1. Assume that $f_{tt}(t,B)$ and $f_{tB}(t,B)$ are bounded. Show as $n \to \infty$, in probability,

$$\sum_{i=1}^{n}\left[\frac{1}{2}f_{tt}(t_{i-1},B_{t_{i-1}})\Delta_{t_i}^2 + f_{tB}(t_{i-1},B_{t_{i-1}})\Delta_{t_i}\Delta_{B_{t_i}}\right] \to 0,$$

where $\Delta_{t_i} = t_i - t_{i-1} = \frac{1}{n}, i = 1,\ldots,n$, and $\Delta_{B_{t_i}} = B_{t_i} - B_{t_{i-1}}$.

$$\sum_{i=1}^{n}\left[\frac{1}{2}f_{tt}(t_{i-1},B_{t_{i-1}})\underbrace{\Delta_{t_i}^2}_{0} + f_{tB}(t_{i-1},B_{t_{i-1}})\underbrace{\Delta_{t_i}\Delta_{B_{t_i}}}_{0}\right]$$

$$= \Sigma[0+0] = 0$$

$\because$

$\times$	dt	dB_t
dt	0	0
dB_t	0	dt

2. Take $t_j = \frac{j}{n}t$. Show that in probability,

$$\lim_{n\to\infty}\sum_{j=1}^{n}|B(t_j) - B(t_{j-1})|^2 = t.$$

$$\lim_{n\to\infty}\sum_{j=1}^{n}|B(t_j)-B(t_{j-1})|^2 = \lim_{n\to\infty}\sum_{j=1}^{n}|B(t_j)^2 - 2B(t_{j-1})B(t_j) + B(t_{j-1})^2|$$

$$= \lim_{n\to\infty}\sum_{j=1}^{n}(t_j - 2t_{j-1} + t_{j-1})$$

$\because W_t W_{t-1} = (W_t - W_{t-1} + W_{t-1})W_{t-1}$

$= (W_t - W_{t-1})W_{t-1} + W_{t-1}^2$

$= t_{-1}$ (W_t는 Brownian Motion)

$$= \lim_{n\to\infty}\sum_{j=1}^{n}(t_j - t_{j-1})$$

$$= \lim_{n\to\infty}(t_n - t_0)$$

$$= \lim_{n\to\infty}(t-0) = t$$

3. We have the following SDE (Vasicek model)

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t. \rightarrow$$ Vasicek stochastic differential equation

Solve the SDE. (Hint: consider $e^{\alpha t}X_t$.)

결국 이 SDE가 원하는 것은 $f(x) = f(\text{without } x)$

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$$

$$dX_t + \alpha X_t dt = \alpha\mu dt + \sigma dB_t$$

$$e^{\int \alpha dt} = e^{\alpha \int 1 dt} = e^{\alpha t} \longrightarrow \because \frac{\partial y}{\partial x} + f(x)y = g(x)$$

integrating factor $\Rightarrow e^{\int f(x)dx}$

integrating factor 이용.

$$e^{\alpha dt}(dX_t + \alpha X_t dt) = e^{\alpha t}(\alpha\mu dt + \sigma dB_t)$$

$$e^{\alpha dt} dX_t + e^{\alpha dt}\alpha X_t dt = e^{\alpha t}\alpha\mu dt + e^{\alpha t}\sigma dB_t$$

$$d(e^{\alpha dt}X_t) = e^{\alpha t}\alpha\mu dt + e^{\alpha t}\sigma dB_t$$

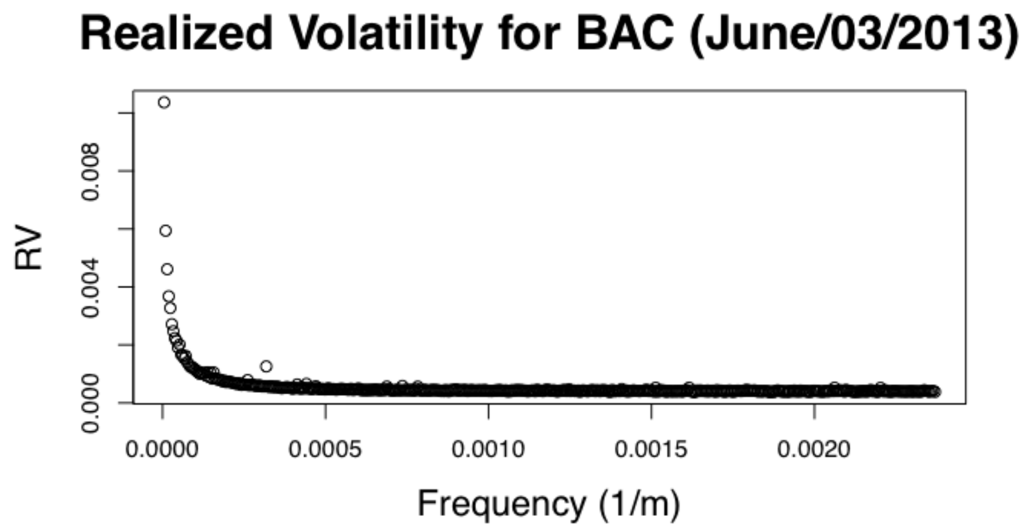
$\because e^{\alpha dt}dX_t + e^{\alpha dt}\alpha X_t dt = d(e^{\alpha dt}X_t)$ by product rule.

$$\int_{t=s}^{t=T} d(e^{\alpha dt}X_t) = \int_s^T e^{\alpha t}\alpha\mu dt + \int_s^T e^{\alpha t}\sigma dW_t$$

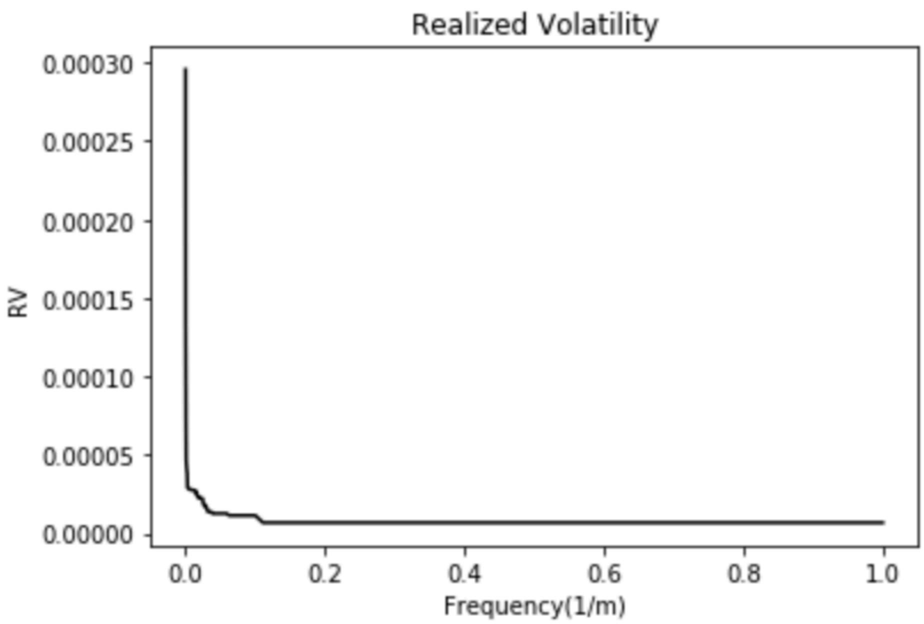
$$e^{\alpha T}X_T - e^{\alpha S}X_S = \int_s^T e^{\alpha t}\alpha\mu dt + \int_s^T e^{\alpha t}\sigma dW_t$$

$$X_T = e^{\alpha(S-T)}X_S + e^{-\alpha T}\int_s^T e^{\alpha t}\alpha\mu dt + e^{-\alpha T}\int_s^T e^{\alpha t}\alpha\mu dt$$

4. Download the high-frequency data (MSFT) from Wharton Research Data Services. Then calculate the realized volatility estimates, $\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2$, as varying the frequency. And make the following plot and

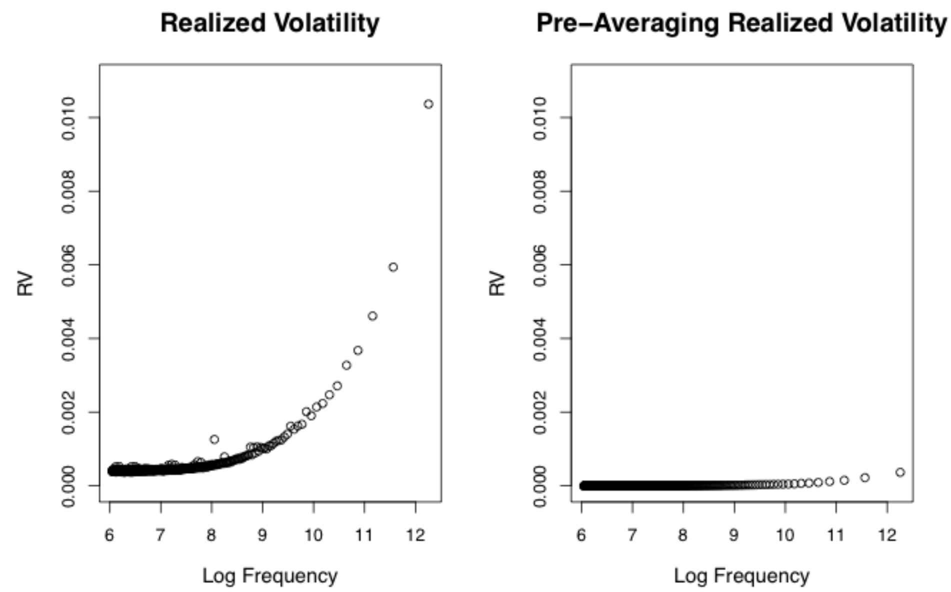


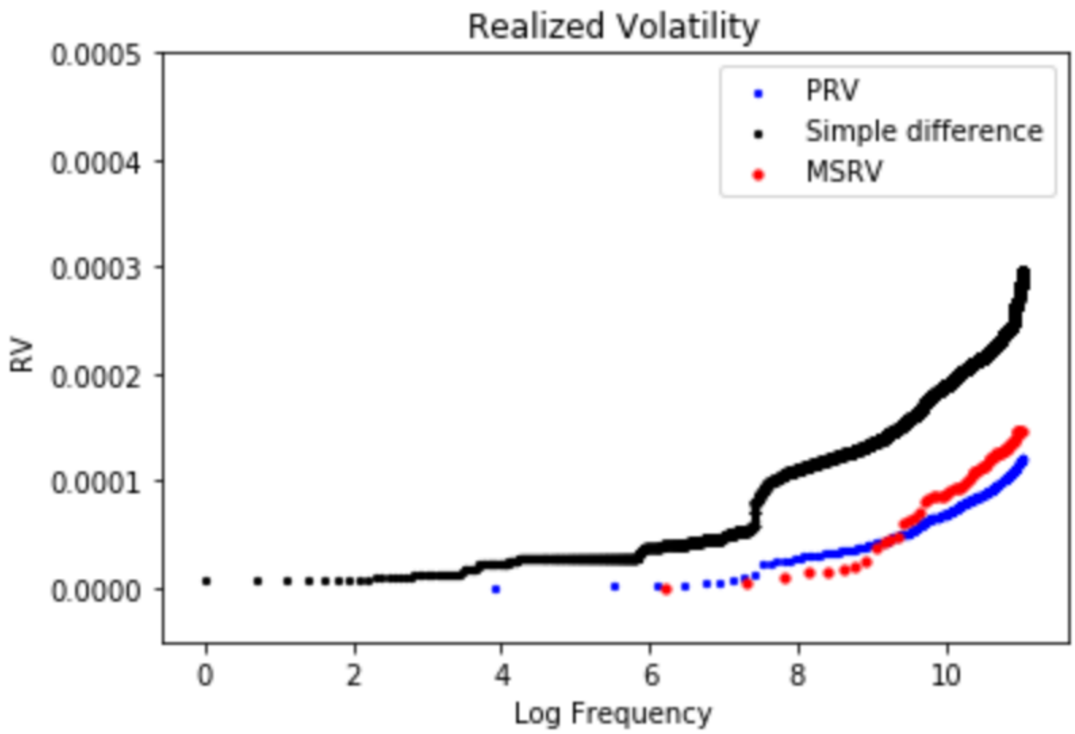
check the inconsistency of the realized volatility.



자세한 내용은 첨부한 Source Code
참고 바랍니다.

5. Download the high-frequency data from Wharton Research Data Services. Then calculate the realized volatility estimates by using MSRV, PRV, and KRV, as varying the frequency. And for each method, make the following plot and compare the consistency of the MSRV, PRV, and KRV procedures.





+ Kernel은 구현하지 못했습니다...

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