1

Conditional probability

A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal.

1. He shuts his eyes, picks a coin at random, and tosses it. What is the probability that the lower face of the coin is a head?
2. He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head? He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?
3. He opens his eyes and sees that the coin isshowing heads; what is the probability that the lower face is a head?
4. He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?

2

Independence

Families. Jane has three children, each of which is equally likely to be a boy or a girl independently of the others. Define the events:

A = {all the children are of the same sex},
B = {there is at most one boy},
C = {the family includes a boy and a girl}.

1) Is A independent of C?
2) Show that A is independent of B, and that B is independent of C.
3) Do these results hold if boys and girls are not equally likely?
4) Do these results hold if Jane has four children?

3

Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace.

What is the probability that it contains exactly one ace given that it contains exactly two kings?

4

Expectation

When you buy a item, there's a small, interesting plastic thing inside. There are c different types of these plastic items, and each time you buy, it's equally likely to have any type. You make one purchase every day.

1. Find out, on average, how many days it takes to get from the jth unique item to the (j + 1)th unique item.
2. Determine the average number of days it takes to collect all the different types of plastic items.

Solutions

1

3. Let M be the event that the first coin is double-headed, R the event that it is double-tailed, and N the event that it is normal. Let H_l^i be the event that the lower face is a head on the ith toss, T_u^i the event that the upper face is a tail on the ith toss, and so on. Then, using conditional probability *ad nauseam*, we find:

(i) $$\mathbb{P}(H_l^1) = \tfrac{2}{5}\mathbb{P}(H_l^1 \mid M) + \tfrac{1}{5}\mathbb{P}(H_l^1 \mid R) + \tfrac{2}{5}\mathbb{P}(H_l^1 \mid N) = \tfrac{2}{5} + 0 + \tfrac{2}{5}\cdot\tfrac{1}{2} = \tfrac{3}{5}.$$

(ii) $$\mathbb{P}(H_l^1 \mid H_u^1) = \frac{\mathbb{P}(H_l^1 \cap H_u^1)}{\mathbb{P}(H_u^1)} = \frac{\mathbb{P}(M)}{\mathbb{P}(H_l^1)} = \tfrac{2}{5}\big/\tfrac{3}{5} = \tfrac{2}{3}.$$

(iii) $$\begin{aligned}\mathbb{P}(H_l^2 \mid H_u^1) &= 1\cdot\mathbb{P}(M \mid H_u^1) + \tfrac{1}{2}\mathbb{P}(N \mid H_u^1)\\ &= \mathbb{P}(H_l^1 \mid H_u^1) + \tfrac{1}{2}\big(1 - \mathbb{P}(H_l^1 \mid H_u^1)\big) = \tfrac{2}{3} + \tfrac{1}{2}\cdot\tfrac{1}{3} = \tfrac{5}{6}.\end{aligned}$$

(iv)

$$\mathbb{P}(H_l^2 \mid H_u^1 \cap H_u^2) = \frac{\mathbb{P}(H_l^2 \cap H_u^1 \cap H_u^2)}{\mathbb{P}(H_u^1 \cap H_u^2)} = \frac{\mathbb{P}(M)}{1\cdot\mathbb{P}(M) + \frac{1}{4}\cdot\mathbb{P}(N)} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{10}} = \frac{4}{5}.$$

(v) From (iv), the probability that he discards a double-headed coin is $\frac{4}{5}$, the probability that he discards a normal coin is $\frac{1}{5}$. (There is of course no chance of it being double-tailed.) Hence, by conditioning on the discard,

$$\mathbb{P}(H_u^3) = \tfrac{4}{5}\mathbb{P}(H_u^3 \mid M) + \tfrac{1}{5}\mathbb{P}(H_u^3 \mid N) = \tfrac{4}{5}\big(\tfrac{1}{4} + \tfrac{1}{2}\cdot\tfrac{1}{2}\big) + \tfrac{1}{5}\big(\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{4}\big) = \tfrac{21}{40}.$$

2

7. (a) $\mathbb{P}(A \cap B) = \frac{1}{8} = \frac{1}{4} \cdot \frac{1}{2} = \mathbb{P}(A)\mathbb{P}(B)$, and $\mathbb{P}(B \cap C) = \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} = \mathbb{P}(B)\mathbb{P}(C)$.

(b) $\mathbb{P}(A \cap C) = 0 \neq \mathbb{P}(A)\mathbb{P}(C)$.

(c) Only in the trivial cases when children are either almost surely boys or almost surely girls.

(d) No.

3

2. Let A be the event of exactly one ace, and KK be the event of exactly two kings. Then $\mathbb{P}(\mathrm{A} \mid \mathrm{KK}) = \mathbb{P}(\mathrm{A} \cap \mathrm{KK})/\mathbb{P}(\mathrm{KK})$. Now, by counting acceptable combinations,

$$\mathbb{P}(\mathrm{A} \cap \mathrm{KK}) = \binom{4}{1}\binom{4}{2}\binom{44}{10}\Big/\binom{52}{13}, \quad \mathbb{P}(\mathrm{KK}) = \binom{4}{2}\binom{48}{11}\Big/\binom{52}{13},$$

so the required probability is

$$\binom{4}{1}\binom{4}{2}\binom{44}{10}\Big/\binom{4}{2}\binom{48}{11} = \frac{7 \cdot 11 \cdot 37}{3 \cdot 46 \cdot 47} \simeq 0.44.$$

4

2. (a) If you have already j distinct types of object, the probability that the next packet contains a different type is $(c-j)/c$, and the probability that it does not is j/c. Hence the number of days required has the geometric distribution with parameter $(c-j)/c$; this distribution has mean $c/(c-j)$.

(b) The time required to collect all the types is the sum of the successive times to collect each new type. The mean is therefore

$$\sum_{j=0}^{c-1} \frac{c}{c-j} = c \sum_{k=1}^{c} \frac{1}{k}.$$

- Geometric distribution
- Binomial distribution