## Working with Binomials

Before advancing to the next section, we must see some useful mathematical relations related to binomials.

We know that

$$C_k^n = C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

We also know that the Pascal triangle is

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

*Pascal (image from Wikipedia)*

And we can easily find a recursion if we write the Pascal triangle in this way:

| n | k | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | - | - | - | - | - |
| 1 | 1 | 1 | - | - | - | - |
| 2 | 1 | 2 | 1 | - | - | - |
| 3 | 1 | 3 | 3 | 1 | - | - |
| 4 | 1 | 4 | 6 | 4 | 1 | - |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |

*Pascal2*

By looking at the table or by a simple mathematical proof we get the following recurrence:

$$C(n,k) = C(n-1,k) + C(n-1,k-1).$$

And the base cases are

$$C(n,0) = \frac{n!}{0!(n-0)!} = 1 \quad \text{and} \quad C(n,n) = \frac{n!}{n!(n-n)!} = 1.$$

## A Better Approach

With this knowledge in hand, let's define a function $f(d,n)$ that represents the number of floors we can cover using with $d$ remaining drops. If the egg breaks, we will be able to cover $f(d-1,n-1)$ floors; otherwise we'll be able to $f(d-1,n)$ floors. Hence, the total number of floors we will be able to cover is

$$f(d,n) = 1 + f(d-1,n-1) + f(d-1,n).$$

We must find a function $f(d,n)$ that's a solution for this recursion. First, we will define an auxiliary function $g(d,n)$:

$$g(d,n) = f(d,n+1) - f(d,n).$$

Plugging it into our first equation gives

$$\begin{aligned}
g(d,n) &= f(d,n+1) - f(d,n) \\
&= f(d-1,n+1) + f(d-1,n) + 1 - f(d-1,n) - f(d-1,n-1) - 1 \\
&= [f(d-1,n+1) - f(d-1,n)] + [f(d-1,n) - f(d-1,n-1)] \\
&= g(d-1,n) + g(d-1,n-1).
\end{aligned}$$

This is precisely the same recursion that we saw in the previous section, and thus the function $g(d, n)$ can be written

$$g(d, n) = \binom{d}{n}.$$

But we have a problem: $f(0, n)$ is 0 for every $n$, as well as $g(0, n)$, according to the relation between $f$ and $g$. Howev contradiction occurs when $n = 0$ because $g(0, 0) = \binom{0}{0} = 1$. But $g(0, n)$ should be 0 for every $n$! We can fix this pro defining $g(d, n)$ as follows:

$$g(d, n) = \binom{d}{n+1}.$$

And the recursion is still valid (you can check it by yourself!).

Now, using a telescopic sum for $f(d, n)$, we can write it as

$$\begin{aligned} f(d, n) = &[f(d, n) - f(d, n-1)] \\ &+[f(d, n-1) - f(d, n-2)] \\ &+\cdots \\ &+[f(d, 1) - f(d, 0)] \\ &+f(d, 0). \end{aligned}$$

We know that $f(d, 0) = 0$, and therefore

$$f(d, n) = g(d, n-1) + g(d, n-2) + \cdots + g(d, 0).$$

And we also know that

$$g(d, n) = \binom{d}{n+1}.$$

Hence,

$$g(d, n-1) + g(d, n-2) + \cdots + g(d, 0) = \binom{d}{n} + \binom{d}{n-1} + \cdots + \binom{d}{1}.$$

Finally,

$$f(d, n) = \sum_{i=1}^{n} \binom{d}{i}.$$

Now that we have a nice formula for $f(d, n)$, how can we find the minimum number of drops?

It's simple! We know that $f(d, N)$ is the number of floors we can cover in the building with $k$ floors using $N$ eggs an than $d$ drops in the worst cases. We simply have to find a value for $d$ such that

$$f(d, N) \geqslant k.$$

Using our last formula,

$$\sum_{i=1}^{N} \binom{d}{i} \geqslant k.$$

This solution is very fast. We can do a linear search to find a value for $d$, or we can binary search it for an even faster so