Contact Analysis of Heterogeneous Media

Lagrangian and Hamiltonian Formulation

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Main Content

- Background
- Lagrangian formulation and surface Green's functions
- Hamiltonian formulation (with multi-field coupling)
  - Exponentially graded model
  - Laminated model : interfacial effect and boundary effect
  - Viscoelastic model : dual Hamiltonian transformation
  - Couple stress model : size effect and local phase transition
  - Arbitrary gradient: generalized dual Hamiltonian transformation
  - Three-dimensional layered model: sub-symplectic structure

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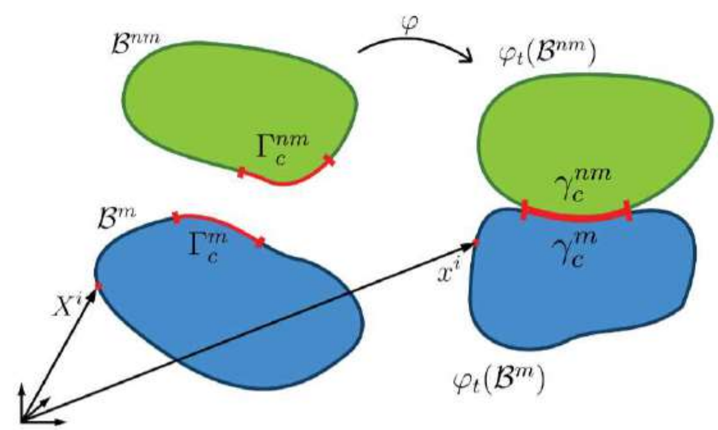
Background

$$\mathcal{X}' = \mathcal{O}\mathcal{X} + \mathcal{T}$$

material frame orthogonal group translation

symmetry breaking → information



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Lagrangian Formulation

Background

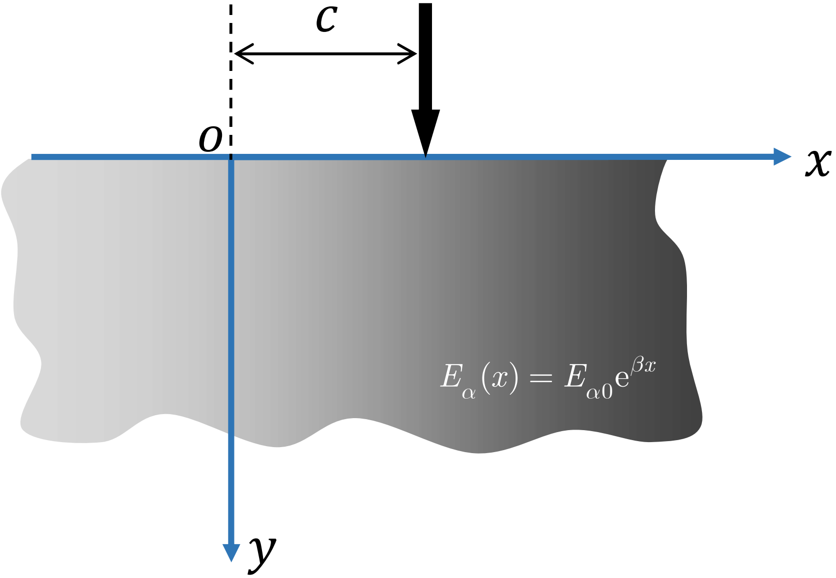


Figure 1. Horizontal graded plane (Normal case)

Methodology (vertical displacement)

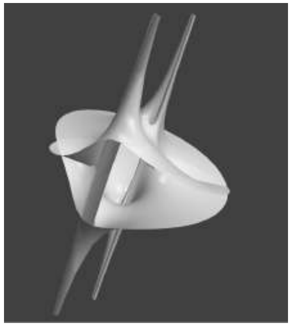
$$\boxed{v\Big|_{y=0} = \mathscr{F}^{-1}\left[\mathcal{V}\right]\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\omega^2-\mathrm{i}\,\beta\omega+\beta^2\nu_{\alpha 0}\,/\,4}}{\omega\left(\omega-\mathrm{i}\,\beta\right)}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega}$$

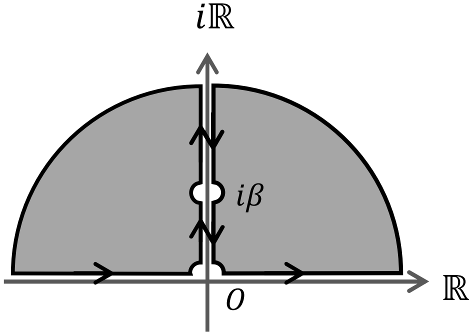
Weak Gradient Assumption $\beta^2 \to 0$

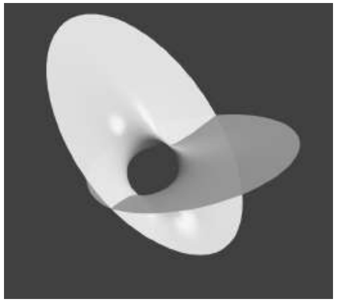
$$v\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\omega^2-\mathrm{i}\,\beta\omega}}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega$$

Homogeneous Case

$$\begin{aligned} v\Big|_{y=0} &= \frac{P}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\omega^2}}{\omega^2}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega \\ &= \frac{P}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{1}{\left|\omega\right|}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega \\ &= -\frac{2P}{E_{\alpha 0}\pi}\left(\ln\left|x-c\right|+\gamma\right) \end{aligned}$$







Analytical Solutions for Normal Case

Methodology (vertical displacement)

$$\boxed{v\Big|_{y=0} = \mathscr{F}^{-1}\left[\mathcal{V}\right]\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega + \beta^2\nu_{\alpha 0}\,/\,4}}{\omega\left(\omega - \mathrm{i}\,\beta\right)}\,\mathrm{e}^{\mathrm{i}\,\omega\left(x-c\right)}\,\mathrm{d}\omega}$$

Weak Gradient Assumption $\quad \beta^2 \to 0$

$$v\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega}}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega$$

$$\int_{0}^{+\infty}\frac{1}{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega}}\,\mathrm{e}^{\mathrm{i}\,\omega\left(x-c\right)}\,\mathrm{d}\omega + \int_{-\infty}^{0}\frac{1}{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega}}\,\mathrm{e}^{\mathrm{i}\,\omega\left(x-c\right)}\,\mathrm{d}\omega$$

$$\omega = s^2$$

Analytical Solutions for Normal Case

Analytical solutions

$$\boxed{v\Big|_{y=0} = \mathscr{F}^{-1}\left[\mathcal{V}\right]\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega + \beta^2\nu_{\alpha 0}\,/\,4}}{\omega\left(\omega - \mathrm{i}\,\beta\right)}\,\mathrm{e}^{\mathrm{i}\,\omega\left(x-c\right)}\,\mathrm{d}\omega}$$

Weak Gradient Assumption

$\beta^2 \to 0$

$$v\Big|_{y=0} = \frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{\omega^2 - \mathrm{i}\,\beta\omega}}\,\mathrm{e}^{\mathrm{i}\,\omega(x-c)}\,\mathrm{d}\omega$$

$$\begin{aligned}\frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\Bigg[\!\!\Bigg[&\frac{1}{2\sqrt{2}\pi}\left\{G^{2,3}_{4,2}\left[\frac{2\,\mathrm{i}}{\beta\left|c-x\right|},\frac{1}{2}\Big|^{\frac{1}{2},1,1,\frac{1}{2}}_{\frac{1}{4},\frac{3}{4}}\right] - \mathrm{i}\,\mathrm{sgn}(c-x)G^{2,3}_{4,2}\left[\frac{2\,\mathrm{i}}{\beta\left|c-x\right|},\frac{1}{2}\Big|^{\frac{1}{2},\frac{1}{2},1,1}_{\frac{1}{4},\frac{3}{4}}\right]\right\}\\ &+ K_0\left(\frac{\beta\left|c-x\right|}{2}\right)\left[\mathrm{sgn}\left(c-x\right)\sinh\left(\frac{\beta\left|c-x\right|}{2}\right) + \cosh\left(\frac{\beta\left|c-x\right|}{2}\right)\right] + \frac{1}{2}\,\mathrm{i}\,\pi[\mathrm{sgn}\left(c-x\right) - 1]I_0\left(\frac{\beta\left|c-x\right|}{2}\right)\left[\cosh\left(\frac{\beta\left|c-x\right|}{2}\right) - \sinh\left(\frac{\beta\left|c-x\right|}{2}\right)\right]\Bigg]\!\!\Bigg]\end{aligned}$$

$$\boxed{G^{m,n}_{p,q}\left[z,r\Big|^{a_1,\cdots,a_p}_{b_1,\cdots,b_q}\right] = \frac{r}{2\pi\,\mathrm{i}}\int_L\frac{\Gamma(1-a_1-rs)\cdots\Gamma(1-a_n-rs)\Gamma(b_1+rs)\cdots\Gamma(b_m+rs)}{\Gamma(a_{n+1}+rs)\cdots\Gamma(a_p+rs)\Gamma(1-b_{m+1}-rs)\cdots\Gamma(1-b_q-rs)}\,z^{-s}\,\mathrm{d}s}$$

Analytical Solutions for Normal Case

Properties of Meijer G-Function & Fox H-Function

$$G_{p,q}^{m,n}\left[z,r\middle|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right]=\frac{r}{2\pi\,\mathrm{i}}\int_L\frac{\Gamma(1-a_1-rs)\cdots\Gamma(1-a_n-rs)\Gamma(b_1+rs)\cdots\Gamma(b_m+rs)}{\Gamma(a_{n+1}+rs)\cdots\Gamma(a_p+rs)\Gamma(1-b_{m+1}-rs)\cdots\Gamma(1-b_q-rs)}z^{-s}\mathrm{d}s$$

$$H_{p,q}^{m,n}\left[z\middle|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right]=\frac{1}{2\pi\,\mathrm{i}}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}z^{-s}\mathrm{d}s$$

Property 1

$$\frac{1}{\kappa}H_{p,q}^{m,n}\left[z\middle|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right]=H_{p,q}^{m,n}\left[z^{\kappa}\middle|_{(b_1,\kappa\beta_1),\cdots,(b_q,\kappa\beta_q)}^{(a_1,\kappa\alpha_1),\cdots,(a_p,\kappa\alpha_p)}\right]$$

$$G_{p,q}^{m,n}\left[z,\tfrac{1}{2}\middle|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right]=\tfrac{1}{2}H_{p,q}^{m,n}\left[z\middle|_{(b_1,\frac{1}{2}),\cdots,(b_q,\frac{1}{2})}^{(a_1,\frac{1}{2}),\cdots,(a_p,\frac{1}{2})}\right]=H_{p,q}^{m,n}\left[z^2\middle|_{(b_1,1),\cdots,(b_q,1)}^{(a_1,1),\cdots,(a_p,1)}\right]=G_{p,q}^{m,n}\left[z^2\middle|_{b_1,\cdots,b_q}^{a_1,\cdots,a_p}\right]$$

Property 2

$$H_{p,q}^{m,n}\left[z\middle|_{(b_1,\beta_1),\cdots,(b_q,\beta_q)}^{(a_1,\alpha_1),\cdots,(a_p,\alpha_p)}\right]=H_{q,p}^{n,m}\left[\frac{1}{z}\middle|_{(1-a_1,\alpha_1),\cdots,(1-a_p,\alpha_p)}^{(1-b_1,\beta_1),\cdots,(1-b_q,\beta_q)}\right]$$

$$\arg(1/z)=-\arg(z)$$

Mathai, Arak. M. "A handbook of generalized special functions for statistical and physical sciences." *Oxford: Oxford University Press*, 1993.

Analytical Solutions for Normal Case

Simplification

$$\boxed{\begin{aligned}\frac{P\,\mathrm{e}^{-\beta c}}{E_{\alpha 0}\pi}\Bigg[\!\!\Bigg[&\frac{1}{2\sqrt{2}\pi}\left\{G_{4,2}^{2,3}\left[\frac{2\,\mathrm{i}}{\beta|c-x|},\frac{1}{2}\middle|\begin{matrix}\frac{1}{2},1,1,\frac{1}{2}\\ \frac{1}{4},\frac{3}{4}\end{matrix}\right]-\mathrm{i}\,\mathrm{sgn}(c-x)G_{4,2}^{2,3}\left[\frac{2\,\mathrm{i}}{\beta|c-x|},\frac{1}{2}\middle|\begin{matrix}\frac{1}{2},\frac{1}{2},1,1\\ \frac{1}{4},\frac{3}{4}\end{matrix}\right]\right\}\\&+K_0\left(\frac{\beta|c-x|}{2}\right)\left[\mathrm{sgn}\left(c-x\right)\sinh\left(\frac{\beta|c-x|}{2}\right)+\cosh\left(\frac{\beta|c-x|}{2}\right)\right]+\frac{1}{2}\,\mathrm{i}\,\pi[\mathrm{sgn}\left(c-x\right)-1]I_0\left(\frac{\beta|c-x|}{2}\right)\left[\cosh\left(\frac{\beta|c-x|}{2}\right)-\sinh\left(\frac{\beta|c-x|}{2}\right)\right]\Bigg]\!\!\Bigg]\end{aligned}}$$

$$G_{2,4}^{3,2}\left[-\frac{\beta^2\left(c-x\right)^2}{4}\middle|\begin{matrix}\frac{3}{4},\frac{1}{4}\\ \frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right]$$

$$\begin{aligned}G_{2,4}^{3,2}\left[-\frac{\beta^2\left(c-x\right)^2}{4}\middle|\begin{matrix}\frac{3}{4},\frac{1}{4}\\ \frac{1}{2},0,0,\frac{1}{2}\end{matrix}\right]=&\Bigg[2\sqrt{2}\pi K_0\left(\frac{\beta\left(c-x\right)}{2}\right)\cosh\left(\frac{\beta\left(c-x\right)}{2}\right)\\&+\mathrm{i}\sqrt{2}\pi^2\,\mathrm{sgn}(c-x)I_0\left(\frac{\beta\left(c-x\right)}{2}\right)\left(\sinh\left(\frac{\beta\left(c-x\right)}{2}\right)-\cosh\left(\frac{\beta\left(c-x\right)}{2}\right)\right)\Bigg]^*\end{aligned}$$

$$G_{2,4}^{3,2}\left[-\frac{\beta^2\left(c-x\right)^2}{4}\middle|\begin{matrix}\frac{3}{4},\frac{1}{4}\\ \frac{1}{2},\frac{1}{2},0,0\end{matrix}\right]$$

$$\begin{aligned}G_{2,4}^{3,2}\left[-\frac{\beta^2\left(c-x\right)^2}{4}\middle|\begin{matrix}\frac{3}{4},\frac{1}{4}\\ \frac{1}{2},\frac{1}{2},0,0\end{matrix}\right]=&\Bigg[-2\sqrt{2}\pi\,\mathrm{i}\,\mathrm{sgn}(c-x)K_0\left(\frac{\beta\left(c-x\right)}{2}\right)\sinh\left(\frac{\beta\left(c-x\right)}{2}\right)\\&+\sqrt{2}\pi^2 I_0\left(\frac{\beta\left(c-x\right)}{2}\right)\left(\cosh\left(\frac{\beta\left(c-x\right)}{2}\right)-\sinh\left(\frac{\beta\left(c-x\right)}{2}\right)\right)\Bigg]^*\end{aligned}$$

Results and comparison

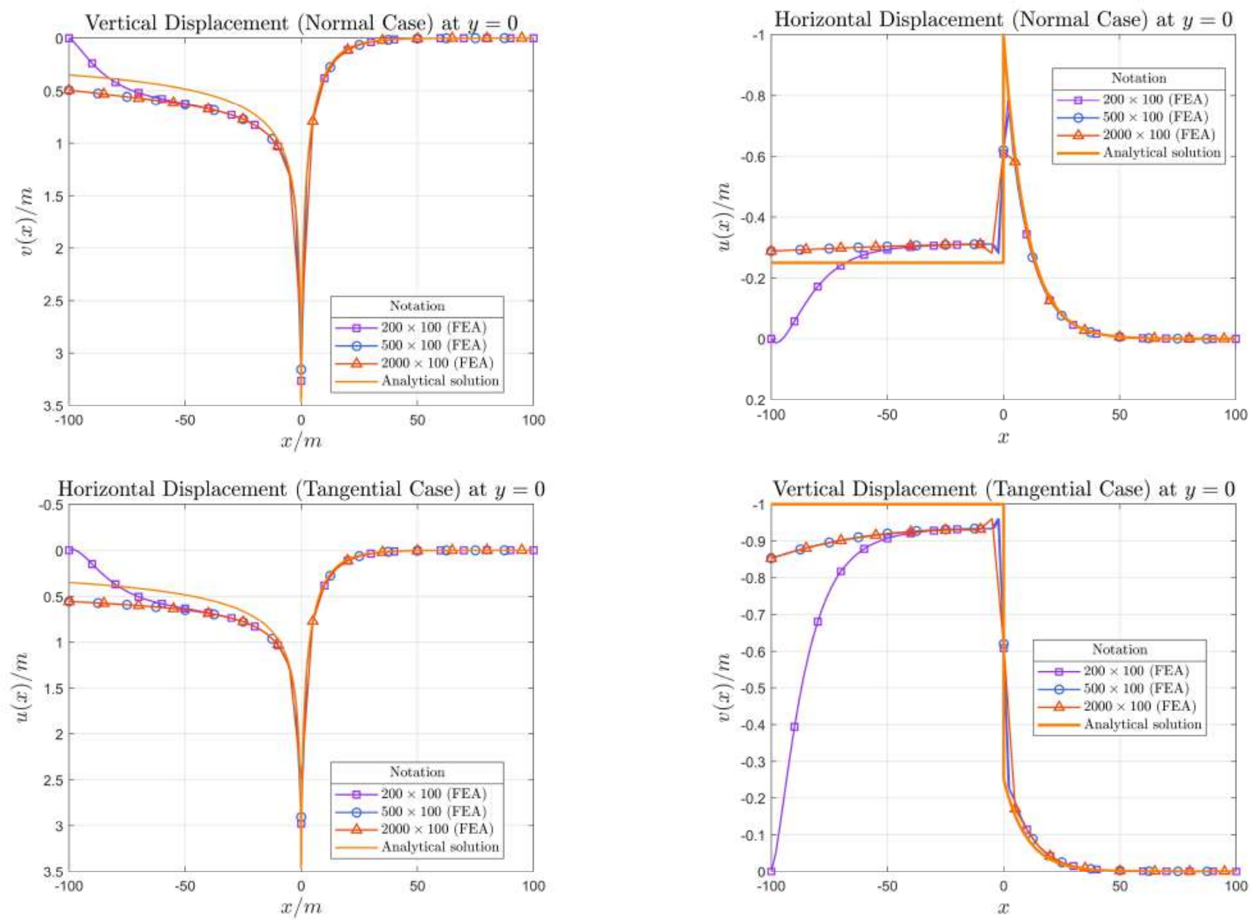


Figure 2. Vertical & horizontal displacements of FEA in different scales (with surface sedimentation correction)

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(QUASI-)HAMILTONIAN OPERATORS

Symmetry & operators

Hamiltonian operators

$$\mathcal{H} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & -\mathbf{A}^{\mathcal{T}} \end{array}\right]$$

quasi-Hamiltonian operators (2D)

$$\mathcal{H} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & -\mathbf{A}_{\alpha}^{\mathcal{T}} \end{array}\right]$$

Shifted Hamiltonian transformation

Gradient (arbitrary direction)

$$\mathcal{H} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & -\mathbf{A}_{\beta}^{\mathcal{T}} \end{array}\right]$$

Dual Hamiltonian transformation

$$\mathcal{H} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C}_{\beta *} & -\mathbf{A}_{\beta}^{\mathcal{T}} \end{array}\right]$$

Dual Hamiltonian transformation

Inherent Asymmetry

$$\mathcal{H}^{x} = \left[\begin{array}{c:c} \mathbf{A}_{x} & \mathbf{B}_{x} \\ \hdashline \mathbf{C}_{\vartheta_x} & -\mathbf{A}_{\vartheta_x}^{\mathcal{T}} \end{array}\right]$$

Generalized dual Hamiltonian transformation

Variable coefficients

skew-Hamiltonian operators

$$\mathcal{H} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{A}^{\mathcal{T}} \end{array}\right]$$

upper triangular/diagonal…… Hamiltonian operators

quasi-Hamiltonian operators (3D)

$$\mathcal{H}^{\dagger} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C}_{\chi} & -\mathbf{A}_{\chi}^{\mathcal{T}} \end{array}\right]$$

Asymmetry induced by odd-dimension

$$\mathcal{H}^{\ddagger} = \left[\begin{array}{c:c} \mathbf{A}_{x} & \mathbf{B}_{x} \\ \hdashline \mathbf{C}_{E_x} & -\mathbf{A}_{E_x}^{\mathcal{T}} \end{array}\right]$$

$$\mathcal{H}^{**} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C}^{**} & -\mathbf{A}^{\mathcal{T}} \end{array}\right]$$

Incremental theory

Symplectic Formulation

Symplectic structure in mechanics

Lagrangian formulation — low dimension & high order

Hamiltonian formulation — high dimension & low order

High order, one-dimensional differential equations

(generalized) Almansi theorem,
potential theory,
(dual/singular) integral equation

→

full state vector, low order, high-dimensional symplectic structure

method of separation of variables,
Hamilton transformation,
Jordan chain,
symplectic orthogonality,
symplectic adjoin,
operator spectrum theorem

Symplectic Formulation

Definitions & properties

State Vector

- Original variable & Dual variable
- $2n$-dimensional phase space

$$\boldsymbol{f} = [\boldsymbol{q}, \boldsymbol{p}]^{\mathrm{T}}$$

Symplectic Inner Product

- $\left\langle \boldsymbol{f}_1, \boldsymbol{f}_2 \right\rangle = -\left\langle \boldsymbol{f}_2, \boldsymbol{f}_1 \right\rangle$
- $\left\langle k\boldsymbol{f}_1, \boldsymbol{f}_2 \right\rangle = k\left\langle \boldsymbol{f}_1, \boldsymbol{f}_2 \right\rangle \qquad k \in \mathbb{R}$
- $\left\langle \boldsymbol{f}_1 + \boldsymbol{f}_3, \boldsymbol{f}_2 \right\rangle = \left\langle \boldsymbol{f}_1, \boldsymbol{f}_2 \right\rangle + \left\langle \boldsymbol{f}_3, \boldsymbol{f}_2 \right\rangle \qquad \boldsymbol{f}_3 \in \mathscr{L}$
- If $\left\langle \boldsymbol{f}_1, \boldsymbol{f}_2 \right\rangle = 0$ for every $\boldsymbol{f}_2 \in \mathscr{L}$, then $\boldsymbol{f}_1 = 0$

Yao, Weian, Zhong, Wanxie, and Lim, Chee Wah. *Symplectic elasticity*. World Scientific, 2009.

Symplectic Formulation

Definitions & properties

Matrix & Transformation

- Symplectic matrix $\mathbf{S}^{\mathrm{T}}\mathbf{J}\mathbf{S} = \mathbf{J}$
- Hamiltonian matrix $\mathbf{H}^{\mathrm{T}} = \mathbf{J}\mathbf{H}\mathbf{J}$
- Hamiltonian transformation $\left\langle \boldsymbol{f}_1, \boldsymbol{\mathcal{H}}\boldsymbol{f}_2 \right\rangle = \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}}\boldsymbol{f}_1 \right\rangle$

$$\mathbf{J}_{2n} = \begin{bmatrix} 0 & \mathbf{I}_n \\ -\mathbf{I}_n & 0 \end{bmatrix} \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

symplectic self-adjoint

$\sigma(\mathcal{H}) = \mathbb{C} \setminus \rho(\mathcal{H})$

spectrum is symmetric about the imaginary axis, and the union of the point spectrum and the remaining spectrum is also symmetric about the imaginary axis.

$\sigma_p(\mathcal{H}) \cup \sigma_r(\mathcal{H})$

Hamiltonian Matrix Properties

- Eigenvalues of $\boldsymbol{\mathcal{H}}$: $\pm\mu$ (with same multiplicity for adjoint eigenvalues)
- Symplectic orthogonality of eigenvectors (with non-adjoint eigenvalues)

$$\left\langle \Phi_i^{(s)}, \Phi_j^{(t)} \right\rangle = 0 \qquad (s = 0, 1, \cdots, m;\ t = 0, 1, \cdots, n)$$

Yao, Weian, Zhong, Wanxie, and Lim, Chee Wah. *Symplectic elasticity*. World Scientific, 2009.

Exponentially Graded Model

Background

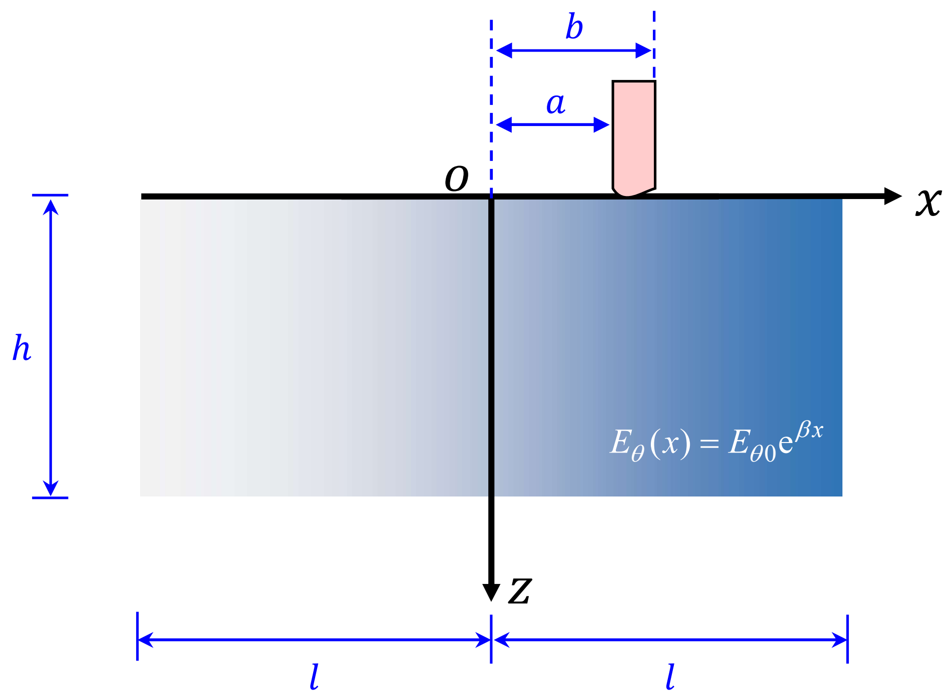


Figure 3. Horizontal graded plane

Advantages (compared with the Lagrangian formulation):

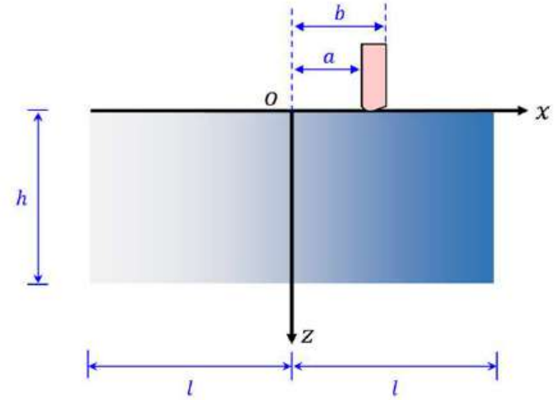
- Method of separation of variables (suitable for arbitrary indenter shape)
- Finite-sized model
- Applicable for inhomogeneous media with multi-field coupling

Horizontal Graded Plane

Multi-field coupling

$$
\begin{cases}
\dfrac{\partial u_x}{\partial z} = -\dfrac{\partial u_z}{\partial x} - a_1\dfrac{\partial \varphi}{\partial x} - a_2\dfrac{\partial \psi}{\partial x} + a_3\hat{\sigma}_{xz} \\
\dfrac{\partial u_z}{\partial z} = -a_4\dfrac{\partial u_x}{\partial x} + a_5\hat{\sigma}_{zz} + a_6\hat{D}_z + a_7\hat{B}_z \\
\dfrac{\partial \varphi}{\partial z} = a_8\dfrac{\partial u_x}{\partial x} + a_6\hat{\sigma}_{zz} - a_9\hat{D}_z + a_{10}\hat{B}_z \\
\dfrac{\partial \psi}{\partial z} = a_{11}\dfrac{\partial u_x}{\partial x} + a_7\hat{\sigma}_{zz} + a_{10}\hat{D}_z - a_{12}\hat{B}_z \\
\dfrac{\partial \hat{\sigma}_{xz}}{\partial z} = a_{13}\dfrac{\partial^2 u_x}{\partial x^2} - a_4\dfrac{\partial \hat{\sigma}_{zz}}{\partial x} + a_8\dfrac{\partial \hat{D}_z}{\partial x} + a_{11}\dfrac{\partial \hat{B}_z}{\partial x} + \beta\left(a_{13}\dfrac{\partial u_x}{\partial x} - a_4\hat{\sigma}_{zz} + a_8\hat{D}_z + a_{11}\hat{B}_z\right) \\
\dfrac{\partial \hat{\sigma}_{zz}}{\partial z} = -\dfrac{\partial \hat{\sigma}_{xz}}{\partial x} - \beta\hat{\sigma}_{xz} \\
\dfrac{\partial \hat{D}_z}{\partial z} = a_{14}\dfrac{\partial^2 \varphi}{\partial x^2} + a_{15}\dfrac{\partial^2 \psi}{\partial x^2} - a_1\dfrac{\partial \hat{\sigma}_{xz}}{\partial x} + \beta\left(a_{14}\dfrac{\partial \varphi}{\partial x} + a_{15}\dfrac{\partial \psi}{\partial x} - a_1\hat{\sigma}_{xz}\right) \\
\dfrac{\partial \hat{B}_z}{\partial z} = a_{15}\dfrac{\partial^2 \varphi}{\partial x^2} + a_{16}\dfrac{\partial^2 \psi}{\partial x^2} - a_2\dfrac{\partial \hat{\sigma}_{xz}}{\partial x} + \beta\left(a_{15}\dfrac{\partial \varphi}{\partial x} + a_{16}\dfrac{\partial \psi}{\partial x} - a_2\hat{\sigma}_{xz}\right)
\end{cases}
$$

It can also be derived *via* Legendre transformation



$$
\begin{cases}
\hat{\sigma}_{xx} = -a_{13}\dfrac{\partial u_x}{\partial x} + a_4\hat{\sigma}_{zz} - a_8\hat{D}_z - a_{11}\hat{B}_z \\
\hat{D}_x = -a_{14}\dfrac{\partial \varphi}{\partial x} - a_{15}\dfrac{\partial \psi}{\partial x} + a_1\hat{\sigma}_{xz} \\
\hat{B}_x = -a_{15}\dfrac{\partial \varphi}{\partial x} - a_{16}\dfrac{\partial \psi}{\partial x} + a_2\hat{\sigma}_{xz}
\end{cases}
$$

$$\frac{\partial}{\partial z}\mathbf{I}_8\boldsymbol{f} = \boldsymbol{\mathcal{H}}\boldsymbol{f}$$

$$\boldsymbol{f} = [\boldsymbol{q}, \boldsymbol{p}]^{\mathrm{T}} = \left[\{u_x, u_z, \varphi, \psi\}, \{\hat{\sigma}_{xz}, \hat{\sigma}_{zz}, \hat{D}_z, \hat{B}_z\}\right]^{\mathrm{T}}$$

$$\boldsymbol{f}(x,z) = \Phi(x)\xi(z) = [u(x), w(x), \phi(x), \psi(x), \tau(x), \sigma(x), D(x), B(x)]^{\mathrm{T}}\xi(z)$$

$$\xi(z) = \mathrm{e}^{\mu z}, \quad \boldsymbol{\mathcal{H}}\Phi(x) = \mu\Phi(x)$$

Horizontal Graded Plane

Multi-field coupling

$$
\frac{\partial}{\partial z}\begin{Bmatrix}\begin{Bmatrix}u_x\\u_z\\\phi\\\psi\end{Bmatrix}\\\begin{Bmatrix}\hat{\sigma}_{xz}\\\hat{\sigma}_{zz}\\\hat{D}_z\\\hat{B}_z\end{Bmatrix}\end{Bmatrix}=\left[\begin{array}{cccc:cccc}
0 & -\frac{\partial}{\partial x} & -a_1\frac{\partial}{\partial x} & -a_2\frac{\partial}{\partial x} & a_3 & 0 & 0 & 0\\
-a_4\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & a_5 & a_6 & a_7\\
a_8\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & a_6 & -a_9 & a_{10}\\
a_{11}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & a_7 & a_{10} & -a_{12}\\
\hdashline
a_{13}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & 0 & 0 & 0 & 0 & -a_4\left(\frac{\partial}{\partial x}+\beta\right) & a_8\left(\frac{\partial}{\partial x}+\beta\right) & a_{11}\left(\frac{\partial}{\partial x}+\beta\right)\\
0 & 0 & 0 & 0 & -\left(\frac{\partial}{\partial x}+\beta\right) & 0 & 0 & 0\\
0 & 0 & a_{14}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & a_{15}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & -a_1\left(\frac{\partial}{\partial x}+\beta\right) & 0 & 0 & 0\\
0 & 0 & a_{15}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & a_{16}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & -a_2\left(\frac{\partial}{\partial x}+\beta\right) & 0 & 0 & 0
\end{array}\right]\begin{Bmatrix}\begin{Bmatrix}u_x\\u_z\\\phi\\\psi\end{Bmatrix}\\\begin{Bmatrix}\hat{\sigma}_{xz}\\\hat{\sigma}_{zz}\\\hat{D}_z\\\hat{B}_z\end{Bmatrix}\end{Bmatrix}
$$

$$
\boldsymbol{\mathcal{H}}=\left[\begin{array}{c:c}\mathbf{A} & \mathbf{B}\\\hdashline\mathbf{C} & -\mathbf{A}_\beta^{\mathcal{T}}\end{array}\right]
$$

infinite dimensional quasi-Hamiltonian operator

$$
-\boldsymbol{\mathcal{H}}_{11}^{\mathcal{T}}=\lim_{\beta\to0}\boldsymbol{\mathcal{H}}_{22},\quad \boldsymbol{\mathcal{H}}_{12}=\boldsymbol{\mathcal{H}}_{12}^{\mathcal{T}},\quad \boldsymbol{\mathcal{H}}_{21}=\boldsymbol{\mathcal{H}}_{21\,\beta\to-\beta}^{\mathcal{T}}
$$

Horizontal Graded Plane

Zero eigenvalue

$$\boxed{\boldsymbol{\mathcal{H}}\Phi_{0,s}^{(1)} = 0 \quad (s = 1, 2, \cdots, 4)}$$

$$\boldsymbol{f}_{0,1}^{(0)} = \Phi_{0,1}^{(0)} = [1,0,0,0,0,0,0,0]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,2}^{(0)} = \Phi_{0,2}^{(0)} = [0,1,0,0,0,0,0,0]^{\mathrm{T}}$$
$$\boldsymbol{f}_{0,3}^{(0)} = \Phi_{0,3}^{(0)} = [0,0,1,0,0,0,0,0]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,4}^{(0)} = \Phi_{0,4}^{(0)} = [0,0,0,1,0,0,0,0]^{\mathrm{T}}$$

$$\boxed{\boldsymbol{\mathcal{H}}\Phi_{0,s}^{(i+1)} = \Phi_{0,s}^{(i)} \quad (s = 1, 2, \cdots, 4)}$$

$$\boldsymbol{f}_{0,1}^{(1)} = \Phi_{0,1}^{(1)} + z\Phi_{0,1}^{(0)} = [z,-x,0,0,0,0,0,0]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,2}^{(1)} = \Phi_{0,2}^{(1)} + z\Phi_{0,2}^{(0)} = [-a_{17}x, z, 0, 0, 0, k_1, k_2, k_3]^{\mathrm{T}}$$
$$\boldsymbol{f}_{0,3}^{(1)} = \Phi_{0,3}^{(1)} + z\Phi_{0,3}^{(0)} = [-a_{18}x, 0, z, 0, 0, k_4, k_5, k_6]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,4}^{(1)} = \Phi_{0,4}^{(1)} + z\Phi_{0,4}^{(0)} = [-a_{19}x, 0, 0, z, 0, k_7, k_8, k_9]^{\mathrm{T}}$$

$$\boldsymbol{f}_{0,1}^{(2)} = \Phi_{0,1}^{(2)} + z\Phi_{0,1}^{(1)} + \frac{z^2}{2}\Phi_{0,1}^{(0)}$$

$$\boldsymbol{f}_{0}^{(3)} = \Phi_{0}^{(3)} + z\Phi_{0,1}^{(2)} + \frac{z^2}{2!}\Phi_{0,1}^{(1)} + \frac{z^3}{3!}\Phi_{0,1}^{(0)} + \zeta_1\left(z\Phi_{0,2}^{(1)} + \frac{z^2}{2!}\Phi_{0,2}^{(0)}\right)$$

The specific form is omitted here for brevity

Horizontal Graded Plane

General eigenvalues

$$
\det\begin{bmatrix}
-\mu & -\eta & -a_1\eta & -a_2\eta & a_3 & 0 & 0 & 0 \\
-a_4\eta & -\mu & 0 & 0 & 0 & a_5 & a_6 & a_7 \\
a_8\eta & 0 & -\mu & 0 & 0 & a_6 & -a_9 & a_{10} \\
a_{11}\eta & 0 & 0 & -\mu & 0 & a_7 & a_{10} & -a_{12} \\
a_{13}\left(\eta^2+\beta\eta\right) & 0 & 0 & 0 & -\mu & -a_4\left(\eta+\beta\right) & a_8\left(\eta+\beta\right) & a_{11}\left(\eta+\beta\right) \\
0 & 0 & 0 & 0 & -\left(\eta+\beta\right) & -\mu & 0 & 0 \\
0 & 0 & a_{14}\left(\eta^2+\beta\eta\right) & a_{15}\left(\eta^2+\beta\eta\right) & -a_1\left(\eta+\beta\right) & 0 & -\mu & 0 \\
0 & 0 & a_{15}\left(\eta^2+\beta\eta\right) & a_{16}\left(\eta^2+\beta\eta\right) & -a_2\left(\eta+\beta\right) & 0 & 0 & -\mu
\end{bmatrix} = 0
$$

symmetry & degeneracy $\eta = \sqrt{\lambda} - \beta / 2$

$$
\Lambda_0\lambda^4 + (\mu^2\Lambda_1 + \beta^2\Lambda_2)\lambda^3 + (\mu^4\Lambda_3 + \mu^2\beta^2\Lambda_4 + \beta^4\Lambda_5)\lambda^2 + (\mu^6\Lambda_6 + \mu^4\beta^2\Lambda_7 + \mu^2\beta^4\Lambda_8 + \beta^6\Lambda_9)\lambda + (\mu^8\Lambda_{10} + \mu^6\beta^2\Lambda_{11} + \mu^4\beta^4\Lambda_{12} + \mu^2\beta^6\Lambda_{13} + \beta^8\Lambda_{14}) = 0
$$

$$
\Phi = \sum_{t=1}^{8} \mathrm{e}^{\eta_t x}[\tilde{A}_t, \tilde{B}_t, \tilde{C}_t, \tilde{D}_t, \tilde{E}_t, \tilde{F}_t, \tilde{G}_t, \tilde{H}_t]^{\mathrm{T}}
$$

$$
\det\left[\varpi_{\left[\frac{i}{2}\right]j}\mathrm{e}^{(-1)^{i+1}\eta_j l}\right] = 0 \quad (i = 1, \cdots, 8; j = 1, \cdots, 8)
$$

$$
\boxed{\boldsymbol{f}_{\mu,i} = \mathrm{e}^{\mu_i z}\Phi_i}
$$

Horizontal Graded Plane

Symplectic expansion

$$\tilde{\boldsymbol{f}} = \sum_{i=1}^{10} m_{0,i} \tilde{\boldsymbol{f}}_{0,i} + \sum_{i=1}^{\infty} \left[\left(m_{\mu,i}^{\mathrm{Re}} \mathrm{Re} \tilde{\boldsymbol{f}}_{\mu,i} + m_{\mu,i}^{\mathrm{Im}} \mathrm{Im} \tilde{\boldsymbol{f}}_{\mu,i} \right) + \left(m_{-\mu,i}^{\mathrm{Re}} \mathrm{Re} \tilde{\boldsymbol{f}}_{-\mu,i} + m_{-\mu,i}^{\mathrm{Im}} \mathrm{Im} \tilde{\boldsymbol{f}}_{-\mu,i} \right) \right] \equiv \sum_{i=1}^{\infty} m_i \tilde{\boldsymbol{f}}_i$$

In the Hilbert space, the symplectic expansion is complete under the Cauchy principal value.

$$\mathrm{p.v.} \sum_{k=-\infty}^{+\infty} a_k \boldsymbol{f}_k = a_0 \boldsymbol{f}_0 + \sum_{m=1}^{+\infty} \left(a_m \boldsymbol{f}_m + a_{-m} \boldsymbol{f}_{-m} \right)$$

Hamiltonian mixed energy variational principle

$$\delta \left\{ \int_0^h \int_{-l}^{l} \left[\boldsymbol{p}^{\mathrm{T}} \frac{\partial \boldsymbol{q}}{\partial z} - H(\boldsymbol{q}, \boldsymbol{p}) \right] \mathrm{d}x \mathrm{d}z - \int_{\Gamma_{\boldsymbol{q}_h}} \left[\boldsymbol{p}^{\mathrm{T}} (\boldsymbol{q} - \bar{\boldsymbol{q}}_h) \right] \mathrm{d}x - \int_{\Gamma_{\boldsymbol{p}_h}} \left[\bar{\boldsymbol{p}}_h^{\mathrm{T}} \boldsymbol{q} \right] \mathrm{d}x + \int_{\Gamma_{\boldsymbol{q}_0}} \left[\boldsymbol{p}^{\mathrm{T}} (\boldsymbol{q} - \bar{\boldsymbol{q}}_0) \right] \mathrm{d}x + \int_{\Gamma_{\boldsymbol{p}_0}} \left[\bar{\boldsymbol{p}}_0^{\mathrm{T}} \boldsymbol{q} \right] \mathrm{d}x \right\} = 0$$

$$\begin{aligned}
\mathscr{A}_{ij} &= \int_{\Gamma_{\boldsymbol{p}_h}} \left[(\boldsymbol{q}_i)^{\mathrm{T}} \boldsymbol{p}_j \right] \mathrm{d}x - \int_{\Gamma_{\boldsymbol{q}_h}} \left[(\boldsymbol{p}_i)^{\mathrm{T}} \boldsymbol{q}_j \right] \mathrm{d}x + \int_{\Gamma_{\boldsymbol{q}_0}} \left[(\boldsymbol{p}_i)^{\mathrm{T}} \boldsymbol{q}_j \right] \mathrm{d}x - \int_{\Gamma_{\boldsymbol{p}_0}} \left[(\boldsymbol{q}_i)^{\mathrm{T}} \boldsymbol{p}_j \right] \mathrm{d}x \\
\mathscr{H}_i &= \int_{\Gamma_{\boldsymbol{p}_h}} \left[(\boldsymbol{q}_i)^{\mathrm{T}} \bar{\boldsymbol{p}}_h \right] \mathrm{d}x - \int_{\Gamma_{\boldsymbol{q}_h}} \left[(\boldsymbol{p}_i)^{\mathrm{T}} \bar{\boldsymbol{q}}_h \right] \mathrm{d}x + \int_{\Gamma_{\boldsymbol{q}_0}} \left[(\boldsymbol{p}_i)^{\mathrm{T}} \bar{\boldsymbol{q}}_0 \right] \mathrm{d}x - \int_{\Gamma_{\boldsymbol{p}_0}} \left[(\boldsymbol{q}_i)^{\mathrm{T}} \bar{\boldsymbol{p}}_0 \right] \mathrm{d}x
\end{aligned}$$

$$m_k = \frac{\det \mathscr{A}_{ij;k}}{\det \mathscr{A}_{ij}}$$

Horizontal Graded Plane

Example

$$\frac{\partial}{\partial z}\mathbf{I}_4 \boldsymbol{f} = \left[\begin{array}{cc|cc} 0 & -\nu_{\alpha 0}\dfrac{\partial}{\partial x} & \dfrac{1-\nu_{\alpha 0}^2}{E_{\alpha 0}} & 0 \\ -\dfrac{\partial}{\partial x} & 0 & 0 & \dfrac{2(1+\nu_{\alpha 0})}{E_{\alpha 0}} \\ \hline 0 & 0 & 0 & -\beta - \dfrac{\partial}{\partial x} \\ 0 & -E_{\alpha 0}\dfrac{\partial^2}{\partial x^2} - E_{\alpha 0}\beta\dfrac{\partial}{\partial x} & -\nu_{\alpha 0}\dfrac{\partial}{\partial x} - \nu_{\alpha 0}\beta & 0 \end{array}\right]\boldsymbol{f}$$

$$\boldsymbol{f}_{0,s}^{(0)} = \Phi_{0,s}^{(0)} = [1,0,0,0]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,a}^{(0)} = \Phi_{0,a}^{(0)} = [0,1,0,0]^{\mathrm{T}}$$

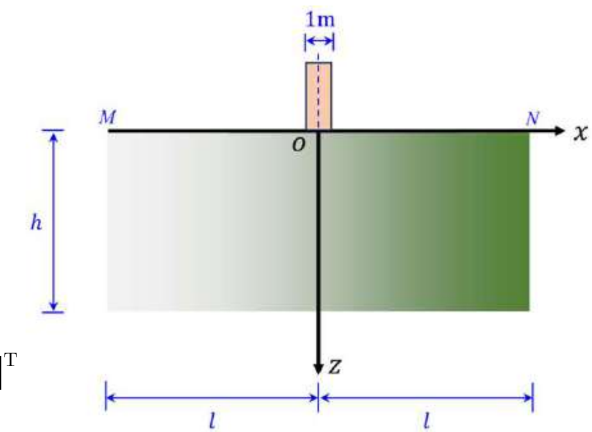
$$\boldsymbol{f}_{0,s}^{(1)} = \Phi_{0,s}^{(1)} + z\Phi_{0,s}^{(0)} = [z, -\nu_{\alpha 0}x, E_{\alpha 0}, 0]^{\mathrm{T}}, \quad \boldsymbol{f}_{0,a}^{(1)} = \Phi_{0,a}^{(1)} + z\Phi_{0,a}^{(0)} = [-x, z, 0, 0]^{\mathrm{T}}$$

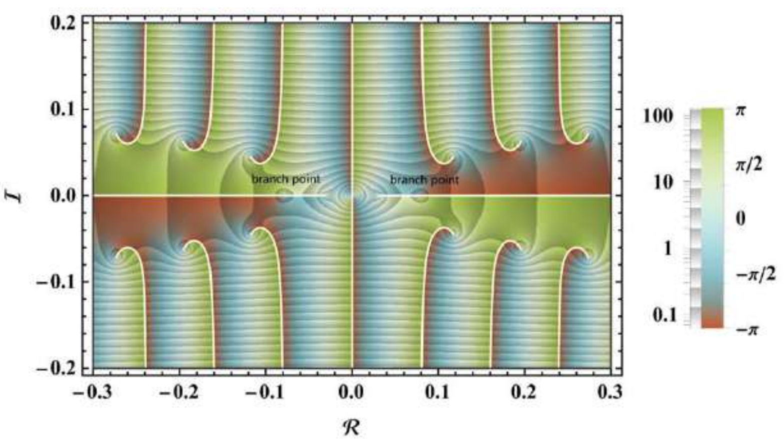
$$\boldsymbol{f}_{0,a}^{(2)} = \Phi_{0,a}^{(2)} + z\Phi_{0,a}^{(1)} + \frac{z^2}{2}\Phi_{0,a}^{(0)} = [-xz, \frac{1}{2}(\nu_{\alpha 0}x^2 + z^2), -E_{\alpha 0}x, 0]^{\mathrm{T}}$$

$$\boldsymbol{f}_{0}^{(3)} = \Phi_{0}^{(3)} + z\Phi_{0,a}^{(2)} + \frac{z^2}{2!}\Phi_{0,a}^{(1)} + \frac{z^3}{3!}\Phi_{0,a}^{(0)} + \zeta_0 z\Phi_{0,s}^{(1)} + \zeta_0\frac{z^2}{2!}\Phi_{0,s}^{(0)}$$

$$\boldsymbol{f}_{\mu,n} = \mathrm{e}^{\mu_n z}\Phi_n = \mathrm{e}^{\mu_n z}\sum_{i=1}^{4}\left(\mathrm{e}^{\eta_{in}x}[\mathcal{A}_{in}, \mathcal{B}_{in}, \mathcal{C}_{in}, \mathcal{D}_{in}]^{\mathrm{T}}\right)$$

$$\Phi_0^{(3)} = \begin{Bmatrix} \dfrac{2(1+\nu_{\alpha 0})}{E_{\alpha 0}}\left[-\dfrac{E_{\alpha 0}l}{\beta^2\sinh(\beta l)}\mathrm{e}^{-\beta x} + \dfrac{E_{\alpha 0}}{\beta}\dfrac{x^2}{2} - (\dfrac{\zeta_0 E_{\alpha 0}}{\beta} + \dfrac{E_{\alpha 0}}{\beta^2})x\right] - \dfrac{1}{6}\nu_{\alpha 0}x^3 + \dfrac{1}{2}\zeta_0\nu_{\alpha 0}x^2 \\ 0 \\ 0 \\ \dfrac{E_{\alpha 0}l}{\beta\sinh(\beta l)}\mathrm{e}^{-\beta x} + \dfrac{E_{\alpha 0}}{\beta}x - (\dfrac{\zeta_0 E_{\alpha 0}}{\beta} + \dfrac{E_{\alpha 0}}{\beta^2}) \end{Bmatrix}$$





Numerical results

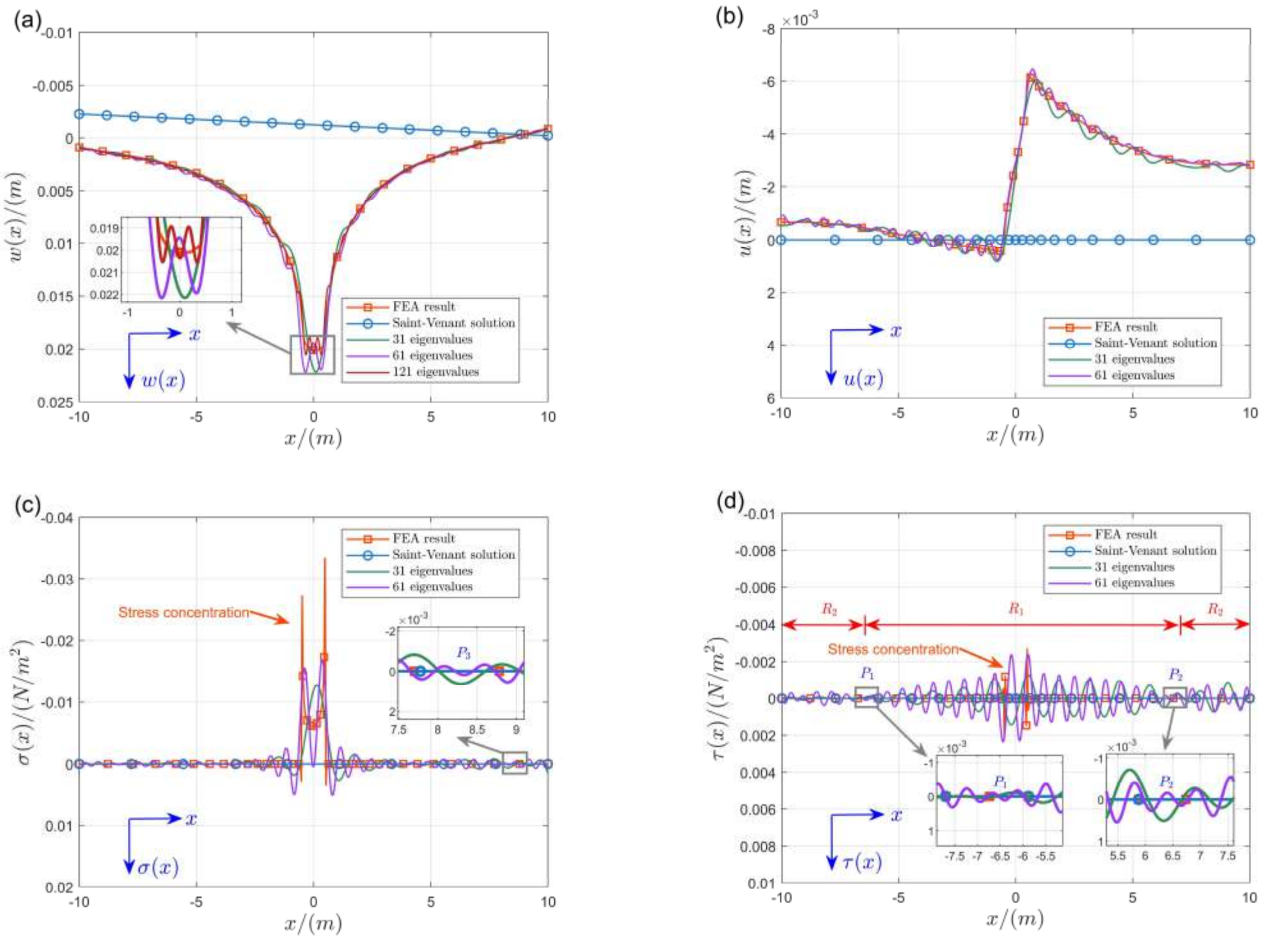


FIGURE 4. Deformations and stress distributions at the surface

Finite element analyses

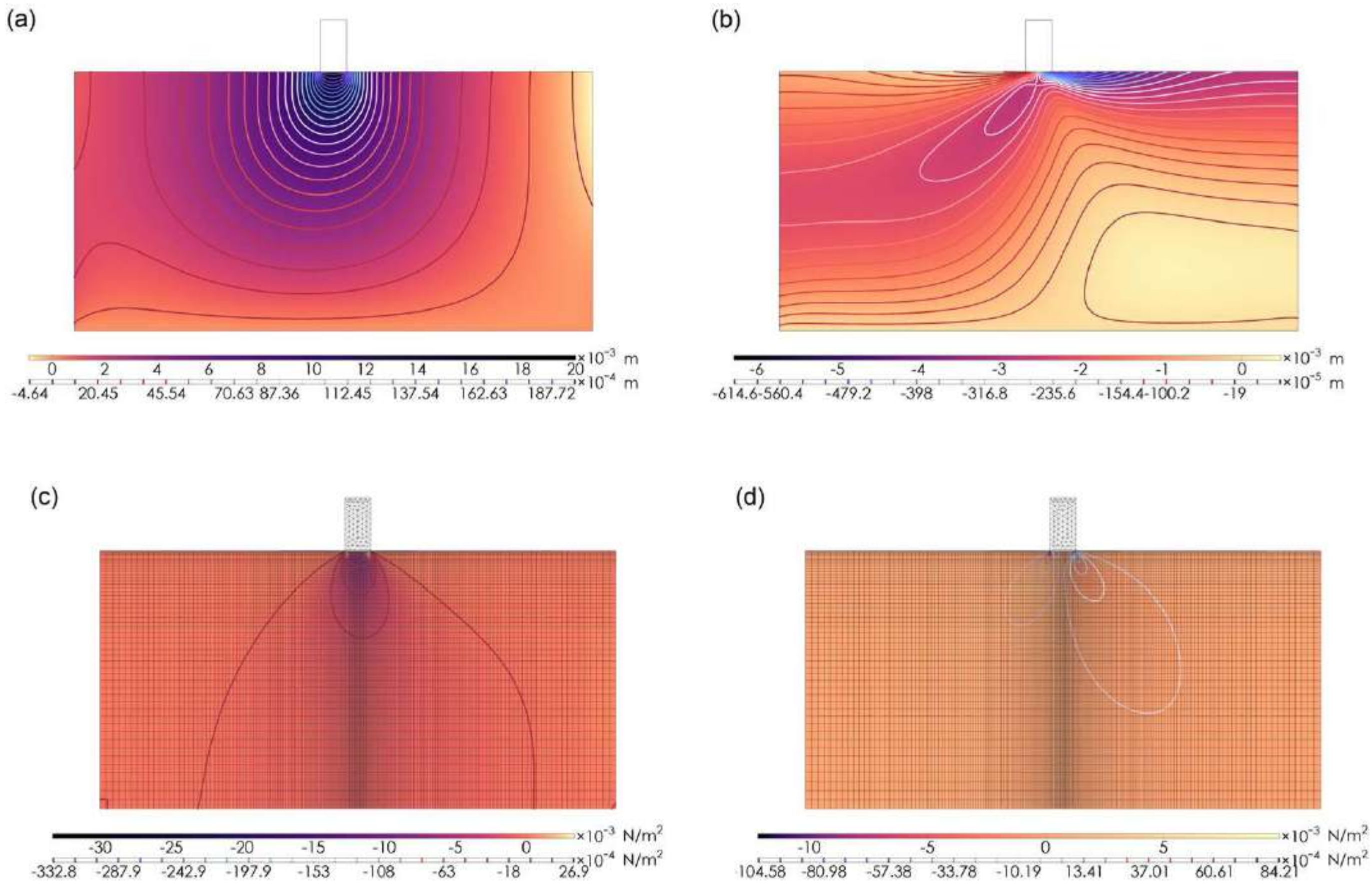


Figure 5. FEA results

Background

Boundary continuity

↓

Transfer matrix

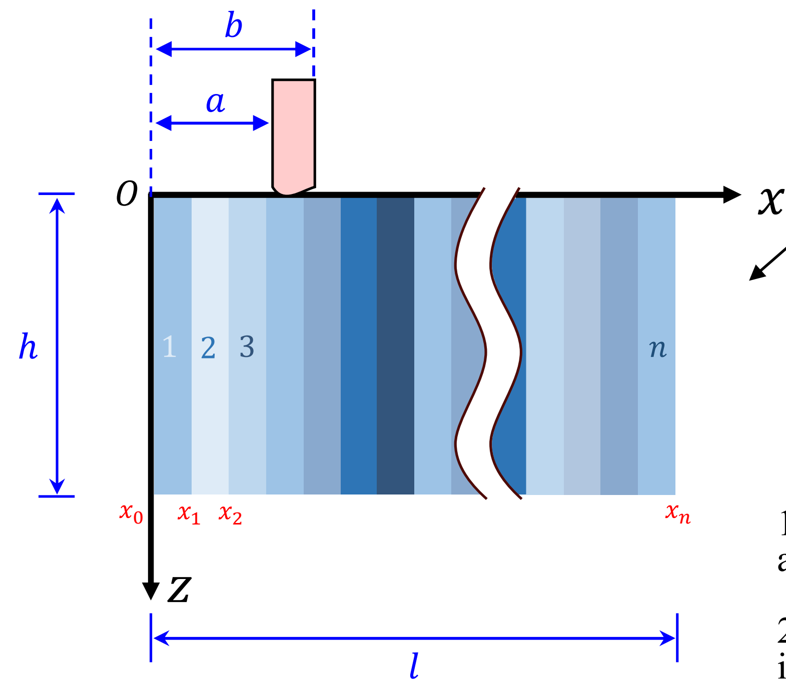


FIGURE 6. Horizontal laminated plane

The lateral boundaries and the state equations can be non-homogeneous, both of which may be homogenized through particular solutions

1. Applicable for the boundary effect and interfacial effect

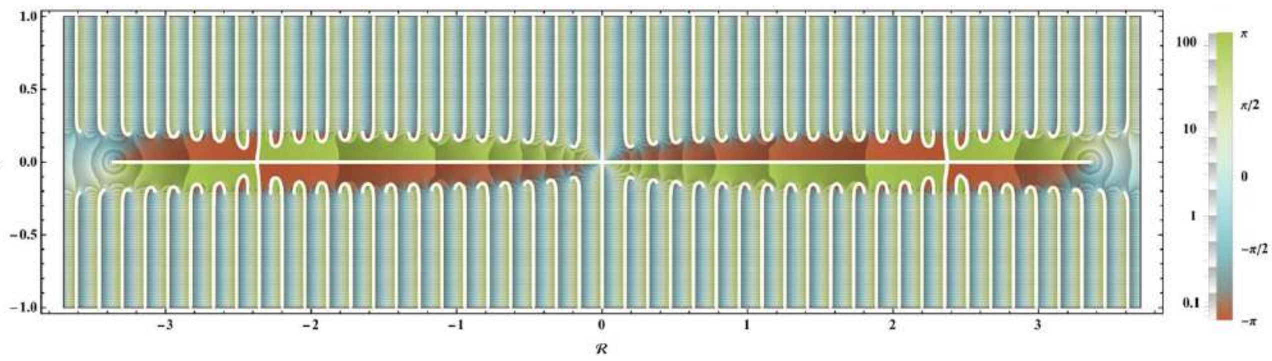
2. If there are enough discrete layers, it is possible to simulate arbitrary gradient changes in the lateral direction of the material

$$
\begin{aligned}
\boldsymbol{f} &= \sum_{i=1}^{10} m_{0,i}\boldsymbol{f}_{0,i} + \sum_{i=1}^{\infty}\left[\left(m_{\mu,i}^{\mathrm{Re}}\mathrm{Re}\boldsymbol{f}_{\mu,i} + m_{\mu,i}^{\mathrm{Im}}\mathrm{Im}\boldsymbol{f}_{\mu,i}\right) + \left(m_{-\mu,i}^{\mathrm{Re}}\mathrm{Re}\boldsymbol{f}_{-\mu,i} + m_{-\mu,i}^{\mathrm{Im}}\mathrm{Im}\boldsymbol{f}_{-\mu,i}\right)\right] \\
&= \sum_{i=1}^{10} m_{0,i}\bigcup_{k=1}^{n}\boldsymbol{f}_{0,i;k} + \sum_{i=1}^{\infty}\left[\left(m_{\mu,i}^{\mathrm{Re}}\bigcup_{k=1}^{n}\mathrm{Re}\boldsymbol{f}_{\mu,i;k} + m_{\mu,i}^{\mathrm{Im}}\bigcup_{k=1}^{n}\mathrm{Im}\boldsymbol{f}_{\mu,i;k}\right) + \left(m_{-\mu,i}^{\mathrm{Re}}\bigcup_{k=1}^{n}\mathrm{Re}\boldsymbol{f}_{-\mu,i;k} + m_{-\mu,i}^{\mathrm{Im}}\bigcup_{k=1}^{n}\mathrm{Im}\boldsymbol{f}_{-\mu,i;k}\right)\right]
\end{aligned}
$$

Symplectic Framework

Interfacial effect

$$\frac{\partial}{\partial z}\mathbf{I}_4 \boldsymbol{f} = \begin{bmatrix} 0 & -\nu\dfrac{\partial}{\partial x} & \dfrac{1-\nu^2}{E_i} & 0 \\ -\dfrac{\partial}{\partial x} & 0 & 0 & \dfrac{2(1+\nu)}{E_i} \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial x} \\ 0 & -E_i\dfrac{\partial^2}{\partial x^2} & -\nu\dfrac{\partial}{\partial x} & 0 \end{bmatrix}\boldsymbol{f}$$



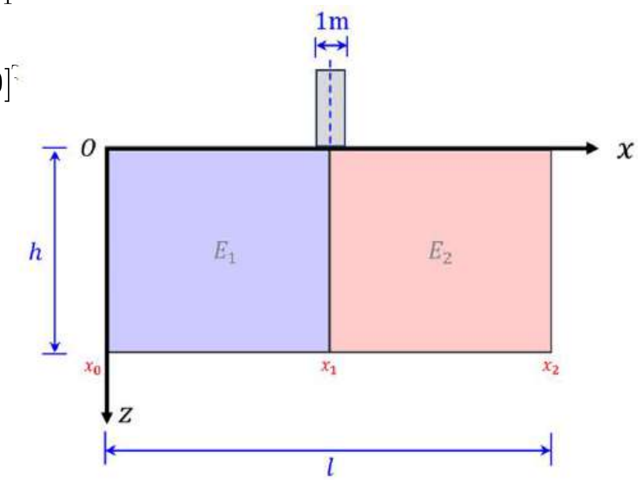
$$\boldsymbol{f}_{0,s;i}^{(0)} = \Phi_{0,s;i}^{(0)} = [1,0,0,0]^{\mathrm{T}}, \qquad \boldsymbol{f}_{0,a;i}^{(0)} = \Phi_{0,a;i}^{(0)} = [0,1,0,0]^{\mathrm{T}}$$

$$\boldsymbol{f}_{0,s;i}^{(1)} = \Phi_{0,s;i}^{(1)} + z\Phi_{0,s;i}^{(0)} = [z, -\nu x + \zeta_1, E_i, 0]^{\mathrm{T}}, \qquad \boldsymbol{f}_{0,a;i}^{(1)} = \Phi_{0,a;i}^{(1)} + z\Phi_{0,a;i}^{(0)} = [-x+\zeta_2, z, 0, 0]^{\mathrm{T}}$$

$$\boldsymbol{f}_{0,a;i}^{(2)} = \Phi_{0,a;i}^{(2)} + z\Phi_{0,a;i}^{(1)} + \frac{z^2}{2}\Phi_{0,a;i}^{(0)} = [(-x+\zeta_2)z, \frac{1}{2}[\nu(x-\zeta_2)^2 + z^2] + \zeta_3, -E_i(x-\zeta_2), 0]^{\mathrm{T}}$$

$$\boldsymbol{f}_{0,a;i}^{(3)} = \Phi_{0,a;i}^{(3)} + z\Phi_{0,a;i}^{(2)} + \frac{z^2}{2!}\Phi_{0,a;i}^{(1)} + \frac{z^3}{3!}\Phi_{0,a;i}^{(0)}$$

$$\Phi_{0,a;i}^{(3)} = \begin{Bmatrix} \dfrac{2(1+\nu)}{6}(x-\zeta_2)^3 + \left[\dfrac{2(1+\nu)}{E_i} - \zeta_3\right](x-\zeta_2) + \beta_{3i} + \zeta_4 \\ 0 \\ 0 \\ \dfrac{E_i}{2}(x-\zeta_2)^2 + \beta_{1i} \end{Bmatrix}$$



Numerical results

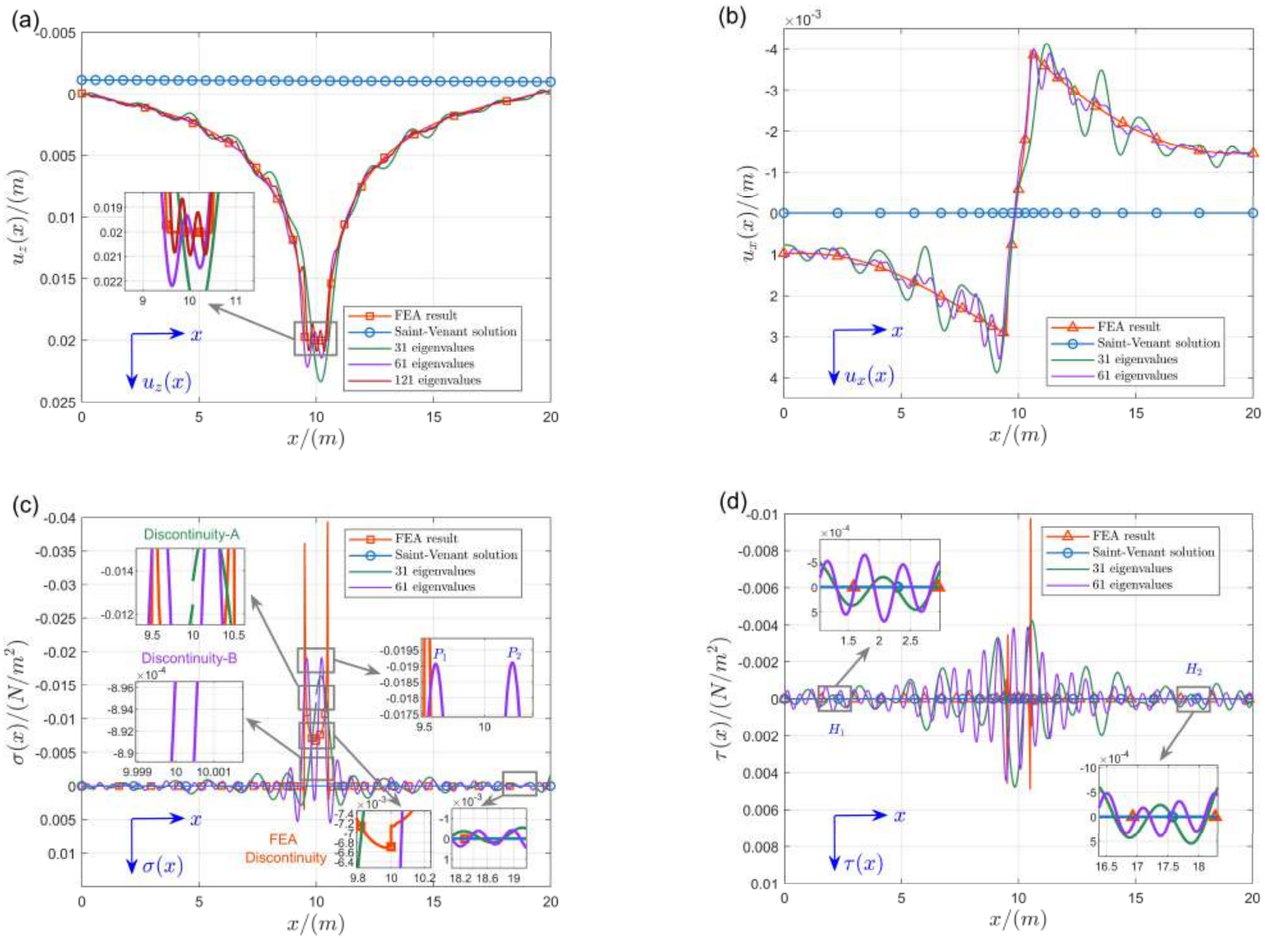


FIGURE 7. Deformations and stress distributions at the surface

Finite element analyses

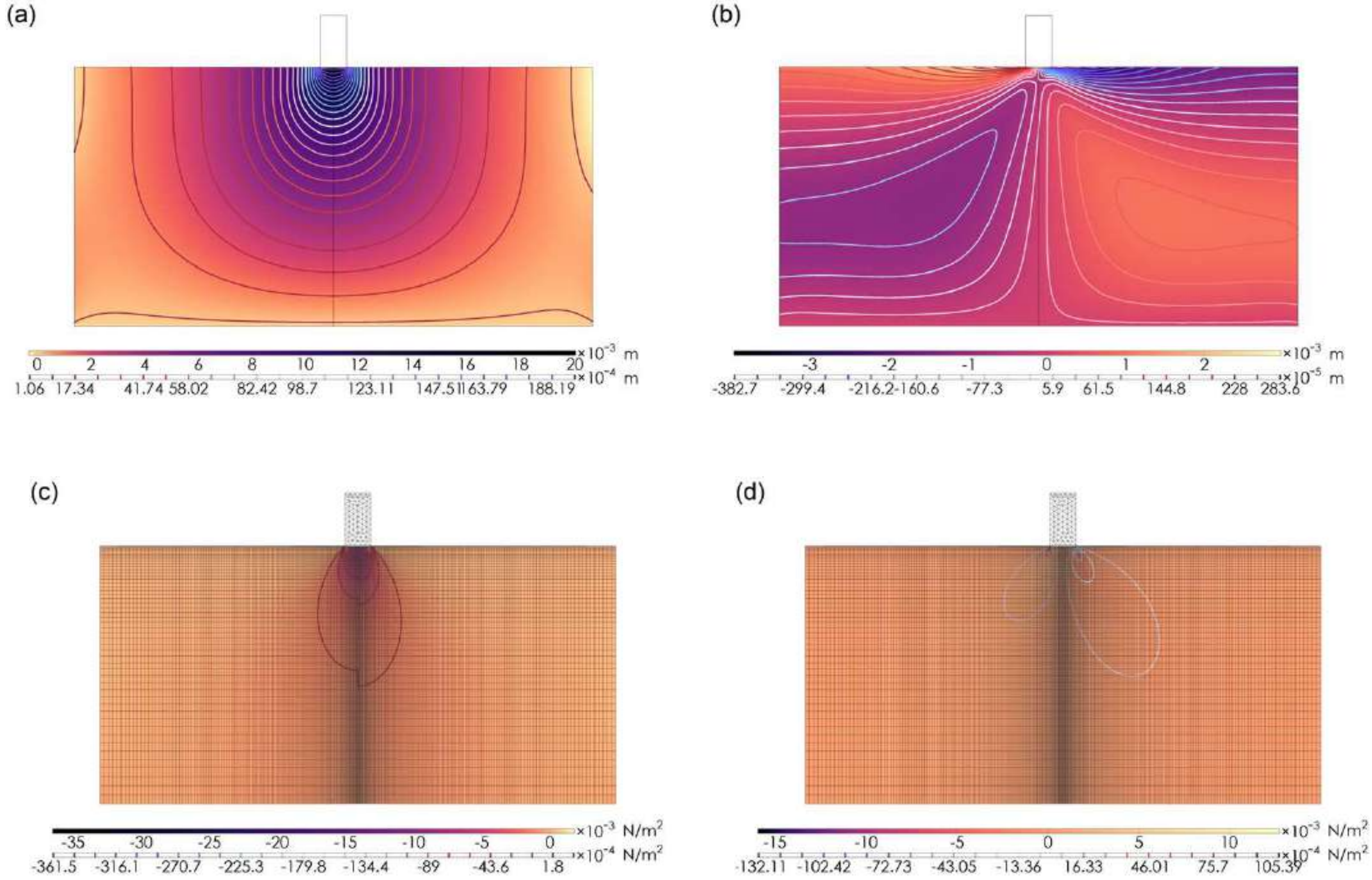


FIGURE 8. FEA results

Laminated Model

Boundary effect

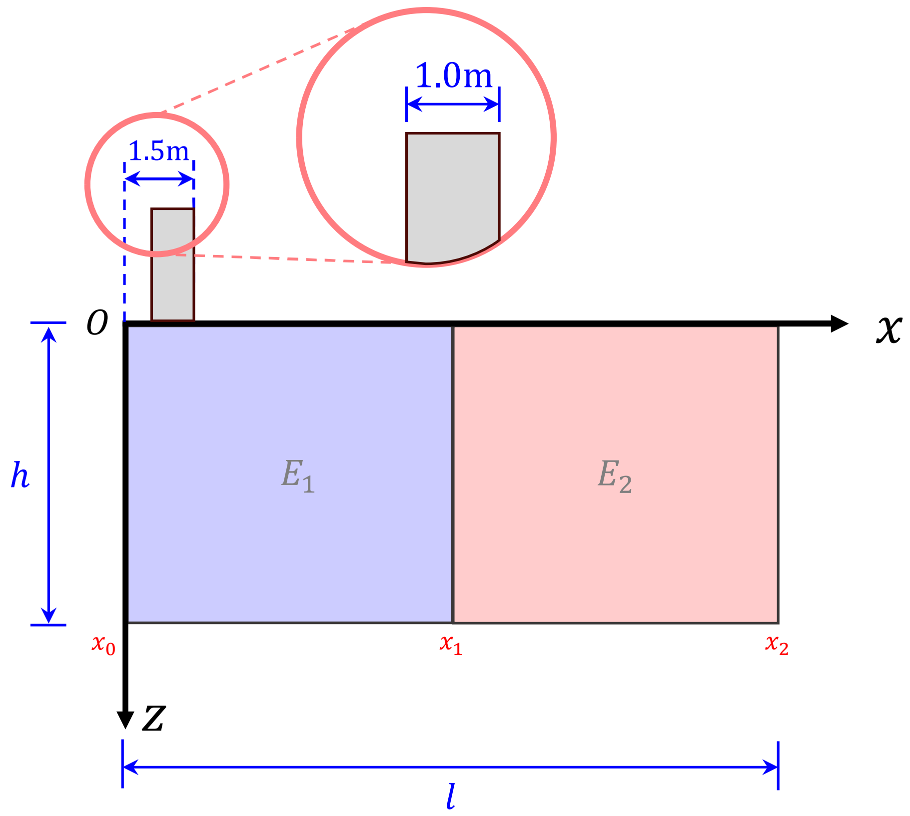


Figure 9. Horizontal layered plane

Numerical results

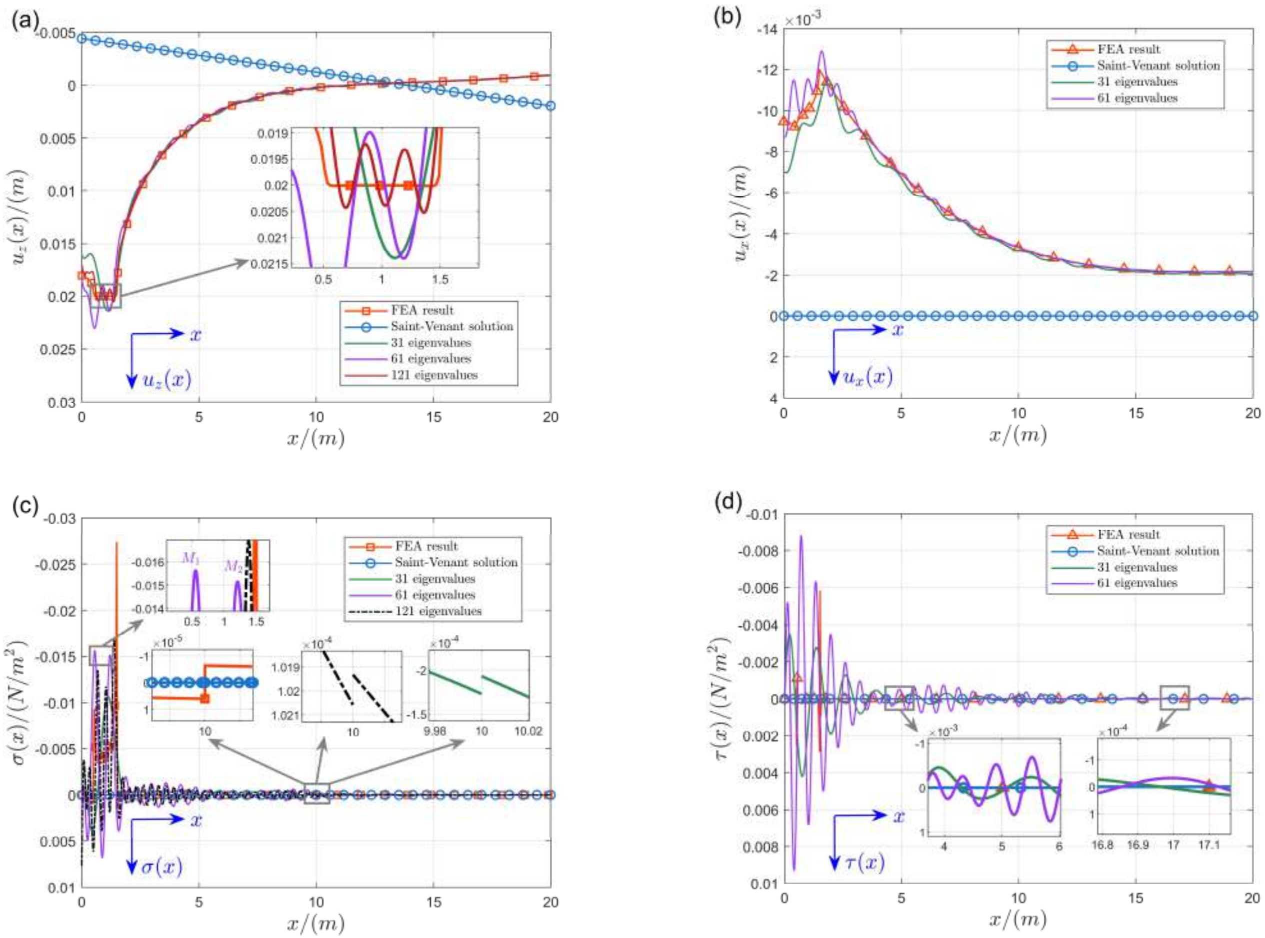


Figure 10. Deformations and stress distributions at the surface

Finite element analyses

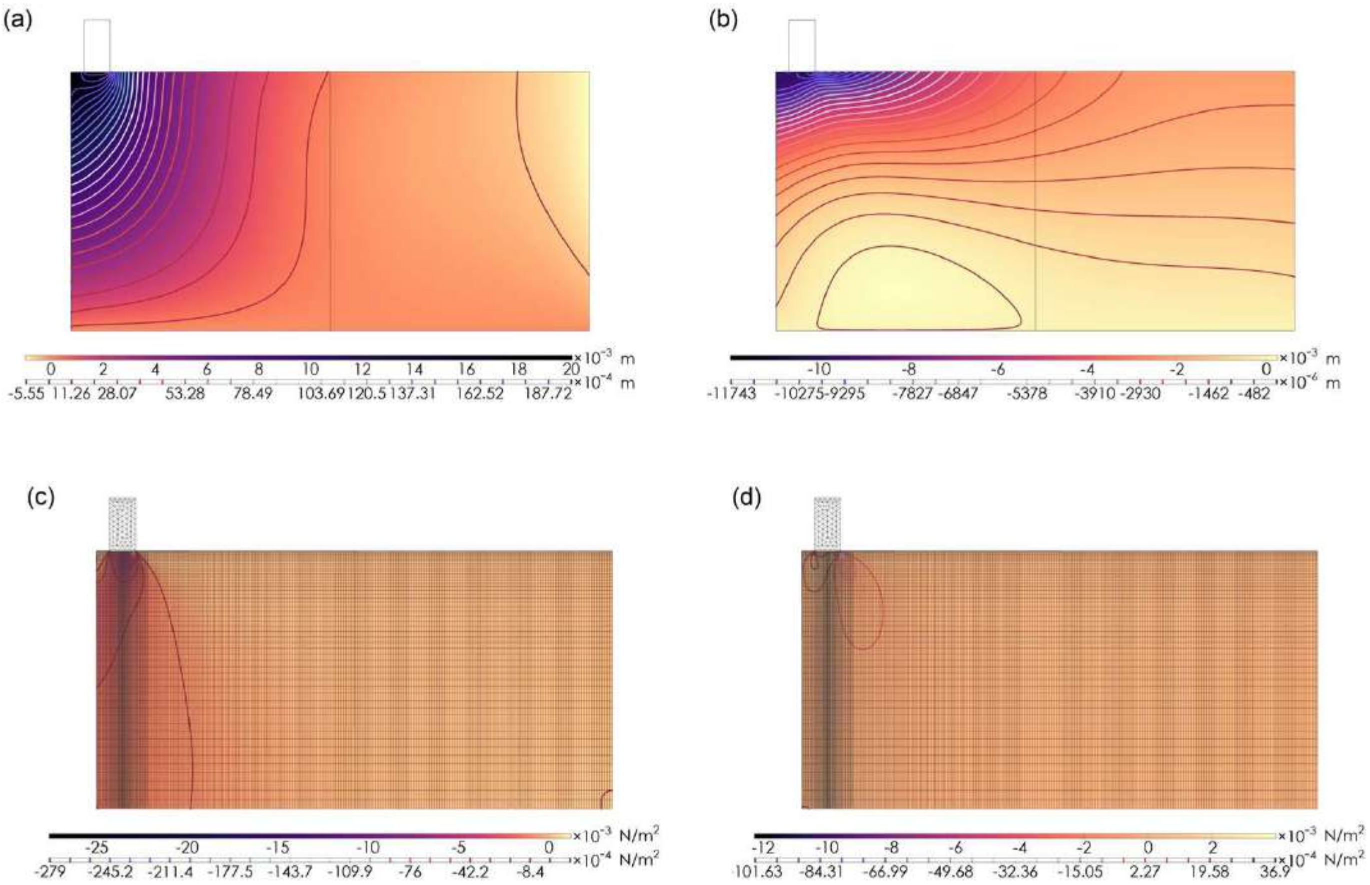
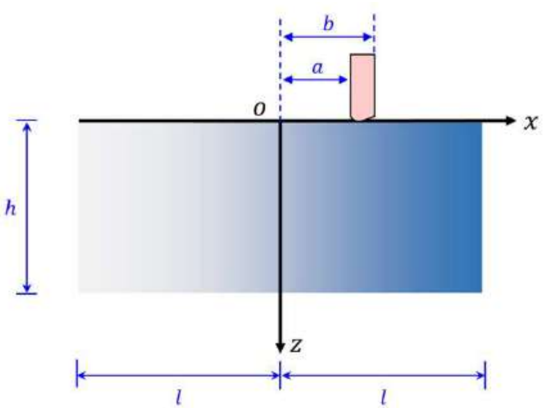


Figure 11. FEA results

Viscoelastic Model

Background

$$
\begin{cases}
\sigma_{xx} = c_{11} * \mathrm{d}\left(\dfrac{\partial u_x}{\partial x}\right) + c_{13} * \mathrm{d}\left(\dfrac{\partial u_z}{\partial z}\right) + e_{31}\dfrac{\partial \varphi}{\partial z} + q_{31}\dfrac{\partial \psi}{\partial z} - \mathfrak{a}_1 T \\
\sigma_{zz} = c_{13} * \mathrm{d}\left(\dfrac{\partial u_x}{\partial x}\right) + c_{33} * \mathrm{d}\left(\dfrac{\partial u_z}{\partial z}\right) + e_{33}\dfrac{\partial \varphi}{\partial z} + q_{33}\dfrac{\partial \psi}{\partial z} - \mathfrak{a}_3 T \\
\sigma_{xz} = c_{44} * \mathrm{d}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) + e_{15}\dfrac{\partial \varphi}{\partial x} + q_{15}\dfrac{\partial \psi}{\partial x} \\
D_x = e_{15}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) - \varepsilon_{11}\dfrac{\partial \varphi}{\partial x} - d_{11}\dfrac{\partial \psi}{\partial x} \\
D_z = e_{31}\dfrac{\partial u_x}{\partial x} + e_{33}\dfrac{\partial u_z}{\partial z} - \varepsilon_{33}\dfrac{\partial \varphi}{\partial z} - d_{33}\dfrac{\partial \psi}{\partial z} + \mathfrak{b}_3 T \\
B_x = q_{15}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) - d_{11}\dfrac{\partial \varphi}{\partial x} - \gamma_{11}\dfrac{\partial \psi}{\partial x} \\
B_z = q_{31}\dfrac{\partial u_x}{\partial x} + q_{33}\dfrac{\partial u_z}{\partial z} - d_{33}\dfrac{\partial \varphi}{\partial z} - \gamma_{33}\dfrac{\partial \psi}{\partial z} + \mathfrak{c}_3 T
\end{cases}
$$



$$
\begin{cases}
\dfrac{\partial \tilde{u}_x}{\partial z} = -\dfrac{\partial \tilde{u}_z}{\partial x} - a_1\dfrac{\partial \tilde{\varphi}}{\partial x} - a_2\dfrac{\partial \tilde{\psi}}{\partial x} + a_3\tilde{\hat{\sigma}}_{xz} \\
\dfrac{\partial \tilde{u}_z}{\partial z} = -a_4\dfrac{\partial \tilde{u}_x}{\partial x} + a_5\tilde{\hat{\sigma}}_{zz} + a_6\tilde{\hat{D}}_z + a_7\tilde{\hat{B}}_z + \iota_1\tilde{T} \\
\dfrac{\partial \tilde{\varphi}}{\partial z} = a_8\dfrac{\partial \tilde{u}_x}{\partial x} + a_6\tilde{\hat{\sigma}}_{zz} - a_9\tilde{\hat{D}}_z + a_{10}\tilde{\hat{B}}_z + \iota_2\tilde{T} \\
\dfrac{\partial \tilde{\psi}}{\partial z} = a_{11}\dfrac{\partial \tilde{u}_x}{\partial x} + a_7\tilde{\hat{\sigma}}_{zz} + a_{10}\tilde{\hat{D}}_z - a_{12}\tilde{\hat{B}}_z + \iota_3\tilde{T} \\
\dfrac{\partial \tilde{\hat{\sigma}}_{xz}}{\partial z} = a_{13}\dfrac{\partial^2 \tilde{u}_x}{\partial x^2} - a_4\dfrac{\partial \tilde{\hat{\sigma}}_{zz}}{\partial x} + a_8\dfrac{\partial \tilde{\hat{D}}_z}{\partial x} + a_{11}\dfrac{\partial \tilde{\hat{B}}_z}{\partial x} + \iota_4\dfrac{\partial \tilde{T}}{\partial x} + \beta\left(a_{13}\dfrac{\partial \tilde{u}_x}{\partial x} - a_4\tilde{\hat{\sigma}}_{zz} + a_8\tilde{\hat{D}}_z + a_{11}\tilde{\hat{B}}_z + \iota_4\tilde{T}\right) \\
\dfrac{\partial \tilde{\hat{\sigma}}_{zz}}{\partial z} = -\dfrac{\partial \tilde{\hat{\sigma}}_{xz}}{\partial x} - \beta\tilde{\hat{\sigma}}_{xz} \\
\dfrac{\partial \tilde{\hat{D}}_z}{\partial z} = a_{14}\dfrac{\partial^2 \tilde{\varphi}}{\partial x^2} + a_{15}\dfrac{\partial^2 \tilde{\psi}}{\partial x^2} - a_1\dfrac{\partial \tilde{\hat{\sigma}}_{xz}}{\partial x} + \beta\left(a_{14}\dfrac{\partial \tilde{\varphi}}{\partial x} + a_{15}\dfrac{\partial \tilde{\psi}}{\partial x} - a_1\tilde{\hat{\sigma}}_{xz}\right) \\
\dfrac{\partial \tilde{\hat{B}}_z}{\partial z} = a_{15}\dfrac{\partial^2 \tilde{\varphi}}{\partial x^2} + a_{16}\dfrac{\partial^2 \tilde{\psi}}{\partial x^2} - a_2\dfrac{\partial \tilde{\hat{\sigma}}_{xz}}{\partial z} + \beta\left(a_{15}\dfrac{\partial \tilde{\varphi}}{\partial x} + a_{16}\dfrac{\partial \tilde{\psi}}{\partial x} - a_2\tilde{\hat{\sigma}}_{xz}\right)
\end{cases}
$$

Laplace transform

Non-homogeneous terms in the state equations

New Form of Symplectic Orthogonality

Dual Hamiltonian transformation

Shift Hamiltonian Transformation

$$\langle \boldsymbol{f}_1, \mathcal{H}\boldsymbol{f}_2 \rangle = \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \alpha \langle \boldsymbol{f}_1, \boldsymbol{f}_2 \rangle$$

proved the shift symplectic self-adjoint property of $\mathcal{H}$ operator

Here, we decide to prove the

Dual Hamiltonian Transformation

$$\langle \boldsymbol{f}_1, \mathcal{H}\boldsymbol{f}_2 \rangle = \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \beta \langle \boldsymbol{f}_1^*, \boldsymbol{f}_2^* \rangle$$

*Original and dual symplectic space

*Note: This condition holds only with the displacement constrained, and the stress-free boundary is not applicable

$$\boldsymbol{f} = \left[\{u_x, u_z, \varphi, \psi\}, \{\hat{\sigma}_{xz}, \hat{\sigma}_{zz}, \hat{D}_z, \hat{B}_z\} \right]^{\mathrm{T}}$$

$$\boldsymbol{f}^* = \left[\{u_x, u_z, \varphi, \psi\}, \{\hat{\sigma}_{xx}, \hat{\sigma}_{xz}, \hat{D}_x, \hat{B}_x\} \right]^{\mathrm{T}}$$

New Form of Symplectic Orthogonality

Dual Hamiltonian transformation

$$
\begin{aligned}
\left\langle \boldsymbol{f}_1, \boldsymbol{\mathcal{H}} \boldsymbol{f}_2 \right\rangle &= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \left\langle \boldsymbol{f}_1^*, \boldsymbol{f}_2^* \right\rangle \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \left\langle \boldsymbol{\mathcal{A}} \boldsymbol{f}_1, \boldsymbol{\mathcal{A}} \boldsymbol{f}_2 \right\rangle \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \int_{-l}^{l} (\boldsymbol{\mathcal{A}} \boldsymbol{f}_1)^{\mathrm{T}} \mathbf{J} (\boldsymbol{\mathcal{A}} \boldsymbol{f}_2) \mathrm{d}x \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \int_{-l}^{l} \boldsymbol{f}_1^{\mathrm{T}} \boldsymbol{\mathcal{A}}^{\mathcal{T}} \mathbf{J} (\boldsymbol{\mathcal{A}} \boldsymbol{f}_2) \mathrm{d}x \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \int_{-l}^{l} \boldsymbol{f}_1^{\mathrm{T}} \mathbf{J} (-\mathbf{J}) \boldsymbol{\mathcal{A}}^{\mathcal{T}} \mathbf{J} (\boldsymbol{\mathcal{A}} \boldsymbol{f}_2) \mathrm{d}x \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \int_{-l}^{l} \boldsymbol{f}_1^{\mathrm{T}} \mathbf{J} \mathbf{J}^{\mathcal{T}} \boldsymbol{\mathcal{A}}^{\mathcal{T}} \mathbf{J} \boldsymbol{\mathcal{A}} \boldsymbol{f}_2 \mathrm{d}x \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \beta \int_{-l}^{l} \boldsymbol{f}_1^{\mathrm{T}} \mathbf{J} (\boldsymbol{\mathcal{A}} \mathbf{J})^{\mathrm{T}} (\mathbf{J} \boldsymbol{\mathcal{A}}) \boldsymbol{f}_2 \mathrm{d}x \\
&= \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle - \left\langle \boldsymbol{f}_1, \beta \boldsymbol{\mathcal{B}} \boldsymbol{f}_2 \right\rangle
\end{aligned}
$$

Boundary conditions: fixed !!

$$
\left\langle \boldsymbol{f}_1, (\boldsymbol{\mathcal{H}} + \beta \boldsymbol{\mathcal{B}}) \boldsymbol{f}_2 \right\rangle = \left\langle \boldsymbol{f}_2, \boldsymbol{\mathcal{H}} \boldsymbol{f}_1 \right\rangle
$$

$$
\boldsymbol{\mathcal{H}}^{\mathcal{T}} = \mathbf{J} (\boldsymbol{\mathcal{H}} + \beta \boldsymbol{\mathcal{B}}) \mathbf{J}
$$

convert to the same symplectic space

$\boldsymbol{\mathcal{A}}^{\mathcal{T}}$ represents adjoint transpose

Theorem 1

If μ is an eigenvalue of operator matrix $\boldsymbol{\mathcal{H}}$, then $-\mu$ is the eigenvalue of $\beta\boldsymbol{\mathcal{B}}+\boldsymbol{\mathcal{H}}$ with the same multiplicity.

Proof

$$\boldsymbol{\mathcal{H}}^{\mathcal{T}} = \mathbf{J}(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\mathbf{J}$$

$$\begin{aligned}
\left|\mu\mathbf{I}-\boldsymbol{\mathcal{H}}\right| &= \left|(\mu\mathbf{I}-\boldsymbol{\mathcal{H}})\mathbf{J}\right| \\
&= \left|\mu\mathbf{J}\mathbf{J}-\mathbf{J}\boldsymbol{\mathcal{H}}\mathbf{J}\right| \\
&= \left|\mu\mathbf{J}\mathbf{J}-\mathbf{J}(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\mathbf{J}+\mathbf{J}\beta\boldsymbol{\mathcal{B}}\mathbf{J}\right| \\
&= \left|\mu\mathbf{J}\mathbf{J}+\mathbf{J}\beta\boldsymbol{\mathcal{B}}\mathbf{J}-\boldsymbol{\mathcal{H}}^{\mathcal{T}}\right| \\
&= \left|-\mu\mathbf{I}-(\beta\boldsymbol{\mathcal{B}})^{\mathcal{T}}-\boldsymbol{\mathcal{H}}^{\mathcal{T}}\right| \\
&= \left|-\mu\mathbf{I}-(\beta\boldsymbol{\mathcal{B}}+\boldsymbol{\mathcal{H}})^{\mathcal{T}}\right| \\
&= \left|-\mu\mathbf{I}-(\beta\boldsymbol{\mathcal{B}}+\boldsymbol{\mathcal{H}})\right|
\end{aligned}$$

Q.E.D

Theorem 2

Let $\Phi_i^{(0)},\Phi_i^{(1)},\Phi_i^{(2)},\cdots,\Phi_i^{(m)}$ and $\Omega_j^{(0)},\Omega_j^{(1)},\Omega_j^{(2)},\cdots,\Omega_j^{(n)}$ are the basic eigenvectors and Jordan form eigenvectors of the eigenvalues μ_i and μ_j, respectively. For $\mu_i+\mu_j\neq 0$, the dual symplectic orthogonality between the eigenvectors are

$$\left\langle \Phi_i^{(\mathfrak{s})},\Omega_j^{(\mathfrak{t})} \right\rangle = 0,(\mathfrak{s}=0,1,\cdots,m;\mathfrak{t}=0,1,\cdots,n)$$

New Form of Symplectic Orthogonality

Dual symplectic orthogonality

Proof $\Phi_i^{(0)},\Phi_i^{(1)},\Phi_i^{(2)},\cdots,\Phi_i^{(m)}$ and $\Omega_j^{(0)},\Omega_j^{(1)},\Omega_j^{(2)},\cdots,\Omega_j^{(n)}$ are the basic eigenvectors and Jordan form eigenvectors of the eigenvalues corresponds to μ_i and μ_j , respectively. When $\mu_i+\mu_j\neq 0$, $\left\langle \Phi_i^{(s)},\Omega_j^{(t)}\right\rangle = 0,(s=0,1,\cdots,m;t=0,1,\cdots,n)$

$r=s+t=0$

$$\boldsymbol{\mathcal{H}}\Phi_i^{(0)}=\mu_i\Phi_i^{(0)} \qquad (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega_j^{(0)}=\mu_j\Omega_j^{(0)}$$

$$\left\langle \Phi_i^{(0)},(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega_j^{(0)}\right\rangle = \left\langle \Phi_i^{(0)},\mu_j\Omega_j^{(0)}\right\rangle = \mu_j\left\langle \Phi_i^{(0)},\Omega_j^{(0)}\right\rangle$$

$$\left\langle \Omega_j^{(0)},\boldsymbol{\mathcal{H}}\Phi_i^{(0)}\right\rangle = \left\langle \Omega_j^{(0)},\mu_i\Phi_i^{(0)}\right\rangle = \mu_i\left\langle \Omega_j^{(0)},\Phi_i^{(0)}\right\rangle = -\mu_i\left\langle \Phi_i^{(0)},\Omega_j^{(0)}\right\rangle$$

$$(\mu_i+\mu_j)\left\langle \Phi_i^{(0)},\Omega_j^{(0)}\right\rangle = 0 \Rightarrow \left\langle \Phi_i^{(0)},\Omega_j^{(0)}\right\rangle = 0$$

$$\boxed{\boldsymbol{\mathcal{H}}^{\mathcal{T}}=\mathbf{J}(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\mathbf{J}}$$

$$\boxed{\left\langle \boldsymbol{f}_1,(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\boldsymbol{f}_2\right\rangle = \left\langle \boldsymbol{f}_2,\boldsymbol{\mathcal{H}}\boldsymbol{f}_1\right\rangle}$$

$r=s+t=k$ If we assume $\left\langle \Phi_i^{(s)},\Omega_j^{(t)}\right\rangle = 0$

$r=s+t=k+1$

$$\boldsymbol{\mathcal{H}}\Phi_i^{(s)}=\mu_i\Phi_i^{(s)}+\Phi_i^{(s-1)}$$

$$(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega_j^{(t)}=\mu_j\Omega_j^{(t)}+\Omega_j^{(t-1)}$$

$$\left\langle \Phi_i^{(s)},(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega_j^{(t)}\right\rangle = \left\langle \Phi_i^{(s)},\mu_j\Omega_j^{(t)}\right\rangle + \left\langle \Phi_i^{(s)},\Omega_j^{(t-1)}\right\rangle = \mu_j\left\langle \Phi_i^{(s)},\Omega_j^{(t)}\right\rangle$$

$$\left\langle \Omega_j^{(t)},\boldsymbol{\mathcal{H}}\Phi_i^{(s)}\right\rangle = \left\langle \Omega_j^{(t)},\mu_i\Phi_i^{(s)}\right\rangle + \left\langle \Omega_j^{(t)},\Phi_i^{(s-1)}\right\rangle = \mu_i\left\langle \Omega_j^{(t)},\Phi_i^{(s)}\right\rangle = -\mu_i\left\langle \Phi_i^{(s)},\Omega_j^{(t)}\right\rangle$$

Q.E.D

Theorem 3

Let μ and $-\mu$ be the dual symplectic adjoint eigenvalues of the operator $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}}$ with multiplicity $\mathfrak{m}$, respectively. The respective adjoint symplectic orthogonal vector sets are $\left\{\Phi_i^{(0)},\Phi_i^{(1)},\Phi_i^{(2)},\cdots,\Phi_i^{(\mathfrak{m}-1)}\right\}$ and $\left\{\Omega_j^{(0)},\Omega_j^{(1)},\Omega_j^{(2)},\cdots,\Omega_j^{(\mathfrak{m}-1)}\right\}$, such that

$$\left\langle \Phi^{(i)},\Omega^{(j)}\right\rangle = \begin{cases}(-1)^i\mathscr{C}\neq 0 & (i+j=\mathfrak{m}-1)\\ 0 & (i+j\neq\mathfrak{m}-1)\end{cases}$$

Dual symplectic adjoint

Proof

If $i=0$ when $j \le \mathfrak{m}-2$

$$\left\langle \Phi^{(0)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega^{(j+1)} \right\rangle = -\mu\left\langle \Phi^{(0)}, \Omega^{(j+1)} \right\rangle + \left\langle \Phi^{(0)}, \Omega^{(j)} \right\rangle$$

$$\left\langle \Omega^{(j+1)}, \boldsymbol{\mathcal{H}}\Phi^{(0)} \right\rangle = \mu\left\langle \Omega^{(j+1)}, \Phi^{(0)} \right\rangle = -\mu\left\langle \Phi^{(0)}, \Omega^{(j+1)} \right\rangle$$

with $\left\langle \Phi^{(0)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega^{(j+1)} \right\rangle = \left\langle \Omega^{(j+1)}, \boldsymbol{\mathcal{H}}\Phi^{(0)} \right\rangle \Rightarrow \left\langle \Phi^{(0)}, \Omega^{(j)} \right\rangle = 0$

So $\left\langle \Phi^{(0)}, \Omega^{(j)} \right\rangle = \mathscr{C} \ne 0$ when $j = \mathfrak{m}-1$ Otherwise $\Phi^{(0)} \equiv 0$

Assume the theorem is valid for $i=\mathfrak{k}$, then for $i=\mathfrak{k}+1$, set $\mathscr{d} = -\frac{1}{\mathscr{C}}\left\langle \Phi^{(\mathfrak{k}+1)}, \Omega^{(\mathfrak{m}-1)} \right\rangle$

$$\widehat{\Phi}^{(\mathfrak{k}+1+\mathfrak{p})} = \Phi^{(\mathfrak{k}+1+\mathfrak{p})} + \mathscr{d}\Phi^{(\mathfrak{p})} \quad (\mathfrak{p}=1,2,\cdots,\mathfrak{m}-\mathfrak{k}-2)$$

$j \le \mathfrak{m}-2$

$$\left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega^{(j+1)} \right\rangle = -\mu\left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, \Omega^{(j+1)} \right\rangle + \left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, \Omega^{(j)} \right\rangle$$

$$\begin{aligned}\left\langle \Omega^{(j+1)}, \boldsymbol{\mathcal{H}}\widehat{\Phi}^{(\mathfrak{k}+1)} \right\rangle &= \mu\left\langle \Omega^{(j+1)}, \widehat{\Phi}^{(\mathfrak{k}+1)} \right\rangle + \left\langle \Omega^{(j+1)}, \Phi^{(\mathfrak{k})} \right\rangle \\ &= -\mu\left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, \Omega^{(j+1)} \right\rangle - \left\langle \Phi^{(\mathfrak{k})}, \Omega^{(j+1)} \right\rangle\end{aligned}$$

$$\left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\Omega^{(j+1)} \right\rangle = \left\langle \Omega^{(j+1)}, \boldsymbol{\mathcal{H}}\widehat{\Phi}^{(\mathfrak{k}+1)} \right\rangle \quad \left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, \Omega^{(j)} \right\rangle = -\left\langle \Phi^{(\mathfrak{k})}, \Omega^{(j+1)} \right\rangle$$

$$\left\langle \widehat{\Phi}^{(\mathfrak{k}+1)}, \Omega^{(j)} \right\rangle = -\left\langle \Phi^{(\mathfrak{k})}, \Omega^{(j+1)} \right\rangle = \begin{cases} (-1)^{(\mathfrak{k}+1)}\mathscr{C} \ne 0 & (\mathfrak{k}+j=\mathfrak{m}-2) \\ 0 & (\mathfrak{k}+j \ne \mathfrak{m}-2) \end{cases}$$

Q.E.D

New Form of Symplectic Orthogonality

Symmetry breaking

Dual Hamiltonian transformation

$$\left\{\Phi_i^{(0)},\Phi_i^{(1)},\Phi_i^{(2)},\cdots,\Phi_i^{(m)}\right\}$$

$$\left\{\Omega_j^{(0)},\Omega_j^{(1)},\Omega_j^{(2)},\cdots,\Omega_j^{(n)}\right\} \qquad \mu_i + \mu_j \neq 0$$

$$\left\langle\Phi_i^{(s)},\Omega_j^{(t)}\right\rangle = 0,(s=0,1,\cdots,m;t=0,1,\cdots,n)$$

$$\boxed{\left\langle \boldsymbol{f}_1,(\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\boldsymbol{f}_2\right\rangle = \left\langle \boldsymbol{f}_2,\boldsymbol{\mathcal{H}}\boldsymbol{f}_1\right\rangle}$$

Shift Hamiltonian transformation

$$\left\{\Phi_i^{(0)},\Phi_i^{(1)},\Phi_i^{(2)},\cdots,\Phi_i^{(m)}\right\} \qquad \mu_i + \mu_j + \alpha \neq 0$$

$$\left\langle\Phi_i^{(s)},\Phi_j^{(t)}\right\rangle = 0,(s=0,1,\cdots,m;t=0,1,\cdots,n)$$

$$\boxed{\left\langle \boldsymbol{f}_1,(\boldsymbol{\mathcal{H}}+\mathbf{I}\alpha)\boldsymbol{f}_2\right\rangle = \left\langle \boldsymbol{f}_2,\boldsymbol{\mathcal{H}}\boldsymbol{f}_1\right\rangle}$$

Two different sets of (infinite-dimensional space) bases are orthogonal, and the expansion also requires another set of bases to obtain the respective coefficients.

$$\left\{\sin x,\cdots,\sin nx\right\} \quad \left\{1,\cos x,\cdots,\cos nx\right\} \quad \left\{J_\nu(x)\right\} \quad \left\{P_n(x)\right\}$$

$$f = \sum a_i \sin ix$$

Viscoelastic Model

Non-homogeneous terms

$$\tilde{\boldsymbol{f}}'=\sum_{i=1}^{\infty}\left[\tilde{g}_{\mu,i}(z)\tilde{\Phi}_{\mu,i}+\tilde{g}_{-\mu,i}(z)\tilde{\Phi}_{-\mu,i}\right]$$

particular solution $\tilde{\boldsymbol{f}}^{p}=\sum_{i=1}^{\infty}\left[\tilde{\mathfrak{g}}_{\mu,i}(z)\tilde{\Phi}_{\mu,i}+\tilde{\mathfrak{g}}_{-\mu,i}(z)\tilde{\Phi}_{-\mu,i}\right]$

$$\begin{cases}\dfrac{\mathrm{d}\tilde{\mathfrak{g}}_{\mu,i}(z)}{\mathrm{d}z}=\mu_i\tilde{\mathfrak{g}}_{\mu,i}(z)+\tilde{g}_{\mu,i}(z)\\ \dfrac{\mathrm{d}\tilde{\mathfrak{g}}_{-\mu,i}(z)}{\mathrm{d}z}=-\mu_i\tilde{\mathfrak{g}}_{-\mu,i}(z)+\tilde{g}_{-\mu,i}(z)\end{cases}\qquad\begin{cases}\tilde{g}_{\mu,i}(z)=\left\langle\tilde{\boldsymbol{f}}',\tilde{\Omega}_{-\mu,i}\right\rangle\\ \tilde{g}_{-\mu,i}(z)=\left\langle\tilde{\boldsymbol{f}}',\tilde{\Omega}_{\mu,i}\right\rangle\end{cases}$$

complete solution

$$\begin{aligned}\tilde{\tilde{\boldsymbol{f}}}&=\tilde{\tilde{\boldsymbol{f}}}^{g}+\tilde{\tilde{\boldsymbol{f}}}^{p}\\&=\sum_{i=1}^{\infty}(\tilde{m}_{\mu,i}\tilde{\tilde{\boldsymbol{f}}}_{\mu,i}+\tilde{m}_{-\mu,i}\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i})+\sum_{i=1}^{\infty}\left[\tilde{\mathfrak{g}}_{\mu,i}(z)\boldsymbol{\mathcal{M}}\tilde{\Phi}_{\mu,i}+\tilde{\mathfrak{g}}_{-\mu,i}(z)\boldsymbol{\mathcal{M}}\tilde{\Phi}_{-\mu,i}\right]\\&=\sum_{i=1}^{\infty}\left\{[\tilde{m}_{\mu,i}+\tilde{m}^{*}_{\mu,i}(z)]\tilde{\tilde{\boldsymbol{f}}}_{\mu,i}+[\tilde{m}_{-\mu,i}+\tilde{m}^{*}_{-\mu,i}(z)]\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i}\right\}\\&=\sum_{i=1}^{\infty}\left[(\tilde{m}^{\mathrm{Re}}_{\mu,i}\mathrm{Re}\tilde{\tilde{\boldsymbol{f}}}_{\mu,i}+\tilde{m}^{\mathrm{Im}}_{\mu,i}\mathrm{Im}\tilde{\tilde{\boldsymbol{f}}}_{\mu,i})+(\tilde{m}^{\mathrm{Re}}_{-\mu,i}\mathrm{Re}\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i}+\tilde{m}^{\mathrm{Im}}_{-\mu,i}\mathrm{Im}\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i})\right]+\tilde{\tilde{\boldsymbol{f}}}^{p}_{\mathrm{R}}\end{aligned}$$

where

$$\begin{cases}\tilde{\mathfrak{g}}_{\mu,i}(z)=\mathrm{e}^{\mu_i z}\tilde{m}^{*}_{\mu,i}(z)=\mathrm{e}^{\mu_i z}\displaystyle\int_0^z\mathrm{e}^{-\mu_i\zeta}\tilde{g}(\zeta)\mathrm{d}\zeta\\ \tilde{\mathfrak{g}}_{-\mu,i}(z)=\mathrm{e}^{-\mu_i z}\tilde{m}^{*}_{-\mu,i}(z)=\mathrm{e}^{-\mu_i z}\displaystyle\int_0^z\mathrm{e}^{\mu_i\zeta}\tilde{g}(\zeta)\mathrm{d}\zeta\end{cases}$$

$$\tilde{\tilde{\boldsymbol{f}}}^{p}_{\mathrm{R}}=\sum_{i=1}^{\infty}\left\{2\,\mathrm{Re}[\tilde{m}^{*}_{\mu,i}(z)]\,\mathrm{Re}\,\tilde{\tilde{\boldsymbol{f}}}_{\mu,i}-2\,\mathrm{Im}[\tilde{m}^{*}_{\mu,i}(z)]\,\mathrm{Im}\,\tilde{\tilde{\boldsymbol{f}}}_{\mu,i}+2\,\mathrm{Re}[\tilde{m}^{*}_{-\mu,i}(z)]\,\mathrm{Re}\,\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i}-2\,\mathrm{Im}[\tilde{m}^{*}_{-\mu,i}(z)]\,\mathrm{Im}\,\tilde{\tilde{\boldsymbol{f}}}_{-\mu,i}\right\}$$

Viscoelastic Model

Hamiltonian mixed energy variational principle

$$\delta\left\{\int_0^h\int_{-l}^l\left[\tilde{\tilde{\boldsymbol{p}}}^{\mathrm{T}}\frac{\partial\tilde{\tilde{\boldsymbol{q}}}}{\partial z}-H(\tilde{\tilde{\boldsymbol{q}}},\tilde{\tilde{\boldsymbol{p}}})\right]\mathrm{d}x\mathrm{d}z-\int_{\Gamma_{\tilde{\tilde{q}}_h}}\left[\tilde{\tilde{\boldsymbol{p}}}^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{q}}}-\bar{\tilde{\tilde{\boldsymbol{q}}}}_h)\right]\mathrm{d}x\right.$$
$$\left.-\int_{\Gamma_{\tilde{\tilde{p}}_h}}\left[\bar{\tilde{\tilde{\boldsymbol{p}}}}_h{}^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}\right]\mathrm{d}x+\int_{\Gamma_{\tilde{\tilde{q}}_0}}\left[\tilde{\tilde{\boldsymbol{p}}}^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{q}}}-\bar{\tilde{\tilde{\boldsymbol{q}}}}_0)\right]\mathrm{d}x+\int_{\Gamma_{\tilde{\tilde{p}}_0}}\left[\bar{\tilde{\tilde{\boldsymbol{p}}}}_0{}^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}\right]\mathrm{d}x\right\}=0$$

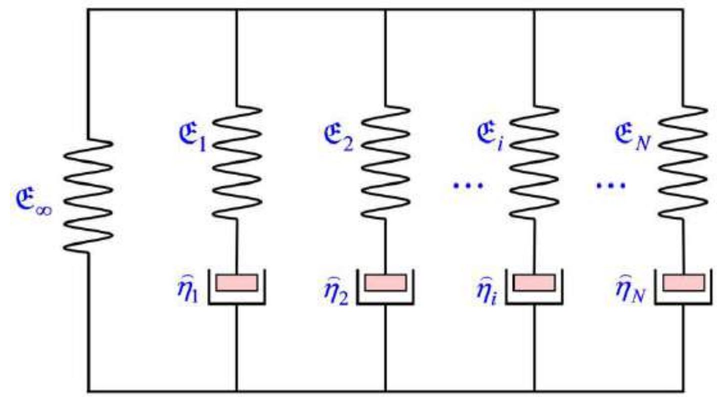
$$\int_{\Gamma_{\tilde{\tilde{p}}_h}}\left[(\sum_{i=1}^{\infty}\delta\tilde{m}_i\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{p}}}_{\mathrm{R}}^p+\sum_{j=1}^{\infty}\tilde{m}_j\tilde{\tilde{\boldsymbol{p}}}_j-\bar{\tilde{\tilde{\boldsymbol{p}}}}_h)\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{q}}_h}}\left[(\sum_{i=1}^{\infty}\delta\tilde{m}_i\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{q}}}_{\mathrm{R}}^p+\sum_{j=1}^{\infty}\tilde{m}_j\tilde{\tilde{\boldsymbol{q}}}_j-\bar{\tilde{\tilde{\boldsymbol{q}}}}_h)\right]\mathrm{d}x$$
$$+\int_{\Gamma_{\tilde{\tilde{q}}_0}}\left[(\sum_{i=1}^{\infty}\delta\tilde{m}_i\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{q}}}_{\mathrm{R}}^p+\sum_{j=1}^{\infty}\tilde{m}_j\tilde{\tilde{\boldsymbol{q}}}_j-\bar{\tilde{\tilde{\boldsymbol{q}}}}_0)\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{p}}_0}}\left[(\sum_{i=1}^{\infty}\delta\tilde{m}_i\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}(\tilde{\tilde{\boldsymbol{p}}}_{\mathrm{R}}^p+\sum_{j=1}^{\infty}\tilde{m}_j\tilde{\tilde{\boldsymbol{p}}}_j-\bar{\tilde{\tilde{\boldsymbol{p}}}}_0)\right]\mathrm{d}x=0$$

$$\tilde{\tilde{\mathscr{A}}}_{ij}=\int_{\Gamma_{\tilde{\tilde{p}}_h}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{p}}}_j\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{q}}_h}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}_j\right]\mathrm{d}x+\int_{\Gamma_{\tilde{\tilde{q}}_0}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}_j\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{p}}_0}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{p}}}_j\right]\mathrm{d}x$$
$$\tilde{\tilde{\mathscr{H}}}_{i}=\int_{\Gamma_{\tilde{\tilde{p}}_h}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\bar{\tilde{\tilde{\boldsymbol{p}}}}_h\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{q}}_h}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\bar{\tilde{\tilde{\boldsymbol{q}}}}_h\right]\mathrm{d}x+\int_{\Gamma_{\tilde{\tilde{q}}_0}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\bar{\tilde{\tilde{\boldsymbol{q}}}}_0\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{p}}_0}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\bar{\tilde{\tilde{\boldsymbol{p}}}}_0\right]\mathrm{d}x$$
$$\tilde{\tilde{\mathscr{K}}}_{i}=\int_{\Gamma_{\tilde{\tilde{p}}_h}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{p}}}_{\mathrm{R}}^p\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{q}}_h}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}_{\mathrm{R}}^p\right]\mathrm{d}x+\int_{\Gamma_{\tilde{\tilde{q}}_0}}\left[(\tilde{\tilde{\boldsymbol{p}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{q}}}_{\mathrm{R}}^p\right]\mathrm{d}x-\int_{\Gamma_{\tilde{\tilde{p}}_0}}\left[(\tilde{\tilde{\boldsymbol{q}}}_i)^{\mathrm{T}}\tilde{\tilde{\boldsymbol{p}}}_{\mathrm{R}}^p\right]\mathrm{d}x$$

Viscoelastic Model

Example

$$
\begin{cases}
s_{ij} = 2G(x,t) * \mathrm{d}\widehat{e}_{ij}(t) = 2\int_0^t G(x,t-\tau)\dfrac{\mathrm{d}\widehat{e}_{ij}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau \\
\sigma_{kk} = 3K(x,t) * \mathrm{d}\widehat{\varepsilon}_{ij}(t) = 3\int_0^t K(x,t-\tau)\dfrac{\mathrm{d}\widehat{\varepsilon}_{kk}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau
\end{cases}
$$



$$
\begin{cases}
G(x,t) = \mathfrak{G}(t)\mathrm{e}^{\beta x} = [\mathfrak{G}_\infty + \sum_{i=1}^{N}\mathfrak{G}_i \mathrm{e}^{-\frac{t}{\widehat{\tau}_{1i}}}]\mathrm{e}^{\beta x} \\
K(x,t) = \mathfrak{K}(t)\mathrm{e}^{\beta x} = [\mathfrak{K}_\infty + \sum_{i=1}^{N}\mathfrak{K}_i \mathrm{e}^{-\frac{t}{\widehat{\tau}_{2i}}}]\mathrm{e}^{\beta x}
\end{cases}
$$

$$
\frac{\partial}{\partial z}\begin{Bmatrix} \tilde{u}_z \\ \tilde{u}_x \\ \tilde{\hat{\sigma}}_{zz} \\ \tilde{\hat{\sigma}}_{xz} \end{Bmatrix} =
\left[\begin{array}{cc|cc}
0 & -\dfrac{\mathfrak{A}_2}{\mathfrak{A}_1}\dfrac{\partial}{\partial x} & \dfrac{1}{\omega\mathfrak{A}_1} & 0 \\
-\dfrac{\partial}{\partial x} & 0 & 0 & \dfrac{1}{\omega\tilde{\mathfrak{G}}} \\
\hline
0 & 0 & 0 & -\left(\dfrac{\partial}{\partial x}+\beta\right) \\
0 & -\omega\dfrac{\mathfrak{A}_1^2-\mathfrak{A}_2^2}{\mathfrak{A}_1}\left(\dfrac{\partial^2}{\partial x^2}+\beta\dfrac{\partial}{\partial x}\right) & -\dfrac{\mathfrak{A}_2}{\mathfrak{A}_1}\left(\dfrac{\partial}{\partial x}+\beta\right) & 0
\end{array}\right]
\begin{Bmatrix} \tilde{u}_z \\ \tilde{u}_x \\ \tilde{\hat{\sigma}}_{zz} \\ \tilde{\hat{\sigma}}_{xz} \end{Bmatrix}
$$

Comparison

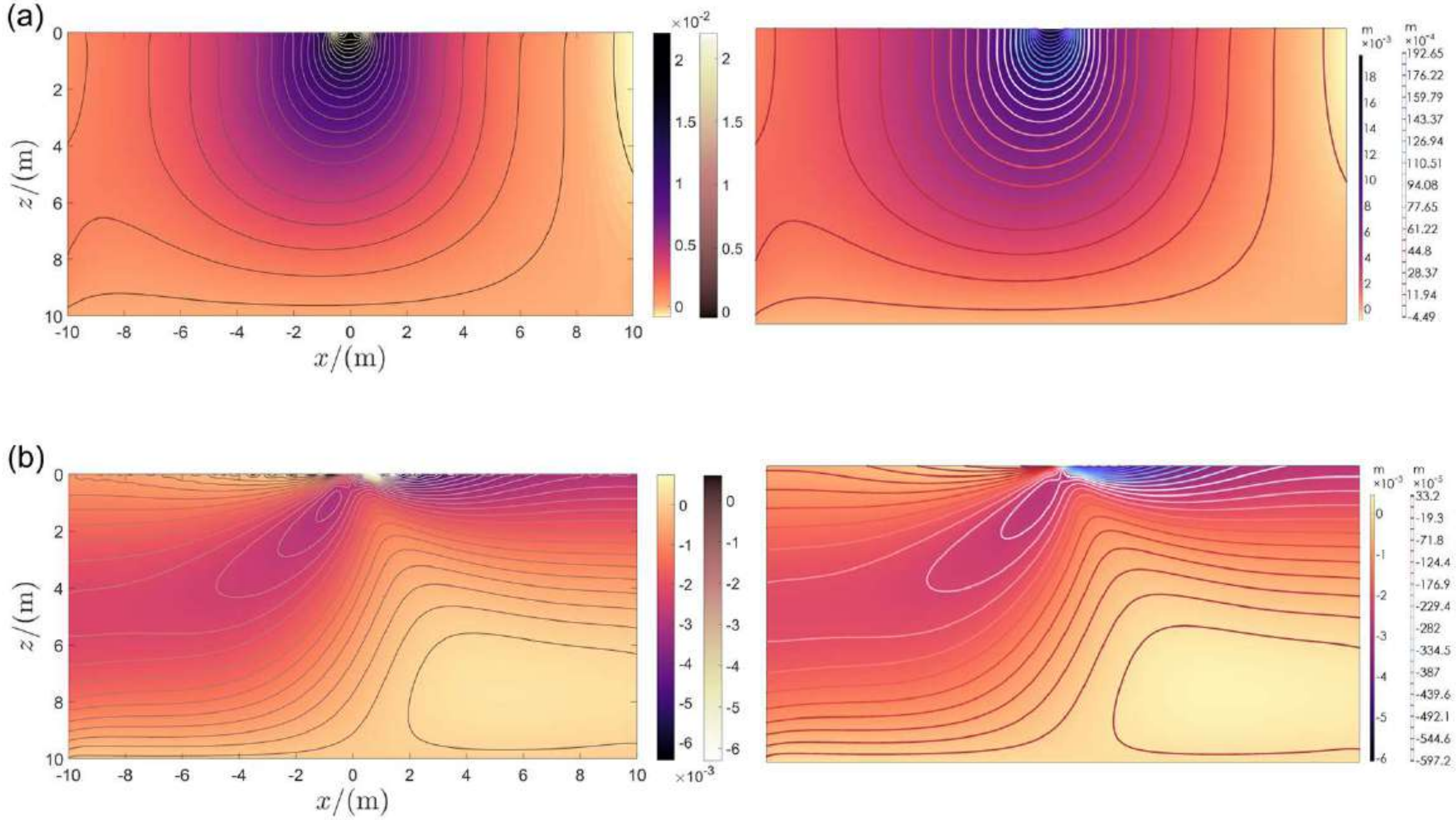


FIGURE 12. Comparison between FEA results and analytical solutions

Comparison

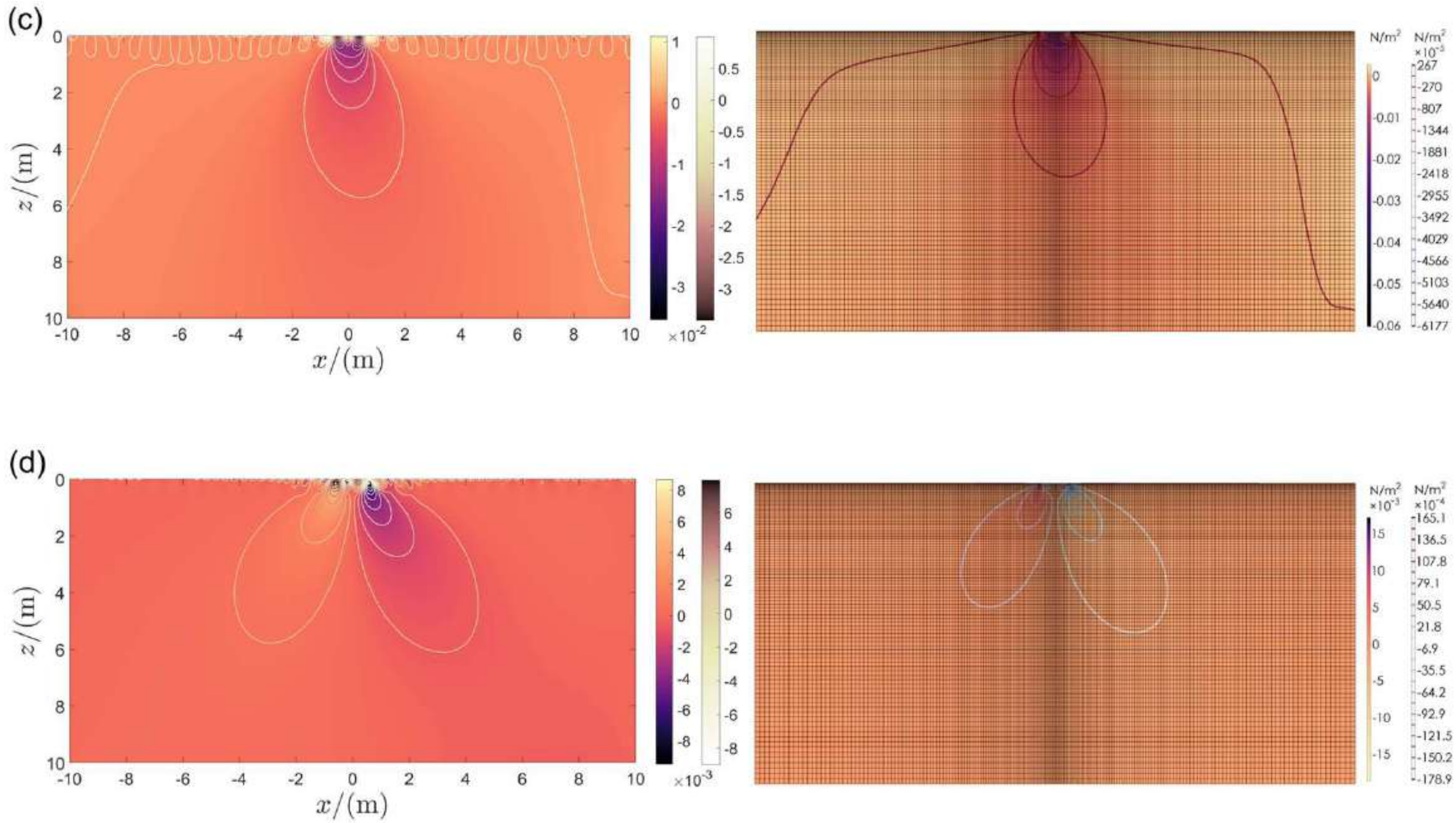


FIGURE 12. Comparison between FEA results and analytical solutions

Numerical results

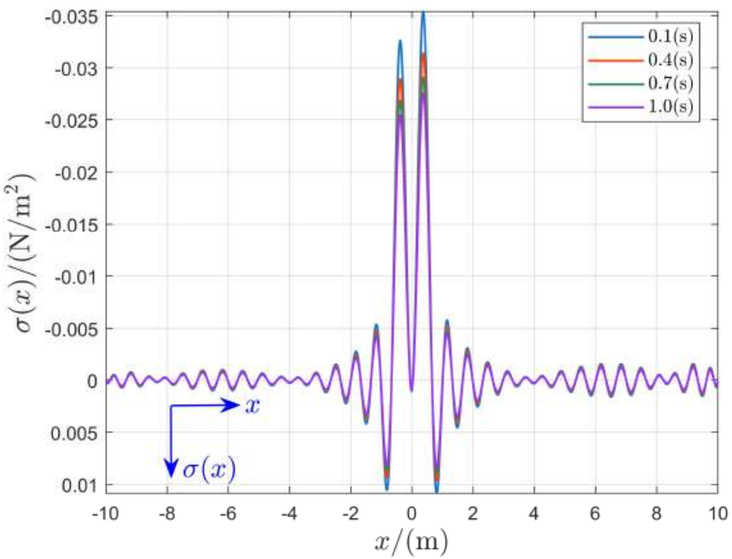


Figure 13. Stress relaxation

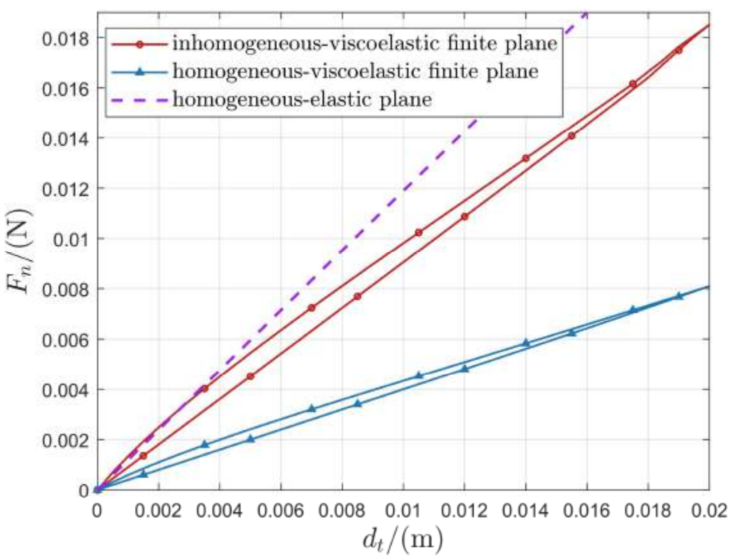


Figure 14. Indentation curve

Numerical results

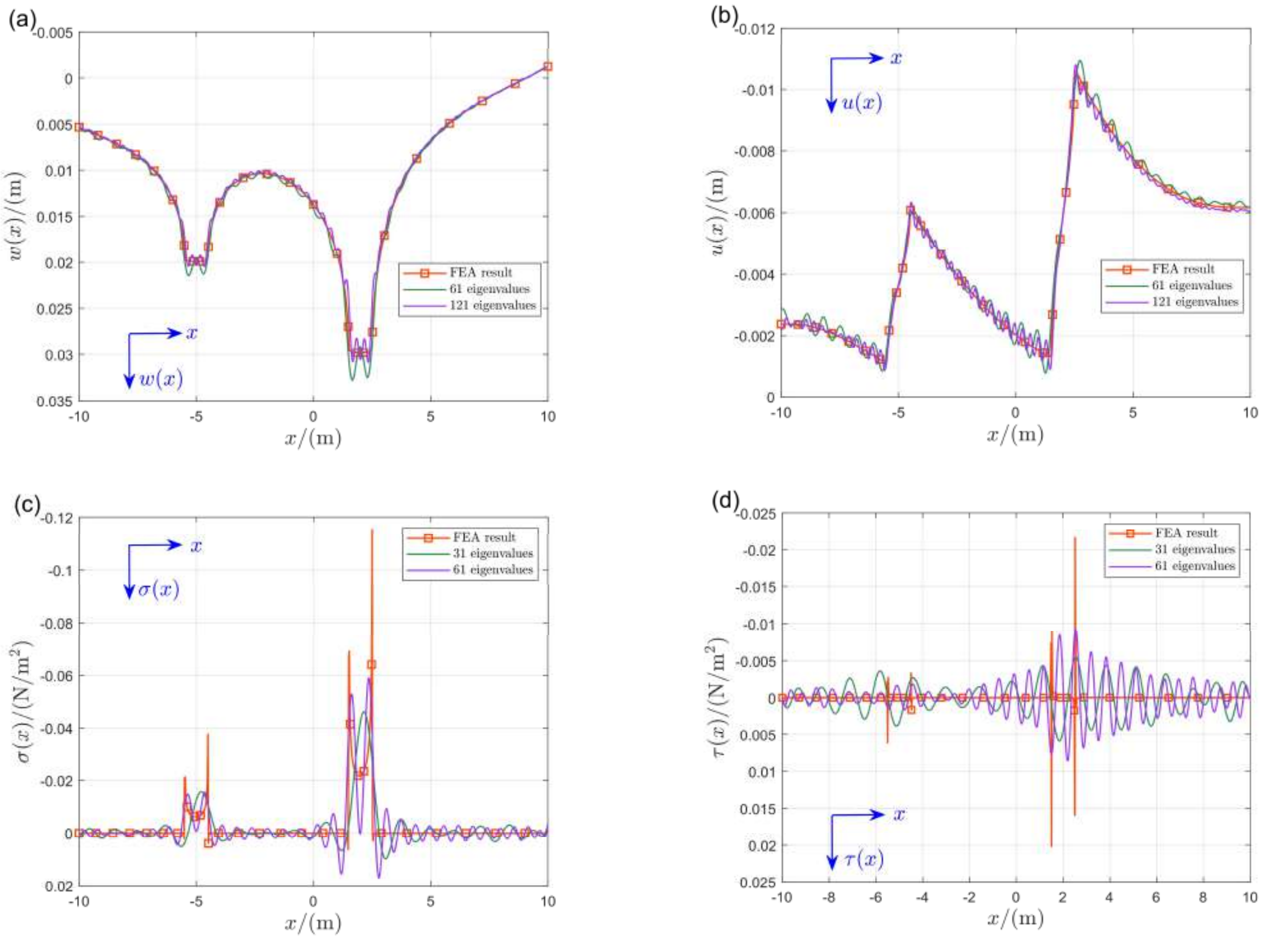


FIGURE 15. Case of multi-indenter

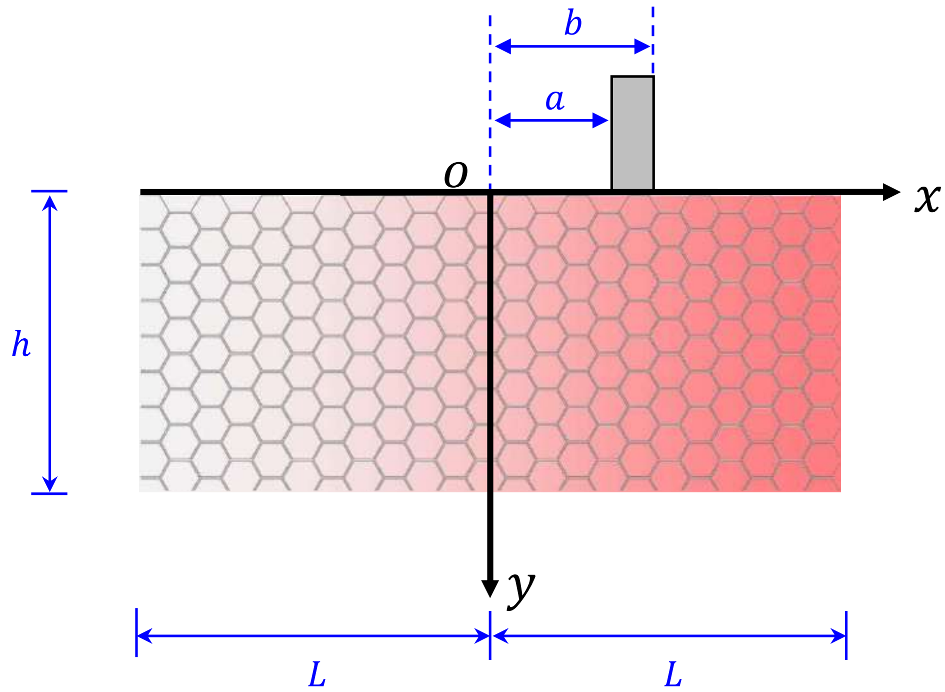


Figure 16. Horizontally graded plane

Couple Stress Model

Basic formulation

$$\frac{\partial}{\partial y}\begin{Bmatrix} u_x \\ u_y \\ \omega_z \\ \hat{\sigma}_{yx} \\ \hat{\sigma}_{yy} \\ \hat{m}_{yz} \end{Bmatrix} = \left[\begin{array}{ccc:ccc} 0 & \frac{\partial}{\partial x} & -2 & 0 & 0 & 0 \\ -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{(1+\nu)(1-2\nu)}{E_0(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\nu}{2E_0\ell^2} \\ \hdashline -\frac{E_0}{1-\nu^2}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial x}+\beta\right) & 0 \\ 0 & -\frac{2E_0}{1+\nu}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right) & \frac{2E_0}{1+\nu}\left(\frac{\partial}{\partial x}+\beta\right) & \left(\frac{\partial}{\partial x}+\beta\right) & 0 & 0 \\ 0 & -\frac{2E_0}{1+\nu}\frac{\partial}{\partial x} & -\frac{2E_0\ell^2}{1+\nu}\left(\frac{\partial^2}{\partial x^2}+\beta\frac{\partial}{\partial x}\right)+\frac{2E_0}{1+\nu} & 2 & 0 & 0 \end{array}\right]\begin{Bmatrix} u_x \\ u_y \\ \omega_z \\ \hat{\sigma}_{yx} \\ \hat{\sigma}_{yy} \\ \hat{m}_{yz} \end{Bmatrix}$$

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c:c} \mathbf{A} & \mathbf{B} \\ \hdashline \mathbf{C} & \mathbf{D} \end{array}\right] \qquad \mathbf{A} = -\lim_{\beta\to 0}\mathbf{D}^{\mathcal{T}}, \mathbf{B} = \mathbf{B}^{\mathcal{T}}, \lim_{\beta\to 0}\mathbf{C} = \lim_{\beta\to 0}\mathbf{C}^{\mathcal{T}}$$

Couple Stress Model

Eigenvectors

$$\boxed{\left\langle \boldsymbol{f}_1, \mathcal{H}\boldsymbol{f}_2 \right\rangle = \left\langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \right\rangle - \beta \left\langle \boldsymbol{f}_1^*, \boldsymbol{f}_2^* \right\rangle}$$

$$\begin{cases} \dfrac{\mathrm{d}\,u_y(x)}{\mathrm{d}\,x} - 2\omega_z(x) = \mu u_x(x) \\ -\dfrac{\nu}{1-\nu}\dfrac{\mathrm{d}\,u_x(x)}{\mathrm{d}\,x} + \dfrac{(1+\nu)(1-2\nu)}{E_0(1-\nu)}\hat{\sigma}_{yy}(x) = \mu u_y(x) \\ \dfrac{1+\nu}{2E_0\ell^2}\hat{m}_{yz}(x) = \mu\omega_z(x) \\ -\dfrac{E_0}{1-\nu^2}\dfrac{\mathrm{d}^2 u_x(x)}{\mathrm{d}x^2} - \dfrac{\nu}{1-\nu}\dfrac{\mathrm{d}\hat{\sigma}_{yy}(x)}{\mathrm{d}x} - \beta\left(\dfrac{E_0}{1-\nu^2}\dfrac{\mathrm{d}u_x(x)}{\mathrm{d}x} + \dfrac{\nu}{1-\nu}\hat{\sigma}_{yy}(x)\right) = \mu\hat{\sigma}_{yx}(x) \\ -\dfrac{2E_0}{1+\nu}\left(\dfrac{\mathrm{d}^2 u_y(x)}{\mathrm{d}x^2} - \dfrac{\mathrm{d}\omega_z(x)}{\mathrm{d}x}\right) + \dfrac{\mathrm{d}\hat{\sigma}_{yx}(x)}{\mathrm{d}x} - \beta\left[\dfrac{2E_0}{1+\nu}\left(\dfrac{\mathrm{d}u_y(x)}{\mathrm{d}x} - \omega_z(x)\right) - \hat{\sigma}_{yx}(x)\right] = \mu\hat{\sigma}_{yy}(x) \\ -\dfrac{2E_0\ell^2}{1+\nu}\dfrac{\mathrm{d}^2\omega_z(x)}{\mathrm{d}x^2} - \beta\dfrac{2E_0\ell^2}{1+\nu}\dfrac{\mathrm{d}\omega_z(x)}{\mathrm{d}x} + \dfrac{2E_0}{1+\nu}\omega_z(x) + 2\hat{\sigma}_{yx}(x) - \dfrac{2E_0}{1+\nu}\dfrac{\mathrm{d}u_y(x)}{\mathrm{d}x} = \mu\hat{m}_{yz}(x) \end{cases}$$

$$\Phi = \sum_{k=1}^{6} \mathrm{e}^{\eta_k x}\left[\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k, \mathcal{E}_k, \mathcal{F}_k\right]^{\mathrm{T}}$$

$$\eta^6 + 3\beta\eta^5 + \left[3\left(\beta^2 + \mu^2\right) - \frac{1}{\ell^2}\right]\eta^4 + \left[\left(\beta^2 + 6\mu^2\right) - \frac{2}{\ell^2}\right]\beta\eta^3 + \left[\beta^2\left(\mu^2\frac{3-2\nu}{1-\nu} - \frac{1}{\ell^2}\right) + \mu^2\left(3\mu^2 - \frac{2}{\ell^2}\right)\right]\eta^2$$
$$+\left(\frac{\beta^2\nu}{1-\nu} + 3\mu^2 - \frac{2}{\ell^2}\right)\beta\mu^2\eta + \frac{\beta^2\mu^2\nu + \mu^4\left(\beta^2\ell^2\nu + \nu - 1\right) + \ell^2\mu^6(1-\nu)}{\ell^2(1-\nu)} = 0$$

Couple Stress Model

Complete solutions

$$\begin{cases} \boldsymbol{\mathcal{H}}\Phi_n^{(0)} = \mu_n \Phi_n^{(0)} \\ \boldsymbol{\mathcal{H}}\Phi_n^{(r+1)} = \mu_n \Phi_n^{(r+1)} + \Phi_n^{(r)} \end{cases} \quad \text{or} \quad \left(\boldsymbol{\mathcal{H}} - \mu_n \mathbf{I}_6\right)^{r+1} \Phi_n^{(r)} = 0$$

$$(r = 0, 1, \cdots, N_n - 1)$$

$$\begin{aligned} \tilde{\boldsymbol{f}} &= \boldsymbol{\mathcal{M}}\boldsymbol{f} \\ &= \sum_{n=1}^{6} \gamma_{0,n} \tilde{\boldsymbol{f}}_{0,n} + \sum_{n=1}^{\infty} \sum_{i=0}^{Nn} \left(\gamma_{\mu,n}^{\mathrm{Re},i} \mathrm{Re}\tilde{\boldsymbol{f}}_{\mu,n}^{(i)} + \gamma_{\mu,n}^{\mathrm{Im},i} \mathrm{Im}\tilde{\boldsymbol{f}}_{\mu,n}^{(i)} + \gamma_{-\mu,n}^{\mathrm{Re},i} \mathrm{Re}\tilde{\boldsymbol{f}}_{-\mu,n}^{(i)} + \gamma_{-\mu,n}^{\mathrm{Im},i} \mathrm{Im}\tilde{\boldsymbol{f}}_{-\mu,n}^{(i)} \right) \\ &\equiv \sum_{n=1}^{\infty} \gamma_n \tilde{\boldsymbol{f}}_n \end{aligned}$$

$$\boxed{\boldsymbol{f}_{\mu,n}^{(i)} = \mathrm{e}^{\mu_n y} \left(\Phi_n^{(i)} + y\Phi_n^{(i-1)} + \cdots + \frac{y^i}{i!} \Phi_n^{(0)} \right)}$$

$$y = h, \quad \begin{cases} u_x = 0 \\ u_y = 0 \\ \omega_z = 0 \end{cases}; \quad y = 0, \quad \begin{cases} u_y = d & x \in [a, b] \\ u_x = 0 & x \in [a, b] \\ \omega_z = 0 & x \in [a, b] \\ \sigma_{yy} = 0 & x \in [-L, a] \cup [b, L] \\ \sigma_{yx} = 0 & x \in [-L, a] \cup [b, L] \\ m_{yz} = 0 & x \in [-L, a] \cup [b, L] \end{cases}$$

No-slip indentation

Couple Stress Model

Local phase transition

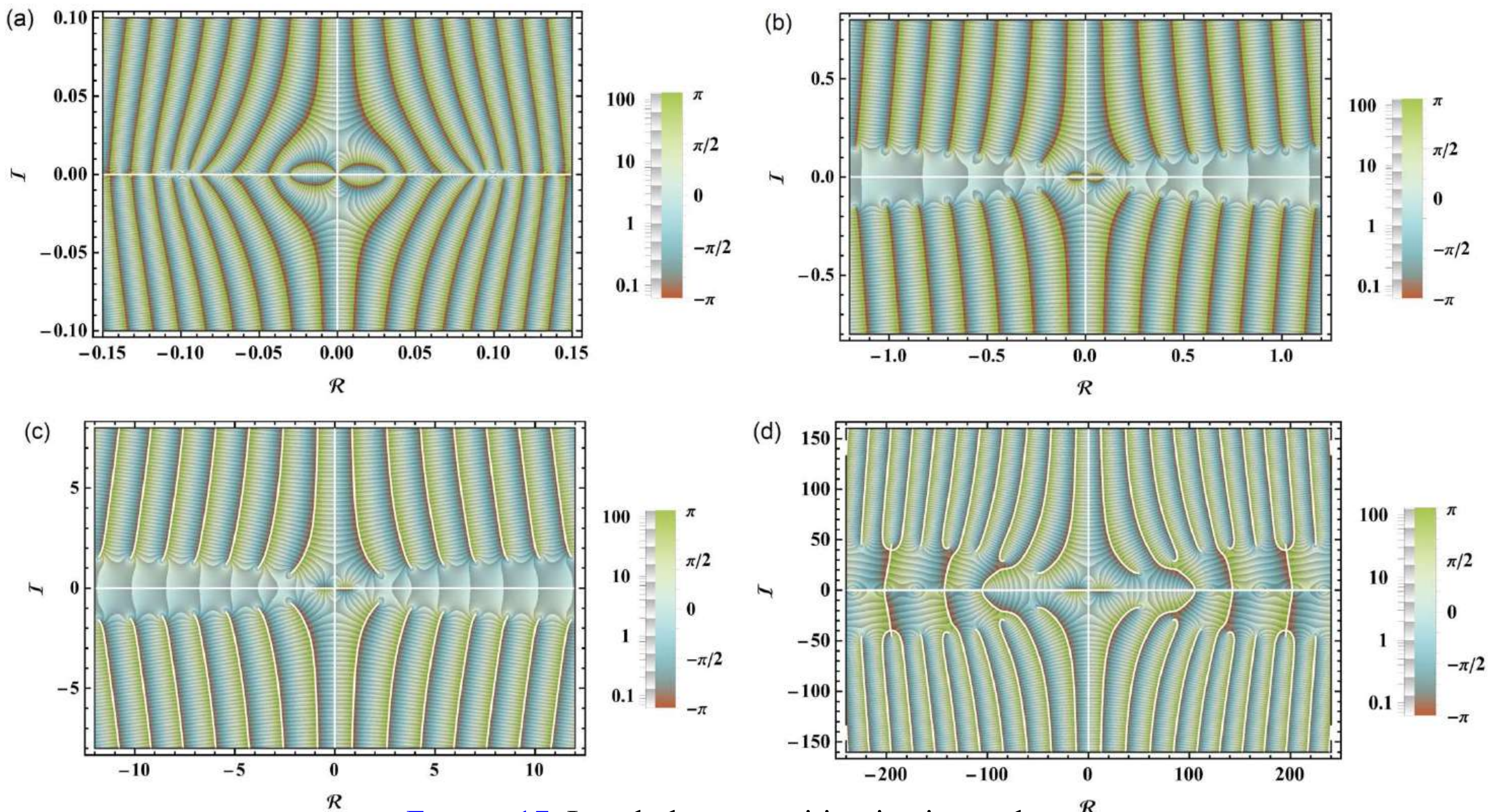


Figure 17. Local phase transition in eigenvalues.

Results

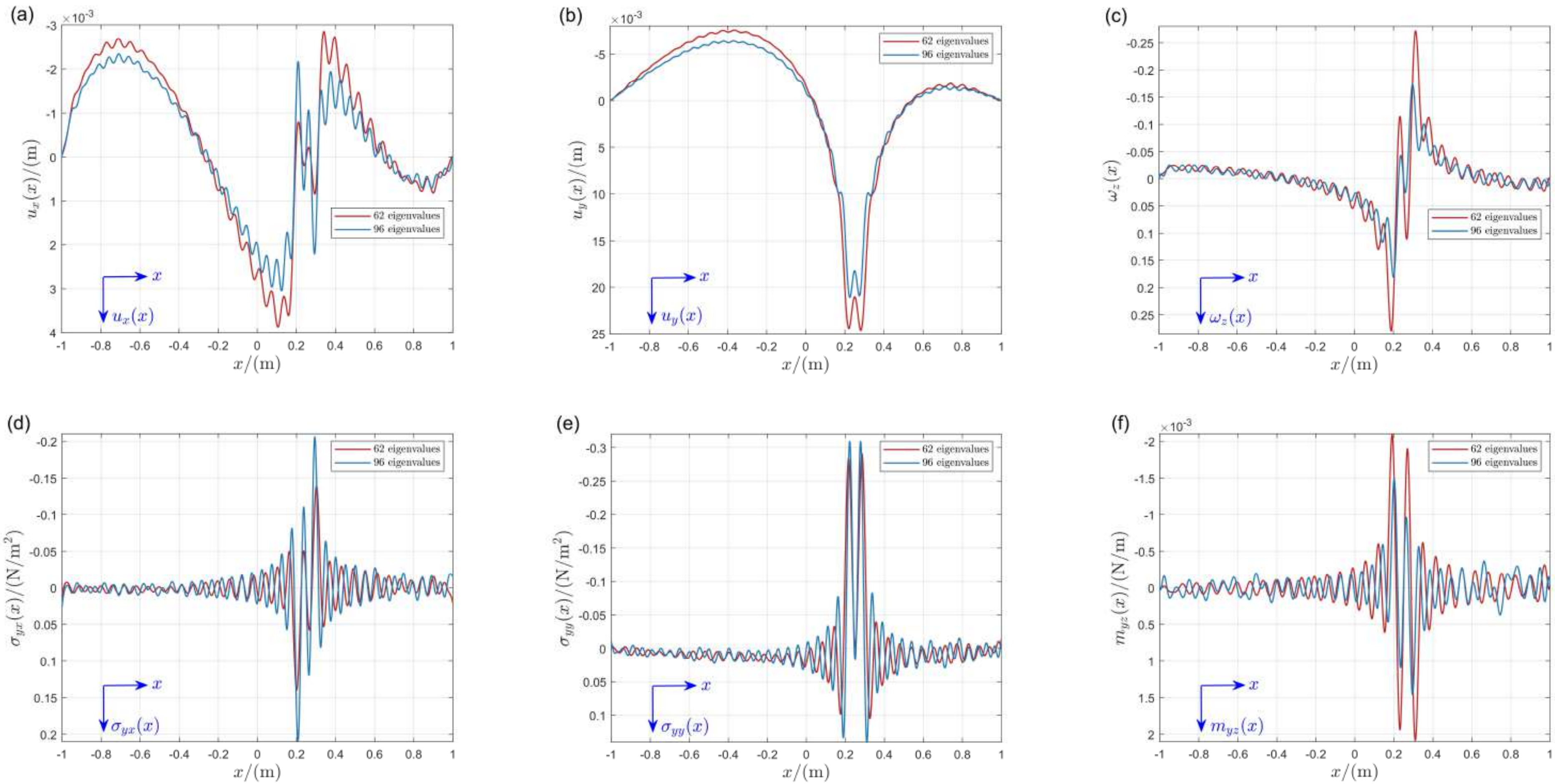


Figure 18. Examples of symplectic solutions for displacements and stresses.

Couple Stress Model

Commutator

If we assume $N_n = 2$

$$\begin{cases} \mathcal{H}\Phi_n^{(0)} = \mu_n \Phi_n^{(0)} \\ \mathcal{H}\Phi_n^{(1)} = \mu_n \Phi_n^{(1)} + \Phi_n^{(0)} \\ \mathcal{H}\Phi_n^{(2)} = \mu_n \Phi_n^{(2)} + \Phi_n^{(1)} \end{cases}$$

$$\begin{cases} \Phi_n^{(1)} = \boldsymbol{Q}_n^{(1)} \Phi_n^{(0)} \\ \Phi_n^{(2)} = \boldsymbol{Q}_n^{(2)} \Phi_n^{(1)} \end{cases}$$

Evolution metrices

$$\Downarrow$$

$$\begin{cases} \boldsymbol{Q}_n^{(1)} \mathcal{H}\Phi_n^{(0)} = \mu_n \boldsymbol{Q}_n^{(1)} \Phi_n^{(0)} \\ \mathcal{H}\boldsymbol{Q}_n^{(1)} \Phi_n^{(0)} = \mu_n \boldsymbol{Q}_n^{(1)} \Phi_n^{(0)} + \Phi_n^{(0)} \end{cases}$$

$$\boxed{\left(\left[\!\left[\mathcal{H}, \boldsymbol{Q}_n^{(1)} \right]\!\right] - \mathbf{I}_6\right)\Phi_n^{(0)} = 0}$$

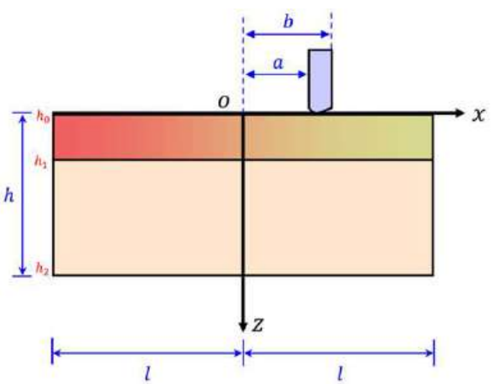
$$\boxed{\left[\!\left[\left(\mathcal{H} - \mu_n \mathbf{I}_6\right), \boldsymbol{Q}_n^{(2)} \right]\!\right]\Phi_n^{(1)} = \left(\boldsymbol{Q}_n^{(1)} - \boldsymbol{Q}_n^{(2)}\right)\Phi_n^{(0)}}$$

where $\left[\!\left[\mathcal{H}, \boldsymbol{Q}_n^{(1)} \right]\!\right] = \mathcal{H}\boldsymbol{Q}_n^{(1)} - \boldsymbol{Q}_n^{(1)}\mathcal{H}$ is a commutator (对易子)

Arbitrary Gradient

Film-substrate system & variable coefficients

$$\boldsymbol{\mathcal{H}}_k^x = \left[\begin{array}{c:c} \mathbf{A}_k^x & \mathbf{B}_k^x \\ \hdashline \mathbf{D}_k^{\vartheta_x} & -\left(\mathbf{A}_k^{\vartheta_x}\right)^{\mathcal{T}} \end{array}\right]$$



$$\left\langle \boldsymbol{f}_k^\alpha, \boldsymbol{\mathcal{H}}_k^x \boldsymbol{f}_k^\beta \right\rangle = \left\langle \boldsymbol{f}_k^\beta, \boldsymbol{\mathcal{H}}_k^x \boldsymbol{f}_k^\alpha \right\rangle$$

$$\mathbf{A}_k^x = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial x} & -a_{1;k}(x)\dfrac{\partial}{\partial x} & -a_{2;k}(x)\dfrac{\partial}{\partial x} \\ -a_{4;k}(x)\dfrac{\partial}{\partial x} & 0 & 0 & 0 \\ a_{8;k}(x)\dfrac{\partial}{\partial x} & 0 & 0 & 0 \\ a_{11;k}(x)\dfrac{\partial}{\partial x} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_k^x = \begin{bmatrix} a_{3;k}(x) & 0 & 0 & 0 \\ 0 & a_{5;k}(x) & a_{6;k}(x) & a_{7;k}(x) \\ 0 & a_{6;k}(x) & -a_{9;k}(x) & a_{10;k}(x) \\ 0 & a_{7;k}(x) & a_{10;k}(x) & -a_{12;k}(x) \end{bmatrix}$$

$$\mathbf{D}_k^{\vartheta_x} = \begin{bmatrix} a_{13;k}(x)\dfrac{\partial^2}{\partial x^2} + \dfrac{\mathrm{d}a_{13;k}(x)}{\mathrm{d}x}\dfrac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{14;k}(x)\dfrac{\partial^2}{\partial x^2} + \dfrac{\mathrm{d}a_{14;k}(x)}{\mathrm{d}x}\dfrac{\partial}{\partial x} & a_{15;k}(x)\dfrac{\partial^2}{\partial x^2} + \dfrac{\mathrm{d}a_{15;k}(x)}{\mathrm{d}x}\dfrac{\partial}{\partial x} \\ 0 & 0 & a_{15;k}(x)\dfrac{\partial^2}{\partial x^2} + \dfrac{\mathrm{d}a_{15;k}(x)}{\mathrm{d}x}\dfrac{\partial}{\partial x} & a_{16;k}(x)\dfrac{\partial^2}{\partial x^2} + \dfrac{\mathrm{d}a_{16;k}(x)}{\mathrm{d}x}\dfrac{\partial}{\partial x} \end{bmatrix}$$

$$-\left(\mathbf{A}_k^{\vartheta_k}\right)^{\mathcal{T}} = \begin{bmatrix} 0 & -a_{4;k}(x)\dfrac{\partial}{\partial x} - \dfrac{\mathrm{d}a_{4;k}(x)}{\mathrm{d}x} & a_{8;k}(x)\dfrac{\partial}{\partial x} + \dfrac{\mathrm{d}a_{8;k}(x)}{\mathrm{d}x} & a_{11;k}(x)\dfrac{\partial}{\partial x} + \dfrac{\mathrm{d}a_{11;k}(x)}{\mathrm{d}x} \\ -\dfrac{\partial}{\partial x} & 0 & 0 & 0 \\ -a_{1;k}(x)\dfrac{\partial}{\partial x} - \dfrac{\mathrm{d}a_{1;k}(x)}{\mathrm{d}x} & 0 & 0 & 0 \\ -a_{2;k}(x)\dfrac{\partial}{\partial x} - \dfrac{\mathrm{d}a_{2;k}(x)}{\mathrm{d}x} & 0 & 0 & 0 \end{bmatrix}$$

According to Weierstrass approximation theorem, the coefficient functions can be uniformly approximated on $x\in[-l,l]$ by polynomials to any degree of accuracy.

$$\vartheta_{ij;k}(x)=\sum_{r=0}^{\infty}\vartheta_{ij;k}^{(r)}x^r \quad (k=1,2)$$

Positive valued assumption: concerning the properties of the materials, we may further assume that no zero of the coefficient functions exists (i.e., coefficient functions remain positive or negative), which indicates the absence of the singularities in that interval.

$$\det\begin{bmatrix} a_{5;k}(0) & a_{6;k}(0) & a_{7;k}(0) \\ a_{6;k}(0) & -a_{9;k}(0) & a_{10;k}(0) \\ a_{7;k}(0) & a_{10;k}(0) & -a_{12;k}(0) \end{bmatrix}\neq 0, \quad \frac{c_{13;k}(0)}{c_{11;k}(0)}\begin{Bmatrix} \frac{c_{11;k}(0)}{c_{13;k}(0)}-a_{4;k}(0) \\ a_{8;k}(0) \\ a_{11;k}(0) \end{Bmatrix}\neq\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Arbitrary Gradient

General eigenvalues & eigen-solutions

$$u_1 = \sum_{r=0}^{\infty} \tilde{A}_1^{(r)} x^r, \qquad w_1 = \sum_{r=0}^{\infty} \tilde{B}_1^{(r)} x^r, \qquad \phi_1 = \sum_{r=0}^{\infty} \tilde{C}_1^{(r)} x^r, \qquad \psi_1 = \sum_{r=0}^{\infty} \tilde{D}_1^{(r)} x^r$$

$$\int \tau_1 \mathrm{d}x = \sum_{r=0}^{\infty} \tilde{E}_1^{(r)} x^r, \quad \int \sigma_1 \mathrm{d}x = \sum_{r=0}^{\infty} \tilde{F}_1^{(r)} x^r, \quad \int D_1 \mathrm{d}x = \sum_{r=0}^{\infty} \tilde{G}_1^{(r)} x^r, \quad \int B_1 \mathrm{d}x = \sum_{r=0}^{\infty} \tilde{H}_1^{(r)} x^r$$

$$\begin{cases}
(r+1)\tilde{A}_1^{(r+1)} = -\mu_1 \left(\sum_{s=0}^{r} a_{17;1}^{(s)} \tilde{B}_1^{(r-s)} + \sum_{s=0}^{r} a_{18;1}^{(s)} \tilde{C}_1^{(r-s)} + \sum_{s=0}^{r} a_{19;1}^{(s)} \tilde{D}_1^{(r-s)} + \sum_{s=0}^{r} a_{20;1}^{(s)} \tilde{E}_1^{(r-s)} \right) \\
(r+1)\tilde{B}_1^{(r+1)} = -\mu_1 \tilde{A}_1^{(r)} + \sum_{s=0}^{r} (r+1-s)\rho_{15;1}^{(s)} \tilde{E}_1^{(r+1-s)} + \mu_1 \sum_{s=0}^{r} \rho_{16;1}^{(s)} \tilde{G}_1^{(r-s)} + \mu_1 \sum_{s=0}^{r} \rho_{17;1}^{(s)} \tilde{H}_1^{(r-s)} \\
(r+1)\tilde{C}_1^{(r+1)} = \sum_{s=0}^{r} (r+1-s)\rho_{10;1}^{(s)} \tilde{E}_1^{(r+1-s)} + \mu_1 \sum_{s=0}^{r} \rho_{14;1}^{(s)} \tilde{G}_1^{(r-s)} - \mu_1 \sum_{s=0}^{r} \rho_{13;1}^{(s)} \tilde{H}_1^{(r-s)} \\
(r+1)\tilde{D}_1^{(r+1)} = \sum_{s=0}^{r} (r+1-s)\rho_{11;1}^{(s)} \tilde{E}_1^{(r+1-s)} + \mu_1 \sum_{s=0}^{r} \rho_{12;1}^{(s)} \tilde{H}_1^{(r-s)} - \mu_1 \sum_{s=0}^{r} \rho_{13;1}^{(s)} \tilde{G}_1^{(r-s)} \\
(r+1)\tilde{E}_1^{(r+1)} = -\mu_1 \tilde{F}_1^{(r)} \\
(r+1)\tilde{F}_1^{(r+1)} = \mu_1 \left(\sum_{s=0}^{r} \rho_{18;1}^{(s)} \tilde{B}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{19;1}^{(s)} \tilde{C}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{20;1}^{(s)} \tilde{D}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{21;1}^{(s)} \tilde{E}_1^{(r-s)} \right) \\
(r+1)\tilde{G}_1^{(r+1)} = \mu_1 \left(\sum_{s=0}^{r} \rho_{2;1}^{(s)} \tilde{B}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{5;1}^{(s)} \tilde{C}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{8;1}^{(s)} \tilde{D}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{22;1}^{(s)} \tilde{E}_1^{(r-s)} \right) \\
(r+1)\tilde{H}_1^{(r+1)} = \mu_1 \left(\sum_{s=0}^{r} \rho_{3;1}^{(s)} \tilde{B}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{6;1}^{(s)} \tilde{C}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{9;1}^{(s)} \tilde{D}_1^{(r-s)} + \sum_{s=0}^{r} \rho_{23;1}^{(s)} \tilde{E}_1^{(r-s)} \right)
\end{cases}$$

$(r \geq 0)$

Arbitrary Gradient

Film-substrate system

$$\boldsymbol{f}_k = \sum_{i=1}^{10} m_{0,i;k}\boldsymbol{f}_{0,i;k} + \sum_{i=1}^{\infty}\Big[(m^{\mathrm{Re}}_{\mu,i;k}\mathrm{Re}\boldsymbol{f}_{\mu,i;k} + m^{\mathrm{Im}}_{\mu,i;k}\mathrm{Im}\boldsymbol{f}_{\mu,i;k}) + (m^{\mathrm{Re}}_{-\mu,i;k}\mathrm{Re}\boldsymbol{f}_{-\mu,i;k} + m^{\mathrm{Im}}_{-\mu,i;k}\mathrm{Im}\boldsymbol{f}_{-\mu,i;k})\Big]$$

$$\triangleq \sum_{i=1}^{\infty} m_{i;k}\boldsymbol{f}_{i;k}$$

$$\boldsymbol{f}_1(x,h_1) = \boldsymbol{f}_2(x,h_1)$$

$$m_{i;2} = \frac{1}{\left\langle \Phi_{i;2},\Phi_{-i;2}\right\rangle}\sum_{j=1}^{\infty} m_{j;1}\mathrm{e}^{(\mu_{j;1}-\mu_{i;2})h_1}\left\langle \Phi_{j;1},\Phi_{-i;2}\right\rangle$$

$$\triangleq \sum_{j=1}^{\infty} m_{j;1}\lambda_{ij}$$

$$\lambda_{ij} = \mathrm{e}^{(\mu_{j;1}-\mu_{i;2})h_1}\frac{\left\langle \Phi_{j;1},\Phi_{-i;2}\right\rangle}{\left\langle \Phi_{i;2},\Phi_{-i;2}\right\rangle}$$

$$\boldsymbol{f}_2 = \sum_{i=1}^{\infty} m_{i;2}\boldsymbol{f}_{i;2} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} m_{j;1}\lambda_{ij}\boldsymbol{f}_{i;2} = \sum_{j=1}^{\infty} m_{j;1}\sum_{i=1}^{\infty}\lambda_{ij}\boldsymbol{f}_{i;2} \triangleq \sum_{j=1}^{\infty} m_{j;1}\tilde{\boldsymbol{f}}_{j;2}$$

Arbitrary Gradient

Generalized dual Hamiltonian transformation

$$\left\langle \boldsymbol{f}^{\alpha}, \boldsymbol{\mathcal{H}}^{x} \boldsymbol{f}^{\beta} \right\rangle = \left\langle \boldsymbol{f}^{\beta}, \boldsymbol{\mathcal{H}}^{x} \boldsymbol{f}^{\alpha} \right\rangle - \left\langle \sqrt{\frac{E'(x)}{E(x)}} \left(\boldsymbol{f}^{\alpha}\right)^{*}, \sqrt{\frac{E'(x)}{E(x)}} \left(\boldsymbol{f}^{\beta}\right)^{*} \right\rangle$$

$$\begin{aligned}
\left\langle \sqrt{\frac{E'(x)}{E(x)}} \left(\boldsymbol{f}^{\alpha}\right)^{*}, \sqrt{\frac{E'(x)}{E(x)}} \left(\boldsymbol{f}^{\beta}\right)^{*} \right\rangle &= \int_{-l}^{l} \left[\left(\boldsymbol{f}^{\alpha}\right)^{*}\right]^{\mathrm{T}} \mathbf{J} \left(\boldsymbol{f}^{\beta}\right)^{*} \frac{E'(x)}{E(x)} \mathrm{d}x \\
&= \left\langle \sqrt{\frac{E'(x)}{E(x)}} \boldsymbol{\mathcal{A}} \boldsymbol{f}^{\alpha}, \sqrt{\frac{E'(x)}{E(x)}} \boldsymbol{\mathcal{A}} \boldsymbol{f}^{\beta} \right\rangle \\
&\triangleq \left\langle \boldsymbol{\mathcal{A}}^{\dagger} \boldsymbol{f}^{\alpha}, \boldsymbol{\mathcal{A}}^{\dagger} \boldsymbol{f}^{\beta} \right\rangle \\
&= \int_{-l}^{l} \left(\boldsymbol{\mathcal{A}}^{\dagger} \boldsymbol{f}^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \left(\boldsymbol{\mathcal{A}}^{\dagger} \boldsymbol{f}^{\beta}\right) \mathrm{d}x \\
&= \int_{-l}^{l} \left(\boldsymbol{f}^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \mathbf{J}^{\mathrm{T}} \left(\boldsymbol{\mathcal{A}}^{\dagger}\right)^{\mathcal{T}} \mathbf{J} \boldsymbol{\mathcal{A}}^{\dagger} \boldsymbol{f}^{\beta} \mathrm{d}x \\
&= \int_{-l}^{l} \left(\boldsymbol{f}^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \left(\boldsymbol{\mathcal{A}}^{\dagger} \mathbf{J}\right)^{\mathrm{T}} \left(\mathbf{J} \boldsymbol{\mathcal{A}}^{\dagger}\right) \boldsymbol{f}^{\beta} \mathrm{d}x \\
&\triangleq \int_{-l}^{l} \left(\boldsymbol{f}^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \boldsymbol{\mathcal{B}}^{\dagger} \boldsymbol{f}^{\beta} \mathrm{d}x
\end{aligned}$$

$$\left\langle \boldsymbol{f}^{\alpha}, (\boldsymbol{\mathcal{H}}^{x} + \boldsymbol{\mathcal{B}}^{\dagger}) \boldsymbol{f}^{\beta} \right\rangle = \left\langle \boldsymbol{f}^{\beta}, \boldsymbol{\mathcal{H}}^{x} \boldsymbol{f}^{\alpha} \right\rangle$$

Arbitrary Gradient

Category (范畴)

$$\frac{\mathrm{d}\,\boldsymbol{Y}(x)}{\mathrm{d}x} = \boldsymbol{\mathcal{A}}(x)\boldsymbol{Y}(x)$$

$$\boldsymbol{\mathcal{A}}(x)\cdot\int_{x_0}^{x}\boldsymbol{\mathcal{A}}(x)\mathrm{d}x = \int_{x_0}^{x}\boldsymbol{\mathcal{A}}(x)\mathrm{d}x\cdot\boldsymbol{\mathcal{A}}(x)$$

$$\begin{tikzcd}
\langle u_i, \phi, \psi, T\rangle \arrow[r, "f"] \arrow[dr, "g\circ f"'] & \langle \sigma_{ij}, D_i, B_i, Q_i\rangle \arrow[d, "g"] \\
 & \langle \mathbf{0}\rangle
\end{tikzcd}$$

(1) The element $\mathcal{X}$ in a set $\mathrm{Ob}(\mathscr{C})$ is defined as an object of $\mathscr{C}$, where $\mathrm{Ob}(\mathscr{C}) = \{\langle u_i, \phi, \psi, T\rangle, \langle \sigma_{ij}, D_i, B_i, Q_i\rangle, \langle \mathbf{0}\rangle\}$, and $\langle\cdot\rangle$ stands for ordered index set, elements of which are in order by index.

(2) For every two objects $\mathcal{X}, \mathcal{Y} \in \mathscr{C}$, there is a set $\mathrm{Hom}_{\mathscr{C}}(\mathcal{X}, \mathcal{Y})$, in which the elements are defined morphisms from $\mathcal{X}$ to $\mathcal{Y}$, let $f : \mathcal{X} \to \mathcal{Y}$ represent a morphism from $\mathcal{X}$ to $\mathcal{Y}$. f is a differential operator containing material constants, and g is a divergence operator.

(3) For every three objects $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{C}$, there is a mapping $\circ$: $\mathrm{Hom}_{\mathscr{C}}(\mathcal{Y}, \mathcal{Z}) \times \mathrm{Hom}_{\mathscr{C}}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Hom}_{\mathscr{C}}(\mathcal{X}, \mathcal{Z})$.

Arbitrary Gradient

Quasi-Hamiltonian operator (variable coefficients)

$$\boldsymbol{\mathcal{H}}^{x}=\left[\begin{array}{c:c}\mathbf{A}^{x} & \mathbf{B}^{x}\\ \hdashline \mathbf{D}^{\vartheta_x} & -\left(\mathbf{A}^{\vartheta_x}\right)^{\mathcal{T}}\end{array}\right]\quad \text{variable coefficients}$$

1. Magnus expansion
2. Frobenius method

$$\boldsymbol{Y}(x)=\exp(\Omega(x,x_0))\,\boldsymbol{Y}_0$$

$$\Omega_n(x)=\sum_{j=1}^{n-1}\frac{B_j}{j!}\sum_{\substack{k_1+\cdots+k_j=n-1\\ k_1\geq 1,\cdots k_j\geq 1}}\int_0^x \mathrm{ad}_{\Omega_{k_1}(X)}\mathrm{ad}_{\Omega_{k_2}(X)}\cdots\mathrm{ad}_{\Omega_{k_j}(X)}\boldsymbol{\mathcal{A}}(X)\mathrm{d}X$$

adjoint endomorphism (伴随自同态)

Arbitrary Gradient

Contact region determination

Shape function $R-\sqrt{R^2-(x-x_0)^2}$ Contact region $x\in[x_0-l_1, x_0+l_2]$

JKR Model

$$\boxed{d_{ac}=d_h-d_{ad}}$$

$$\tilde{\boldsymbol{f}}(x,z;l_1,l_2)=\sum_{i=1}^{\infty}m_i(l_1,l_2)\tilde{\boldsymbol{f}}_i(x,z)=\sum_{i=1}^{\infty}m_i(l_1,l_2)\boldsymbol{\mathcal{M}}\boldsymbol{f}_i(x,z)$$

$$\begin{aligned}U&=\int_{-l}^{l}\int_0^h\frac{1}{2}(\sigma_{xx}\varepsilon_x+\sigma_{zz}\varepsilon_z+\sigma_{xz}\gamma_{xz})\mathrm{d}z\mathrm{d}x\\&=\int_{-l}^{l}\int_0^h\frac{1}{2}\left[\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{|2}}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{|3}\right)\frac{\partial\tilde{\boldsymbol{f}}_{|2}}{\partial x}+\tilde{\boldsymbol{f}}_{|3}\frac{\partial\tilde{\boldsymbol{f}}_{|1}}{\partial z}+\tilde{\boldsymbol{f}}_{|4}\left(\frac{\partial\tilde{\boldsymbol{f}}_{|2}}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{|1}}{\partial x}\right)\right]\mathrm{d}z\mathrm{d}x\end{aligned}$$

$$\frac{\partial U}{\partial l_1}=\Delta\gamma,\quad\frac{\partial U}{\partial l_2}=\Delta\gamma$$

Arbitrary Gradient

Contact region determination

$$
\begin{aligned}
\frac{\partial U}{\partial l_1} &= \sum_{s=1}^{\infty} \frac{\partial U}{\partial m_s} \frac{\partial m_s}{\partial l_1} \\
&= \sum_{s=1}^{\infty} \frac{\partial \left\{ \int_{-l}^{l} \int_{0}^{h} \frac{1}{2} \left[\left(E(x) \sum_{i=1}^{\infty} m_i \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} + \nu_0 \sum_{i=1}^{\infty} m_i \tilde{\boldsymbol{f}}_{i|3} \right) \left(\sum_{i=1}^{\infty} m_i \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} \right) + \left(\sum_{i=1}^{\infty} m_i \tilde{\boldsymbol{f}}_{i|3} \right) \left(\sum_{i=1}^{\infty} m_i \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial z} \right) + \left(\sum_{i=1}^{\infty} m_i \tilde{\boldsymbol{f}}_{i|4} \right) \left(\sum_{i=1}^{\infty} m_i \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial z} + \sum_{i=1}^{\infty} m_i \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial x} \right) \right] \mathrm{d}z \mathrm{d}x \right\}}{\partial m_s} \frac{\partial m_s}{\partial l_1} \\
&= \frac{1}{2} \sum_{s=1}^{\infty} \frac{\partial \left\{ \int_{-l}^{l} \int_{0}^{h} \left[\sum_{i=1}^{\infty} \left(E(x) \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} + \nu_0 \tilde{\boldsymbol{f}}_{i|3} \right) m_i \cdot \sum_{i=1}^{\infty} \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} m_i + \sum_{i=1}^{\infty} \tilde{\boldsymbol{f}}_{i|3} m_i \cdot \sum_{i=1}^{\infty} \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial z} m_i + \sum_{i=1}^{\infty} \tilde{\boldsymbol{f}}_{i|4} m_i \cdot \sum_{i=1}^{\infty} \left(\frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial z} + \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial x} \right) m_i \right] \mathrm{d}z \mathrm{d}x \right\}}{\partial m_s} \frac{\partial m_s}{\partial l_1} \\
&= \frac{1}{2} \sum_{s=1}^{\infty} \left\{ \int_{-l}^{l} \int_{0}^{h} \left[\left(E(x) \frac{\partial \tilde{\boldsymbol{f}}_{s|2}}{\partial x} + \nu_0 \tilde{\boldsymbol{f}}_{s|3} \right) \cdot \sum_{i=1}^{\infty} \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} m_i + \frac{\partial \tilde{\boldsymbol{f}}_{s|2}}{\partial x} \cdot \sum_{i=1}^{\infty} \left(E(x) \frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial x} + \nu_0 \tilde{\boldsymbol{f}}_{i|3} \right) m_i + \tilde{\boldsymbol{f}}_{s|3} \cdot \sum_{i=1}^{\infty} \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial z} m_i + \frac{\partial \tilde{\boldsymbol{f}}_{s|1}}{\partial z} \cdot \sum_{i=1}^{\infty} \tilde{\boldsymbol{f}}_{i|3} m_i + \tilde{\boldsymbol{f}}_{s|4} \cdot \sum_{i=1}^{\infty} \left(\frac{\partial \tilde{\boldsymbol{f}}_{i|2}}{\partial z} + \frac{\partial \tilde{\boldsymbol{f}}_{i|1}}{\partial x} \right) m_i + \left(\frac{\partial \tilde{\boldsymbol{f}}_{s|2}}{\partial z} + \frac{\partial \tilde{\boldsymbol{f}}_{s|1}}{\partial x} \right) \cdot \sum_{i=1}^{\infty} \tilde{\boldsymbol{f}}_{i|4} m_i \right] \mathrm{d}z \mathrm{d}x \cdot \frac{\partial m_s}{\partial l_1} \right\} \\
&= \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} \varpi_{si} m_i \frac{\partial m_s}{\partial l_1}
\end{aligned}
$$

Arbitrary Gradient

Contact region determination

$$\varpi_{si}=\frac{1}{2}\int_{-l}^{l}\int_{0}^{h}\left[\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{s|2}}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{s|3}\right)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}}{\partial x}+\frac{\partial\tilde{\boldsymbol{f}}_{s|2}}{\partial x}\left(E(x)\frac{\partial\tilde{\boldsymbol{f}}_{i|2}}{\partial x}+\nu_0\tilde{\boldsymbol{f}}_{i|3}\right)+\tilde{\boldsymbol{f}}_{s|3}\frac{\partial\tilde{\boldsymbol{f}}_{i|1}}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}}{\partial z}\tilde{\boldsymbol{f}}_{i|3}+\tilde{\boldsymbol{f}}_{s|4}\left(\frac{\partial\tilde{\boldsymbol{f}}_{i|2}}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{i|1}}{\partial x}\right)+\left(\frac{\partial\tilde{\boldsymbol{f}}_{s|2}}{\partial z}+\frac{\partial\tilde{\boldsymbol{f}}_{s|1}}{\partial x}\right)\tilde{\boldsymbol{f}}_{i|4}\right]\mathrm{d}z\mathrm{d}x$$

$$\begin{aligned}\frac{\partial m_s}{\partial l_1}&=\frac{\partial\mathscr{A}_{si}^{-1}}{\partial l_1}\mathscr{H}_i+\mathscr{A}_{si}^{-1}\frac{\partial\mathscr{H}_i}{\partial l_1}\\&=-\mathscr{A}_{si}^{-1}\frac{\partial\mathscr{A}_{is}}{\partial l_1}\mathscr{A}_{si}^{-1}\mathscr{H}_i+\mathscr{A}_{si}^{-1}\frac{\partial\mathscr{H}_i}{\partial l_1}\end{aligned}$$

$$\begin{cases}\dfrac{\partial\mathscr{A}_{is}}{\partial l_1}=\dfrac{\partial}{\partial l_1}\left\{\displaystyle\int_{x_0-l_1}^{x_0+l_2}\left[(\hat{\sigma}_{zz})_i(u_z)_s\right]\Big|_{z=0}\mathrm{d}x-\int_{-l}^{x_0-l_1}\left[(u_z)_i(\hat{\sigma}_{zz})_j\right]\Big|_{z=0}\mathrm{d}x\right\}\\ \qquad=\left[\hat{\sigma}_{zz}(x_0-l_1,0)\right]_i\left[u_z(x_0-l_1,0)\right]_s+\left[u_z(x_0-l_1,0)\right]_i\left[\hat{\sigma}_{zz}(x_0-l_1,0)\right]_s\\ \dfrac{\partial\mathscr{H}_i}{\partial l_1}=\dfrac{\partial}{\partial l_1}\left\{\displaystyle\int_{x_0-l_1}^{x_0+l_2}\left[\left\{d_{ac}-\left[R-\sqrt{R^2-(x-x_0)^2}\right]\right\}(\hat{\sigma}_{zz})_i\right]\Big|_{z=0}\mathrm{d}x\right\}\\ \qquad=\left[d_{ac}-R+\sqrt{R^2-l_1^{\,2}}\right]\left[\hat{\sigma}_{zz}(x_0-l_1,0)\right]_i\end{cases}$$

Arbitrary Gradient

Contact region determination

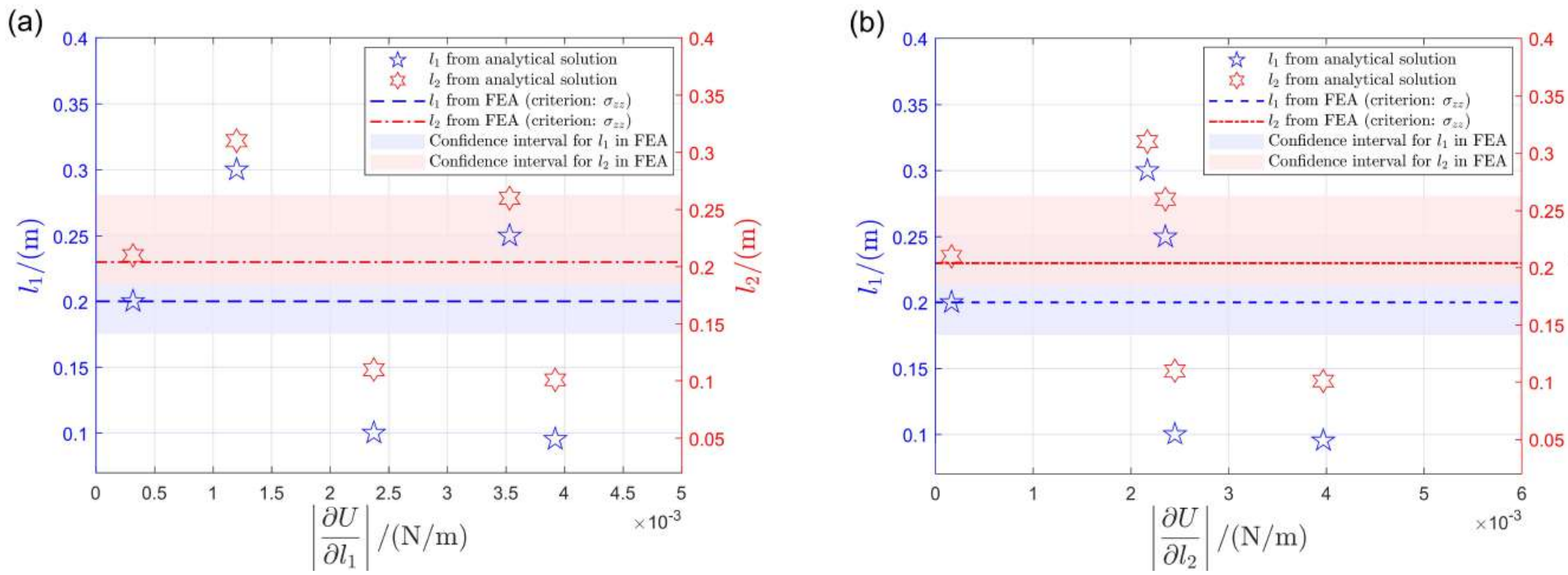
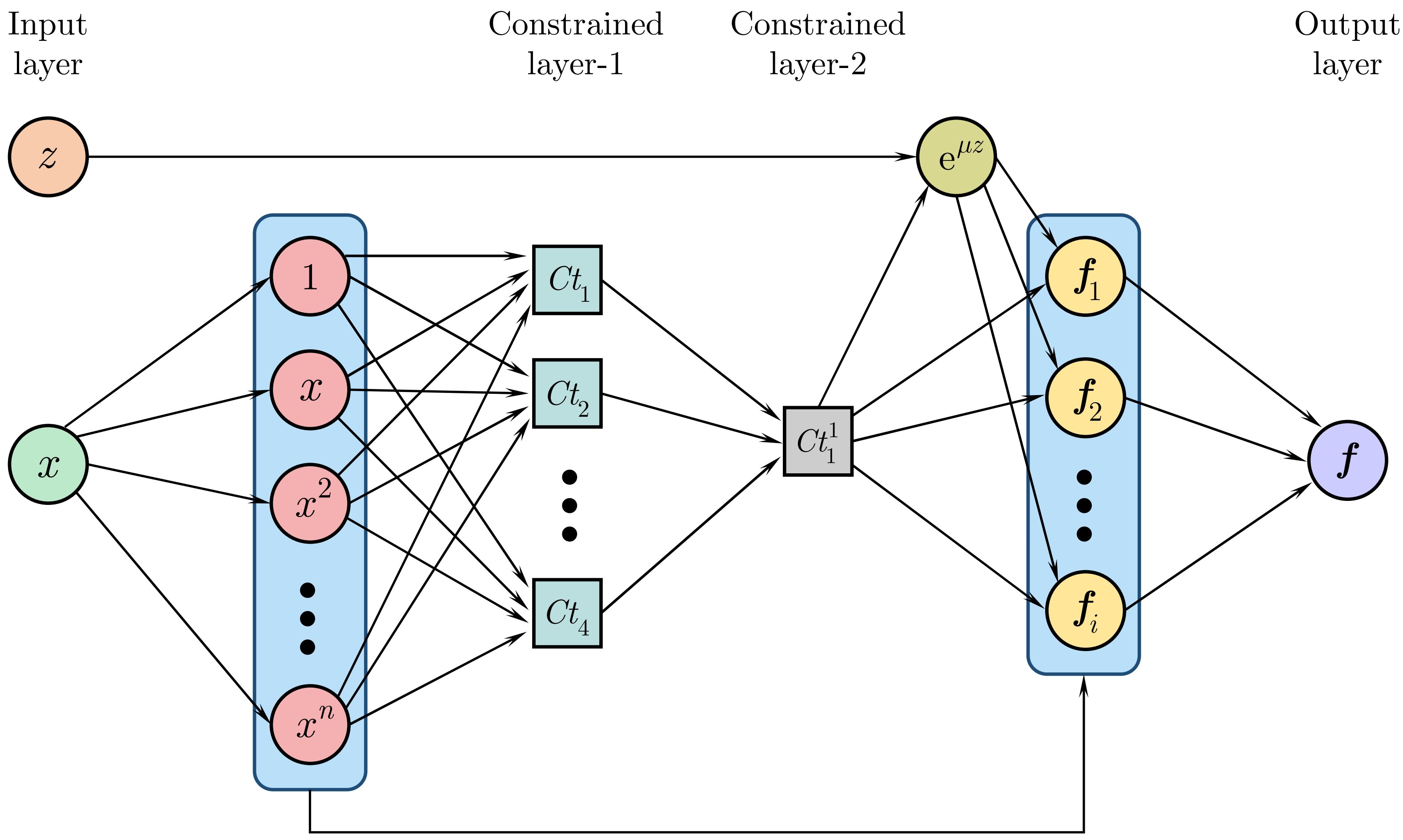


FIGURE 19. Comparison of contact region between FEA results and analytical solutions.

Arbitrary Gradient

Network structure of symplectic analysis

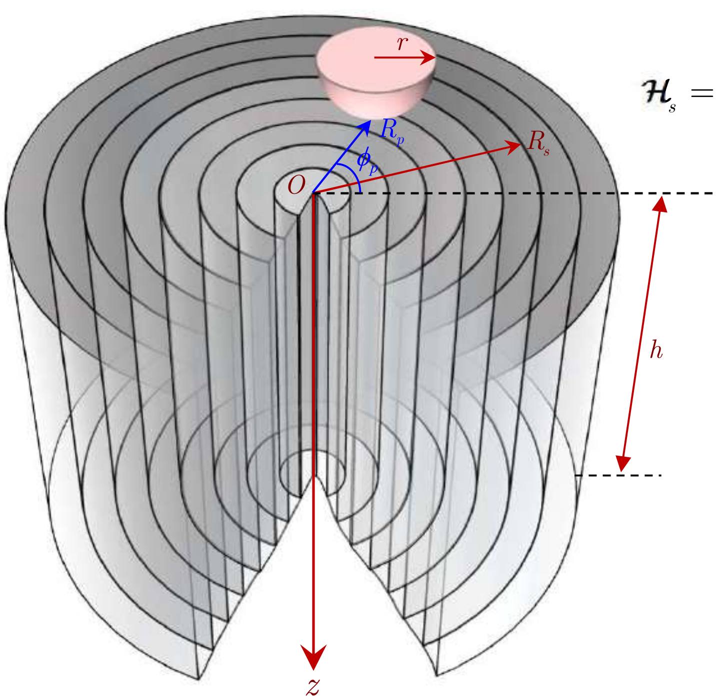


$$\mathcal{L} = \mathcal{L}_{\text{boundary}} + \hat{\lambda}_1 \left\| (\boldsymbol{\mathcal{H}}^x - \mu_n \mathbf{I}) \boldsymbol{\Phi}_n \right\|_1 + \hat{\lambda}_2 \det \boldsymbol{\mathcal{Q}}$$

Quantum computing for symplectic analysis

3D Laminated Model

Basic formulations



$$
\boldsymbol{\mathcal{H}}_s = \left[\begin{array}{ccc|ccc}
0 & 0 & -\dfrac{\partial}{\partial\rho} & \dfrac{2(1+\nu)}{E_s\rho} & 0 & 0 \\
0 & 0 & -\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & \dfrac{2(1+\nu)}{E_s\rho} & 0 \\
-\dfrac{\nu}{1-\nu}\left(\dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho}\right) & -\dfrac{\nu}{1-\nu}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & 0 & 0 & \dfrac{(1+\nu)(1-2\nu)}{E_s(1-\nu)\rho} \\
\hline
\mathcal{D}_{1;s} & \mathcal{D}_{2;s} & 0 & 0 & 0 & -\dfrac{\nu}{1-\nu}\left(\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho}\right) \\
\mathcal{D}_{3;s} & \mathcal{D}_{4;s} & 0 & 0 & 0 & -\dfrac{\nu}{1-\nu}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\
0 & 0 & 0 & -\dfrac{\partial}{\partial\rho} & -\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0
\end{array}\right]
$$

$$
\begin{cases}
\mathcal{D}_{1;s} = -\dfrac{E_s}{1-\nu^2}\left(\rho\dfrac{\partial^2}{\partial\rho^2}+\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho}\right)-\dfrac{E_s}{2(1+\nu)}\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2} \\
\mathcal{D}_{2;s} = \dfrac{E_s(3-\nu)}{2(1-\nu^2)}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi}-\dfrac{E_s}{2(1-\nu)}\dfrac{\partial^2}{\partial\rho\partial\varphi} \\
\mathcal{D}_{3;s} = -\dfrac{E_s}{2(1-\nu)}\dfrac{\partial^2}{\partial\rho\partial\varphi}-\dfrac{E_s(3-\nu)}{2(1-\nu^2)}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\
\mathcal{D}_{4;s} = -\dfrac{E_s}{2(1+\nu)}\left(\rho\dfrac{\partial^2}{\partial\rho^2}+\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho}\right)-\dfrac{E_s}{1-\nu^2}\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2}
\end{cases}
$$

3D Laminated Model

Hamiltonian transformation

$$\left\langle \boldsymbol{f}^{\alpha}, \boldsymbol{\mathcal{H}} \boldsymbol{f}^{\beta} \right\rangle = \left\langle \boldsymbol{f}^{\beta}, \boldsymbol{\mathcal{H}} \boldsymbol{f}^{\alpha} \right\rangle - \cancel{\left\langle \frac{\partial}{\partial \rho} \widetilde{\boldsymbol{f}}^{\alpha}, \widetilde{\boldsymbol{f}}^{\beta} \right\rangle} - \cancel{\left\langle \widetilde{\boldsymbol{f}}^{\alpha}, \frac{\partial}{\partial \rho} \widetilde{\boldsymbol{f}}^{\beta} \right\rangle} - \cancel{\left\langle \frac{1}{\rho} \frac{\partial}{\partial \varphi} \widehat{\boldsymbol{f}}^{\alpha}, \widehat{\boldsymbol{f}}^{\beta} \right\rangle} - \cancel{\left\langle \widehat{\boldsymbol{f}}^{\alpha}, \frac{1}{\rho} \frac{\partial}{\partial \varphi} \widehat{\boldsymbol{f}}^{\beta} \right\rangle}$$

$$\begin{cases} \widetilde{\boldsymbol{f}} = [u_{\rho}, u_{\varphi}, u_{z}, \rho\sigma_{\rho\rho}, \rho\sigma_{\rho\varphi}, \rho\sigma_{\rho z}]^{\mathrm{T}} \\ \widehat{\boldsymbol{f}} = [u_{\rho}, u_{\varphi}, u_{z}, \rho\sigma_{\varphi\rho}, \rho\sigma_{\varphi\varphi}, \rho\sigma_{\varphi z}]^{\mathrm{T}} \end{cases}$$

$$\boxed{\sum_{s=1}^{n} \iint_{\Omega^s} \left(\boldsymbol{f}_s^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \boldsymbol{\mathcal{H}}_s \boldsymbol{f}_s^{\beta} \mathrm{d}\rho \mathrm{d}\varphi = \sum_{s=1}^{n} \iint_{\Omega^s} \left(\boldsymbol{f}_s^{\beta}\right)^{\mathrm{T}} \mathbf{J} \boldsymbol{\mathcal{H}}_s \boldsymbol{f}_s^{\alpha} \mathrm{d}\rho \mathrm{d}\varphi - \sum_{s=1}^{n} \iint_{\Omega^s} \frac{\partial}{\partial \rho} \left[\left(\widetilde{\boldsymbol{f}}_s^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_s^{\beta} \right] \mathrm{d}\rho \mathrm{d}\varphi - \sum_{s=1}^{n} \iint_{\Omega^s} \frac{1}{\rho} \frac{\partial}{\partial \varphi} \left[\left(\widehat{\boldsymbol{f}}_s^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widehat{\boldsymbol{f}}_s^{\beta} \right] \mathrm{d}\rho \mathrm{d}\varphi}$$

1. If $O_p(\Omega_c) = O_p(\Omega)$

$$\begin{aligned} \sum_{s=1}^{n} \iint_{\Omega^s} \frac{\partial}{\partial \rho} \left[\left(\widetilde{\boldsymbol{f}}_s^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_s^{\beta} \right] \mathrm{d}\rho \mathrm{d}\varphi &= \sum_{s=1}^{n} \int_0^{2\pi} \int_{R_{s-1}}^{R_s} \frac{\partial}{\partial \rho} \left[\left(\widetilde{\boldsymbol{f}}_s^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_s^{\beta} \right] \mathrm{d}\rho \mathrm{d}\varphi \\ &= \underbrace{\int_0^{2\pi} \left\{ \left[\left(\widetilde{\boldsymbol{f}}_1^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_1^{\beta} \right] \Bigg|_{\rho=R_0}^{\rho=R_1} + \cdots + \left[\left(\widetilde{\boldsymbol{f}}_n^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_n^{\beta} \right] \Bigg|_{\rho=R_{n-1}}^{\rho=R_n} \right\} \mathrm{d}\varphi}_{\text{continuity conditions}} \\ &= \underbrace{\cancel{\int_0^{2\pi} \left[\left(\widetilde{\boldsymbol{f}}_n^{\alpha}\right)^{\mathrm{T}} \mathbf{J} \widetilde{\boldsymbol{f}}_n^{\beta} \right]_{\rho=R_n}} \mathrm{d}\varphi}_{\text{boundary conditions}} - \underbrace{\cancel{\int_0^{2\pi} \left[-\rho\sigma_{\rho z;1}^{\alpha} u_{z;1}^{\beta} - \rho\sigma_{\rho\rho;1}^{\alpha} u_{\rho;1}^{\beta} - \rho\sigma_{\rho\varphi;1}^{\alpha} u_{\varphi;1}^{\beta} + u_{z;1}^{\alpha} \rho\sigma_{\rho z;1}^{\beta} + u_{\rho;1}^{\alpha} \rho\sigma_{\rho\rho;1}^{\beta} + u_{\varphi;1}^{\alpha} \rho\sigma_{\rho\varphi;1}^{\beta} \right]_{\rho=R_0} \mathrm{d}\varphi}}_{R_0=0} \\ &= 0 \end{aligned}$$

Hamiltonian transformation

$$\begin{aligned}\sum_{s=1}^{n}\iint_{\Omega^s}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi &= \sum_{s=1}^{n}\int_0^{2\pi}\int_{R_{s-1}}^{R_s}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi\\ &= \sum_{s=1}^{n}\int_{R_{s-1}}^{R_s}\int_0^{2\pi}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\varphi\mathrm d\rho\\ &= \underbrace{\cancel{\sum_{s=1}^{n}\int_{R_{s-1}}^{R_s}\left[-\sigma_{\varphi z;s}^{\alpha}u_{z;s}^{\beta}-\sigma_{\varphi\rho;s}^{\alpha}u_{\rho;s}^{\beta}-\sigma_{\varphi\varphi;s}^{\alpha}u_{\varphi;s}^{\beta}+u_{z;s}^{\alpha}\sigma_{\varphi z;s}^{\beta}+u_{\rho;s}^{\alpha}\sigma_{\varphi\rho;s}^{\beta}+u_{\varphi;s}^{\alpha}\sigma_{\varphi\varphi;s}^{\beta}\right]\Big|_{\varphi=0}^{\varphi=2\pi}\mathrm d\rho}}_{u_{i;s}(\rho,2\pi)=u_{i;s}(\rho,0)\,\&\,\sigma_{ij;s}(\rho,2\pi)=\sigma_{ij;s}(\rho,0)}\\ &= 0\end{aligned}$$

2. If $O_p(\Omega_c)\in\Omega$ but $O_p(\Omega_c)\neq O_p(\Omega)$

$$\begin{aligned}\sum_{s=1}^{n}\iint_{\Omega^s}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi &= \sum_{s=1}^{n}\left\{\int_{\theta_{2s}}^{\theta_{2s+1}}\int_{\omega_{2s}(\varphi)}^{\omega_{2s+1}(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi-\int_{\theta_{2s-2}}^{\theta_{2s-1}}\int_{\omega_{2s-2}(\varphi)}^{\omega_{2s-1}(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi\right\}\\ &= \underbrace{\cancel{-\int_{\theta_0}^{\theta_1}\int_{\omega_0(\varphi)}^{\omega_1(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho\mathrm d\varphi}}_{\theta_0=\theta_1\,\&\,\omega_0(\varphi)=\omega_1(\varphi)=0}+\underbrace{\cancel{\int_{\theta_2}^{\theta_3}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho\mathrm d\varphi-\int_{\theta_2}^{\theta_3}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_2^{\beta}\right]\mathrm d\rho\mathrm d\varphi}}_{\text{continuity conditions}}\\ &\quad+\cdots+\underbrace{\int_0^{2\pi}\int_0^{\omega_{2r+1}(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_r^{\beta}\right]\mathrm d\rho\mathrm d\varphi+\int_0^{2\pi}\int_{\omega_{2r+1}(\varphi)}^{\omega_{2r+3}(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_{r+1}^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_{r+1}^{\beta}\right]\mathrm d\rho\mathrm d\varphi+\cdots+\int_0^{2\pi}\int_{\omega_{2n-1}(\varphi)}^{\omega_{2n+1}(\varphi)}\frac{\partial}{\partial\rho}\left[\left(\widetilde{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_n^{\beta}\right]\mathrm d\rho\mathrm d\varphi}_{\text{continuity conditions}}\\ &= \int_0^{2\pi}\left\{\left[\left(\widetilde{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_n^{\beta}\right]_{\rho=\omega_{2n+1}(\varphi)}-\left[\left(\widetilde{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widetilde{\boldsymbol f}_r^{\beta}\right]_{\rho=0}\right\}\mathrm d\varphi\\ &= \underbrace{\cancel{\int_0^{2\pi}\left[-\rho\sigma_{\rho\rho;n}^{\alpha}u_{\rho;n}^{\beta}-\rho\sigma_{\rho\varphi;n}^{\alpha}u_{\varphi;n}^{\beta}-\rho\sigma_{\rho z;n}^{\alpha}u_{z;n}^{\beta}+u_{\rho;n}^{\alpha}\rho\sigma_{\rho\rho;n}^{\beta}+u_{\varphi;n}^{\alpha}\rho\sigma_{\rho\varphi;n}^{\beta}+u_{z;n}^{\alpha}\rho\sigma_{\rho z;n}^{\beta}\right]_{\rho=\omega_{2n+1}(\varphi)}\mathrm d\varphi}}_{\text{homogeneous boundary conditions}}\\ &\quad\underbrace{\cancel{-\int_0^{2\pi}\left[-\rho\sigma_{\rho\rho;r}^{\alpha}u_{\rho;r}^{\beta}-\rho\sigma_{\rho\varphi;r}^{\alpha}u_{\varphi;r}^{\beta}-\rho\sigma_{\rho z;r}^{\alpha}u_{z;r}^{\beta}+u_{\rho;r}^{\alpha}\rho\sigma_{\rho\rho;r}^{\beta}+u_{\varphi;r}^{\alpha}\rho\sigma_{\rho\varphi;r}^{\beta}+u_{z;r}^{\alpha}\rho\sigma_{\rho z;r}^{\beta}\right]_{\rho=0}\mathrm d\varphi}}_{\rho=0}\\ &= 0\end{aligned}$$

3D Laminated Model

Hamiltonian transformation

$$
\sum_{s=1}^{n}\iint_{\Omega^s}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi=\sum_{s=1}^{n}\left\{\int_{\theta_{2s}}^{\theta_{2s+1}}\int_{\omega_{2s}(\varphi)}^{\omega_{2s+1}(\varphi)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi-\int_{\theta_{2s-2}}^{\theta_{2s-1}}\int_{\omega_{2s-2}(\varphi)}^{\omega_{2s-1}(\varphi)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\left[\left(\widehat{\boldsymbol f}_s^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_s^{\beta}\right]\mathrm d\rho\mathrm d\varphi\right\}
$$

$$
=\underbrace{\cancel{-\int_{\theta_0}^{\theta_1}\int_{\omega_0(\varphi)}^{\omega_1(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho\mathrm d\varphi}}_{\theta_0=\theta_1\,\&\,\omega_0(\varphi)=\omega_1(\varphi)=0}+\int_{\theta_2}^{\theta_3}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho\mathrm d\varphi-\int_{\theta_2}^{\theta_3}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_2^{\beta}\right]\mathrm d\rho\mathrm d\varphi
$$

$$
+\cdots+\int_0^{2\pi}\int_0^{\omega_{2r+1}(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_r^{\beta}\right]\mathrm d\rho\mathrm d\varphi+\int_0^{2\pi}\int_{\omega_{2r+1}(\varphi)}^{\omega_{2r+3}(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_{r+1}^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_{r+1}^{\beta}\right]\mathrm d\rho\mathrm d\varphi+\cdots+\int_0^{2\pi}\int_{\omega_{2n-1}(\varphi)}^{\omega_{2n+1}(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_n^{\beta}\right]\mathrm d\rho\mathrm d\varphi
$$

$$
=\int_{\theta_2}^{\theta_3}\left\{\frac{\mathrm d}{\mathrm d\varphi}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho-\frac{\mathrm d}{\mathrm d\varphi}\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_2^{\beta}\right]\mathrm d\rho\right.
$$

$$
\left.-\underbrace{\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]_{\rho=\omega_3(\varphi)}\cdot\frac{\mathrm d\omega_3(\varphi)}{\mathrm d\varphi}}+\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_2^{\beta}\right]_{\rho=\omega_3(\varphi)}\cdot\frac{\mathrm d\omega_3(\varphi)}{\mathrm d\varphi}}}_{\text{continuity conditions}}+\underbrace{\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]_{\rho=\omega_2(\varphi)}\cdot\frac{\mathrm d\omega_2(\varphi)}{\mathrm d\varphi}}-\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_2^{\beta}\right]_{\rho=\omega_2(\varphi)}\cdot\frac{\mathrm d\omega_2(\varphi)}{\mathrm d\varphi}}}_{\text{continuity conditions}}\right\}\mathrm d\varphi
$$

$$
+\cdots+\int_0^{2\pi}\left\{\frac{\mathrm d}{\mathrm d\varphi}\int_0^{\omega_{2r+1}(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_r^{\beta}\right]\mathrm d\rho-\underbrace{\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_r^{\beta}\right]_{\rho=\omega_{2r+1}(\varphi)}\cdot\frac{\mathrm d\omega_{2r+1}(\varphi)}{\mathrm d\varphi}}}_{\text{continuity conditions}}\right\}\mathrm d\varphi
$$

$$
+\cdots+\int_0^{2\pi}\left\{\frac{\mathrm d}{\mathrm d\varphi}\int_{\omega_{2n-1}(\varphi)}^{\omega_{2n+1}(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_n^{\beta}\right]\mathrm d\rho-\underbrace{\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_n^{\beta}\right]_{\rho=\omega_{2n+1}(\varphi)}\cdot\frac{\mathrm d\omega_{2n+1}(\varphi)}{\mathrm d\varphi}}}_{\text{homogeneous boundary conditions}}+\underbrace{\cancel{\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_n^{\beta}\right]_{\rho=\omega_{2n-1}(\varphi)}\cdot\frac{\mathrm d\omega_{2n-1}(\varphi)}{\mathrm d\varphi}}}_{\text{continuity conditions}}\right\}\mathrm d\varphi
$$

To be continued

$$
=\underbrace{\cancel{\left\{\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_1^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_1^{\beta}\right]\mathrm d\rho-\int_{\omega_2(\varphi)}^{\omega_3(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_2^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_2^{\beta}\right]\mathrm d\rho\right\}\Bigg|_{\varphi=\theta_2}^{\varphi=\theta_3}}}_{\omega_3(\theta_3)=\omega_2(\theta_3)\,\&\,\omega_3(\theta_2)=\omega_2(\theta_2)}+\cdots+\left\{\int_0^{\omega_{2r+1}(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_r^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_r^{\beta}\right]\mathrm d\rho\right\}\Bigg|_{\varphi=0}^{\varphi=2\pi}+\cdots+\left\{\int_{\omega_{2n-1}(\varphi)}^{\omega_{2n+1}(\varphi)}\left[\frac{1}{\rho}\left(\widehat{\boldsymbol f}_n^{\alpha}\right)^{\mathrm T}\mathbf J\widehat{\boldsymbol f}_n^{\beta}\right]\mathrm d\rho\right\}\Bigg|_{\varphi=0}^{\varphi=2\pi}
$$

3D Laminated Model

Hamiltonian transformation

$$
\begin{aligned}
&= \left\{\int_0^{\omega_{2r+1}(\varphi)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]\mathrm{d}\rho\right\}\Bigg|_{\varphi=0}^{\varphi=2\pi} \\
&\quad+\cdots+\left\{\int_{\omega_{2n-1}(\varphi)}^{\omega_{2n+1}(\varphi)}\left[-\sigma^{\alpha}_{\varphi\rho;n}u^{\beta}_{\rho;n}-\sigma^{\alpha}_{\varphi\varphi;n}u^{\beta}_{\varphi;n}-\sigma^{\alpha}_{\varphi z;n}u^{\beta}_{z;n}+u^{\alpha}_{\rho;n}\sigma^{\beta}_{\varphi\rho;n}+u^{\alpha}_{\varphi;n}\sigma^{\beta}_{\varphi\varphi;n}+u^{\alpha}_{z;n}\sigma^{\beta}_{\varphi z;n}\right]\mathrm{d}\rho\right\}\Bigg|_{\varphi=0}^{\varphi=2\pi} \\
&= \int_0^{\omega_{2r+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]_{\varphi=2\pi}\mathrm{d}\rho-\int_0^{\omega_{2r+1}(0)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]_{\varphi=0}\mathrm{d}\rho \\
&\quad+\cdots+\int_{\omega_{2n-1}(2\pi)}^{\omega_{2n+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;n}u^{\beta}_{\rho;n}-\sigma^{\alpha}_{\varphi\varphi;n}u^{\beta}_{\varphi;n}-\sigma^{\alpha}_{\varphi z;n}u^{\beta}_{z;n}+u^{\alpha}_{\rho;n}\sigma^{\beta}_{\varphi\rho;n}+u^{\alpha}_{\varphi;n}\sigma^{\beta}_{\varphi\varphi;n}+u^{\alpha}_{z;n}\sigma^{\beta}_{\varphi z;n}\right]_{\varphi=2\pi}\mathrm{d}\rho-\int_{\omega_{2n+1}(0)}^{\omega_{2n+1}(0)}\left[-\sigma^{\alpha}_{\varphi\rho;n}u^{\beta}_{\rho;n}-\sigma^{\alpha}_{\varphi\varphi;n}u^{\beta}_{\varphi;n}-\sigma^{\alpha}_{\varphi z;n}u^{\beta}_{z;n}+u^{\alpha}_{\rho;n}\sigma^{\beta}_{\varphi\rho;n}+u^{\alpha}_{\varphi;n}\sigma^{\beta}_{\varphi\varphi;n}+u^{\alpha}_{z;n}\sigma^{\beta}_{\varphi z;n}\right]_{\varphi=0}\mathrm{d}\rho \\
&= \int_0^{\omega_{2r+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]_{\varphi=2\pi}\mathrm{d}\rho-\underbrace{\int_0^{\omega_{2r+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]_{\varphi=2\pi}\mathrm{d}\rho}_{\omega_{2r+1}(2\pi)=\omega_{2r+1}(0)\&u_{i;r}(\rho,2\pi)=u_{i;r}(\rho,0)\&\sigma_{ij;r}(\rho,2\pi)=\sigma_{ij;r}(\rho,0)} \\
&\quad+\cdots+\underbrace{\cancel{\int_{\omega_{2n-1}(2\pi)}^{\omega_{2n+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;n}u^{\beta}_{\rho;n}-\sigma^{\alpha}_{\varphi\varphi;n}u^{\beta}_{\varphi;n}-\sigma^{\alpha}_{\varphi z;n}u^{\beta}_{z;n}+u^{\alpha}_{\rho;n}\sigma^{\beta}_{\varphi\rho;n}+u^{\alpha}_{\varphi;n}\sigma^{\beta}_{\varphi\varphi;n}+u^{\alpha}_{z;n}\sigma^{\beta}_{\varphi z;n}\right]_{\varphi=2\pi}\mathrm{d}\rho-\int_{\omega_{2n+1}(2\pi)}^{\omega_{2n+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;n}u^{\beta}_{\rho;n}-\sigma^{\alpha}_{\varphi\varphi;n}u^{\beta}_{\varphi;n}-\sigma^{\alpha}_{\varphi z;n}u^{\beta}_{z;n}+u^{\alpha}_{\rho;n}\sigma^{\beta}_{\varphi\rho;n}+u^{\alpha}_{\varphi;n}\sigma^{\beta}_{\varphi\varphi;n}+u^{\alpha}_{z;n}\sigma^{\beta}_{\varphi z;n}\right]_{\varphi=2\pi}\mathrm{d}\rho}}_{\omega_{2n+1}(2\pi)=\omega_{2n+1}(0)\&u_{i;n}(\rho,2\pi)=u_{i;n}(\rho,0)\&\sigma_{ij;n}(\rho,2\pi)=\sigma_{ij;n}(\rho,0)} \\
&= \cancel{\int_{\omega_{2r+1}(2\pi)}^{\omega_{2r+1}(2\pi)}\left[-\sigma^{\alpha}_{\varphi\rho;r}u^{\beta}_{\rho;r}-\sigma^{\alpha}_{\varphi\varphi;r}u^{\beta}_{\varphi;r}-\sigma^{\alpha}_{\varphi z;r}u^{\beta}_{z;r}+u^{\alpha}_{\rho;r}\sigma^{\beta}_{\varphi\rho;r}+u^{\alpha}_{\varphi;r}\sigma^{\beta}_{\varphi\varphi;r}+u^{\alpha}_{z;r}\sigma^{\beta}_{\varphi z;r}\right]_{\varphi=2\pi}\mathrm{d}\rho} \\
&= 0
\end{aligned}
$$

$$
\text{where}\quad \begin{cases}\omega_0(\varphi)=\omega_1(\varphi)=0\\ \theta_0=\theta_1\\ \theta_{2s}=0;\theta_{2s+1}=2\pi;\omega_{2s}(\varphi)=0\ (s\ge r)\end{cases}
$$

3D Laminated Model

Saint-Venant solution

$$\Phi_{0,1;s}^{(0)} = \left[\sin\varphi, \cos\varphi, 0, 0, 0, 0\right]^{\mathrm{T}}, \quad \Phi_{0,2;s}^{(0)} = \left[-\cos\varphi, \sin\varphi, 0, 0, 0, 0\right]^{\mathrm{T}}$$

$$\Phi_{0,3;s}^{(0)} = \left[0, 0, 1, 0, 0, 0\right]^{\mathrm{T}}, \quad \Phi_{0,4;s}^{(0)} = \left[0, \rho, 0, 0, 0, 0\right]^{\mathrm{T}}$$

$$k = 1: \begin{cases} \Phi_{0,1;s}^{(1)} = \left[0, 0, -\rho\sin\varphi, 0, 0, 0\right]^{\mathrm{T}} \\ \Phi_{0,2;s}^{(1)} = \left[0, 0, \rho\cos\varphi, 0, 0, 0\right]^{\mathrm{T}} \\ \Phi_{0,3;s}^{(1)} = \left[-\nu\rho, 0, 0, 0, 0, E_s\rho\right]^{\mathrm{T}} \\ \Phi_{0,4;s}^{(1)} = \left[0, 0, \psi_s, \dfrac{E_s}{2(1+\nu)}\rho\dfrac{\partial\psi_s}{\partial\rho}, \dfrac{E_s}{2(1+\nu)}\left(\dfrac{\partial\psi_s}{\partial\varphi} + \rho^2\right), 0\right]^{\mathrm{T}} \end{cases}$$

$$k = 2: \begin{cases} \Phi_{0,1;s}^{(2)} = \left[\dfrac{1}{2}\nu\rho^2\sin\varphi, -\dfrac{1}{2}\nu\rho^2\cos\varphi, 0, 0, 0, -E_s\rho^2\sin\varphi\right]^{\mathrm{T}} \\ \Phi_{0,2;s}^{(2)} = \left[-\dfrac{1}{2}\nu\rho^2\cos\varphi, -\dfrac{1}{2}\nu\rho^2\sin\varphi, 0, 0, 0, E_s\rho^2\cos\varphi\right]^{\mathrm{T}} \end{cases}$$

$$k = 3: \begin{cases} \Phi_{0,1;s}^{(3)} = \left[0, 0, \widehat{\psi}_s + \dfrac{\rho^3\sin\varphi}{4}, \dfrac{E_s}{2(1+\nu)}\left(\rho\dfrac{\partial\widehat{\psi}_s}{\partial\rho} + \dfrac{(3+2\nu)}{4}\rho^3\sin\varphi\right), \dfrac{E_s}{2(1+\nu)}\left(\dfrac{\partial\widehat{\psi}_s}{\partial\varphi} + \dfrac{(1-2\nu)}{4}\rho^3\cos\varphi\right), 0\right]^{\mathrm{T}} \\ \Phi_{0,2;s}^{(3)} = \left[0, 0, -\widetilde{\psi}_s - \dfrac{\rho^3\cos\varphi}{4}, -\dfrac{E_s}{2(1+\nu)}\left(\rho\dfrac{\partial\widetilde{\psi}_s}{\partial\rho} + \dfrac{(3+2\nu)}{4}\rho^3\cos\varphi\right), -\dfrac{E_s}{2(1+\nu)}\left(\dfrac{\partial\widetilde{\psi}_s}{\partial\varphi} - \dfrac{(1-2\nu)}{4}\rho^3\sin\varphi\right), 0\right]^{\mathrm{T}} \end{cases}$$

3D Laminated Model

Saint-Venant solution

$$\begin{cases} \nabla^2 \psi_1 = 0 \\ E_1 \left.\dfrac{\partial \psi_1}{\partial \rho}\right|_{\rho=R_1} = E_2 \left.\dfrac{\partial \psi_2}{\partial \rho}\right|_{\rho=R_1} \end{cases}, \quad \begin{cases} \nabla^2 \psi_s = 0 \\ \begin{cases} E_s \left.\dfrac{\partial \psi_s}{\partial \rho}\right|_{\rho=R_s} = E_{s+1} \left.\dfrac{\partial \psi_{s+1}}{\partial \rho}\right|_{\rho=R_s} \\ \left.\psi_{s-1}\right|_{\rho=R_{s-1}} = \left.\psi_s\right|_{\rho=R_{s-1}} \end{cases} \\ (2 \le s \le n-1) \end{cases}, \quad \begin{cases} \nabla^2 \psi_n = 0 \\ \begin{cases} E_n \left.\dfrac{\partial \psi_n}{\partial \rho}\right|_{\rho=R_n} = 0 \\ \left.\psi_{n-1}\right|_{\rho=R_{n-1}} = \left.\psi_n\right|_{\rho=R_{n-1}} \end{cases} \end{cases}$$

$$\begin{cases} \nabla^2 \hat{\psi}_1 = 0 \\ E_1 \left.\dfrac{\partial \hat{\psi}_1}{\partial \rho}\right|_{\rho=R_1} = E_2 \left.\dfrac{\partial \hat{\psi}_2}{\partial \rho}\right|_{\rho=R_1} + (E_2 - E_1)\dfrac{(3+2\nu)}{4} R_1^2 \sin\varphi \end{cases} \qquad \begin{cases} \nabla^2 \tilde{\psi}_1 = 0 \\ E_1 \left.\dfrac{\partial \tilde{\psi}_1}{\partial \rho}\right|_{\rho=R_1} = E_2 \left.\dfrac{\partial \tilde{\psi}_2}{\partial \rho}\right|_{\rho=R_1} + (E_2 - E_1)\dfrac{(3+2\nu)}{4} R_1^2 \cos\varphi \end{cases}$$

$$\begin{cases} \nabla^2 \hat{\psi}_s = 0 \\ \begin{cases} E_s \left.\dfrac{\partial \hat{\psi}_s}{\partial \rho}\right|_{\rho=R_s} = E_{s+1} \left.\dfrac{\partial \hat{\psi}_{s+1}}{\partial \rho}\right|_{\rho=R_s} + (E_{s+1} - E_s)\dfrac{(3+2\nu)}{4} R_s^2 \sin\varphi \\ \left.\hat{\psi}_{s-1}\right|_{\rho=R_{s-1}} = \left.\hat{\psi}_s\right|_{\rho=R_{s-1}} \end{cases} \\ (2 \le s \le n-1) \end{cases} \qquad \begin{cases} \nabla^2 \tilde{\psi}_s = 0 \\ \begin{cases} E_s \left.\dfrac{\partial \tilde{\psi}_s}{\partial \rho}\right|_{\rho=R_s} = E_{s+1} \left.\dfrac{\partial \tilde{\psi}_{s+1}}{\partial \rho}\right|_{\rho=R_s} + (E_{s+1} - E_s)\dfrac{(3+2\nu)}{4} R_s^2 \cos\varphi \\ \left.\tilde{\psi}_{s-1}\right|_{\rho=R_{s-1}} = \left.\tilde{\psi}_s\right|_{\rho=R_{s-1}} \end{cases} \\ (2 \le s \le n-1) \end{cases}$$

$$\begin{cases} \nabla^2 \hat{\psi}_n = 0 \\ \begin{cases} E_n \left.\dfrac{\partial \hat{\psi}_n}{\partial \rho}\right|_{\rho=R_n} = -E_n \dfrac{(3+2\nu)}{4} R_n^2 \sin\varphi \\ \left.\hat{\psi}_{n-1}\right|_{\rho=R_{n-1}} = \left.\hat{\psi}_n\right|_{\rho=R_{n-1}} \end{cases} \end{cases} \qquad \begin{cases} \nabla^2 \tilde{\psi}_n = 0 \\ \begin{cases} E_n \left.\dfrac{\partial \tilde{\psi}_n}{\partial \rho}\right|_{\rho=R_n} = -E_n \dfrac{(3+2\nu)}{4} R_n^2 \cos\varphi \\ \left.\tilde{\psi}_{n-1}\right|_{\rho=R_{n-1}} = \left.\tilde{\psi}_n\right|_{\rho=R_{n-1}} \end{cases} \end{cases}$$

$$\boldsymbol{f}_{0,i;s}^{(k)} = \Phi_{0,i;s}^{(k)} + z\Phi_{0,i;s}^{(k-1)} + \cdots + \frac{z^k}{k!}\Phi_{0,i;s}^{(0)} \quad (s = 1,2,\cdots,n; i = 1,2,\cdots,4; k \ge 0)$$

3D Laminated Model

General solutions

Papkovich-Neuber form

$$\Phi_s = \left\{ \begin{array}{c} B_{\rho;s} - \dfrac{1}{4(1-\nu)} \dfrac{\partial}{\partial \rho}\left(B_{0;s} + \rho B_{\rho;s}\right) \\ B_{\varphi;s} - \dfrac{1}{4(1-\nu)} \dfrac{1}{\rho} \dfrac{\partial}{\partial \varphi}\left(B_{0;s} + \rho B_{\rho;s}\right) \\ -\mu \dfrac{1}{4(1-\nu)}\left(B_{0;s} + \rho B_{\rho;s}\right) \\ \mu \dfrac{E_s}{2(1+\nu)} \rho \left[B_{\rho;s} - \dfrac{1}{2(1-\nu)} \dfrac{\partial}{\partial \rho}\left(B_{0;s} + \rho B_{\rho;s}\right)\right] \\ \mu \dfrac{E_s}{2(1+\nu)} \rho \left[B_{\varphi;s} - \dfrac{1}{2(1-\nu)} \dfrac{1}{\rho} \dfrac{\partial}{\partial \varphi}\left(B_{0;s} + \rho B_{\rho;s}\right)\right] \\ \dfrac{E_s \nu}{2(1-\nu^2)} \rho \left[\dfrac{\partial B_{\rho;s}}{\partial \rho} + \dfrac{B_{\rho;s}}{\rho} + \dfrac{1}{\rho} \dfrac{\partial B_{\varphi;s}}{\partial \varphi}\right] - \dfrac{E_s}{4(1-\nu^2)} \mu^2 \rho \left(B_{0;s} + \rho B_{\rho;s}\right) \end{array} \right\}$$

$$\left(B_{0;s}, B_{\rho;s}, B_{\varphi;s}\right) = \sum_{m=-\infty}^{\infty} \left(\mathcal{R}_{0;s}^{\{m\}}, \mathcal{R}_{\rho;s}^{\{m\}}, \mathrm{i}\mathcal{R}_{\varphi;s}^{\{m\}}\right) \mathrm{e}^{\mathrm{i}m\varphi}$$

$$\begin{aligned} &\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \mu^2 - \frac{m^2}{\rho^2}\right)\mathcal{R}_{0;s}^{\{m\}} = 0 \\ &\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \mu^2 - \frac{m^2+1}{\rho^2}\right)\mathcal{R}_{\rho;s}^{\{m\}} + \frac{2m}{\rho^2}\mathcal{R}_{\varphi;s}^{\{m\}} = 0 \\ &\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \mu^2 - \frac{m^2+1}{\rho^2}\right)\mathcal{R}_{\varphi;s}^{\{m\}} + \frac{2m}{\rho^2}\mathcal{R}_{\rho;s}^{\{m\}} = 0 \end{aligned}$$

$$\left\{ \begin{aligned} \mathcal{R}_{0;s}^{\{m\}} &= C_{1;s}^{\{m\}} J_m(\mu\rho) + C_{2;s}^{\{m\}} N_m(\mu\rho) \\ \mathcal{R}_{\rho;s}^{\{m\}} &= C_{3;s}^{\{m\}} J_{m+1}(\mu\rho) + C_{4;s}^{\{m\}} J_{m-1}(\mu\rho) + C_{5;s}^{\{m\}} N_{m+1}(\mu\rho) + C_{6;s}^{\{m\}} N_{m-1}(\mu\rho) \\ \mathcal{R}_{\varphi;s}^{\{m\}} &= -C_{3;s}^{\{m\}} J_{m+1}(\mu\rho) + C_{4;s}^{\{m\}} J_{m-1}(\mu\rho) - C_{5;s}^{\{m\}} N_{m+1}(\mu\rho) + C_{6;s}^{\{m\}} N_{m-1}(\mu\rho) \end{aligned} \right.$$

$$(s = 2, 3, \cdots, n)$$

3D Laminated Model

Sub-symplectic structure

$$
\begin{aligned}
L_{\mu} &= \bigcup_{s=1}^{n} L_{\mu;s} \\
&= \frac{1}{2}\bigcup_{s=1}^{n} \rho \left\{ \frac{E_s(1-\nu)}{(1+\nu)(1-2\nu)} \left[\left(\frac{\partial u_{\rho;s}}{\partial \rho} \right)^2 + \frac{1}{\rho^2}\left(\frac{\partial u_{\varphi;s}}{\partial \varphi} + u_{\rho;s} \right)^2 + \mu^2 u_{z;s}^2 \right] + 2\frac{E_s \nu}{(1+\nu)(1-2\nu)} \left[\frac{1}{\rho}\frac{\partial u_{\rho;s}}{\partial \rho}\left(\frac{\partial u_{\varphi;s}}{\partial \varphi} + u_{\rho;s} \right) + \mu u_{z;s}\frac{\partial u_{\rho;s}}{\partial \rho} + \mu u_{z;s}\frac{1}{\rho}\left(\frac{\partial u_{\varphi;s}}{\partial \varphi} + u_{\rho;s} \right) \right] \right. \\
&\quad \left. + \frac{E_s}{2(1+\nu)} \left[\left(\frac{1}{\rho}\frac{\partial u_{z;s}}{\partial \varphi} + \mu u_{\varphi;s} \right)^2 + \left(\frac{\partial u_{z;s}}{\partial \rho} + \mu u_{\rho;s} \right)^2 + \left(\frac{\partial u_{\varphi;s}}{\partial \rho} + \frac{1}{\rho}\frac{\partial u_{\rho;s}}{\partial \varphi} - \frac{u_{\varphi;s}}{\rho} \right)^2 \right] \right\}
\end{aligned}
$$

$$
\widehat{H}_{\mu;s}(\boldsymbol{q}_s, \widehat{\boldsymbol{p}}_s) = \left(\widehat{\boldsymbol{p}}_s\right)^{\mathrm{T}} \frac{\partial \boldsymbol{q}_s}{\partial \varphi} - L_{\mu;s}\left(\boldsymbol{q}_s, \frac{\partial \boldsymbol{q}_s}{\partial \varphi}\right)
$$

$$
\frac{\partial}{\partial \varphi}\begin{Bmatrix} \boldsymbol{q}_s \\ \widehat{\boldsymbol{p}}_s \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \widehat{H}_{\mu;s}}{\partial \widehat{\boldsymbol{p}}_s} \\ -\dfrac{\partial \widehat{H}_{\mu;s}}{\partial \boldsymbol{q}_s} \end{Bmatrix} = \left[\begin{array}{ccc|ccc}
0 & 1-\rho\dfrac{\partial}{\partial \rho} & 0 & \dfrac{2(1+\nu)}{E_s}\rho & 0 & 0 \\
-1-\dfrac{\nu}{1-\nu}\rho\dfrac{\partial}{\partial \rho} & 0 & -\dfrac{\nu}{1-\nu}\mu\rho & 0 & \dfrac{(1+\nu)(1-2\nu)}{E_s(1-\nu)}\rho & 0 \\
0 & -\mu\rho & 0 & 0 & 0 & \dfrac{2(1+\nu)}{E_s}\rho \\
\hline
-\dfrac{E_s}{1-\nu^2}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial}{\partial \rho}\right) & 0 & -\dfrac{E_s\nu}{1-\nu^2}\mu\dfrac{\partial}{\partial \rho}(\rho\,\cdot) & 0 & 1-\dfrac{\nu}{1-\nu}\dfrac{\partial}{\partial \rho}(\rho\,\cdot) & 0 \\
0 & 0 & 0 & -1-\dfrac{\partial}{\partial \rho}(\rho\,\cdot) & 0 & \mu\rho \\
\dfrac{E_s\nu}{1-\nu^2}\mu\rho\dfrac{\partial}{\partial \rho} & 0 & \dfrac{E_s}{1-\nu^2}\rho\mu^2 & 0 & \dfrac{\nu}{1-\nu}\mu\rho & 0
\end{array}\right]\begin{Bmatrix} u_{\rho;s} \\ u_{\varphi;s} \\ u_{z;s} \\ \sigma_{\varphi\rho;s} \\ \sigma_{\varphi\varphi;s} \\ \sigma_{\varphi z;s} \end{Bmatrix}
$$

Complete solution

$$\begin{aligned} \boldsymbol{f} &= \sum_{i=1}^{12} \varpi_{0,i} \boldsymbol{f}_{0,i} + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} \left(\varpi_{\mu,i,\mathrm{Re}}^{\{m\}} \mathrm{Re} \boldsymbol{f}_{\mu,i}^{\{m\}} + \varpi_{\mu,i,\mathrm{Im}}^{\{m\}} \mathrm{Im} \boldsymbol{f}_{\mu,i}^{\{m\}} + \varpi_{-\mu,i,\mathrm{Re}}^{\{-m\}} \mathrm{Re}\, \boldsymbol{f}_{-\mu,i}^{\{-m\}} + \varpi_{-\mu,i,\mathrm{Im}}^{\{-m\}} \mathrm{Im}\, \boldsymbol{f}_{-\mu,i}^{\{-m\}} \right) \\ &\triangleq \sum_{i=1}^{\infty} \varpi_i \boldsymbol{f}_i \end{aligned}$$

$$\begin{aligned} \delta \Bigg\{ & \int_0^h \left(\sum_{s=1}^n \int_0^{2\pi} \int_{R_{s-1}}^{R_s} \left[\boldsymbol{p}_s^{\mathrm{T}} \frac{\partial \boldsymbol{q}_s}{\partial z} - H_s(\boldsymbol{q}, \boldsymbol{p}) \right] \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}z \right) - \sum_{s=1}^n \iint_{\Omega_{\boldsymbol{q}_h}^s} \left[\boldsymbol{p}_s^{\mathrm{T}} (\boldsymbol{q}_s - \bar{\boldsymbol{q}}_{h;s}) \right] \mathrm{d}\rho \mathrm{d}\varphi \\ & - \sum_{s=1}^n \iint_{\Omega_{\boldsymbol{p}_h}^s} \left[\bar{\boldsymbol{p}}_{h;s}^{\mathrm{T}} \boldsymbol{q}_s \right] \mathrm{d}\rho \mathrm{d}\varphi + \sum_{s=1}^n \iint_{\Omega_{\boldsymbol{q}_0}^s} \left[\boldsymbol{p}_s^{\mathrm{T}} (\boldsymbol{q}_s - \bar{\boldsymbol{q}}_{0;s}) \right] \mathrm{d}\rho \mathrm{d}\varphi + \sum_{s=1}^n \iint_{\Omega_{\boldsymbol{p}_0}^s} \left[\bar{\boldsymbol{p}}_{0;s}^{\mathrm{T}} \boldsymbol{q}_s \right] \mathrm{d}\rho \mathrm{d}\varphi \Bigg\} = 0 \end{aligned}$$

$$\begin{cases} \left\langle \bigcup_{s=1}^n \left(\Phi_{\mu,i;s}^{\{m\}} \mathrm{e}^{\mathrm{i}m\varphi} \right), \bigcup_{s=1}^n \left(\Phi_{-\mu,j;s}^{\{-k\}} \mathrm{e}^{-\mathrm{i}k\varphi} \right) \right\rangle = \delta_{ij} \delta_{mk} \\ \left\langle \bigcup_{s=1}^n \left(\Phi_{\mu,i;s}^{\{m\}} \mathrm{e}^{\mathrm{i}m\varphi} \right), \bigcup_{s=1}^n \left(\Phi_{\mu,j;s}^{\{k\}} \mathrm{e}^{\mathrm{i}k\varphi} \right) \right\rangle = \left\langle \bigcup_{s=1}^n \left(\Phi_{-\mu,i;s}^{\{-m\}} \mathrm{e}^{-\mathrm{i}m\varphi} \right), \bigcup_{s=1}^n \left(\Phi_{-\mu,j;s}^{\{-k\}} \mathrm{e}^{-\mathrm{i}k\varphi} \right) \right\rangle = 0 \\ \left\langle \bigcup_{s=1}^n \left(\Phi_{\mu,i;s}^{\{m\}} \mathrm{e}^{\mathrm{i}m\varphi} \right), \bigcup_{s=1}^n \Phi_{0,j;s}^{(k)} \right\rangle = 0 \end{cases}$$

3D Laminated Model

Cantor pairing diagram

$$
\begin{cases}
\varpi^{\{m\}}_{\mu,i,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{m\}}_{\mu,i} \mapsto \varpi_j \boldsymbol{f}_j & (i = 1,2,\cdots,+\infty; m = 0,\cdots,+\infty; j = 13,17,21,\cdots,+\infty) \\
\varpi^{\{m\}}_{\mu,i,\mathrm{Im}}\mathrm{Im}\boldsymbol{f}^{\{m\}}_{\mu,i} \mapsto \varpi_j \boldsymbol{f}_j & (i = 1,2,\cdots,+\infty; m = 0,\cdots,+\infty; j = 14,18,22,\cdots,+\infty) \\
\varpi^{\{-m\}}_{-\mu,i,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{-m\}}_{-\mu,i} \mapsto \varpi_j \boldsymbol{f}_j & (i = 1,2,\cdots,+\infty; m = 0,\cdots,+\infty; j = 15,19,23,\cdots,+\infty) \\
\varpi^{\{-m\}}_{-\mu,i,\mathrm{Im}}\,\mathrm{Im}\,\boldsymbol{f}^{\{-m\}}_{-\mu,i} \mapsto \varpi_j \boldsymbol{f}_j & (i = 1,2,\cdots,+\infty; m = 0,\cdots,+\infty; j = 16,20,24,\cdots,+\infty)
\end{cases}
$$

$\boxed{\aleph_0}$

$$
\begin{tikzcd}
\varpi^{\{0\}}_{\mu,1,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{0\}}_{\mu,1} \arrow[r] & \varpi^{\{1\}}_{\mu,1,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{1\}}_{\mu,1} \arrow[dl] & \varpi^{\{2\}}_{\mu,1,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{2\}}_{\mu,1} \arrow[r] & \varpi^{\{3\}}_{\mu,1,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{3\}}_{\mu,1} \arrow[dl] & \varpi^{\{4\}}_{\mu,1,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{4\}}_{\mu,1} & \cdots \\
\varpi^{\{0\}}_{\mu,2,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{0\}}_{\mu,2} \arrow[d] & \varpi^{\{1\}}_{\mu,2,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{1\}}_{\mu,2} \arrow[ur] & \varpi^{\{2\}}_{\mu,2,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{2\}}_{\mu,2} \arrow[dl] & \varpi^{\{3\}}_{\mu,2,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{3\}}_{\mu,2} \arrow[ur] & & \cdots \\
\varpi^{\{0\}}_{\mu,3,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{0\}}_{\mu,3} \arrow[ur] & \varpi^{\{1\}}_{\mu,3,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{1\}}_{\mu,3} \arrow[dl] & \varpi^{\{2\}}_{\mu,3,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{2\}}_{\mu,3} \arrow[ur] & & & \cdots \\
\varpi^{\{0\}}_{\mu,4,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{0\}}_{\mu,4} \arrow[d] & \varpi^{\{1\}}_{\mu,4,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{1\}}_{\mu,4} \arrow[ur] & & & & \cdots \\
\varpi^{\{0\}}_{\mu,5,\mathrm{Re}}\mathrm{Re}\boldsymbol{f}^{\{0\}}_{\mu,5} \arrow[ur] & & & & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}
$$

3D Laminated Model

Contact region determination

$$
\begin{aligned}
U &= \frac{1}{2}\int_0^h \iint_\Omega \left(\sigma_{\rho\rho}\varepsilon_{\rho\rho} + \sigma_{\varphi\varphi}\varepsilon_{\varphi\varphi} + \sigma_{zz}\varepsilon_{zz} + \sigma_{\rho\varphi}\gamma_{\rho\varphi} + \sigma_{\rho z}\gamma_{\rho z} + \sigma_{\varphi z}\gamma_{\varphi z}\right)\rho \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}z \\
&= \frac{1}{2}\int_0^h \iint_\Omega \left[\left(\frac{1}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{|1}}{\partial\rho} + \frac{\nu}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{|2}}{\partial\varphi} + E\boldsymbol{f}_{|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{|6}\right)\frac{\partial\boldsymbol{f}_{|1}}{\partial\rho} + \boldsymbol{f}_{|4}\left(\frac{\partial\boldsymbol{f}_{|1}}{\partial z} + \frac{\partial\boldsymbol{f}_{|3}}{\partial\rho}\right)\right. \\
&\quad + \left(\frac{\nu}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{|1}}{\partial\rho} + \frac{1}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{|2}}{\partial\varphi} + E\boldsymbol{f}_{|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{|6}\right)\left(\frac{1}{\rho}\frac{\partial\boldsymbol{f}_{|2}}{\partial\varphi} + \frac{\boldsymbol{f}_{|1}}{\rho}\right) + \boldsymbol{f}_{|6}\frac{\partial\boldsymbol{f}_{|3}}{\partial z} \\
&\quad \left. + \frac{1}{2(1+\nu)}\left(\rho\frac{\partial E\boldsymbol{f}_{|2}}{\partial\rho} + \frac{\partial E\boldsymbol{f}_{|1}}{\partial\varphi} - E\boldsymbol{f}_{|2}\right)\left(\frac{\partial\boldsymbol{f}_{|2}}{\partial\rho} + \frac{1}{\rho}\frac{\partial\boldsymbol{f}_{|1}}{\partial\varphi} - \frac{\boldsymbol{f}_{|2}}{\rho}\right) + \boldsymbol{f}_{|5}\left(\frac{1}{\rho}\frac{\partial\boldsymbol{f}_{|3}}{\partial\varphi} + \frac{\partial\boldsymbol{f}_{|2}}{\partial z}\right)\right]\mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}z
\end{aligned}
$$

$$
\sum_{j=1}^{\infty}\frac{\partial U}{\partial\varpi_j}\delta\varpi_j - \Delta\tilde{\gamma}\int_0^{2\pi} g(\varphi)\delta g \mathrm{d}\varphi = 0
$$

$$
\delta U_f = \delta\left(U - \Delta\tilde{\gamma}\int_0^{2\pi}\int_0^{g(\varphi)}\rho \mathrm{d}\rho \mathrm{d}\varphi\right) = 0
$$

$$
\begin{aligned}
\delta U - \Delta\tilde{\gamma}\delta\int_{\alpha_1}^{\alpha_2}\int_{g_1(\varphi)}^{g_2(\varphi)}\rho \mathrm{d}\rho \mathrm{d}\varphi &= \delta U - \Delta\tilde{\gamma}\underbrace{\cancel{\int_{g_1(\alpha_2)}^{g_2(\alpha_2)}\rho \mathrm{d}\rho\delta\alpha_2}}_{g_2(\alpha_2)=g_1(\alpha_2)} + \Delta\tilde{\gamma}\underbrace{\cancel{\int_{g_1(\alpha_1)}^{g_2(\alpha_1)}\rho \mathrm{d}\rho\delta\alpha_1}}_{g_2(\alpha_1)=g_1(\alpha_1)} \\
&\quad - \Delta\tilde{\gamma}\int_{\alpha_1}^{\alpha_2} g_2(\varphi)\delta g_2 \mathrm{d}\varphi + \Delta\tilde{\gamma}\int_{\alpha_1}^{\alpha_2} g_1(\varphi)\delta g_1 \mathrm{d}\varphi
\end{aligned}
$$

3D Laminated Model

Contact region determination

$$
\begin{aligned}
\frac{\partial U}{\partial \varpi_j} = \frac{1}{2}\int_0^h \iint_\Omega \Bigg[& \left(\frac{1}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{j|1}}{\partial \rho} + \frac{\nu}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{j|2}}{\partial \varphi} + E\boldsymbol{f}_{j|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{j|6}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\frac{\partial \boldsymbol{f}_{i|1}}{\partial \rho} + \frac{\partial \boldsymbol{f}_{j|1}}{\partial \rho}\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{1}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{i|1}}{\partial \rho} + \frac{\nu}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{i|2}}{\partial \varphi} + E\boldsymbol{f}_{i|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{i|6}\right) \\
& + \boldsymbol{f}_{j|4}\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{\partial \boldsymbol{f}_{i|1}}{\partial z} + \frac{\partial \boldsymbol{f}_{i|3}}{\partial \rho}\right) + \left(\frac{\partial \boldsymbol{f}_{j|1}}{\partial z} + \frac{\partial \boldsymbol{f}_{j|3}}{\partial \rho}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\boldsymbol{f}_{i|4} + \boldsymbol{f}_{j|6}\cdot\sum_{i=1}^{\infty}\varpi_i\frac{\partial \boldsymbol{f}_{i|3}}{\partial z} + \frac{\partial \boldsymbol{f}_{j|3}}{\partial z}\cdot\sum_{i=1}^{\infty}\varpi_i\boldsymbol{f}_{i|6} \\
& + \left(\frac{\nu}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{j|1}}{\partial \rho} + \frac{1}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{j|2}}{\partial \varphi} + E\boldsymbol{f}_{j|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{j|6}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{1}{\rho}\frac{\partial \boldsymbol{f}_{i|2}}{\partial \varphi} + \frac{\boldsymbol{f}_{i|1}}{\rho}\right) \\
& + \left(\frac{1}{\rho}\frac{\partial \boldsymbol{f}_{j|2}}{\partial \varphi} + \frac{\boldsymbol{f}_{j|1}}{\rho}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{\nu}{1-\nu^2}\rho\frac{\partial E\boldsymbol{f}_{i|1}}{\partial \rho} + \frac{1}{1-\nu^2}\left(\frac{\partial E\boldsymbol{f}_{i|2}}{\partial \varphi} + E\boldsymbol{f}_{i|1}\right) + \frac{\nu}{1-\nu}\boldsymbol{f}_{i|6}\right) + \frac{1}{2(1+\nu)}\left(\rho\frac{\partial E\boldsymbol{f}_{j|2}}{\partial \rho} + \frac{\partial E\boldsymbol{f}_{j|1}}{\partial \varphi} - E\boldsymbol{f}_{j|2}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{\partial \boldsymbol{f}_{i|2}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \boldsymbol{f}_{i|1}}{\partial \varphi} - \frac{\boldsymbol{f}_{i|2}}{\rho}\right) \\
& + \frac{1}{2(1+\nu)}\left(\frac{\partial \boldsymbol{f}_{j|2}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \boldsymbol{f}_{j|1}}{\partial \varphi} - \frac{\boldsymbol{f}_{j|2}}{\rho}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\left(\rho\frac{\partial E\boldsymbol{f}_{i|2}}{\partial \rho} + \frac{\partial E\boldsymbol{f}_{i|1}}{\partial \varphi} - E\boldsymbol{f}_{i|2}\right) + \boldsymbol{f}_{j|5}\cdot\sum_{i=1}^{\infty}\varpi_i\left(\frac{1}{\rho}\frac{\partial \boldsymbol{f}_{i|3}}{\partial \varphi} + \frac{\partial \boldsymbol{f}_{i|2}}{\partial z}\right) + \left(\frac{1}{\rho}\frac{\partial \boldsymbol{f}_{j|3}}{\partial \varphi} + \frac{\partial \boldsymbol{f}_{j|2}}{\partial z}\right)\cdot\sum_{i=1}^{\infty}\varpi_i\boldsymbol{f}_{i|5}\Bigg]\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
-\Delta\tilde{\gamma}g(\varphi) + \sum_{j=1}^{\infty}\Bigg\{\frac{\partial U}{\partial \varpi_j}\Bigg[& \mathscr{A}_{ji}^{-1}\left[(\rho\sigma_{zz})_i(u_z)_j + (u_z)_i(\rho\sigma_{zz})_j\right]_{\rho=g(\varphi)}\mathscr{A}_{ji}^{-1}\mathscr{H}_i \\
& + \mathscr{A}_{ji}^{-1}\left[\left\{d - r + \sqrt{r^2 - \left[\rho^2 + R_p^2 - 2\rho R_p\cos\left(\varphi - \phi_p\right)\right]}\right\}\left(\rho\sigma_{zz}\right)_i\right]_{\rho=g(\varphi)}\Bigg]\Bigg\} = 0
\end{aligned}
$$

3D Laminated Model

Dual Hamiltonian transformation

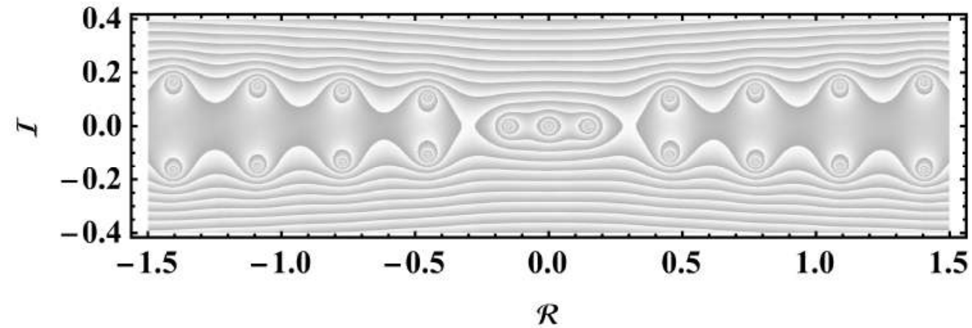
$$\frac{\partial}{\partial z}\mathbf{I}_6\boldsymbol{f}^{\dagger}=\boldsymbol{\mathcal{H}}^{\dagger}\boldsymbol{f}^{\dagger} \qquad \sigma_{ij}^{\dagger}=\sigma_{ij}\mathrm{e}^{-\chi\rho}$$

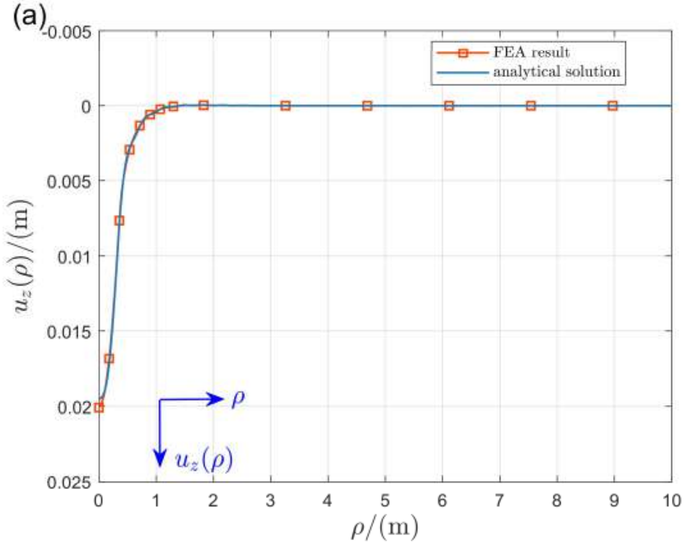
$$\boldsymbol{\mathcal{H}}^{\dagger}=\left[\begin{array}{ccc:ccc}
0 & 0 & -\frac{\partial}{\partial\rho} & \frac{2(1+\nu)}{E_0\rho} & 0 & 0 \\
0 & 0 & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & \frac{2(1+\nu)}{E_0\rho} & 0 \\
-\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right) & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & \frac{(1+\nu)(1-2\nu)}{E_0(1-\nu)\rho} \\
\hdashline
-\frac{E_0}{1-\nu^2}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{E_0}{2(1+\nu)}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\chi\frac{E_0}{1-\nu^2}\left(\rho\frac{\partial}{\partial\rho}+\nu\right) & -\frac{E_0}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{E_0(3-\nu)}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\chi\frac{E_0\nu}{1-\nu^2}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}+\chi\right) \\
-\frac{E_0}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}-\frac{E_0(3-\nu)}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\chi\frac{E_0}{2(1+\nu)}\frac{\partial}{\partial\varphi} & -\frac{E_0}{2(1+\nu)}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{E_0}{1-\nu^2}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\chi\frac{E_0}{2(1+\nu)}\left(\rho\frac{\partial}{\partial\rho}-1\right) & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} \\
0 & 0 & 0 & -\left(\frac{\partial}{\partial\rho}+\chi\right) & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0
\end{array}\right]$$

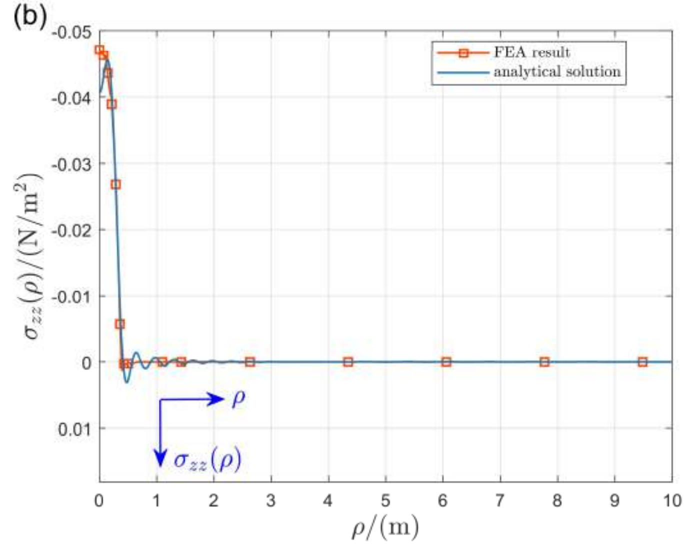
$$\left\langle\left(\boldsymbol{f}^{\alpha}\right)^{\dagger},\boldsymbol{\mathcal{H}}^{\dagger}\left(\boldsymbol{f}^{\beta}\right)^{\dagger}\right\rangle=\left\langle\left(\boldsymbol{f}^{\beta}\right)^{\dagger},\boldsymbol{\mathcal{H}}^{\dagger}\left(\boldsymbol{f}^{\alpha}\right)^{\dagger}\right\rangle-\chi\left\langle\left(\widetilde{\boldsymbol{f}}^{\alpha}\right)^{\dagger},\left(\widetilde{\boldsymbol{f}}^{\beta}\right)^{\dagger}\right\rangle$$

3D Laminated Model

Numerical example







(QUASI-)HAMILTONIAN OPERATORS

Symmetry & operators

Hamiltonian operators

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & -\mathbf{A}^{\mathcal{T}} \end{array}\right]$$

skew-Hamiltonian operators

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{A}^{\mathcal{T}} \end{array}\right]$$

upper triangular/diagonal······
Hamiltonian operators

quasi-Hamiltonian operators (2D)

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & -\mathbf{A}_{\alpha}^{\mathcal{T}} \end{array}\right]$$

Shifted Hamiltonian transformation

$\updownarrow$ Gradient
(arbitrary direction)

Dual Hamiltonian transformation

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & -\mathbf{A}_{\beta}^{\mathcal{T}} \end{array}\right]$$

$$\boldsymbol{\mathcal{H}} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C}_{\beta *} & -\mathbf{A}_{\beta}^{\mathcal{T}} \end{array}\right]$$

Dual Hamiltonian transformation

Inherent Asymmetry

$$\boldsymbol{\mathcal{H}}^{x} = \left[\begin{array}{c|c} \mathbf{A}_{x} & \mathbf{B}_{x} \\ \hline \mathbf{C}_{\vartheta_{x}} & -\mathbf{A}_{\vartheta_{x}}^{\mathcal{T}} \end{array}\right]$$

Generalized dual Hamiltonian transformation

Variable coefficients

quasi-Hamiltonian operators (3D)

$$\boldsymbol{\mathcal{H}}^{\dagger} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C}_{\chi} & -\mathbf{A}_{\chi}^{\mathcal{T}} \end{array}\right]$$

Asymmetry induced by odd-dimension

$$\boldsymbol{\mathcal{H}}^{\ddagger} = \left[\begin{array}{c|c} \mathbf{A}_{x} & \mathbf{B}_{x} \\ \hline \mathbf{C}_{E_{x}} & -\mathbf{A}_{E_{x}}^{\mathcal{T}} \end{array}\right]$$

$$\boldsymbol{\mathcal{H}}_{\oplus} = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C}_{\oplus} & -\mathbf{A}^{\mathcal{T}} \end{array}\right]$$

Incremental theory

Summary

Hypothesis

- 推广到三维直角坐标辛体系的难点：边界条件混合与控制方程混合（而柱坐标较易推广，均匀情况下最终直接化为Bessel函数叠加）
  - Hypothesis1：三维是否需要三类变量？
    - 三类变量广义变分原理在构建辛失效（三类变量分别是位移、应力、应变）
    - 对应于3n维相空间，依据Nambu力学推广Hamilton力学体系［南部阳一郎，经典弦理论］，但是需要引入两个Hamiltonian，以及两个Lagrangian

$$\dot{q} = r \quad \dot{p} = \frac{\partial L}{\partial q} \quad p = \frac{\partial L}{\partial r}$$

  - Hypothesis2：三维是否需要构造新空间以区别辛体系？
    - Clifford体系（四元数是Clifford的一种特殊情况）引入外微分形式
    - 想法的来源与依据：
      - 第一，辛内积区别于欧氏内积，具有面积度量，辛矩阵就是旋转90度的旋转矩阵，而三维是不是需要定义体积度量，考虑三维旋转（四元数用于旋转恰可以避免万向结自锁）
      - 第二，借助于算子矩阵理论，我们发现，倘若需要将三维的调和算子分解，需要引入Clifford代数，分解为Dirac算子

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H_1, H_2\} \equiv \frac{\partial(f, H_1, H_2)}{\partial(q, p, r)}$$

$$dH_1 \wedge dH_2 = \frac{1}{\dot{q}} d(p\dot{q} - L_1) \wedge d(r\dot{q} - L_2)$$

$$\Delta = \mathcal{D}^2 = \left(e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right)\left(e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right)$$

      - 第三，从命名而言，symplectic来源于希腊语，是复（complex）的含义，辛空间对应于复域分析，而缺少四元域分析体系，需要区别于闵可夫斯基空间（不存在三元数，域不封闭）

Summary

Hypothesis

- Hypothesis3: 不改变辛，能不能引入多辛（multi-symplectic）体系？
  - 确实存在多辛体系，但大多对非线性双变元情况

$$
Jz_t = \nabla_z H(z) \qquad Kz_t + Lz_x = \nabla_z S(z)
$$

- Hypothesis4: 构建直角形式的二次辛化（升维）

$$
\left[\begin{array}{cccc|cccc}
-\mu & -\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 & \frac{(1+\nu)(1-2\nu)}{E_0(1-\nu)} & 0 & 0 & 0 \\
-\frac{\partial}{\partial y} & -\mu & 0 & 0 & 0 & \frac{2(1+\nu)}{E_0} & 0 & 0 \\
-\frac{\partial}{\partial x} & 0 & -\mu & 0 & 0 & 0 & \frac{2(1+\nu)}{E_0} & 0 \\
0 & -\frac{1+\nu}{1-\nu}\frac{\partial}{\partial x} & -\frac{1+\nu}{1-\nu}\frac{\partial}{\partial y} & -1 & 0 & 0 & 0 & \frac{2}{1-\nu}\frac{2(1+\nu)}{E_0} \\
\hline
0 & 0 & 0 & 0 & -\mu & -\frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \\
0 & -\frac{E_0}{1-\nu^2}\frac{\partial^2}{\partial y^2}+\frac{E_0\nu}{1-\nu^2}\frac{\partial^2}{\partial x^2} & 0 & 0 & -\frac{\nu}{1-\nu}\frac{\partial}{\partial y} & -\mu & 0 & -\frac{1+\nu}{1-\nu}\frac{\partial}{\partial x} \\
0 & 0 & -\frac{E_0}{1-\nu^2}\frac{\partial^2}{\partial x^2}+\frac{E_0\nu}{1-\nu^2}\frac{\partial^2}{\partial y^2} & 0 & -\frac{\nu}{1-\nu}\frac{\partial}{\partial x} & 0 & -\mu & -\frac{1+\nu}{1-\nu}\frac{\partial}{\partial y} \\
0 & 0 & 0 & \frac{E_0}{2(1+\nu)} & 0 & 0 & 0 & -1
\end{array}\right]
\begin{Bmatrix} u_z \\ u_y \\ u_x \\ \gamma_{xy} \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = 0
$$

Acknowledgments

Thank You !

It is a great honor to invite you!
I am wondering if I could have some feedbacks.

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