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Apr. 11, 2025

Appendix for "Contact analysis for a horizontally graded soft electroactive material under uniform biasing fields in symplectic approach"

Appendix A: Independent invariants

According to the Hamilton-Cayley theorem,

$$\mathbf{C}^3 - I_1^\dagger \mathbf{C}^2 + I_2^\dagger \mathbf{C} - I_3^\dagger \mathbf{I}_3 = 0, \tag{A.1}$$

which leads to

$$\mathbf{C}^{-1} = \frac{1}{I_3^\dagger}\left(\mathbf{C}^2 - I_1^\dagger \mathbf{C} + I_2^\dagger \mathbf{I}_3\right), \quad \mathbf{C}^{-2} = \frac{1}{I_3^\dagger}\left(\mathbf{C} - I_1^\dagger \mathbf{I}_3 + I_2^\dagger \mathbf{C}^{-1}\right), \quad \mathbf{C}^{-3} = \frac{1}{I_3^\dagger}\left(\mathbf{I}_3 - I_1^\dagger \mathbf{C}^{-1} + I_2^\dagger \mathbf{C}^{-2}\right), \tag{A.2}$$

then the invariants are

$$I_1^\ddagger = \mathrm{tr}\mathbf{C}^{-1} = \frac{1}{I_3^\dagger}\left(\mathrm{tr}\mathbf{C}^2 - I_1^\dagger \mathrm{tr}\mathbf{C} + 3I_2^\dagger\right) = \frac{1}{I_3^\dagger}\left[\left(I_1^\dagger\right)^2 - 2I_2^\dagger - \left(I_1^\dagger\right)^2 + 3I_2^\dagger\right] = \frac{I_2^\dagger}{I_3^\dagger} \tag{A.3}$$

$$I_2^\ddagger = \frac{1}{2}\left[\left(\mathrm{tr}\mathbf{C}^{-1}\right)^2 - \mathrm{tr}\mathbf{C}^{-2}\right] = \frac{1}{2}\left[\left(I_1^\ddagger\right)^2 - \frac{1}{I_3^\dagger}\left(\mathrm{tr}\mathbf{C} - 3I_1^\dagger + I_2^\dagger \mathrm{tr}\mathbf{C}^{-1}\right)\right] = \frac{1}{2}\left[\left(\frac{I_2^\dagger}{I_3^\dagger}\right)^2 - \frac{1}{I_3^\dagger}\left(I_1^\dagger - 3I_1^\dagger + I_2^\dagger \frac{I_2^\dagger}{I_3^\dagger}\right)\right] = \frac{I_1^\dagger}{I_3^\dagger} \tag{A.4}$$

$$I_3^\ddagger = \det \mathbf{C}^{-1} = \frac{1}{6}\left[2\mathrm{tr}\mathbf{C}^{-3} - 3\mathrm{tr}\mathbf{C}^{-1}\mathrm{tr}\mathbf{C}^{-2} + \left(\mathrm{tr}\mathbf{C}^{-1}\right)^3\right] = \frac{1}{6}\left[\frac{2}{I_3^\dagger}\left(3 - I_1^\dagger \mathrm{tr}\mathbf{C}^{-1} + I_2^\dagger \mathrm{tr}\mathbf{C}^{-2}\right) - 3\frac{I_2^\dagger}{\left(I_3^\dagger\right)^2}\left(I_1^\dagger - 3I_1^\dagger + I_2^\dagger \frac{I_2^\dagger}{I_3^\dagger}\right) + \left(\frac{I_2^\dagger}{I_3^\dagger}\right)^3\right] = \frac{1}{I_3^\dagger} \tag{A.5}$$

$$I_4^{\ddagger} = \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{D}} = I_4^{\dagger} \tag{A.6}$$

$$I_5^{\ddagger} = \boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}^{-1}\boldsymbol{\mathcal{D}}\right) = \frac{1}{I_3^{\dagger}}\left[\boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}^{2}\boldsymbol{\mathcal{D}}\right) - I_1^{?}\boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}\boldsymbol{\mathcal{D}}\right) + I_2^{\dagger}\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{D}}\right] = \frac{1}{I_3^{\dagger}}\left(I_6^{\dagger} - I_1^{\dagger} I_5^{\dagger} + I_2^{\dagger} I_4^{\dagger}\right) \tag{A.7}$$

$$I_6^{\ddagger} = \boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}^{-2}\boldsymbol{\mathcal{D}}\right) = \frac{1}{I_3^{\dagger}}\left[\boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}\boldsymbol{\mathcal{D}}\right) - I_1^{?}\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\mathcal{D}} + I_2^{\dagger}\boldsymbol{\mathcal{D}} \cdot \left(\mathbf{C}^{-1}\boldsymbol{\mathcal{D}}\right)\right] = \frac{1}{I_3^{\dagger}}\left(I_5^{?} - I_1^{\dagger} I_4^{\dagger} + I_2^{\dagger} I_5^{\ddagger}\right) \tag{A.8}$$

Appendix B: Parameters and derivations in section 2.2

B.1 Material parameters

Material parameters shown in Eqs. (14) and (15) are generally derived from $c_{ij} = c_{ij}^0 \mathrm{e}^{\beta x}$, $e_{ij} = e_{ij}^0 \mathrm{e}^{\beta x}$, and $\epsilon_{ij} = \epsilon_{ij}^0 \mathrm{e}^{\beta x}$, where

$$
\begin{aligned}
& c_{11} = \mathrm{A}_{1111}^* - \frac{\left(\mathrm{M}_{113}^*\right)^2}{\mathrm{K}_{33}^*}, && c_{12} = \mathrm{A}_{1122}^* - \frac{\mathrm{M}_{223}^*\mathrm{M}_{113}^*}{\mathrm{K}_{33}^*}, && c_{13} = \mathrm{A}_{1133}^* - \frac{\mathrm{M}_{113}^*\mathrm{M}_{333}^*}{\mathrm{K}_{33}^*}, \\
& c_{23} = \mathrm{A}_{2233}^* - \frac{\mathrm{M}_{223}^*\mathrm{M}_{333}^*}{\mathrm{K}_{33}^*}, && c_{33} = \mathrm{A}_{3333}^* - \frac{\left(\mathrm{M}_{333}^*\right)^2}{\mathrm{K}_{33}^*}, && c_{55} = \mathrm{A}_{1313}^* - \frac{\left(\mathrm{M}_{131}^*\right)^2}{\mathrm{K}_{11}^*}, \\
& c_{58} = \mathrm{A}_{1331}^* - \frac{\left(\mathrm{M}_{131}^*\right)^2}{\mathrm{K}_{11}^*}, && c_{88} = \mathrm{A}_{3131}^* - \frac{\left(\mathrm{M}_{131}^*\right)^2}{\mathrm{K}_{11}^*}, && e_{15} = -\frac{\mathrm{M}_{131}^*}{\mathrm{K}_{11}^*}, \\
& e_{31} = -\frac{\mathrm{M}_{113}^*}{\mathrm{K}_{33}^*}, && e_{32} = -\frac{\mathrm{M}_{223}^*}{\mathrm{K}_{33}^*}, && e_{33} = -\frac{\mathrm{M}_{333}^*}{\mathrm{K}_{33}^*}, \\
& \epsilon_{11} = \frac{1}{\mathrm{K}_{11}^*}, && \epsilon_{33} = \frac{1}{\mathrm{K}_{33}^*}.
\end{aligned}
$$

Essential components of the effective material tensors appeared above are expressed as

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$$JA^*_{1111} = 4\left[\Omega_{11}\lambda_1^4 + \Omega_{22}\lambda_1^4\left(1+\lambda_3^2\right)^2 + \Omega_{33}\lambda_1^4\lambda_3^4 + 2\Omega_{12}\lambda_1^4\left(1+\lambda_3^2\right) + 2\Omega_{13}\lambda_1^4\lambda_3^2 + 2\Omega_{23}\lambda_1^4\lambda_3^2\left(1+\lambda_3^2\right)\right] + 2\left[\Omega_1\lambda_1^2 + \Omega_2\lambda_1^2\left(1+\lambda_3^2\right) + \Omega_3\lambda_1^2\lambda_3^2\right],$$

$$JA^*_{1122} = 4\left[\Omega_{11}\lambda_1^2 + \Omega_{22}\left(\lambda_1^2+\lambda_1^2\lambda_3^2\right)\left(\lambda_1^2+\lambda_3^2\right) + \Omega_{33}\lambda_1^4\lambda_3^4 + \Omega_{12}\left(\lambda_1^4 + 2\lambda_1^2\lambda_3^2 + \lambda_1^2\right) + \Omega_{13}\lambda_1^2\lambda_3^2\left(1+\lambda_1^2\right) + \Omega_{23}\lambda_1^2\lambda_3^2\left(2\lambda_1^2+\lambda_3^2+\lambda_1^2\lambda_3^2\right)\right] + 2\left(2\Omega_2\lambda_1^2 + 2\Omega_3\lambda_1^2\lambda_3^2\right),$$

$$\begin{aligned} JA^*_{1133} = 4\Big\{ & \Omega_{11}\lambda_1^2\lambda_3^2 + \Omega_{22}\left(\lambda_1^2+\lambda_1^2\lambda_3^2\right)\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \Omega_{33}\lambda_1^4\lambda_3^4 + \Omega_{12}\left[\lambda_1^2\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \lambda_3^2\left(\lambda_1^2+\lambda_1^2\lambda_3^2\right)\right] + \Omega_{13}\lambda_1^2\lambda_3^2\left(\lambda_1^2+\lambda_3^2\right) + \Omega_{23}\lambda_1^2\lambda_3^2\left(\lambda_1^2+\lambda_3^2+2\lambda_1^2\lambda_3^2\right) + \Omega_{15}\lambda_1^4\lambda_3^2D_3^2 \\ & + \Omega_{35}\lambda_1^4\lambda_3^4D_3^2 + \Omega_{25}\lambda_1^2\lambda_3^2\left(\lambda_1^2+\lambda_1^2\lambda_3^2\right)D_3^2 + 2\Omega_{16}\lambda_1^4\lambda_3^4D_3^2 + 2\Omega_{26}\lambda_1^2\lambda_3^4\left(\lambda_1^2+\lambda_1^2\lambda_3^2\right)D_3^2 + 2\Omega_{36}\lambda_1^4\lambda_3^6D_3^2\Big\} + 2\left(2\Omega_2\lambda_1^2\lambda_3^2 + 2\Omega_3\lambda_1^2\lambda_3^2\right), \end{aligned}$$

$$\begin{aligned} JA^*_{2233} = 4\Big\{ & \Omega_{11}\lambda_3^2 + \Omega_{22}\left(\lambda_1^2+\lambda_3^2\right)\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \Omega_{33}\lambda_1^4\lambda_3^4 + \Omega_{12}\left[\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \lambda_3^2\left(\lambda_1^2+\lambda_3^2\right)\right] + \Omega_{13}\lambda_1^2\lambda_3^2\left(1+\lambda_3^2\right) + \Omega_{23}\lambda_1^2\lambda_3^2\left(\lambda_1^2+2\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \Omega_{15}\lambda_1^2\lambda_3^2D_3^2 + \Omega_{35}\lambda_1^4\lambda_3^4D_3^2 \\ & + \Omega_{25}\lambda_1^2\lambda_3^2\left(\lambda_1^2+\lambda_3^2\right)D_3^2 + 2\Omega_{16}\lambda_1^2\lambda_3^4D_3^2 + 2\Omega_{26}\lambda_1^2\lambda_3^4\left(\lambda_1^2+\lambda_3^2\right)D_3^2 + 2\Omega_{36}\lambda_1^4\lambda_3^6D_3^2\Big\} + 2\left(2\Omega_2\lambda_3^2 + 2\Omega_3\lambda_1^2\lambda_3^2\right), \end{aligned}$$

$$\begin{aligned} JA^*_{3333} = 4\Big\{ & \Omega_{11}\lambda_3^4 + \Omega_{22}\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right)^2 + \Omega_{33}\lambda_1^4\lambda_3^4 + 2\Omega_{12}\lambda_3^2\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + 2\Omega_{13}\lambda_1^2\lambda_3^4 + 2\Omega_{23}\lambda_1^2\lambda_3^2\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right) + \Omega_{55}\lambda_1^4\lambda_3^4D_3^4 + 2\Omega_{15}\lambda_1^2\lambda_3^4D_3^2 + 2\Omega_{35}\lambda_1^4\lambda_3^4D_3^2 + 2\Omega_{25}\lambda_1^2\lambda_3^2\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right)D_3^2 \\ & + 4\Omega_{66}\lambda_1^4\lambda_3^8D_3^4 + 4\Omega_{16}\lambda_1^2\lambda_3^6D_3^2 + 4\Omega_{26}\lambda_1^2\lambda_3^4\left(\lambda_3^2+\lambda_1^2\lambda_3^2\right)D_3^2 + 4\Omega_{36}\lambda_1^4\lambda_3^6D_3^2 + 4\Omega_{56}\lambda_1^4\lambda_3^6D_3^4\Big\} + 2\left[\Omega_1\lambda_3^2 + \Omega_2\left(1+\lambda_1^2\right)\lambda_3^2 + \Omega_3\lambda_1^2\lambda_3^2 + \Omega_5\lambda_1^2\lambda_3^2D_3^2 + 6\Omega_6\lambda_1^2\lambda_3^4D_3^2\right], \end{aligned}$$

$$JA^*_{1331} = 2\left(-\Omega_2\lambda_1^2\lambda_3^2 - \Omega_3\lambda_1^2\lambda_3^2 + \Omega_6\lambda_1^4\lambda_3^2D_3^2\right),$$

$$JA^*_{1313} = 2\left(\Omega_1\lambda_1^2 + \Omega_2\lambda_1^2 + \Omega_6\lambda_1^4\lambda_3^2D_3^2\right),$$

$$JA^*_{3131} = 2\left[\Omega_1\lambda_3^2 + \Omega_2\lambda_3^2 + \Omega_5\lambda_1^2\lambda_3^2D_3^2 + \Omega_6\left(2\lambda_1^2\lambda_3^4D_3^2 + \lambda_1^4\lambda_3^2D_3^2\right)\right],$$

$$M^*_{113} = 4J\left[\Omega_{14}\frac{\lambda_1^2}{\lambda_3^2}D_3 + \Omega_{24}\frac{\lambda_1^2}{\lambda_3^2}\left(1+\lambda_3^2\right)D_3 + \Omega_{34}\lambda_1^2D_3 + \Omega_{15}\lambda_1^2D_3 + \Omega_{25}\lambda_1^2\left(1+\lambda_3^2\right)D_3 + \Omega_{35}\lambda_1^2\lambda_3^2D_3 + \Omega_{16}\lambda_1^2\lambda_3^2D_3 + \Omega_{26}\lambda_1^2\lambda_3^2\left(1+\lambda_3^2\right)D_3 + \Omega_{36}\lambda_1^2\lambda_3^4D_3\right],$$

$$\mathrm{M}_{223}^{*}=4J\left[\Omega_{14}\frac{1}{\lambda_3^2}D_3+\Omega_{24}\frac{\lambda_1^2+\lambda_3^2}{\lambda_3^2}D_3+\Omega_{34}\lambda_1^2D_3+\Omega_{15}D_3+\Omega_{25}\left(\lambda_1^2+\lambda_3^2\right)D_3+\Omega_{35}\lambda_1^2\lambda_3^2D_3+\Omega_{16}\lambda_3^2D_3+\Omega_{26}\left(\lambda_1^2+\lambda_3^2\right)\lambda_3^2D_3+\Omega_{36}\lambda_1^2\lambda_3^4D_3\right],$$

$$\begin{aligned}\mathrm{M}_{333}^{*}=4J\Big[&\Omega_{14}D_3+\Omega_{24}\left(1+\lambda_1^2\right)D_3+\Omega_{34}\lambda_1^2D_3+\Omega_{45}\lambda_1^2D_3^3+2\Omega_{46}\lambda_1^2\lambda_3^2D_3^3+\Omega_{15}\lambda_3^2D_3+\Omega_{25}\lambda_3^2\left(1+\lambda_1^2\right)D_3+\Omega_{35}\lambda_1^2\lambda_3^2D_3+\Omega_{55}\lambda_1^2\lambda_3^2D_3^3+3\Omega_{56}\lambda_1^2\lambda_3^4D_3^3+\Omega_{16}\lambda_3^4D_3\\&+\Omega_{26}\lambda_3^4\left(1+\lambda_1^2\right)D_3+\Omega_{36}\lambda_1^2\lambda_3^4D_3+2\Omega_{66}\lambda_1^2\lambda_3^6D_3^3\Big]+2J\left(2\Omega_5D_3+4\Omega_6\lambda_3^2D_3\right),\end{aligned}$$

$$\mathrm{M}_{131}^{*}=2J\left[\Omega_5D_3+\Omega_6\left(\lambda_3^2D_3+\lambda_1^2D_3\right)\right],$$

$$\mathrm{M}_{232}^{*}=2J\left[\Omega_5D_3+\Omega_6\left(\lambda_3^2D_3+D_3\right)\right],$$

$$\mathrm{K}_{11}^{*}=2J\left(\Omega_4\frac{1}{\lambda_1^2}+\Omega_5+\Omega_6\lambda_1^2\right),$$

$$\mathrm{K}_{22}^{*}=2J\left(\Omega_4+\Omega_5+\Omega_6\right),$$

$$\mathrm{K}_{33}^{*}=4J^3\left(\Omega_{44}D_3^2+\Omega_{55}D_3^2+\Omega_{66}\lambda_3^4D_3^2+2\Omega_{45}\frac{1}{\lambda_3^2}D_3^2+2\Omega_{46}D_3^2+2\Omega_{56}\lambda_3^2D_3^2\right)+2J\left(\Omega_4\frac{1}{\lambda_3^2}+\Omega_5+\Omega_6\lambda_3^2\right).$$

B.2 Symplectic formulations

The symplectic form is constructed via the following process: we first rearrange Eqs. (13) and (14):

$$\begin{aligned}
&\frac{\partial u_x}{\partial z} = -a_1 \frac{\partial u_z}{\partial x} - a_2 \left(\frac{\partial \hat{\dot{\varphi}}}{\partial x} + \beta \hat{\dot{\varphi}} \right) + a_3 \hat{\dot{\tau}}_{zx}^*, \\
&\frac{\partial u_z}{\partial z} = -a_4 \frac{\partial u_x}{\partial x} + a_5 \hat{\dot{\tau}}_{zz}^* + a_6 \dot{\mathrm{D}}_z^*, \\
&\frac{\partial \hat{\dot{\varphi}}}{\partial z} = -a_7 \frac{\partial u_x}{\partial x} + a_6 \hat{\dot{\tau}}_{zz}^* - a_8 \dot{\mathrm{D}}_z^*,
\end{aligned} \tag{B.1}$$

together with supplementary Eq. (16) and an additional equation:

$$\hat{\dot{\tau}}_{yy}^* = \left(c_{12}^0 + \frac{c_{33}^0 e_{31}^0 e_{32}^0 - c_{13}^0 c_{23}^0 \epsilon_{33}^0 - c_{13}^0 e_{32}^0 e_{33}^0 - c_{23}^0 e_{31}^0 e_{33}^0}{a_0} \right) \frac{\partial u_x}{\partial x} + \frac{c_{23}^0 \epsilon_{33}^0 + e_{32}^0 e_{33}^0}{a_0} \hat{\dot{\tau}}_{zz}^* + \frac{c_{23}^0 e_{33}^0 - c_{33}^0 e_{32}^0}{a_0} \dot{\mathrm{D}}_z^*. \tag{B.2}$$

Then Eq. $(7)_{1,2}$ can be rewritten as

$$\begin{aligned}
&\frac{\partial \hat{\dot{\tau}}_{zx}^*}{\partial z} = -\frac{\partial \hat{\dot{\tau}}_{xx}^*}{\partial x} - \beta \hat{\dot{\tau}}_{xx}^* = -a_9 \left(\frac{\partial^2 u_x}{\partial x^2} + \beta \frac{\partial u_x}{\partial x} \right) - a_4 \left(\frac{\partial \hat{\dot{\tau}}_{zz}^*}{\partial x} + \beta \hat{\dot{\tau}}_{zz}^* \right) - a_7 \left(\frac{\partial \dot{\mathrm{D}}_z^*}{\partial x} + \beta \dot{\mathrm{D}}_z^* \right), \\
&\frac{\partial \hat{\dot{\tau}}_{zz}^*}{\partial z} = -\frac{\partial \hat{\dot{\tau}}_{xz}^*}{\partial x} - \beta \hat{\dot{\tau}}_{xz}^* = -a_{10} \left(\frac{\partial^2 u_z}{\partial x^2} + \beta \frac{\partial u_z}{\partial x} \right) - a_{11} \left(\frac{\partial^2 \hat{\dot{\varphi}}}{\partial x^2} + 2\beta \frac{\partial \hat{\dot{\varphi}}}{\partial x} + \beta^2 \hat{\dot{\varphi}} \right) - a_1 \left(\frac{\partial \hat{\dot{\tau}}_{zx}^*}{\partial x} + \beta \hat{\dot{\tau}}_{zx}^* \right), \\
&\frac{\partial \dot{\mathrm{D}}_z^*}{\partial z} = -\frac{\partial \dot{\mathrm{D}}_x^*}{\partial x} = -a_{11} \frac{\partial^2 u_z}{\partial x^2} + a_{12} \left(\frac{\partial^2 \hat{\dot{\varphi}}}{\partial x^2} + \beta \frac{\partial \hat{\dot{\varphi}}}{\partial x} \right) - a_2 \frac{\partial \hat{\dot{\tau}}_{zx}^*}{\partial x},
\end{aligned} \tag{B.3}$$

where

$$
\begin{aligned}
&a_0 = c_{33}^0 \epsilon_{33}^0 + \left(e_{33}^0\right)^2, && a_1 = c_{58}^0 \big/ c_{88}^0, && a_2 = e_{15}^0 / c_{88}^0, \\
&a_3 = 1 \big/ c_{88}^0, && a_4 = \left(c_{13}^0 \epsilon_{33}^0 + e_{31}^0 e_{33}^0\right) \big/ a_0, && a_5 = \epsilon_{33}^0 \big/ a_0, \\
&a_6 = e_{33}^0 \big/ a_0, && a_7 = \left(c_{13}^0 e_{33}^0 - c_{33}^0 e_{31}^0\right) \big/ a_0, && a_8 = c_{33}^0 \big/ a_0, \\
&a_9 = c_{11}^0 - c_{13}^0 a_4 - e_{31}^0 a_7, && a_{10} = c_{55}^0 - \left(c_{58}^0\right)^2 \big/ c_{88}^0, && a_{11} = e_{15}^0 - \left(c_{58}^0 e_{15}^0\right) \big/ c_{88}^0, \\
&a_{12} = \epsilon_{11}^0 + \left(e_{15}^0\right)^2 \big/ c_{88}^0.
\end{aligned}
$$

Combining Eqs. (B.1) and (B.3) together, we will arrive at Eq. (15), where

$$
\boldsymbol{\mathcal{H}} = \left[\begin{array}{ccc:ccc}
0 & -a_1 \dfrac{\partial}{\partial x} & -a_2\left(\dfrac{\partial}{\partial x} + \beta\right) & a_3 & 0 & 0 \\
-a_4 \dfrac{\partial}{\partial x} & 0 & 0 & 0 & a_5 & a_6 \\
-a_7 \dfrac{\partial}{\partial x} & 0 & 0 & 0 & a_6 & -a_8 \\
\hdashline
-a_9\left(\dfrac{\partial^2}{\partial x^2} + \beta \dfrac{\partial}{\partial x}\right) & 0 & 0 & 0 & -a_4\left(\dfrac{\partial}{\partial x} + \beta\right) & -a_7\left(\dfrac{\partial}{\partial x} + \beta\right) \\
0 & -a_{10}\left(\dfrac{\partial^2}{\partial x^2} + \beta \dfrac{\partial}{\partial x}\right) & -a_{11}\left(\dfrac{\partial^2}{\partial x^2} + 2\beta \dfrac{\partial}{\partial x} + \beta^2\right) & -a_1\left(\dfrac{\partial}{\partial x} + \beta\right) & 0 & 0 \\
0 & -a_{11} \dfrac{\partial^2}{\partial x^2} & a_{12}\left(\dfrac{\partial^2}{\partial x^2} + \beta \dfrac{\partial}{\partial x}\right) & -a_2 \dfrac{\partial}{\partial x} & 0 & 0
\end{array}\right].
$$

In addition, the symplectic form can also be derived via canonical equations, which originate in a Legendre transformation of a Lagrangian (Zhong, 1995).

B.3 Derivation of Eq. (23)

After substituting Eq. (22) into Eq. (15), we arrive at

$$\begin{aligned}\frac{\partial}{\partial z}\xi(z)\boldsymbol{\Phi}(x) &= \boldsymbol{\mathcal{H}}\xi(z)\boldsymbol{\Phi}(x)\\ &= \xi(z)\boldsymbol{\mathcal{H}}\boldsymbol{\Phi}(x),\end{aligned} \tag{B.4}$$

from which we have

$$\frac{\frac{\partial}{\partial z}\xi(z)}{\xi(z)}\boldsymbol{\Phi}(x) = \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}(x). \tag{B.5}$$

According to the definition of the pseudoinverse of a state vector as $\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1}\boldsymbol{\Phi} = \boldsymbol{\Phi}$, then

$$\frac{\frac{\partial}{\partial z}\xi(z)}{\xi(z)}\boldsymbol{\Phi} = \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1}\boldsymbol{\Phi}, \tag{B.6}$$

which results in

$$\left[\frac{\frac{\partial}{\partial z}\xi(z)}{\xi(z)} - \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}\boldsymbol{\Phi}^{-1}\right]\boldsymbol{\Phi} = 0. \tag{B.7}$$

Since $\boldsymbol{\Phi} \neq 0$, Eq. (23) is obtained.

Appendix C: Constants and roots in section 2.3

Constants in Eq. (31):

$$\Pi_6 = \left(a_1^2 a_{12} + 2a_1 a_2 a_{11} + a_3 a_{10} a_{12} - a_2^2 a_{10} + a_3 a_{11}^2\right)\left(a_4^2 a_8 + 2a_4 a_6 a_7 - a_5 a_7^2 + a_5 a_8 a_9 + a_6^2 a_9\right),$$

$$\Pi_5 = 3\beta\left(a_1^2 a_{12} + 2a_1 a_2 a_{11} + a_3 a_{10} a_{12} - a_2^2 a_{10} + a_3 a_{11}^2\right)\left(a_4^2 a_8 + 2a_4 a_6 a_7 - a_5 a_7^2 + a_5 a_8 a_9 + a_6^2 a_9\right),$$

$$\begin{aligned}\Pi_4 = {} & 3\beta^2\left(a_1^2 a_{12} + 2a_1 a_2 a_{11} + a_3 a_{10} a_{12} - a_2^2 a_{10} + a_3 a_{11}^2\right)\left(a_4^2 a_8 + 2a_4 a_6 a_7 - a_5 a_7^2 + a_5 a_8 a_9 + a_6^2 a_9\right) + \mu^2\Big[a_1^2\left(a_4^2 + a_5 a_9\right) + a_5 a_8 a_{11}^2 + a_6^2 a_{11}^2 - 2a_2 a_4 a_8 a_{11} \\ & + a_{10}\left(a_5 a_8 a_{12} + a_6^2 a_{12} + 2a_2 a_4 a_6 - 2a_2 a_5 a_7 + a_3 a_4^2 + a_3 a_5 a_9\right) + 2a_1\left(-a_4 a_6 a_{11} + a_5 a_7 a_{11} - a_4 a_8 a_{12} - a_6 a_7 a_{12} + a_2 a_4 a_7 + a_2 a_6 a_9\right) \\ & - 2a_2 a_6 a_7 a_{11} + 2a_3 a_4 a_7 a_{11} + 2a_3 a_6 a_9 a_{11} - a_3 a_7^2 a_{12} + a_3 a_8 a_9 a_{12} + a_2^2 a_7^2 - a_2^2 a_8 a_9\Big],\end{aligned}$$

$$\begin{aligned}\Pi_3 = {} & \beta^3\left(a_1^2 a_{12} + 2a_1 a_2 a_{11} + a_3 a_{10} a_{12} - a_2^2 a_{10} + a_3 a_{11}^2\right)\left(a_4^2 a_8 + 2a_4 a_6 a_7 - a_5 a_7^2 + a_5 a_8 a_9 + a_6^2 a_9\right) + 2\beta\mu^2\Big[a_1^2\left(a_4^2 + a_5 a_9\right) + a_5 a_8 a_{11}^2 + a_6^2 a_{11}^2 - 2a_2 a_4 a_8 a_{11} \\ & + a_{10}\left(a_5 a_8 a_{12} + a_6^2 a_{12} + 2a_2 a_4 a_6 - 2a_2 a_5 a_7 + a_3 a_4^2 + a_3 a_5 a_9\right) + 2a_1\left(-a_4 a_6 a_{11} + a_5 a_7 a_{11} - a_4 a_8 a_{12} - a_6 a_7 a_{12} + a_2 a_4 a_7 + a_2 a_6 a_9\right) \\ & - 2a_2 a_6 a_7 a_{11} + 2a_3 a_4 a_7 a_{11} + 2a_3 a_6 a_9 a_{11} - a_3 a_7^2 a_{12} + a_3 a_8 a_9 a_{12} + a_2^2 a_7^2 - a_2^2 a_8 a_9\Big],\end{aligned}$$

$$\begin{aligned}\Pi_2 = {} & \beta^2\mu^2\Big\{a_1^2\left(a_4^2 + a_5 a_9\right) + a_1\left[a_{11}\left(2a_5 a_7 - 2a_4 a_6\right) - 3a_{12}\left(a_4 a_8 + a_6 a_7\right) + 3a_2\left(a_4 a_7 + a_6 a_9\right)\right] + a_{10}\left[a_5 a_8 a_{12} + a_6^2 a_{12} + 2a_2 a_4 a_6 - 2a_2 a_5 a_7 + a_3\left(a_4^2 + a_5 a_9\right)\right] \\ & + a_{11}^2\left(a_5 a_8 + a_6^2\right) + 3a_{11}\left(-a_2 a_4 a_8 - a_2 a_6 a_7 + a_3 a_4 a_7 + a_3 a_6 a_9\right) + \left(a_2^2 - a_3 a_{12}\right)\left(a_7^2 - a_8 a_9\right)\Big\} + \mu^4\left(-2a_1 a_4 + a_5 a_{10} + 2a_6 a_{11} + a_8 a_{12} - 2a_2 a_7 + a_3 a_9\right),\end{aligned}$$

$$\Pi_1 = \beta\mu^4\left(-2a_1 a_4 + a_5 a_{10} + 2a_6 a_{11} + a_8 a_{12} - 2a_2 a_7 + a_3 a_9\right) + \beta^3\mu^2\Big\{a_1\left[a_2\left(a_4 a_7 + a_6 a_9\right) - a_{12}\left(a_4 a_8 + a_6 a_7\right)\right] + a_{11}\left[a_3\left(a_4 a_7 + a_6 a_9\right) - a_2\left(a_4 a_8 + a_6 a_7\right)\right]\Big\},$$

$$\Pi_0 = \left(a_6 a_{11} - a_1 a_4\right)\beta^2\mu^4 + \mu^6.$$

Roots of Eq. (31): $\eta_{k-3\left\lfloor \frac{k-1}{3} \right\rfloor} = -\frac{\beta}{2} + (-1)^{\left\lfloor \frac{k-1}{3} \right\rfloor} \sqrt{\vartheta_k}$ ($\lfloor \# \rfloor$ is a floor function, which maps # to the largest integer smaller than or equal to #)

ϑ_k fulfill $R_3\vartheta^3 + R_2\vartheta^2 + R_1\vartheta + R_0 = 0$, which are derived as

$$\begin{aligned}
\vartheta_1 ={}& \frac{\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}}{3\sqrt[3]{2}R_3} \\
&- \frac{\sqrt[3]{2}\left(3R_1R_2 - R_2^2\right)}{3R_3\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}} - \frac{R_2}{3R_3}, \\
\vartheta_2 ={}& -\frac{\left(1 - \mathrm{i}\sqrt{3}\right)\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}}{6\sqrt[3]{2}R_3} \\
&+ \frac{\left(1 + \mathrm{i}\sqrt{3}\right)\left(3R_1R_3 - R_2^2\right)}{3\ 2^{2/3}R_3\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}} - \frac{R_2}{3R_3}, \\
\vartheta_3 ={}& -\frac{\left(1 + \mathrm{i}\sqrt{3}\right)\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}}{6\sqrt[3]{2}R_3} \\
&+ \frac{\left(1 - \mathrm{i}\sqrt{3}\right)\left(3R_1R_3 - R_2^2\right)}{3\ 2^{2/3}R_3\sqrt[3]{\sqrt{\left(27R_0R_3^2 - 9R_1R_2R_3 + 2R_2^3\right)^2 - 4\left(R_2^2 - 3R_1R_3\right)^3} - 27R_0R_3^2 + 9R_1R_2R_3 - 2R_2^3}} - \frac{R_2}{3R_3}.
\end{aligned}$$

where

$$R_3 = \left(a_1^2 a_{12} + 2a_1 a_2 a_{11} + a_3 a_{10} a_{12} - a_2^2 a_{10} + a_3 a_{11}^2\right)\left(a_4^2 a_8 + 2a_4 a_6 a_7 - a_5 a_7^2 + a_5 a_8 a_9 + a_6^2 a_9\right),$$

$$
\begin{aligned}
R_2 = {} & \mu^2\Big[a_1^2\left(a_4^2+a_5a_9\right)+2a_1\left(-a_4a_6a_{11}+a_5a_7a_{11}-a_4a_8a_{12}-a_6a_7a_{12}+a_2a_4a_7+a_2a_6a_9\right)+a_{10}\left(a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3a_4^2+a_3a_5a_9\right) \\
& +a_5a_8a_{11}^2+a_6^2a_{11}^2-2a_2a_4a_8a_{11}-2a_2a_6a_7a_{11}+2a_3a_4a_7a_{11}+2a_3a_6a_9a_{11}-a_3a_7^2a_{12}+a_3a_8a_9a_{12}+a_2^2a_7^2-a_2^2a_8a_9\Big] \\
& -\frac{3}{4}\beta^2\left(a_1^2a_{12}+2a_1a_2a_{11}+a_3a_{10}a_{12}-a_2^2a_{10}+a_3a_{11}^2\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right),
\end{aligned}
$$

$$
\begin{aligned}
R_1 = {} & \frac{3}{16}\beta^4\left(a_1^2a_{12}+2a_1a_2a_{11}+a_3a_{10}a_{12}-a_2^2a_{10}+a_3a_{11}^2\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right) \\
& -\frac{1}{2}\beta^2\mu^2\Big[a_1^2\left(a_4^2+a_5a_9\right)+2a_1a_{11}\left(a_5a_7-a_4a_6\right)+a_{10}\left(a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3a_4^2+a_3a_5a_9\right)+a_{11}^2\left(a_5a_8+a_6^2\right)+a_7^2\left(a_2^2-a_3a_{12}\right)+a_8a_9\left(a_3a_{12}-a_2^2\right)\Big] \\
& +\mu^4\left(-2a_1a_4+a_5a_{10}+2a_6a_{11}+a_8a_{12}-2a_2a_7+a_3a_9\right),
\end{aligned}
$$

$$
\begin{aligned}
R_0 = {} & -\frac{1}{64}\beta^6\left(a_1^2a_{12}+2a_1a_2a_{11}+a_3a_{10}a_{12}-a_2^2a_{10}+a_3a_{11}^2\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right) \\
& +\frac{1}{16}\beta^4\mu^2\Big[a_1^2\left(a_4^2+a_5a_9\right)+2a_1\left(-a_4a_6a_{11}+a_5a_7a_{11}+a_4a_8a_{12}+a_6a_7a_{12}-a_2a_4a_7-a_2a_6a_9\right)+a_{10}\left(a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3a_4^2+a_3a_5a_9\right) \\
& +a_5a_8a_{11}^2+a_6^2a_{11}^2+2a_2a_4a_8a_{11}+2a_2a_6a_7a_{11}-2a_3a_4a_7a_{11}-2a_3a_6a_9a_{11}-a_3a_7^2a_{12}+a_3a_8a_9a_{12}+a_2^2a_7^2-a_2^2a_8a_9\Big] \\
& -\frac{1}{4}\beta^2\mu^4\left(2a_1a_4+a_5a_{10}-2a_6a_{11}+a_8a_{12}-2a_2a_7+a_3a_9\right)+\mu^6.
\end{aligned}
$$

Parameters in Eq. (32):

$$
\begin{aligned}
t_{1k} = {} & -\mu\Big\{(\beta+\eta_k)\left(a_2a_5a_8+a_2a_6^2+a_3a_4a_6-a_3a_5a_7\right)\Big[a_1a_2a_6\eta_k^2+a_3\left(a_6a_{11}\eta_k^2+a_8a_{12}\beta\eta_k+a_8a_{12}\eta_k^2+\mu^2\right)-a_2^2a_8\eta_k\left(\beta+\eta_k\right)\Big] \\
& -\Big[a_1a_6\eta_k+(\beta+\eta_k)\left(a_3a_7-a_2a_8\right)\Big]\Big[\eta_k(\beta+\eta_k)\left(a_2^2-a_3a_{12}\right)\left(a_5a_8+a_6^2\right)-a_3a_5\mu^2\Big]\Big\} \\
& /\Big\{a_3a_6\Big[-\eta_k^2\left(\beta+\eta_k\right)^2\left(a_2^2-a_3a_{12}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)+\mu^2\eta_k\left(\beta+\eta_k\right)\left(a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3a_4^2+a_3a_5a_9\right)+a_5\mu^4\Big]\Big\},
\end{aligned}
$$

$$\begin{aligned}\iota_{2k} = &\Big\{\eta_k^3\left(\beta+\eta_k\right)\left(a_1a_2+a_3a_{11}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)\\ &+\eta_k\mu^2\left[\eta_k\left(-a_1a_4a_6+a_1a_5a_7+a_5a_8a_{11}+a_6^2a_{11}\right)-a_2\left(\beta+\eta_k\right)\left(a_4a_8+a_6a_7\right)+a_3\left(\beta+\eta_k\right)\left(a_4a_7+a_6a_9\right)\right]+a_6\mu^4\Big\}\\ &/\Big\{-\eta_k^2\left(\beta+\eta_k\right)^2\left(a_2^2-a_3a_{12}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)+\mu^2\eta_k\left(\beta+\eta_k\right)\left[a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3\left(a_4^2+a_5a_9\right)\right]+a_5\mu^4\Big\},\end{aligned}$$

$$\begin{aligned}\iota_{3k} = &\left(\beta+\eta_k\right)\Big\{\eta_k^3\left(\beta+\eta_k\right)\left(a_1a_{12}+a_2a_{11}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)\\ &+\eta_k\mu^2\left[\eta_k\left(a_1a_4^2+a_1a_5a_9-a_4a_6a_{11}+a_5a_7a_{11}\right)-a_{12}\left(\beta+\eta_k\right)\left(a_4a_8+a_6a_7\right)+a_2\left(\beta+\eta_k\right)\left(a_4a_7+a_6a_9\right)\right]-a_4\mu^4\Big\}\\ &/\Big\{\mu^2\eta_k\left(\beta+\eta_k\right)\left[a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3\left(a_4^2+a_5a_9\right)\right]-\eta_k^2\left(\beta+\eta_k\right)^2\left(a_2^2-a_3a_{12}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)+a_5\mu^4\Big\},\end{aligned}$$

$$\begin{aligned}\iota_{4k} = &\mu\Big\{\eta_k^2\left(\beta+\eta_k\right)\Big[a_2\eta_k\left(a_1\left(a_4a_7+a_6a_9\right)-a_{11}\left(a_4a_8+a_6a_7\right)\right)-a_{12}\left(a_1\eta_k\left(a_4a_8+a_6a_7\right)+a_3(\beta+\eta)\left(\mathrm{a7}^2-\mathrm{a8a9}\right)\right)+a_3a_{11}\eta_k\left(a_4a_7+a_6a_9\right)+a_2^2\left(\beta+\eta_k\right)\left(a_7^2-a_8a_9\right)\Big]\\ &+\eta_k\mu^2\left[-a_1a_4\eta_k+a_6a_{11}\eta_k+\left(\beta+\eta_k\right)\left(a_3a_9+a_8a_{12}\right)-2a_2a_7\left(\beta+\eta_k\right)\right]+\mu^4\Big\}\\ &/\Big\{\eta_k\mu^2\left(\beta+\eta_k\right)\left[a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3\left(a_4^2+a_5a_9\right)\right]-\eta_k^2\left(\beta+\eta_k\right)^2\left(a_2^2-a_3a_{12}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)+a_5\mu^4\Big\},\end{aligned}$$

$$\begin{aligned}\iota_{5k} = &-\mu\eta_k\Big\{a_2\left(\beta+\eta_k\right)\left[\eta_k^2\left(a_1a_4^2+a_1a_5a_9+a_4a_6a_{11}-a_5a_7a_{11}\right)-a_4\mu^2\right]-a_{12}\left(\beta+\eta_k\right)\left[a_1\eta_k^2\left(a_5a_7-a_4a_6\right)+a_3\eta_k\left(\beta+\eta_k\right)\left(a_4a_7+a_6a_9\right)+a_6\mu^2\right]\\ &+a_{11}\eta_k\left[a_3\eta_k\left(\beta+\eta_k\right)\left(a_4^2+a_5a_9\right)+a_5\mu^2\right]+a_2^2\eta_k\left(\beta+\eta_k\right)^2\left(a_4a_7+a_6a_9\right)\Big\}\\ &/\Big\{\eta_k\mu^2\left(\beta+\eta_k\right)\left[a_5a_8a_{12}+a_6^2a_{12}+2a_2a_4a_6-2a_2a_5a_7+a_3\left(a_4^2+a_5a_9\right)\right]-\eta_k^2\left(\beta+\eta_k\right)^2\left(a_2^2-a_3a_{12}\right)\left(a_4^2a_8+2a_4a_6a_7-a_5a_7^2+a_5a_8a_9+a_6^2a_9\right)+a_5\mu^4\Big\}.\end{aligned}$$

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