Lines and ratios in a triangle

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1 Problem description

In $\triangle ABC$ (the points A, B, C are non-colinear) P is the midpoint of the segment BC and R is the point on the line AB such that A is the midpoint of the segment BR. Use vectors to determine the point of intersection Q of lines PR and AC, and show that $AQ : QC = 1 : 2$.

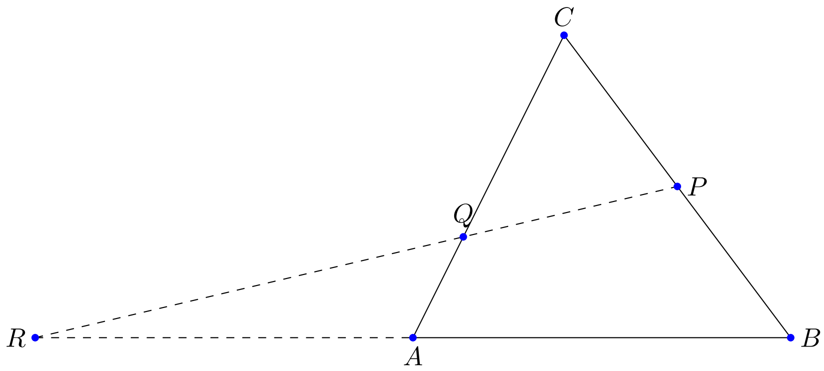

Figure 1: Example contructions

2 Solution

In solving the earlier described problem we will mainly use vector techniques.

In geometric exercises that envolve vectors we need to first choose where to put the origin if it's not explicitly specified. Without loss of generality, we will choose point A as our origin and assign vectors $\underline{b}, \underline{c}, \underline{p}, \underline{q}, \underline{r}, \underline{q}$ to the points B, C, P, Q, R, Q respectively.

Since A is the midpoint of the segment RB and the origin, we get $\underline{r} = -\underline{b}$. Additionally, since P is the midpoint of the segment BC, we get $\underline{p} = \underline{b} + \frac{1}{2}(\underline{c} - \underline{b}) = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$.
The line RP can be defined as $RP : \underline{x} = \underline{r} + \lambda(\underline{p} - \underline{r})$ [1], using our above defined substitutions we thus get $RP : \underline{x} = -\underline{b} + \lambda(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c})$. We then find the intersection of RP and $AC : \underline{x} = \mu\underline{c}$.

$$-\underline{b} + \lambda(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}) = \mu\underline{c}$$
$$-\underline{b} + \lambda(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}) - \mu\underline{c} = \underline{0}$$
$$(-1 + \frac{3}{2}\lambda)\underline{b} + (\frac{1}{2}\lambda - \mu)\underline{c} = \underline{0}$$

Since $\underline{b}$ and $\underline{c}$ are linearly independant [2] we get

$$\begin{cases} -1 + \frac{3}{2}\lambda = 0 \\ \frac{1}{2}\lambda - \mu = 0 \end{cases} \iff \begin{cases} \frac{3}{2}\lambda = 1 \\ \mu = \frac{1}{2}\lambda \end{cases} \iff \begin{cases} \lambda = \frac{2}{3} \\ \mu = \frac{1}{3} \end{cases}$$

Since $\underline{q} \in AC$, we can write $\underline{q} = \mu\underline{c} = \frac{1}{3}\underline{c}$. From this we get $|AQ| = ||\frac{1}{3}\underline{c}|| = \frac{1}{3}||\underline{c}||$ and $|QC| = |AC - AQ| = 7|\underline{c} - \frac{1}{3}\underline{c}|| = ||\frac{2}{3}\underline{c}|| = \frac{2}{3}||\underline{c}||$. Having calculated the lengths of AQ and QC we see that $|QC| = 2 \cdot |AQ|$.

3 Conclusion

Indeed, we find that the intersection Q of lines RP and AC divides the segment AC into two segments AQ and QC where QC is twice the length of AQ ($AQ : QC = 1 : 2$).

Remark

Of course there are different ways that we could have solved this exercis

4 Role of Homework group members

References

[1] Hans Sterk. *Linear Algebra 1*, chapter 1.2.1. Technische Universiteit Eindhoven, 2023-2024.

[2] Hans Sterk. *Linear Algebra 1*, chapter 3.2.10. Tecnhische Universiteit Eindhoven, 2023-2024.