Department of Mathematics and Computer Science
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Lines and Ratios in a Triangle

Jiaqi Wang (1986619)
Mil Majerus
Jean Nguyen
Long Pham

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1 Problem description

In $\triangle ABC$ (the points A, B, C are non-colinear) P is the midpoint of the segment BC and R is the point on the line AB such that A is the midpoint of the segment BR. Use vectors to determine the point of intersection Q of lines PR and AC, and show that $AQ : QC = 1 : 2$.

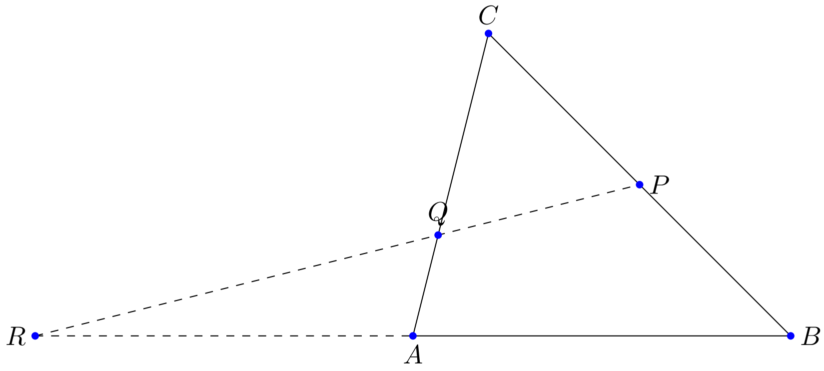

Figure 1: Example contructions

2 Solution

In solving the earlier described problem we will mainly use vector techniques.

In geometric exercises that envolve vectors we need to first choose where to put the origin if it's not explicitly specified. Without loss of generality, we can choose point A as our origin and assign vectors $\underline{b}, \underline{c}, \underline{p}, \underline{q}, \underline{r}, \underline{q}$ to the points B, C, P, Q, R, Q respectively.

Since A is the midpoint of the segment RB and the origin, we get $\underline{r} = -\underline{b}$. Additionally, since P is the midpoint of the segment BC, we get $\underline{p} = \underline{b} + \frac{1}{2}(\underline{c} - \underline{b}) = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$.

The line RP can be defined as $RP : \underline{x} = \underline{r} + \lambda\left(\underline{p} - \underline{r}\right)$ [**?**], using our above defined substitutions we thus get $RP : \underline{x} = -\underline{b} + \lambda\left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}\right)$. We then find the intersection of RP and $AC : \underline{x} = \mu\underline{c}$.

$$-\underline{b} + \lambda\left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}\right) = \mu\underline{c}$$
$$-\underline{b} + \lambda\left(\frac{3}{2}\underline{b} + \frac{1}{2}\underline{c}\right) - \mu\underline{c} = \underline{0}$$
$$\left(-1 + \frac{3}{2}\lambda\right)\underline{b} + \left(\frac{1}{2}\lambda - \mu\right)\underline{c} = \underline{0}$$

Since $\underline{b}$ and $\underline{c}$ are linearly independant [**?**] we get

$$\begin{cases} -1 + \frac{3}{2}\lambda = 0 \\ \frac{1}{2}\lambda - \mu = 0 \end{cases}$$
$$\iff \begin{cases} \frac{3}{2}\lambda = 1 \\ \mu = \frac{1}{2}\lambda \end{cases}$$
$$\iff \begin{cases} \lambda = \frac{2}{3} \\ \mu = \frac{1}{3} \end{cases}$$

Since $\underline{q} \in AC$, we can write $\underline{q} = \mu\underline{c} = \frac{1}{3}\underline{c}$. From this we get $|AQ| = \|\frac{1}{3}\underline{c}\| = \frac{1}{3}\|\underline{c}\|$ and $|QC| = |AC - AQ| = 72\|\underline{c} - \frac{1}{3}\underline{c}\| = \|\frac{2}{3}\underline{c}\| = \frac{2}{3}\|\underline{c}\|$. Having calculated the lengths of AQ and QC we see that $|QC| = 2 \cdot |AQ|$.

3 Conclusion

Indeed, we find that the intersection Q of lines RP and AC divides the segment AC into two segments AQ and QC where QC is twice the length of AQ (i.e. $AQ : QC = 1 : 2$).

Remark

Of course there are different ways that we could have solved this exercise. For instance, instead of choosing A as origin we could have chosen any other given point or even an arbitrary point in the plane. This, however, may lead to more complicated expressions for the lines RP and AC, thus making it more prone to computational errors.

4 Roles of Homework group members

- Jiaqi Wang - document organization and visual aspects
- Mil Majorus - writing the solution
- Jean Nguyen
- Long Pham