

Concurrent Balanced Augmented Trees

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Abstract

Augmentation makes search trees tremendously more versatile, allowing them to support efficient aggregation queries, order-statistic queries, and range queries in addition to insertion, deletion, and lookup. In this paper, we present the first lock-free augmented *balanced* search tree. Our algorithmic ideas build upon a recent augmented *unbalanced* search tree presented by Fatourou and Ruppert [DISC, 2024]. We implement both data structures, solving some memory reclamation challenges in the process, and provide an experimental performance analysis of them. We also present optimized versions of our balanced tree that use delegation to achieve better scalability and performance (by more than 2x in some workloads). Our experiments show that our augmented balanced tree is 2.2 to 30 times faster than the unbalanced augmented tree, and up to several orders of magnitude faster than unaugmented trees on 120 threads.

1 Introduction

Sets and Key-Value Stores—which support insertions, deletions, and queries—are among the most fundamental and widely used data objects. Sequentially, two efficient classical data structures are used for these objects: Hash Tables and Balanced Search Trees. Hash Tables are fast; but balanced trees are significantly more versatile, since they preserve ordering of keys and support *augmentation* of the nodes with extra information, to enable many more operations efficiently, including **aggregation queries**, such as *size*, *sum of values*, *maximum*, *minimum*, and *average*; **order statistics**, such as finding the *i*th smallest (or largest) key in the set, or the rank of a given key; and **range queries** that list or aggregate keys in a given range. While augmentation of sequential balanced trees is an indispensable and widely-used technique that is discussed in standard undergraduate algorithms textbooks [10, 16, 27, 28], the technique has evaded concurrent implementation until now. We design and implement the first lock-free *augmented* balanced search tree and demonstrate the efficiency of our data structure empirically.

1.1 Approach and Challenges

In the single-process setting, a binary search tree (BST) is often augmented by adding information to each node to support additional operations. For example, in an order-statistic tree, each node is augmented with a *size* field that stores the number of keys in the subtree rooted at that node. This facilitates the order-statistic queries mentioned above. More generally, augmentation adds *supplementary* fields to each node, whose values can be computed using information in the node and its children. Augmented search trees are building blocks for many other data structures, including measure trees [15], priority search trees [22] and link/cut trees [30].

An update to an augmented search tree must often modify the supplementary fields of many nodes. For example, in an order-statistic tree, an insertion or deletion must update the *size* field of all ancestors of the inserted or deleted node. This gives rise to two key challenges in designing a concurrent augmented tree: all changes to the tree required by an update must appear to take place atomically, and nodes close to the root become hot spots of contention since many operations must modify their supplementary fields.

Recently, Fatourou and Ruppert [13] described a scheme for augmenting concurrent search trees. In particular, they applied the technique to a lock-free unbalanced leaf-oriented BST [11]. Their technique stores multiple versions of the supplementary fields. Operations that update the tree propagate information about the update to each node along the path from the location of the update to the root, step by step. To ensure all changes appear atomic, the changes only become visible to the operations that use the supplementary fields when this propagation reaches the root. Propagation is done *cooperatively*: if several processes try to update a node’s supplementary fields, they need not all succeed, because one update can propagate information about many others. As a bonus, the augmentation scheme’s multiversioning provides simple snapshots of the set of keys stored in the search tree.

Most BST operations take time proportional to the tree’s height, which can be linear in the number of keys in the tree. Hence, *balanced* BSTs, which guarantee the height is

logarithmic in the number of keys, are often preferable. We show how to extend Fatourou and Ruppert’s augmentation technique to get a lock-free augmented *balanced* BST. This requires coping with rotations (rebalancing operations), which can change the structure of the tree at any location, whereas the original paper dealt only with insertions and deletions of leaves. We apply our extension of the augmentation technique to Brown, Ellen and Ruppert’s lock-free implementation [7] of a chromatic BST [25], which provides balancing guarantees. We call our data structure *BAT* (*Balanced Augmented Tree*).

We also show how to add memory reclamation to both the augmented unbalanced BST [13] and the BAT. This required solving several novel challenges, since some shared objects can be reached via multiple paths, so it requires care to track when they have been fully removed from the data structure. We present a lightweight method for this tracking that avoids the overhead of reference counters. A crucial observation is that some objects are safe to free even while still reachable from the root if we can guarantee that no operation will access them.

Using the above ideas, we provide the first C++ implementation of both the augmented unbalanced BST [13] and of BAT. We provide an empirical performance analysis of the augmented BSTs. These experiments show that our BAT scales well and provides order-statistic queries that are, in some cases, orders of magnitude faster than previous concurrent set data structures, which could only handle them by a brute-force traversal of large portions of a snapshot of the data structure.

Naturally, updating supplementary fields in augmented BSTs adds overhead to insertions and deletions. We describe a novel mechanism to significantly reduce this overhead by having processes delegate the work of propagating information about updates up the tree to one another. Roughly speaking, when several updates are trying to propagate information along the same path up the tree, one update propagates information about all of the updates to the root, while the other updates wait for it to complete. In our experiments, this mechanism improves BAT’s performance by up to a factor of 3. It can also be applied to speed up the original augmented (unbalanced) BST of Fatourou and Ruppert [13].

1.2 Our Contributions

- **Lock-Free BAT.** We design the first lock-free balanced augmented search tree data structure.
- **Implementation.** We implement our algorithm in C++ and provide a lightweight memory reclamation scheme.
- **Optimization.** We design two delegation schemes that reduce contention between processes that propagate augmenting values along intersecting paths up the tree.
- **Performance.** Our experiments show that our BAT is between 2.2 and 30 times faster than the augmented unbalanced tree [13] across all our experiments. Compared

to the fastest unaugmented concurrent tree [4], BAT is up to several orders of magnitude faster in workloads with order-statistic queries or large range queries.

2 Related Work

Three papers on augmenting concurrent (unbalanced) search trees appeared in 2024: Fatourou and Ruppert (FR) [13], Kokorin, Yudov, Aksenen and Alistarh (KYAA) [21], and Sela and Petrank (SP) [29]. Our approach extends that of FR (detailed in Section 3.2), which is the most general of the three and also the only one that results in a lock-free data structure. Our safe memory reclamation technique from Section 6 can be applied to their approach too.

KYAA use a lock-based approach in which each node has an associated FIFO queue, and before accessing a node, an operation must join the corresponding queue and help all operations ahead of it before reading or writing to it. KYAA’s approach is specifically designed for order-statistic trees, and it is not clear how to generalize it to other augmentations.

SP also gave a lock-based augmented tree that supports aggregating functions formed using Abelian group operators (i.e., a generalization of augmenting nodes with the sizes of their subtrees). In their approach, update operations announce themselves with timestamps, and each query must gather information from ongoing updates with smaller timestamps than the query using the multiversioning approach of Wei et al. [32].

Our augmentation scheme, like FR’s, has the bonus property of providing simple atomic snapshots of the set of keys in the BST. Taking snapshots of shared data structures has received much attention recently [3, 12, 18–20, 24, 26, 32]. Naïve but inefficient algorithms for order-statistic queries can use such snapshots. For example, one can count the keys in a given range by taking a snapshot and traversing all keys in the range. This takes time linear in the number of keys in the range. Our augmented trees can answer these queries much more efficiently by traversing just two paths of the BST in time proportional to the tree’s height. In our experimental analysis we compare the performance of these two approaches to answering such queries, where the snapshots are provided by the general technique of Wei et al. [32].

Our BAT data structure builds on Nurmi and Soisalon-Soininen’s chromatic tree [25], which was implemented in a lock-free manner by Brown, Ellen and Ruppert [7]. The latter implementation is described in Section 3.1.

3 Background

Chromatic BSTs [25] are a variant of red-black trees [17] that separate the steps that balance the tree from the updates that insert or delete nodes, which makes them more suitable for concurrent implementations. In Section 3.1, we discuss a lock-free implementation of chromatic trees [7], which our BAT data structure builds upon. Section 3.2 describes

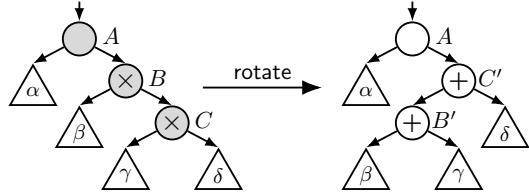


Figure 1. A rotation implemented using LLX and SCX.

Fatourou and Ruppert’s augmentation technique [13], which we extend so that it can be applied to chromatic trees.

3.1 Lock-free Chromatic Trees

Brown, Ellen and Ruppert [7] gave a lock-free implementation of a chromatic BST. It uses LLX and SCX operations, which are an extension of load-link and store-conditional operations. LLX and SCX can be implemented from single-word CAS instructions [6] and provide a simpler way of synchronizing concurrent updates to a data structure.

LLX and SCX operate on a collection of *records*, each consisting of several words of memory, called the record’s *fields*. Records can be *finalized* to prevent further changes to their fields. A process p can perform LLXs on a set V of records and then do an SCX that atomically updates one field of a record in V and finalizes records in a specified subset of V . The SCX succeeds *only if* no SCX has modified any record in V between p ’s LLX of that record and p ’s SCX.

Brown, Ellen and Ruppert [7] described a general technique for implementing lock-free tree data structures using LLX and SCX. The trees have child pointers, but no parent pointers. Each tree node is a record. The record for a node is finalized when the node is removed from the tree. Starting from the tree’s root, an update reads child pointers to arrive at the location in the tree where it must perform the update. An update can either be an insertion, a deletion or a rotation. Each update to the tree can be thought of as replacing a small group of neighbouring nodes in the tree (which we call a *patch*) by a new patch, containing newly created nodes.

For example, Figure 1 depicts one of the 22 possible rotations, called RB1; the nodes of the old patch, marked with \times ’s, are removed from the tree and the new patch consists of two new nodes, marked with $+$ ’s. The rotation does not change the subtrees labelled by Greek letters (some may be empty subtrees). To perform this rotation, a process first does an LLX on and reads the shaded nodes A , B , C . (If any node of these nodes is FINALIZED, the rotation is aborted, since some other concurrent process has updated this portion of the tree.) The process creates new nodes B' and C' , with the same keys as B and C , using information returned by the LLX operations on B and C . Finally, the process uses SCX to atomically update the child pointer of A to point to C' and simultaneously finalize nodes B and C , which have been removed from the tree. If this SCX succeeds, no other updates have modified A , B or C after the LLX operations on them,

ensuring the modification makes the atomic change shown in Figure 1. Figure 2 shows other examples of tree updates.

The lock-free chromatic tree [7] is kept balanced by maintaining balance properties that generalize the properties of red-black trees [17]. The tree is a *leaf-oriented* BST, meaning keys of the set being represented are stored in the leaves of the tree; internal nodes serve only to direct searches to a leaf. A few sentinel nodes, each with key ∞ are included at the top of the tree to simplify updates and ensure that the root node never changes.

Our BAT operations use the insert and delete operations of [7] for the chromatic BST, denoted CTInsert(k), CTDelete(k). They use an SCX to insert or delete a leaf with key k , as shown in Figure 2. If this creates a violation of a chromatic tree balance property, the update operation is responsible for fixing it before it terminates by applying rebalancing steps (like the one shown in Figure 1), again using SCX. There is at most one balance violation per pending update operation, and it follows that the height of a tree containing n keys with c pending operations is $O(\log n + c)$. CTInsert returns true if k was not already present, and CTDelete returns true if it succeeds in deleting k ; otherwise they return false.

3.2 Augmenting Search Trees

Next, we describe Fatourou and Ruppert’s technique for augmenting search trees [13], in particular, their augmentation of a lock-free (unbalanced) leaf-oriented BST [11].

Each node of the search tree has a pointer to a *version* object, which stores a version of the supplementary fields of the node. The version of a node x is updated by performing a *refresh* on x . The refresh reads x ’s children’s versions v_ℓ and v_r , computes the value of x ’s supplementary fields, creates a new version object v' constructed using this information and finally performs a CAS to swing x ’s version pointer to v' . In addition to the supplementary fields, v' stores the key of x and pointers to v_ℓ and v_r . Thus, the version objects themselves form a BST (called the *version tree*) that mirrors the structure of the original tree (called the *node tree*). See Figure 4a.

After inserting or deleting a leaf of the BST, an update must modify the supplementary fields of nodes along the path from the leaf to the root. To do so, it performs a refresh (at most) twice at each node along the path. If a refresh successfully updates the node’s version, then information about the operation has propagated to the node. If both attempts fail, it is guaranteed that another process has already propagated information about the operation to the node. Thus, processes cooperate to carry information about all updates to the root. (This cooperative propagation technique originated in a universal construction [1].)

The search tree stores child pointers but no parent pointers (which would be hard to maintain). To refresh each node on the leaf-to-root path, an update stores the nodes it traversed to reach the leaf from the root on a thread-local stack. Then,

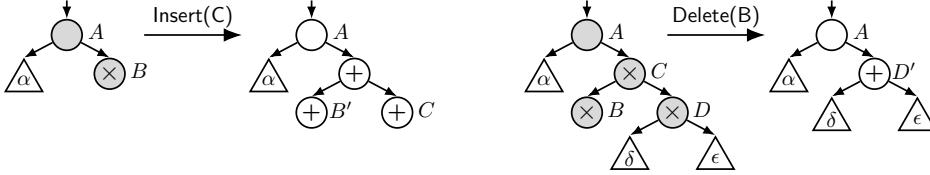


Figure 2. An insertion of C and a deletion of B . In both cases, B is a leaf. B' and D' are new copies of B and D .

it can refresh each node as it pops nodes off the stack to propagate information about the update to the root.

The proof of correctness for the augmented BST defines the *arrival point* of an update operation at a node to be the moment when information about the update has been transmitted to the supplementary fields of the node. For example, the SCX step that adds C into the tree as shown in Figure 2 is the arrival point of the insertion at C and C 's parent, because the version objects of those new nodes are initialized to reflect the insertion. The first successful CAS by a refresh on some ancestor X of C that reads X 's child Y after the operation has arrived at Y is the arrival point of the operation at X . The arrival point of the update operation at the root serves as the linearization point of the update. Much of the proof of correctness is concerned with showing that operations do arrive at the root before they terminate, and the key invariant that the version tree rooted at a node's version object accurately reflects all updates that have arrived at that node.

Whenever a refresh updates a node's supplementary fields, it creates a new version, so the contents of versions are immutable. Thus, when a (read-only) query operation reads the root's version pointer, it essentially obtains a snapshot of the entire version tree. Any query designed for a sequential augmented BST can therefore be executed on this snapshot without any adjustment to cope with concurrency. The query is linearized when it reads the root's version.

4 Lock-Free Balanced Augmented Tree

We now present BAT. An update operation first executes the chromatic tree routine CTInsert or CTDelete [7] to add or delete a leaf of the chromatic BST (see Figure 2). These two routines also perform rotations to eliminate any balance violations introduced by the update. Then, the update must modify the supplementary fields of nodes along the path from the affected leaf to the root to accurately reflect the update. This is done using a routine called Propagate.

To support augmentation, each BAT node has a *version* field, which points to a version object that stores the node's supplementary fields, as in the augmented unbalanced BST [13] (see Section 3.2). A node's *version* field is not included as part of the LLX/SCX record that makes up the rest of the node's contents; the *version* field can be manipulated directly by CAS instructions. This separation ensures that our augmentation does not interfere with the original chromatic tree operations.

Recall that all changes to the chromatic tree of [7] are performed by an SCX that replaces one patch of the BST with a new patch consisting of new nodes and simultaneously finalizes the nodes of the replaced patch. Whenever a new patch is created for an insertion, deletion or rebalancing step, it uses the following rules to initialize its nodes' versions. (We use *size* as an example augmentation; supplementary fields required by any other augmentation could be used instead.)

Definition 1 (Version Initialization Rules). Whenever BAT creates a new node x , its *version* field is initialized as follows.

1. If x is a non-sentinel leaf (i.e., x 's key is not ∞), $x.\text{version}$ initially points to a version with *size* 1.
2. If x is a sentinel leaf, $x.\text{version}$ initially points to a version with *size* 0.
3. If x is an internal node, $x.\text{version}$ is initially nil.

Moreover, the key of every newly-allocated version object is the same as that of the node pointing to it.

If a node's *version* is nil, it indicates that information that should be in the node's supplementary fields is missing.

We now describe how the Propagate routine called by an update operation modifies supplementary fields of nodes starting from the leaf ℓ that was inserted or deleted and moving up to the root. The supplementary fields are stored in the nodes' *version* fields. As in Section 3.2, the basic mechanism for updating a node x 's version is a *refresh* that attempts to install a new version for x that is created using information read from x 's children's *version* fields. The goal of Propagate is to ensure refreshes successfully install new versions at each of a sequence of nodes x_1, x_2, \dots, x_r , where x_1 is the parent of the leaf ℓ and x_r is the root so that the successful refresh on x_i reads information from x_{i-1} 's child x_{i-1} after the successful refresh on x_{i-1} . This way, information about the update operation is propagated all the way to the root.

To facilitate this, Propagate uses a thread-local stack to keep track of the nodes that it should refresh. The operation first pushes the internal nodes that were visited to get from the root to the leaf ℓ . If there were no concurrent modifications to the node tree, Propagate could simply retrace its steps, refreshing each node it pops off the stack. However, concurrent operations on the node tree may have added nodes or replaced nodes that appear in this stack. Since each refresh transmits information only from children to parent, any gap in the chain of refreshed nodes would prevent information about the update from reaching the root. Before

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1: type Node           ▷ used to store nodes of chromatic tree
2:   LLX/SCX record containing the following fields:
3:     Node *left, *right      ▷ pointers to children
4:     Key key                ▷ tree is sorted based on key field
5:     int weight              ▷ used for balancing tree
6:     bool finalized          ▷ node marked as removed
7:     Version* version        ▷ pointer to current Version

8: type Version          ▷ stores a node's supplementary fields
9:   Version *left, *right    ▷ pointers to children Versions
10:  Key key               ▷ key of node for which this is a version
11:  int size               ▷ number of leaf descendants

12: Node *Root             ▷ shared pointer to tree root

13: Insert(Key key) : Boolean
14:   Boolean result ← CTInsert(key) with this change:
15:     Whenever allocating a new Node, apply the
16:       Version Initialization Rules to initialize its version.
17:     Propagate(key)
18:   return result
19: end Insert

20: Delete(Key key) : Boolean
21:   Boolean result ← CTDelete(key) with this change:
22:     Whenever allocating a new Node, apply the
23:       Version Initialization Rules to initialize its version.
24:     Propagate(key)
25:   return result
26: end Delete

27: Find(Key key) : Boolean ▷ do standard BST search in version tree
28:   Version* v ← Root.version           ▷ Start at the root
29:   while v has non-nil children do
30:     v ← (key < v.key ? v.left : v.right)
31:   end while
32:   return (v.key = key)
33: end Find

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32: Propagate(Key key)
33:   Set refreshed ← {}                      ▷ stores refreshed nodes
34:   Stack stack initialized to contain Root      ▷ thread-local
35:   repeat
36:     Node *next ← stack.Top()
37:     loop                                ▷ go down tree until child is refreshed
38:       next ← (key < next.key ? next.left : next.right)
39:       exit loop when next ∈ refreshed or next is a leaf
40:       stack.Push(next)
41:   end loop
42:   Node *top ← stack.Pop()
43:   if ¬Refresh(top) then                  ▷ if first refresh fails
44:     Refresh(top)                         ▷ refresh again
45:   end if
46:   refreshed ← refreshed ∪ {top}
47:   until Root ∈ refreshed
48: end Propagate

49: Refresh(Node* x) : Boolean
50:   Version* old ← x.version
51:   repeat      ▷ get consistent view of left child and its version
52:     Node* x_l ← x.left
53:     Version* v_l ← x_l.version
54:     if v_l = nil then
55:       Refresh(x_l)
56:       v_l ← x_l.version
57:     end if
58:   until x_l = x.left
59:   repeat      ▷ do the same thing for the right child
60:     Node* x_r ← x.right
61:     Version* v_r ← x_r.version
62:     if v_r = nil then
63:       Refresh(x_r)
64:       v_r ← x_r.version
65:     end if
66:   until x_r = x.right
67:   Version* new ← new Version(key ← x.key, left ← v_l,
68:                               right ← v_r, size ← v_l.size + v_r.size)
69:   return (CAS(x.version, old, new) = old)
70: end Refresh

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Figure 3. Pseudocode for BAT. The details of CTInsert and CTDelete on the chromatic tree are provided in [7].

proceeding to the top node x on the stack, Propagate checks whether it has already refreshed x 's current child. If so, it pops x and refreshes it. Otherwise, it traverses down the tree from x (in the direction towards the update's key), pushing nodes onto the stack until it pushes a node y whose child has been refreshed (or is a leaf). It then pops y and refreshes it. Propagate repeats this process until the root is refreshed.

As in Section 3.2, Propagate refreshes each node a second time if the first attempted refresh fails. If the second attempt fails, it is guaranteed that some other successful refresh was performed entirely during the interval of the Propagate's double refresh. That other refresh is guaranteed to have written information into the node that includes the update the Propagate is attempting to complete.

All newly created internal nodes are initialized with nil version pointers (Definition 1) to indicate that their supplementary fields have not yet been computed. If an update op ever reads a node x 's nil version pointer, op performs a *refresh* to fix x .*version* using information read from x 's children's versions. However, the version pointers of x 's children may themselves be nil. In this case, op recursively tries to fix the *version* pointers of the children by reading x 's grandchildren. This process goes down the tree recursively until it finds nodes whose *version* pointers are non-nil. (This is guaranteed to happen, since leaf nodes never have nil *version* pointers.) Then, the recursion stops and the version pointers of all nodes visited during the recursion are set to non-nil values.

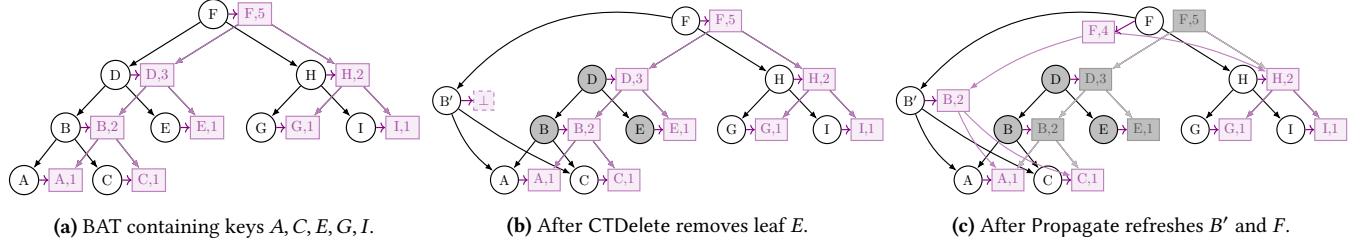


Figure 4. Stages of $\text{Delete}(E)$. Circles are nodes, rectangles are versions. Elements shaded gray are unreachable from the root node F .

As in [13], (read-only) query operations simply read the root’s version to take a snapshot of the version tree, and run a *sequential* algorithm on this snapshot, unaffected by concurrent updates. For example, any order-statistic query in [10] for sequential BSTs can be used verbatim on BAT.

Description of the Pseudocode. Figure 3 presents the pseudocode for BAT. It uses two types of objects. A *Node* object represents a tree node and stores a key *key*, a *weight* (used for rebalancing), the pointers *left* and *right* to the *Node*’s children, and the *version* pointer. The first four fields of a *node* constitute an LLX/SCX record. A *Version* object represents one version of the supplementary fields of a *node*. It stores a key *key*, the *size* of the subtree rooted at the *version*, and the pointers to the *left* and *right* children in the *version* tree.

BAT’s Insert and Delete operations simply call *CTInsert* and *CTDelete* to perform the update on the node tree, with one change: the *version* field of any newly-allocated node is initialized using the version initialization rules of Definition 1 (line 15 or line 21). Then, the operation invokes *Propagate* (line 16 or line 22). Even unsuccessful updates must call *Propagate*. For example, a process p ’s $\text{Delete}(\text{key})$ might fail because *key* was deleted from the node tree by a concurrent process q that has not propagated its deletion to the root, so p must ensure q ’s deletion is propagated before p returns false.

The loop at lines 35–47 of *Propagate* ensures that information about an update operation reaches the root by performing a double Refresh (lines 43–44) on a sequence of nodes, as described above. It uses the thread-local *stack* to determine which nodes to refresh. Initially, *stack* is empty. In the first iteration of the outer loop, the inner loop at lines 37–40 pushes all nodes visited when searching for key *key* onto *stack*. Since *CTInsert* and *CTDelete* also execute a search for *key*, we could have those routines store the nodes they visit in *stack* before calling *Propagate* as an optimization for performance.

If *Propagate* only refreshes the nodes on the *stack*, it may skip a new ancestor, for example one that was rotated onto the search path by another update’s rebalancing step. To ensure that no ancestor is skipped, *Propagate* stores the set of nodes that it has refreshed in its local variable *refreshed* (line 46). If, at any iteration of the outer loop, the child of the

top node x on the *stack* is not in *refreshed*, the inner loop at lines 37–40 traverses down the tree from x until reaching a node whose child has been refreshed (or is a leaf).

A Refresh on node x reads x ’s left and right child pointers and the children’s *version* pointers (lines 52–53 and 60–61). If either child’s *version* is nil (lines 54 and 62), it recursively refreshes that child (lines 55 and 63). When all the recursive calls return, Refresh has all the needed information to refresh x . Then, it allocates a new *Version* object initializing its fields appropriately (line 67), and attempts to change the *version* pointer of x to point to this new *Version* (line 68).

The Find routine does a standard sequential BST search on the version tree rooted at $\text{Root}.\text{version}$. Any other read-only query operation can be done in the same way.

4.1 Linearizability

To prove that BAT is linearizable, we extend the arguments used for the augmented unbalanced tree [13]. The proof appears in Appendix B; we sketch it here. The goal is to define arrival points of update operations at nodes, ensuring the tree of versions rooted at a node’s *version* reflects all the operations that have arrived at that node so far. This is formalized as follows.

Invariant 2. *For any node x in the tree that has a non-nil *version* v , the version tree rooted at v is a legal augmented BST whose leaves store the set of keys that would be obtained by executing the operations that have arrived at x in the order of their arrival points at x (starting with an empty set).*

Invariant 2 lets us linearize operations as follows. An update operation is linearized at its arrival point at the root. Invariant 2 implies that the version tree rooted at the root’s *version* is a legal augmented BST that reflects all updates linearized so far. Thus, we linearize each query when it reads the root’s *version* and gets a snapshot of the version tree.

Intuitively, an update’s arrival point at a node is when information about the operation is taken into account in the version tree rooted at the node’s *version*. The *Insert(C)* shown in Figure 2 arrives at the leaf *C* when the SCX adds it to the node tree, since *C*’s *version* is initialized to have key *C* and size 1, according to Definition 1. The *Delete(B)* in Figure 2 arrives at *D'* when the SCX changes *A*’s child pointer. This SCX is also the *Delete*’s arrival point in all nodes of δ on

the search path for B . All of those nodes' versions reflect the absence of key B , since B could not have been counted when constructing their versions. Arrival points are transferred from a node x to its parent y when a refresh successfully updates $y.version$: the refresh's CAS is the arrival point at y of all operations that arrived at x before the refresh read x 's version and that do not already have an arrival point at y .

We must ensure that Invariant 2 is preserved by each modification of the node tree that uses an SCX to replace one patch by another. As an example, consider the rotation shown in Figure 1. This SCX will succeed even if B 's *version* field has been updated by another process during the time the replacement patch was being constructed. It would be difficult to ensure that the *version* field of the new node C' is initialized to be as up-to-date as the *version* field of B . When the SCX changes A 's child from B to C' , information about operations that had arrived at B (and hence at A) might not be included in the version for C' . If A is then refreshed using information from C' , A may lose information about operations that had already previously arrived at A , which would violate Invariant 2. Avoiding this bad scenario is the reason we initialize the *version* of all new internal nodes (like C') to nil, indicating that their supplementary fields must be recomputed when they are needed. This exempts C' from the requirements of Invariant 2, which must hold only for nodes with non-nil *version* fields. On the other hand, it is trivial to initialize the *version* field of leaf nodes to accurately reflect the single key in the leaf.

We must ensure that Invariant 2 is restored when a node's *version* pointer is first set to a non-nil value. Consider the node C' for the rotation shown in Figure 1. To avoid violating Invariant 2 as described above, we must ensure that when $C'.version$ is changed from nil to a version object v , v reflects all operations that had previously arrived at A from B . For this, we use the fact that all such operations must have arrived at B via the roots of one of the subtrees β , γ or δ . When the nil *version* pointer of C' is fixed, the recursive refresh routine will ensure that $B'.version$ is fixed first, and thus the new version installed at C' will draw upon the latest information from the roots of β , γ or δ , ensuring that all operations that had previously arrived at B will be included in the new version installed at C' . (In the full proof of correctness, we must also consider the possibility that the roots of these subtrees have also been replaced by other modifications to the tree after the rotation's SCX, but a similar argument applies in this case.) Thus, we define the SCX that performs the rotation shown in Figure 1 to be the arrival point at B' of all operations that had arrived at the roots of β and γ , and the arrival point at C' of all operations that had arrived at the roots of β , γ and δ . Even though these operations are not reflected in the (nil) *version* pointers of B' and C' , we know that when those pointer are changed to non-nil values, all of the operations will be reflected, restoring Invariant 2. It is as if the information about the operations is effectively already

in the versions of B' and C' as soon as the SCX performs the rotation, because any operation that reads their *version* fields must first fix them to include that information.

The proof of the analogue of Invariant 2 for the augmented unbalanced BST [13] uses another invariant: if an update operation op with key key has arrived at a node x , then op has also arrived at the child of x on the search path for key . Thus, the set of nodes that op has arrived at form a suffix of the search path for key in the BST. Our definition of arrival points also has this property. To prove Invariant 2, we also show that the operation is reflected in the version objects of all nodes of this suffix that have non-nil version pointers.

5 Reducing Contention via Delegation

A performance bottleneck of BAT, and the original augmented unbalanced BST [13], is that all updates propagate their changes all the way to the root. This causes more cache misses and high contention in the upper levels of the tree. We propose a way to alleviate these drawbacks by having instances of Propagate delegate their work to concurrent Propagate instances working along the same path. To ensure linearizability, a Propagate that delegates its work cannot return until the Propagate to which it delegates finishes. We propose two implementations of this idea. Detailed pseudocode is in Appendix A.

In our first implementation, called **BAT-Del**, an instance P of Propagate delegates its work after failing *both* of its attempts to refresh a node x at lines 43 and 44 of Figure 3. This means some other successful Refresh on x occurs between the start of P 's first failed Refresh and the end of P 's second failed Refresh. Let R_ℓ be the last such successful Refresh and P_ℓ be the Propagate that called R_ℓ . After P 's second failed Refresh, if x is not finalized (i.e., it is still in the tree), P delegates the rest of its work to P_ℓ by waiting for P_ℓ to complete before returning. When P_ℓ completes, all of operations P was attempting to propagate will have reached the root.

We next describe how P synchronizes with P_ℓ . Each call to Propagate creates a PropStatus object, which stores a boolean value *done* indicating whether or not the Propagate has finished, and a pointer to another PropStatus if the Propagate has delegated its work (if not, this pointer is nil). Each version stores a pointer to the PropStatus of the Propagate that created it. When a Propagate P fails the CAS of its second Refresh on a node, the CAS returns the version written by the last successful Refresh R_ℓ , and P can delegate to that version's PropStatus, which belongs to P_ℓ . P waits for P_ℓ to finish by spinning on the *done* field in the PropStatus object. There may be a chain of delegations, so to avoid waiting for the *done* flag to propagate down the chain, P can find the head of the chain using the pointers in the PropStatus objects, and directly wait on the head of the chain.

For correctness, we distinguish between *top-level* Refreshes called by Propagate and *recursive* Refreshes called by Refresh

to fix nil version pointers. We need to ensure that a top-level Refresh cannot fail (and thus delegate) due to a recursive Refresh. We do this by making the CAS in recursive Refreshes only change version pointers from nil to non-nil and the CAS in top-level Refreshes change version pointers only from non-nil to non-nil. We accomplish this by creating two separate refresh functions that only differ in their first few steps. A recursive Refresh begins by reading the node’s version pointer and returning if it is non-nil. A top-level Refresh begins by reading the node’s version pointer and, if it is nil, calling a recursive refresh and rereading the node’s version pointer (which is now guaranteed to no longer be nil). The remaining steps for both versions of Refresh are the same as in Figure 3, with all calls to Refresh going to the recursive version. Delegating due to a recursive Refresh is dangerous because a Propagate may perform recursive Refreshes on nodes outside of its search path. For example, when a new patch is installed, a Propagate might recursively Refresh every node in the new patch, even though many are on different search paths.

In general, it is safe for a Propagate P_1 to delegate to another Propagate P_2 at node x if (1) x is still reachable from the root, (2) x is on both their search paths, and (3) the Refresh of P_2 saw all the arrival points A that P_1 was attempting to propagate to x . Properties (1) and (2) are important for arguing that P_2 will perform a sequence of Refreshes from x (or from a new node replacing x , which inherits all the arrival points in A) to the root. This sequence of refreshes has the same effect as continuing P_1 because property (3) ensures they will bring the arrival points in A to the root.

In our experiments, BAT-Del improves performance by more than a factor of two in update-heavy workloads. We develop a more optimized version called **BAT-EagerDel**, which further increases the frequency of delegation. BAT-EagerDel uses the same delegation mechanism, but does so after just *one* failed Refresh. To make this safe, we had to modify a successful top-level Refresh on a node x to re-read the version pointers in x ’s children and make sure they have not changed after they were last read on line 53 or 56, and line 61 or 64 of the Refresh. If they have changed, the successful Refresh repeats from the beginning (line 50) until either it fails a CAS and sees that x has not been removed (in which case it delegates) or it succeeds in a CAS and sees that the version pointers in x ’s children have not changed.

Both delegation techniques described above are blocking: if a thread that has undertaken the work of other threads stalls, it prevents the other threads from completing. We can make both BAT-Del and BAT-EagerDel non-blocking by adding a timeout, after which the waiting process resumes its propagation to the root itself. Our implementations in the experiments section include this timeout, tuned for the common case where there are no stalled threads.

6 Memory Reclamation

In this section, we describe how to apply Epoch-Based Reclamation (EBR) [14] to free the three types of shared objects used by BAT: nodes, versions, and, when using delegation, PropStatus objects. EBR tracks the beginning and end of each *high-level* operation (e.g. Insert, Delete, RangeQuery). It provides a retire function, which takes as input an object to be freed and delays freeing that object until all high-level operations active during the retire have completed.

Applying EBR (or any other memory reclamation scheme) to BAT poses several novel challenges. Traditionally, EBR retires objects when they are removed from the data structure. This is easy to track for tree nodes since they can be reached by only a single path from the root. For example, nodes marked by \times in Figures 1 and 2 can be retired after the SCX replaces them. Safely retiring versions and PropStatus objects, which can be reached via multiple paths from the root, is more difficult. To do so, we use the property of EBR that an object is safe to retire at time T if it will not be accessed by any high-level operation that starts after time T .

We first create separate functions for top-level Refreshes and recursive Refreshes, defined in Section 5. Each time a propagate performs a successful top-level Refresh, it keeps track of the old version that it replaced in a *toRetire* list. No node points to this old version, but it is not safe to retire yet, since it could still be reachable from the root of the version tree. Once the Propagate reaches the root, all versions in its *toRetire* list are guaranteed to be unreachable from the root of the version tree and they can therefore be safely retired.

We have shown how to reclaim old versions replaced by newer ones. What remains is to reclaim the final version stored in each node. A Refresh can change a node’s version pointer even after the node is retired, but the version pointer stops changing when the node is safe to free. Even then, the final version may still be accessed by a long-running query via an old version tree. However, newly started queries will not access the final version, so it can be safely retired immediately before freeing the node.

Safely reclaiming PropStatus objects also poses a challenge because each one can be pointed to by multiple versions, and it is not clear when they all become unreachable. To avoid complex reachability checks, we observe that a PropStatus object can be safely retired at the end of the Propagate operation P that created it, even while the object is still reachable. This is because the only operations that access a PropStatus object are those whose work is delegated to P (directly or indirectly). This delegation can only happen while P is running. Therefore, any high-level operation that starts after P completes will never access P ’s PropStatus object.

7 Experimental Results

We implemented BAT using the chromatic tree implementation in [7], which uses LLX/SCX primitives from [6], in

the publicly available SetBench [31] microbenchmark. Our experiments show that (1) delegation can improve BAT’s insert and delete throughput by over 100%, (2) BAT scales well with thread count and data structure size, and (3) BAT performs significantly faster than previous data structures in workloads with more than 2% order-statistic queries and in workloads with range queries of size larger than 2K-10K.

Data Structures. The following table summarizes the key properties of the data structures we compare.

	Augmented	Balanced	Fanout	Lock-free
BAT	yes	yes	2	yes
BAT-Del	yes	yes	2	yes
BAT-EagerDel	yes	yes	2	yes
FR-BST [13]	yes	no	2	yes
Bundled [24] CitrusTree	no	no	2	no
VcasBST [32]	no	no	2	yes
VerlibBTree [4]	no	yes	4-22	yes

All are linearizable, written in C++, integrated into SetBench, and use the same memory reclamation scheme. Data structures that do not maintain augmented values achieve linearizable range queries by taking a snapshot and iterating over the range. Each implementation (other than FR-BST, which we implemented) was taken from its original paper.

Among the three existing concurrent augmented trees [13, 21, 29], we chose to compare with FR-BST [13] because it has the best theoretical bounds. [21] has a Kotlin implementation that is several orders of magnitude slower than our C++ implementation of FR-BST, according to the results in their paper. This cannot fully be attributed to language differences. We are not aware of any implementations of [29].

Setup. Our experiments ran on a 96-core Dell PowerEdge R940 machine with 4x Intel(R) Xeon(R) Platinum 8260 CPUs (24 cores, 2.4GHz and 143MB L3 cache each), and 3.7TB memory. Each core is 2-way hyperthreaded, giving 192 hyperthreads. We used `numactl -i all`, evenly spreading the memory pages across the sockets in a round-robin fashion. The machine runs Ubuntu 22.04.5 LTS. The C++ code was compiled with `g++ 11.4.0` with `-O3`. For scalable memory allocation, we used `mimalloc` [23]. Memory was reclaimed using DEBRA [8], an optimized implementation of epoch-based reclamation [14] (see Section 6 for more details). BAT used the original LLX/SCX implementation as described in [6]. The optimized LLX/SCX implementation of [2] could provide additional improvements.

All of our experiments (other than Figure 5b, which has no prefilling) began with a prefilling phase where random inserts and deletes ran until the structure contained half the keys in the key range. Then, threads perform operations chosen randomly (using various distributions, described below) for 3 seconds. We report the average of 5 runs. The variance within trials of the same experiment was relatively consistent between experiments of different parameters, with the lowest throughput of a trial being within around 8–10% of

the highest throughput of a trial. Since most of the plots we show are on a log scale, this difference is hardly visible.

Workloads. We varied the following parameters:

Total Threads (TT): Number of threads concurrently executing operations on the data structure.

Max Key (MK): The maximum integer key that can be inserted into the data structure. Since all our experiments (except Figure 5b) were run with the same fraction of inserts as deletes and the trees are pre-filled with half the key range, the size of the data structure remained around half the size of this parameter.

Range Query Size (RQ): The size of each range query performed. The lower bound of the range query interval was generated uniformly from the range of valid lower bounds and is added to this parameter to get the upper bound.

Workload (i%-d%-f%-rq%): The probability of choosing each operation (insert, delete, find, range query) as the next one a thread executes. In Figure 5c, rq% is replaced with the percentage of the given query (rank, select, or rangeQuery). In Figure 7, rq% is replaced with rank%.

Key Distribution: Either uniform, sorted or Zipfian with parameter 0.95. The sorted and Zipfian workloads result in high contention as updates are routed to the same parts of the tree. The distribution was uniform unless otherwise specified.

The sorted distribution inserted keys in roughly increasing order to evaluate the benefits of balancing (Figure 5b). Threads acquired keys to insert from an increasing global counter. To reduce contention on the counter, threads incremented it by 100 each time to acquire a batch of 100 keys.

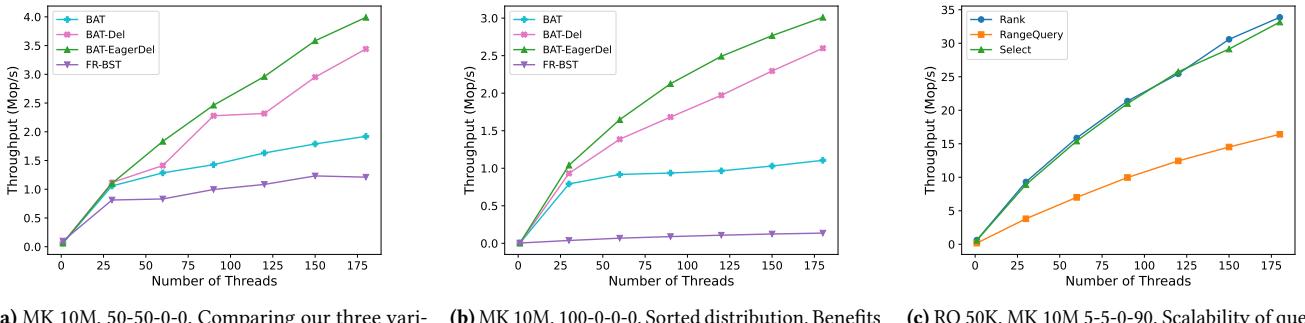
Results. We summarize our experiments in Table 1. Each entry describes one type of experiment and its purpose.

Comparing Augmented Trees. Figures 5a and 5b show the performance improvements for updates that we get from variants of our algorithm under two different workloads. As expected, balancing allows BAT to significantly outperform the unbalanced FR-BST, especially when using a sorted workload. The average number of nodes seen by a Propagate decreases from 31 (in FR-BST) to 25 (in BAT) in the uniform workload (Figure 5a, 180 threads) and 2300 to 56 in the sorted workload (Figure 5b, 180 threads).

Adding delegation also improves throughput by around 100% in the case of BAT-Del and 120% in the case of BAT-EagerDel for update-only uniform workloads on 180 threads. This is because delegation reduces the average number of nodes a propagate visits in a tree with 5M keys by around 3 for delegation after two failed Refresh in BAT-Del and 4.5 for delegation after single failed Refresh in BAT-EagerDel. Since these nodes are usually close to the top of the tree, this greatly reduces the bottleneck at these levels. We therefore focus on the BAT-EagerDel variant for the remaining comparisons.

Queries. In Figure 5c, we see the performance of several order-statistic queries on BAT-EagerDel scales well. Rank queries return the number of keys in the set that are less than or equal to a given key. Select queries return the kth

Experiment	Figure	Description	Purpose
Improvement Study	5a, 5b	Throughput vs number of threads on an update only workload	Compare our variants against an unbalanced augmented tree under uniform and skewed workloads
Query Scalability	5c	Throughput vs number of threads for queries on BAT-EagerDel	Compare the scalability of different queries on our balanced augmented tree
Range Query Size	6	Throughput vs range query size for a 20% update workload	Show the benefits of augmenting a concurrent tree
Rank Query Percentage	7	Throughput vs percentage of rank queries for varied percentage of updates	Compare the performance of different trees under varying amounts of rank queries
Thread Scalability	8	Throughput vs number of threads for different workloads	Show how various trees scale to higher numbers of threads
Isolated Performance	9	Average update/range query time vs range query size for a 20% update workload	Show the individual performance of updates and queries
Size Scalability	10	Throughput vs data structure size for uniform and Zipfian distributions	Show how various trees scale to larger data structure sizes

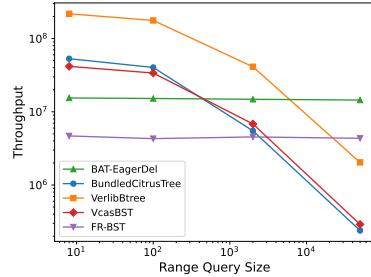
Table 1. Description of our experiments**Figure 5.** Performance of variants of our BAT.

smallest key in the set, for a given k . Range queries return the number of keys in a given range. They are slower than rank and select queries due to having to traverse two paths (for the lower and upper bound of the range) in the BST instead of just one.

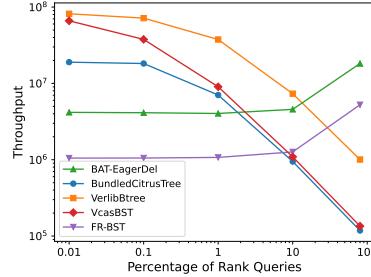
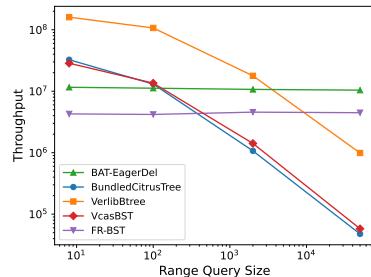
Range Query Size. Figure 6 shows the performance of various structures under varying range query size. Figures 6a and 6b show results for small and large trees. Since the unaugmented trees perform work proportional to the number of keys in the range, their performance drops off sharply for larger range queries. In contrast, in the augmented trees (FR-BST and ours), queries only perform work proportional to the height of the BST, so their performance stays consistent no matter the range query size. VerlibBtree outperforms the other non-augmented trees since it uses higher fanout trees for better cache efficiency, but loses out to the augmented structures after range query sizes reach 2000–4000. After reaching range query size 2M for a tree of size 10M, BAT-EagerDel is 400x as fast as the closest non-augmented tree. However, the added overhead for inserts and deletes causes the augmented structures to lose out heavily when range

queries only traverse a few keys. For range queries of only 8 keys, BAT-EagerDel is 15x slower than VerlibBtree. BAT-EagerDel is around 3x faster than FR-BST because balancing reduces the average depth of the leaves and delegation reduces contention at higher levels of the tree.

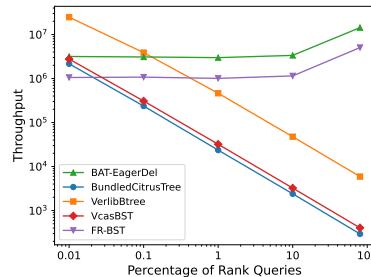
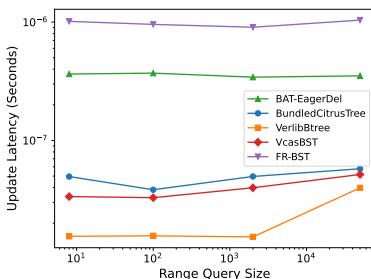
Rank Queries. Figure 7 compares the performance of concurrent trees with different percentages of rank queries. We only use rank and not select here since the similarity in their algorithms would produce an identical graph for select. We vary the percentage of rank queries and the remaining operations are split evenly between inserts and deletes (e.g., 1% rank, 49.5% insert, 49.5% delete). Since non-augmented rank queries take time proportional to the number of keys less than the selected key, the downside of non-augmented trees is less pronounced in smaller trees (Figure 7a). However, BAT-EagerDel still performs best for more than 11% rank queries. In larger trees (Figure 7b), we can see BAT-EagerDel outperforms the other structures even for 0.15% rank queries. We see a large improvement in BAT-EagerDel and FR-BST when going from 10% to 80% rank queries since there is a



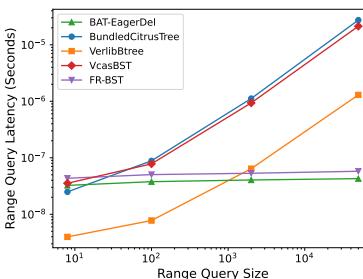
(a) TT 120, MK 100K, 10-10-40-40. Benefits of augmenting BST, small tree.

(b) TT 120, MK 100K, $\frac{1}{2}(100-x)-\frac{1}{2}(100-x)-0-x$. Performance on small tree for x% of rank queries.

(c) TT 120, MK 10M, 10-10-40-40. Benefits of augmenting BST, large tree.

(d) TT 120, MK 10M, $\frac{1}{2}(100-x)-\frac{1}{2}(100-x)-0-x$. Performance on large tree for x% of rank queries.**Figure 6.** Performance of our top performing BAT with respect to range query size.**Figure 7.** Performance of our top performing BAT on different workloads of rank queries.

(a) TT 120, MK 10M, 10-10-40-40. Average update latency.



(b) TT 120, MK 10M, 10-10-40-40. Average range query latency.

Figure 9. Performance of updates and range queries with respect to range query size on a mixed workload.

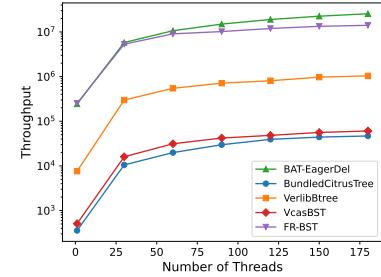
significant drop in the number of inserts and deletes, which are the worst performing operations for these structures.

Thread Scalability. Figures 8a and 8b show scalability under low (5%) and high (50%) update percentages, respectively. These update percentages were selected according to YCSB workloads A and B [9]. FR-BST scales less well since it has higher contention close to the root, making updates perform worse when more threads are involved. The scaling of the other data structures are similar to each other, however BAT-EagerDel outperforms the closest unaugmented competitor

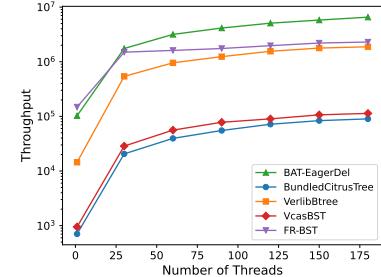
by around 4x on the high update workload and 30x on the low update workload for all thread counts.

Isolated Performance. Figure 9 shows the average latency in seconds for updates and range queries under the same workload as Figure 6. In the update graph, we see that the performance of inserts and deletes on BAT remains relatively constant. Furthermore, BAT-EagerDel has a lower range query latency than all unaugmented competitors when at least 2000 elements are being queried.

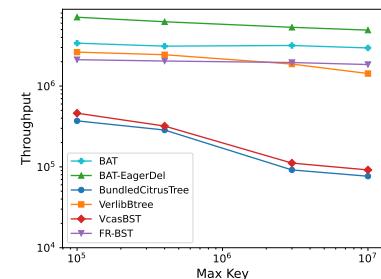
Size Scalability and Zipfian Distribution. Figure 10 shows the effect of increasing the data structure size (by varying the



(a) RQ 50K, MK 10M, 2.5-2.5-47.5-47.5. Thread scalability, low update workload.



(b) RQ 50K, MK 10M, 25-25-25-25. Thread scalability, high update workload.

Figure 8. Performance of our top performing BAT with respect to number of threads.

(a) TT 120, RQ 50K, 25-25-25-25. Size scalability, high update Zipfian workload.

Figure 10. Comparison of BAT variants to other trees with respect to data structure size.

maximum key). We include the BAT variant with no delegation in these graphs to show it is still worse than the versions with delegation in the case of a Zipfian distribution. Overall, we see that BAT-EagerDel scales slightly better with size compared to VerlibBtree, BundledCitrusTree and VcasBST. VerlibBtree performs 15% worse on the Zipfian distribution (Figure 10) for small trees, while the others stay relatively constant.

Why Balancing Improves Throughput. BAT performs extra work to balance the tree, and calls to propagate occasionally have to traverse backwards or fill in nil versions. Nevertheless, our results show that BAT variants consistently outperform FR-BST, even on uniform and Zipfian key distributions, where FR-BST would be fairly balanced. We provide some key statistics to explain this. We measured on a workload with 120 threads, 100K max key, 50K range query size and an even percentage of inserts, deletes, range queries and finds on both a uniform and Zipfian distribution with parameter 0.99. On the BAT variants, each propagate only traverses 6.4% (5.9%) more nodes beyond the initial search path for the uniform (Zipfian) distribution. A call to propagate fills in only 0.075 (0.03) nil versions on average. Lastly, the average number of CASes attempted during a propagate call is 22.2 (22.4) for BAT, 13.9 (13.2) for BAT-EagerDel and 26.8 (27.5) for FR-BST. Thus, the extra costs incurred by rebalancing are minimal compared to the advantages of maintaining a more carefully balanced tree.

8 Conclusion

Augmentation makes search trees significantly more versatile by extending the set interface to enable support for aggregation queries, order-statistic queries, and range queries. In this paper, we designed, implemented, and empirically validated BAT, the first lock-free Balanced Augmented Tree. While we emphasized our augmentation scheme as applied to a chromatic tree, our scheme is general—adaptable to concurrent search trees where updates modify the tree by replacing one patch by another patch of new nodes.

Our experiments show that BAT and its optimized versions are scalable. For applications where augmentation is essential, BAT is the only efficient, concurrent option to date. Some queries, like finding the predecessor of a given key, can be answered by exploring a small part of a snapshot of the tree. In such cases, snapshots, e.g. [20, 32], provide a sufficiently good solution because they avoid the overhead of augmentation. However, our experiments show that queries that have to traverse many nodes of the tree—like range queries, rank queries or selection queries—are vastly faster with BAT than with other snapshot-based approaches. Thus, even if the workload is mostly updates with occasional queries, BAT outperforms other approaches.

There remain interesting open directions in designing concurrent search tree data structures. Complex sequential data structures like link/cut trees [30], measure trees [15],

and priority search trees [22] rely on balanced augmented trees. Now that we have designed a *concurrent* balanced augmented tree, we can ponder the possibility of concurrent versions of these more complex data structures.

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```

1: type Version
2:   Version* left, right
3:   Key k
4:   int size
5:   PropStatus* status
6: type PropStatus
7:   Boolean done
8:   PropStatus* delegatee

```

Figure 11. Modification to Version object for BAT-Deland BAT-EagerDel, and the new PropStatus object. Nodes are as described in Figure 3.

A Algorithm with Delegate Mechanism

Here, we give the details of the delegation mechanisms described in Section 5. Figure 11 and 12 shows code that is common to both BAT-Del and BAT-EagerDel. The Propagate functions of BAT-Del and BAT-EagerDel are shown in Figures 13 and 14 respectively.

```

1: WaitForDelegatee(PropStatus* d)
2:   while  $\neg d.done$  do
3:     if d.delegatee  $\neq \text{nil}$  then
4:       d  $\leftarrow d.delegatee
5:     end if
6:   end while
7: end WaitForDelegatee
8: ReadVersion(Node* x) : Version*
9:    $\triangleright$  Sets x.version if nil and then returns x.version.
10:  Version* v  $\leftarrow x.version
11:  if v = nil then
12:    RefreshNil(x)
13:    v  $\leftarrow x.version
14:  end if
15:  return v
16: end ReadVersion
17: RefreshNil(Node* x)
18:    $\triangleright$  Recursive refresh for setting nil versions.
19: repeat
20:   Node* xl  $\leftarrow x.left
21:   Version* vl  $\leftarrow$  ReadVersion(xl)
22: until xl = x.left
23: repeat
24:   Node* xr  $\leftarrow x.right
25:   Version* vr  $\leftarrow$  ReadVersion(xr)
26: until xr = x.right
27: Version* new  $\leftarrow$  new Version(k  $\leftarrow x.k, left  $\leftarrow v_l,
28:                                right  $\leftarrow v_r, size  $\leftarrow v_l.size + v_r.size, status  $\leftarrow \perp$ )
29: CAS(x.version, nil, new)
29: end RefreshNil
30: Refresh(Node* x, PropStatus* ps) : Boolean, PropStatus*, Version*, Version*
31:    $\triangleright$  Return True if Refresh succeeds, False otherwise
32:    $\triangleright$  Also returns PropStatus of propagate that blocked
33:    $\triangleright$  the CAS (or nil if successful).
34:    $\triangleright$  Also returns left and right versions that were read
35: Version* old  $\leftarrow$  ReadVersion(x)
36: repeat
37:   Node* xl  $\leftarrow x.left
38:   Version* vl  $\leftarrow$  ReadVersion(xl)
39: until xl = x.left
40: repeat
41:   Node* xr  $\leftarrow x.right
42:   Version* vr  $\leftarrow$  ReadVersion(xr)
43: until xr = x.right
44: Version* new  $\leftarrow$  new Version(k  $\leftarrow x.k, left  $\leftarrow v_l,
45:                                right  $\leftarrow v_r, size  $\leftarrow v_l.size + v_r.size, status  $\leftarrow ps)
46: Version* res  $\leftarrow$  CAS(x.version, old, new)
47: Boolean success  $\leftarrow$  (res = old)
47: return success, (success ? nil : res.status), vl, vr
48: end Refresh$$$$$$$$$$$$$$$$ 
```

Figure 12. Helper functions for BAT-Del and BAT-EagerDel

```

1: Propagate(Key k)
2:   Set refreshed  $\leftarrow \{\}$             $\triangleright$  stores refreshed nodes
3:   Stack stack initialized to contain Root  $\triangleright$  thread-local
4:   PropStatus* ps  $\leftarrow$  new PropStatus(done  $\leftarrow$  false,
                                         delegatee  $\leftarrow$  nil)
5:   repeat
6:     Node* next  $\leftarrow$  stack.Top()
7:     loop       $\triangleright$  go down tree until child is refreshed
8:       next  $\leftarrow$  (k < next.key? next.left:next.right)
9:       exit when next  $\in$  refreshed or next is a leaf
10:      stack.Push(next)
11:   end loop
12:   Node* top  $\leftarrow$  stack.Pop()
13:   success, *, *, *  $\leftarrow$  Refresh(top, ps)
14:   if  $\neg$ success then           $\triangleright$  if try1 fails
15:     success, del, *, *  $\leftarrow$  Refresh(top, ps)
16:     if  $\neg$ success and  $\neg$ top.finalized then
17:       ps.delegatee  $\leftarrow$  del
18:       WaitForDelegatee(ps.delegatee)
19:        $\triangleright$  Can be made lock-free by resuming
20:        $\triangleright$  from line 13 after waiting exceeds
21:        $\triangleright$  a time limit.
22:       ps.done  $\leftarrow$  true
23:     return
24:   end if
25: end if
26:   refreshed  $\leftarrow$  refreshed  $\cup \{top\}$ 
27: until Root  $\in$  refreshed
28:   ps.done  $\leftarrow$  true
29: end Propagate

```

Figure 13. BAT-Del

```

1: Propagate(Key k)
2:   Set refreshed  $\leftarrow \{\}$             $\triangleright$  stores refreshed nodes
3:   Stack stack initialized to contain Root  $\triangleright$  thread-local
4:   PropStatus* ps  $\leftarrow$  new PropStatus(done  $\leftarrow$  false,
                                         delegatee  $\leftarrow$  nil)
5:   repeat
6:     Node* next  $\leftarrow$  stack.Top()
7:     loop       $\triangleright$  go down tree until child is refreshed
8:       next  $\leftarrow$  (k < next.key? next.left:next.right)
9:       exit when next  $\in$  refreshed or next is a leaf
10:      stack.Push(next)
11:   end loop
12:   Node* top  $\leftarrow$  stack.Pop()
13:   repeat
14:     success, del, vl, vr  $\leftarrow$  Refresh(top, ps)
15:     if  $\neg$ success and  $\neg$ top.finalized then
16:       ps.delegatee  $\leftarrow$  del
17:       WaitForDelegatee(ps.delegatee)
18:        $\triangleright$  Can be made lock-free by resuming
19:        $\triangleright$  from line 13 after waiting exceeds
20:        $\triangleright$  a time limit.
21:       ps.done  $\leftarrow$  true
22:     return
23:   end if
24:   until success and vl = top.left.version and
25:           vr = top.right.version
26:           refreshed  $\leftarrow$  refreshed  $\cup \{top\}$ 
27:   until Root  $\in$  refreshed
28:   ps.done  $\leftarrow$  true
29: end Propagate

```

Figure 14. BAT-EagerDel, only lines 13-24 changed relative to Figure 13.

B Correctness

We follow the arguments similar to those for augmented binary search trees in [13] and extend them for the augmented chromatic trees presented in this paper. We first present the proof for the non-delegating version.

B.1 Facts About the Unaugmented Chromatic Tree

We first summarize some facts from [5] about the original, unaugmented, lock-free chromatic tree. Since our augmentation does not affect the *node tree*, these facts remain true in the augmented chromatic tree.

In the chromatic tree, the coordination of updates using LLX/SCX primitives ensure the following claims.

The following is a consequence of Lemma 3.94 and Lemma 5.1, claim 3, and Corollary 5.2 of [5]. A Node is considered reachable if it can be accessed by traversing pointers starting from *root*.

Lemma 3. *A Node’s child pointer can change only when the Node is not finalized and it is reachable.*

The following is a consequence of Lemma 6.3.3. of [5].

Lemma 4. *If a Node is on the search path for key k in one configuration and is still reachable in some later configuration, then it is still on the search path for k in the later configuration.*

The chromatic tree uses an ordinary BST search and the following lemma is a direct consequence of Lemma 6.3.4 of [5].

Lemma 5. *If an insert, delete or rebalance operation visits a Node x during its search for the location of key k , then there was a configuration between the beginning of the operation and the time it reaches x when x was on the search path for k in the node tree.*

As shown in Figure 6.3 of [5], *entry* is a special pointer serving as the immutable root of the *node tree*.

Let T_C be the *node tree* in configuration C . Let n be the number of keys in the *node tree* and c be the number of pending update operations.

The following directly follows from Lemma 6.3.7 of [5]. The Lemma 6.3.7 of [5] implies that the *node tree* is a BST with additional properties required for a chromatic tree.

Lemma 6. *For all configurations C , T_C is a balanced BST of height $O(\log n + c)$.*

For a *node tree*, how the augmentation information propagates up the tree is crucial for correctness. To achieve this, we introduce the notion that describes when an update operation’s augmentation information is reflected at a Node in the *node tree*, referred to as the *arrival point* of the update.

B.2 Linearization Respects Real-Time Order.

In this section, we begin by formally defining arrival point of an update at a Node. Then, for an update operation on a

given key, we show that the update’s Propagate ensures that the update has an arrival points at each reachable Node on which it performs a double Refresh. Moreover, that arrival point is during the update’s execution interval. Eventually, if the call to Propagate completes, the update is assigned an arrival point at the root, before it returns. The arrival point of an update at the root is the linearization point of the update. Each query is also assigned a linearization point when it reads the version pointer of the root Node to get an immutable snapshot of the *version tree* rooted at *root.version*. Since arrival at the root serves as the linearization point of the update, and each query is also assigned a linearization point during the query, it follows that the linearization respects the real-time order of operations.

Intuitively, the arrival point of an update operation op on key k at a Node x is the moment in time during its execution when both (a) x is on the search path for k and (b) the effect of op is reflected in the *version tree* rooted at $x.version$.

We now formally define arrival points of insert and delete operations over an execution α of the implementation.

Definition 7. The base case defines the arrival points of unsuccessful Insert and Delete operations at a leaf.

1. A Delete(k) whose traversal of the tree ends at a leaf ℓ that does not contain k returns false. Similarly, an Insert(k) that reaches a leaf ℓ containing k returns false. In both cases, Insert and Delete return without modifying the *node tree*. Their arrival point at ℓ is the last configuration during their execution in which ℓ is on the search path for k . Such a configuration exists by Lemma 5.

We define inductively the arrival points of update cases that modify the *node tree*. Assume the arrival points are defined for a prefix of the execution α . Let s be the next step that modifies the *node tree*. The possible cases are as follows.

2. Consider an Insert(k) that executes a successful SCX step s to replace a leaf ℓ by an internal Node new with two leaf children, $newLeaf$ and ℓ' , that contain k and ℓ ’s key, respectively. This SCX is the arrival point at new and either the left or right child of new (depending on whether the operation’s key is less than $new.key$ or not) of all operations whose arrival points at ℓ precede the SCX, in the order of their arrival points at ℓ , followed by the Insert(k) at both $newLeaf$ and new .
3. Consider a Delete(k) that performs a SCX step s to modify the *node tree*. This step replaces an internal Node p (whose children are a leaf ℓ containing k and its sibling sib) by a new copy sib' of sib . For each operation on any key k' whose arrival point at ℓ precedes s , s is the operation’s arrival point at sib' and all of its descendants that are on the search path for k' . Additionally, for each operation whose arrival point at sib precedes s , s is the operation’s arrival point at sib' . If multiple operations on k' are assigned arrival points

- at the same node, they occur in the same order as their arrival points at ℓ . Finally, s is also the arrival point of the Delete(k) at sib' and all its descendants that are on the search path for k .
4. Consider a rebalancing operation that performs a successful SCX step s . Let G_{old} be a patch of nodes in the *node tree*, rooted at node old . Let F be the set of nodes that are the children of nodes at the last level of the patch. This step s atomically modifies a *node tree* by replacing the patch G_{old} , with a new patch G_{new} , rooted at a node new and the same fringe F , in the *node tree* (as shown in the rebalancing diagrams in [5, Figure 6.5]).
 - For each operation that arrived at a fringe node prior to s , s serves as the operation's arrival point at every ancestor of that fringe node in G_{new} . The order of arrival points is same as it is in the fringe node. Operations from different fringe nodes are ordered according to their left-to-right position in the tree: operations from the left fringe node precede those from the right. If the SCX replaces a leaf ℓ (if any) with new copy ℓ' , then this SCX serves as the arrival point at ℓ' of all operations that had an arrival point at ℓ prior to the SCX, in the same order.
 5. Consider a successful CAS performed by a Refresh R on the *version* field of an internal Node x at line 68. Let x_L and x_R be the children of x read by R at line 52 or 60. The CAS is the arrival point at x of
 - a. all operations that have an arrival point at x_L prior to R 's last read at line 53 or 56 and do not already have an arrival point at x prior to the CAS, in the order of their arrival points at x_L , followed by
 - b. all operations that have an arrival point at x_R prior to R 's last read at line 61 or 64 and do not already have an arrival point at x prior to the CAS, in the order of their arrival points at x_R . Note again that this preserves the order of operations on same key; the order of operations across the key need not be preserved and is irrelevant.

The arrival points at an arbitrary Node x form a sequence of operations, referred to as *Ops* sequence. Each element in this sequence is of the form $\langle operation(k) : response \rangle$, where operation is either an insert or a delete with a boolean response attached. *Ops* sequences track the operations that have arrived at a Node.

For deletes and inserts whose arrival point is defined by Part 1, the associated response is *false*. For inserts and deletes whose arrival points are defined by Part 2 and 3, the associated response is *true*. Finally, whenever arrival points are copied from a removed or replaced node to another, as in Part 2, 3, 4 and 5, the associated responses are copied as well.

Definition 8. For each configuration C and Node x ,

1. Let $Ops(C, x)$ be the sequence of update operations with arrival points at x that are at or before C , in order of their arrival points at x .
2. Let $Ops^*(C, x)$ be the sequence of update operations with arrival points at x that are strictly before C , in the order of their arrival points at x .
3. Let $Ops(C, x, k)$ be the subsequence of $Ops(C, x)$ consisting of operations with key k .

Observation 9. *The CAS on line 68 never attempts to store a value in a node's version field that has previously been stored in it.*

Lemma 10. *If two calls to Refresh on the same Node perform successful CAS steps, then one performs the read on line 50 after the CAS on line 68 of the other.*

Proof. Without loss of generality, let x be an internal Node, and let $T1$ and $T2$ be two arbitrary processes. Suppose $T1$ reads the value old from $x.version$ at line 50 and successfully performs a CAS at line 68, updating $x.version$ from old to new , where new is a pointer to a Version object allocated at line 67. Now, assume that $T2$ reads old from $x.version$ immediately before $T1$'s CAS, and $T2$ also succeeds in its CAS at line 68.

For $T2$ to succeed, at the time of its CAS, it must see that $x.version$ is same as old . However, since $T2$ changes $x.version$ to new and from observation 9, $x.version$ cannot be old , a contradiction. Therefore, $T2$'s read at line 50 must be after $T1$'s successful CAS. \square

The claims of the following Invariant ensure that no update is dropped by the propagation if a Node where it has arrived is removed from the *node tree*. In other words, propagation ensures two properties: the upward consistency of propagation, such that a *Ops* sequence of a parent Node (on a particular key) is always a prefix of its children's and descendants'; and the monotonic growth of all *Ops* sequences of all nodes.

Refresh operation on a node x updates $x.version$ using the versions of x 's children. Reading the version of a child (between lines 51-58 and 59-66) is a three-step process:

1. read pointer to x 's child y ;
2. read the pointer to y 's version; and
3. verify that y is still a child of x .

these steps repeat until the last step succeeds. Then following observation directly follows from the code of Refresh.

Observation 11. *For a node x with child y , Refresh ensures that y was the child of x at the time it read y 's version. This implies that Refresh can add to the Ops sequence of x only those operations from y that were added in y 's Ops sequence before y was replaced.*

This helps avoid the violation of Invariant 12 (part 1), where a concurrent Refresh on x uses child y 's version after

y was replaced by a new child y' such that y' contains operations of y from before it was deleted and since then y got a new version which was read by the Refresh at x . In this case, the Invariant 12 (part 1) is violated between x and y' .

Invariant 12. *For any configuration C , any key k , and any internal node x with a child y in C :*

1. *Every operation in $\text{Ops}(C, x, k)$ is also in $\text{Ops}(C, y, k)$.*
2. *If an SCX changes the child pointer of x from y to some node y' at configuration C' , then every operation in $\text{Ops}(C, y, k)$ is also in $\text{Ops}(C', y', k)$.*

Proof. We will use induction on the sequence of configurations and argue that every possible modification to Ops sequence of nodes preserves the claims.

The invariant holds vacuously in the initial configuration because no operations have arrival points. Assume that the claims hold up to some configuration C . We show that they hold up to the next configuration C' . The following cases can occur:

1. Part 1 of Definition 7. It adds the update operation only to the Ops sequence of a leaf Node. Since a leaf does not have children, claim 1 is not violated. Additionally, since it does not modify the *node tree*, claim 2 is trivially satisfied.
2. Part 2 of Definition 7. It ensures that each operation op that have an arrival point at the replaced leaf ℓ is moved to the Ops sequences of the two new leaves. op with key $k < \text{new}.key$ is moved to the left leaf and the op with $k \geq \text{new}.key$ is moved to the right leaf. The insert itself is appended to the Ops sequence of the appropriate leaf based on whether its key is less than or greater than equal to $\text{new}.key$. Additionally, op including the current insert is also added to the Ops sequence of new . Therefore, claim 1 is preserved. Moreover, no violation of the invariant is created at the node whose child pointer is changed to point to the new internal node new , since all arrival points of the replaced leaf are transferred to new . Also, since every operation in the Ops sequence of the old leaf ℓ is moved to new and its appropriate leaf child Claim 2 is also preserved.
3. Part 3 of Definition 7. This has two cases.
First, if sib' is a leaf, then satisfying Claim 1, is trivial as a leaf has no children. Moreover, since all arrival points of the replaced leaf nodes (ℓ and sib) are transferred to the newly created leaf sib' , no violation of the invariant is introduced at the Node whose child pointer is updated to point to sib' . Every operation in p is in either ℓ or sib , by induction hypothesis of claim 1. sib' has all operations from the Ops sequence of ℓ and sib and also includes the current delete(k) operation. Hence, sib' has strictly larger Ops sequence than

p , implying every operation in Ops sequence of p is also in sib' preserving Claim 2.

Second, if sib' is an internal Node, then the delete operation ensures that all operations that arrived at the removed leaf ℓ are transferred to the Ops sequence of sib' and to all its descendants along the search path of those operations. Additionally, all operations from sib are moved to the Ops sequence of sib' . By induction hypothesis all operation at sib should already be there in descendants of sib' as they do not change. This ensures that Claim 1 is preserved at sib' .

Moreover, all arrival points of the replaced Nodes are transferred to sib' and to all its descendants along the corresponding search paths. Therefore, no violation of the invariant is introduced at the Node whose child pointer is updated from p to sib' . In fact, as explained in the first case above, Ops sequence of sib' is strictly larger than the Ops sequence of removed p . Thus, Claim 2 is preserved.

4. Part 4 of Definition 7. A rebalance operation ensures that all operations whose arrival points are at fringe Nodes of the replaced sub graph are added to the Ops sequences of the newly created ancestors of those fringe Nodes. If the replaced subgraph contains leaves (leaves do not have any fringe Nodes), then all operations with a key k' that have their arrival points at those leaves are added to the newly created leaf copies and their appropriate ancestors in the search path of k' . Meaning that every operation in the Ops sequence of a newly created parent in the new patch also appears in the Ops sequence its children (if they exist). Thus, Claim 1 is preserved. which lazily propagate the operations to the Ops sequence of the new ancestors when their version pointers become non-nil.
Similarly, no violation of the invariant is introduced at the Node whose child pointer is updated, since all arrival points at the replaced child old are transferred to the new node that replaces it. As a result, all operations in the Ops sequence old are incorporated into the Ops sequence of new . Consequently, Claim 2 is also preserved.
5. Part 5 of Definition 7. Consider new operations added to the Ops sequence of Node x by a Refresh when it changes $x.version$ (at Line 68). There are two cases.
First, suppose the children of x have not changed since their versions were read before C . Then, these new operations were in the Ops sequence of x 's children last time their (non-nil) version pointers were read. Thus, Claim 1 is preserved at x in a later configuration C' .
Second, suppose the children of x have changed since their versions were read before C . There can be two possibilities within this case. Either new operations were not added to the Ops sequences of the replaced

children before C or they were added. In both cases, by Observation 11, Refresh propagates only those operations O that were in Ops sequences of children before they were replaced. Additionally, by claim 2 of the induction hypothesis, all such operations O are also present in the Ops sequences of the new children. Therefore, in C' , Claim 1 is preserved between x and its new children.

□

Observation 13. *For any leaf, version pointer field is never nil.*

This directly follows from Definition 7.

Lemma 14 and Lemma 15 are identical to Lemmas 21 and 22 of FR-BST [13].

The following lemma shows that a successful Refresh operation on a node propagates all operations that arrived at its children before the refresh read their version pointers.

Lemma 14. *Suppose Node y is a child of Node x at a configuration C and that an update operation op has an arrival point at y at or before C . If Refresh(x) reads $x.version$ at line 50 after C and performs a successful CAS on line 68 then op has an arrival point at x at or before the CAS.*

Proof. Assume y is the left child of x in C . (The case where y is the right child of x is symmetric.)

To ensure that op is propagated to x , we consider the following cases:

1. If the Refresh reads y as the left child of x at line 52 after C , then op already has arrival point at y , by the assumption of this lemma. Now, when op reads $y.version$ at line 53, by Definition 7, part 5a, op has an arrival point at x at or before the CAS at line 68.
2. If the Refresh reads a different Node y' as the left child of x at line 52 after C , then by the second claim of Invariant 12, op has an arrival point at y' before this read occurs.

In either case, op has an arrival point at the left child of x no later than line 53 or line 56 (whichever occurs last), and strictly before the successful CAS performed by Refresh at line 68. Therefore, by Definition 5a and recursive Refresh mechanism, op has an arrival point at x at or before the successful CAS of the Refresh. □

Lemma 15. *Suppose Node y is a child of Node x at a configuration C and that an update operation op has an arrival point at y before C . If a process executes the double refresh at lines 43-44 on x after C then op has an arrival point at x at or before the end of the double refresh.*

Proof. If either call to Refresh successfully performs the CAS step at line 68, then the claim directly follows from Lemma 14.

Now consider the case where both Refresh operations, R_1 and R_2 , fail their CAS steps. This can only occur if two other Refresh operations performed successful CAS steps, c_1 and c_2 , during the execution of R_1 and R_2 , respectively.

Let R be the Refresh operation that executes the CAS step c_2 . By Lemma 10, the read at line 50 in R must occur after the successful CAS step c_1 , implying that R started after configuration C . Now, if op has an arrival point at y in C , by Invariant 12, part 2, op still has arrival point at a child of x , when R reads a pointer to its child.

Applying Lemma 14 to R implies that op has an arrival point at x no later than c_2 , which is before the end of R_2 . □

Lemma 16. *If op has arrived at the leaf at the end of the search path for a key k in the node tree T_c of configuration C , then in any configuration C' later than C , op has arrived at the leaf at the end of the search path for k in T'_c .*

This directly follows from Invariant 12, part 2.

Lemma 17. *If y is a child of x in some configuration C and y' is the child of x in some later configuration C' , the $Ops(C, y, k)$ is a prefix of $Ops(C', y', k)$.*

This directly follows from Invariant 12, part 2.

In this section, we consider an update operation op on a key k and show that it has an arrival point at the root before it terminates. Let C_0 be the first configuration at or before op calls Propagate. Let x_1, \dots, x_m be the Nodes on the local stack (from the newest pushed to the oldest) in C_0 . Since x_1 is on the stack, it is an internal Node. Let x_0 be the left child of x_1 in C_0 if $k < x_1.key$ or the right child of x_1 otherwise. We show by induction on i that before op adds x_i to its refresh set, op has an arrival point at x_i . We first prove the base case by showing that op has an arrival point at x_0 before Propagate is called. Then move on to show that when op adds some node x_i from x_1, \dots, x_m to its refresh set, in some later configuration, then the op has an arrival point at x and at all its descendants in its search path.

Lemma 18. *op has an arrival point at x_0 at or before C_0 , where C_0 is a configuration before op invokes Propagate.*

Proof. We consider several cases.

- Suppose op is either a delete or an insert operation, as described in part 1 of Definition 7. Then op reaches a leaf Node ℓ from some internal node x and determines that ℓ does not contain key k . The arrival point of op at ℓ is the latest configuration C prior to C_0 in which op reads a pointer to ℓ . By Lemma 16, in the later configuration C_0 , the arrival point of op is at the leaf x_0 in the search path of k .
- Suppose op is an update operation, as described in part 2 of Definition 7. By definition, the SCX that adds a new leaves Node ℓ and $newLeaf$ is the arrival point

of the op at $newLeaf$. Let C be the configuration immediately after this SCX. Then ℓ' lies on the search path for key k in configuration C .

To see this, let x be the node whose child pointer is changed by the SCX to add ℓ' . By Lemma 5, x was reachable in some earlier configuration prior to the SCX. By Lemma 3, x remains reachable at the time the SCX is performed. As a result, By Lemma 4, x lies on the search path for k when SCX occurs. Therefore, ℓ' is on the search path for k in configuration C .

Now, consider a later configuration C_0 in which the search path for k ends at a (possibly different) leaf Node x_0 . Then, by Lemma 16, the arrival point of op lies at x_0 , the leaf in the search path for k at C_0 , whether or not $x_0 = \ell'$.

- If op is a delete(k) operation, as described in part 3 of Definition 7, then at the time SCX modifies the tree, op has an arrival point at a descendant leaf of sib' that lies on the search path of k . Further, any subsequent modification to the tree still ensures that op has an arrival point at x_0 before C_0 , by arguments similar to those give above.
- Similarly, if op is a rebalance operation, as described in part 4 of Definition 7. Then the same reasoning as above applies to show that the arrival point of op is at the leaf x_0 in configuration C_0 .

□

Lemma 19. op has an arrival point at root before it terminates.

Proof. We prove by the induction on the sequence of nodes, x_1, \dots, x_m , op adds to its refreshed set during execution of Propagate.

Base case: op has an arrival point at the leaf x_0 in the search path of op 's key k . This follows from lemma 18.

Induction step: Suppose op adds Node x_i to its refreshed set and has an arrival point at the Node. We need to show that before op adds a node x_{i+1} to its refreshed set it will have an arrival point at x_{i+1} . Note, in order to add x_{i+1} to its refreshed set, op first reads x_{i+1} from top of the stack, reads a child x_j of x_{i+1} . *W.l.o.g.*, assume x_j is the left child of x_{i+1} such that $x_j.key < x_{i+1}.key$. (The argument when x_j is the right child, i.e., $x_j.key \geq x_{i+1}.key$ is symmetric.) Note that by the way Propagation adds Nodes in its refresh set, there are following cases for x_j :

Case 1: Node $x_j = x_i$. Then x_j could either be a leaf or an internal node in the refreshed set. In the former case, op will execute a double refresh before adding x_{i+1} to its refresh set followed by removing x_{i+1} from the stack. This means from lemma 15, op will also have its arrival point at x_{i+1} , which is before the double refresh ends and thus is definitely before x_{i+1} is removed from the stack.

In the latter case, x_i is the left child of x_{i+1} already in the refreshed set of op . Therefore, op will execute a double refresh on x_{i+1} ensuring it arrives at x_{i+1} before the double refresh returns and then removes x_{i+1} from the stack followed by adding x_{i+1} to its refresh set. Therefore, the op has an arrival point at x_{i+1} before it adds x_{i+1} to its refresh set and therefore definitely before propagation terminates.

Case 2: Node $x_j \neq x_i$. Then, x_j could either be a leaf or the internal node **not** in the refreshed set of op . In the former case, op will have its arrival point moved to the leaf x_j from lemma 16. Therefore, by lemma 15, before op returns from the double refresh at x_{i+1} , op has its arrival point at x_{i+1} .

In the latter case, x_{i+1} is not the next node to be added to the refreshed set, contradicting the choice of x_{i+1} . Infact, op will traverse down the new search path for k adding the nodes it encounters to its stack until the traversal hits a leaf node or an internal node already in the refresh set (refer to Propagate() in Figure 3). In either case, the inductive argument restarts and follows the same reasoning as in Case 1.

Since Propagate procedure only terminates when the stack is empty, this implies that the root must have been added to the refresh set and subsequently removed from the stack. The double refresh at the root ensures that op has an arrival point there prior to its removal from the stack. So, it follows by induction that op has an arrival point at the root before op terminates. □

B.3 Linearization is consistent with responses

The following invariant ensures that the Ops sequences associated with nodes in a Node x 's left subtree only contain operations on keys less than $x.key$. Similarly, Ops sequences for nodes in the right subtree only contain operations on keys greater than or equal to $x.key$. This is crucial for maintaining the BST property in the Version tree, even as the *node tree* undergoes structural changes.

Invariant 20. (*BST Property of Ops.*) For every internal Node x with any Node x_L and x_R in its left or right subtree, respectively, in any configuration C ,

1. every operation in $Ops(C, x_L)$ has key less than $x.key$
2. every operation in $Ops(C, x_R)$ has key greater than or equal to $x.key$

Proof. Initially, all Ops sequences are empty, so the both claims are trivially true.

Assume both claims hold up to some configuration C . We show that both claims will hold in a later configuration C' . Let s be a step that adds or removes a key from the left subtree of Node x , leading to the configuration C' . The argument for the right subtree is symmetric. We consider various types of steps that can add or remove keys, or modify Ops sequences within the left subtree of x .

Case 1: s is a successful SCX by a *delete(k)* operation described in part 1 of Definition 7 (The argument for the

insert operation is similar). In this case, the Operation is added to the Ops sequence of ℓ at the time when it read the left child pointer of x . Since, the tree is not modified, ℓ remains reachable by x 's left child pointer, and $l.key = k < x.key$, preserving the Invariant.

by Definition, when s replaces ℓ by ℓ' all operations in the Ops sequence of ℓ are transferred to ℓ' . Such that $Ops(C', \ell') = Ops(C', \ell') \langle insert(k) : false \rangle$. By lemma 6, the chromatic tree is still a BST after s . Since ℓ and ℓ' are left child of x , all keys in Ops sequence of ℓ' are strictly less than $x.key$.

Case 2: s is a SCX due to $insert(k)$ described in part 2 of Definition 7.

In this case, s replaces a leaf ℓ by a new internal Node new with two new leaf children, $newLeaf$ and ℓ' . $newLeaf$ contains k , and ℓ' contains $l.key$

Lemma 6 implies that the *node tree* is a chromatic tree in all configurations and therefore always satisfies the BST property. This implies that when new is added to left subtree of x in C' , then $new.key < x.key$.

By Definition 7, part 2, all previously arrived operations (including the current insert) with keys less than $new.key$ are transferred to new 's left child, and those with keys greater than or equal to $new.key$ are transferred to new 's right child.

As a result, all operations in the Ops sequence of new will have keys less than $x.key$. Further, within the subtree at new , the operations in Ops sequence of ℓ' and $newLeaf$ are distributed according to $new.key$, preserving the invariant.

Case 3: s is a SCX for $delete(k)$ described in part 3 of Definition 7.

In this case, in the subtree of x , step s replaces an internal Node p with key k (whose children are a leaf ℓ with key k' and its sibling sib with key k'') by a new copy sib' of sib . By lemma 6, the resulting tree in configuration C' after the step s remains a chromatic tree. Therefore all keys for the subtree rooted at the new internal node sib' are less than $x.key$.

Moreover, from part 3 of Definition 7, this step s is the arrival point of all operations that arrived at ℓ and p before s at sib' and at all its descendants who are in the search path of these operations. As a result, the Ops sequence of sib' is updated to contain all operations who arrived at ℓ , p or sib before step s with keys less than $x.key$. Additionally, all descendants of sib' whose Ops sequence is updated is appended with all operations which arrived at ℓ or p before s , such that for any descendant d with d_L and d_R as its left and right child, in configuration C' , the Ops sequence at d_L contains all operations with keys less than $d.key$ and the Ops sequence at d_R contains all operations with keys greater than equal to $d.key$. Consequently, in configuration C' , the updated ops sequences at sib' and all its descendants follow claim 1 and claim 2 and all operations in any x_L in the left subtree of Node x have keys less than $x.key$.

Case 4: s is a successful SCX for rebalancing operation described in part 4 of Definition 7. It atomically replaces a subgraph of Nodes G_{old} with a new subgraph G_{new} , where

old and new are roots of their respective subgraphs and are the left child of x . By lemma 6, the resulting subtree remains a chromatic tree in C' , infact, s does not add any new key and maintains the overall BST structure. Only the Ops sequences of the new Nodes in G_{new} are updated. By Definition 7, part 4, operation from fringe Nodes of G_{old} , which is same as the fringe Nodes in G_{new} , are transferred to the Ops sequences of appropriate new ancestors in the search path of their keys. Therefore, all keys in Ops sequences of operation in the left subtree of x remain less than $x.key$ as they were before the rebalance step.

Case 5: s is a successful CAS by a refresh operation described in part 5 of Definition 7. Consider that s changes a version field of a Node y in the left subtree of x . Let, y_L and y_R be the left and right child of y read by its Refresh, respectively. Since lemma 6 implies that in all configurations BST property is preserved, at the time s occurs at y , keys of all operations added to the Ops sequence of y in C' are strictly less than $x.key$. Precisely, by Definition 7, part 5, operations added to $Ops(C', y)$ come from the Ops sequence of y_L and y_R .

□

Observation 21. For each leafNode x , $x.version$ always points to a Version with key $x.key$ and no children.

Invariant 22. (prefix property of Ops) For all configurations and all Nodes x that are reachable in C , if x_L and x_R are the left and right child of x in C , then

1. for keys $k < x.key$, $Ops(C, x, k)$ is a prefix of $Ops(C, x_L, k)$, and
2. for keys $k \geq x.key$, $Ops(C, x, k)$ is a prefix of $Ops(C, x_R, k)$.

Proof. In the initial configuration, Ops sequences of all Nodes are empty so the claim is trivially satisfied.

Assume that the claim is satisfied in some configuration C . We want to show that after a step s that changes Ops sequence of a Node x leading to a configuration C' , the claim still holds. The step s could occur due to an insert, delete, rebalance and refresh. we consider them one by one and argue that the invariants holds after a step s .

First, consider a delete case that reaches a leaf ℓ and return $false$ without modifying the *node tree*, as described in part 1 in Definition 7. Let ℓ be the left child of its parent p (case when ℓ is a right child is symmetric). Here in configuration C since delete reached ℓ , it is reachable from p , by lemma 5. Additionally, since the *node tree* is a BST in all configurations, key k' for delete is less than $p.key$ (Lemma 6). In configuration C , by induction hypothesis, for all keys $k < p.key$, $Ops(C, p, k)$ is a prefix of $Ops(C, l, k)$. This delete adds its arrival point for $k' < p.key$ to ℓ . Such that, $Ops(C', l, k) = Ops(C, l, k) \langle delete(k) : false \rangle$. Thus, Ops sequence of ℓ has only grown, ensuring in C' , $Ops(C', p, k)$ is a prefix of $Ops(C', l, k)$.

The insert case that returns false is similar.

Consider the $\text{insert}(k)$ that replaces a leaf ℓ by a new internal Node new with two new leaf children ℓ' and $newLeaf$, as described in part 2 in Definition 7. Let new be the left child of p and $newLeaf$ be the left child of new . The arguments for other cases are similar.

First we will tackle the relation between new and the two new leaves. In C' , all the arrival point at ℓ are transferred to new and $newLeaf$, plus the arrival point of insert. Therefore, by Definition 7, part 2, in C' , for all keys $k', k' < new.key$, $\text{Ops}(C', new, k')$ is a prefix of $\text{Ops}(C', newLeaf, k')$. The key k' , also includes k . Similarly, in C' , for all keys $k'' \geq new.key$, $\text{Ops}(C', new, k'')$ is a prefix of $\text{Ops}(C', \ell', k'')$. Note that $k'' \neq k$. Therefore, $\text{Ops}(C', \ell', k)$ is empty.

Now, consider the relation between new , p and ℓ . In C , for all keys $k' < p.key$ (k' also includes k), $\text{Ops}(C, p, k')$ is a prefix of $\text{Ops}(C, l, k')$ (By induction hypothesis).

In C' , note for a delete that returns fail, its arrival point is at C' and not s . Therefore, for all keys $k' \neq k$ and $k' < p.key$, $|\text{Ops}(C', new, k')| \geq |\text{Ops}(C, l, k')|$. For $k' = k$, $|\text{Ops}(C', new, k')| = |\text{Ops}(C, l, k')|$. In sum, the Ops sequence at new can only grow, combined with induction hypothesis, for all keys $k' < p.key$, $\text{Ops}(C', p, k')$ is a prefix of $\text{Ops}(C', new, k')$. Thus, the invariant is preserved.

Consider the delete that replaces an internal Node p whose children are a leaf ℓ and its sibling sib by a new node sib' , as described in part 3 in Definition 7. Assume p is left child of gp and ℓ is left child of p in C , therefore, sib' is the left child of gp in C' . The other case is symmetric.

The delete transfers all prior arrival point from p , ℓ and sib to sib' and to the descendants of sib' . Precisely, the prior arrival point for a key k in C is transferred to all descendant nodes that are in the search path for k in C' . In addition, the delete also is assigned an arrival point at sib' and all the descendants in its search path in C' . As a result, every op on a key $k < gp.key$ in the $\text{Ops}(C, p, k) \cup \text{Ops}(C, l, k) \cup \text{Ops}(C, sib, k)$ is appended to Nodes in the search path for k in the subtree rooted at sib' in C' . Let, d be an arbitrary node in the search path for k and dp be its parent, then for each such d and dp pair, $\text{Ops}(C', dp, k)$ is a prefix of $\text{Ops}(C', d, k)$. By lemma 6, such a search path exists in the subtree rooted at sib' and nowhere else.

What remains for the delete case is showing that the invariants are preserved for sib' and gp in C' . By part 3 of Definition 7, for all keys $k < gp.key$, $|\text{Ops}(C', sib', k)| \geq |\text{Ops}(C, sib, k)| + |\text{Ops}(C, l, k)|$. Note for the delete returning false, the arrival point can be at C' , hence, at C' , sib' can have more operations than those transferred from leaves in C .

By induction hypothesis, $\text{Ops}(C, gp, k)$ is a prefix of $\text{Ops}(C, p, k)$ which is a prefix of $\text{Ops}(C, sib, k) \cup \text{Ops}(C, l, k)$. Note that the Ops sequence of $\text{Ops}(C', sib', k)$ has only grown, such that $|\text{Ops}(C', sib', k)| \geq |\text{Ops}(C', gp, k)|$. Consequently, $\text{Ops}(C', gp, k)$ is a prefix of $\text{Ops}(C', sib', k)$.

Consider a rebalance operation RB1 in [5, Chap 6, Fig. 6.5]. All arrival points from the fringe nodes in configuration C are transferred to the newly created nodes in the configuration C' , by part 4 in Definition 7. Let n and n_R be the two newly added nodes, such that n_R is right child of n .

- For all keys $k \geq n_R.key \geq n.key$, $\text{Ops}(C', u_{xr}, k) = \text{Ops}(C', n_R, k) = \text{Ops}(C', n, k)$.
- For all keys k , such that $n_R.key > k \geq n.key$, $\text{Ops}(C', u_{xlr}, k) = \text{Ops}(C', n_R, k) = \text{Ops}(C', n, k)$.
- For all keys $k < n.key$, $\text{Ops}(C', u_{xll}, k) = \text{Ops}(C', n, k)$.

Let, u_x be the left child of u , since in configuration C , for all keys $k < u.key$, $\text{Ops}(C, u, k)$ is a prefix $\text{Ops}(C, u_{xll}, k) \cup \text{Ops}(C, u_{xlr}, k) \cup \text{Ops}(C, u_{xr}, k)$. Further in C' , Ops sequence of n can only grow, implying $\text{Ops}(C', u, k)$ is a prefix of $\text{Ops}(C', n, k)$.

It remains to check that a successful CAS step of a Refresh(x) that updates $x.version$ preserves the invariant. Parts 5 of Definition 7 appends new operations to $\text{Ops}(C', x, k)$. By Invariant 20, the new operations can only come from x_L if $k < x.key$ or x_R if $k \geq x.key$. Thus, in case x_L and x_R were still the child in C' , $\text{Ops}(C', x, k)$ simply becomes a longer prefix of $\text{Ops}(C', x_L, k)$ or $\text{Ops}(C', x_R, k)$.

Suppose, x 's children changed from x_L and x_R to x'_L and x'_R , respectively, at some point before the CAS in Refresh but after the versions of x_L and x_R were read. Even in this case, the invariant holds because, by Claim 2 of Invariant 12, every operation that arrived in the Ops sequences of x_L and x_R before they were replaced is transferred to x'_L and x'_R . Further, by Observation 11, only those operations that arrived at x_L and x_R before they were replaced by the new children are added to the Ops sequence of x . \square

Lemma 23. *In every configuration C , for every Node x that is reachable in C and has a non-nil version,*

1. *the Version tree rooted at $x.version$ is a BST whose leaves contains exactly the keys that would be in a set after performing $\text{Ops}(C, x)$ sequentially, and*
2. *the responses recorded in $\text{Ops}(C, x)$ are consistent with executing the operations sequentially.*

Proof. In the initial configuration, both claims hold trivially because each Node's version tree is empty, and thus all corresponding Ops sequences are empty.

Lets consider a step s of an update operation that either change the number of keys or the Ops sequence of a Node. The step s leads from a configuration C to a configuration C' . Lets assume that both the claims hold up to C and we will prove that both the claims will still hold in C' that results from the execution of step s . The step s could be due to several cases and we will show by induction that in each of the case the claims hold.

Case 1: Let s be due to either $\text{delete}(k)$ or $\text{insert}(k)$ in part 1 of Definition 7. In this case, s modifies the Ops sequence of leaf ℓ with some key k' . Specifically,

$$\text{Ops}(C', l) = \text{Ops}(C, l). \langle \text{delete}(k), \text{false} \rangle \text{ and}$$

$\text{Ops}(C', l) = \text{Ops}(C, l). \langle \text{insert}(k), \text{false} \rangle$, for the delete and insert operation, respectively. Since both operations return false, it does not change the number of keys stored at ℓ in C and C' . In both the configurations the set remains $\{k'\}$. So the claims hold.

Case 2: Let s be due to the insert on key k in part 2 of Definition 7. In this case, s replaces ℓ having key k' by a Node new whose two children are ℓ' and newLeaf . *W.l.o.g.*, assume ℓ' is the left child of new and newLeaf is the right child. Then, new and newLeaf are assigned the key k and ℓ' gets the key k' .

For all keys $k'' < k$, $\text{Ops}(C', \ell', k'') \geq \text{Ops}(C, l, k'')$. Note that delete ops returning false can have arrival points at C' and not s . The Ops sequence at ℓ' , for $k'' = k'$ can either result in $\langle \text{Ops}(\text{insert}(k'')) : \text{true} \rangle$ or $\langle \text{Ops}(\text{insert}(k'')) : \text{false} \rangle$. For $k'' \neq k'$, it will end in $\langle \text{Ops}(\text{delete}(k'')) : \text{true} \rangle$. Since the claim holds for $\text{Ops}(C, l, k'')$ and since the set of keys do not change in $\text{Ops}(C', l, k'')$, the claim holds in C' , where the set of keys in C and C' is $\{k'\}$.

For all keys $k''' \geq k$, $\text{Ops}(C', \text{newLeaf}, k''') \geq \text{Ops}(C, l, k''')$. $\langle \text{insert}(k) : \text{true} \rangle$. For $k''' \neq k$, performing $\text{Ops}(C, \text{newLeaf}, k''')$ will yield same set as performing $\text{Ops}(C', \text{newLeaf}, k''')$, by induction hypothesis. The corresponding $\text{Ops}(C', \text{newLeaf}, k''')$ will end in $\langle \text{delete}(k''') : \text{false} \rangle$ or $\langle \text{delete}(k''') : \text{true} \rangle$. Appending $\langle \text{insert}(k) : \text{true} \rangle$ to $\text{Ops}(C', \text{newLeaf}, k''')$, will yield a set $\{k\}$, which would be same as if $\text{Ops}(C, l, k''') \cdot \langle \text{insert}(k) : \text{true} \rangle$ is performed sequentially.

Remaining is proving that claim holds at new as well. By the definition, at new , $\text{Ops}(C', \text{new}) = \text{Ops}(C, l) \cdot \langle \text{insert}(k) : \text{true} \rangle$. The $\text{Ops}(C', \text{new})$ for all keys k'' other than k' and k , will end with either $\langle \text{delete}(k'') : \text{false} \rangle$ or $\langle \text{delete}(k'') : \text{true} \rangle$. For k' , $\text{Ops}(C', \text{new})$ will either end $\langle \text{Ops}(\text{insert}(k')) : \text{true} \rangle$ or $\langle \text{Ops}(\text{insert}(k')) : \text{false} \rangle$. For k , $\text{Ops}(C', \text{new})$ will end in $\langle \text{Ops}(\text{insert}(k)) : \text{true} \rangle$. Performing $\text{Ops}(C', \text{new})$ sequentially, will yield a set $\{k', k\}$. Thus the claims hold.

Case 3: Let s be due to the delete on key k in part 3 of Definition 7. In this case, s replaces p , which has two children, ℓ with key k and sib with key k' , with sib' having key k' . *W.l.o.g.*, assume ℓ is left child of p and p is left child of its parent (other cases can be argued similarly).

For any key k'' , all operations in the Ops sequences of the removed nodes p , ℓ and sib in C are moved to the Ops sequence of sib' and all its descendants up to a leaf in the search path of k'' in C' . Note that such a search path exists by Lemma 6. Let z be any node in the search path for $k'' = k$ from the sub tree rooted at sib' in C' . Then, the Ops sequence of z in C' , ends with $\langle \text{delete}(k) : \text{true} \rangle$. For the node z in the search path of $k'' \neq k$, the $\text{Ops}(C', z, k'')$ either remains same as $\text{Ops}(C, z, k'')$ or ends with either $\langle \text{delete}(k'') : \text{true} \rangle$ or $\langle \text{delete}(k'') : \text{false} \rangle$. Thus, for all keys, k'' , $\text{Ops}(C', \text{sib}', k'')$

yields a set of keys which is same as $\text{Ops}(C, \text{sib}, k'')$. This reflects the set of keys is the leaves of the version tree at sib' . Additionally, since the set of responses move along with the operations in the Ops sequences claim 2 also holds.

Case 4: Let s be due to the rebalancing in part 4 of Definition 7. In this case, s replaces a subtree rooted at a Node old (a child of Node u) by a new subtree rooted at new . This may rearrange the keys of the replaced Nodes but does not change the set of keys in the subtree(see proof in [5, Theorem 6.4]. From Invariant 20, the Ops sequences of all the replaced nodes are transferred to newly created nodes in a way that for each newly created node z with any Node z_L and z_R in its left or right subtree, all operations in $\text{Ops}(C', z_L)$ have key less than $z.\text{key}$ and all operations in $\text{Ops}(C', z_R)$. Moreover, from Invariant 22, for all keys k less than $z.\text{key}$, $\text{Ops}(C', z, k)$ is a prefix of $\text{Ops}(C', z_L, k)$ and for all k greater than or equal to $z.\text{key}$, $\text{Ops}(C', z, k)$ is a prefix of $\text{Ops}(C', z_R, k)$. By observation, rebalance step's net effect is that it changes the search paths of existing keys at the leaf level without changing the number of keys at the leaf level. Therefore, in C' , both the claims hold.

Case 5: Let s be a successful CAS step of a refresh in part 4 of Definition 7. Let σ_L be all operations in $\text{Ops}(C, x_L)$ that do not already have an arrival point at x and σ_R be all operations in $\text{Ops}(C, x_R)$ that do not have an arrival point at x . In this case, s is the arrival point of all operations in σ_L and σ_R at x , such that $\text{Ops}(C', x) = \text{Ops}(C, x) \cdot \sigma_L \cdot \sigma_R$.

In C , Let v_L and v_R be the Versions stored in $x_L.\text{version}$ and $x_R.\text{version}$, respectively. These two version trees represent sets κ_L and κ_R . By Observation 11, v_L was read at some configuration in or before C when x_L was reachable from x . Consequently, the set κ_L consists of keys obtained from sequentially executing operations for the keys in x_L 's previous version and for the new keys in v_L . Same applies for v_R and κ_R .

When s changes $x.\text{version}$ to a new Version v with children v_L and v_R , then in C' , v represents a set of keys $\kappa = \kappa_L \cup \kappa_R$. This is the set of keys which will result in sequentially executing operation in $\text{Ops}(C', x)$. Additionally, since the step s does not change responses of operations that arrived from its children the claim 2 is preserved.

□

Corollary. *Each update that terminates returns a response consistent with the linearization ordering.*

Proof. By Definition 7, for an insert or delete that returns false, the response value associated with it is false. Similarly, for an insert or delete that returns true, the response value associated with it is true. In both cases, the responses are carried up along with their corresponding operations when they arrive at root. From Invariant 23, it follows that responses of operations are consistent with performing the operations sequentially in their linearization order. □

The following invariant follows from the way Refresh works. Essentially, all the Version nodes are immutable and they satisfy the Invariant at the time they are created at Line 67 in Refresh.

Invariant 24. *For every Version v that has children, $v.size = v.left.size + v.right.size$*

Corollary 25. *For every Version v , $v.size$ is the number of leaves in the tree rooted at v that contains key from the universe of keys.*

Lemma 26. *The result returned by each query operation is consistent with the linearization.*

Proof. Queries are defined to be linearized at the moment they read $root.version$. This provides a query with an immutable snapshot of the Version tree. When Invariant 23 is applied at root, the Version tree obtained by the query is a BST and accurately reflects the precise set of keys that would result from sequentially performing all update operations in the Ops sequence of the root. Additionally, Corollary 25 ensures that the number of leaves at a Version tree rooted at any node v taken from the snapshot is same as the value of $size$ in v .

□