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Supplementary material for 3D symplectic laminated contact analysis

Appendix E. Dual Hamiltonian Transformation for a 3D Inhomogeneous Layer of a Cylinder

We may assume that Young's modulus varies exponentially as $E(\rho)=E_0\mathrm{e}^{\chi\rho}$ within a layer, then the state equations are

$$\frac{\partial}{\partial z}\mathbf{I}_6\boldsymbol{f}^{\dagger}=\boldsymbol{\mathcal{H}}^{\dagger}\boldsymbol{f}^{\dagger} \tag{E.1}$$

which is detailed as

$$\frac{\partial}{\partial z}\begin{Bmatrix} u_\rho \\ u_\varphi \\ u_z \\ \rho\sigma_{\rho z}^{\dagger} \\ \rho\sigma_{\varphi z}^{\dagger} \\ \rho\sigma_{zz}^{\dagger} \end{Bmatrix}=\left[\begin{array}{ccc|ccc} 0 & 0 & -\frac{\partial}{\partial\rho} & \frac{2(1+\nu)}{E_0\rho} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & \frac{2(1+\nu)}{E_0\rho} & 0 \\ -\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right) & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & \frac{(1+\nu)(1-2\nu)}{E_0(1-\nu)\rho} \\ \hline -\frac{E_0}{1-\nu^2}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{E_0}{2(1+\nu)}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\chi\frac{E_0}{1-\nu^2}\left(\rho\frac{\partial}{\partial\rho}+\nu\right) & -\frac{E_0}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{E_0(3-\nu)}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\chi\frac{E_0\nu}{1-\nu^2}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}+\chi\right) \\ -\frac{E_0}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}-\frac{E_0(3-\nu)}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\chi\frac{E_0}{2(1+\nu)}\frac{\partial}{\partial\varphi} & -\frac{E_0}{2(1+\nu)}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{E_0}{1-\nu^2}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\chi\frac{E_0}{2(1+\nu)}\left(\rho\frac{\partial}{\partial\rho}-1\right) & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} \\ 0 & 0 & 0 & -\left(\frac{\partial}{\partial\rho}+\chi\right) & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 \end{array}\right]\begin{Bmatrix} u_\rho \\ u_\varphi \\ u_z \\ \rho\sigma_{\rho z}^{\dagger} \\ \rho\sigma_{\varphi z}^{\dagger} \\ \rho\sigma_{zz}^{\dagger} \end{Bmatrix} \tag{E.2}$$

where χ is a constant, E_0 is Young's modulus at $\rho=0$, and $\sigma_{ij}^{\dagger}=\sigma_{ij}\mathrm{e}^{-\chi\rho}$. If the boundary conditions are homogeneous, we can prove that

$$\left\langle\left(\boldsymbol{f}^{\alpha}\right)^{\dagger},\boldsymbol{\mathcal{H}}^{\dagger}\left(\boldsymbol{f}^{\beta}\right)^{\dagger}\right\rangle=\left\langle\left(\boldsymbol{f}^{\beta}\right)^{\dagger},\boldsymbol{\mathcal{H}}^{\dagger}\left(\boldsymbol{f}^{\alpha}\right)^{\dagger}\right\rangle-\chi\left\langle\left(\widetilde{\boldsymbol{f}}^{\alpha}\right)^{\dagger},\left(\widetilde{\boldsymbol{f}}^{\beta}\right)^{\dagger}\right\rangle \tag{E.3}$$

which is a dual Hamiltonian transformation. The notations here are the same as those in Eqs. (10) and (11). Following a set of proofs similar to those presented in appendix B of Chen et al. (2025a) together with fixed lateral boundary conditions, we can also prove the dual adjoint symplectic orthogonality in 3D.

We may further generalize the dual Hamiltonian transformation in Eq. (E.3) as that in section 3 of Chen et al. (2025b) as

$$\left\langle \left(\boldsymbol{f}^{\alpha}\right)^{\ddagger}, \boldsymbol{\mathcal{H}}^{\ddagger}\left(\boldsymbol{f}^{\beta}\right)^{\ddagger} \right\rangle = \left\langle \left(\boldsymbol{f}^{\beta}\right)^{\ddagger}, \boldsymbol{\mathcal{H}}^{\ddagger}\left(\boldsymbol{f}^{\alpha}\right)^{\ddagger} \right\rangle - \left\langle \sqrt{\frac{E'(\rho)}{E(\rho)}}\left(\tilde{\boldsymbol{f}}^{\alpha}\right)^{\ddagger}, \sqrt{\frac{E'(\rho)}{E(\rho)}}\left(\tilde{\boldsymbol{f}}^{\beta}\right)^{\ddagger} \right\rangle \tag{E.4}$$

where $\sigma_{ij}^{\ddagger} = \sigma_{ij}/E(\rho)$, $E'(\rho) = \mathrm{d}E(\rho)/\mathrm{d}\rho$, and

$$\boldsymbol{\mathcal{H}}^{\ddagger} = \left[\begin{array}{ccc:ccc}
0 & 0 & -\frac{\partial}{\partial\rho} & \frac{2(1+\nu)}{\rho} & 0 & 0 \\
0 & 0 & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & \frac{2(1+\nu)}{\rho} & 0 \\
-\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right) & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & \frac{(1+\nu)(1-2\nu)}{(1-\nu)\rho} \\
\hdashline
-\frac{1}{1-\nu^2}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{1}{2(1+\nu)}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\frac{E'(\rho)}{E(\rho)}\frac{1}{1-\nu^2}\left(\rho\frac{\partial}{\partial\rho}+\nu\right) & -\frac{1}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{3-\nu}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\frac{E'(\rho)}{E(\rho)}\frac{\nu}{1-\nu^2}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}+\frac{E'(\rho)}{E(\rho)}\right) \\
-\frac{1}{2(1-\nu)}\frac{\partial^2}{\partial\rho\partial\varphi}-\frac{3-\nu}{2(1-\nu^2)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}-\frac{E'(\rho)}{E(\rho)}\frac{1}{2(1+\nu)}\frac{\partial}{\partial\varphi} & -\frac{1}{2(1+\nu)}\left(\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)-\frac{1}{1-\nu^2}\frac{1}{\rho}\frac{\partial^2}{\partial\varphi^2}-\frac{E'(\rho)}{E(\rho)}\frac{1}{2(1+\nu)}\left(\rho\frac{\partial}{\partial\rho}-1\right) & 0 & 0 & 0 & -\frac{\nu}{1-\nu}\frac{1}{\rho}\frac{\partial}{\partial\varphi} \\
0 & 0 & 0 & -\left(\frac{\partial}{\partial\rho}+\frac{E'(\rho)}{E(\rho)}\right) & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0
\end{array}\right] \tag{E.5}$$

As we have already seen in Chen et al. (2025a), the dual Hamiltonian transformation is interesting since the respective (two sets of) bases of two pairs of dual operators, are orthogonal and conjugate to each other, which characterize the symplectic dual adjoint quasi-Hamiltonian operator, as distinct from the only one set of self-orthogonal/conjugate bases of a conventional self-adjoint operator satisfying the Sturm-Liouville condition.

Appendix F. Symplectic Formulism for Coupled Physical Multi-fields

We will establish symplectic forms for a linear transversely isotropic piezoelectric cylinder and a compressible soft electroactive cylinder under different fields here.

F.1 Linear piezoelectric cylinder

The constitutive relations and equilibrium equations are respectively expressed as

$$\left\{\begin{aligned}
&\sigma_{\rho\rho}=c_{11}^{*}\frac{\partial u_{\rho}}{\partial\rho}+c_{12}^{*}\left(\frac{1}{\rho}\frac{\partial u_{\varphi}}{\partial\varphi}+\frac{u_{\rho}}{\rho}\right)+c_{13}^{*}\frac{\partial u_{z}}{\partial z}+e_{31}^{*}\frac{\partial\phi_{\odot}}{\partial z}\\
&\sigma_{\varphi\varphi}=c_{12}^{*}\frac{\partial u_{\rho}}{\partial\rho}+c_{11}^{*}\left(\frac{1}{\rho}\frac{\partial u_{\varphi}}{\partial\varphi}+\frac{u_{\rho}}{\rho}\right)+c_{13}^{*}\frac{\partial u_{z}}{\partial z}+e_{31}^{*}\frac{\partial\phi_{\odot}}{\partial z}\\
&\sigma_{zz}=c_{13}^{*}\frac{\partial u_{\rho}}{\partial\rho}+c_{13}^{*}\left(\frac{1}{\rho}\frac{\partial u_{\varphi}}{\partial\varphi}+\frac{u_{\rho}}{\rho}\right)+c_{33}^{*}\frac{\partial u_{z}}{\partial z}+e_{33}^{*}\frac{\partial\phi_{\odot}}{\partial z}\\
&\sigma_{\rho\varphi}=c_{66}^{*}\left(\frac{\partial u_{\varphi}}{\partial\rho}-\frac{u_{\varphi}}{\rho}+\frac{1}{\rho}\frac{\partial u_{\rho}}{\partial\varphi}\right)\\
&\sigma_{\rho z}=c_{44}^{*}\left(\frac{\partial u_{z}}{\partial\rho}+\frac{\partial u_{\rho}}{\partial z}\right)+e_{15}^{*}\frac{\partial\phi_{\odot}}{\partial\rho}\\
&\sigma_{\varphi z}=c_{44}^{*}\left(\frac{1}{\rho}\frac{\partial u_{z}}{\partial\varphi}+\frac{\partial u_{\varphi}}{\partial z}\right)+e_{15}^{*}\frac{1}{\rho}\frac{\partial\phi_{\odot}}{\partial\varphi}\\
&D_{\rho}=e_{15}^{*}\left(\frac{\partial u_{z}}{\partial\rho}+\frac{\partial u_{\rho}}{\partial z}\right)-\epsilon_{11}^{*}\frac{\partial\phi_{\odot}}{\partial\rho}\\
&D_{\varphi}=e_{15}^{*}\left(\frac{1}{\rho}\frac{\partial u_{z}}{\partial\varphi}+\frac{\partial u_{\varphi}}{\partial z}\right)-\epsilon_{11}^{*}\frac{1}{\rho}\frac{\partial\phi_{\odot}}{\partial\varphi}\\
&D_{z}=e_{31}^{*}\frac{\partial u_{\rho}}{\partial\rho}+e_{31}^{*}\left(\frac{1}{\rho}\frac{\partial u_{\varphi}}{\partial\varphi}+\frac{u_{\rho}}{\rho}\right)+e_{33}^{*}\frac{\partial u_{z}}{\partial z}-\epsilon_{33}^{*}\frac{\partial\phi_{\odot}}{\partial z}
\end{aligned}\right.,\quad
\left\{\begin{aligned}
&\frac{\partial\sigma_{\rho\rho}}{\partial\rho}+\frac{1}{\rho}\frac{\partial\sigma_{\rho\varphi}}{\partial\varphi}+\frac{\partial\sigma_{\rho z}}{\partial z}+\frac{\sigma_{\rho\rho}-\sigma_{\varphi\varphi}}{\rho}=0\\
&\frac{\partial\sigma_{\rho\varphi}}{\partial\rho}+\frac{1}{\rho}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi}+\frac{\partial\sigma_{\varphi z}}{\partial z}+\frac{2\sigma_{\rho\varphi}}{\rho}=0\\
&\frac{\partial\sigma_{\rho z}}{\partial\rho}+\frac{1}{\rho}\frac{\partial\sigma_{\varphi z}}{\partial\varphi}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{\rho z}}{\rho}=0\\
&\frac{\partial D_{\rho}}{\partial\rho}+\frac{1}{\rho}\left(\frac{\partial D_{\varphi}}{\partial\varphi}+D_{\rho}\right)+\frac{\partial D_{z}}{\partial z}=0
\end{aligned}\right. \tag{F.1}$$

which lead to the matrix form as $\frac{\partial}{\partial z}\mathbf{I}_8\boldsymbol{f}^*=\boldsymbol{\mathcal{H}}^*\boldsymbol{f}^*$, where c_{ij}^*, e_{ij}^*, and ϵ_{ij}^* are constants,

$$\boldsymbol{f}^*=\left[u_\rho,u_\varphi,u_z,\phi_\odot,\rho\sigma_{\rho z},\rho\sigma_{\varphi z},\rho\sigma_{zz},\rho D_z\right]^{\mathrm{T}},\quad \boldsymbol{\mathcal{H}}^*=\left[\begin{array}{c|c}\mathbf{A}^* & \mathbf{B}^*\\ \hline \mathbf{C}^* & -\mathbf{A}^{*\mathcal{T}}\end{array}\right] \tag{F.2}$$

where

$$\mathbf{A}^{*}=\begin{bmatrix} 0 & 0 & -\frac{\partial}{\partial\rho} & -\frac{e_{15}^{*}}{c_{44}^{*}}\frac{\partial}{\partial\rho} \\ 0 & 0 & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & -\frac{e_{15}^{*}}{c_{44}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} \\ -\frac{\epsilon_{33}^{*}c_{13}^{*}+e_{33}^{*}e_{31}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right) & -\frac{\epsilon_{33}^{*}c_{13}^{*}+e_{33}^{*}e_{31}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 \\ \frac{c_{33}^{*}e_{31}^{*}-c_{13}^{*}e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right) & \frac{c_{33}^{*}e_{31}^{*}-c_{13}^{*}e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 \end{bmatrix},\quad \mathbf{B}^{*}=\begin{bmatrix} \frac{1}{c_{44}^{*}}\frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \frac{1}{c_{44}^{*}}\frac{1}{\rho} & 0 & 0 \\ 0 & 0 & \frac{\epsilon_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho} & \frac{e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho} \\ 0 & 0 & \frac{e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho} & -\frac{c_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho} \end{bmatrix} \tag{F.3}$$

$$\mathbf{C}^{*}=\begin{bmatrix} \mathcal{D}_{1}^{*} & \mathcal{D}_{2}^{*} & 0 & 0 \\ \mathcal{D}_{3}^{*} & \mathcal{D}_{4}^{*} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\epsilon_{11}^{**}+\frac{\left(e_{15}^{*}\right)^{2}}{c_{44}^{*}}\right)\rho\nabla^{2} \end{bmatrix},\quad -\mathbf{A}^{*\mathcal{T}}=\begin{bmatrix} 0 & 0 & -\frac{\epsilon_{33}^{*}c_{13}^{*}+e_{33}^{*}e_{31}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right) & \frac{c_{33}^{*}e_{31}^{*}-c_{13}^{*}e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right) \\ 0 & 0 & -\frac{\epsilon_{33}^{*}c_{13}^{*}+e_{33}^{*}e_{31}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & \frac{c_{33}^{*}e_{31}^{*}-c_{13}^{*}e_{33}^{*}}{\left(e_{33}^{*}\right)^{2}+c_{33}^{*}\epsilon_{33}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} \\ -\frac{\partial}{\partial\rho} & -\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 \\ -\frac{e_{15}^{*}}{c_{44}^{*}}\frac{\partial}{\partial\rho} & -\frac{e_{15}^{*}}{c_{44}^{*}}\frac{1}{\rho}\frac{\partial}{\partial\varphi} & 0 & 0 \end{bmatrix} \tag{F.4}$$

$$\begin{cases} \mathcal{D}_1^* = -\left[c_{11}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\left(\rho\dfrac{\partial^2}{\partial\rho^2} + \dfrac{\partial}{\partial\rho} - \dfrac{1}{\rho}\right) - c_{66}^*\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2} \\ \mathcal{D}_2^* = -\left[c_{12}^* + c_{66}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\dfrac{\partial^2}{\partial\rho\partial\varphi} + \left[c_{11}^* + c_{66}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\ \mathcal{D}_3^* = -\left[c_{12}^* + c_{66}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\dfrac{\partial^2}{\partial\rho\partial\varphi} - \left[c_{11}^* + c_{66}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\ \mathcal{D}_4^* = -c_{66}^*\left(\rho\dfrac{\partial^2}{\partial\rho^2} + \dfrac{\partial}{\partial\rho} - \dfrac{1}{\rho}\right) - \left[c_{11}^* + \dfrac{c_{33}^*\left(e_{31}^*\right)^2 - \epsilon_{33}^*\left(c_{13}^*\right)^2 - 2c_{13}^* e_{31}^* e_{33}^*}{\left(e_{33}^*\right)^2 + c_{33}^*\epsilon_{33}^*}\right]\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2} \end{cases} \tag{F.5}$$

From the sub-symplectic space representation, we can obtain the solutions with Bessel functions (Xu et al., 2008).

F.2 Compressible soft electroactive cylinder

In this section, the formulations and constants are generally adopted as in Su et al. (2016) except for replacing incompressible governing equations with compressible ones. According to the linear theory of incremental field, the governing equations are

$$\begin{cases}
\dot{T}_{0\rho\rho} = c_{11}^{**}\dfrac{\partial u_\rho}{\partial \rho} + c_{12}^{**}\left(\dfrac{1}{\rho}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_\rho}{\rho}\right) + c_{13}^{**}\dfrac{\partial u_z}{\partial z} + e_{31}^{**}\dfrac{\partial \phi_\odot}{\partial z} \\
\dot{T}_{0\varphi\varphi} = c_{12}^{**}\dfrac{\partial u_\rho}{\partial \rho} + c_{11}^{**}\left(\dfrac{1}{\rho}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_\rho}{\rho}\right) + c_{13}^{**}\dfrac{\partial u_z}{\partial z} + e_{31}^{**}\dfrac{\partial \phi_\odot}{\partial z} \\
\dot{T}_{0zz} = c_{13}^{**}\dfrac{\partial u_\rho}{\partial \rho} + c_{13}^{**}\left(\dfrac{1}{\rho}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_\rho}{\rho}\right) + c_{33}^{**}\dfrac{\partial u_z}{\partial z} + e_{33}^{**}\dfrac{\partial \phi_\odot}{\partial z} \\
\dot{T}_{0\rho\varphi} = c_{14}^{**}\dfrac{\partial u_\varphi}{\partial \rho} + c_{15}^{**}\left(\dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \varphi} - \dfrac{u_\varphi}{\rho}\right), \quad \dot{T}_{0\varphi\rho} = c_{14}^{**}\left(\dfrac{1}{\rho}\dfrac{\partial u_\rho}{\partial \varphi} - \dfrac{u_\varphi}{\rho}\right) + c_{15}^{**}\dfrac{\partial u_\varphi}{\partial \rho} \\
\dot{T}_{0\rho z} = c_{551}^{**}\dfrac{\partial u_\rho}{\partial z} + c_{552}^{**}\dfrac{\partial u_z}{\partial \rho} + e_{15}^{**}\dfrac{\partial \phi_\odot}{\partial \rho}, \quad \dot{T}_{0z\rho} = c_{553}^{**}\dfrac{\partial u_\rho}{\partial z} + c_{551}^{**}\dfrac{\partial u_z}{\partial \rho} + e_{15}^{**}\dfrac{\partial \phi_\odot}{\partial \rho} \\
\dot{T}_{0\varphi z} = c_{552}^{**}\dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \varphi} + c_{551}^{**}\dfrac{\partial u_\varphi}{\partial z} + e_{15}^{**}\dfrac{1}{\rho}\dfrac{\partial \phi_\odot}{\partial \varphi}, \quad \dot{T}_{0z\varphi} = c_{553}^{**}\dfrac{\partial u_\varphi}{\partial z} + c_{551}^{**}\dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \varphi} + e_{15}^{**}\dfrac{1}{\rho}\dfrac{\partial \phi_\odot}{\partial \varphi} \\
\dot{D}_{l0\rho} = e_{15}^{**}\left(\dfrac{\partial u_\rho}{\partial z} + \dfrac{\partial u_z}{\partial \rho}\right) - \epsilon_{11}^{**}\dfrac{\partial \phi_\odot}{\partial \rho} \\
\dot{D}_{l0\varphi} = e_{15}^{**}\left(\dfrac{\partial u_\varphi}{\partial z} + \dfrac{1}{\rho}\dfrac{\partial u_z}{\partial \varphi}\right) - \epsilon_{11}^{**}\dfrac{1}{\rho}\dfrac{\partial \phi_\odot}{\partial \varphi} \\
\dot{D}_{l0z} = e_{31}^{**}\left(\dfrac{\partial u_\rho}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_\rho}{\rho}\right) + e_{33}^{**}\dfrac{\partial u_z}{\partial z} - \epsilon_{33}^{**}\dfrac{\partial \phi_\odot}{\partial z}
\end{cases}, \quad
\begin{cases}
\dfrac{\partial \dot{T}_{0\rho\rho}}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial \dot{T}_{0\varphi\rho}}{\partial \varphi} + \dfrac{\partial \dot{T}_{0zr}}{\partial z} + \dfrac{\dot{T}_{0\rho\rho} - \dot{T}_{0\varphi\varphi}}{\rho} = 0 \\
\dfrac{\partial \dot{T}_{0\rho\varphi}}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial \dot{T}_{0\varphi\varphi}}{\partial \varphi} + \dfrac{\partial \dot{T}_{0z\varphi}}{\partial z} + \dfrac{\dot{T}_{0\rho\varphi} + \dot{T}_{0\varphi\rho}}{\rho} = 0 \\
\dfrac{\partial \dot{T}_{0\rho z}}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial \dot{T}_{0\varphi z}}{\partial \varphi} + \dfrac{\partial \dot{T}_{0zz}}{\partial z} + \dfrac{\dot{T}_{0\rho z}}{\rho} = 0 \\
\dfrac{\partial \dot{D}_{l0\rho}}{\partial \rho} + \dfrac{1}{\rho}\left(\dfrac{\partial \dot{D}_{l0\varphi}}{\partial \varphi} + \dot{D}_{l0\rho}\right) + \dfrac{\partial \dot{D}_{l0z}}{\partial z} = 0
\end{cases} \tag{F.6}$$

which lead to $\frac{\partial}{\partial z}\mathbf{I}_8\boldsymbol{f}^{**} = \boldsymbol{\mathcal{H}}^{**}\boldsymbol{f}^{**}$, where c_{ij}^{**}, c_{55k}^{**}, e_{ij}^{**}, and ϵ_{ij}^{**} are constants, and

$$\boldsymbol{f}^{**} = \left[u_\rho, u_\varphi, u_z, \phi_\odot, \rho\dot{T}_{0z\rho}, \rho\dot{T}_{0z\varphi}, \rho\dot{T}_{0zz}, \rho\dot{D}_{l0z}\right]^{\mathrm{T}}, \quad \boldsymbol{\mathcal{H}}^{**} = \left[\begin{array}{c|c} \mathbf{A}^{**} & \mathbf{B}^{**} \\ \hline \mathbf{C}^{**} & -\mathbf{A}^{**T} \end{array}\right] \tag{F.7}$$

where

$$
\mathbf{A}^{**}=\begin{bmatrix}
0 & 0 & -\dfrac{c_{551}^{**}}{c_{553}^{**}}\dfrac{\partial}{\partial\rho} & -\dfrac{e_{15}^{**}}{c_{553}^{**}}\dfrac{\partial}{\partial\rho} \\
0 & 0 & -\dfrac{c_{551}^{**}}{c_{553}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & -\dfrac{e_{15}^{**}}{c_{553}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\
-\dfrac{\epsilon_{33}^{**}c_{13}^{**}+e_{33}^{**}e_{31}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\left(\dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho}\right) & -\dfrac{\epsilon_{33}^{**}c_{13}^{**}+e_{33}^{**}e_{31}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & 0 \\
\dfrac{c_{33}^{**}e_{31}^{**}-c_{13}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\left(\dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho}\right) & \dfrac{c_{33}^{**}e_{31}^{**}-c_{13}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & 0
\end{bmatrix},\quad
\mathbf{B}^{**}=\begin{bmatrix}
\dfrac{1}{c_{553}^{**}}\dfrac{1}{\rho} & 0 & 0 & 0 \\
0 & \dfrac{1}{c_{553}^{**}}\dfrac{1}{\rho} & 0 & 0 \\
0 & 0 & \dfrac{\epsilon_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho} & \dfrac{e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho} \\
0 & 0 & \dfrac{e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho} & -\dfrac{c_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho}
\end{bmatrix}
\tag{F.8}
$$

$$
\mathbf{C}^{**}=\begin{bmatrix}
\mathcal{D}_{1}^{**} & \mathcal{D}_{2}^{**} & 0 & 0 \\
\mathcal{D}_{3}^{**} & \mathcal{D}_{4}^{**} & 0 & 0 \\
0 & 0 & -\left(c_{552}^{**}-\dfrac{\left(c_{551}^{**}\right)^{2}}{c_{553}^{**}}\right)\rho\nabla^{2} & -\left(e_{15}^{**}-\dfrac{e_{15}^{**}c_{551}^{**}}{c_{553}^{**}}\right)\rho\nabla^{2} \\
0 & 0 & -\left(e_{15}^{**}-\dfrac{e_{15}^{**}c_{551}^{**}}{c_{553}^{**}}\right)\rho\nabla^{2} & \left(\epsilon_{11}^{**}+\dfrac{\left(e_{15}^{**}\right)^{2}}{c_{553}^{**}}\right)\rho\nabla^{2}
\end{bmatrix},\quad
-\mathbf{A}^{**\mathcal{T}}=\begin{bmatrix}
0 & 0 & -\dfrac{\epsilon_{33}^{**}c_{13}^{**}+e_{33}^{**}e_{31}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\left(\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho}\right) & \dfrac{c_{33}^{**}e_{31}^{**}-c_{13}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\left(\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho}\right) \\
0 & 0 & -\dfrac{\epsilon_{33}^{**}c_{13}^{**}+e_{33}^{**}e_{31}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & \dfrac{c_{33}^{**}e_{31}^{**}-c_{13}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^{2}+c_{33}^{**}\epsilon_{33}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\
-\dfrac{c_{551}^{**}}{c_{553}^{**}}\dfrac{\partial}{\partial\rho} & -\dfrac{c_{551}^{**}}{c_{553}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & 0 \\
-\dfrac{e_{15}^{**}}{c_{553}^{**}}\dfrac{\partial}{\partial\rho} & -\dfrac{e_{15}^{**}}{c_{553}^{**}}\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} & 0 & 0
\end{bmatrix}
\tag{F.9}
$$

$$\begin{cases} \mathcal{D}_1^{**} = -\left[c_{11}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\left(\rho\dfrac{\partial^2}{\partial\rho^2} + \dfrac{\partial}{\partial\rho} - \dfrac{1}{\rho}\right) - c_{14}^{**}\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2} \\ \mathcal{D}_2^{**} = -\left[c_{12}^{**} + c_{15}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\dfrac{\partial^2}{\partial\rho\partial\varphi} + \left[c_{11}^{**} + c_{14}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\ \mathcal{D}_3^{**} = -\left[c_{12}^{**} + c_{15}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\dfrac{\partial^2}{\partial\rho\partial\varphi} - \left[c_{11}^{**} + c_{14}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\dfrac{1}{\rho}\dfrac{\partial}{\partial\varphi} \\ \mathcal{D}_4^{**} = -c_{14}^{**}\left(\rho\dfrac{\partial^2}{\partial\rho^2} + \dfrac{\partial}{\partial\rho} - \dfrac{1}{\rho}\right) - \left[c_{11}^{**} + \dfrac{c_{33}^{**}\left(e_{31}^{**}\right)^2 - \epsilon_{33}^{**}\left(c_{13}^{**}\right)^2 - 2c_{13}^{**}e_{31}^{**}e_{33}^{**}}{\left(e_{33}^{**}\right)^2 + c_{33}^{**}\epsilon_{33}^{**}}\right]\dfrac{1}{\rho}\dfrac{\partial^2}{\partial\varphi^2} \end{cases} \tag{F.10}$$

It is noteworthy that the lower-left block of the novel operator matrix $\boldsymbol{\mathcal{H}}^{**}$ (i.e., $\mathbf{C}^{**}$) obtained here is distinct from that (i.e., $\mathbf{C}^{*}$) in Section F.1, which is a consequence of the asymmetric incremental stresses. Additionally, the analytical solutions can be derived through the same procedure as that stated in Section F.1.

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