2IT80 Discrete Structures

2023-24 Q2

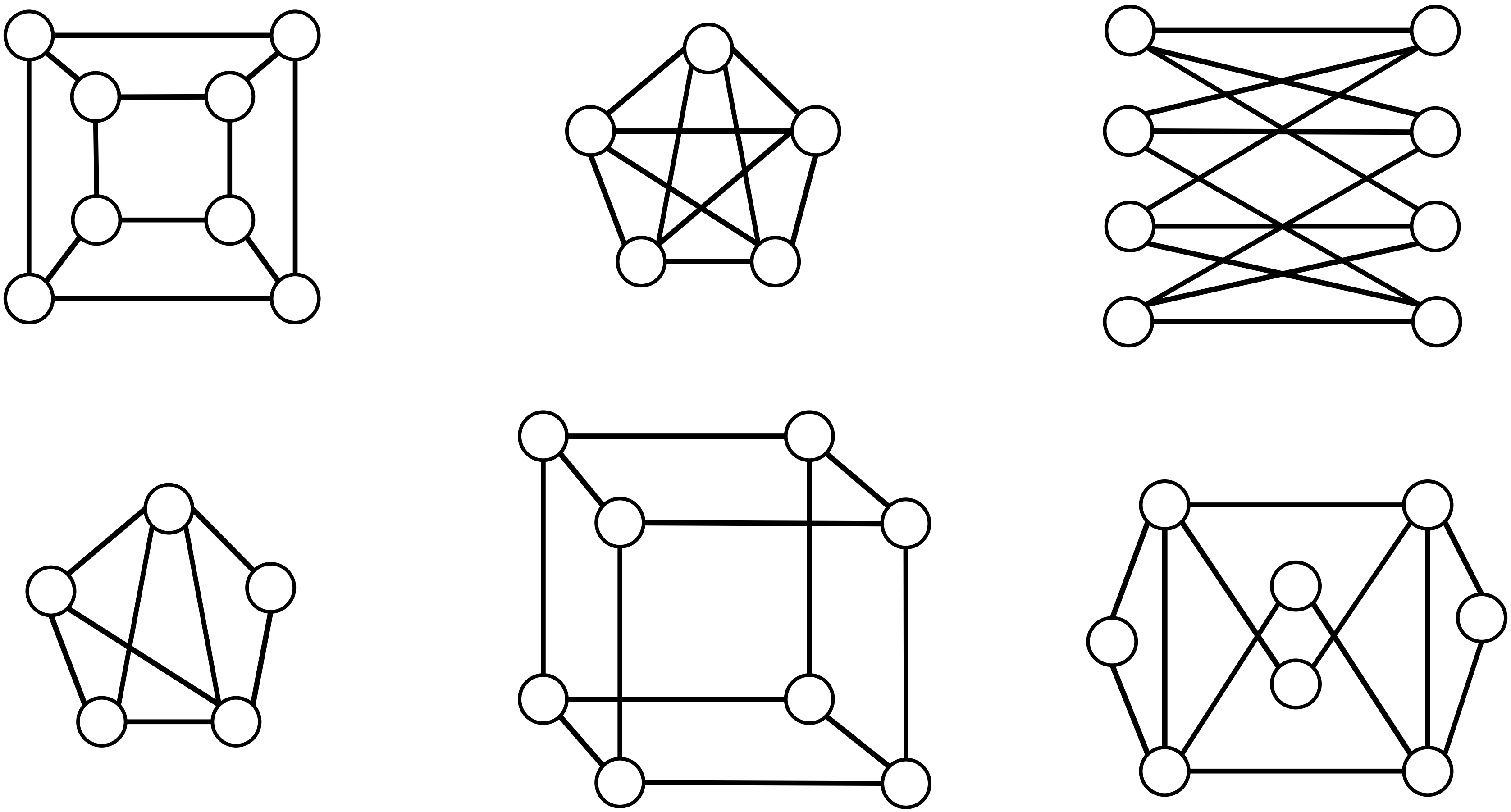
Lecture 11: Planar Graphs

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Graph drawing

A graph defines only connectivity, but we can draw it.

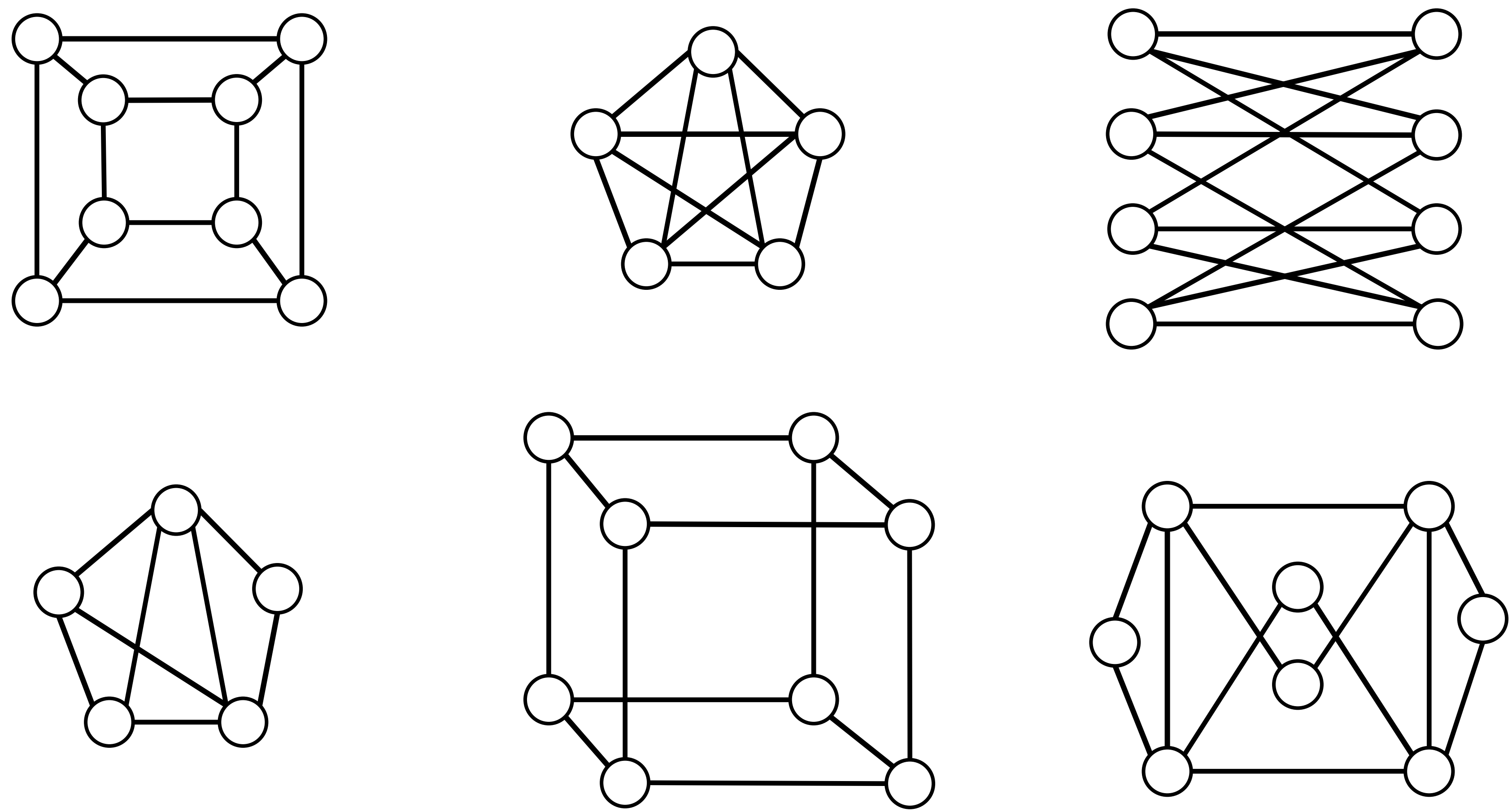
Planar graphs can be drawn in the plane without edge crossings.



Graph drawing

Differently drawn graphs may actually have the same structure.

What precisely is a drawing of a graph?

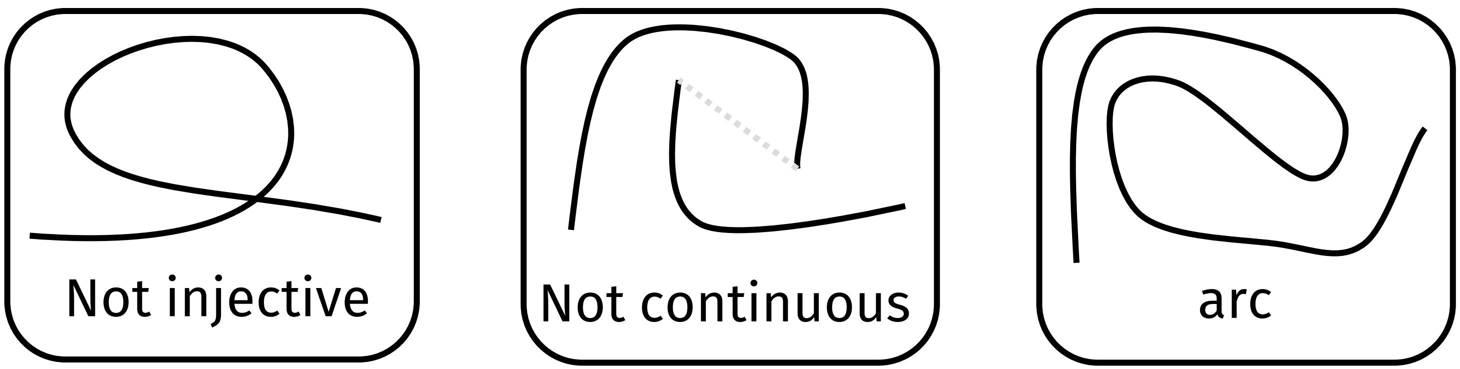


Arc

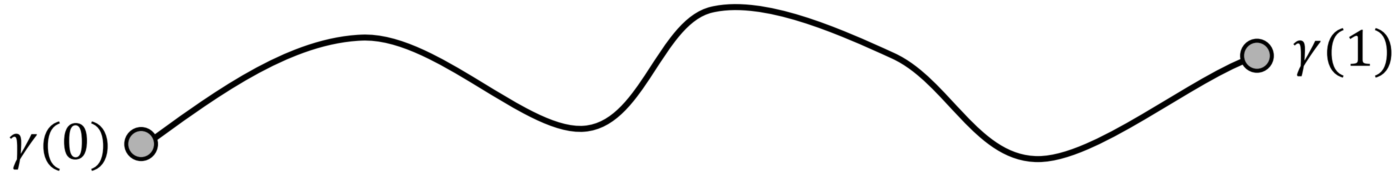
An arc is a subset α of the plane

$\alpha = \gamma([0,1]) = \{\gamma(x) : x \in [0,1]\}$

where $\gamma : [0,1] \to \mathbb{R}^2$ is an injective continuous map



$\gamma(0)$ and $\gamma(1)$ are called the endpoints of the arc α



Drawing

A drawing of a graph $G = (V, E)$ is an assignment:

- to every vertex $v \in V$ assign a point $b(v)$ of the plane
- to every edge $e = \{v, v'\} \in E$, assign an arc $\alpha(e)$ in the plane with endpoints $b(v)$ and $b(v')$

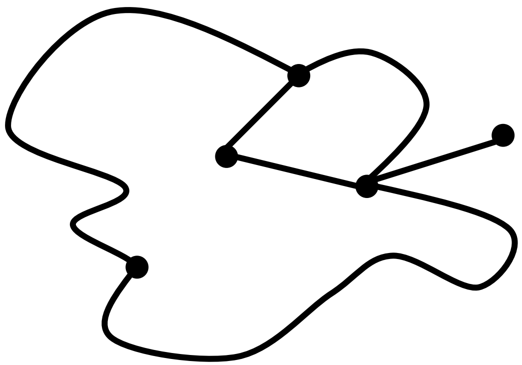
Such that

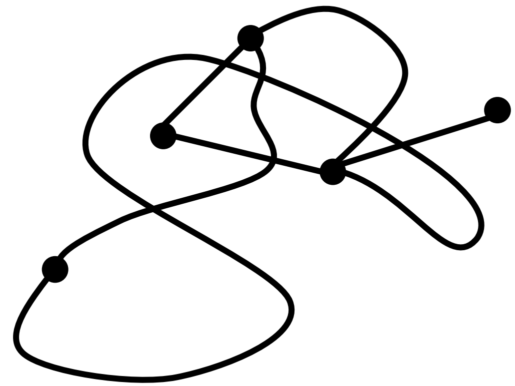
- the mapping b is injective
  (*different vertices are assigned distinct points in the plane*)
- no point $b(v)$ lies on any of the arcs $\alpha(e)$ unless it is an endpoint of that arc.

Topological graph: graph together with a drawing

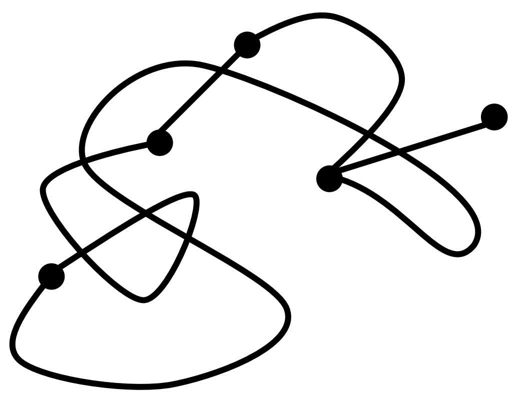
Examples

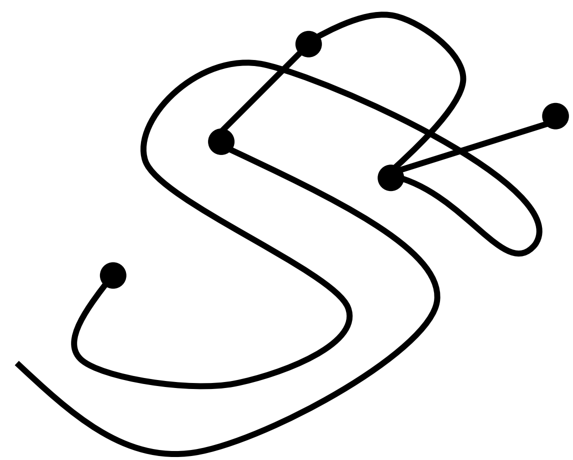
Drawings of a graph:



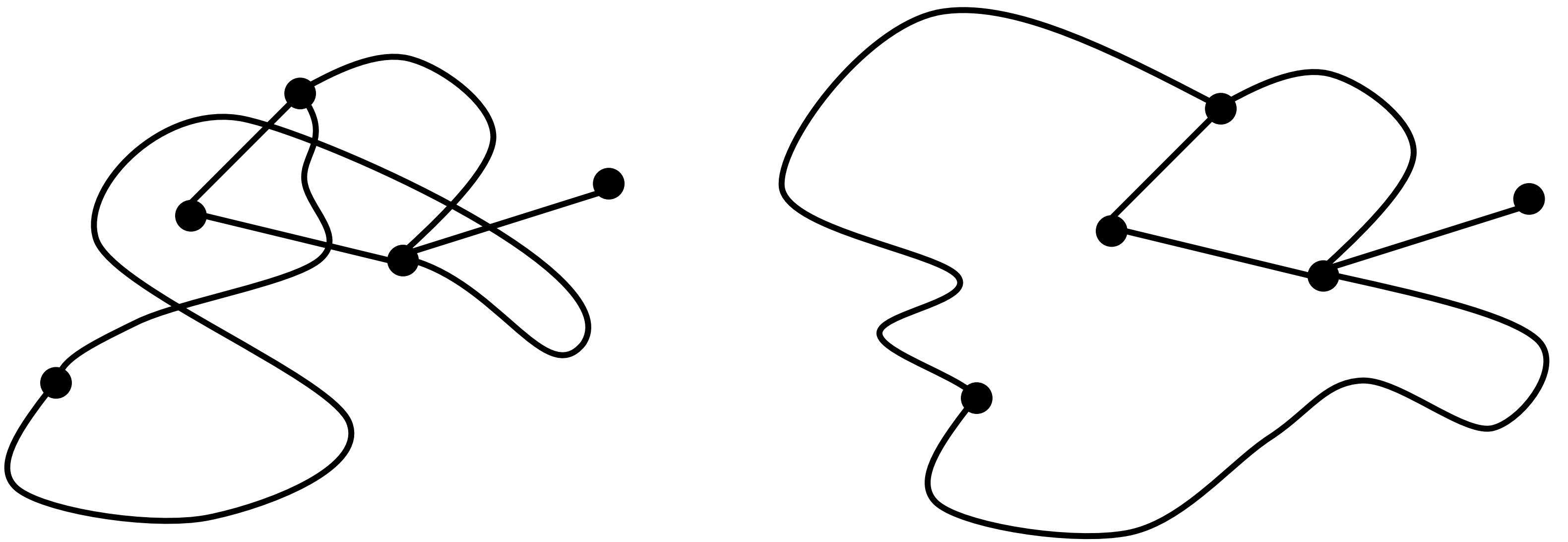


NOT drawings of a graph:





Topological graphs



Note: these graphs are isomorphic,
but they are different topological graphs

Planarity

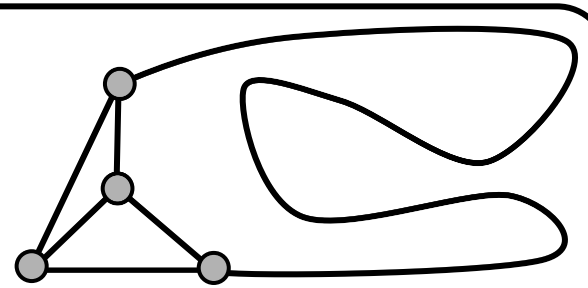
A drawing is planar if any two arcs corresponding to distinct edges either have no intersection or only share an endpoint.

A graph is planar if it has a planar drawing.

Example: K_4

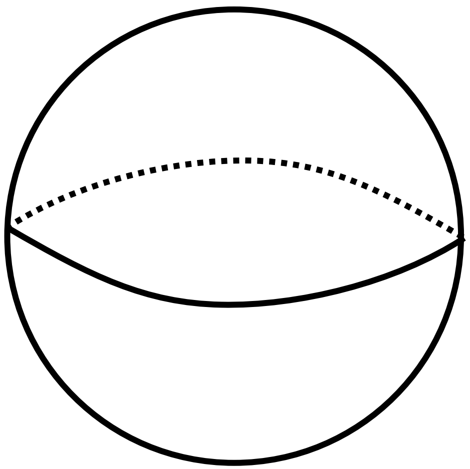
A topological planar graph or plane graph is a graph together with a planar drawing.

Example:

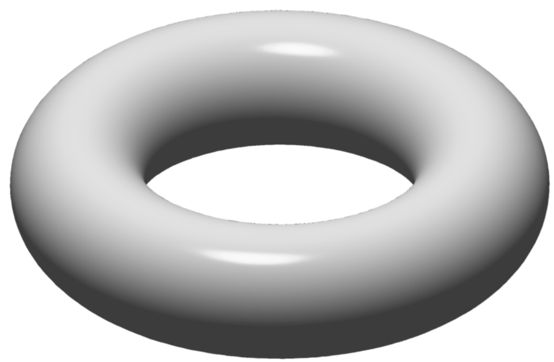


Drawing on surfaces

We can also draw graphs on other surfaces:

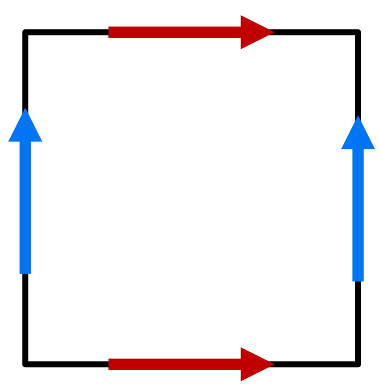


sphere



torus

=

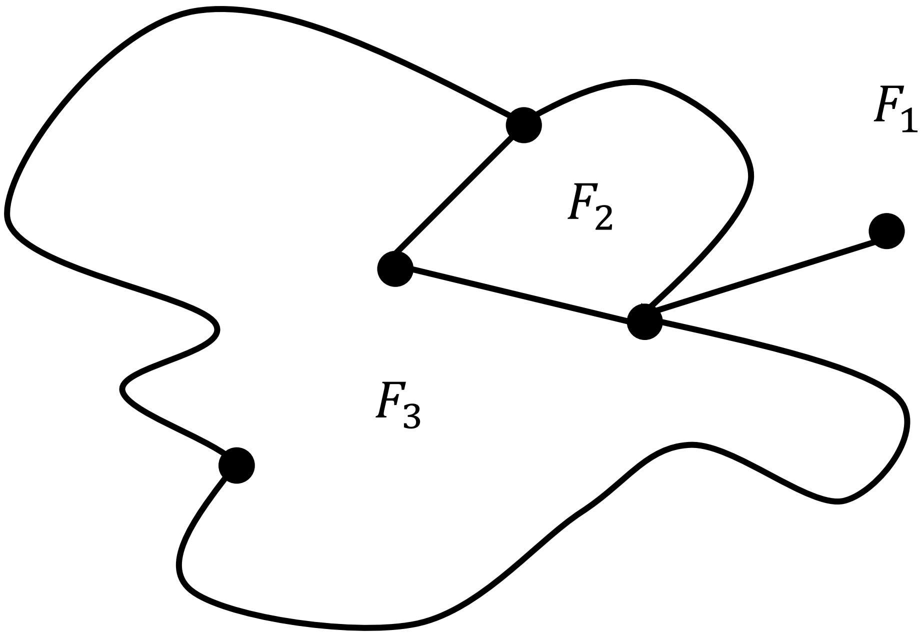


glue left/right and
top/bottom side

double torus

Planar Drawings and Faces

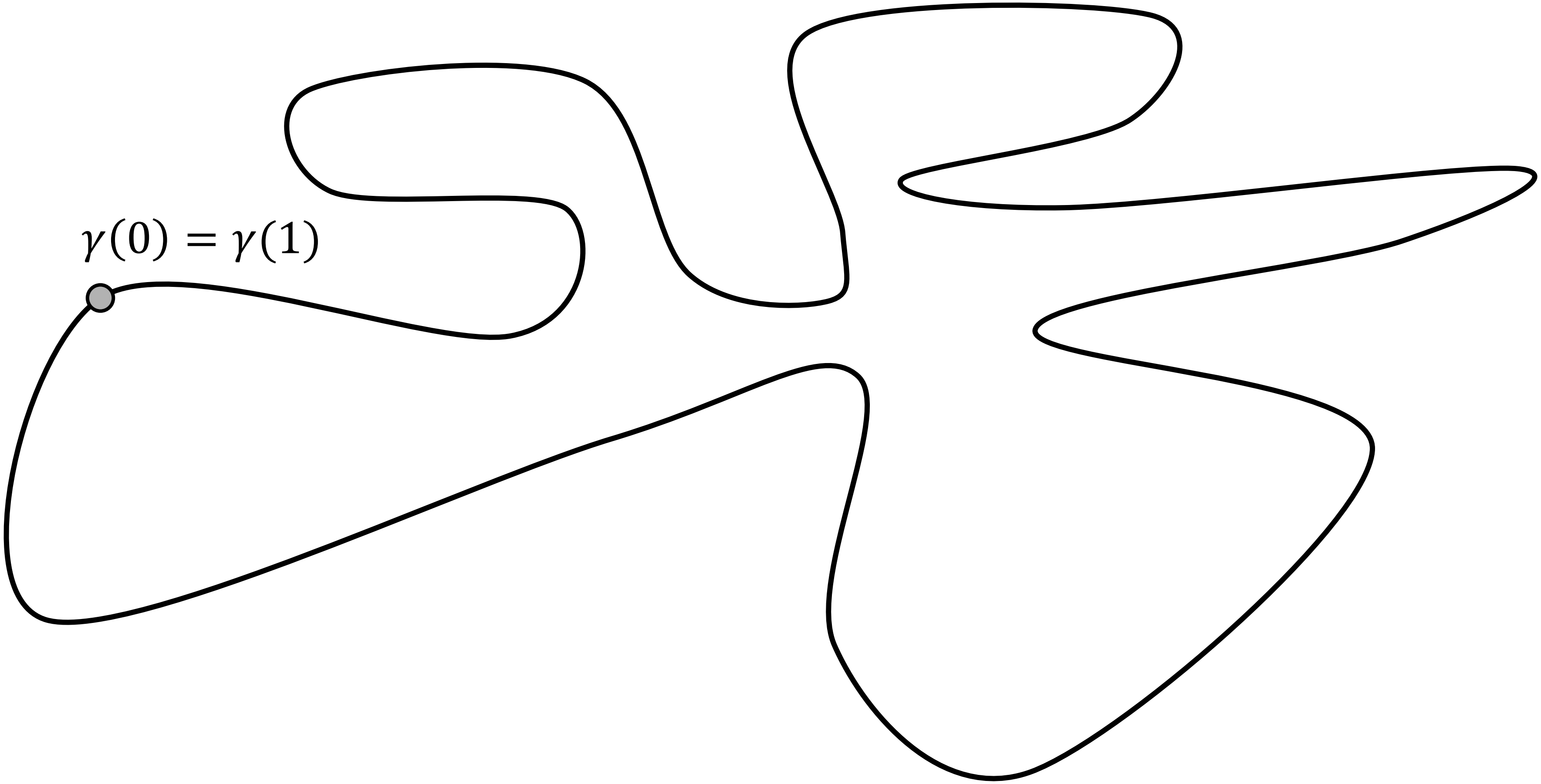
A drawing partitions the plane into finitely many connected regions: the faces of the drawing



The outer face (F_1) stretches out to infinity
all other faces are inner faces (F_2, F_3).

The Jordan Curve Theorem

A Jordan curve is an arc whose endpoints coincide.
That is: the image of a continuous mapping $\gamma : [0,1] \to \mathbb{R}^2$ that is injective except for $\gamma(0) = \gamma(1)$.



The Jordan Curve Theorem

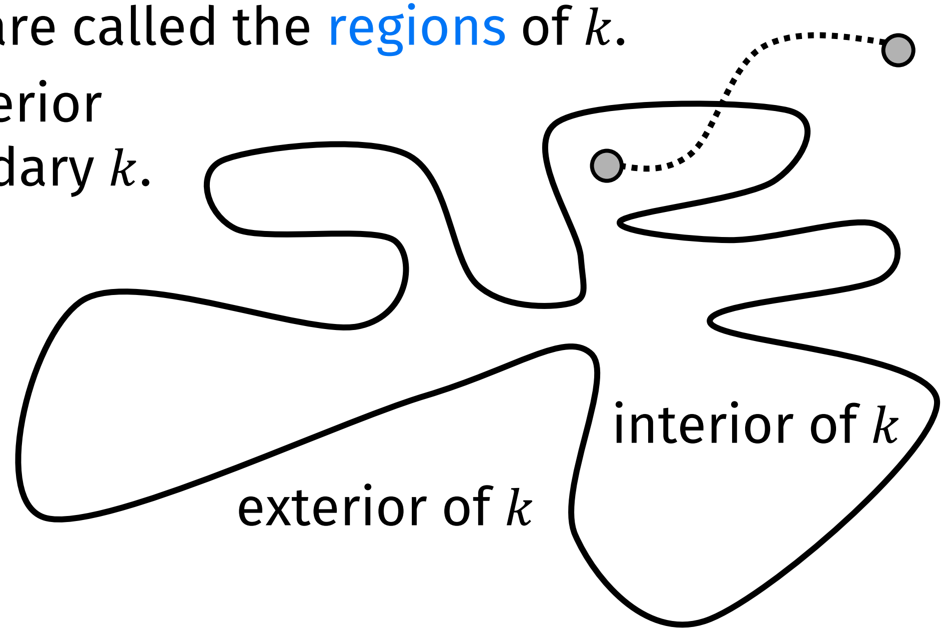
A Jordan curve is an arc whose endpoints coincide.
That is: the image of a continuous mapping $\gamma : [0,1] \to \mathbb{R}^2$ that is injective except for $\gamma(0) = \gamma(1)$.

Theorem: Any Jordan curve k divides the plane into exactly two connected parts, the “interior” and the “exterior” of k, and k is the boundary of both the interior and the exterior.

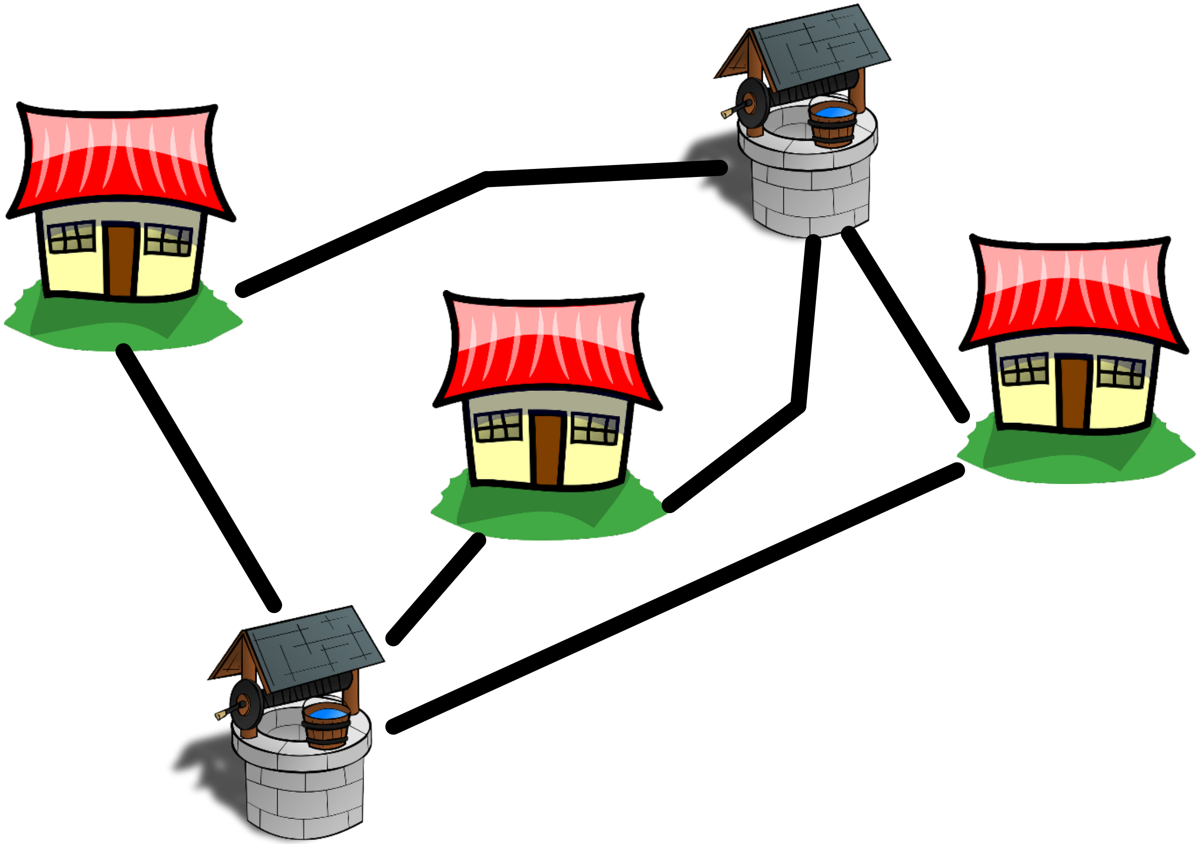
The interior and the exterior of k are called the regions of k.

Any arc between a point in the interior and the exterior crosses the boundary k.

The proof is beyond the scope of this course.

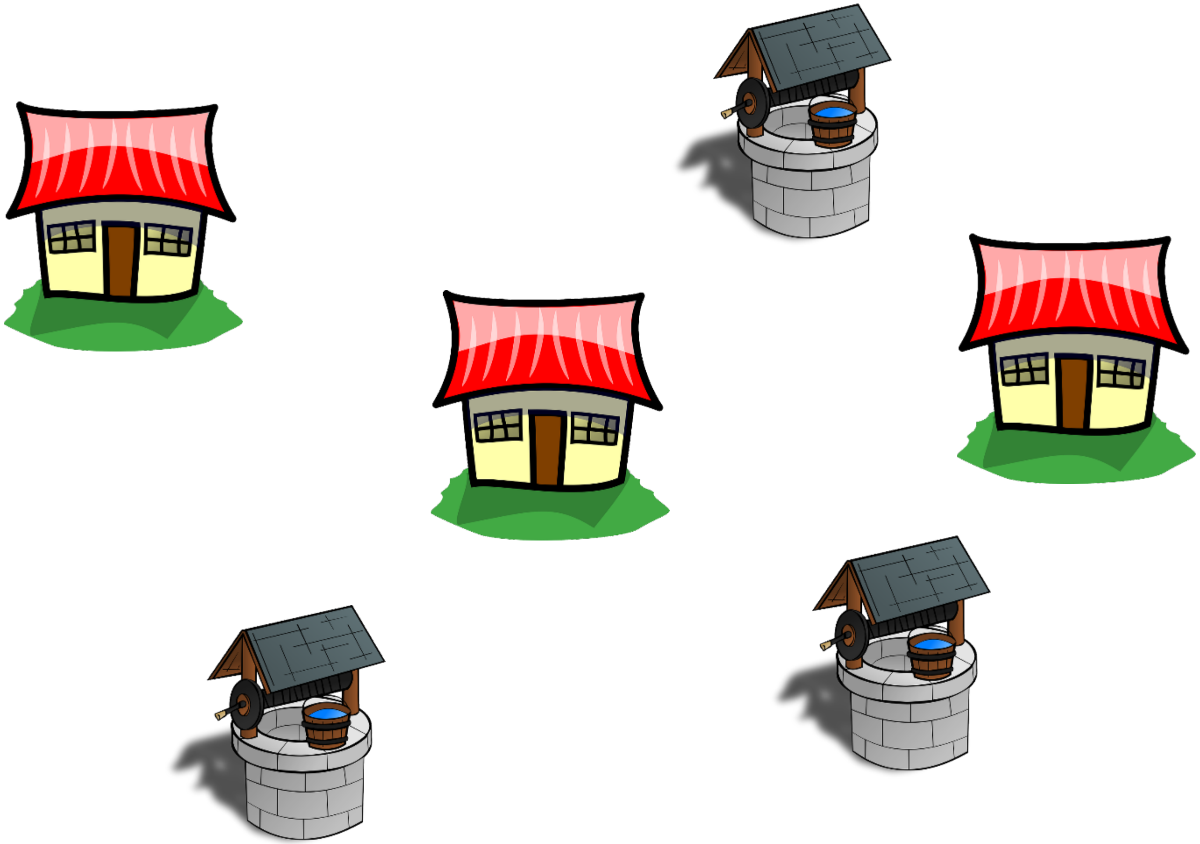


Which graphs are planar?



Is it possible to draw a path from each house to each well without any pair of paths intersecting?

Which graphs are planar?

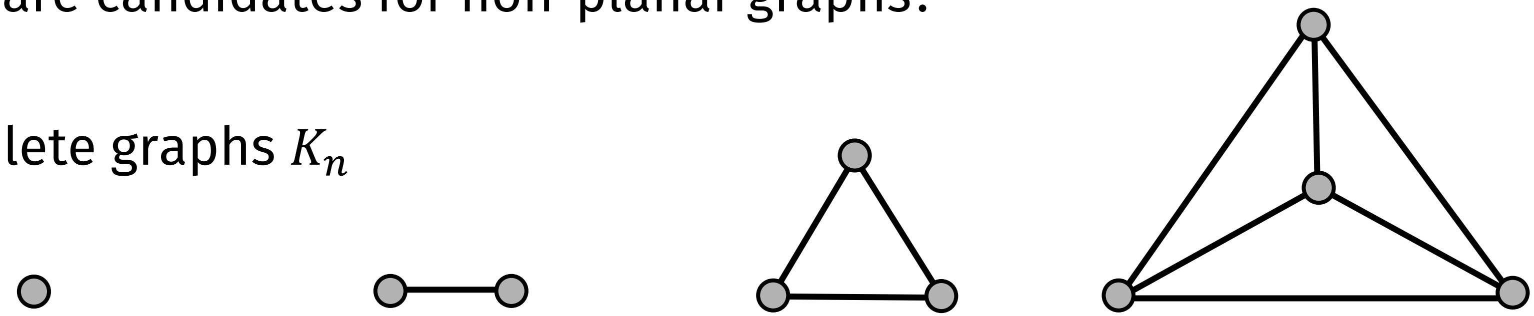


Is it possible to draw a path from each house to each well without any pair of paths intersecting?

Which graphs are planar?

What are candidates for non-planar graphs?

Complete graphs K_n



K_1 K_2 K_3 K_4

Proposition: K_5 is not planar.

A non-planar graph

Proposition: K_5 is not planar.

Proof: We use a proof by contradiction.

Assume that K_5 is planar. Then it has a planar drawing.

Let b_1, b_2, b_3, b_4, b_5 be the points corresponding to the vertices of K_5.

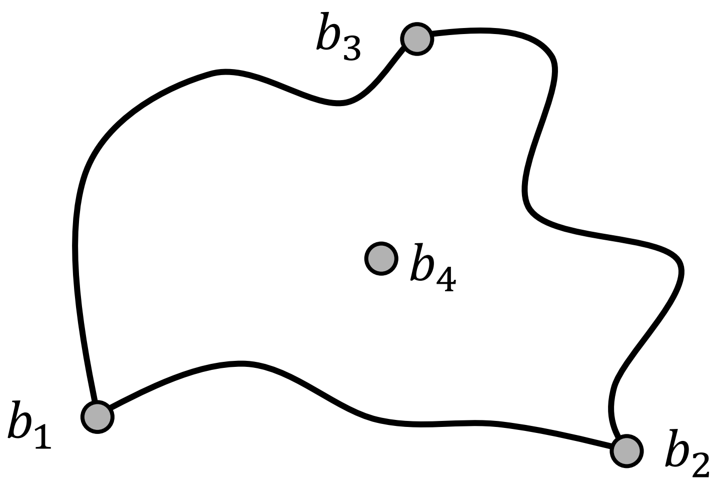
Denote the arc connecting the points b_i and b_j by $\alpha(i,j)$.

b_1, b_2, b_3 are vertices of a cycle of the graph K_5

➡ the arcs $\alpha(1,2)$, $\alpha(2,3)$, and $\alpha(3,1)$ form a Jordan curve k

b_4 and b_5 either lie both inside or both outside k
(*otherwise* $\alpha(4,5)$ *would cross* k)

Case 1: b_4 lies inside k.



A non-planar graph

Proposition: K_5 is not planar.

Proof: (continued)

Case 1: b_4 lies inside k.

➡ b_5 lies inside one of the Jordan curves formed by

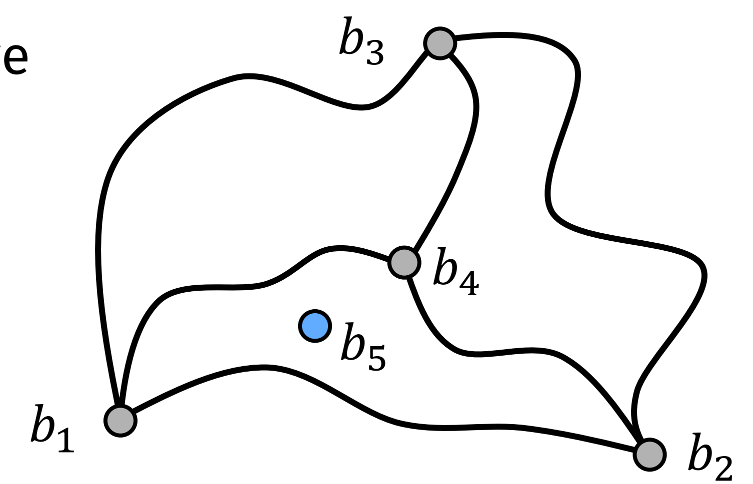
- $\alpha(1,4)$, $\alpha(2,4)$, $\alpha(1,2)$, or
- $\alpha(2,3)$, $\alpha(3,4)$, $\alpha(2,4)$, or
- $\alpha(1,3)$, $\alpha(3,4)$, $\alpha(1,4)$.

In the first case:

$\alpha(3,5)$ has to intersect the Jordan curve formed by arcs $\alpha(1,4)$, $\alpha(2,4)$, $\alpha(1,2)$.

Other cases work similarly.

If b_4 and b_5 lie both outside k, we proceed analogously. ∎

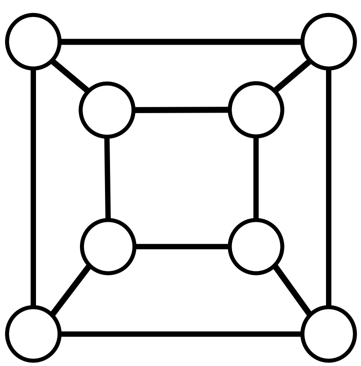


A non-planar graph

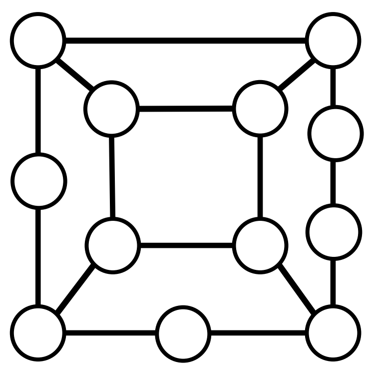
Proposition: $K_{3,3}$ is not planar (proof is similar as for K_5)

Observation:
A graph G is planar if and only if each subdivision of G is planar.

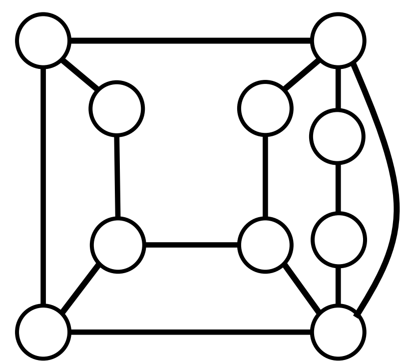
Subdivision of a graph
Graph G' is a subdivision of G if and only if G' can be obtained by repeatedly subdivision edges starting from G.



G



Subdivision of G



Not a subdivision of G

Kuratowski's theorem

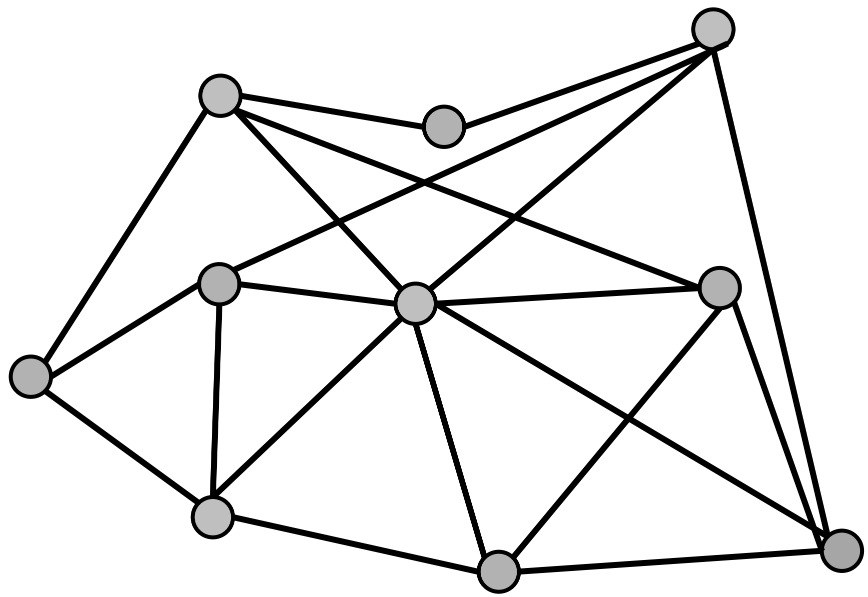
Theorem:

A graph G is planar if and only if it has no subgraph isomorphic to a subdivision of $K_{3,3}$ or to a subdivision of K_5.

One direction is easy to prove, the other one...
would take the rest of today and the next lecture.

Example

This graph is not planar, find a subdivision of K_5 or $K_{3,3}$



Example

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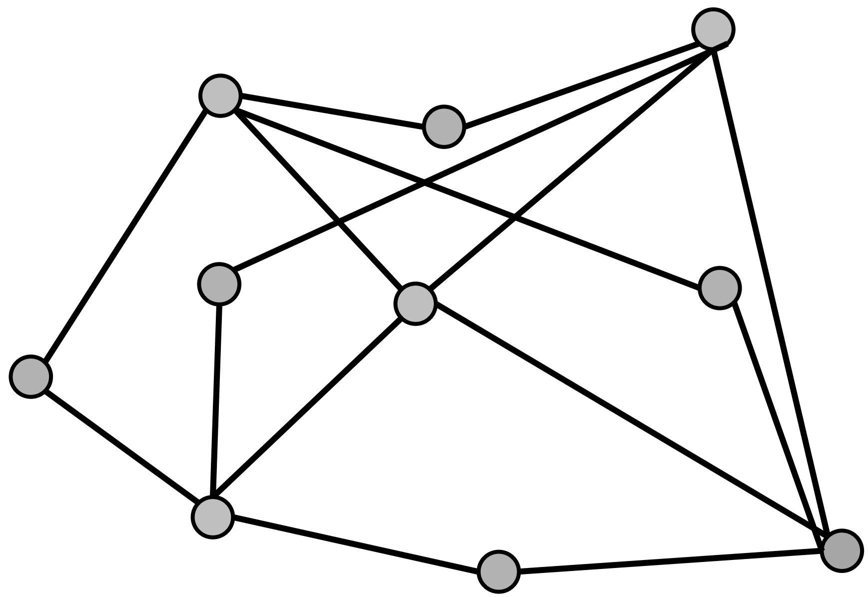
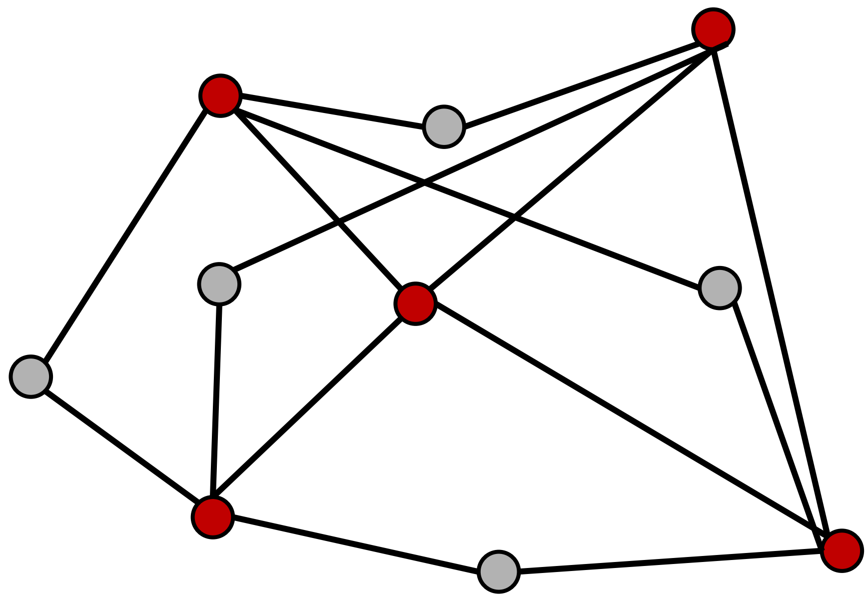


Example

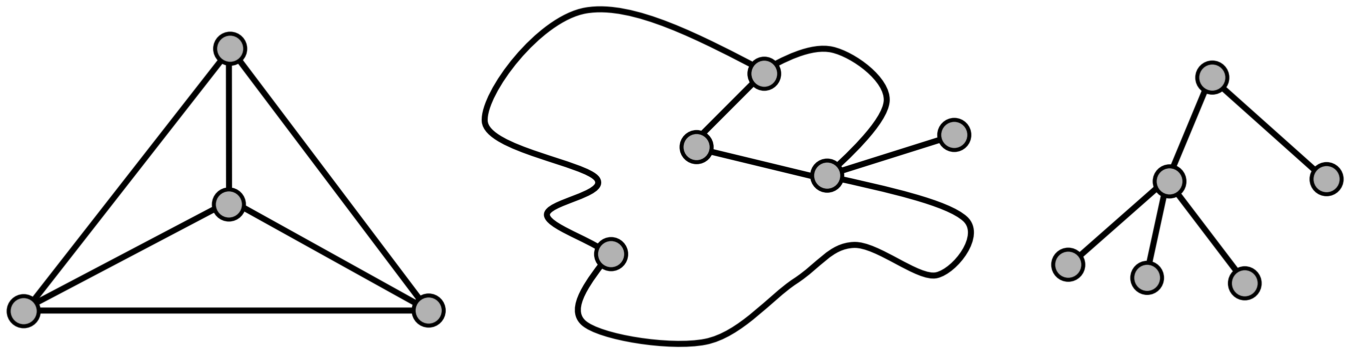
This graph is not planar, find a subdivision of K_5 or $K_{3,3}$



Basic Properties of Planar Graphs

Every planar graph has...

Euler's formula



#vertices:	4	5	6
#edges:	6	6	5
#faces:	4	3	1

Can you spot the pattern?

#vertices - #edges + #faces = ???

Euler's formula

Theorem (Euler's formula)
Let $G = (V, E)$ be a connected planar graph, and let f be the number of faces of some planar drawing of G. Then we have

$$|V| - |E| + f = 2.$$

The number of faces does not depend on the particular drawing.

Proof by induction in the book

The single most important formula for planar graphs.

The density of planar graphs

How many edges can a planar graph with n vertices have?

Not too many.
When adding more and more edges, any graph necessarily becomes non-planar.

Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices.
Then $|E| \leq 3|V| - 6$.

The density of planar graphs

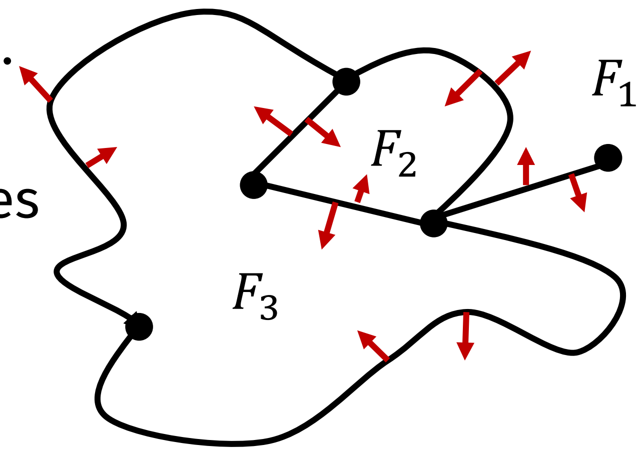
Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then $|E| \leq 3|V| - 6$.

Proof: W.l.o.g. assume G is connected.

Consider a planar drawing of G with k faces.

Denote the faces by $F_1, \dots, F_k$.

Denote by f_i the number of edge adjacencies that face F_i has.



We have $\sum_{i=1}^{k} f_i = 2 \cdot |E|$.

Observe that $f_i \geq 3$ since we do not allow parallel edges.

Therefore $3k \leq 2|E|$.

Euler's formula: $|V| - |E| + k = 2$.

The density of planar graphs

Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then $|E| \leq 3|V| - 6$.

Proof: (continued)

$$3k \leq 2|E| \Leftrightarrow k \leq \frac{2}{3}|E|$$

$$\text{Euler } \Rightarrow |V| - |E| + k = 2 \Rightarrow |V| - |E| + \frac{2}{3}|E| \geq 2$$

$$\Leftrightarrow |V| - \frac{1}{3}|E| \geq 2$$

$$\Leftrightarrow \frac{1}{3}|E| \leq |V| - 2$$

$$\Leftrightarrow |E| \leq 3|V| - 6 \qquad \blacksquare$$

Another proof that K_5 is not planar

Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices.
Then $|E| \leq 3|V| - 6$.

K_5 has 5 vertices

K_5 has $\frac{4 \cdot 5}{2} = 10$ edges.

A planar graph on 5 vertices has at most $3 \cdot 5 \ - 6 = 9$ edges.

Hence K_5 cannot be planar.

More consequences

Theorem: Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then $|E| \leq 3|V| - 6$.

Theorem: Every planar graph contains a vertex of degree at most 5.

Proof: Let $G = (V, E)$ be a planar graph.

If $|V| \leq 2$, then the claim holds. So assume $|V| \geq 3$.

By double counting the edges and theorem above we have

$$\sum_{v \in V} \deg(v) = 2|E| \leq 6|V| - 12.$$

the sum of all degrees is less than $6|V|$

at least one vertex must have degree less than 6 ■

Coloring Maps and Planar Graphs

Coloring maps

How many colors do we need to color a map so that no two adjacent countries get the same color?

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How many colors do we need to color a map so that no two adjacent countries get the same color?

In this case four colors suffice.

What is the relation to planar graphs?

Coloring maps

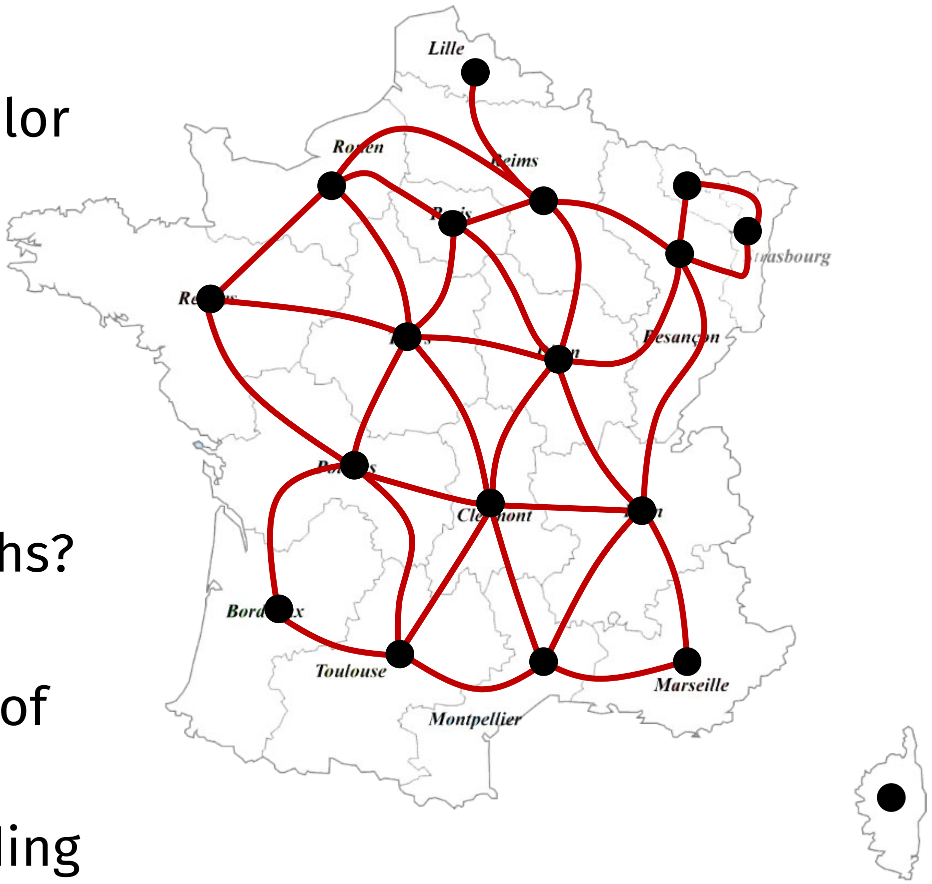
How many colors do we need to color a map so that no two adjacent countries get the same color?

In this case four colors suffice.

What is the relation to planar graphs?

Construct a conflict graph G by

- placing a vertex in each region of the map
- connecting vertices corresponding to adjacent regions.



Coloring maps

How many colors do we need to color a map so that no two adjacent countries get the same color?

In this case four colors suffice.

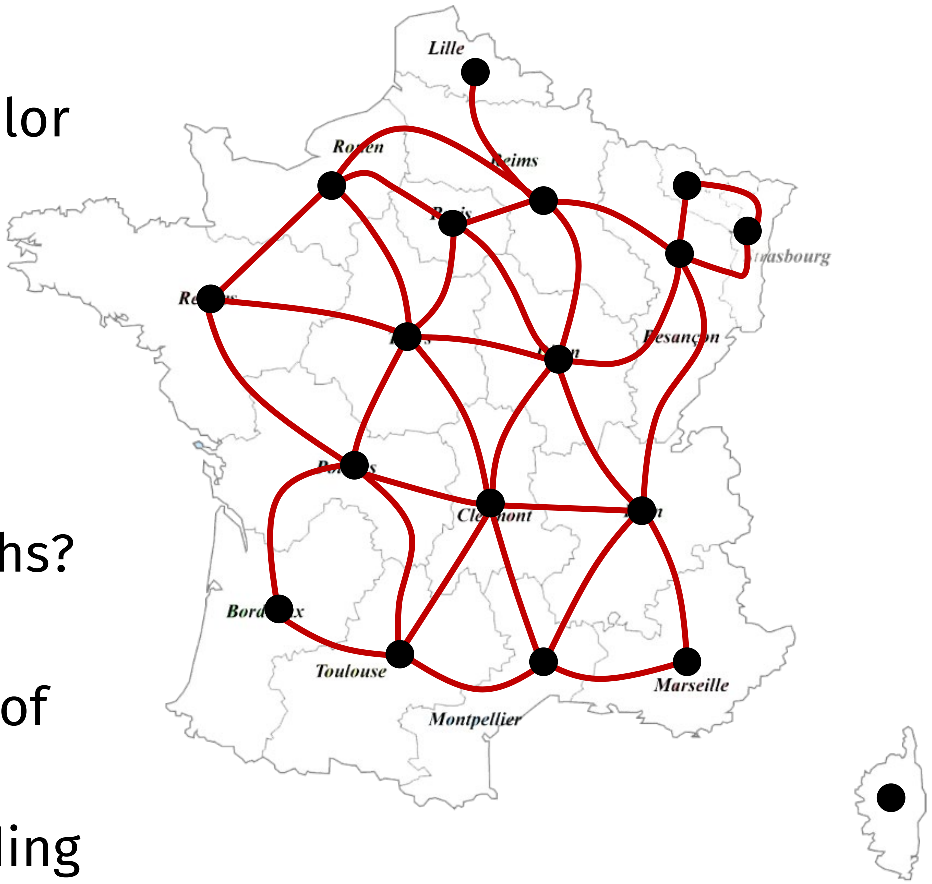
What is the relation to planar graphs?

Construct a conflict graph G by

placing a vertex in each region of the map

connecting vertices corresponding to adjacent regions.

The dual graph G is planar.

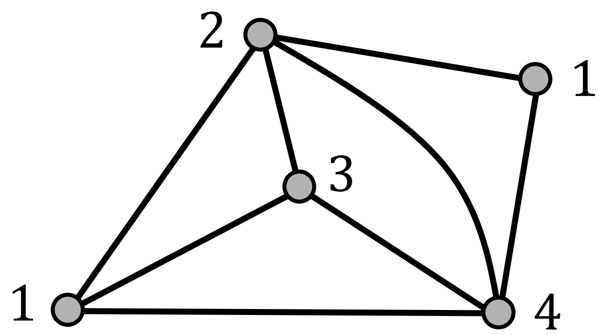


Coloring the map ≈ coloring the vertices of G such that adjacent vertices receive different colors.

Chromatic number

Let $G = (V, E)$ be a graph, and let k be a natural number.

A mapping $c : V \to \{1, 2, \dots, k\}$ is called a coloring of the graph G if $c(x) \neq c(y)$ of every edge $\{x, y\} \in E$.



The chromatic number of G, denoted by $\chi(G)$, is the smallest k such that there exists a coloring $c : V \to \{1, 2, \dots, k\}$.

Warm up: Other graphs

The chromatic number of G, denoted by $\chi(G)$, is the smallest k such that there exists a coloring $c : V \to \{1,2, \dots, k\}$.

What is the chromatic number of

- ☐ The complete graph K_8
- ☐ The complete bipartite graph $K_{n,m}$
- ☐ A cycle C_n
- ☐ A tree with n vertices

Onward to planar graphs

Coloring planar graphs

Proposition: Any planar graph satisfies $\chi(G) \leq 6$.

Proof: We use induction on the number of vertices of $G = (V, E)$.

Base: For $|V| \leq 6$ the claim holds trivially.

Coloring planar graphs

Proposition: Any planar graph satisfies $\chi(G) \leq 6$.

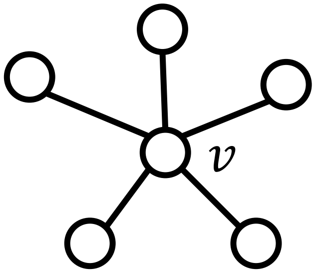
Proof: (continued)

Step: Assume $|V| \geq 7$.

IH: Planar graphs with fewer than |V| vertices have $\chi(G) \leq 6$

G contains a vertex v of degree at most 5.

Consider $G' = G - v$, apply the inductive hypothesis
➡ coloring c' of G' using at most 6 colors.



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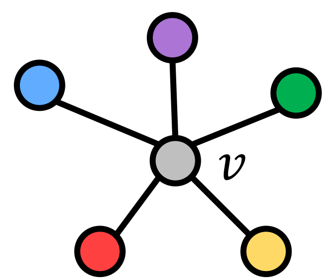

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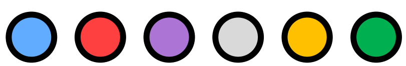

since v has at most degree 5, there is a color
$C \in \{1,2,\ldots,6\}$ that is not used by any neighbor of v.

define coloring c of G by $c(v) = C$
and $c(v') = c'(v')$ for $v' \in V \setminus \{v\}$.

Coloring planar graphs

Proposition: Any planar graph satisfies $\chi(G) \leq 6$.

Proof: (continued)

Step: Assume $|V| \geq 7$.

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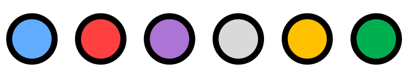

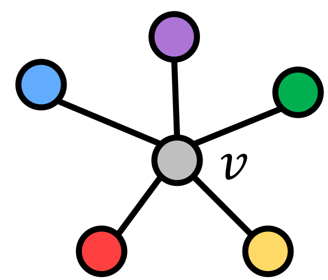

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$C \in \{1,2, \dots, 6\}$ that is not used by any neighbor of v.

define coloring c of G by $c(v) = C$
and $c(v') = c'(v')$ for $v' \in V \setminus \{v\}$. ■

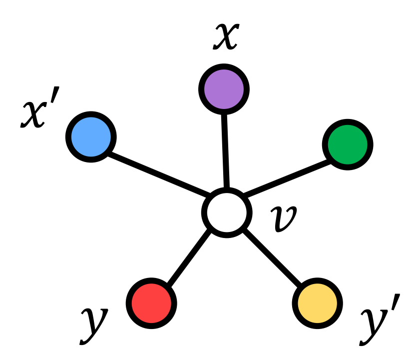
Where does this approach fail when we try to use only 5 colors?

Coloring planar graphs

Proposition: Any planar graph satisfies $\chi(G) \leq 5$.

Proof: (sketch) Use the same proof as before.
The only case where it fails is when the removed vertex v

1. has degree 5, and
2. the coloring c' uses all five colors for the neighbors of v.



W.l.o.g. coloring as in illustration
(we can arbitrarily exchange the colors)

Let $V_{x,y}$ be the vertices of $G - v$ having color $c'(x)$ or $c'(y)$.

Distinguish cases based on whether there exists a path from x to y in $G - v$ using only vertices of $V_{x,y}$.

Coloring planar graphs

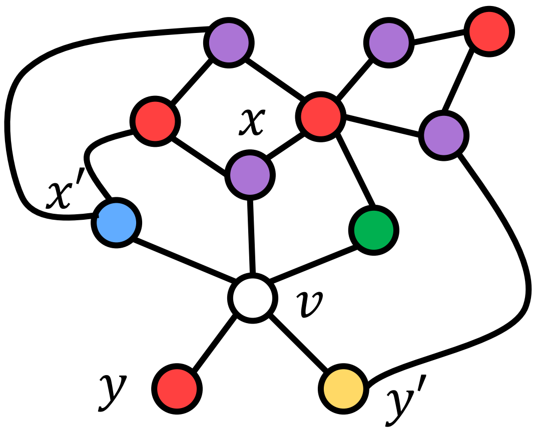
Proposition: Any planar graph satisfies $\chi(G) \leq 5$.

Proof: (continued)

Let $V_{x,y}$ be the vertices of $G - v$ having color $c'(x)$ or $c'(y)$.

Distinguish cases based on whether there exists a path from x to y in $G - v$ using only vertices of $V_{x,y}$.

1. no such path exists



Let $V_{x,y}'$ be the set of vertices $s \in V(G - v)$ that can be reached from x by a path using only the vertices from $V_{x,y}$. We have $y \notin V_{x,y}'$.

Define a new coloring c by

$$c(s) = \begin{cases} c'(s) & \text{if } s \notin V_{x,y}' \\ c'(y) & \text{if } s \in V_{x,y}' \text{and } c'(s) = c'(x) \\ c'(x) & \text{if } s \in V_{x,y}' \text{and } c'(s) = c'(y) \end{cases}$$

Coloring planar graphs

Proposition: Any planar graph satisfies $\chi(G) \leq 5$.

Proof: (continued)

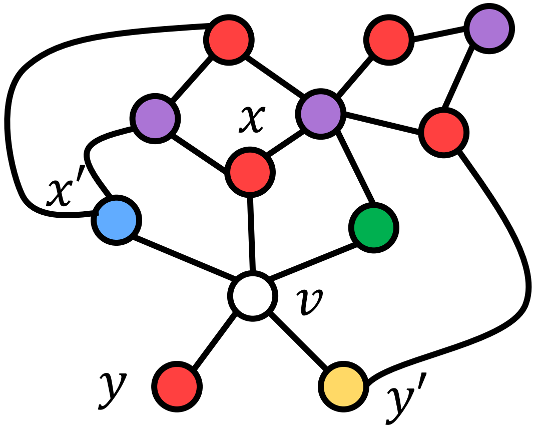
Let $V_{x,y}$ be the vertices of $G - v$ having color $c'(x)$ or $c'(y)$.

Distinguish cases based on whether there exists a path from x to y in $G - v$ using only vertices of $V_{x,y}$.

1. no such path exists

Let $V_{x,y}{}'$ be the set of vertices $s \in V(G - v)$ that can be reached from x by a path using only the vertices from $V_{x,y}$. We have $y \notin V_{x,y}{}'$.

Define a new coloring c by



$$c(s) = \begin{cases} c'(s) & \text{if } s \notin V_{x,y}{}' \\ c'(y) & \text{if } s \in V_{x,y}{}' \text{and } c'(s) = c'(x) \\ c'(x) & \text{if } s \in V_{x,y}{}' \text{and } c'(s) = c'(y) \end{cases}$$

This swaps colors in $V_{x,y}{}'$.

Now there is an unused color for v.

Coloring planar graphs

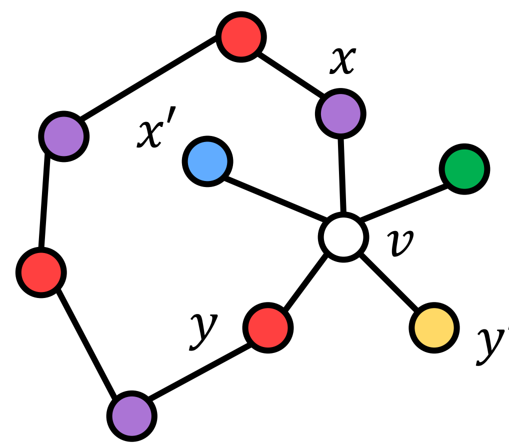
Proposition: Any planar graph satisfies $\chi(G) \leq 5$.

Proof: (continued)

Let $V_{x,y}$ be the vertices of $G - v$ having color $c'(x)$ or $c'(y)$.

Distinguish cases based on whether there exists a path from x to y in $G - v$ using only vertices of $V_{x,y}$.

2. there exist a path P from x to y with all vertices in $V_{x,y}$



consider vertices x', y' and the set $V_{x',y'}$ of vertices in $G - v$ with colors $c(x')$, $c(y')$

path P together with $\{x, v\}, \{y, v\}$ forms cycle

one of x', y' lies inside, the other outside

➡ there is no path from x' to y' using only colors from $V_{x',y'}$.

➡ can use Case 1 with x', y' instead of x, y ■

Even fewer colors?

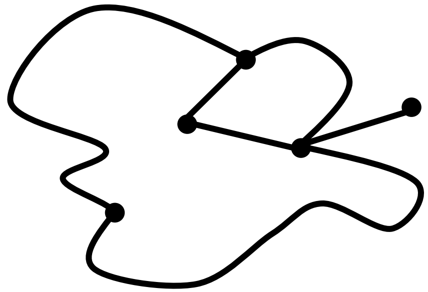
Can we use a similar trick to reduce to only four colors?

Theorem: Any planar graph satisfies $\chi(G) \leq 4$.

very complicated proof, even after significant simplification
requires computer assistance to check hundreds of cases
sparked huge discussion about what a (mathematical) proof is

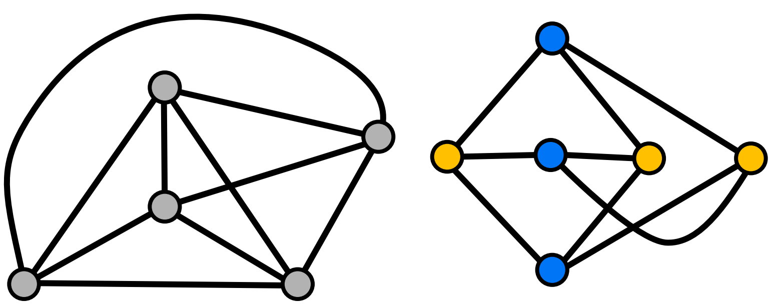
Summary

Planar graphs



Eulers formula: $|V| - |E| + f = 2$.

Kuratowski's theorem:



K_5 and $K_{3,3}$ are not planar

Chromatic number

Planar graphs can be colored with 4 colors.

Practical stuff

- Practice set
  - Ex. 3 together in discussion group today (with 2 that you prepared about directed graphs)
  - Prepare Ex. 4+5 for Thursday

- Last test next Monday
  - Assignment grade is $\frac{\#points\ for\ best\ 3}{4.5}$ rounded to 1 decimal
  - So #points for best 3 should be at least 24.525