# Linear Algebra 2

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December 26, 2023

## 1 From Linear Algebra 1

### 1.1 Field

**Definition 1.1.1** − **Field** A field is a set $\mathbb{K}$ that contains two special elements, 0 and 1, together with two binary operations, addition and multiplication $+, \cdot : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, such that addition and multiplication obey the following axioms:

1. $a + b = b + a$          (addition is commutative)

2. $(a + b) + c = a + (b + c)$          (addition is associative)

3. $a + 0 = a$          (0 is the additive identity)

4. $\forall a, \exists (-a)[a + (-a) = 0]$          (existence of an inverse for addition)
   In short, i-iv express that $(\mathbb{K}, +, 0)$ is an abelian group.

5. $a \cdot b = b \cdot a$ (or simply $ab$)          (multiplication is commutative)

6. $(ab)c = a(bc)$          (multiplication is associative)

7. $a \cdot 1 = a$          (1 is the multiplicative identity)

8. $\forall a \neq 0, \exists a^{-1}[aa^{-1} = 1]$          (existence of an inverse for multiplication)
   In short, v-viii express that $(\mathbb{K} \setminus \{0\}, \cdot, 1)$ is an abelian group.

9. $a(b + c) = ab + ac$          (multiplication is distributive over addition)

### 1.2 Vector Space and Subspace

**Definition 1.2.1** − **Vector space** Let $\mathbb{K}$ be a field. A $\mathbb{K}$-vector space is a set $V$ that contains a special element, $\underline{0}$, called the zero vector, together with two operations:

1. addition: $V \times V \to V$,

2. scalar multiplication: $\mathbb{K} \times V \to V$,

such that addition and and multiplication obey the following axioms:

i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$          (addition is commutative)

ii) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{a} + \underline{c})$          (addition is associative)

iii) $\underline{a} + \underline{0} = \underline{a}$          ($\underline{0}$ is the additive identity)

iv) $\forall \underline{a}, \exists (-\underline{a})[\underline{a} + (-\underline{a}) = \underline{0}]$          (existence of an inverse for addition)
   In short, i-iv express that $(V, +, \underline{0})$ is an abelian group.

v)  $1 \cdot \underline{a} = \underline{a}$                                                     (1 is the multiplicative identity)

vi)  $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$                          (multiplication is distributive over vector addition)

vii)  $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$                          (multiplication is distributive over scalar addition)

viii)  $(\lambda\mu)\underline{a} = \lambda(\mu\underline{a})$             (associativity of combining field multiplication with scalar multiplication)

**Definition 1.2.2 −** **Subspace** A subset $W$ of a vector space $V$ is a (linear) subspace of $V$ if

1.  $\underline{0} \in W$.

2.  $\lambda\underline{u} + \mu\underline{v} \in W$ for all $\underline{u}, \underline{v} \in W$ and $\lambda, \mu \in \mathbb{K}$.

**Definition 1.2.3 −** **Span** Let $V$ be a vector space over $\mathbb{K}$. Let $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n \in V$. The span of $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n$ is the set of all linear combinations $\lambda_1\underline{v}_1 + \lambda_2\underline{v}_2 + \cdot\lambda_n\underline{v}_n$. We denote it by $< \underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n >$.

**Definition 1.2.4 −** **Bases** A set of vectors $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ is a basis of a vector space $V$ if

1.  $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ is linearly independent.

2.  $\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n\}$ spans $V$.

**Definition 1.2.5 −** **Dimension** All bases of a vector space have the same size (cardinality), called the dimension of $V$ and denoted by $\dim V$.

## 1.3   Inner product spaces

**Definition 1.3.1 −** **Standard inner product of** $\mathbb{K}^n$ Let $\mathbb{K}$ be a field. For two vectors $\underline{v}, \underline{u} \in \mathbb{K}^n$, the standard inner product $(\cdot, \cdot) : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ is defined by

$$(\underline{v}, \underline{u}) = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n.$$

Let $\underline{u} = (u_1, u_2, \cdots, u_n)$ and $\underline{v} = (v_1, v_2, \cdots, v_n)$ be vectors in $\mathbb{R}^n$. The standard inner product of $\underline{u}$ and $\underline{v}$ is defined as
(Other popular notations are $\langle \underline{u}, \underline{v} \rangle$ or $\underline{u} \cdot \underline{v}$)

**Definition 1.3.2 −** **Inner product** Let $V$ be a vector space. We say that $V$ together with map $(\cdot, \cdot) : V \times V \to \mathbb{K}$ is an inner product space if the following properties are satisfied:

1.  $(\underline{u}, \underline{v}) = (\underline{v}, \underline{u})$                                                                (symmetry)

2.  $(\lambda\underline{u}, \underline{v} + \underline{w}) = \lambda(\underline{u}, \underline{v}) + \lambda(\underline{u}, \underline{w})$                                        (linearity)

3.  $(\underline{v}, \underline{v}) \geq 0$ and $(\underline{v}, \underline{v}) = 0 \iff \underline{v} = \underline{0}$                                   (positive definite)

**Definition 1.3.3 −** **Orthogonal** Let $V$ be an inner product space. Two vectors $\underline{u}, \underline{v} \in V$ are orthogonal if $(\underline{u}, \underline{v}) = 0$. We write $\underline{u} \perp \underline{v}$.

**Definition 1.3.4 −** **Length** Let $V$ be an inner product space. The length of a vector $\underline{v} \in V$ is defined as
$$\|\underline{v}\| = \sqrt{(\underline{v}, \underline{v})}.$$

**Definition 1.3.5 −** **Angle between vectors** Let $V$ be an inner product space. The angle between two vectors $\underline{u}, \underline{v} \in V$ is defined as
$$\cos\theta = \frac{(\underline{u}, \underline{v})}{\|\underline{u}\|\|\underline{v}\|}.$$

## 1.4   Linear Map

**Definition 1.4.1** − **Linear Map** A map $\mathcal{A} : V \to W$ ($V, W$ vector spaces) is linear if

1. $\mathcal{A}(\underline{u} + \underline{v}) = \mathcal{A}(\underline{u}) + \mathcal{A}(\underline{v})$.

2. $\mathcal{A}(\lambda\underline{u}) = \lambda\mathcal{A}(\underline{u})$.

Combined we have $\mathcal{A}(\lambda\underline{u} + \mu\underline{v}) = \lambda\mathcal{A}\underline{u} + \mu\mathcal{A}\underline{v}$.

**Example 1.4.2** Reflections, rotations, projections, identity map, zero map, etc.

## 1.5   Multiplication with Matrices

$\mathcal{A}(\underline{v})) = A \cdot \underline{v}$, where $A$ is a matrix.

## 1.6   Orthogonal projection

**Definition 1.6.1** − **Orthogonal projection** Let $V$ be a vector space with inner product $\langle \cdot, \cdot \rangle$. Let $W$ be a subspace of $V$. The orthogonal projection of $V$ onto $W$ is the linear map $\mathcal{P}_W : V \to W$ such that

1. $\mathcal{P}_W(\underline{v}) \in W$.

2. $\underline{v} - \mathcal{P}_W(\underline{v}) \in W^\perp$.

**Theorem 1.6.2** − **Addition** For $\mathcal{A}, \mathcal{B}$ linear maps, we define $(\mathcal{A} + \mathcal{B})(\underline{v}) = \mathcal{A}(\underline{v}) + \mathcal{B}(\underline{v})$.

**Theorem 1.6.3** − **Scalar multiplication** For $\mathcal{A}$ linear map, we define $(\lambda\mathcal{A})(\underline{v}) = \lambda\mathcal{A}(\underline{v})$.

**Theorem 1.6.4** − **Composition** For $\mathcal{A}, \mathcal{B}$ linear maps, we define $(\mathcal{A} \circ \mathcal{B})(\underline{v}) = \mathcal{A}(\mathcal{B}(\underline{v}))$.

**Theorem 1.6.5** − **Inverse** For $\mathcal{A}$ linear map, we define $\mathcal{A}^{-1}$ such that $\mathcal{A}^{-1} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{A}^{-1} = \mathcal{I}$.

### 1.6.6   Powers of maps

1. $\mathcal{A}^2 = \mathcal{A}\mathcal{A}$

2. $\mathcal{A}^n = \mathcal{A}\mathcal{A}^{n-1}$

3. $\mathcal{A}^{-n} = (\mathcal{A}^{-1})^n$

### 1.6.7   Null space and range

**Definition 1.6.8** − **Null space** The null space of a linear map $\mathcal{A} : V \to W$ is the set of vectors $\underline{v} \in V$ such that $\mathcal{A}(\underline{v}) = \underline{0}$.

**Definition 1.6.9** − **Range** The range of a linear map $\mathcal{A} : V \to W$ is the set of vectors $\underline{w} \in W$ such that $\exists \underline{v} \in V$ such that $\mathcal{A}(\underline{v}) = \underline{w}$.

**Theorem 1.6.10** − Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is injective if and only if $\mathcal{N}(\mathcal{A}) = \{\underline{0}\}$.

**Theorem 1.6.11** − Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is surjective if and only if $\mathcal{R}(\mathcal{A}) = W$.

**Theorem 1.6.12** − Let $\mathcal{A} : V \to W$ be a linear map. Then $\mathcal{A}$ is bijective if and only if $\mathcal{N}(\mathcal{A}) = \{\underline{0}\}$ and $\mathcal{R}(\mathcal{A}) = W$.

### 1.6.13   Null space / Range for matrix multiplication

> **Theorem 1.6.14** − Let $A$ be an $m \times n$ matrix. Then $\mathcal{N}(A) = \{\underline{v} \in V \mid A\underline{v} = \underline{0}\}$ and $\mathcal{R}(A) = \{\underline{w} \in V \mid \exists \underline{v} \in V$ such that $A\underline{v} = \underline{w}\}$.

### 1.6.15   Quotient spaces

> **Definition 1.6.16** − **Quotient space** Let $V$ be a vector space and $W$ a subspace of $V$. The quotient space $V/W$ is the set of cosets of $W$ in $V$. I.e. $V/W = \{\underline{v} + W \mid \underline{v} \in V\}$.

> **Theorem 1.6.17** − **Noether's fundamental theorem on homomorphisms** For any linear map $\mathcal{A} : V \to W$, there exists a linear bijection between its range $\mathcal{R}$ and the quottient space $V/\mathcal{N}$.

### 1.6.18   Example

Take $P : \mathbb{R}^3 \to \mathbb{R}^3, (x, a, b) \mapsto (0, a, b)$ and $\mathcal{R}(P) = < (0, a, 0), (0, 0, b) >$

*Proof.* ... $\bar{\mathcal{A}} : v/\mathcal{N}(\mathcal{A}) \to W, \underline{v} \mapsto \mathcal{A}\underline{v}$ Restrict target space of $\bar{\mathcal{A}}$ to $\mathcal{R}(A)$. Homework : show that $\bar{\mathcal{A}}$ is linear and injective. So $\bar{\mathcal{A}}$ is a linear bijection. $\qquad\square$

$\bar{\mathcal{A}}^{-1}$ :

# 2   Transition Matrices

We can find matrices for linear maps $\mathbb{K}^n \to \mathbb{K}^m$ by using the standard basis of $\mathbb{K}^n$ and $\mathbb{K}^m$. However, working with abstract vector spaces, we do not have a standard basis. For that reason we look at transition matrices.

## 2.1   Coordinates

> **Definition 2.1.1** − **coordinates** Let $V$ be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$. Every vector $\underline{v} \in V$ can be expressed as a linear combination of the basis vectors in exactly one way:
>
> $$\underline{v} = \sum_{i=1}^{n} \lambda_i \underline{a}_i. \tag{1}$$
>
> The coordinates of $\underline{v}$ with respect to $\alpha$ are the numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$.

*Remark* 2.1.2. Clearly, the coordinates depend on the choice of the basis $\alpha$.

> **Theorem 2.1.3** − Let $V$ be an n-dimensional vector space with basis $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$. We will denote the map sending each vector $\underline{v}$ to its coordinates with respect to $\alpha$ by $\alpha$.
> Then $\alpha$ is an invertible linear map from $V$ to $\mathbb{K}^n$.

With this notation, $\alpha(\underline{v})$ is the coordinate vector of the vector $\underline{v} \in V$ with respect to the basis $\alpha$.

> **Definition 2.1.4** − **Coordinate transformation (map)** Let $\alpha$ and $\beta$ be bases of an n-dimensional vector space $V$. The map $\beta\alpha^{-1} : \mathbb{K}^n \to \mathbb{K}^n$ is called the coordinate transformation (map) from $\alpha$ to $\beta$.

## 2.2   Basis transition matrix

> **Definition 2.2.1** − **Transition matrix** Let $\alpha$ and $\beta$ be bases of an n-dimensional vector space $V$. We call the $n \times n$-matrix associated to the linear map $\beta\alpha^{-1}$ the transition matrix from basis $\alpha$ to basis $\beta$ and denote it by $_\beta S_\alpha$.

The following theorem states that multiplication with the matrix $_\beta S_\alpha$ translates $\alpha$- into $\beta$-coordinates, and gives a direct description of how the matrix looks, entry-wise.

> **Theorem 2.2.2** − Let $\alpha$ and $\beta$ be bases of an n-dimensional vector space $V$ and let $_\beta S_\alpha$ be the basis transition matrix, i.e. the matrix of $\beta\alpha^{-1}$. Let $\underline{x} := \alpha(\underline{v})$ be the $\alpha$-coordinate vector of a vector $\underline{v} \in V$. Then the $\beta$-coordinate vector of $\underline{v}$ is equal to the product $_\beta S_\alpha \underline{x}$. Furthermore, the columns of matrix $_\beta S\alpha$ are the $\beta$-coordinate vectors of the $\alpha$-basis vectors.

*Remark* 2.2.3.
$$_\alpha S_{\beta\,\beta} S_\alpha = I, \text{ so } _\alpha S_\beta =_\beta S_\alpha^{-1}$$

> **Theorem 2.2.4** − Let $\alpha, \beta$ and $\gamma$ be bases of an n-dimensional vector space $V$, with respective basis transition matrices $_\beta S_\alpha$ and $_\gamma S_\beta$. Then the basis transition matrix from $\alpha$ to $\gamma$ is $_\gamma S_\alpha = _\gamma S_{\beta\,\beta} S_\alpha$.

*Remark* 2.2.5. It is important to distinguish between calculating with vectors (so elements of the vectors space $V$) and calculating with coordinates (so sequences of elements from $\mathbb{K}^n$).

## 2.3   Generalizing the map-matrix connection for spaces that aren't $\mathbb{K}^n$

> **Definition 2.3.1** − **Matrix of a linear map** Let $V$ and $W$ be vector spaces with bases $\alpha$ and $\beta$ respectively. Let $\mathscr{A} : V \to W$ be a linear map. We denote the matrix of the linear map $\beta\mathscr{A}\alpha^{-1}$ by $_\beta A_\alpha$ and call it the matrix of $\mathscr{A}$ with respect to the bases $\alpha$ and $\beta$.

*Remark* 2.3.2. If $V = W$ and $\alpha = \beta$, then we simplify notation by denoting the corresponding matrix by $A_\alpha$. We call it the matrix of $\mathscr{A}$ with respect to the basis $\alpha$.

*Remark* 2.3.3 (How does the matrix look?). The columns of $_\beta A\alpha$ are

$$(\beta\mathscr{A}\alpha^{-1})(\underline{e}_i) = \beta(\mathscr{A}\underline{a}_i), \qquad i = 1, \ldots, n,$$

meaning the i-th column consists of the $\beta$-coordinates of the image $\mathscr{A}\underline{a}_i$ of the i-th basis vector $\underline{a}_i$.

*Remark* 2.3.4. To find the image of a vector $\underline{v} \in V$, we can:

1. Determine the coordinate vector $\alpha(\underline{v})$ of $\underline{v}$;

2. Multiply $\alpha(\underline{v})$ with the representation matrix $_\beta A_\alpha$, yielding the coordinate vector of $\mathscr{A}\underline{v}$;

3. Translate the coordinate vector of $\mathscr{A}\underline{v}$ back to the corresponding vector in $W$.

## 2.4   How do base changes affect the matrix of a linear map?

> **Theorem 2.4.1** − **Effect of change of basis** Choose in a finite-dimensional space $V$ two bases $\alpha$ and $\beta$, and suppose $\mathscr{A} : V \to V$ is linear. Then
>
> $$A_\beta = {}_\beta S_\alpha A_{\alpha\,\alpha} S_\beta.$$

# 3   Eigenvalues and Eigenvectors

## 3.1   Diagonalization of matrices

**Definition 3.1.1** $-$ A square matrix $A$ has *diagonal form* if all elements $a_{ij}$ with $i \neq j$ are zero.

**Theorem 3.1.2** $-$ Let $\mathcal{A} : V \to V$ be a linear map and let $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$ be a basis of $V$. The matrix $A_\alpha$ has a diagonal form

$$A_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}$$

if and only if $\mathcal{A}\underline{a}_1 = \lambda_i \underline{a}_i$ for all $i \in \{1, \dots, n\}$.

## 3.2   Eigenvalues and eigenvectors

**Definition 3.2.1** $-$ **Eigenvector and eigenvalue** Let $\mathcal{A} : V \to V$ be a linear map from a $\mathbb{K}$-vector space $V$ to itself. A vector $\underline{v} \neq \underline{0} \in V$ is called an *eigenvector* of $\mathcal{A}$ with *eigenvalue* $\lambda$ if $\mathcal{A}\underline{v} = \lambda\underline{v}$. We denote the set of all eigenvalues of $\mathcal{A}$ by $\text{spec}\mathcal{A}$ and call it the *spectrum* of $\mathcal{A}$.

**Theorem 3.2.2** $-$ Let $\mathcal{A} : V \to V$ be a linear map with representation matrix $A_\alpha$ for a basis $\alpha$. Then $A_\alpha$ is in diagonal form if and only if $\alpha$ is a basis of eigenvectors of $\mathcal{A}$. In this case, the diagonal entries of $A_\alpha$ are the eigenvalues of $\mathcal{A}$.

**Definition 3.2.3** $-$ **Eigenspace** Let $\mathcal{A} : V \to V$ be a linear map. For any scalar $\lambda \in \mathbb{K}$, we denote

$$E_\lambda := \mathcal{N}(\mathcal{A} - \lambda\mathcal{I})$$

Since *null spaces* are subspaces, $E_\lambda$ is a subspace, called *the eigenspace* of $\mathcal{A}$ for $\lambda$.

*Remark* 3.2.4. Eigenspaces *indeed are spaces of eigenvectors for a given eigenvalue*: $E_\lambda$ is the null space of the linear map $\mathcal{A} - \lambda\mathcal{I}$.

- $\underline{v} \in E_\lambda \iff (\mathcal{A} - \lambda\mathcal{I})\underline{v} = \underline{0}$.

- $\underline{v} \in E_\lambda \iff \mathcal{A}\underline{v} - \lambda\underline{v} = \underline{0}$.

So any vector $\underline{v}$ lies in $E_\lambda$ if and only if $(\mathcal{A} - \lambda\mathcal{I})\underline{v} = \underline{0}$, which is equivalent to $\mathcal{A}\underline{v} - \lambda\underline{v} = \underline{0}$.

*Remark* 3.2.5 (null space as eigenspace). We can also write the null space of $\mathcal{A}$ as an eigenspace: $E_0$ consists of vectors that are mapped to 0 times itself, so on $\underline{0}$.

## 3.3   Computing eigenvalues and eigenspaces

**Theorem 3.3.1** $-$ $\lambda$ is an eigenvalue if and only if $det(A - \lambda\mathcal{I}) = 0$. Let $\alpha = \{\underline{a}_1, \dots, \underline{a}_n\}$ be a basis for $V$, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. this basis. Then the eigenvectors for eigenvalue $\lambda$, in $\alpha$-coordinates, are the

non-zero solutions of the system

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## 3.4   Characteristic polynomial

**Definition 3.4.1** − **Characteristic polynomial** Let $\mathcal{A} : V \to V$ be a linear map and let $A_\alpha$ be the matrix of $\mathcal{A}$ w.r.t. a basis $\alpha$. We call the equation $\det(A_\alpha - \lambda\mathcal{I}) = 0$ the *characteristic equation* of $A_\alpha$, and the left-hand side off this equation, $\det(A_\alpha - \lambda\mathcal{I})$, the *characteristic polynomial* of $A_\alpha$.

We also call them characteristic equation/polynomial of $\mathcal{A}$, and denote the characteristic polynomial by $\chi_\mathcal{A}$.

**Theorem 3.4.2** − Let $\mathcal{A} : V \to V$ be a linear map, $\alpha$ and $\beta$ be two bases for $V$ and let $A_\alpha/A_\beta$ be the matrix of $\mathcal{A}$ w.r.t. a basis $\alpha/\beta$. Then $\det(A_\alpha - \lambda\mathcal{I}) = \det(A_\beta - \lambda\mathcal{I})$.

*Remark* 3.4.3. The characteristic polynomial is independent of the choice of basis.

**Theorem 3.4.4** − Let $\mathcal{A} : V \to V$ be a linear map on a vector space $V$ of dimension $n$, and let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

be the matrix of $\mathcal{A}$ w.r.t. a basis $\alpha$. Then the characteristic polynomial $\chi_\mathcal{A}$ is a polynomial of degree (exactly) $n$, and of the following shape:

$$\chi_\mathcal{A}(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

for some coefficients $c_0, c_1, \cdots \in \mathbb{K}$.

**Definition 3.4.5** − **Trace** The sum of the diagonal elements of a square matrix $A$ is called the *trace* of the matrix $A$. We denote it by $\text{tr}(A)$.

**Theorem 3.4.6** − Let $\mathcal{A} : V \to V$ be a linear map with $\dim(V) < \infty$. For every basis $\alpha$, the matrix $A_\alpha$

1. has the same trace, which we therefore also call the *trace* of $\mathcal{A}$, and denote it by $\text{tr}(\mathcal{A})$.

2. has the same determinant, which we therefore also call the *determinant* of $\mathcal{A}$, and denote it by $\det(\mathcal{A})$. We have the identity $\det(\mathcal{A}) = c_0$, where $c_0$ is the constant coefficient of the characteristic equation.

**Theorem 3.4.7** − Let $A$ be a square matrix with entries in $\mathbb{K}$, where $\mathbb{K} \in \{\mathbb{C}, \mathbb{C}\}$, with characteristic polynomial $\chi_A(\lambda)$. Then the

- trace of the matrix is the sum of the roots of $\chi_A$; and the

- determinant of the matrix is the product of the roots of $\chi_A$.

## 3.5   Linear independence of eigenvectors

> **Theorem 3.5.1** − Let $\mathcal{A} : V \to V$ be a linear map and let $\underline{v}_1, \ldots, \underline{v}_n$ be eigenvectors of $\mathcal{A}$ for *mutually different* eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent.

# 4   Invariant subspaces

## 4.1   Invariant subspace

> **Definition 4.1.1** − **Invariant subspace** Let $W$ be a subspace of $V$. $W$ is called *invariant under linear map* $\mathcal{A} : V \to V$ if $\mathcal{A}\underline{w} \in W$ for all $\underline{w} \in W$.

**Example 4.1.2   Null space and range**
- The null space $\mathcal{N}$ pf a linear map $\mathcal{A}$ is always invariant: if $\underline{x} \in \mathcal{N}$, then $\mathcal{A}\underline{x} = \underline{0}$, and $\underline{0} \in \mathcal{N}$.

- The range $\mathcal{R}$ of a linear map $\mathcal{A}$ is invariant if and only if $\mathcal{A}$ is surjective: if $\underline{y} \in \mathcal{R}$, then $\underline{y} = \mathcal{A}\underline{x}$ for some $\underline{x} \in V$, and $\mathcal{A}\underline{x} \in \mathcal{R}$.

**Example 4.1.3   Counterexample, rotation in two-dimension space** Let $\mathcal{A}$ be a 90° rotation map. Then let $W = <e_1>$. $W$ is not invariant

> **Theorem 4.1.4** − Let $\mathcal{A} : V \to V$ be linear and let $W = <\underline{a}_1, \ldots, \underline{a}_n>$. W is invariant under $\mathcal{A}$ if and only if $\mathcal{A}\underline{a}_i \in W$ for $i = 1, \ldots, n$.

## 4.2   Restriction unto an invariant subspace

> **Definition 4.2.1** − **Restriction unto an invariant subspace** If $W$ is invariant under $\mathcal{A}$, then all image vectors $\mathcal{A}\underline{w}$ with $\underline{w} \in W$ are again in $W$. So if we restrict $\mathcal{A}$ to $W$, we obtain a well-defined linear map $W \to W$, the *restriction of the map* $\mathcal{A}$ *unto* $W$, which we denote by $\mathcal{A}|_W$.

*Invariant spaces give us a simpler matrix shape*, because the matrix contains a block of the restriction:

> **Theorem 4.2.2** − Suppose $\alpha = \{\underline{a}_1, \ldots, \underline{a}_2\}$ is a basis for $V$ such that $W = <\underline{a}_1, \ldots, \underline{a}_m>$ is invariant under $\mathcal{A}$. Then the matrix $A_\alpha$ has the following form:
>
> $$\begin{pmatrix} & & & * & \cdots & * \\ & M_1 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \vdots & & \vdots \\ & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix},$$
>
> The $m \times m$-matrix $M_1$ is the matrix of the restriction $\mathcal{A}|_W : W \to W$ w.r.t. the basis $\{\underline{a}_1, \ldots, \underline{a}_m\}$.

**Example 4.2.3   Proving invariance and analysing a map without even knowing its full map description** Consider in $\mathbb{R}^4$ the (independent) vectors

$$\underline{a} = (1, -1, 1, -1) \text{ and } \underline{b} = (1, 1, 1, 1).$$

Say we have a linear map $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^4$ of which we only know that

$$\mathcal{A}\underline{a} = (4, -6, 4, -6) \text{ and } \mathcal{A}\underline{b} = (4, 6, 4, 6).$$

Even without knowing the full description of $\mathcal{A}$, we will now show that $W = <\underline{a}, \underline{b}>$ is invariant and determine a matrix of the restriction unto $W$ - $\mathcal{A}|_W : W \to W$.

To show the invariance of $< \underline{a}, \underline{b} >$, we must verify that $\mathcal{A}\underline{a}$ and $\mathcal{A}\underline{b}$ are linear combinations of $\underline{a}$ and $\underline{b}$. We do this by simultaneously solving the systems of equations with columns $\underline{a}, \underline{b}, \mathcal{A}\underline{a}$ and $\mathcal{A}\underline{b}$:

$$\left( \begin{array}{cc|cc} 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & -6 \\ 1 & 1 & 4 & 4 \\ -1 & 1 & -6 & -6 \end{array} \right)$$

After row reduction and deleting zero rows, the system reduces to

$$\left( \begin{array}{cc|cc} 1 & 0 & 5 & -1 \\ 0 & 1 & -1 & 5 \end{array} \right),$$

which tells us that $\mathcal{A}\underline{a} = 5\underline{a} - \underline{b}$ and $\mathcal{A}\underline{b} = -\underline{a} + 5\underline{b}$, so $W$ is invariant under $\mathcal{A}$. This also tells us how the matrix of the restriction $A_{|W} W \to W$ w.r.t. the basis $\{\underline{a}, \underline{b}\}$ looks like:

$$\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}.$$

Using the restriction matrix, we can now even *determine some eigenvectors without knowing the full map.* The characteristic polynomial of the restriction is $\chi_{A_{|W}}(\lambda) = (5-\lambda)^2 - 1 = 25 - 10\lambda + \lambda^2 - 1 = (\lambda - 4)(\lambda - 6)$. We find that the matrix has eigenvalues 4 and 6. In coordinates, we compute the respective $E_4 = < (1,1) >$ and $E_6 = < (1,-1) >$. In this basis, the restriction map is simply the diagonal map with the eigenvalues on the diagonal:

$$\begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}.$$

We transform the coordinate vectors back into elements of $\mathbb{R}^4$ : $\underline{a} + \underline{b} = (2, 0, 2, 0)$ and $\underline{a} - \underline{b} = (0, 2, 0, 2)$. So the eigenvector basis of $W$ is $\{(2, 0, 2, 0), (0, 2, 0, 2)\}$.

*We now can simplify the representation of the full map:* if we pick any basis $\alpha$ of $\mathbb{R}^4$ such that the first two basis vectors are the eigenvectors $(2, 0, 2, 0)$ and $(0, 2, 0, 2)$, then the full matrix has the shape

$$A_\alpha = \begin{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix} & \begin{matrix} * & \dots & * \\ \vdots & & \vdots \end{matrix} \\ \begin{matrix} 0 \dots 0 \\ \vdots\vdots \\ 0 \dots 0 \end{matrix} & \begin{matrix} \vdots & & \vdots \\ \vdots & & \vdots \\ * & \dots & * \end{matrix} \end{pmatrix}$$

*Remark* 4.2.4. The characteristic polynomial of a restriction always divides the characteristic polynomial of the larger map.

**Theorem 4.2.5 −** If $W$ is an invariant subspace for the linear map $\mathcal{A}V \to V$, then $\chi_{A_{|W}}$, the characteristic polynomial of $\mathcal{A}$'s restriction unto $W$, $\mathcal{A}_{|W} : W \to W$, is a factor of $\chi_\mathcal{A}$, the characteristic polynomial of the map $\mathcal{A} : V \to V$.

**Lemma 4.2.6 −** Let $A$ be a $p \times p$-matrix, and let B be a $q \times q$-matrix. Then

$$\det \begin{pmatrix} A & * \\ O & B \end{pmatrix} = \det(A) \cdot \det(B),$$

where $*$ stands for na arbitrary $(p \times q)$-matrix and $O$ for the $q \times p$-zero matrix.

## 4.3  Nice results for combinations of invariant subspaces

> **Theorem 4.3.1** $-$ Let $\alpha = \{\underline{a}_1, \ldots, \underline{a}_n\}$ be a basis for $V$ such that $W_1 =< \underline{a}_1, \ldots, \underline{a}_m >$ and $W_2 =< \underline{a}_{m+1}, \ldots, \underline{a}_n >$ are invariant under $\mathcal{A} : V \to V$. Then the matrix $A_\alpha$ has the following form:
>
> $$A_\alpha = \begin{pmatrix} & & 0 & \cdots & 0 \\ M_1 & & \vdots & & \vdots \\ & & 0 & \cdots & 0 \\ 0 \ldots 0 & & & & \\ \vdots & & & M_2 & \\ 0 \ldots 0 & & & & \end{pmatrix}$$
>
> Here $M_1$ and $M_2$ are the $m \times m$ and $(n-m) \times (n-m)$ matrices of the two restrictions $\mathcal{A}_{|W_1} : W_1 \to W_1$ and $\mathcal{A}_{|W_2} : W_2 \to W_2$.
> In addition we have that
> $$\det(A_\alpha) = \det(M_1)\det(M_2),$$
> and that the characteristic polynomial of $\mathcal{A}$ is the product of the characteristic polynomials of the two restrictions:
> $$\chi_{\mathcal{A}} = \chi_{A_{|W_1}} \chi_{A_{|W_2}}.$$

*Remark* 4.3.2. We remark that this result can be generalised further such that it holds for an arbitrary number of invariant subspaces: if $V$ can be broken down into invariant subspaces $W_1, \ldots, W_p$, we can pick a basis $\alpha$ whose $i$-th section is a basis for $W_i$. Let $A_i : W_i \to W_i$ denote the restriction of $\mathcal{A}$ unto the subspace $W_i$.

Then the matrix $A_\alpha$ has the form.

$$A_\alpha = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_p \end{pmatrix},$$

where $M_i$ is the matrix of the respective restriction $\mathcal{A}_i$ w.r.t. the respective basis.

# 5  Orthogonal and symmetric maps

## 5.1  Orthogonal maps

> **Definition 5.1.1** $-$ **Orthogonal map** Let $v$ be a real inner product space. A linear map $\mathcal{A} : V \to V$ is called *orthogonal* if
> $$\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$$
> for all vectors $\underline{x} \in V$. In other words, a linear map $\mathcal{A} : V \to V$ is orthogonal if the *length is invariant* under $\mathcal{A}$.

> **Theorem 5.1.2** $-$ **Polarization formula** In a real inner product space $V$, we always have
>
> $$(\underline{x}, \underline{y}) = \frac{1}{2}\left((\underline{x} + \underline{y}, \underline{x} + \underline{y}) - (\underline{x}, \underline{x}) - (\underline{y}, \underline{y})\right)$$
>
> As a consequence, we can express *inner products between vectors in terms of vector lengths*:
>
> $$(\underline{x}, \underline{y}) = \frac{1}{2}\left(\|\underline{x} + \underline{y}\| - \|x\| - \|y\|\right).$$

**Theorem 5.1.3** − Let $V$ ba a finite real inner product space, and let $\mathcal{A} : V \to V$ be linear. Then the following are equivalent:

1. $\mathcal{A}$ is orthogonal.

2. $\|\mathcal{A}\underline{x}\| = \|\underline{x}\|$ for all $\underline{x} \in V$.

3. $(\mathcal{A}\underline{x}, \mathcal{A}\underline{y}) = (\underline{x}, \underline{y})$ for all $\underline{x}, \underline{y} \in V$.

4. For every orthonormal system $\underline{a}_1, \ldots, \underline{a}_n$ in $V$, the system $\mathcal{A}\underline{a}_1, \ldots, \mathcal{A}\underline{a}_n$ is again orthonormal.

5. For every orthonormal basis $\alpha$ of $V$, the basis $\mathcal{A}\alpha$ is again orthonormal.

**Theorem 5.1.4** − Let $V$ be a finite real inner product space, and let $\mathcal{A} : V \to V$ and $\mathcal{B} : v \to V$ be orthogonal linear maps.

1. The composition $\mathcal{A}\mathcal{B} : V \to V$ is orthogonal.

2. $\mathcal{A}$ is invertible and $\mathcal{A}^{-1}$ is orthogonal.

*Remark* 5.1.5. As a consequence, powers of orthogonal maps are orthogonal. However, in infinite dimensional spaces, there are exist orthogonal maps that are not invertible.

## 5.2 Orthogonal matrices

**Corollary 5.2.1** − We now consider $\mathbb{R}^n$ with the standard inner product. A linear map $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if the matrix $\mathcal{A}e_1, \ldots, \mathcal{A}e_n$ is an orthonormal system.

**Definition 5.2.2** − **Orthogonal matrix** A real $n \times n$-matrix $A$ is called *orthogonal* if the columns of $A$ form an orthonormal system in $\mathbb{R}^n$.

**Theorem 5.2.3** − Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with representation matrix $A$. The following statements are equivalent:

1. $\mathcal{A}$ is orthogonal.

2. $A$ is orthogonal.

3. $A^\top A = \mathcal{I}_n$. In other words, the transpose the inverse.

4. Thr rows of $A$ form an orthonormal system in $\mathbb{R}^n$.

**Lemma 5.2.4** − Let $V$ be an n-dimensional real inner product space, with its inner product denoted as $(\cdot, \cdot)_V$, and $\alpha$ an orthonormal basis of $V$. Let $\mathcal{A} : V \to V$ be an orthogonal map. We denote $\|\cdot\|_{\text{st}}$ the standard length in $\mathbb{R}^n$ and by $\|\cdot\|_V$ the length implied by $V$'s inner product.

1. $\|\alpha \mathcal{A}\underline{v}\|_{\text{st}} = \|\underline{v}\|_V$

2. $\|\mathcal{A}\alpha^{-1}\underline{x}\| = \|\underline{x}\|_{\text{st}}$

3. $\alpha \mathcal{A}\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal.

4. If $\mathcal{B} : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, the so is $\alpha^{-1}\mathcal{B}\alpha : V \to V$.

**Theorem 5.2.5** − If $\alpha, \beta$ are two *orthonormal* bases of in a real inner product space, then the transition matrix $\beta S\alpha$ is orthogonal.

**Theorem 5.2.6** − If $\alpha$ and $\beta$ are two orthonormal bases in a real inner product space, then

$$\alpha S\beta = \beta S\alpha^{-1} = \beta S\alpha^{\top}$$

**Theorem 5.2.7** − Let $\alpha$ be an orthonormal basis for a finite-dimensional real inner product space $V$, and let $\mathcal{A} : V \to V$ be a linear map and $A_\alpha$ the matrix of $\mathcal{A}$ (with respect to basis $\alpha$). Then the map $\mathcal{A}$ is orthogonal if and only if its matrix $A_\alpha$ is orthogonal.

## 5.3   Symmetric maps