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Adhesive Case in Symplectic Contact Analysis

We discuss the contact between a cylindrical indenter and a finite-sized isotropic, adhesive plane here. Consider a horizontally graded elastic plane with Young's modulus in an exponential form, with the parameters being the same as the first numerical example in Chen and Chen (2025), except for the indenter shape and contact condition. The shape function of the indenter is $R-\sqrt{R^2-(x-x_0)^2}$, and the contact region is assumed as $x\in[x_0-l_1,x_0+l_2]$, where l_1 and l_2 are to be determined, x_0 is the x-coordinate of the center of the circle, and R is the radius of the circle. The contact region is different from the size of the rigid frictionless punch (i.e., $b-a$). For contact analysis, the JKR theory (Johnson et al., 1971) is adopted to model the adhesive effect. The actual indentation depth is d_{ac}, which fulfills the relation

$$d_{ac}=d_h-d_{ad} \tag{1}$$

where d_h and d_{ad} are the indentation depths of the Hertz contact case and the adhesive contact case, respectively. It should be emphasized that the adhesive effect in the JKR theory is considered by analogizing a flat punch applied (upward) on the surface, the stress distribution of which is asymmetrical with respect to $x=x_0$ (rather than uniform distribution in the classical JKR model). In addition, the indenter shape function may be simplified into the form of a quadratic function via Taylor expansion when the indentation depth is relatively small.

The coefficients for superposition are obtained through

$$\begin{gathered} \mathscr{A}_{ij}' m_j' = \mathscr{H}_i' \\ \begin{cases} \mathscr{A}_{ij}^{(1)} m_j^{(1)} = \mathscr{H}_i^{(1)} \\ \mathscr{A}_{ij}^{(2)} m_j^{(2)} = \mathscr{H}_i^{(2)} \end{cases} \end{gathered} \tag{2}$$

where

$$\begin{aligned} \mathscr{A}_{ij}' &= -\int_{-l}^{l} \left[\left(\hat{\sigma}_{zz}' \right)_i \left(u_z' \right)_j + \left(\hat{\tau}_{xz}' \right)_i \left(u_z' \right)_j \right] \Big|_{z=h} \mathrm{d}x + \int_{x_0 - l_1'}^{x_0 + l_2'} \left[\left(\hat{\sigma}_{zz}' \right)_i \left(u_z' \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ &\quad - \int_{-l}^{l} \left[\left(u_x' \right)_i \left(\hat{\tau}_{xz}' \right)_j \right] \Big|_{z=0} \mathrm{d}x - \left(\int_{-l}^{x_0 - l_1'} + \int_{x_0 + l_2'}^{l} \right) \left[\left(u_z' \right)_i \left(\hat{\sigma}_{zz}' \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ \mathscr{H}_i' &= \int_{x_0 - l_1'}^{x_0 + l_2'} \left[\left\{ d_h - \left[R - \sqrt{R^2 - (x - x_0)^2} \right] \right\} \left(\hat{\sigma}_{zz}' \right)_i \right] \Big|_{z=0} \mathrm{d}x \end{aligned} \tag{3}$$

$$\begin{aligned} \mathscr{A}_{ij}^{(1)} &= -\int_{-l}^{l} \left[\left(\hat{\sigma}_{zz}^{(1)} \right)_i \left(u_z^{(1)} \right)_j + \left(\hat{\tau}_{xz}^{(1)} \right)_i \left(u_z^{(1)} \right)_j \right] \Big|_{z=h} \mathrm{d}x + \int_{x_0 - l_1}^{x_0 + l_2} \left[\left(\hat{\sigma}_{zz}^{(1)} \right)_i \left(u_z^{(1)} \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ &\quad - \int_{-l}^{l} \left[\left(u_x^{(1)} \right)_i \left(\hat{\tau}_{xz}^{(1)} \right)_j \right] \Big|_{z=0} \mathrm{d}x - \left(\int_{-l}^{x_0 - l_1} + \int_{x_0 + l_2}^{l} \right) \left[\left(u_z^{(1)} \right)_i \left(\hat{\sigma}_{zz}^{(1)} \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ \mathscr{H}_i^{(1)} &= \int_{x_0 - l_1}^{x_0 + l_2} \left[\left\{ d_h - \left[R - \sqrt{R^2 - (x - x_0)^2} \right] \right\} \left(\hat{\sigma}_{zz}^{(1)} \right)_i \right] \Big|_{z=0} \mathrm{d}x \end{aligned} \tag{4}$$

$$\begin{aligned} \mathscr{A}_{ij}^{(2)} &= -\int_{-l}^{l} \left[\left(\hat{\sigma}_{zz}^{(2)} \right)_i \left(u_z^{(2)} \right)_j + \left(\hat{\tau}_{xz}^{(2)} \right)_i \left(u_z^{(2)} \right)_j \right] \Big|_{z=h} \mathrm{d}x + \int_{x_0 - l_1}^{x_0 + l_2} \left[\left(\hat{\sigma}_{zz}^{(2)} \right)_i \left(u_z^{(2)} \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ &\quad - \int_{-l}^{l} \left[\left(u_x^{(2)} \right)_i \left(\hat{\tau}_{xz}^{(2)} \right)_j \right] \Big|_{z=0} \mathrm{d}x - \left(\int_{-l}^{x_0 - l_1} + \int_{x_0 + l_2}^{l} \right) \left[\left(u_z^{(2)} \right)_i \left(\hat{\sigma}_{zz}^{(2)} \right)_j \right] \Big|_{z=0} \mathrm{d}x \\ \mathscr{H}_i^{(2)} &= \int_{x_0 - l_1}^{x_0 + l_2} \left[-d_{ad} \left(\hat{\sigma}_{zz}^{(2)} \right)_i \right] \Big|_{z=0} \mathrm{d}x \end{aligned} \tag{5}$$

from which we arrive at

$$\begin{cases} \tilde{\boldsymbol{f}}'(x,z;l_1',l_2') = \sum_{i=1}^{\infty} m_i'(l_1',l_2')\tilde{\boldsymbol{f}}_i'(x,z) = \sum_{i=1}^{\infty} m_i'(l_1',l_2')\boldsymbol{\mathcal{M}}\boldsymbol{f}_i'(x,z) \\ \tilde{\boldsymbol{f}}(x,z;l_1,l_2) = \sum_{i=1}^{\infty} m_i^{(1)}(l_1,l_2)\tilde{\boldsymbol{f}}_i^{(1)}(x,z) + \sum_{i=1}^{\infty} m_i^{(2)}(l_1,l_2)\tilde{\boldsymbol{f}}_i^{(2)}(x,z) = \sum_{i=1}^{\infty} m_i^{(1)}(l_1,l_2)\boldsymbol{\mathcal{M}}\boldsymbol{f}_i^{(1)}(x,z) + \sum_{i=1}^{\infty} m_i^{(2)}(l_1,l_2)\boldsymbol{\mathcal{M}}\boldsymbol{f}_i^{(2)}(x,z) \end{cases} \tag{6}$$

The strain energy is established as

$$\begin{cases} U' = \int_{-l}^{l}\int_0^h \frac{1}{2}\left(\sigma_{xx}'\varepsilon_x' + \sigma_{zz}'\varepsilon_z' + \sigma_{xz}'\gamma_{xz}'\right)\mathrm{d}z\mathrm{d}x \\ \quad = \int_{-l}^{l}\int_0^h \frac{1}{2}\left[\left(E(x)\frac{\partial \tilde{\boldsymbol{f}}_{|2}'}{\partial x} + \nu_0 \tilde{\boldsymbol{f}}_{|3}'\right)\frac{\partial \tilde{\boldsymbol{f}}_{|2}'}{\partial x} + \tilde{\boldsymbol{f}}_{|3}'\frac{\partial \tilde{\boldsymbol{f}}_{|1}'}{\partial z} + \tilde{\boldsymbol{f}}_{|4}'\left(\frac{\partial \tilde{\boldsymbol{f}}_{|2}'}{\partial z} + \frac{\partial \tilde{\boldsymbol{f}}_{|1}'}{\partial x}\right)\right]\mathrm{d}z\mathrm{d}x \\ U = \int_{-l}^{l}\int_0^h \frac{1}{2}\left(\sigma_{xx}\varepsilon_x + \sigma_{zz}\varepsilon_z + \sigma_{xz}\gamma_{xz}\right)\mathrm{d}z\mathrm{d}x \\ \quad = \int_{-l}^{l}\int_0^h \frac{1}{2}\left[\left(E(x)\frac{\partial \tilde{\boldsymbol{f}}_{|2}}{\partial x} + \nu_0 \tilde{\boldsymbol{f}}_{|3}\right)\frac{\partial \tilde{\boldsymbol{f}}_{|2}}{\partial x} + \tilde{\boldsymbol{f}}_{|3}\frac{\partial \tilde{\boldsymbol{f}}_{|1}}{\partial z} + \tilde{\boldsymbol{f}}_{|4}\left(\frac{\partial \tilde{\boldsymbol{f}}_{|2}}{\partial z} + \frac{\partial \tilde{\boldsymbol{f}}_{|1}}{\partial x}\right)\right]\mathrm{d}z\mathrm{d}x \end{cases} \tag{7}$$

where $\tilde{\boldsymbol{f}}_{|j}$ represent the j-th variable in the state vector $\tilde{\boldsymbol{f}}$. According to fracture mechanics, we have

$$-\left.\frac{\partial U'}{\partial l_1'}\right|_{d_h} = 0, \quad \left.\frac{\partial U'}{\partial l_2'}\right|_{d_h} = 0 \tag{8}$$

$$-\left.\frac{\partial U}{\partial l_1}\right|_{d_{ac}} = \Delta\gamma_1, \quad \left.\frac{\partial U}{\partial l_2}\right|_{d_{ac}} = \Delta\gamma_2 \tag{9}$$

where $\Delta\gamma_1$ are the surface energy. To elucidate, if the flat punch solution is established in a homogeneous material (rather than a graded material), the surface energy will be a constant. If the adhesive effect is not considered (i.e., in Hertz contact), the surface energy function in Eq. (9) should be replaced by zero, as displayed in Eq. (8). Then, the relations between l_1', l_2' and d_h are derived accordingly. Consequently, we can obtain l_1 and l_2 from Eq. (9) with the basic assumptions of $l_1 = l_1'$ and $l_2 = l_2'$. However, the two relations shown in Eq. (9) are not independent, which indicate that we can deduce the surface energy from one point to another (the surface energy at the left or right critical contact point should be measured before). It is important to note that this derivation is valid only in cases that JKR model is acceptable. Additionally, we may prove $l_1 = l_2$ in the case of homogeneous materials.

For efficient use of program coding to obtain l_1' and l_2' (also for l_1 and l_2, we just take the non-adhesive case for instance here), we may rewrite the left-hand side of the first equation in Eq. (8) as

$$
\begin{aligned}
\frac{\partial U'}{\partial l_1'} &= \sum_{s=1}^{\infty}\frac{\partial U'}{\partial m_s'}\frac{\partial m_s'}{\partial l_1'} \\
&= \sum_{s=1}^{\infty}\frac{\partial\left\{\int_{-l}^{l}\int_0^h \frac{1}{2}\left[\left(E(x)\sum_{i=1}^{\infty}m_i'\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}+\nu_0\sum_{i=1}^{\infty}m_i'\tilde{\boldsymbol{f}}_{i|3}'\right)\left(\sum_{i=1}^{\infty}m_i'\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}\right)+\left(\sum_{i=1}^{\infty}m_i'\tilde{\boldsymbol{f}}_{i|3}'\right)\left(\sum_{i=1}^{\infty}m_i'\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial z}\right)+\left(\sum_{i=1}^{\infty}m_i'\tilde{\boldsymbol{f}}_{i|4}'\right)\left(\sum_{i=1}^{\infty}m_i'\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial z}+\sum_{i=1}^{\infty}m_i'\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial x}\right)\right]\mathrm{d}z\mathrm{d}x\right\}}{\partial m_s'}\frac{\partial m_s'}{\partial l_1'} \\
&= \frac{1}{2}\sum_{s=1}^{\infty}\frac{\partial\left\{\int_{-l}^{l}\int_0^h\left[\sum_{i=1}^{\infty}\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{i|3}'\right)m_i'\cdot\sum_{i=1}^{\infty}\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}m_i'+\sum_{i=1}^{\infty}\tilde{\boldsymbol{f}}_{i|3}'m_i'\cdot\sum_{i=1}^{\infty}\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial z}m_i'+\sum_{i=1}^{\infty}\tilde{\boldsymbol{f}}_{i|4}'m_i'\cdot\sum_{i=1}^{\infty}\left(\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial x}\right)m_i\right]\mathrm{d}z\mathrm{d}x\right\}}{\partial m_s'}\frac{\partial m_s'}{\partial l_1'} \\
&= \frac{1}{2}\sum_{s=1}^{\infty}\left\{\int_{-l}^{l}\int_0^h\left[\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{s|3}'\right)\cdot\sum_{i=1}^{\infty}\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}m_i'+\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial x}\cdot\sum_{i=1}^{\infty}\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{i|3}'\right)m_i'+\tilde{\boldsymbol{f}}_{s|3}'\cdot\sum_{i=1}^{\infty}\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial z}m_i'+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}'}{\partial z}\cdot\sum_{i=1}^{\infty}\tilde{\boldsymbol{f}}_{i|3}'m_i'+\tilde{\boldsymbol{f}}_{s|4}'\cdot\sum_{i=1}^{\infty}\left(\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial x}\right)m_i'+\left(\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}'}{\partial x}\right)\cdot\sum_{i=1}^{\infty}\tilde{\boldsymbol{f}}_{i|4}'m_i'\right]\mathrm{d}z\mathrm{d}x\cdot\frac{\partial m_s'}{\partial l_1'}\right\} \\
&= \sum_{s=1}^{\infty}\sum_{i=1}^{\infty}\varpi_{si}'m_i'\frac{\partial m_s'}{\partial l_1'}
\end{aligned}
\tag{10}
$$

where $\tilde{\boldsymbol{f}}_i'$ is shown in symplectic expansion Eq. (44) in Chen et al. (2025); and

$$
\varpi_{si}' = \frac{1}{2}\int_{-l}^{l}\int_0^h\left[\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{s|3}'\right)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}+\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial x}\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{i|3}'\right)+\tilde{\boldsymbol{f}}_{s|3}'\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}'}{\partial z}\tilde{\boldsymbol{f}}_{i|3}'+\tilde{\boldsymbol{f}}_{s|4}'\left(\frac{\partial\tilde{\boldsymbol{f}}_{i|2}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{i|1}'}{\partial x}\right)+\left(\frac{\partial\tilde{\boldsymbol{f}}_{s|2}'}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}'}{\partial x}\right)\tilde{\boldsymbol{f}}_{i|4}'\right]\mathrm{d}z\mathrm{d}x
\tag{11}
$$

Then, $\partial m_s'/\partial l_1'$ is derived from Eqs. (2) and (3), as

$$\begin{aligned}\frac{\partial m_s'}{\partial l_1'} &= \frac{\partial \mathscr{A}_{si}'^{-1}}{\partial l_1'}\mathscr{H}_i' + \mathscr{A}_{si}'^{-1}\frac{\partial \mathscr{H}_i'}{\partial l_1'} \\ &= -\mathscr{A}_{si}'^{-1}\frac{\partial \mathscr{A}_{is}'}{\partial l_1'}\mathscr{A}_{si}'^{-1}\mathscr{H}_i' + \mathscr{A}_{si}'^{-1}\frac{\partial \mathscr{H}_i'}{\partial l_1'}\end{aligned} \tag{12}$$

where

$$\begin{aligned}\frac{\partial \mathscr{A}_{is}'}{\partial l_1'} &= \frac{\partial}{\partial l_1'}\left\{\int_{x_0-l_1'}^{x_0+l_2'}\left[\left(\hat{\sigma}_{zz}'\right)_i\left(u_z'\right)_s\right]\Big|_{z=0}\,\mathrm{d}x - \int_{-l}^{x_0-l_1'}\left[\left(u_z'\right)_i\left(\hat{\sigma}_{zz}'\right)_j\right]\Big|_{z=0}\,\mathrm{d}x\right\} \\ &= \left[\hat{\sigma}_{zz}'(x_0-l_1',0)\right]_i\left[u_z'(x_0-l_1',0)\right]_s + \left[u_z'(x_0-l_1',0)\right]_i\left[\hat{\sigma}_{zz}'(x_0-l_1',0)\right]_s\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial \mathscr{H}_i'}{\partial l_1'} &= \frac{\partial}{\partial l_1'}\left\{\int_{x_0-l_1'}^{x_0+l_2'}\left[\left\{d_h - \left[R-\sqrt{R^2-(x-x_0)^2}\right]\right\}\left(\hat{\sigma}_{zz}'\right)_i\right]\Big|_{z=0}\,\mathrm{d}x\right\} \\ &= \left[d_h - R + \sqrt{R^2-l_1^2}\right]\left[\hat{\sigma}_{zz}'(x_0-l_1',0)\right]_i\end{aligned} \tag{14}$$

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