Homework: Week 1

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1 2.7.1

Problem 1.1
Let (Y, dist_Y) be a metric space. Let X be a set and $f : X \to Y$ be injective. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) := \text{dist}_Y(f(x), f(y)), \text{ for all } x, y \in X.$$

Show that the function d is a distance on X.

Proof. To show that d is a distance on X, we need to show that d satisfies the following three properties:

1. Positivity: $d(x, y) \geq 0$ for all $x, y \in X$.
2. Non-degeneracy: $d(x, y) = 0$ if and only if $x = y$.
3. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
5. Reflexivity: $d(x, x) = 0$ for all $x \in X$.

We show each of these properties in turn.
1. Positivity:
Take $x \in X$.
Take $y \in X$.
It holds that $f(x) \in Y$.
It holds that $f(y) \in Y$.
By positivity of dist_Y it holds that $\text{dist}_Y(f(x), f(y)) \geq 0$.
It holds that $d(x, y) = \text{dist}_Y(f(x), f(y)) \geq 0$.
We conclude that $d(x, y) \geq 0$ for all $x, y \in X$.

2. Non-degeneracy:
We need to show both directions.
2.1 Forward direction:
Take $x \in X$.
Take $y \in X$.
Assume $d(x, y) = 0$.
It holds that $d(x, y) = \text{dist}_Y(f(x), f(y)) = 0$.
By non-degeneracy of dist_Y it holds that $f(x) = f(y)$.
By injectivity of f it holds that $x = y$.

We conclude that $d(x, y) = 0 \implies x = y$.
2.2 Backward direction:
Take $x \in X$.
Take $y \in X$.
Assume $x = y$.
It holds that $f(x) = f(y)$.
By non-degeneracy of dist_Y it holds that $\mathrm{dist}_Y(f(x), f(y)) = 0$.
It holds that $d(x, y) = \mathrm{dist}_Y(f(x), f(y)) = 0$.
We conclude that $x = y \implies d(x, y) = 0$.
We conclude that $d(x, y) = 0 \iff x = y$.

3. Symmetry:
Take $x \in X$.
Take $y \in X$.
It holds that $d(x, y) = \mathrm{dist}_Y(f(x), f(y))$.
By symmetry of dist_Y it holds that $\mathrm{dist}_Y(f(x), f(y)) = \mathrm{dist}_Y(f(y), f(x))$.
It holds that $d(x, y) = \mathrm{dist}_Y(f(x), f(y)) = \mathrm{dist}_Y(f(y), f(x)) = d(y, x)$.
We conclude that $d(x, y) = d(y, x)$ for all $x, y \in X$.

4. Triangle inequality:
Take $x \in X$.
Take $y \in X$.
Take $z \in X$.
It holds that $d(x, y) = \mathrm{dist}_Y(f(x), f(y))$.
It holds that $d(x, z) = \mathrm{dist}_Y(f(x), f(z))$.
It holds that $d(z, y) = \mathrm{dist}_Y(f(z), f(y))$.
By triangle inequality of dist_Y it holds that $\mathrm{dist}_Y(f(x), f(z)) \leq \mathrm{dist}_Y(f(x), f(y)) + \mathrm{dist}_Y(f(y), f(z))$.
It holds that $d(x, z) = \mathrm{dist}_Y(f(x), f(z)) \leq \mathrm{dist}_Y(f(x), f(y)) + \mathrm{dist}_Y(f(y), f(y)) = d(x, y) + d(y, z)$.
We conclude that $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

5. Reflexivity:
Take $x \in X$.
It holds that $d(x, x) = \mathrm{dist}_Y(f(x), f(x))$.
By reflexivity of dist_Y it holds that $\mathrm{dist}_Y(f(x), f(x)) = 0$.
It holds that $d(x, x) = \mathrm{dist}_Y(f(x), f(x)) = 0$.
We conclude that $d(x, x) = 0$ for all $x \in X$.

We conclude that d is a distance on X. $\square$

2 2.7.3

Problem 2.1
Consider the function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by

$$d(a,b) = \begin{cases} 0, & \text{if } a = b \\ 3, & \text{if } a \neq b \end{cases}$$

Show that d is a distance function on $\mathbb{Z}$.

Proof. To show that d is a distance on $\mathbb{Z}$, we need to show that d satisfies the following 5 properties:

1. Positivity: $d(a,b) \geq 0$ for all $a, b \in \mathbb{Z}$.
2. Non-degeneracy: $d(a,b) = 0$ if and only if $a = b$.
3. Symmetry: $d(a,b) = d(b,a)$ for all $a, b \in \mathbb{Z}$.
4. Triangle inequality: $d(a,c) \leq d(a,b) + d(b,c)$ for all $a, b, c \in \mathbb{Z}$.
5. Reflexivity: $d(a,a) = 0$ for all $a \in \mathbb{Z}$.

We show each of these properties in turn.
1. Positivity:
Take $a \in \mathbb{Z}$.
Take $b \in \mathbb{Z}$.
Either $a = b$ or $a \neq b$.
- Case 1: $a = b$.
 It holds that $d(a,b) = 0$.
 We conclude that $d(a,b) = 0 \geq 0$.
- Case 2: $a \neq b$.
 It holds that $d(a,b) = 3$.
 We conclude that $d(a,b) = 3 \geq 0$.

We conclude that $d(a,b) \geq 0$ for all $a, b \in \mathbb{Z}$.

2. Non-degeneracy:
We need to show both directions.
2.1 Forward direction:
Take $a \in \mathbb{Z}$.
Take $b \in \mathbb{Z}$.
Assume $d(a,b) = 0$.
It holds that $a = b$.
We conclude that $d(a,b) = 0 \implies a = b$.
2.2 Backward direction:
Take $a \in \mathbb{Z}$.
Take $b \in \mathbb{Z}$.
Assume $a = b$.
It holds that $d(a,b) = 0$.
We conclude that $a = b \implies d(a,b) = 0$.
We conclude that $d(a,b) = 0 \iff a = b$.

3. Symmetry:

Take $a \in \mathbb{Z}$.
Take $b \in \mathbb{Z}$.
Either $a = b$ or $a \neq b$.

- Case 1: $a = b$.
  It holds that $d(a,b) = 0$.
  We conclude that $d(a,b) = 0 = 0 = d(b,a)$.
- Case 2: $a \neq b$.
  It holds that $d(a,b) = 3$.
  We conclude that $d(a,b) = 3 = 3 = d(b,a)$.

We conclude that $d(a,b) = d(b,a)$ for all $a, b \in \mathbb{Z}$.

4. Triangle inequality:

Take $a \in \mathbb{Z}$.
Take $b \in \mathbb{Z}$.
Take $c \in \mathbb{Z}$.
Either $a = b$ or $a \neq b$.
Either $a = c$ or $a \neq c$.
Either $b = c$ or $b \neq c$.

- Case 1: $a = b$.
  It holds that $d(a,b) = 0$.
  Either $a = c$ or $a \neq c$.
  + Case 1.1: $a = c$.
    It holds that $d(a,c) = 0$.
    It holds that $b = c$
    It holds that $d(b,c) = 0$.
    It holds that $d(a,c) = 0 \leq 0 = d(a,b) + d(b,c)$.
    We conclude that $d(a,c) \leq d(a,b) + d(b,c)$.
  + Case 1.2: $a \neq c$.
    It holds that $d(a,c) = 3$.
    Either $b = c$ or $b \neq c$.
    ++ Case 1.2.1: $b = c$.
      It holds that $d(b,c) = 0$.
      It holds that $d(a,c) = 3 \leq 3 = d(a,b) + d(b,c)$.
      We conclude that $d(a,c) \leq d(a,b) + d(b,c)$.
    ++ Case 1.2.2: $b \neq c$.
      It holds that $d(b,c) = 3$.
      It holds that $d(a,c) = 3 \leq 6 = d(a,b) + d(b,c)$.
      We conclude that $d(a,c) \leq d(a,b) + d(b,c)$.
- Case 2: $a \neq b$.
  It holds that $d(a,b) = 3$.
  Either $a = c$ or $a \neq c$.
  + Case 2.1: $a = c$.
    It holds that $d(a,c) = 0$.
    Either $b = c$ or $b \neq c$.
    ++ Case 2.1.1: $b = c$.
      It holds that $d(b,c) = 0$.
      It holds that $d(a,c) = 0 \leq 3 = d(a,b) + d(b,c)$.
      We conclude that $d(a,c) \leq d(a,b) + d(b,c)$.

++ Case 2.1.2: $b \neq c$.

It holds that $d(b, c) = 3$.

It holds that $d(a, c) = 0 \leq 6 = d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$.

+ Case 2.2: $a \neq c$.

It holds that $d(a, c) = 3$.

Either $b = c$ or $b \neq c$.

++ Case 2.2.1: $b = c$.

It holds that $d(b, c) = 0$.

It holds that $d(a, c) = 3 \leq 3 = d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$.

++ Case 2.2.2: $b \neq c$.

It holds that $d(b, c) = 3$.

It holds that $d(a, c) = 3 \leq 6 = d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$.

We conclude that $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in \mathbb{Z}$.

5. Reflexivity:

Take $a \in \mathbb{Z}$.

It holds that $d(a, a) = 0$.

We conclude that $d(a, a) = 0$ for all $a \in \mathbb{Z}$.

$\square$

3 2.7.4

Problem 3.1
Let (X, dist) be a metric space. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \sqrt{\text{dist}(x, y)}$$

Show that d is a distance function on X.

Proof. To show that d is a distance on X, we need to show that d satisfies the following 5 properties:

1. Positivity: $d(x, y) \geq 0$ for all $x, y \in X$.
2. Non-degeneracy: $d(x, y) = 0$ if and only if $x = y$.
3. Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.
4. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
5. Reflexivity: $d(x, x) = 0$ for all $x \in X$.

We show each of these properties in turn.
1. Positivity:
Take $x \in X$.
Take $y \in X$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)}$.
By positivity of dist it holds that $\text{dist}(x, y) \geq 0$.
By positivity of square root it holds that $\sqrt{\text{dist}(x, y)} \geq 0$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)} \geq 0$.
We conclude that $d(x, y) \geq 0$ for all $x, y \in X$.

2. Non-degeneracy:
We need to show both directions.
2.1 Forward direction:
Take $x \in X$.
Take $y \in X$.
Assume $d(x, y) = 0$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)} = 0$.
It holds that $\text{dist}(x, y) = 0$.
By non-degeneracy of dist it holds that $x = y$.
We conclude that $d(x, y) = 0 \implies x = y$.
2.2 Backward direction:
Take $x \in X$.
Take $y \in X$.
Assume $x = y$.
By non-degeneracy of dist it holds that $\text{dist}(x, y) = 0$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)} = 0$.
We conclude that $x = y \implies d(x, y) = 0$.
We conclude that $d(x, y) = 0 \iff x = y$.

3. Symmetry:
Take $x \in X$.
Take $y \in X$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)}$.
It holds that $d(y, x) = \sqrt{\text{dist}(y, x)}$.
By symmetry of dist it holds that $\text{dist}(x, y) = \text{dist}(y, x)$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)} = \sqrt{\text{dist}(y, x)} = d(y, x)$.
We conclude that $d(x, y) = d(y, x)$ for all $x, y \in X$.

4. Triangle inequality:
Take $x \in X$.
Take $y \in X$.
Take $z \in X$.
It holds that $d(x, y) = \sqrt{\text{dist}(x, y)}$.
It holds that $d(x, z) = \sqrt{\text{dist}(x, z)}$.
It holds that $d(z, y) = \sqrt{\text{dist}(z, y)}$.
By triangle inequality of dist it holds that $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.
It holds that $d(x, z) = \sqrt{\text{dist}(x, z)} \leq \sqrt{\text{dist}(x, y) + \text{dist}(y, z)}$.
We need to show that $\sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\text{dist}(x, y)} + \sqrt{\text{dist}(y, z)}$.
By positivity of square root it holds that $\sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\text{dist}(x, y) + 2\sqrt{\text{dist(x,y)}}\sqrt{\text{dist}(y, z)} + \text{dist}(y, z)}$
It holds that $\sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\left(\sqrt{\text{dist}(x, y)}\right)^2 + 2\sqrt{\text{dist}(x, y)}\sqrt{\text{dist}(y, z)} + \left(\sqrt{\text{dist}(y, z)}\right)^2}$
It holds that $\sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\left(\sqrt{\text{dist}(x, y)} + \sqrt{\text{dist}(y, z)}\right)^2}$
It holds that $\sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\text{dist}(x, y)} + \sqrt{\text{dist}(y, z)}$
It holds that $\sqrt{\text{dist}(x, z)} \leq \sqrt{\text{dist}(x, y) + \text{dist}(y, z)} \leq \sqrt{\text{dist}(x, y)} + \sqrt{\text{dist}(y, z)}$
It holds that $d(x, z) \leq d(x, y) + d(y, z)$
We conclude that $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

5. Reflexivity:
Take $x \in X$.
It holds that $d(x, x) = \sqrt{\text{dist}(x, x)}$.
By reflexivity of dist it holds that $\text{dist}(x, x) = 0$.
It holds that $d(x, x) = \sqrt{\text{dist}(x, x)} = \sqrt{0} = 0$.
We conclude that $d(x, x) = 0$ for all $x \in X$.

$\square$

4 2.7.5

Problem 4.1
Let $(V, \|\cdot\|)$ be a normed vector space. We say a subset $U \subseteq V$ is convex if

$$\begin{aligned} &\text{for all } x, y \in U \\ &\quad \text{for all } \lambda \in (0,1) \\ &\lambda x + (1-\lambda) y \in U \end{aligned}$$

Let $z \in V$ and $r > 0$. Define $B(z, r) := \{x \in V : \|x - z\| < r\}$. Show that $B(z, r)$ is convex.

Proof. To show that $B(z, r)$ is convex, we need to show that $B(z, r)$ satisfies the following property:

$$\text{For all } x, y \in B(z,r) \text{ and for all } \lambda \in (0,1),\ \lambda x + (1-\lambda) y \in B(z,r).$$
$$\text{i.e. } \|\lambda x + (1-\lambda) y - z\| < r$$

Take $x \in B(z, r)$.
Take $y \in B(z, r)$.
Take $\lambda \in (0, 1)$.
It holds that $\|x - z\| < r$.
It holds that $\|y - z\| < r$.
It holds that $\lambda x + (1 - \lambda)y = \lambda(x - z) + (1 - \lambda)(y - z) + z$.
It suffices to show that $\|\lambda(x - z) + (1 - \lambda)(y - z) + z - z\| < r$.
By the triangle inequality it holds that $\|\lambda(x - z) + (1 - \lambda)(y - z)\| \leq \|\lambda(x - z)\| + \|(1 - \lambda)(y - z)\|$
Since $\lambda \in (0, 1)$ it holds that $\|\lambda(x - z) + (1 - \lambda)(y - z)\| \leq \lambda\|x - z\| + (1 - \lambda)\|y - z\|$.
It holds that $\|\lambda(x - z) + (1 - \lambda)(y - z)\| \leq \lambda r + (1 - \lambda)r = r$
We conclude that $\|\lambda(x - z) + (1 - \lambda)(y - z) + z - z\| < r$.
We conclude that $B(z, r)$ is convex.

□

5 3.11.1

Problem 5.1
Show that

$$\begin{gathered}\exists M \in \mathbb{R},\\ \forall x \in [0,5],\\ x \leq M\end{gathered}$$

Proof. Choose $M = 5$.
Let $x \in [0,5]$.
It holds that $x \leq 5$.
We conclude that $\exists M \in \mathbb{R}, \forall x \in [0,5], x \leq M$.

□

6 3.11.2

Problem 6.1
Show that

$$\begin{gathered}\forall x \in \mathbb{R},\\ \exists y \in \mathbb{R},\\ \forall u \in \mathbb{R},\\ u > 0 \implies \exists v \in \mathbb{R},\\ v > 0 \land x + u < y + v.\end{gathered}$$

Proof. Let $x \in \mathbb{R}$.
Choose $y = x$
Let $u \in \mathbb{R}$.
Assume $u > 0$.
Choose $v = u + 1$.
It holds that $v > 0$.
It holds that $x + u < x + (u + 1) = y + v$.
We conclude that $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall u \in \mathbb{R}, [u > 0 \implies \exists v \in \mathbb{R}[v > 0 \land x + u < y + v]]$.

□