Homework: Week 2

Jiaqi Wang

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1 Exercise 6.8.1

Problem 1.1 Let $a : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below. Prove it.

Proof. We prove both directions of the if and only if statement. $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below.

1. We prove the forward direction.

Suppose $a : \mathbb{N} \to \mathbb{R}$ is bounded.
Then

$$\begin{aligned}&\text{there exists } M_0 > 0,\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad |a_n| \leq M_0.\end{aligned}$$

Obtain such M_0.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad -M_0 \leq a_n \leq M_0.\end{aligned}$$

Choose $m = -M_0$, then $m \in \mathbb{R}$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad m \leq a_n.\end{aligned}$$

We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded from below.

Choose $M = M_0$, then $M \in \mathbb{R}$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad a_n \leq M.\end{aligned}$$

We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded from above.

2. Now we prove the reverse direction.

Assume $a : \mathbb{N} \to \mathbb{R}$ is bounded from above and bounded from below.
By definition of bounded from above, we have

$$\begin{aligned}&\text{there exists } M_1 \in \mathbb{R},\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad a_n \leq M_1.\end{aligned}$$

By definition of bounded from below, we have

$$\begin{aligned}&\text{there exists } M_2 \in \mathbb{R},\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad M_2 \le a_n.\end{aligned}$$

We need to show that

$$\begin{aligned}&\text{there exists } M_0 > 0,\\ &\quad \text{for all } n \in \mathbb{N},\\ &\qquad |a_n| \le M_0.\end{aligned}$$

Choose $M_0 = \max\{|M_1|, |M_2|\}$, then $M_0 \in \mathbb{R}$.
It holds that

$$\begin{aligned}&\text{for all } n \in \mathbb{N},\\ &\quad -M_0 \le a_n \text{ and } a_n \le M_0.\end{aligned}$$

Then it holds that $|a_n| \le M_0$.
We conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded.

Since both directions hold, we conclude that $a : \mathbb{N} \to \mathbb{R}$ is bounded if and only if it is both bounded above and bounded below. □

2 Exercise 6.8.2

Problem 2.1 Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be real-valued sequences. Prove that

$$\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (-a_n) = -\infty$$

Proof. We show both directions of the if and only if statement.

$$\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (-a_n) = -\infty$$

1. First we prove the forward direction.

Suppose $\lim_{n\to\infty} a_n = \infty$.
Then

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{there exists } N_0 \in \mathbb{N},\\ &\qquad \text{for all } n \ge N_0,\\ &\qquad\quad a_n > M.\end{aligned}$$

Obtain such N_0.
We need to show that

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\qquad \text{for all } n \ge N,\\ &\qquad\quad -a_n < M.\end{aligned}$$

Take $M \in \mathbb{R}$ Choose $N = N_0$, then $N \in \mathbb{N}$.
It holds that

$$\begin{aligned}&\text{for all } M \in \mathbb{R},\\ &\quad \text{for all } n \ge N,\\ &\qquad a_n > M.\\ &\qquad\quad -a_n < -M.\end{aligned}$$

Because M is arbitrary, we conclude that

$$
\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } N \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N,\\
&\quad\quad\quad -a_n < M.
\end{aligned}
$$

$\square$

3 Exercise 6.8.3

Problem 3.1 Let $a : \mathbb{N} \to \mathbb{R}$ and $b : \mathbb{N} \to (0, \infty)$ be real-valued sequences. Prove that

$$\lim_{n\to\infty} b_n = \infty \iff \lim_{n\to\infty} \frac{1}{b_n} = 0$$

Proof. We need to show both directions of the if and only if statement.

$$\lim_{n\to\infty} b_n = \infty \iff \lim_{n\to\infty} \frac{1}{b_n} = 0$$

1. First we prove the forward direction.

We need to show that $\lim_{n\to\infty} b_n = \infty \implies \lim_{n\to\infty} \frac{1}{b_n} = 0$.
I.e.

$$
\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{there exists } N \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N,\\
&\quad\quad\quad |\tfrac{1}{b_n} - 0| < \epsilon.
\end{aligned}
$$

Take $\epsilon > 0$.
Suppose $\lim_{n\to\infty} b_n = \infty$.
Then

$$
\begin{aligned}
&\text{for all } M \in \mathbb{R},\\
&\quad \text{there exists } N_0 \in \mathbb{N},\\
&\quad\quad \text{for all } n \geq N_0,\\
&\quad\quad\quad b_n > M.
\end{aligned}
$$

Choose $M = \frac{1}{\epsilon}$, then $M \in \mathbb{R}$.
Obtain such N_0.
Take $\epsilon > 0$.
Choose $N = N_0$, then $N \in \mathbb{N}$.
It holds that

$$
\begin{aligned}
&\text{for all } \epsilon > 0,\\
&\quad \text{for all } n \geq N,\\
&\quad\quad b_n > M.\\
&\quad\quad\quad \tfrac{1}{b_n} < \tfrac{1}{M}.\\
&\quad\quad\quad\quad |\tfrac{1}{b_n} - 0| < \tfrac{1}{M}.\\
&\quad\quad\quad\quad\quad |\tfrac{1}{b_n} - 0| < \epsilon.
\end{aligned}
$$

2. Now we prove the reverse direction.

We need to show that $\lim_{n\to\infty} \frac{1}{b_n} = 0 \implies \lim_{n\to\infty} b_n = \infty$.
I.e.

$$\begin{aligned}&\text{for all } M \in (0,\infty),\\ &\quad \text{there exists } N \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N,\\ &\quad\quad\quad b_n > M.\end{aligned}$$

Take $M \in \mathbb{R}$.
Assume $\lim_{n\to\infty} \frac{1}{b_n} = 0$.
Then

$$\begin{aligned}&\text{for all } \epsilon > 0,\\ &\quad \text{there exists } N_0 \in \mathbb{N},\\ &\quad\quad \text{for all } n \geq N_0,\\ &\quad\quad\quad |\tfrac{1}{b_n} - 0| < \epsilon.\end{aligned}$$

Choose $\epsilon = \frac{1}{M}$, then $\epsilon > 0$.
Obtain such N_0.
Choose $N = N_0$, then $N \in \mathbb{N}$.
It holds that

$$\begin{aligned}&\text{for all } n \geq N,\\ &\quad |\tfrac{1}{b_n} - 0| < \epsilon.\\ &\quad\quad \tfrac{1}{b_n} < \epsilon.\\ &\quad\quad\quad \tfrac{1}{b_n} < \tfrac{1}{M}.\\ &\quad\quad\quad\quad b_n > M.\end{aligned}$$

□

4 Exercise 6.8.5

Problem 4.1 Define the sequence $x : \mathbb{N} \to \mathbb{R}$ recursively by

$$x_{n+1} := \frac{2 + x_n^2}{2x_n}$$

for $n \in \mathbb{N}$ while $x_0 = 2$. Prove that the sequence $x : \mathbb{N} \to \mathbb{R}$ converges and determine its limit.

Proof.

$$x_{n+1} := \frac{2 + x_n^2}{2x_n} = \frac{1}{x_n} + \frac{x_n}{2}$$

Since we are starting with $x_0 = 2$, and never substracting to get the following x, we know $x_n \geq 0$.
This gives us a lower bound for the sequence.
Notice, that for $x_n > 1$ we have $x_n = \frac{x_n}{2} + \frac{x_n}{2} > \frac{1}{x_n} + \frac{x_n}{2}$.
And $\frac{1}{x_n} + \frac{x_n}{2} > 1 \iff \frac{2+x_n^2}{2x_n} > 1 \iff x_n^2 - 2x_n + 1 > 0$, which is always true for every $x > 0$.
Thus since we start at $x_0 = 2$, our sequence decreases.
Hence $x_n > x_{n+1} \iff x_n > \frac{2+x_n^2}{2x_n} \iff 2x_n^2 - x_n^2 > 2 \iff x_n^2 > 2$ This gives us that x_n is either larger than $\sqrt{2}$ or smaller than $-\sqrt{2}$. The latter is impossible since x_n is always positive, hence our new lower bound is $\sqrt{2}$. This lower bound is in fact our infimum.
Since our infimum is $\sqrt{2}$ and the function is decreasing, the sequence converges to $\sqrt{2}$. □

5 Exercise 6.8.6

Problem 5.1 Determine whether the following sequences converge, diverge to $+\infty$, diverge to $-\infty$, or diverge in a different way. In case the sequence converges, determine the limit.

- $a_n := \frac{1}{n^3} - 3$
- $b_n := \frac{5n^5+2n^2}{3n^5+7n^3+4}$
- $c_n := n - \sqrt{n}$
- $d_n := \frac{2^n}{n^{100}}$
- $e_n := \sqrt{n^2+n} - n$
- $f_n := \sqrt[n]{3n^2}$
- $g_n := \frac{2^n+5n^200}{3^n+n^10}$
- $h_n := (-1)^n 3^n$
- $i_n := \sqrt[n]{5^n+n^2}$

5.1 a)

$$a_n := \frac{1}{n^3} - 3$$

By limit laws and standard limits, we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\frac{1}{n^3} - 3\right) \\
&= \lim_{n\to\infty} \frac{1}{n^3} - \lim_{n\to\infty} 3 \\
&= \left(\lim_{n\to\infty} \frac{1}{n}\right)^3 - 3 \\
&= 0^3 - 3 \\
&= -3
\end{aligned}$$

So the sequence converges to -3.

5.2 b)

$$b_n := \frac{5n^5+2n^2}{3n^5+7n^3+4}$$

By limit laws and standard limits, we have

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\left(\frac{5n^5+2n^2}{3n^5+7n^3+4}\right)\\
&= \lim_{n\to\infty}\left(\frac{n^5(5+\frac{2}{n^3})}{n^5(3+\frac{7}{n^2}+\frac{4}{n^5})}\right)\\
&= \lim_{n\to\infty}\left(\frac{5+\frac{2}{n^3}}{3+\frac{7}{n^2}+\frac{4}{n^5}}\right)\\
&= \frac{\lim_{n\to\infty}(5+\frac{2}{n^3})}{\lim_{n\to\infty}(3+\frac{7}{n^2}+\frac{4}{n^5})}\\
&= \frac{\lim_{n\to\infty}5+\lim_{n\to\infty}\frac{2}{n^3}}{\lim_{n\to\infty}3+\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{4}{n^5}}\\
&= \frac{5+0}{3+0+0}\\
&= \frac{5}{3}
\end{aligned}$$

So the sequence converges to $\frac{5}{3}$.

5.3 c)

$$c_n := n-\sqrt{n}$$

By limit laws and standard limits, we have

$$
\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} (n - \sqrt{n}) \\
&= \lim_{n\to\infty} \left(\frac{(n-\sqrt{n})(n+\sqrt{n})}{n+\sqrt{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{n^2 - n}{n+\sqrt{n}} \right) \\
&= \lim_{n\to\infty} \left(\frac{n(n-1)}{n(1+\frac{1}{\sqrt{n}})} \right) \\
&= \lim_{n\to\infty} \left(\frac{n-1}{1+\frac{1}{\sqrt{n}}} \right) \\
&= \frac{\lim_{n\to\infty}(n-1)}{\lim_{n\to\infty}(1+\frac{1}{\sqrt{n}})} \\
&= \frac{\lim_{n\to\infty} n - \lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{\sqrt{n}}} \\
&= \frac{\lim_{n\to\infty} n - 1}{1 + \lim_{n\to\infty} \frac{1}{\sqrt{n}}} \\
&= \frac{\lim_{n\to\infty} n - 1}{1+0} \\
&= \lim_{n\to\infty} n - 1 \\
&= \infty
\end{aligned}
$$

So the sequence diverges to ∞.

5.4 d)

$$d_n := \frac{2^n}{n^{100}}$$

By limit laws and standard limits, we have

$$
\begin{aligned}
\lim_{n\to\infty} d_n &= \lim_{n\to\infty} (\frac{2^n}{n^{100}}) \\
&= \lim_{n\to\infty} \left(\frac{2^n}{n^{100}} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{\frac{n^{100}}{2^n}} \right) \\
&= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{n^{100}}{2^n}} \\
&= \frac{1}{\lim_{n\to\infty} \frac{n^{100}}{2^n}} \\
&= \infty
\end{aligned}
$$

So the sequence diverges to ∞.

5.5 e)

$$e_n := \sqrt{n^2+n} - n$$

By limit laws and standard limits, we have

$$\begin{aligned}
\lim_{n\to\infty} e_n &= \lim_{n\to\infty} \sqrt{n^2+n} - n \\
&= \lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} \\
&= \lim_{n\to\infty} \frac{1}{\frac{\sqrt{n^2+n}}{n} + 1} \\
&= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n} + \lim_{n\to\infty} 1} \\
&= \frac{1}{\lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n} + 1} \\
&= \frac{1}{\lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2}} + 1} \\
&= \frac{1}{\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} + 1} \\
&= \frac{1}{\sqrt{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} + 1} \\
&= \frac{1}{\sqrt{1+0} + 1} \\
&= \frac{1}{2}
\end{aligned}$$

So the sequence converges to $\frac{1}{2}$.

5.6 f)

$$f_n := \sqrt[n]{3n^2}$$

By limit laws and standard limits, we have

$$\begin{aligned}\lim_{n\to\infty} f_n &= \lim_{n\to\infty} \sqrt[n]{3n^2} \\ &= \lim_{n\to\infty} \sqrt[n]{3}\sqrt[n]{n^2} \\ &= \lim_{n\to\infty} \sqrt[n]{3}(\sqrt[n]{n})^2 \\ &= \lim_{n\to\infty} \sqrt[n]{3}(\lim_{n\to\infty} \sqrt[n]{n})^2 \\ &= 1 \cdot 1^2 \\ &= 1\end{aligned}$$

So the sequence converges to 1.

5.7 g)

$$g_n := \frac{2^n + 5n^200}{3^n + n^10}$$

By limit laws and standard limits, we have

$$\begin{aligned}\lim_{n\to\infty} g_n &= \lim_{n\to\infty} \frac{2^n + 5n^{200}}{3^n + n^{10}} \\ &= \lim_{n\to\infty} \frac{\frac{2^n}{3^n} + \frac{5n^{200}}{3^n}}{1 + \frac{n^{10}}{3^n}} \\ &= \frac{\lim_{n\to\infty}(\frac{2^n}{3^n} + \frac{5n^{200}}{3^n})}{\lim_{n\to\infty}(1 + \frac{n^{10}}{3^n})} \\ &= \frac{\lim_{n\to\infty}(\frac{2}{3})^n + 5\lim_{n\to\infty}\frac{n^{200}}{3^n}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{n^{10}}{3^n}} \\ &= \frac{0 + 5 \cdot 0}{1 + 0} \\ &= 0\end{aligned}$$

So the sequence converges to 0.

5.8 h)

$$h_n := (-1)^n 3^n$$

Since the sequence 3^n diverges to ∞ and $(-1)^n$ osilates between -1 and 1, the sequence h_n doesn't converge, but neither diverges to ∞ or to $-\infty$.

5.9 i)

$$i_n := \sqrt[n]{5^n + n^2}$$

By limit laws and standard limits, we have

$$
\begin{aligned}
\lim_{n\to\infty} i_n &= \lim_{n\to\infty} \sqrt[n]{5^n + n^2} \\
&= \lim_{n\to\infty} \left(5^n + \left(1 + \frac{n^2}{5^n}\right)\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \exp\left(\ln\left(5^n + \left(1 + \frac{n^2}{5^n}\right)\right)^{\frac{1}{n}}\right) \\
&= \lim_{n\to\infty} \exp\left(\frac{1}{n}\ln\left(5^n + \frac{n^2}{5^n}\right)\right) \\
&= \exp\left(\lim_{n\to\infty} \frac{\ln\left(5^n + \frac{n^2}{5^n}\right)}{n}\right) \\
&= \exp\left(\lim_{n\to\infty} \frac{\ln 5^n + \ln\left(1 + \frac{n^2}{5^n}\right)}{n}\right) \\
&= \exp\left(\lim_{n\to\infty} \frac{n\ln 5}{n} + \lim_{n\to\infty} \frac{\ln\left(1 + \frac{n^2}{5^n}\right)}{n}\right) \\
&= \exp\left(\ln 5 + \frac{\lim_{n\to\infty} \ln\left(1 + \frac{n^2}{5^n}\right)}{\lim_{n\to\infty} n}\right) \\
&= \exp\left(\ln 5 + \frac{\ln(\lim_{n\to\infty} \left(1 + \frac{n^2}{5^n}\right))}{\lim_{n\to\infty} n}\right) \\
&= \exp\left(\ln 5 + \frac{\ln\left(\lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{n^2}{5^n}\right)\right)}{\lim_{n\to\infty} n}\right) \\
&= \exp\left(\ln 5 + \frac{\ln 1}{\lim_{n\to\infty} n}\right) \\
&= \exp\left(\ln 5 + 0\right) \\
&= 5
\end{aligned}
$$

So the sequence converges to 5.