CUMC3D User Guide

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February 23, 2023

1 Preliminary

The one-to-one conversion between primitive and conservative variables are:

$$\begin{pmatrix} \rho \\ v_x \\ v_y \\ v_z \\ p \\ \rho\epsilon \\ \psi_i \\ B_x \\ B_y \\ B_z \end{pmatrix} \longleftrightarrow \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ \tau \\ \rho\epsilon \\ \rho\psi_i \\ B_x \\ B_y \\ B_z \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{irho2} \\ \text{ivel2}_\text{x} \\ \text{ivel2}_\text{y} \\ \text{ivel2}_\text{z} \\ \text{itau2} \\ \text{ieps2} \\ \text{iphi2} \\ \text{ibx} \\ \text{iby} \\ \text{ibz} \end{pmatrix} \tag{1}$$

Note that magnetic fields are always placed as the last three indicies

2 Ideal Magnetohydrodynamics (MHD)

We solve the ideal MHD equation in the conservative form:

$$\begin{aligned} \partial_t\rho + \nabla\cdot(\rho\vec{v}) &= 0, \\ \partial_t(\rho\vec{v}) + \nabla\cdot[\rho(\vec{v}\otimes\vec{v}) - (\vec{B}\otimes\vec{B})] + \nabla(P + \frac{1}{2}\vec{B}\cdot\vec{B}) &= -\rho\nabla\Phi, \\ \partial_t\tau + \nabla\cdot[(\tau + P + \frac{1}{2}\vec{B}\cdot\vec{B})\vec{v} - \vec{B}(\vec{v}\cdot\vec{B})] &= -\rho\vec{v}\cdot\nabla\Phi, \\ \partial_t(\rho\epsilon) + \nabla\cdot\{[(\rho\epsilon) + P]\vec{v}\} &= \vec{v}\cdot\nabla(P), \\ \partial_t(\rho\psi) + \nabla\cdot(\rho\psi\vec{v}) &= 0, \\ \partial_t(\vec{B}) &= -\nabla\times\vec{E}, \\ \vec{E} &= -\vec{v}\times\vec{B} \end{aligned} \tag{2}$$

With $\tau = \rho\epsilon + \frac{1}{2}\rho v^2 + \frac{1}{2}B^2$. Note that we set $\mu_0 = 1$, which alters the ampere definition and the magnetic field units. The ideal MHD equations in Cartesian coordinate:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_x\\ \rho v_y\\ \rho v_z\\ \tau\\ \rho\epsilon\\ \rho\psi\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho v_x\\ \rho v_x^2+P+B^2/2-B_x^2\\ \rho v_x v_y-B_xB_y\\ \rho v_x v_z-B_xB_z\\ (\tau+P+B^2/2)v_x-B_x(\vec{v}\cdot\vec{B})\\ (\rho\epsilon+P)v_x\\ \rho\psi v_x\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v_y\\ \rho v_x v_y-B_xB_y\\ \rho v_y^2+P+B^2/2-B_y^2\\ \rho v_z v_y-B_zB_y\\ (\tau+P+B^2/2)v_y-B_y(\vec{v}\cdot\vec{B})\\ (\rho\epsilon+P)v_y\\ \rho\psi v_y\end{pmatrix}$$
$$+\frac{\partial}{\partial z}\begin{pmatrix}\rho v_z\\ \rho v_x v_z-B_xB_z\\ \rho v_y v_z-B_yB_z\\ \rho v_z^2+P+B^2/2-B_z^2\\ (\tau+P+B^2/2)v_z-B_z(\vec{v}\cdot\vec{B})\\ (\rho\epsilon+P)v_z\\ \rho\psi v_z\end{pmatrix}=-\begin{pmatrix}0\\ \rho\partial_x\Phi\\ \rho\partial_y\Phi\\ \rho\partial_z\Phi\\ \rho(v_x\partial_x\Phi+v_y\partial_y\Phi+v_z\partial_z\Phi)\\ -(v_x\partial_xP+v_y\partial_yP+v_z\partial_zP)\\ 0\end{pmatrix} \tag{3}$$

The magnetic fields are solved separately:

$$\frac{\partial}{\partial t}\begin{pmatrix}B_x\\ B_y\\ B_z\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}0\\ v_xB_y-v_yB_x\\ v_xB_z-v_zB_x\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}v_yB_x-v_xB_y\\ 0\\ v_yB_z-v_zB_y\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}v_zB_x-v_xB_z\\ v_zB_y-v_yB_z\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{4}$$

which translates to:

$$\frac{\partial}{\partial t}\begin{pmatrix}B_x\\ B_y\\ B_z\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}0\\ -E_z\\ E_y\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}E_z\\ 0\\ -E_x\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}-E_y\\ E_x\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \tag{5}$$

The electric fields are:

$$\begin{pmatrix}E_x\\ E_y\\ E_z\end{pmatrix}=\begin{pmatrix}v_zB_y-v_yB_z\\ v_xB_z-v_zB_x\\ v_yB_x-v_xB_y\end{pmatrix} \tag{6}$$

In the current version, I assumed you solved MHD. If not, set all B-fields to

0. Now, in cylindrical coordinates:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_r\\ \rho v_\phi\\ \rho v_z\\ \tau\\ \rho\epsilon\\ \rho\psi\end{pmatrix}
+\frac{\partial}{r\partial r}\begin{pmatrix} r\rho v_r\\ r(\rho v_r^2+P+B^2/2-B_r^2)\\ r(\rho v_r v_\phi - B_r B_\phi)\\ r(\rho v_r v_z - B_r B_z)\\ r[(\tau+P+B^2/2)v_r - B_r(\vec{v}\cdot\vec{B})]\\ r(\rho\epsilon+P)v_r\\ r\rho\psi v_r\end{pmatrix}
+\frac{\partial}{r\partial \phi}\begin{pmatrix}\rho v_\phi\\ \rho v_r v_\phi - B_r B_\phi\\ \rho v_\phi^2+P+B^2/2-B_\phi^2\\ \rho v_z v_\phi - B_z B_\phi\\ (\tau+P+B^2/2)v_\phi - B_\phi(\vec{v}\cdot\vec{B})\\ (\rho\epsilon+P)v_\phi\\ \rho\psi v_\phi\end{pmatrix}\\
&+\frac{\partial}{\partial z}\begin{pmatrix}\rho v_z\\ \rho v_r v_z - B_r B_z\\ \rho v_\phi v_z - B_\phi B_z\\ \rho v_z^2+P+B^2/2-B_z^2\\ (\tau+P+B^2/2)v_z - B_z(\vec{v}\cdot\vec{B})\\ (\rho\epsilon+P)v_z\\ \rho\psi v_z\end{pmatrix}
=-\frac{1}{r}\begin{pmatrix}0\\ -(p+B^2/2+\rho v_\phi^2-B_\phi^2)\\ \rho v_r v_\phi - B_r B_\phi\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\\
&\qquad-\begin{pmatrix}0\\ \rho\partial_r\Phi\\ \rho\partial_\phi\Phi/r\\ \rho\partial_z\Phi\\ \rho(v_r\partial_r\Phi+v_\phi(\partial_\phi\Phi)/r+v_z\partial_z\Phi)\\ -(v_r\partial_r P+v_\phi(\partial_\phi P)/r+v_z\partial_z P)\\ 0\end{pmatrix}
\end{aligned}
\tag{7}
$$

The magnetic fields are solved separately:

$$
\begin{aligned}
\frac{\partial}{\partial t}B_r+\frac{\partial}{r\partial\phi}(v_\phi B_r - v_r B_\phi)+\frac{\partial}{\partial z}(v_z B_r - v_r B_z)&=0\\
\frac{\partial}{\partial t}B_\phi+\frac{\partial}{\partial r}(v_r B_\phi - v_\phi B_r)+\frac{\partial}{\partial z}(v_z B_\phi - v_\phi B_z)&=0\\
\frac{\partial}{\partial t}B_z+\frac{\partial}{r\partial r}[r(v_r B_z - v_z B_r)]+\frac{\partial}{r\partial\phi}(v_\phi B_z - v_z B_\phi)&=0
\end{aligned}
\tag{8}
$$

which translates to:

$$
\begin{aligned}
\frac{\partial}{\partial t}B_r+\frac{\partial}{r\partial\phi}(E_z)+\frac{\partial}{\partial z}(-E_\phi)&=0\\
\frac{\partial}{\partial t}B_\phi+\frac{\partial}{\partial r}(-E_z)+\frac{\partial}{\partial z}(E_r)&=0\\
\frac{\partial}{\partial t}B_z+\frac{\partial}{r\partial r}[r(E_\phi)]+\frac{\partial}{r\partial\phi}(-E_r)&=0
\end{aligned}
\tag{9}
$$

The electric fields are:

$$
\begin{pmatrix}E_r\\ E_\phi\\ E_z\end{pmatrix}=\begin{pmatrix}v_z B_\phi - v_\phi B_z\\ v_r B_z - v_z B_r\\ v_\phi B_r - v_r B_\phi\end{pmatrix}
\tag{10}
$$

In spherical coordinates:

$$
\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_r\\ \rho v_\theta\\ \rho v_\phi\\ \tau\\ \rho\epsilon\\ \rho\psi\end{pmatrix}+\frac{\partial}{r^2\partial r}\begin{pmatrix}r^2\rho v_r\\ r^2(\rho v_r^2+P+B^2/2-B_r^2)\\ r^2(\rho v_r v_\theta-B_rB_\theta)\\ r^2(\rho v_r v_\phi-B_rB_\phi)\\ r^2[(\tau+P+B^2/2)v_r-B_r(\vec{v}\cdot\vec{B})]\\ r^2(\rho\epsilon+P)v_r\\ r^2\rho\psi v_r\end{pmatrix}
$$

$$
+\frac{\partial}{r\sin\theta\partial\theta}\begin{pmatrix}\sin\theta\rho v_\theta\\ \sin\theta(\rho v_r v_\theta-B_rB_\theta)\\ \sin\theta(\rho v_\theta^2+P+B^2/2-B_\theta^2)\\ \sin\theta(\rho v_\theta v_\phi-B_\theta B_\phi)\\ \sin\theta[(\tau+P+B^2/2)v_\theta-B_\theta(\vec{v}\cdot\vec{B})]\\ \sin\theta(\rho\epsilon+P)v_\theta\\ \sin\theta\rho\psi v_\theta\end{pmatrix}+\frac{\partial}{r\sin\theta\partial\phi}\begin{pmatrix}\rho v_\phi\\ \rho v_r v_\phi-B_rB_\phi\\ \rho v_\theta v_\phi-B_\theta B_\phi\\ \rho v_\phi^2+P+B^2/2-B_\phi^2\\ [(\tau+P+B^2/2)v_\phi-B_\phi(\vec{v}\cdot\vec{B})]\\ (\rho\epsilon+P)v_\phi\\ \rho\psi v_\phi\end{pmatrix}=
$$

$$
-\frac{1}{r}\begin{pmatrix}0\\ -[2(p+B^2/2)+[\rho(v_\phi^2+v_\theta^2)-(B_\phi^2+B_\theta^2)]]\\ -(P+B^2/2+\rho v_\phi^2-B_\phi^2)/\tan\theta+[\rho v_r v_\theta-B_rB_\theta]\\ [\rho v_r v_\phi-B_rB_\phi]+[\rho v_\phi v_\theta-B_\phi B_\theta]/\tan\theta\\ 0\\ 0\\ 0\end{pmatrix}-\begin{pmatrix}0\\ \rho\partial_r\Phi\\ \rho(\partial_\theta\Phi)/r\\ \rho(\partial_\phi\Phi)/(r\sin\theta)\\ \rho(v_r\partial_r\Phi+v_\theta(\partial_\theta\Phi)/r+v_\phi(\partial_\phi\Phi)/(r\sin\theta)\\ -(v_r\partial_rP+v_\theta(\partial_\theta P)/r+v_\phi(\partial_\phi P)/(r\sin\theta)\\ 0\end{pmatrix} \tag{11}
$$

The magnetic fields are solved separately:

$$
\begin{aligned}
\frac{\partial}{\partial t}B_r+\frac{\partial}{r\sin\theta\partial\theta}[\sin\theta(v_\theta B_r-B_\theta v_r)]+\frac{\partial}{r\sin\theta\partial\phi}(v_\phi B_r-B_\phi v_r)&=0\\
\frac{\partial}{\partial t}B_\theta+\frac{\partial}{\partial r}[r(v_rB_\theta-B_rv_\theta)]+\frac{\partial}{r\sin\theta\partial\phi}(v_\phi B_\theta-B_\phi v_\theta)&=0\\
\frac{\partial}{\partial t}B_\phi+\frac{\partial}{r\partial r}[r(v_rB_\phi-B_rv_\phi)]+\frac{\partial}{r\partial\theta}(v_\theta B_\phi-B_\theta v_\phi)&=0
\end{aligned} \tag{12}
$$

which translates to:

$$
\begin{aligned}
\frac{\partial}{\partial t}B_r+\frac{\partial}{r\sin\theta\partial\theta}[\sin\theta(E_\phi)]+\frac{\partial}{r\sin\theta\partial\phi}(-E_\theta)&=0\\
\frac{\partial}{\partial t}B_\theta+\frac{\partial}{r\partial r}[r(-E_\phi)]+\frac{\partial}{r\sin\theta\partial\phi}(E_r)&=0\\
\frac{\partial}{\partial t}B_\phi+\frac{\partial}{r\partial r}[r(E_\phi)]+\frac{\partial}{r\partial\theta}(-E_r)&=0
\end{aligned} \tag{13}
$$

The electric fields are:

$$
\begin{pmatrix}E_r\\ E_\theta\\ E_\phi\end{pmatrix}=\begin{pmatrix}v_\phi B_\theta-v_\theta B_\phi\\ v_rB_\phi-v_\phi B_r\\ v_\theta B_r-v_rB_\theta\end{pmatrix} \tag{14}
$$