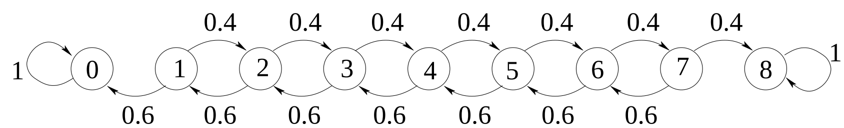
1.

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6. Find the probability that he wins 8 dollars before losing all of his money if (a) he bets 1 dollar each time (timid strategy). (b) he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy). (c) Which strategy gives Smith the better chance of getting out of jail?

Solution. (a) The Markov chain $(X_n, n = 0, 1, \ldots)$ representing the evolution of Smith's money has diagram



Let $\varphi(i)$ be the probability that the chain reaches state 8 before reaching state 0, starting from state i. In other words, if S_j is the first $n \geq 0$ such that $X_n = j$,

$$\varphi(i) = P_i(S_8 < S_0) = P(S_8 < S_0 | X_0 = i).$$

Using first-step analysis (viz. the Markov property at time $n = 1$), we have

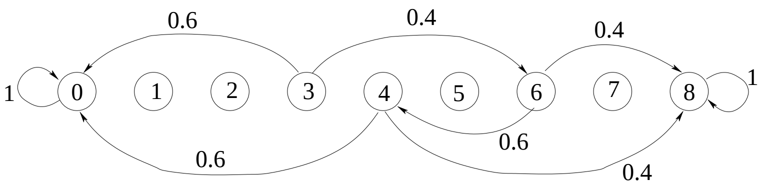
$$\begin{aligned}
&\varphi(i) = 0.4\varphi(i+1) + 0.6\varphi(i-1), \quad i = 1, 2, 3, 4, 5, 6, 7 \\
&\varphi(0) = 0 \\
&\varphi(8) = 1.
\end{aligned}$$

We solve this system of linear equations and find

$$\begin{aligned}
\varphi &= (\varphi(1),\ \varphi(2),\ \varphi(3),\ \varphi(4),\ \varphi(5),\ \varphi(6),\ \varphi(7)) \\
&= (0.0203,\ 0.0508,\ 0.0964,\ 0.1649,\ 0.2677,\ 0.4219,\ 0.6531,\ 1).
\end{aligned}$$

E.g., the probability that the chain reaches state 8 before reaching state 0, starting from state 3 is the third component of this vector and is equal to 0.0964. Note that $\varphi(i)$ is increasing in i, which was expected.

(b) Now the chain is



and the equations are:

$$
\begin{aligned}
\varphi(3) &= 0.4\varphi(6) \\
\varphi(6) &= 0.4\varphi(8) + 0.6\varphi(4) \\
\varphi(4) &= 0.4\varphi(8) \\
\varphi(0) &= 0 \\
\varphi(8) &= 1.
\end{aligned}
$$

We solve and find

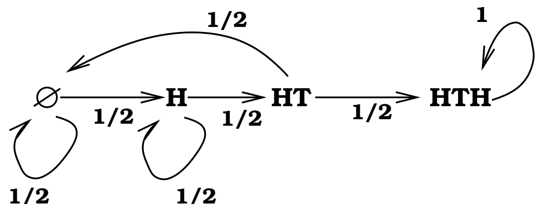
$$\varphi(3) = 0.256, \varphi(4) = 0.4, \varphi(6) = 0.64.$$

(c) By comparing the third components of the vector φ we find that the bold strategy gives Smith a better chance to get out jail.

2.

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

Solution. Call HTH our target. Consider a chain that starts from a state called nothing $\emptyset$ and is eventually absorbed at HTH. If we first toss H then we move to state H because this is the first letter of our target. If we toss a T then we move back to $\emptyset$ having expended 1 unit of time. Being in state H we either move to a new state HT if we bring T and we are 1 step closer to the target or, if we bring H, we move back to H: we have expended 1 unit of time, but the new H can be the beginning of a target. When in state HT we either move to HTH and we are done or, if T occurs then we move to $\emptyset$. The transition diagram is



Rename the states $\emptyset$, H, HT, HTH as $0, 1, 2, 3$, respectively. Let $\psi(i)$ be the expected number of steps to reach HTH starting from i. We have

$$\begin{aligned}
\psi(2) &= 1 + \frac{1}{2}\psi(0) \\
\psi(1) &= 1 + \frac{1}{2}\psi(1) + \frac{1}{2}\psi(2) \\
\psi(0) &= 1 + \frac{1}{2}\psi(0) + \frac{1}{2}\psi(1).
\end{aligned}$$

We solve and find $\psi(0) = 10$.

3.

A gambler plays a game in which on each play he wins one dollar with probability p and loses one dollar with probability $q = 1 - p$. The **Gambler's Ruin Problem** is the problem of finding

$$\begin{aligned}\varphi(x) :=&\text{the probability of winning an amount } b\\ &\text{before losing everything, starting with state } x\\ =&P_x(S_b < S_0).\end{aligned}$$

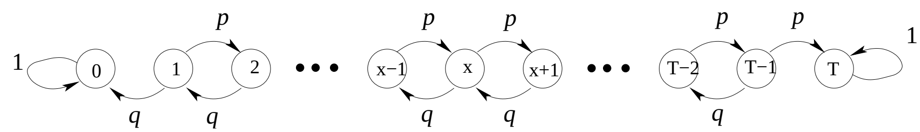
1. Show that this problem may be considered to be an absorbing Markov chain with states $0, 1, 2, \ldots, b$, with 0 and b absorbing states.
2. Write down the equations satisfied by $\varphi(x)$.
3. If $p = q = 1/2$, show that

$$\varphi(x) = x/b.$$

4. If $p \neq q$, show that

$$\varphi(x) = \frac{(q/p)^x - 1}{(q/p)^b - 1}.$$

Solution. 1. If the current fortune is x the next fortune will be either $x + 1$ or $x - 1$, with probability p or 1, respectively, as long as x is not b or x is not 0. We assume independence between games, so the next fortune will not depend on the previous ones; whence the Markov property. If the fortune reaches 0 then the gambler must stop playing. So 0 is absorbing. If it reaches b then the gambler has reached the target hence the play stops again. So both 0 and T are absorbing states. The transition diagram is:



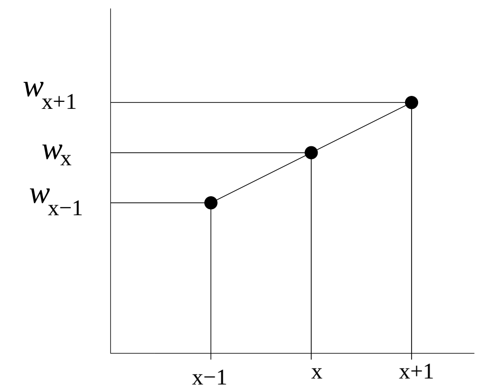
2. The equations are:

$$\begin{aligned}
\varphi(0) &= 0 \\
\varphi(b) &= 1 \\
\varphi(x) &= p\varphi(x+1) + q\varphi(x-1), \quad x = 1, 2, \ldots, b-1.
\end{aligned}$$

3. If $p = q = 1/2$, we have

$$\varphi(x) = \frac{\varphi(x+1) + \varphi(x-1)}{2}, \quad x = 1, 2, \ldots, b-1.$$

This means that the point $(x, \varphi(x))$ in the plane is in the middle of the segment with endpoints $(x-1, \varphi(x-1))$, $(x+1, \varphi(x+1))$. Hence the graph of the function $\varphi(x)$ must be on a straight line (Thales' theorem):



In other words,

$$\varphi(x) = Ax + B.$$

We determine the constants A, B from $\varphi(0) = 0$, $\varphi(b) = 1$. Thus, $\varphi(x) = x/b$.

4. If $p \neq q$, then this nice linear property does not hold. However, if we substitute the given function to the equations, we see that they are satisfied.

4.

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability a that a person will change the answer from yes to no when transmitting it to the next person and a probability b that he or she will change it from no to yes. We choose as states the message, either yes or no. The transition probabilities are

$$p_{yes,no} = a, \quad p_{no,yes} = b.$$

The initial state represents the President's choice. Suppose $a = 0.5$, $b = 0.75$.
(a) Assume that the President says that he or she will run. Find the expected length of time before the first time the answer is passed on incorrectly.
(b) Find the mean recurrence time for each state. In other words, find the expected amount of time r_i, for $i = yes$ and $i = no$ required to return to that state.
(c) Write down the transition probability matrix P and find $\lim_{n\to\infty} \mathsf{P}^n$.

(d) Repeat (b) for general a and b.
(e) Repeat (c) for general a and b.

Solution. (a) The expected length of time before the first answer is passed on incorrectly, i.e. that the President will **not** run in the next election, equals the mean of the geometrically distributed random variable with parameter $1 - p_{yes,no} = 1 - a = 0.5$. Thus, the expected length of time before the first answer is passed on incorrectly is 2. What is found can be viewed as the *mean first passage time* from the state *yes* to the state *no*. By making the corresponding ergodic Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix} \tag{8}$$

absorbing (with absorbing state being *no*), check that the time until absorption will be 2. This is nothing but the mean first passage time from *yes* to *no* in the original Markov chain.
(b) We use the following result to find mean recurrence time for each state:
for an ergodic Markov chain, the mean recurrence time for state i is

$$r_i = E_i T_i = \frac{1}{\pi(i)},$$

where $\pi(i)$ is the ith component of the stationary distribution for the transition probability matrix.
The transition probability matrix (8) has the following stationary distribution:

$$\pi = (.6, .4)\,,$$

from which we find the mean recurrence time for the state *yes* is $\frac{5}{3}$ and for the state *no* is $\frac{5}{2}$.
(c) The transition probability matrix is specified in (8)—it has no zero entries and the corresponding chain is irreducible and aperiodic. For such a chain

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{pmatrix}.$$

Thus,

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}.$$

(d) We apply the same arguments as in (b) and find that the transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

has the following fixed probability vector:

$$\pi = \left(\tfrac{b}{a+b}, \tfrac{a}{a+b}\right),$$

so that the mean recurrence time for the state *yes* is $1 + \frac{a}{b}$ and for the state *no* is $1 + \frac{b}{a}$.
(d) Suppose $a \neq 0$ and $b \neq 0$ to avoid absorbing states and achieve regularity. Then the corresponding Markov chain is regular. Thus,

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}.$$