Practice Midterm

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Exercise 1

Consider the set

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 - \tfrac{1}{10} < y < x^2 + \tfrac{1}{10}\}$$

Suppose that $f : \Omega \to W$ is differentiable on Ω and that for every $a \in \Omega$,

$$\|(Df)_a\|_{\mathbb{R}^2 \to W} \leq 2.$$

and suppose $\|f((-1, 1))\|_W = 3$.

Show that

$$\|f((2, 4))\|_W \leq 30.$$

Solution:

Consider the function

$$\gamma : [-1, 2] \to \Omega$$

given by

$$\gamma(t) = (t, t^2).$$

Then γ is differentiable on the interval $(-1, 2)$ because its components are differentiable because they are polynomials.

We define $g : [-1, 2] \to W$ by

$$g := f \circ \gamma$$

It follows by the chain rule that g is differentiable and that

$$g'(t) = (Dg)_t(1) = \big((Df)_{\gamma(t)} \circ (D\gamma)_t\big)(1) = (Df)_{\gamma(t)}(\gamma'(t)).$$

Note moreover that for $t \in (-1, 2)$,

$$\gamma'(t) = (1, 2t)$$

and therefore

$$\|\gamma'(t)\|_2 = \sqrt{1 + (2t)^2} \leq \sqrt{1 + 16} = \sqrt{17}$$

By the Mean-Value Inequality, we find that

$$\begin{aligned}
\|f((-1,1)) - f((2,4))\|_W &\leq \sup_{\tau \in (-1,2)} \|g'(\tau)\| (2 - (-1)) \\
&= 3 \sup_{\tau \in (-1,2)} \|(Df)_{\gamma(\tau)}(\gamma'(\tau))\|_W \\
&\leq 3 \sup_{\tau \in (-1,2)} \|(Df)_{\gamma(\tau)}\|_{\mathbb{R}^2 \to W} \|\gamma'(\tau)\|_2 \\
&\leq 6\sqrt{17} = 3\sqrt{68} \leq 3 \cdot 9 = 27
\end{aligned}$$

Therefore

$$\begin{aligned}
\|f((2,4))\|_W &= \|f((2,4)) - f((-1,1)) + f((-1,1))\|_W \\
&\leq \|f((2,4)) - f((-1,1))\|_W + \|f((-1,1))\|_W \\
&\leq 27 + 3 = 30.
\end{aligned}$$

Exercise 2

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f((x_1, x_2)) = \begin{cases} (x_1)^2 (x_2)^2, & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

a. Prove that f is differentiable on $\mathbb{R}$.

The previous version of this exercise said ”twice differentiable”, but that was a typo and is not true. In any case, the solution of this exercise is a bit too much work for a midterm.

Solution:

We first determine the first-order partial derivatives of f.

Let $x \in \mathbb{R}^2$. If $x_1 > 0$ and $x_2 > 0$, then

$$\frac{\partial f}{\partial x_1}(x) = 2x_1(x_2)^2 \qquad \frac{\partial f}{\partial x_2}(x) = 2(x_1)^2 x_2$$

If $x_1 < 0$ or $x_2 < 0$, then

$$\frac{\partial f}{\partial x_1}(x) = 0 \qquad \frac{\partial f}{\partial x_2}(x) = 0$$

If $x_1 = 0$, and $x_2 \leq 0$, then we need to go back to the definition of the partial derivative and find

$$\frac{\partial f}{\partial x_1}(x) = \lim_{h \to 0} \frac{f(x + he_1) - f(x)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$
$$\frac{\partial f}{\partial x_2}(x) = \lim_{h \to 0} \frac{f(x + he_2) - f(x)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

and if $x_1 = 0$ and $x_2 > 0$,

$$\lim_{h \downarrow 0} \frac{f(x + he_1) - f(x)}{h} = \lim_{h \downarrow 0} \frac{(x_2)^2 h^2}{h} = 0$$

while

$$\lim_{h \uparrow 0} \frac{f(x + he_1) - f(x)}{h} = \lim_{h \uparrow 0} \frac{0}{h} = 0$$

so that

$$\frac{\partial f}{\partial x_1}(x) = \lim_{h \to 0} \frac{f(x + he_1) - f(x)}{h} = \lim_{h \to 0} \frac{(x_2)^2 h^2}{h} = 0.$$

Also

$$\frac{\partial f}{\partial x_2}(x) = \lim_{h \to 0} \frac{f(x + he_2) - f(x)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

Similarly, if $x_2 = 0$ then also

$$\frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_2}(x) = 0$$

It follows that for $x \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x_1}(x) = \begin{cases} 2x_1(x_2)^2 & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\frac{\partial f}{\partial x_2}(x) = \begin{cases} 2(x_1)^2 x_2 & \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

We now claim that these functions are continuous. We only prove continuity of $\frac{\partial f}{\partial x_1}$, the proof of continuity of $\frac{\partial f}{\partial x_2}$ is similar. It is clear that the functions are on the domains $(0, \infty) \times (0, \infty)$ and the domains $(-\infty, 0) \times \mathbb{R}$ and $\mathbb{R} \times (-\infty, 0)$, because the functions are equal to polynomials on these domains. It remains to check continuity in the cases that $x_1 = 0$ and $x_2 \geq 0$ and the case $x_2 = 0$ and $x_1 \geq 0$. The latter case is similar to the first case, so we will only prove the first case.

Let therefore $(x_1, x_2) \in \mathbb{R}^2$ be such that $x_1 = 0$ and $x_2 > 0$. We need to show that f is continuous in (x_1, x_2). Let $\epsilon > 0$. Choose

$$\delta := \min\left(\frac{\epsilon}{8(x_2)^2}, x_2\right)$$

Let $z \in \mathbb{R}^2$ and assume $0 < \|z - x\|_2 < \delta$. Then

$$\left|\frac{\partial f}{\partial x_1}(z) - \frac{\partial f}{\partial x_1}(x)\right| \leq \left|2z_1(z_2)^2\right| < |2z_1| \left|(2x_2)^2\right| < \epsilon$$

Finally consider the case $(x_1, x_2) = (0, 0)$. Then choose $\delta = \frac{1}{2}\min(\epsilon, 1)$. Let $z \in \mathbb{R}^2$ and assume $0 < \|z - x\|_2 < \delta$. Then

$$\left|\frac{\partial f}{\partial x_1}(z) - \frac{\partial f}{\partial x_1}(x)\right| \leq \left|2z_1(z_2)^2\right| = 2\left|z_1(z_2)^2\right| \leq 2\delta^3 < 2\delta < \epsilon.$$

Since the partial derivatives are continuous, it follows that f is differentiable on $\mathbb{R}^2$.

b. Give the first order Taylor order polynomial of f in $0 \in \mathbb{R}^2$.

The previous version of this exercise said "second order Taylor polynomial".

Solution:

The Taylor polynomial is given by

$$x \mapsto f((0,0)) + \frac{\partial f}{\partial x_1}((0,0))x_1 + \frac{\partial f}{\partial x_2}((0,0))x_2 = 0,$$

i.e. in this case it is the zero polynomial.

Exercise 3

Determine whether the following limits exist, and if so, determine their value.

a.

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x)+\frac{1}{2}x^2+\sin(y)^4}{x^4+y^4}$$

Solution:
This limit does not exist. We argue by contradiction. Suppose the limit exists. Consider the sequence $z^{(n)} := (1/n, 0)$. Then the sequence converges to zero and therefore (by sequence characterizations of limits) the limit

$$\lim_{n\to\infty} f(z^{(n)})$$

needs to exist, where $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is defined by

$$f((x,y)) = \frac{1-\cos(x)+\frac{1}{2}x^2+\sin(y)^4}{x^4+y^4}$$

By Taylor's theorem we know that

$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathsf{E}_1(x)$$

for a function $\mathsf{E}_1 : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\lim_{x\to 0} \frac{\mathsf{E}_1(x)}{|x|^3} = 0$$

so

$$\begin{aligned}
f(z^{(n)}) &= \frac{1 - \cos(1/n) + (1/2)(1/n)^2 + \sin(0)^4}{(1/n)^4 + 0} \\
&= \frac{(1/n)^2 - E_1(1/n)}{(1/n)^4} \\
&= n^2(1 - (1/n)n^3E_1(1/n)).
\end{aligned}$$

Because

$$\lim_{n\to\infty} n^3E_1(1/n) = 0,$$

it follows by limit laws that

$$\lim_{n\to\infty}(1 - (1/n)n^3E_1(1/n)) = 1$$

and therefore, as $\lim_{n\to\infty} n^2 = \infty$ it follows by limit laws for sequences diverging to infinity,

$$\lim_{n\to\infty} f(z^{(n)}) = \lim_{n\to\infty} n^2(1 - (1/n)n^3E_1(1/n)) = \infty,$$

i.e. this limit does not exist.

b.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)\cos(y) - x}{x^4 + y^4}$$

Solution:
This limit also doesn't exist. We argue by contradiction again. Suppose the limit does exist. Consider again the sequence $z^{(n)} := (1/n, 0)$. Then the sequence converges to zero and therefore (by sequence characterizations of limits) the limit

$$\lim_{n\to\infty} g(z^{(n)})$$

needs to exist, where $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is defined as

$$g((x,y)) := \frac{\sin(x)\cos(y) - x}{x^4 + y^4}$$

By Taylor's theorem we know that

$$\sin(y) = y - \frac{1}{6}y^3 + \mathsf{E}_2(y)$$

for a function $\mathsf{E}_2 : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\lim_{y\to 0} \frac{\mathsf{E}_2(y)}{|y|^4} = 0$$

Therefore

$$\begin{aligned} g(z^{(n)}) &= \frac{\sin(1/n)\cos(0) - 1/n}{(1/n)^4 + 0^4} \\ &= \frac{\frac{1}{n} - \frac{1}{6}\frac{1}{n^3} + E_2(1/n) - \frac{1}{n}}{(1/n)^4} \\ &= -\frac{1}{6}n - n^4 E_2(1/n). \end{aligned}$$

We know that

$$\lim_{n\to\infty} n^4 E_2(1/n) = 0$$

and

$$\lim_{n\to\infty} n = \infty$$

so that by limit laws for sequences diverging to infinity or minus infinity we find that

$$\lim_{n\to\infty} g(z^{(n)}) = \lim_{n\to\infty} -\frac{1}{6}n - n^4 E_2(1/n) = -\infty$$

or in other words this limit does not exist.

Exercise 4

Does there exist a three times differentiable function $f : \mathbb{R}^4 \to \mathbb{R}$ such that for all $u \in \mathbb{R}^4$,

$$(D^3 f)_0(u, u, u) = u_1 + 4u_4$$

Either give an example of such a function or show why such a function does not exist.

Solution:
Such a function does not exist. We argue by contradiction. Suppose such a function exists. Take $u = e_1 = (1, 0, 0, 0)$. Then, because of multilinearity of $(D^3 f)_0$ we would have that

$$\begin{aligned} 2 = (D^3 f)_0(2u, 2u, 2u) &= 2(D^3 f)_0(u, 2u, 2u) = 4(D^3 f)_0(u, u, 2u) \\ &= 8(D^3 f)_0(u, u, u) = 8. \end{aligned}$$

This is a contradiction.