Multi-Agent Systems

Homework Assignment 2

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2 Game Theory: Nash Equilibrium

2.1 Dining out

Questions:

1. The pay-off matrix can be seen below.

Alice \ Bob	**C**	**E**
C	2, 2	-3, 3
E	3, -3	-2, -2

2. The best response for Alice is **E**, regardless of what Bob picks. Similarly, the best response for Bob is also **E**. Thus, they will both order the expensive option.

3. The updated pay-off matrix can be seen below.

Alice \ Bob	**C** (q)	**E** ($1-q$)
C (p)	2+s, 2	-3, 3+s
E ($1-p$)	3, s-3	s-2, -2

If Bob picks **C**, the best response for Alice is **C** if $s > 1$ and **E** otherwise. If Bob picks **E**, the best response for Alice is always **E**.

If Alice picks **C**, the best response for Bob is always **E**. If Alice picks **E**, the best response for Bob is **C** if $s > 1$ and **E** otherwise.

The mixed Nash Equilibrium can be calculated as a function of s.

$$2p + (s-3)(1-p) = (3+s)p - 2(1-p) \Rightarrow p = \frac{s-1}{2s}$$
$$q(2+s) - 3(1-q) = 3q + (s-2)(1-q) \Rightarrow q = \frac{1+s}{2s}$$

Thus, if $s < 1$, the NE remains the same. If $s > 1$ there is a cycle and the NE is mixed with $p = \frac{s-1}{2s}$ and $q = \frac{1+s}{2s}$. Finally, if $s = 1$ then there is a new pure Nash equilibrium where Alice always plays **E** ($p = 0$) and Bob always plays **C** ($q = 1$).

2.2 Hawk versus Dove

Questions:

- The pay-off matrix can be seen below with $\widetilde{v} = \frac{v}{2}$.

A \ B	**H** (q)	**D** $(1-q)$
H (p)	$\widetilde{v}-c$, $\widetilde{v}-c$	v, 0
D $(1-p)$	0, v	$\widetilde{v}$, $\widetilde{v}$

- To find the Nash equilibrium, the probabilities that make each other indifferent can be calculated, $p(\widetilde{v}-c)+2\widetilde{v}(1-p)=(1-p)\widetilde{v} \Rightarrow p=\frac{\widetilde{v}}{c}$, and because of symmetry $q=p$. If $\widetilde{v}>c$, there is no mixed NE and the best response is to always play **H**. If $\widetilde{v}=c$, the probability $p=1$, so the Nash equilibrium remains the same with **H** as always the best response. If $\widetilde{v}<c$, then the proportion of hawk responses decreases inversely with the cost, that is, the greater the cost, the less often it is beneficial to play hawk.

2.3 Investment in recycling

Questions:

1. A country's best response can be found by taking the derivative with respect to the allocated resources and setting it to 0:

$$\frac{\partial u_i(r_i,r_j)}{\partial r_i} = b_i(r_i,r_j)+r_i\frac{\partial b_i(r_i,r_j)}{r_i} - 4 = 10 - r_i + \frac{r_j}{2} - r_i - 4$$
$$= 6 - 2r_i + \frac{r_j}{2} = 0 \Rightarrow r_i = \frac{6-r_j}{2}$$

2. Figure 1 shows the best response curves for both companies. It can be seen that the Nash equilibrium is at $(r_i^*, r_j^*) = (2,2)$.

3. The arrows in figure 1 show what happens if the benefit intersect is reduced for the first country. The dashed line has $b_i(r_i,r_j) = 7 - r_i = \frac{r_j}{2}$ and the Nash equilibrium is now at $(r_i^*, r_j^*) = (0,3)$, which means that country 1 has to do less, from 2 to 0, while country 2 now does more, from 2 to 3. The utilities go from $u_1 = 10$ to $u_1 = 0$ and from $u_2 = 10$ to $u_2 = 9$ for countries 1 and 2 respectively.

2.4 Tragedy of the Commons

Questions:

- A Nash equilibrium can be found by examining the best responses. The best response can be found by taking the derivative of the utility with respect to the player's share and setting it to 0:

$$\frac{\partial u_i(x_i,x_j)}{\partial x_i} = 1 - x_i - x_j - x_i = 0 \Rightarrow x_i = \frac{1-x_j}{2}$$

  Setting $x_i = x_j$ gives the NE, $x_i = \frac{1-x_i}{2} \Rightarrow x_i = x_j = \frac{1}{3}$

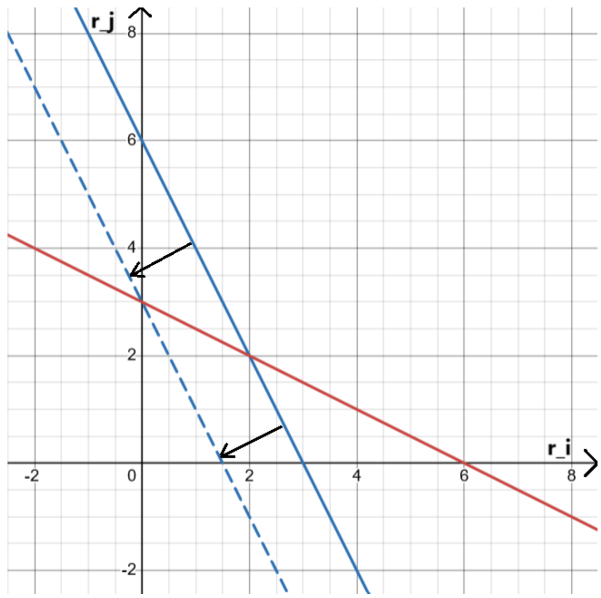

Figure 1: The best response lines for both companies plotted against the actions of one another. The arrows show how the best response for the first country would change as the benefit intercept is reduced. The dashed line represents a reduction from 10 to 7.

- The aggregated utility in this case is $U = u_1 + u_2 = 2\frac{1}{3^2} = \frac{2}{9}$. But it is not the maximal possible aggregated utility as can be easily seen if the fraction if the slack is increased from $\frac{1}{3}$ to $\frac{1}{2}$. Then $x_i = x_j = \frac{1}{4}$ and the aggregated utility is $U = u_1 + u_2 = 2\frac{1}{4 \cdot 2} = \frac{1}{4}$. This alas is not a Nash equilibrium as x_i could increase his share to get a higher utility. With the same reasoning x_j would also attempt to increase his utility and the combined effect would lead to a reduced utility for both agents.

- If there are n agents, the best response becomes:

$$\frac{\partial u_i(x_i, x_{-i})}{\partial x_i} = 1 - 2x_i - \sum_{j \neq i} x_j = 0 \Rightarrow x_i = \frac{1 - \sum_{j \neq i} x_j}{2}$$

  Because the best response is identical for all agents, in the case of a Nash equilibrium $\forall j : x_i = x_j$. Thus, the Nash equilibrium can be found by $x_i = \frac{1 - \sum^{n-1} x_i}{2} = \frac{1-(n-1)x_i}{2} \Rightarrow x_i = \frac{1}{1+n}$. This means that with an increasing number of agents, the amount of slack approaches 0 in the NE and the aggregated utility moves farther and farther away from its maximal value.