1

The following gambling game, known as the wheel of fortune (or chuck-a-luck), is quite popular at many carnivals and gambling casinos:

A player bets on one of the numbers 1 through 6.

Three dice are then rolled, and if the number bet by the player appears i times, i = 1, 2, 3, then the player wins i units;

if the number bet by the player does not appear on any of the dice, then the player loses 1 unit.

Is this game fair to the player?

2

To encourage Elmer's promising tennis career, his father offers him a prize

if he wins (at least) two tennis sets in a row in a three-set series to be played with his father and the club champion alternately: father-champion-father or champion-father-champion, according to Elmer's choice.

The champion is a better player than Elmer's father.

Which series should Elmer choose?

3

An urn contains N white and M black balls.

Balls are randomly selected, one at a time, until a black one is obtained.

If we assume that each ball selected is replaced before the next one is drawn, what is the probability that

(a) exactly n draws are needed?

(b) at least k draws are needed?

4

Consider a jury trial in which it takes 8 of the 12 jurors to convict the defendant;

that is, in order for the defendant to be convicted, at least 8 of the jurors must vote him guilty.

If we assume that jurors act independently and that whether or not thedefendant is guilty, each makes the right decision with probability Θ,

what is the probability that the jury renders a correct decision?

Solutions

1

Solution If we assume that the dice are fair and act independently of one another, then the number of times that the number bet appears is a binomial random variable with parameters $\left(3, \frac{1}{6}\right)$. Hence, letting X denote the player's winnings in the game, we have

$$P\{X = -1\} = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$P\{X = 1\} = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216}$$

$$P\{X = 2\} = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{216}$$

$$P\{X = 3\} = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216}$$

In order to determine whether or not this is a fair game for the player, let us calculate $E[X]$. From the preceding probabilities, we obtain

$$\begin{aligned} E[X] &= \frac{-125 + 75 + 30 + 3}{216} \\ &= \frac{-17}{216} \end{aligned}$$

Hence, in the long run, the player will lose 17 units per every 216 games he plays. ■

Solution for Successive Wins

Since the champion plays better than the father, it seems reasonable that fewer sets should be played with the champion. On the other hand, the middle set is the key one, because Elmer cannot have two wins in a row without winning the middle one. Let C stand for champion, F for father, and W and L for a win and a loss by Elmer. Let f be the probability of Elmer's winning any set from his father, c the corresponding probability of winning from the champion. The table shows the only possible prize-winning sequences together with their probabilities, given independence between sets, for the two choices.

	Father first				Champion first			
Set with:	F	C	F	Probability	C	F	C	Probability
	W	W	W	fcf	W	W	W	cfc
	W	W	L	$fc(1-f)$	W	W	L	$cf(1-c)$
	L	W	W	$(1-f)cf$	L	W	W	$(1-c)fc$
Totals				$fc(2-f)$				$fc(2-c)$

Since Elmer is more likely to best his father than to best the champion, f is larger than c, and $2-f$ is smaller than $2-c$, and so Elmer should choose CFC. For example, for $f = 0.8$, $c = 0.4$, the chance of winning the prize with FCF is 0.384, that for CFC is 0.512. Thus the importance of winning the middle game outweighs the disadvantage of playing the champion twice.

Many of us have a tendency to suppose that the higher the expected number of successes, the higher the probability of winning a prize, and often this supposition is useful. But occasionally a problem has special conditions that destroy this reasoning by analogy. In our problem the expected number of wins under CFC is $2c + f$, which is less than the expected number of wins for FCF, $2f + c$. In our example with $f = 0.8$ and $c = 0.4$, these means are 1.6 and 2.0 in that order. This opposition of answers gives the problem its flavor. The idea of independent events is explained in PWSA, pp. 81–84.

3

The Geometric Random Variable

$$P\{X = n\} = (1 - p)^{n-1}p \qquad n = 1, 2, \ldots \tag{8.1}$$

Solution If we let X denote the number of draws needed to select a black ball, then X satisfies Equation (8.1) with $p = M/(M + N)$. Hence,

(a)

$$P\{X = n\} = \left(\frac{N}{M + N}\right)^{n-1} \frac{M}{M + N} = \frac{MN^{n-1}}{(M + N)^n}$$

(b)

$$\begin{aligned} P\{X \geq k\} &= \frac{M}{M + N} \sum_{n=k}^{\infty} \left(\frac{N}{M + N}\right)^{n-1} \\ &= \left(\frac{M}{M + N}\right)\left(\frac{N}{M + N}\right)^{k-1} \Big/ \left[1 - \frac{N}{M + N}\right] \\ &= \left(\frac{N}{M + N}\right)^{k-1} \end{aligned}$$

Of course, part (b) could have been obtained directly, since the probability that at least k trials are necessary to obtain a success is equal to the probability that the first $k - 1$ trials are all failures. That is, for a geometric random variable,

$$P\{X \geq k\} = (1 - p)^{k-1}$$

■

Solution The problem, as stated, is incapable of solution, for there is not yet enough information. For instance, if the defendant is innocent, the probability of the jury rendering a correct decision is

$$\sum_{i=5}^{12} \binom{12}{i} \theta^i (1 - \theta)^{12-i}$$

whereas, if he is guilty, the probability of a correct decision is

$$\sum_{i=8}^{12} \binom{12}{i} \theta^i (1 - \theta)^{12-i}$$

Therefore, if α represents the probability that the defendant is guilty, then, by conditioning on whether or not he is guilty, we obtain the probability that the jury renders a correct decision:

$$\alpha \sum_{i=8}^{12} \binom{12}{i} \theta^i (1 - \theta)^{12-i} + (1 - \alpha) \sum_{i=5}^{12} \binom{12}{i} \theta^i (1 - \theta)^{12-i} \qquad \blacksquare$$