

EE2703 : Applied Programming Lab
Assignment 8
Discrete Fourier Transform

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1 Aim

The aim of this assignment is to:

1. Learn Digital Fourier Transform (DFT) and it's properties.
2. Use the in-built Python library to generate DFT of various discrete time signals using the Fast Fourier Transform (FFT) algorithm.
3. Analyze the continuous time Fourier Transform (CTFT) of continuous time signals by plotting the DFT of their sampled DT versions.

2 Theory

2.1 Sampling

Sampling is an important concept in digital signal processing. It is the process of creating a discrete-time signal from a known continuous time signal by sampling it at regular intervals. Also all digital devices and computers we use can perform arithmetic and algorithmic analysis only on discrete data. Hence sampling is a very important concept in signal processing.

Let's say the CT signal is $x_c(t)$. Now the sampled DT signal will be:

$$x[n] = x_c(t)|_{t=nT_s} = x_c(nT_s)$$

where T_s is the sampling period. The associated sampling frequency is:

$$\Omega_s = \frac{2\pi}{T_s}$$

2.2 Nyquist-Shannon Sampling Theorem

Whenever we sample a CT signal $x_c(t)$ and obtain a DT signal $x[n]$, their spectra: $X_c(\Omega)$ and $X(\omega)$ are related as:

$$X(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_s}\right)$$

Now the Nyquist-Shannon sampling theorem states that for a bandlimited signal $x_c(t)$, the sampling frequency Ω_s should be atleast twice the maximum frequency present in the spectrum of $x_c(t)$ in order to prevent overlapping (or Aliasing). Only if this condition is satisfied, the DTFT of $x[n]$ will preserve the original information about the CT signal.

2.3 Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT) is an important concept in digital signal processing. As we already saw, we have a continuous variable spectrum pertaining to DT signals known as the Discrete-Time Fourier Transform (DTFT) which is given by:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

This is a real variable function and is defined for a continuum of values. DFT, on the other hand is a discrete variable signal and is defined only for certain integer values. Let's assume we have an N-periodic DT signal, i.e., if we know its first N samples ($x[0], x[1], \dots, x[N-1]$), we know the full signal. Now its DTFT is also periodic with period as N samples and is defined as:

$$a[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$
$$x[n] = \sum_{k=0}^{N-1} \frac{a[k]}{N} e^{j\frac{2\pi}{N}kn}$$

and both k and n belong to the set $0, 1, 2, \dots, N-1$. Also, the DT signal $x[n]$ calculated by the inverse DFT formula is inherently periodic with period of N samples. Hence DFT is usually used to analyze periodic DT signals. Their values present in one single period contains all information about that signal and its DFT. Note that both $x[n]$ and its DFT $a[k]$ are periodic with period as 'N' samples.

2.4 Connection between CTFT and DFT

Let's say we have a finite length CT signal and this signal, when periodically extended and sampled, gives us a DT signal. Let us define the following functions:

1. $x(t)$ = Finite length CT signal of length T_o
2. $X_{fin}(\Omega)$ = CTFT of $x(t)$
3. $x_c(t)$ = Periodic extension of $x(t)$ with period T_o
4. Ω_o = Fundamental frequency of $x_c(t) = \frac{2\pi}{T_o}$
5. a_k = k^{th} CTFS coefficient of $x_c(t)$
6. $X_c\Omega$ = CTFT of periodic CT signal $x_c(t)$
7. $x[n]$ = Sampled version of $x_c(t)$, i.e., $x_c(nT_s)$
8. Ω_s = Sampling frequency = $\frac{2\pi}{T_s}$
9. ω_o = DFT resolution frequency = $\frac{2\pi}{N}$

10. $X(\omega)$ = DTFT of the signal $x[n]$
11. $a[k] = k^{th}$ DFT coefficient of signal $x[n]$

Let's assume the first N samples that define $x[n]$ are given by sampling the T_o period signal $x(t)$. Hence $T_s = \frac{T_o}{N}$. We have the following important equations which relate the spectra of CT signal and the sampled DT signal:

$$X_{fin}(k\Omega_o) = T_o a_k \quad (1)$$

$$X_c\Omega = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_o) \quad (2)$$

$$X(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_s}\right) \quad (3)$$

$$X(\omega) = \sum_{k=0}^{N-1} \frac{2\pi a[k]}{N} \delta(\omega - k\omega_o) \quad (4)$$

The last equation relates the DTFT and DFT of periodic DT signal $x[n]$. Now let's assume that the CTFT of the periodic CT signal $x_c(t)$ is bandlimited or atleast has a decreasing trend in the fourier series coefficients such that we can assume that the spectrum is approximately bandlimited after a particular value of k . Now we can choose a sampling frequency Ω_s such that the Nyquist criteria is met, i.e., $\Omega_s > 2\Omega_m$ where Ω_m is the maximum frequency component present in $X_c(\Omega)$.

Any CT frequency Ω we have will become ΩT_s in the DT domain. Hence a frequency of $\frac{\Omega}{2}$ in CT domain becomes $\frac{\Omega T_s}{2} = \pi$ in the DT domain. So a range of $[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2})$ in the DT domain. Also since the Nyquist criteria is assumed to be satisfied, we can safely say:

$$X(\omega) = \frac{1}{T_s} X_c\left(\frac{\omega}{T_s}\right) \quad (5)$$

for $\omega \in [-\pi, \pi)$. We are able to get this equation only because Nyquist criteria is satisfied. Else the shifted spectrum terms will also contribute to $X(\omega)$. With this assumption, let's see how $X(\omega)$ will be related to DFT coefficients. From equation (4),(5), we can say:

$$X_c\left(\frac{\omega}{T_s}\right) = \sum_{k=0}^{N-1} \frac{2\pi T_s a[k]}{N} \delta(\omega - k\omega_o) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\frac{\omega}{T_s} - k\frac{2\pi}{T_o}\right) = \sum_{k=-\infty}^{\infty} 2\pi T_s a_k \delta\left(\omega - k\frac{2\pi}{N}\right) \quad (6)$$

Since we have assumed that the CTFS coefficients are limited within $[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2})$ in the CT domain, they will be restricted in $[-\pi, \pi)$ on the DT domain or the ω -axis. So our eqn(6) reduces to:

$$\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{2\pi T_s a[k]}{N} \delta(\omega - k\omega_o) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} 2\pi T_s a_k \delta\left(\omega - k\frac{2\pi}{N}\right)$$

Clearly we have:

$$a_k = \frac{a[k]}{N} \quad (7)$$

Now also we have:

$$X_{fin}(k\Omega_o) = T_o \frac{a[k]}{N} = T_s a[k] \quad (8)$$

The final takeaway point is that if we can safely assume the signal is bandlimited and the sampling frequency satisfies the Nyquist criteria, then the CTFT of the finite length CT signal $x(t)$ can be computed at integer values of frequency Ω_o by multiplying the N-point DFT of the sampled version by T_s .

- CT frequency interval : $[-\frac{\pi N}{T_s}, \frac{\pi N}{T_s}]$
- CT frequency least count: $\Omega_o = \frac{2\pi}{T_s}$

Hence to get the CTFT for a higher range, we can either increase N or decrease T_o or do both. However the resolution can be improved only by increasing T_o . Hence, when we aim to increase the resolution the range automatically gets shortened. To maintain same range, we need to proportionately increase N.

3 Assignment

3.1 Spectrum of $\sin(5t)$

$\sin(5t)$ is a very simple continuous time function. Let's analyze some of its parameters and properties.

- Has a single CT frequency of $\Omega = 5$ rad/s.
- Spectrum is bandlimited i.e., only two impulses are present at $\Omega = \pm 5$ rad/s.
- Since it is periodic, its CTFT will have impulses. So its better to plot its CTFS coefficients

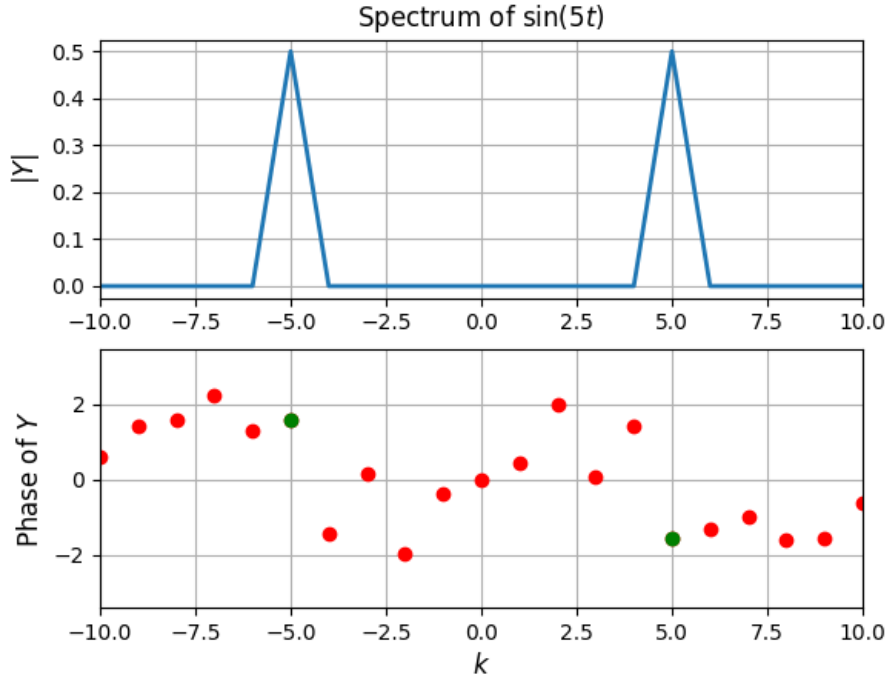
We can write $\sin(5t)$ as:

$$\sin(5t) = \frac{e^{j5t} - e^{-j5t}}{2j} = -0.5je^{j5t} + 0.5je^{-j5t}$$

Clearly the CTFS coefficients are $-0.5j$ and $0.5j$. To get this plot using DFT of the sampled version, we need to satisfy the Nyquist criteria. Also, for the Fast Fourier Transform (FFT) algorithm to work faster, we need to choose an N which a power of 2. Hence let's choose $N = 128$. Also let the finite length signal whose periodic representation is $\sin(5t)$ have a length of 2π i.e., it is $\sin(5t)$ for $t \in [0, 2\pi)$ and zero outside. Hence $T_o = 2\pi$. Thus, $T_s = \frac{2\pi}{128}$ and $\Omega_s = 128$. Clearly $128 > 2 * 5 = 10$. Hence Nyquist criteria is also satisfied. Now, we have:

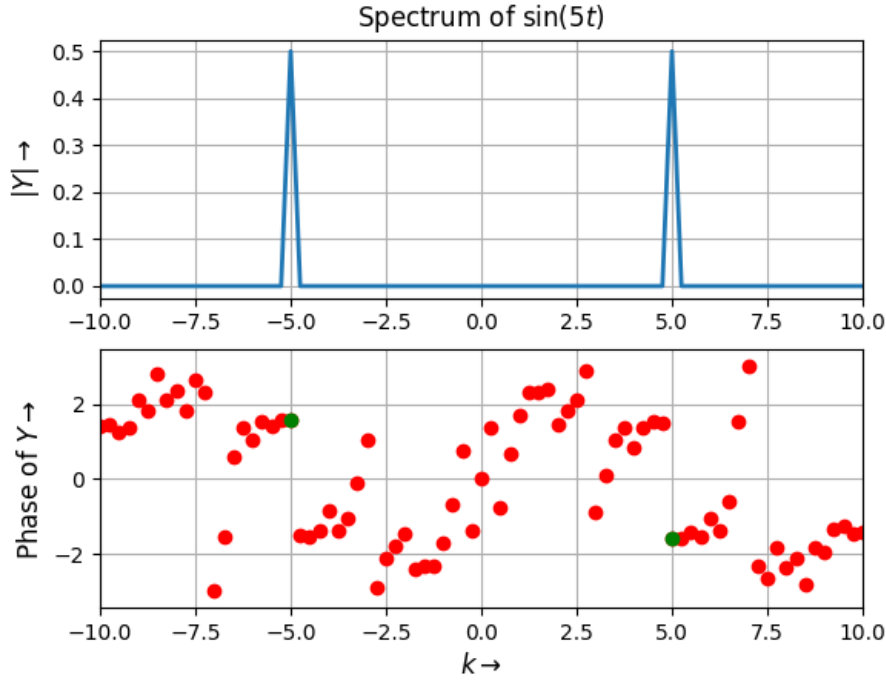
$$a_k = \frac{a[k]}{N}$$

So in our code, after finding the DFT coefficients, we should divide the vector by N to get the CTFS coefficients. after doing all that and plotting the magnitude and phase response plots, we have the following plot:



There are peaks of length 0.5 at $\Omega_o = \pm 5$ as expected

The phase at $\frac{\pi}{2}$ at $\Omega_o = -5$ and $\frac{\pi}{2}$ at $\Omega_o = 5$ as expected. Hence, the plot is correct. To get a more precise plot, we need to increase T_o as resolution will increase but at the same time, we should increase N to maintain the same frequency range for which the CTFT (or CTFS) is calculated. The plot given is for $N = 1024$ and $T_o = 8\pi$:



A much better plot with higher resolution

3.2 Spectrum of amplitude modulated signal

AM modulation is a common technique used in communication of signals. A signal which is to be transmitted, will be modulated over a carrier frequency so that the modulated signal can be transmitted using antennas of small length. The modulated signal can again be converted back to the original signal using signal processing techniques on the receiver end. Let us consider the following example:

$$x(t) = (1 + 0.1 \cos(t)) \cos(10t)$$

The properties of this signal are:

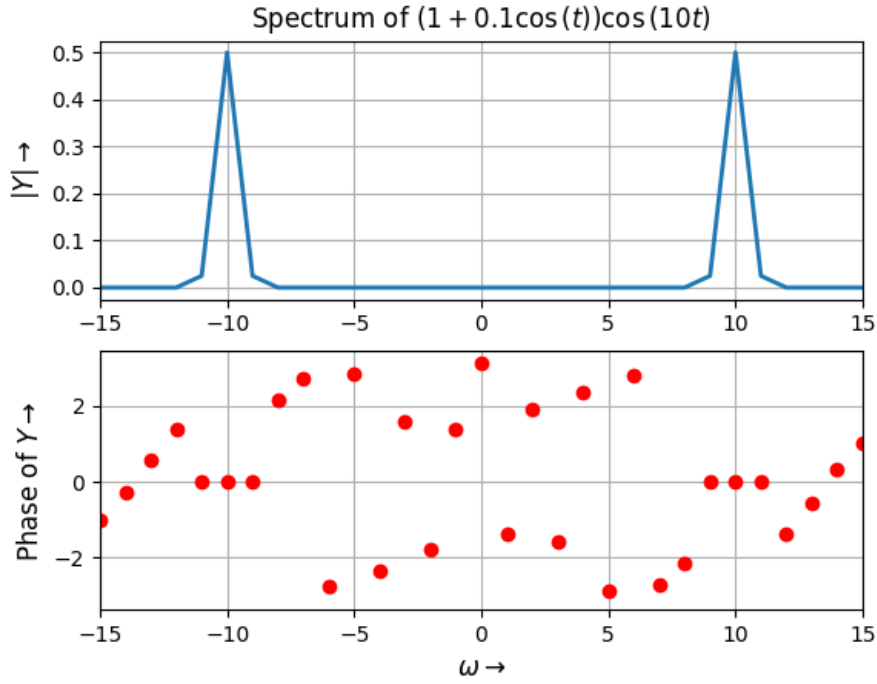
- Has 3 CT frequencies at $\Omega = 9, 10, 11$ rad/s.
- Spectrum is bandlimited i.e., only 2 impulses are present at $\Omega = \pm 9, \pm 10, \pm 11$ rad/s.
- Since it is periodic, its CTFT will have impulses. So its better to plot its CTFS coefficients

We can write this signal as:

$$x(t) = 0.025e^{j9t} + 0.025e^{-j9t} + 0.5e^{j10t} + 0.5e^{-j10t} + 0.025e^{j11t} + 0.025e^{-j11t}$$

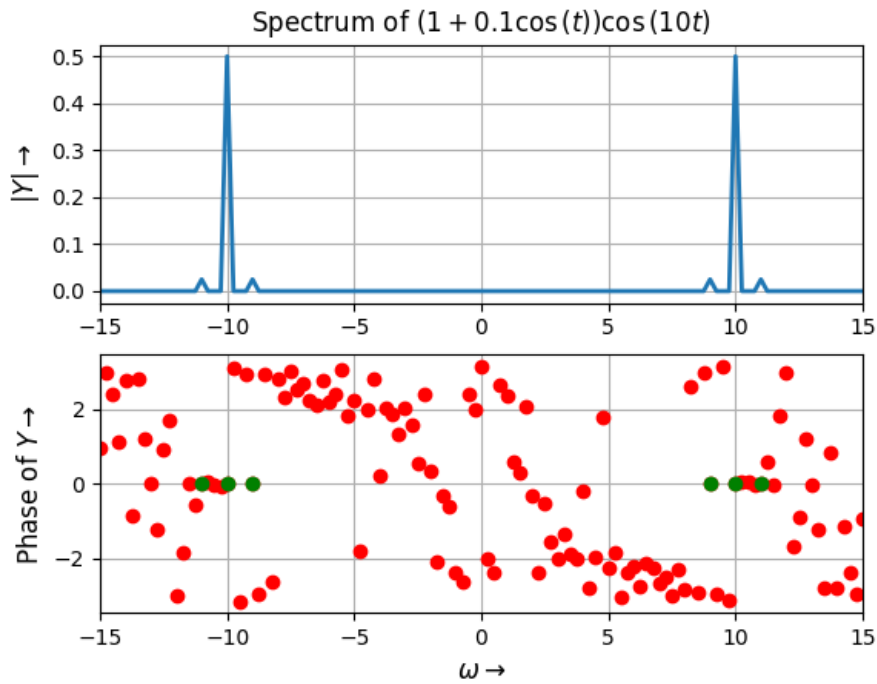
In this question, let's assume we choose $T_o = 2\pi$ and $N = 128$. Now the least count is $\Omega_o = \frac{2\pi}{T_o} = 1$ rad/s. And the CT frequencies we have also differ by 1 rad/s. Hence when we

plt, we won't get distinct impulses. We will get an interpolated plot which is wrong. The plot is given below:



We get interpolated plot which is not correct

To resolve this problem, we need to set the least count less than 1. Our previous least count was $\Omega_o = \frac{2\pi}{T_o} = 1$ rad/s. Now, let's say $T_o = 8\pi$ and $N = 1024$. Now the new least count is $\Omega_o = \frac{2\pi}{T_o} = 0.25$ rad/s. Clearly, now we won't have any wrong interpolation and the graph will be much better and visually appealing. The plot is as given below:



The plot contains magnitudes which are exactly same as the mathematical results we got and the phases are also zero at those frequencies

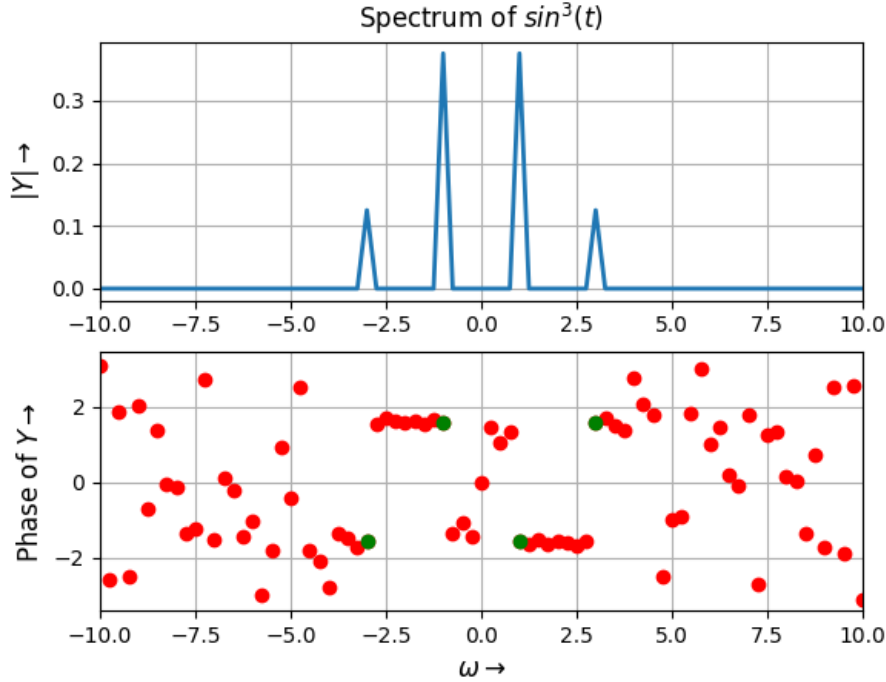
3.3 Spectrum of $\sin^3(t)$, $\cos^3(t)$

Both signals $\sin^3(t)$ and $\cos^3(t)$ are periodic with bandlimited spectrum. We can write them as:

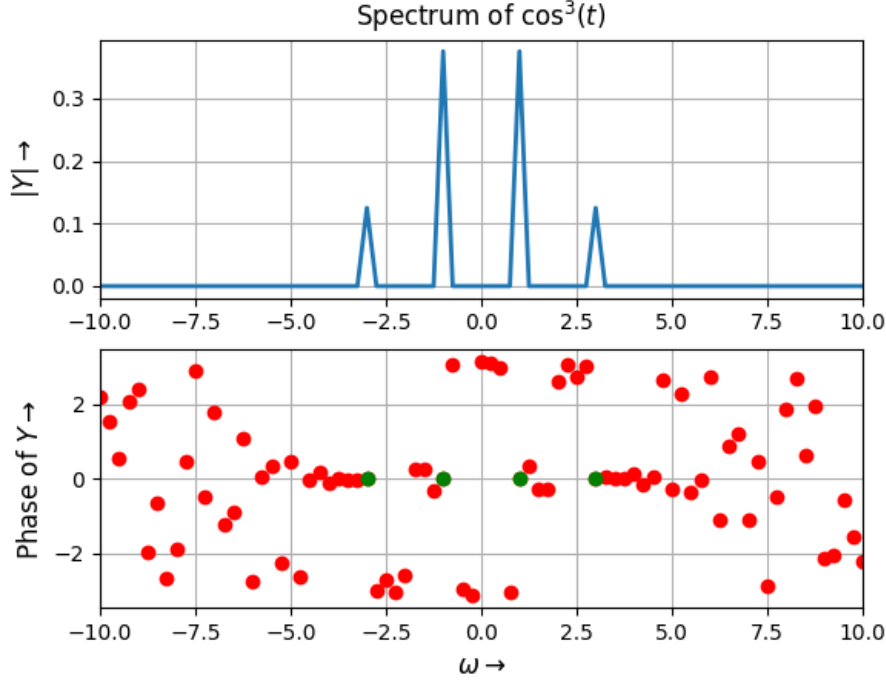
$$\sin^3(t) = \frac{3}{4} \sin(t) - \frac{1}{4} \sin(3t) = -0.375je^{jt} + 0.375je^{-jt} + 0.125je^{j3t} - 0.125je^{-j3t}$$

$$\cos^3(t) = \frac{3}{4} \cos(t) + \frac{1}{4} \cos(3t) = 0.375je^{jt} + 0.375je^{-jt} + 0.125je^{j3t} + 0.125je^{-j3t}$$

Hence the spectrum of both the signals will contain distinct impulses at frequencies 1 and 3 rad/s. The plots are given below:



As expected we have peaks of lengths 0.375 and 0.125. Also, we have phases as $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ at the correct frequencies



Similar to previous graph, we have peaks of lengths 0.375 and 0.125. However, now the phases are purely 0 as expected

3.4 Spectrum of frequency modulated signal

Note: Only in this section, both Ω and ω denote CT frequencies.

Sometimes, signals are also frequency modulated before transmission. One such example is:

$$x(t) = \cos(20t + 5 \cos(t))$$

We can write this function as a sum of complex exponentials based on **Laurent's expansion of Bessel's function**.

$$\exp\left(\frac{\beta}{2}\left(z - \frac{1}{z}\right)\right) = \sum_{k=-\infty}^{\infty} J_k(\beta) z^k$$

Substituting $z = j\exp(jw_m t)$, we have:

$$\exp(j\beta \cos(w_m t)) = \sum_{k=-\infty}^{\infty} J_k(\beta) j^k \exp(jkw_m t)$$

Now we have:

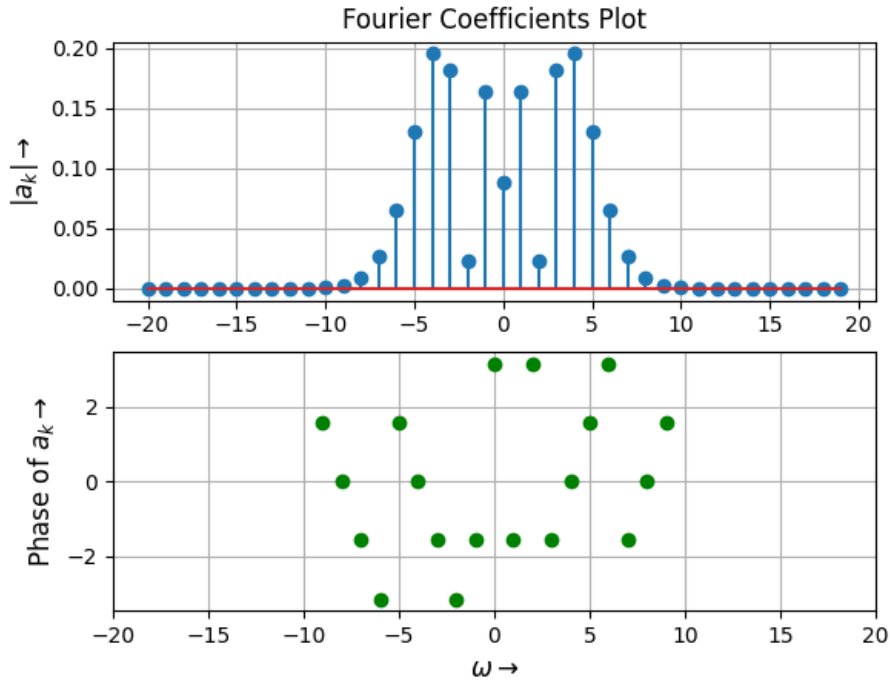
$$\cos(w_c t + \beta \cos(w_m t)) = \text{Re}[\exp(jw_c t + j\beta \cos(w_m t))] = \text{Re}[e^{jw_c t} \sum_{k=-\infty}^{\infty} J_k(\beta) e^{j(w_m t + \frac{\pi}{2})k}]$$

$$\cos(w_c t + \beta \cos(w_m t)) = \sum_{k=-\infty}^{\infty} J_k(\beta) \cos(w_c t + k w_m t + k \frac{\pi}{2})$$

Writing this as a sum of sinusoids, we have:

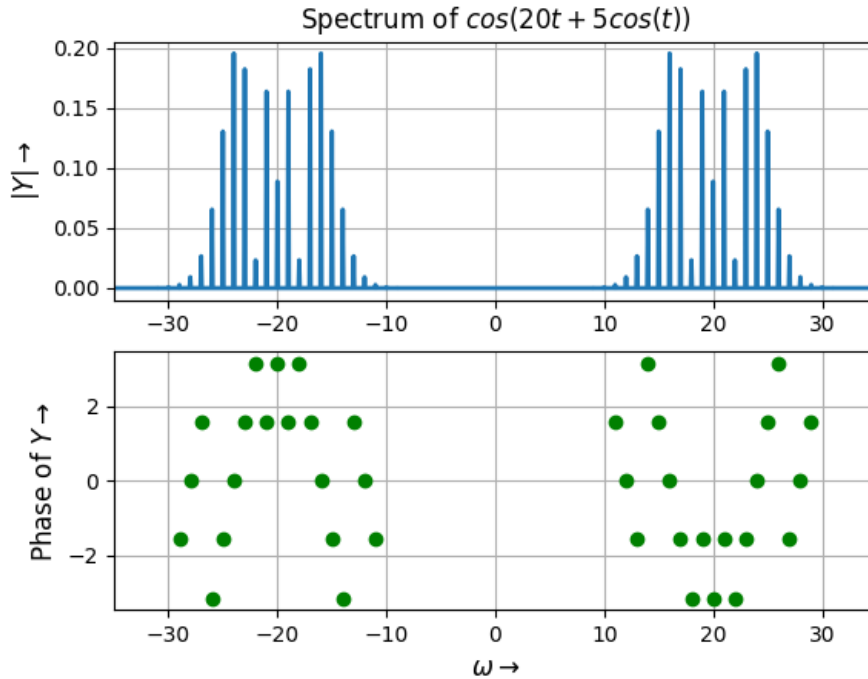
$$\cos(w_c t + \beta \cos(w_m t)) = \sum_{k=-\infty}^{\infty} \frac{J_k(\beta)}{2} e^{j(w_c t + k w_m t + k \frac{\pi}{2})} + \sum_{k=-\infty}^{\infty} \frac{J_k(\beta)}{2} e^{-j(w_c t + k w_m t + k \frac{\pi}{2})}$$

Hence when we plot the CTFT, we will have impulses of length $\frac{J_k(\beta)}{2}$ centered around w_c and $-w_c$. Also the phase of the impulses will depend on the k value. The fourier coefficients i.e., $\frac{J_k(\beta)}{2} j^k$ are plotted for $\beta = 5$ and $w_m = 1$ as given in the question.



Now we have an idea about the fourier coefficients of the FM signal. We see that the signal has a bandlimited spectrum with the maximum frequency at $w = \pm 10$ rad/s. So in order to prevent aliasing, we just need to choose a sufficient N . Let's assume $T_o = 32\pi$ and $N = 4096$. Hence maximum frequency is $\frac{\Omega}{2} = 4096 \frac{\pi}{32\pi} = 128 > (20 + 10) = 30$. Hence we can safely assume that the Nyquist criteria is satisfied. We considered $20 + 10$ and not just 10 since now,

the impulses will be around $w_c = 10$. Hence, we need to check whether $w_c + 10$ is less than $\frac{\Omega}{2}$.



We have peaks at frequency $\Omega = 20$ and a few frequencies surrounding it. Also the magnitude response is even and the phase response is odd since the CT signal is real.

The plot obtained from the DFT exactly matches the Bessel's function plot generated manually. Hence, the DFT gave an accurate plot of the CTFS of the FM signal.

Note: The theory about CTFS of FM modulated signal and the relation with Bessel's functions was referred from this link: https://www.dsprelated.com/freebooks/mdft/Sinusoidal_Frequency_Modulation_FM.html

3.5 Spectrum of the Gaussian signal

This Gaussian function is a special signal. Unlike the previous signals, this is a non-periodic signal and also the spectrum is not bandlimited. This is because Gaussian is a **self-function** i.e., it is its own fourier transform.

Let the signal be of the form:

$$x(t) = e^{-\frac{t^2}{2\sigma^2}}$$

Then the continuous time fourier tranform of this signal will be:

$$X(\Omega) = \sqrt{2\pi\sigma^2}e^{-\frac{\Omega^2\sigma^2}{2}}$$

In the example given in this assignment, $\sigma = 1$. Hence the expected plot should be:

$$X(\Omega) = \sqrt{2\pi}e^{-\frac{\Omega^2}{2}}$$

Now before proceeding forward, we must first analyze some important properties of the signal and make some valid approximations based on that. The following properties are important:

- It is not a bandlimited signal - Satisfying Nyquist criteria exactly is impossible
- It is a non-periodic signal
- It is a steeply decreasing positive even function and almost looks like a finite length signal

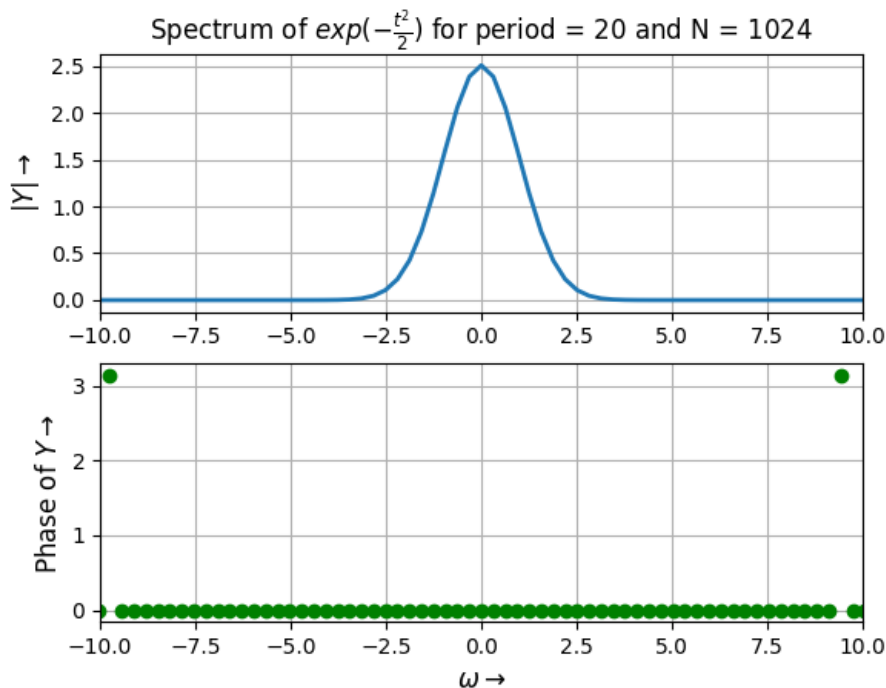
Since the gaussian is a non-periodic signal, its CTFT won't necessarily have impulses (Infact it doesn't have any as we have already derived its CTFT). So, in order to apply the DFT concepts on it, we must first make a periodic CT signal out of it and sample it. Let's first define a new finite length signal like this:

$$x_{finite}(t) = \exp(-\frac{t^2}{2}), t \in [-\frac{T_o}{2}, \frac{T_o}{2})$$

and the signal is zero outside this interval. This new function is not the same gaussian function but a finite piece (or window) of it is taken. Now, let's find its CTFT:

$$X(\Omega) = \int_{-\infty}^{\infty} \exp(-\frac{t^2}{2}) \exp(-j\Omega t) dt \approx \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} \exp(-\frac{t^2}{2}) \exp(-j\Omega t) dt$$

Hence the CTFT of the new finite length function is approximately equal to the CTFT of the Gaussian function. **This is the first assumption.** Now this signal is periodically repeated and a new periodic signal $x_c(t)$ is generated and sampled.



This is the first period of the periodic repetition of the finite length signal ($T_o = 20$)

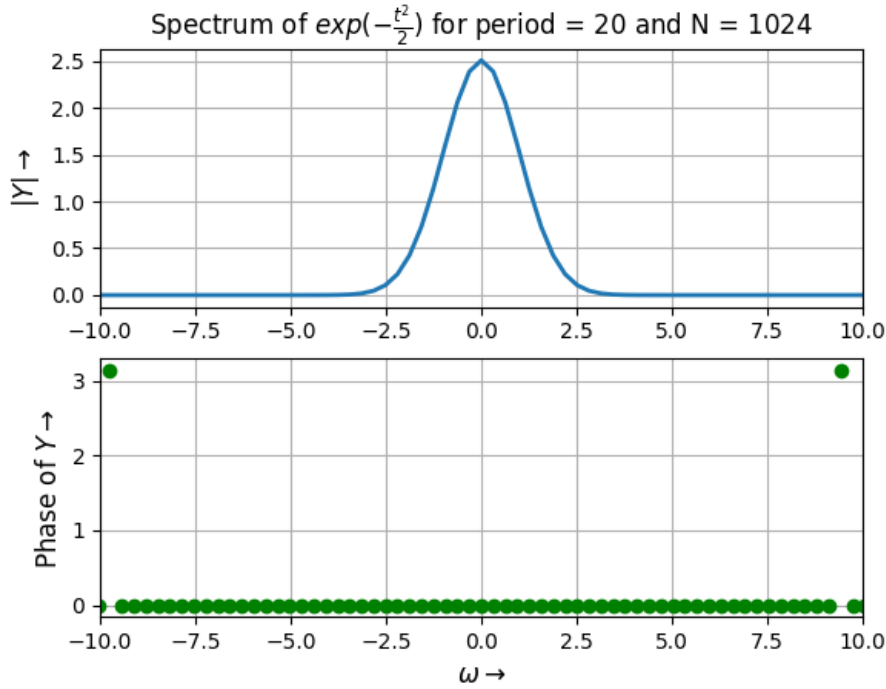
This new periodic function has a period of T_o and hence a fundamental frequency of $\Omega_o = \frac{2\pi}{T_o}$. Hence the signal will have CTFS coefficients at frequencies which are multiples of Ω_o . Now we have already proved in section (2.4) that:

$$X(k\Omega_o) = T_o a_k$$

and we know $X(\Omega)$ decreases as Ω increases. Hence we can safely say the CTFS coefficients are nearly 0 at high values of k . Now, the final step is to choose a sampling frequency such that we can safely assume the CTFT of this periodic signal is bandlimited within that range. Let's choose $N = 1024$ and $T_o = 20$. Hence, $\Omega_o = \frac{2\pi}{T_o} = \frac{\pi}{10}$ and $\Omega_s = N\frac{2\pi}{T_o} = 1024\frac{\pi}{10}$. Clearly Ω_s is sufficiently high to assume that the Nyquist criteria is satisfied. **This is the second assumption.** With these two assumptions in mind, we can use all the results we got in section (2.4). The main result which we want is the following equation:

$$X(k\Omega_o) = T_s a[k]$$

Hence, whatever DFT coefficients vector we are getting, we need to multiply it by T_s . The resulting coefficients are nothing but the CTFT values of the windowed Gaussian signal at integral values of Ω_o . The plot given below is for $T_o = 20$ and $N = 1024$



The curve looks very much similar to the CTFT formula we derived

3.6 Effect of T_o and N on the CTFT of the Gaussian

We already derived that the least count of CT frequency is $\Omega_o = \frac{2\pi}{T_o}$ and the frequency range is $[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}] = [-N\frac{2\pi}{T_o}, N\frac{2\pi}{T_o}]$

With these facts in mind, let's understand the important points to improve the spectrum plot of the Gaussian function:

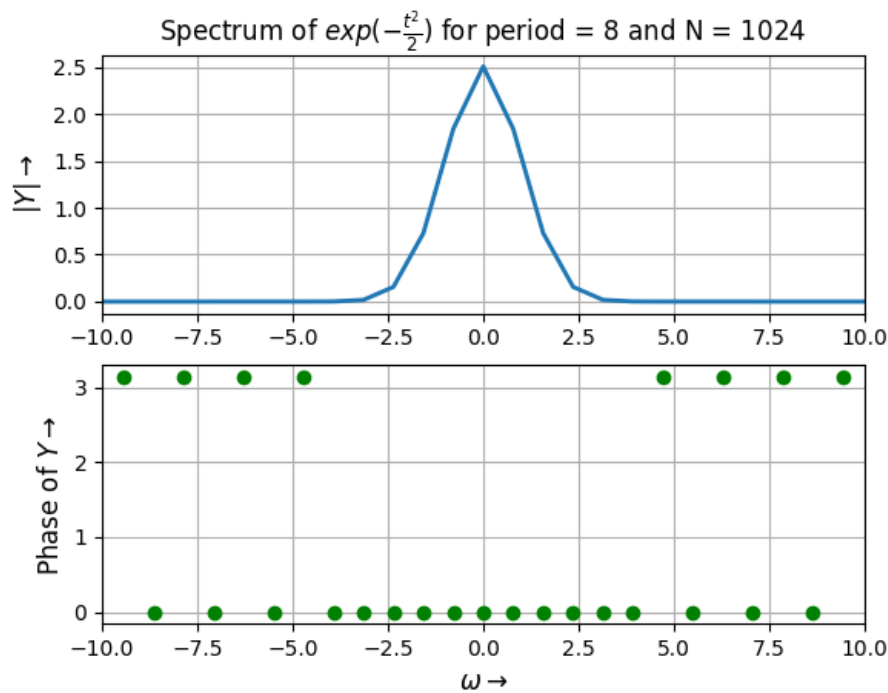
1. Increasing T_o has two benefits:

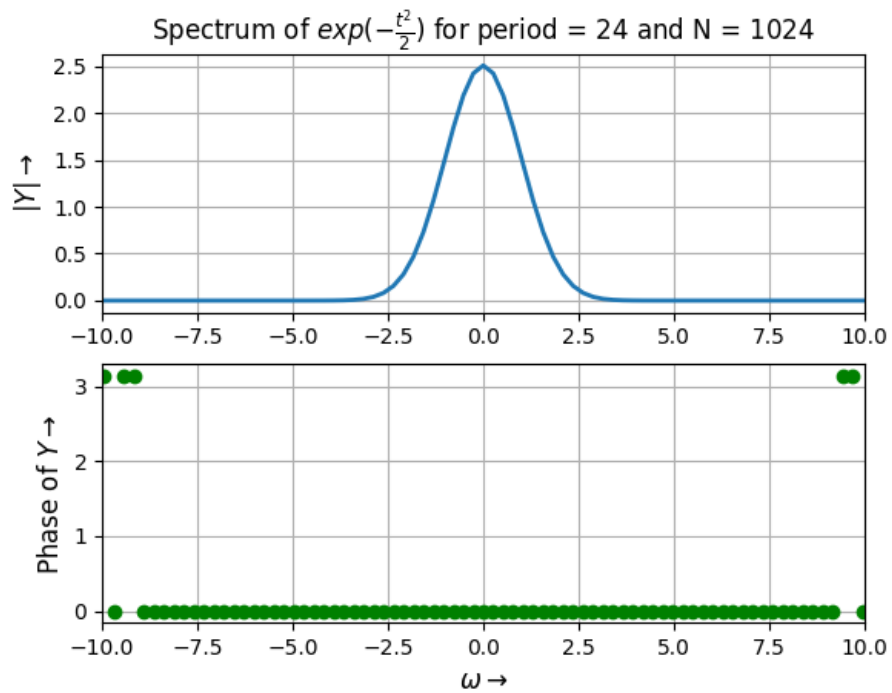
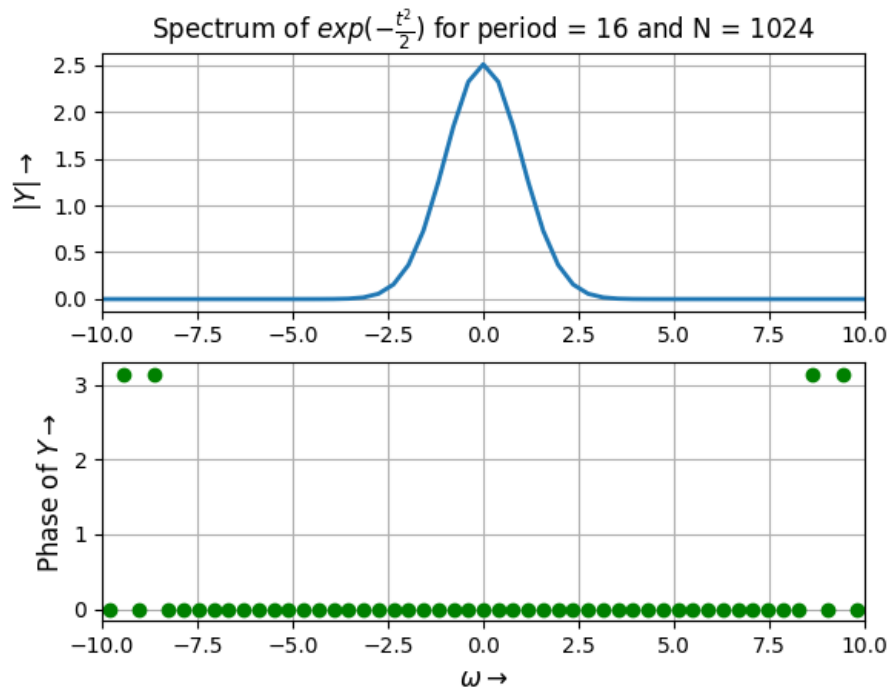
- The window of the finite length signal increases and hence the approximation becomes more accurate.
- The resolution improves and hence, the plot will be much more smoother.

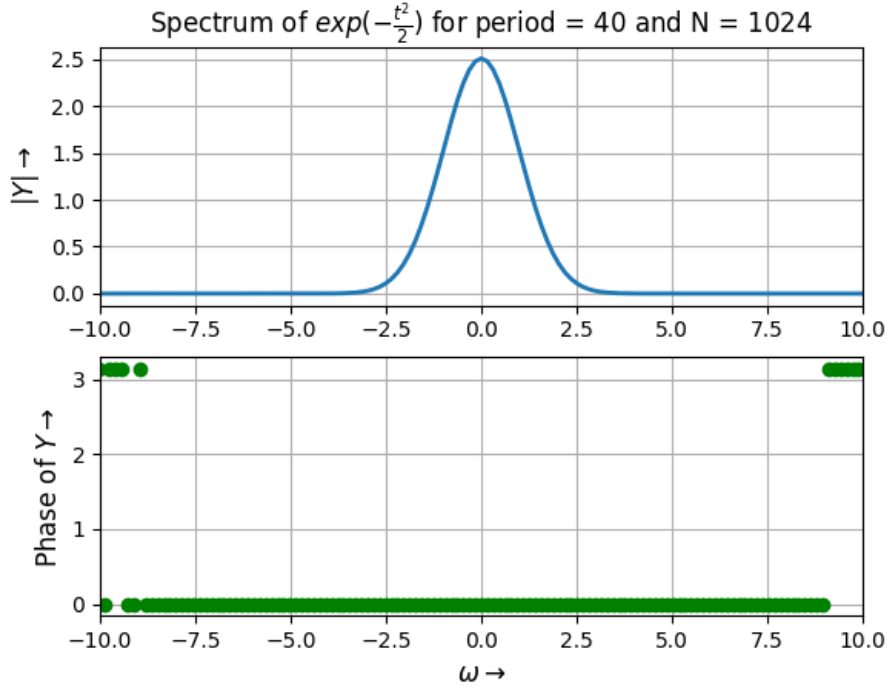
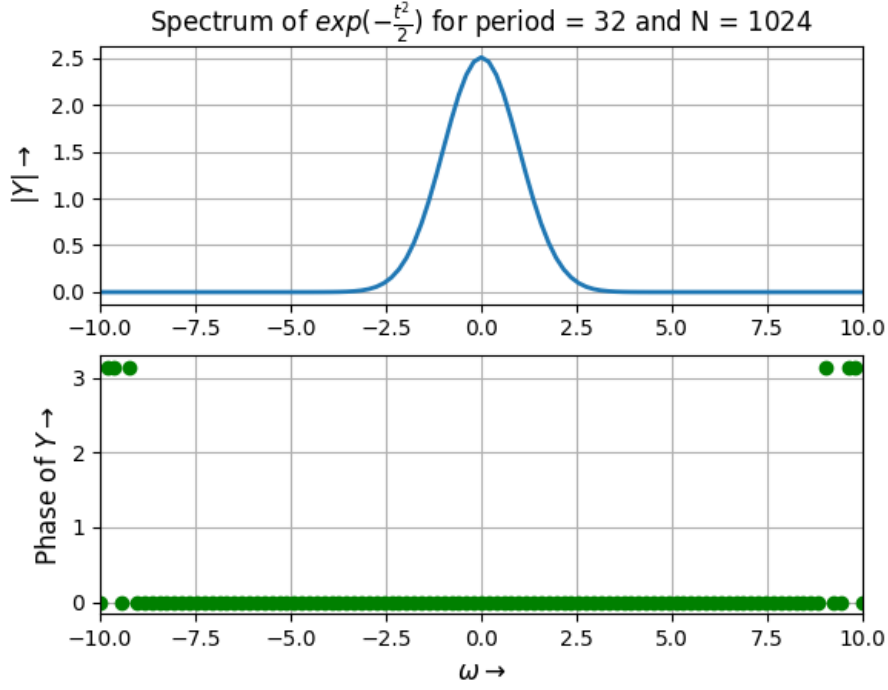
2. Increasing N also has 2 benefits:

- The sampling frequency $\Omega_s = N \frac{2\pi}{T_o}$ increases and hence, the assumption that the Nyquist criteria is satisfied is much more credible
- We can get the CTFT for a broader range of frequency as the range is determined by N .

Hence we expect the graphs to become smoother with increase in period. Let's see if that's the case:

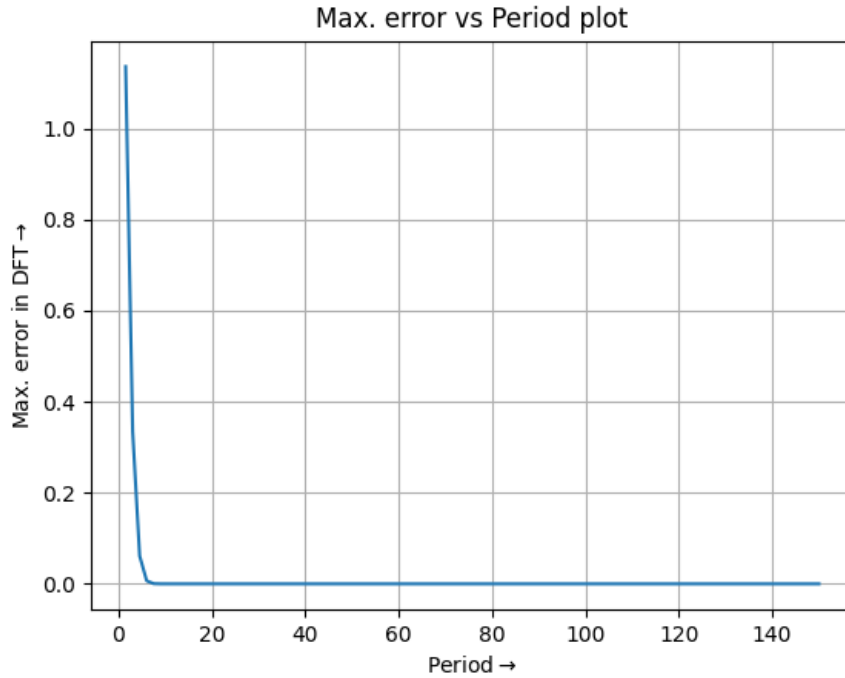






Clearly the graph becomes smoother with increasing T_o value. Also note that the maximum frequency range for which the CTFT is calculated is also decreasing because Ω_s inversely depends on T_o for a given value of N. However in all the plots, the x-limit is set to $[-10, 10)$. And the lowest frequency range in these set of plots is $\frac{\Omega_s}{2} = 512 \frac{2\pi}{40} > 10$. Hence we aren't able to see the decreasing trend in the range of frequencies.

Error plot: As already said, whatever plots we are getting is an approximation and not the exact spectrum of the gaussian. That's why we are even getting phase values of π at high frequencies (Could be due to computer precision or some other non-idealities). However, the graph we get is fairly accurate to the significant range of frequencies i.e., where the spectrum magnitude is more than 0.001. Now, let's see how the maximum error between the computed CTFT values and actual CTFT values change with the period. We expect a decreasing trend with increasing period as we have already seen higher period means better approximation. Let's see if the plot obeys this fact:



In this example, the period T_o starts from 1.5 and ranges till 150 in steps of 1.5. As expected, the error is decreasing with increasing T_o

4 Conclusions

1. The spectrum plot of $\sin(5t)$ is exact and the CTFS coefficients are correctly computed.
2. The spectrum plot of amplitude modulated signal is also exact. Only the resolution had to be improved to get an accurate plot.
3. The spectra plots of $\sin^3(t)$ and $\cos^3(t)$ are also exact.
4. The spectrum plot of frequency modulated signal is very much accurate when the chosen sampling frequency is sufficiently large and it exactly matched the Bessel's function plot generated manually.
5. In all the above cases, the Nyquist criteria is satisfied and hence, the plots are exact.

6. In the gaussian example, the CTFT spectrum plot obtained is very accurate when the window chosen is high because:
 - High resolution
 - Better approximation of the windowed signal with respect to the actual gaussian signal
7. To improve the resolution, the only method is to increase T_o . However, we can control the maximum frequency range (For which the CTFT is computed) by both N as well as T_o .
8. The maximum error between the computed CTFT and actual CTFT decreases with increasing T_o as expected.
9. The window length T_o must be choose so that the periodic repetition of the windowed gaussian almost mimics the sinusoid. In that case, the frequency components in the periodic signal would be less and hence the Nyquist criteria will also be satisfied much more accurately.
10. However we have a trade-off here. Any increase in the period T_o (ideal) will improve the approximation that the CTFT of the windowed signal is same as the CTFT of the original gaussian. However, now the periodic signal will have more frequency components and hence, the Nyquist criteria may not be obeyed fully. To resolve this issue, we must increase the value of N .