Assignment 9

Group 1

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1 Exercise 12.4.4

Problem Let (X, dist) be a metric space and let $K \subseteq X$ be a compact subset. Let $a : \mathbb{N} \to X$ be a sequence with values in X, such that

$$\begin{aligned} &\text{for all } N \in \mathbb{N}, \\ &\quad \text{there exists } \ell \geq N, \\ &\quad\quad a_\ell \in K \end{aligned} \tag{$*$}$$

1. Use $(*)$ to inductively define an index sequence $n : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$, $a_{n_k} \in K$.
2. Use the fact that K is compact to show that there is a point $p \in K$ and a subsequence of $a : \mathbb{N} \to X$ converging to p.

Proof. 1. We define an index sequence $n : \mathbb{N} \to \mathbb{N}$ inductively as follows: **Base:**
Choose $N = 0$ in $(*)$, then there exists $\ell \geq 0$, such that $a_\ell \in K$.
Obtain such ℓ
Set $n_0 = \ell$

Inductive step:
Suppose we have defined $n_0, n_1, \ldots, n_k$ for some $k \in \mathbb{N}$, such that for all $0 \leq i \leq k$, $a_{n_i} \in K$.
Choose $N = n_k + 1$ in $(*)$, then there exists $\ell \geq N$, such that $a_\ell \in K$.
Obtain such ℓ
Set $n_{k+1} = \ell$
Then it holds that $n_k < n_{k+1}$ and $a_{n_{k+1}} \in K$.

2. Since K is compact, it holds that

$$\begin{aligned} &\text{for all sequences } b : \mathbb{N} \to \mathbb{K}, \\ &\quad \text{there exists a subsequence } b' : \mathbb{N} \to \mathbb{K}, \\ &\quad\quad \text{such that } b' \text{ converges to a point } p \in K \end{aligned} \tag{$**$}$$

Choose $b = a \circ n$ in $(**)$, then there exists a subsequence $b' : \mathbb{N} \to \mathbb{K}$, such that b' converges to a point $p \in K$.
I.e. there exists an index sequence $m : \mathbb{N} \to \mathbb{N}$, such that $b \circ m$ converges to p.
The the subsequence $a \circ n \circ m$ converges to $p \in K$.

□

2 Exercise 12.4.5

Problem Consider the sets

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 1\}$$

and

$$B := \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 \le 1\}$$

Prove that the set $A \cap B$ is compact (as a subset of the normed vector space $(\mathbb{R}^2, \|\cdot\|_2)$).

Proof. Note: B is the closed unit ball in $(\mathbb{R}^2, \|\cdot\|_2)$.
Since $A \cap B \subseteq (\mathbb{R}^2, \|\cdot\|_2)$ by the Heine-Borel Theorem, it suffices to show that $A \cap B$ is closed and bounded.
Note: $A \cap B = \{(x_1, x_2) \mid x_1 - x_2 = 1 \wedge x_1^2 + x_2^2 \le 1\}$ which is the closed line segment from $(0, -1)$ to $(1, 0)$.

Closed:
By the sequence characterization of closedness, it suffices to show that

for all sequences $a : \mathbb{N} \to A \cap B$,
if a converges to a point $p \in \mathbb{R}^2$,
then $p \in A \cap B$

Let $a : \mathbb{N} \to A \cap B$ be a sequence, such that a converges to a point $p \in \mathbb{R}^2$.
We need to show that $p \in A \cap B$.
Since $a_n \in A \cap B$ for all $n \in \mathbb{N}$, it holds that $p = \lim_{n\to\infty} a_n \in A \cap B$.
Hence $A \cap B$ is closed.

Bounded:
Need to show that $A \cap B$ is bounded, i.e.

there exists $q \in A \cap B$,
there exists $M > 0$,
for all $p \in A \cap B$,
$\|p - q\| \le M$

Choose $q = (1, 0)$, then $q \in A \cap B$,
Choose $M = 2$, then $M > 0$,
Let $p = (p_1, p_2) \in A \cap B$,
Need to show that $\|p - q\| \le M$
$\|p - q\| = \sqrt{(p_1 - 1)^2 + p_2^2} = \sqrt{p_1^2 + p_2^2 - 2p_1 + 1} \le \sqrt{2 - 2p_1} \le \sqrt{2} < 2 = M$
Since $A \cap B$ is closed and bounded, by the Heine-Borel Theorem, $A \cap B$ is compact. □

3 Exercise 13.11.1

Problem Let $(X, \text{dist}_X) := (\mathbb{R}^2, \text{dist}_{\|\cdot\|_2})$ and $(Y, \text{dist}_Y) := (\mathbb{R}, \text{dist}_{\mathbb{R}})$. Let $D = B(0,1) \subseteq \mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be defined as

$$f(x) := \begin{cases} x_1^2 + x_2^2 & \text{if } x \neq (0,0) \\ 185 & \text{if } x = (0,0). \end{cases}$$

Show that

$$\lim_{x \to (0,0)} f(x) = 0$$

Proof. **Method 1: ($\epsilon - \delta$ proof)**
We need to show that

$$\begin{aligned} &\text{for all } \epsilon > 0, \\ &\quad \text{there exists } \delta > 0, \\ &\quad\quad \text{for all } x \in D, \\ &\quad\quad\quad 0 < \|x - (0,0)\| < \delta \implies |f(x) - 0| < \epsilon \end{aligned}$$

Let $\epsilon > 0$,
Choose $\delta = \sqrt{\epsilon}$,
Let $x \in D$,
Assume $0 < \|x - (0,0)\| < \delta$, i.e. $0 < \sqrt{x_1^2 + x_2^2} < \delta$
Then $x \neq (0,0)$ and $f(x) = x_1^2 + x_2^2$
Need to show that $|f(x) - 0| < \epsilon$
Indeed $|f(x) - 0| = |x_1^2 + x_2^2| < \delta^2 = \epsilon$
Therefore,

$$\lim_{x \to (0,0)} f(x) = 0$$

Method 2: (Sequence characterization proof)
By the sequence characterization of limits, it suffices to show that,

$$\begin{aligned} &\text{for all sequences } (x^n) \text{ in } D \setminus \{(0,0)\} \text{ converging to } (0,0), \\ &\quad \lim_{n\to\infty} f(x^n) = 0 \end{aligned}$$

Let (x^n) be a sequence in $D \setminus \{(0,0)\}$ converging to $(0,0)$.
It holds that $\lim_{n\to\infty} x^n = (0,0)$.
Since $x^n \neq (0,0)$ for all $n \in \mathbb{N}$, we know $f(x^n) = (x^n)_1^2 + (x^n)_2^2$ for all $n \in \mathbb{N}$.
Hence $\lim_{n\to\infty} f(x^n) = \lim_{n\to\infty} (x^n)_1^2 + (x^n)_2^2 = 0^2 + 0^2 = 0$.
Since $\lim_{n\to\infty} f(x^n) = 0$ for all (x^n) in $D \setminus \{(0,0)\}$ converging to $(0,0)$,

$$\lim_{x \to (0,0)} f(x) = 0$$

$\square$

4 Exercise 13.11.2

Problem Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x) = x \quad \text{for } x \in \mathbb{R}$$

where $D = \mathbb{R}$.
Prove that for every $a \in D$, the function f is continuous at a.

Proof. We need to show that for every $a \in D$, f is continuous at a.
Take $a \in D$.
By the sequence chracterization of continuity, it suffices to show that

$$\begin{array}{l}\text{for all sequences } x_n \text{ in } D \text{ converging to } a \in D, \\ \quad \lim_{n\to\infty} f(x_n) = f(a)\end{array}$$

Let $x_n : \mathbb{N} \to D$ be a sequence converging to $a \in D$.
It holds that $f(x_n) = x_n$ for all $n \in \mathbb{N}$.
Therefore, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = a = f(a)$.
Thus, f is continuous at a.

$\square$