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Question 1: Let $T(1) = c$. Find the closed form of each of the following recurrence equations:

1. $T(n) = 2T\left(\frac{n}{2}\right) + \log n$
2. $T(n) = 8T\left(\frac{n}{2}\right) + n^2$
3. $T(n) = 16T\left(\frac{n}{2}\right) + (n\log n)^4$
4. $T(n) = 7T\left(\frac{n}{3}\right) + n$
5. $T(n) = 9T\left(\frac{n}{3}\right) + n^3 \log n$

1. $T(n) = 2T\left(\frac{n}{2}\right) + \log n$

$T(2) = 2T(1) + \log 2$
$T(2) = 2c + 1 = 2c + [4 - (2 + \log 2)]$
$T(4) = 2T(2) + \log 4$
$T(4) = 4c + 4 = 4c + [8 - (2 + \log 4)]$
$T(8) = 2T(4) + \log 8$
$T(8) = 8c + 11 = 8c + [16 - (2 + \log 8)]$
$T(16) = 2T(8) + \log 16$
$T(16) = 16c + 26 = 16c + [32 - (2 + \log 16)]$

$T(n) = n * c + 2n - (2 + \log n)$

2. $T(n) = 8T\left(\frac{n}{2}\right) + n^2$

$= 8[8T\left(\frac{n}{4}\right) + (\frac{n}{2})^2] + n^2$
$= 8^2T\left(\frac{n}{4}\right) + 3n^2$
$= 8^2[8T\left(\frac{n}{8}\right) + (\frac{n}{4})^2] + 3n^2$
$= 8^3T\left(\frac{n}{8}\right) + 7n^2$
$= 8^3[8T\left(\frac{n}{16}\right) + (\frac{n}{8})^2] + 7n^2$
$= 8^4T\left(\frac{n}{16}\right) + 15n^2$
…
$= 8^kT\left(\frac{n}{2^k}\right) + (2^k - 1)n^2$
$\frac{n}{2^k} = 1$ => $k = \log n$

$T(n) = n^3T(1) + (n-1)n^2 = c * n^3 + (n-1)n^2$

3. $T(n) = 16T\left(\frac{n}{2}\right) + (n\log n)^4$

$= 16[16T\left(\frac{n}{4}\right) + (\frac{n}{2}\log\frac{n}{2})^4] + (n\log n)^4$
$= 16^2T\left(\frac{n}{4}\right) + n^4\,[(\log\frac{n}{2})^4 + (\log n)^4]$
$= 16^2[16T\left(\frac{n}{8}\right) + (\frac{n}{4}\log\frac{n}{4})^4] + n^4\,[(\log\frac{n}{2})^4 + (\log n)^4]$

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$$= 16^3T\left(\frac{n}{8}\right) + n^4[(\log\frac{n}{4})^4 + (\log\frac{n}{2})^4 + (\log n)^4]$$
$$= 16^3[16T\left(\frac{n}{16}\right) + (\frac{n}{8}\log\frac{n}{8})^4] + n^4[(\log\frac{n}{4})^4 + (\log\frac{n}{2})^4 + (\log n)^4]$$
$$= 16^4T\left(\frac{n}{16}\right) + n^4[(\log\frac{n}{8})^4 + (\log\frac{n}{4})^4 + (\log\frac{n}{2})^4 + (\log n)^4]$$
…
$$= 16^kT\left(\frac{n}{2^k}\right) + n^4[(\log\frac{n}{2^{k-1}})^4 + (\log\frac{n}{2^{k-2}})^4 + (\log\frac{n}{2^{k-3}})^4 + \cdots + (\log n)^4]$$

$\frac{n}{2^k} = 1 \;\; => k = \log n$

$T(n)$= $16^{\log n} * c + n^4 \; [\sum_{i=0}^{n-1} \left(\log\frac{n}{2^i}\right)^4]$

4. $T(n) = 7T\left(\frac{n}{3}\right) + n$
$$= 7[7T\left(\frac{n}{9}\right) + \frac{n}{3}] + n$$
$$= 7^2T\left(\frac{n}{9}\right) + [n + \frac{7^1 n}{3}]$$
$$= 7^2[7T\left(\frac{n}{27}\right) + \frac{n}{9}] + [n + \frac{7^1 n}{3}]$$
$$= 7^3T\left(\frac{n}{27}\right) + [n + \frac{7^1 n}{3} + \frac{7^2 n}{9}]$$
$$= 7^3[7T\left(\frac{n}{81}\right) + \frac{n}{27}] + [n + \frac{7^1 n}{3} + \frac{7^2 n}{9}]$$
$$= 7^4T\left(\frac{n}{81}\right) + [\frac{7^0 n}{3^0} + \frac{7^1 n}{3^1} + \frac{7^2 n}{3^2} + \frac{7^3 n}{3^3}]$$
…
$$= 7^kT\left(\frac{n}{3^k}\right) + n[\sum_{i=0}^{n-1}\left(\frac{7}{3}\right)^i]$$
$$\left(\frac{n}{3^k}\right) = 1 => k = log_3 n$$

$$T(n) = 7^{log_3 n} * c + n[\sum_{i=0}^{n-1}\left(\frac{7}{3}\right)^i]$$

5. $T(n) = 9T\left(\frac{n}{3}\right) + n^3 \log n$
$$= 9[9T\left(\frac{n}{9}\right) + (\frac{n}{3})^3 \log(\frac{n}{3})] + n^3 \log n$$
$$= 9^2T\left(\frac{n}{9}\right) + n^3\,(\frac{4}{3}\log n - \frac{1}{3})$$
$$= 9^2[9T\left(\frac{n}{27}\right) + (\frac{n}{9})^3 \log(\frac{n}{9})] + n^3\,(\frac{4}{3}\log n - \frac{1}{3})$$
note $\frac{1}{1} + \frac{1}{3} = \frac{4}{3}$
$$= 9^3T\left(\frac{n}{27}\right) + n^3\,(\frac{13}{9}\log n - \frac{5}{9})$$
note $\frac{1}{1} + \frac{1}{3} + \frac{1}{9} = \frac{13}{9}$ and $\frac{1}{3} + \frac{2}{9} = \frac{5}{9}$

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$$= 9^3 \left[9T\left(\frac{n}{81}\right) + \left(\frac{n}{27}\right)^3 \log\left(\frac{n}{27}\right) \right] + n^3\left(\frac{13}{9}\log n - \frac{5}{9}\right)$$

$$= 9^4 T\left(\frac{n}{81}\right) + n^3\left(\frac{40}{27}\log n - \frac{2}{3}\right)$$

note $\frac{1}{1} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} = \frac{40}{27}$ and $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} = \frac{2}{3}$

…

$$= 9^k T\left(\frac{n}{3^k}\right) + n^3 \left[\; log_3 n \; \sum_{i=0}^{n-1}\left(\frac{1}{3^i}\right) - \sum_{i=0}^{n-1}\left(\frac{i}{3}\right)^i \; \right]$$

$$\frac{n}{3^k} = 1 \;\; => k = log_3 n$$

$$T(n) = 9^{log_3 n} * c + n^3 \left[\; log_3 n \; \sum_{i=0}^{n-1}\left(\frac{1}{3^i}\right) - \sum_{i=0}^{n-1}\left(\frac{i}{3}\right)^i \; \right]$$

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Question 2: Given two complex numbers, $x = a + bi, y = c + di$, where a,b,c,d are real numbers. Compute $x \times y$ by using only three real-number multiplications.

A = ac, B = bd, C = (a+b)(c+d) = (ac + ad + bc +bd)

D = A – B = ac –bd

E = C – B – A = (ac + ad + bc +bd)– bd –ac = ad +bc

X x Y = D + iE = (ac-bd) + i(ad+bc)

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Question 3: Show the steps of Strassen's algorithm to multiply the following two 4 X 4 matrices:

$$X = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 5 & 8 & 3 & 2 \\ 3 & 3 & 5 & 9 \\ 1 & 3 & 4 & 2 \end{bmatrix},\ Y = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 7 & 1 & 4 & 4 \\ 4 & 8 & 6 & 3 \\ 5 & 7 & 4 & 2 \end{bmatrix}.$$

To keep your answer shorter, you do not have to recursively apply Strassen's algorithm to the subproblem on 2X2 matrices.

- Divide each matrix into 4 parts. Let X_1 = top left matrix, X_2 = top right matrix, X_3 = bottom left matrix, and X_4= bottom right matrix. Y is split into Y_1, Y_2, Y_3, Y_4 similarly

$$X = \left[\begin{array}{cc|cc} 2 & 2 & 2 & 1 \\ 5 & 8 & 3 & 2 \\ \hline 3 & 3 & 5 & 9 \\ 1 & 3 & 4 & 2 \end{array}\right],\ Y = \left[\begin{array}{cc|cc} 5 & 4 & 2 & 1 \\ 7 & 1 & 4 & 4 \\ \hline 4 & 8 & 6 & 3 \\ 5 & 7 & 4 & 2 \end{array}\right].$$

- According to Strassen's algorithm,
  - For a matrix X = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - For a matrix Y = $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$
  - P1= a * (f – h)
  - P2 = (a + b)*h
  - P3 = (c + d)*e
  - P4 = d * (g – e)
  - P5 = (a + d) * (e + h)
  - P6 = (b – d) * (g + h)
  - P7 = (a – c) * (e + f)
  - X * Y = Z
  - Z = $\begin{bmatrix} P6 + P5 + P4 - P2 & P1 + P2 \\ P3 + P4 & P1 + P5 - P7 - P3 \end{bmatrix}$

- Our result matrix will be Z
- Using Strassen's algorithm to get Z,
  - P1 = $X_1 * (Y_2 - Y_4)$
    - P1 = $\begin{bmatrix} 2 & 2 \\ 5 & 8 \end{bmatrix} * \left(\begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \right) = \begin{bmatrix} 2 & 2 \\ 5 & 8 \end{bmatrix} * \begin{bmatrix} -4 & -2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -8 & 0 \\ -20 & 6 \end{bmatrix}$
  - P2 = $(X_1 + X_2) * Y_4$
    - P2 = $\left(\begin{bmatrix} 2 & 2 \\ 5 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \right) * \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 10 \end{bmatrix} * \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 36 & 18 \\ 88 & 44 \end{bmatrix}$

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- P3 = $(X_3 + X_4) * Y_1$
  - P3 = $\left(\begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 4 & 2 \end{bmatrix} \right) * \begin{bmatrix} 5 & 4 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 5 & 5 \end{bmatrix} * \begin{bmatrix} 5 & 4 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 124 & 44 \\ 60 & 25 \end{bmatrix}$
- P4 = $X_4 * (Y_3 - Y_1)$
  - P4 = $\begin{bmatrix} 5 & 9 \\ 4 & 2 \end{bmatrix} * \left(\begin{bmatrix} 4 & 8 \\ 5 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 7 & 1 \end{bmatrix} \right) = \begin{bmatrix} 5 & 9 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} -1 & 4 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} -23 & 74 \\ -8 & 28 \end{bmatrix}$
- P5 = $(X_1 + X_4) * (Y_1 + Y_4)$
  - P5 = $\left(\begin{bmatrix} 2 & 2 \\ 5 & 8 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 4 & 2 \end{bmatrix} \right) * \left(\begin{bmatrix} 5 & 4 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \right) = \begin{bmatrix} 7 & 11 \\ 9 & 10 \end{bmatrix} * \begin{bmatrix} 11 & 7 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 198 & 82 \\ 209 & 93 \end{bmatrix}$
- P6 = $(X_2 - X_4) * (Y_3 + Y_4)$
  - P6 = $\left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 9 \\ 4 & 2 \end{bmatrix} \right) * \left(\begin{bmatrix} 4 & 8 \\ 5 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \right) = \begin{bmatrix} -3 & -8 \\ -1 & 0 \end{bmatrix} * \begin{bmatrix} 10 & 11 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} -102 & -105 \\ -10 & -11 \end{bmatrix}$
- P7 = $(X_1 - X_3) * (Y_1 + Y_2)$
  - P7 = $\left(\begin{bmatrix} 2 & 2 \\ 5 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 1 & 3 \end{bmatrix} \right) * \left(\begin{bmatrix} 5 & 4 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 4 \end{bmatrix} \right) = \begin{bmatrix} -1 & -1 \\ 4 & 5 \end{bmatrix} * \begin{bmatrix} 7 & 5 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} -18 & -10 \\ 83 & 45 \end{bmatrix}$

- Z_1 = P6 + P5 + P4 – P2
  - $Z_1 = \begin{bmatrix} -102 & -105 \\ -10 & -11 \end{bmatrix} + \begin{bmatrix} 198 & 82 \\ 209 & 93 \end{bmatrix} + \begin{bmatrix} -23 & 74 \\ -8 & 28 \end{bmatrix} - \begin{bmatrix} 36 & 18 \\ 88 & 44 \end{bmatrix} = \begin{bmatrix} 37 & 33 \\ 103 & 66 \end{bmatrix}$
- Z_2 = P1 + P2
  - $Z_2 = \begin{bmatrix} -8 & 0 \\ -20 & 6 \end{bmatrix} + \begin{bmatrix} 36 & 18 \\ 88 & 44 \end{bmatrix} = \begin{bmatrix} 28 & 18 \\ 68 & 50 \end{bmatrix}$
- Z_3 = P3 + P4
  - $Z_3 = \begin{bmatrix} 124 & 44 \\ 60 & 25 \end{bmatrix} + \begin{bmatrix} -23 & 74 \\ -8 & 28 \end{bmatrix} = \begin{bmatrix} 101 & 118 \\ 52 & 53 \end{bmatrix}$
- Z_4 = P1 + P5 – P7 – P3
  - $Z_4 = \begin{bmatrix} -8 & 0 \\ -20 & 6 \end{bmatrix} + \begin{bmatrix} 198 & 82 \\ 209 & 93 \end{bmatrix} - \begin{bmatrix} -18 & -10 \\ 83 & 45 \end{bmatrix} - \begin{bmatrix} 124 & 44 \\ 60 & 25 \end{bmatrix} = \begin{bmatrix} 84 & 48 \\ 46 & 29 \end{bmatrix}$

- Z = $\begin{bmatrix} 37 & 33 & 28 & 18 \\ 103 & 66 & 68 & 50 \\ 101 & 118 & 84 & 48 \\ 52 & 53 & 46 & 29 \end{bmatrix}$

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Question 4: Problem C-11.3 in the GT textbook

C-11.3 There is a sorting algorithm, "Stooge-sort," which is named after the comedy team, "The Three Stooges." if the input size, n, is 1 or 2, then the algorithm sorts the input immediately. Otherwise, it recursively sorts the first $2n/3$ elements, then the last $2n/3$ elements, and then the first $2n/3$ elements again. The details are shown in Algorithm 11.5. Show that Stooge-sort is correct and characterize the running time, $T(n)$, for Stooge-sort, using a recurrence equation, and use the master theorem to determine an asymptotic bound for $T(n)$.

```
Algorithm StoogeSort(A, i, j):
  Input: An array, A, and two indices, i and j, such that 1 ≤ i ≤ j ≤ n
  Output: Subarray, A[i..j], sorted in nondecreasing order
  n ← j − i + 1        // The size of the subarray we are sorting
  if n = 2 then
      if A[i] > A[j] then
          Swap A[i] and A[j]
  else if n > 2 then
      m ← ⌊n/3⌋
      StoogeSort(A, i, j − m)      // Sort the first part
      StoogeSort(A, i + m, j)      // Sort the last part
      StoogeSort(A, i, j − m)      // Sort the first part again
  return A
```

Algorithm 11.5: Stooge-sort.

If n <= 2, the algorithm clearly works. If we call the first third of the array X, the second third Y, and the final third Z, then after sorting the first 2/3 of the array recursively, then we know that all elements in X are less than all elements in Y. After recursively sorting the final 2/3 elements, all the elements in Y are less than all the elements in Z. Since all the elements in Y were greater than the elements in X, then all the elements in Z are greater than all the elements in X. Therefore, the elements in Z are in the correct location. Then after sorting the first 2/3 of the array again, we know get the all the elements in X and Y are now in the correct location. Now we have a sorted array and we see that StoogeSort works.

$T(n) = 3 * T\left(\frac{2n}{3}\right) + O(1)$, since StoogeSort is recursively called 3 times on 2/3 of the size of the original input.

Master's Theorm:

f(n) = 1 = $O(n^c)\ if\ c = 0.$

$a = 3, b = 3/2$

$log_b(a) = log_{\frac{3}{2}}(3) = \sim 2.71$

c < 2.71, so T(n) = $O\left(n^{log_{\frac{3}{2}}(3)}\right) = O(n^{2.71})$