{R}, \mathrm{dist}_\mathbb{R})$. Let $q \in Y$. We say that $f(x)$ *converges to q as x approaches a from the left (or from below)*, and write

$$\lim_{x \uparrow a} f(x) = q (\lim_{x \to a^-} f(x) = q)$$

if

$$\begin{aligned}
&\text{for all } \varepsilon > 0,\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D \cap (-\infty, a),\\
&\quad\quad\quad 0 < \mathrm{dist}_\mathbb{R}(x, a) < \delta \implies \mathrm{dist}_Y(f(x), q) < \varepsilon
\end{aligned}$$

Definition 12.4.2 (Limit from the right) Let (Y, dist_Y) be a metric space, and let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. Let $f : D \to Y$ be a function. Let $a \in \mathbb{R}$ be such that $a \in ((a, \infty) \cap D)'$, i.e. such that a is an accumulation point in the set $(a, \infty) \cap D$ in the metric space $(\mathbb{R}, \mathrm{dist}_\mathbb{R})$. Let $q \in Y$. We say that $f(x)$ *converges to q as x approaches a from the right (or from above)*, and write

$$\lim_{x \downarrow a} f(x) = q (\lim_{x \to a^+} f(x) = q)$$

if

$$\begin{aligned}
&\text{for all } \varepsilon > 0,\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D \cap (a, \infty),\\
&\quad\quad\quad 0 < \mathrm{dist}_\mathbb{R}(x, a) < \delta \implies \mathrm{dist}_Y(f(x), q) < \varepsilon
\end{aligned}$$

12.5 The extended real line

Definition 12.5.1 (The extended real line) The extended real line $\mathbb{R}_{\text{ext}}$ is the union -f the set $\mathbb{R}$ and two symbols, "∞" and "$-\infty$". That is $\mathbb{R}_{\text{ext}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$.

To turn $\mathbb{R}_{\text{ext}}$ into a metric space, we need to define a distance function. First, we define a map $\iota : \mathbb{R}_{\text{ext}} \to [-1, 1]$ by

$$\iota(x) = \begin{cases} -1 & \text{if } x = -\infty \\ \frac{x}{1+x} & \text{if } x \in \mathbb{R} \wedge x \geq 0 \\ \frac{x}{1-x} & \text{if } x \in \mathbb{R} \wedge x < 0 \\ 1 & \text{if } x = \infty \end{cases}$$

Because this function is injective, we can now build a distance on $\mathbb{R}_{\text{ext}}$.

Definition 12.5.2 (Distance on extended real line) Given the definition of the injective function $\iota : \mathbb{R}_{\text{ext}} \to [-1, 1]$ above, we define the distance on $\mathbb{R}_{\text{ext}}$ by

$$\mathrm{dist}_{\mathbb{R}_{\text{ext}}}(x, y) := dist_\mathbb{R}(\iota(x), \iota(y)) \qquad \text{for } x, y \in \mathbb{R}_{\text{ext}}$$

12.6 Limits to ∞ or $-\infty$

Definition 12.6.1 (Divergence to ∞) Let (X, dist_X) be a metric space and $D \subseteq X$ and assume $a \in D'$. Let $f : D \to \mathbb{R}$. We say that f diverges to ∞ in a if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad 0 < \operatorname{dist}_X(x, a) < \delta \implies f(x) > M
\end{aligned}$$

Definition 12.6.2 (Divergence to $-\infty$) Let $(X, \operatorname{dist}_X)$ be a metric space and $D \subseteq X$ and assume $a \in D'$. Let $f : D \to \mathbb{R}$. We say that f diverges to $-\infty$ in a if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } \delta > 0,\\
&\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad 0 < \operatorname{dist}_X(x, a) < \delta \implies f(x) < M
\end{aligned}$$

Proposition 12.6.3 (Alternative characterization of divergence to ∞) Let $(X, \operatorname{dist}_X)$ be a metric space and $D \subseteq X$ and assume $a \in D'$. Let $f : D \to \mathbb{R}$. Then f diverges to ∞ in a if and only if f *converges* in a to the element $\infty \in \mathbb{R}_{\text{ext}}$ when viewed as a function mapping from D as a subset of $(X, \operatorname{dist}_X)$ to the extended real line $(\mathbb{R}_{\text{ext}}, \operatorname{dist}_{\mathbb{R}_{\text{ext}}})$.

12.7 Limits at ∞ and $-\infty$

Definition 12.7.1 (Limit at ∞) Let $(Y, \operatorname{dist}_Y)$ be a metric space and let D be a subset of $\mathbb{R}$ that is unbounded from above. Let $q \in Y$ and $f : D \to Y$ be a function. We say that $f(x)$ converges to q as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = q$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } z \in \mathbb{R},\\
&\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad x > z \implies \operatorname{dist}_Y(f(x), q) < \epsilon
\end{aligned}$$

Definition 12.7.2 (Limit at $-\infty$) Let $(Y, \operatorname{dist}_Y)$ be a metric space and let D be a subset of $\mathbb{R}$ that is unbounded from below. Let $q \in Y$ and $f : D \to Y$ be a function. We say that $f(x)$ converges to q as $x \to -\infty$, and write

$$\lim_{x\to-\infty} f(x) = q$$

if

$$\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } z \in \mathbb{R},\\
&\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad x < z \implies \operatorname{dist}_Y(f(x), q) < \epsilon
\end{aligned}$$

We can also combine divergence to and at infinity.

Definition 12.7.3 (Divergence to ∞ at ∞) Let $D \subseteq \mathbb{R}$ be unbounded from above. Let $f : D \to \mathbb{R}$ be a function. We say that f diverges to ∞ as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = \infty$$

if

$$\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } z \in \mathbb{R},\\
&\quad\quad \text{for all } x \in D,\\
&\quad\quad\quad x > z \implies f(x) > M
\end{aligned}$$

Definition 12.7.4 **(Divergence to $-\infty$ at ∞)** Let $D \subseteq \mathbb{R}$ be unbounded from above. Let $f : D \to \mathbb{R}$ be a function. We say that f diverges to $-\infty$ as $x \to \infty$, and write

$$\lim_{x\to\infty} f(x) = -\infty$$

if

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad\text{there exists } z \in \mathbb{R},\\ &\quad\quad\text{for all } x \in D,\\ &\quad\quad\quad x > z \implies f(x) < M\end{aligned}$$

12.8 The Intermediate Value Theorem

Theorem 12.8.1 **(Intermediate Value Theorem)** Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$ be a value between $f(a)$ and $f(b)$. Then, there exists an $x \in [a, b]$ such that $f(x) = c$.

12.9 The Extreme Value Theorem

The *Extreme Value Theorem* states that a continuous, real-valued function defined on a non-empty, compact domain K always attains both a maximum and a minimum on K.

Theorem 12.9.1 **(Extreme Value Theorem)** Let (X, dist_X) be a metric space, $K \subseteq X$ be a non-empty compact subset and $f : K \to \mathbb{R}$ be continuous. Then f attains a maximum and a minimum on K.

12.10 Equivalence of norms

Definition 12.10.1 **(Equivalent norms)** Let V be a vector space and let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two different norms on V. We say that the norms $\|\cdot\|_A$ and $\|\cdot\|$_B are *equivalent* if there exists a constant $c_1 > 0$ and $c_2 > 0$ such that for all $v \in V$

$$c_1\|x\|_A \leq \|x\|_B \leq c_2\|x\|_A.$$

Theorem 12.10.2 **(Equivalence of norms on finite-dimensional vector spaces)** Let V be a finite-dimensional vector space and let $\|\cdot\|_A$ and $\|\cdot\|_B$ be two norms on V. Then the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ are equivalent.

Theorem 12.10.3 Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space. Then $(V, \|\cdot\|)$ is complete.

Theorem 12.10.4 **(Heine-Borel Theorem for finite-dimensional normed vector spaces)** Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space. Then a subset $A \subseteq V$ is compact if and only if A is closed and bounded.

12.11 Bounded linear maps and operator norms

Definition 12.11.1 **(Linear map)** Let V and W be two vector spaces. A function $L : V \to W$ is called a *linear map* if both

1. for all $a, b \in V$,
$$L(a+b) = L(a) + L(b)$$
2. for all $\lambda \in \mathbb{R}$ and $a \in V$,
$$L(\lambda a) = \lambda L(a)$$

Definition 12.11.2 **(Bounded linear map)** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. We say that a linear map $L : V \to W$ is *bounded* if the image under L of the closed unit ball

$$\bar{B}_V(0,1) = \{v \in V \mid \|v\|_V \leq 1\}$$

is a bounded subset of $(W, \|\cdot\|_W)$, i.e. if
$$L(\bar{B}_V(0,1))$$
is a bounded subset of $(W, \|\cdot\|_W)$.

Proposition 12.11.3 **(Alternative characterization of bounded linear maps)** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. A linear map $L : V \to W$ is bounded if and only if there exists an $M > 0$ such that for all $v \in V$,
$$\|L(v)\|_W \leq M\|v\|_V$$

Proposition 12.11.4 The space of bounded linear maps between one normed vector space to another is itself again a vector space, that we denote by $\mathrm{BLin}(V, W)$. Addition and scalar multiplication are defined pointwise, that means that if $L : V \to W$ and $K : V \to W$ are two linear maps and $\lambda \in \mathbb{R}$ is a scalar, then the linear map $L + K : V \to W$ is defined by

$$(L+K)(v) = L(v) + K(v)$$

and the map
$$(\lambda L)(v) = \lambda(L(v)).$$
The zero-element in this vector space $\mathrm{BLin}(V, W)$ is the map that maps every vector to the zero-element of W.

We now define the operator norm on the space of bounded linear maps.

Proposition 12.11.5 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Consider the vector space $\mathrm{BLin}(V, W)$ of bounded linear maps $L : V \to W$. Then the function $\|\cdot\|_{V\to W} : \mathrm{BLin}(V, W) \to \mathbb{R}$ defined by
$$\|L\|_{V\to W} := \sup_{x \in \bar{B}_V(0,1)} \|L(x)\|_W$$
is a norm on $\mathrm{BLin}(V, W)$.

Definition 12.11.6 **(Operator norm)** The norm $\|\cdot\|_{V\to W}$ on the vectors space $\mathrm{BLin}(V, W)$ is called the *operator norm.*

Proposition 12.11.7 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $L : V \to W$ be a bounded linear map. Then for all $v \in V$,

$$\|L(v)\|_W \leq \|L\|_{V\to W}\|v\|_V$$

and in fact
$$\|L\|_{L\to W} = \min\{C \geq 0 \mid \forall v \in V, \|L(v)\|_W \leq C\|v\|_V\}$$

Theorem 12.11.8 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces and assume that V is finite-dimensional. Let $L : V \to W$ be a linear map. Then L is bounded.

Theorem 12.11.9 Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. Let $L : V \to W$ be a linear map. The function L is continuous if and only if it is bounded.