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Proof of dual-shifted Hamiltonian transformation

With fixed boundary conditions at $x = \pm L$, $\langle \boldsymbol{f}_1, \mathcal{H}\boldsymbol{f}_2 \rangle = \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \langle \boldsymbol{f}_1^*, \boldsymbol{f}_2^* \rangle$ may be rearranged as

$$
\begin{aligned}
\langle \boldsymbol{f}_1, \mathcal{H}\boldsymbol{f}_2 \rangle &= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \langle \boldsymbol{f}_1^*, \boldsymbol{f}_2^* \rangle \\
&= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \langle \mathcal{A}\boldsymbol{f}_1, \mathcal{A}\boldsymbol{f}_2 \rangle \\
&= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \int_{-L}^{L} (\mathcal{A}\boldsymbol{f}_1)^{\mathrm{T}} \mathbf{J} (\mathcal{A}\boldsymbol{f}_2) \mathrm{d}x \\
&= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \int_{-L}^{L} \boldsymbol{f}_1^{\mathrm{T}} \mathbf{J}\mathbf{J}^{\mathrm{T}} \mathcal{A}^{\mathcal{T}} \mathbf{J} \mathcal{A}\boldsymbol{f}_2 \mathrm{d}x \\
&= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \beta \int_{-L}^{L} \boldsymbol{f}_1^{\mathrm{T}} \mathbf{J} (\mathcal{A}\mathbf{J})^{\mathrm{T}} (\mathbf{J}\mathcal{A}) \boldsymbol{f}_2 \mathrm{d}x \\
&= \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle - \langle \boldsymbol{f}_1, \alpha \mathbf{I}_6 \boldsymbol{f}_2 \rangle - \langle \boldsymbol{f}_1, \beta \mathcal{B}\boldsymbol{f}_2 \rangle,
\end{aligned}
\tag{1}
$$

which leads to $\langle \boldsymbol{f}_1, (\mathcal{H} + \alpha \mathbf{I}_6 + \beta \mathcal{B}) \boldsymbol{f}_2 \rangle = \langle \boldsymbol{f}_2, \mathcal{H}\boldsymbol{f}_1 \rangle$ after moving items to the left-hand side of the equal sign, where $(\#)^{\mathcal{T}}$ represents the adjoint transpose of an operator matrix. To elucidate, the fixed boundary conditions are essential to deal with additional terms which appear during integration by parts. Therefore, special attention should be paid for the type of boundary conditions.

Proposition. 1. *If μ is an eigenvalue of operator matrix $\mathcal{H}$, then $-\mu - \alpha$ is the eigenvalue of $\mathcal{H} + \beta \mathcal{B}$ with the same multiplicity.*

Proof. Since

$$\boldsymbol{\mathcal{B}} = \mathbf{J}^{\mathrm{T}} \boldsymbol{\mathcal{A}}^{\mathcal{T}} \mathbf{J} \boldsymbol{\mathcal{A}} = \begin{bmatrix} 0 & a_1 & a_2 & 0 & 0 & 0 \\ a_4 & 0 & 0 & 0 & 0 & 0 \\ a_7 & 0 & 0 & 0 & 0 & 0 \\ 2a_9 \dfrac{\partial}{\partial x} & 0 & 0 & 0 & a_4 & a_7 \\ 0 & 2a_{10} \dfrac{\partial}{\partial x} & a_{11}\left(2\dfrac{\partial}{\partial x} + \beta\right) & a_1 & 0 & 0 \\ 0 & a_{11}\left(2\dfrac{\partial}{\partial x} - \beta\right) & -2a_{12}\dfrac{\partial}{\partial x} & a_2 & 0 & 0 \end{bmatrix}, \tag{2}$$

it is convenient to verify that $\boldsymbol{\mathcal{H}}^{\mathcal{T}} = \mathbf{J}(\boldsymbol{\mathcal{H}} + \alpha \mathbf{I}_6 + \beta \boldsymbol{\mathcal{B}})\mathbf{J}$. Considering eigenvalues are involved in the characteristic polynomial $|\mu \mathbf{I}_6 - \boldsymbol{\mathcal{H}}|$, we have

$$\begin{aligned} |\mu \mathbf{I}_6 - \boldsymbol{\mathcal{H}}| &= |(\mu \mathbf{I}_6 - \boldsymbol{\mathcal{H}})\mathbf{J}| \\ &= |\mu \mathbf{J}\mathbf{J} - \mathbf{J}\boldsymbol{\mathcal{H}}\mathbf{J}| \\ &= |(\mu+\alpha)\mathbf{J}\mathbf{J} - \mathbf{J}(\boldsymbol{\mathcal{H}} + \alpha \mathbf{I}_6 + \beta\boldsymbol{\mathcal{B}})\mathbf{J} + \mathbf{J}\beta\boldsymbol{\mathcal{B}}\mathbf{J}| \\ &= \left|(\mu+\alpha)\mathbf{J}\mathbf{J} + \mathbf{J}\beta\boldsymbol{\mathcal{B}}\mathbf{J} - \boldsymbol{\mathcal{H}}^{\mathcal{T}}\right| \\ &= \left|-(\mu+\alpha)\mathbf{I}_6 - (\beta\boldsymbol{\mathcal{B}})^{\mathcal{T}} - \boldsymbol{\mathcal{H}}^{\mathcal{T}}\right| \\ &= \left|-(\mu+\alpha)\mathbf{I}_6 - (\beta\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{H}})^{\mathcal{T}}\right| \\ &= |-(\mu+\alpha)\mathbf{I}_6 - (\beta\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{H}})|. \end{aligned} \tag{3}$$

$\square$

Proposition. 2. *Let* $\{\boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Phi}_i^{(1)}, \boldsymbol{\Phi}_i^{(2)}, \cdots, \boldsymbol{\Phi}_i^{(m)}\}$ *and* $\{\boldsymbol{\Psi}_j^{(0)}, \boldsymbol{\Psi}_j^{(1)}, \boldsymbol{\Psi}_j^{(2)}, \cdots, \boldsymbol{\Psi}_j^{(n)}\}$ *are eigenvectors (including basic and Jordan form eigenvectors) of* *eigenvalues* μ_i *and* μ_j *, respectively. If* $\mu_i + \mu_j + \alpha \neq 0$ *, the eigenvectors are mutually symplectic orthogonal:* $\left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle = 0,\ (\rho = 0,1,\cdots,m; \theta = 0,1,\cdots,n)$.

Proof. Mathematical induction is adopted here: let $\sigma = \rho + \theta$, when $\rho, \theta = 0$ ($\sigma = 0$ in this case),

$$\begin{cases} \mathcal{H}\boldsymbol{\Phi}_i^{(0)} = \mu_i \boldsymbol{\Phi}_i^{(0)}, \\ (\mathcal{H} + \beta\mathcal{B})\boldsymbol{\Psi}_j^{(0)} = \mu_j \boldsymbol{\Psi}_j^{(0)}. \end{cases} \tag{4}$$

The corresponding symplectic inner products are

$$\begin{cases} \left\langle \boldsymbol{\Phi}_i^{(0)}, (\mathcal{H} + \beta\mathcal{B})\boldsymbol{\Psi}_j^{(0)} \right\rangle = \left\langle \boldsymbol{\Phi}_i^{(0)}, \mu_j \boldsymbol{\Psi}_j^{(0)} \right\rangle = \mu_j \left\langle \boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Psi}_j^{(0)} \right\rangle, \\ \left\langle \boldsymbol{\Psi}_j^{(0)}, \mathcal{H}\boldsymbol{\Phi}_i^{(0)} \right\rangle = \left\langle \boldsymbol{\Psi}_j^{(0)}, \mu_i \boldsymbol{\Phi}_i^{(0)} \right\rangle = \mu_i \left\langle \boldsymbol{\Psi}_j^{(0)}, \boldsymbol{\Phi}_i^{(0)} \right\rangle = -\mu_i \left\langle \boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Psi}_j^{(0)} \right\rangle. \end{cases} \tag{5}$$

Concerning $\left\langle \boldsymbol{\Phi}_i^{(0)}, (\mathcal{H} + \beta\mathcal{B})\boldsymbol{\Psi}_j^{(0)} \right\rangle = \left\langle \boldsymbol{\Psi}_j^{(0)}, \mathcal{H}\boldsymbol{\Phi}_i^{(0)} \right\rangle - \alpha \left\langle \boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Psi}_j^{(0)} \right\rangle$, then $(\mu_i + \mu_j + \alpha)\left\langle \boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Psi}_j^{(0)} \right\rangle = 0$, which indicates $\left\langle \boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Psi}_j^{(0)} \right\rangle = 0$ since $\mu_i + \mu_j + \alpha \neq 0$.

Subsequently, we assume the orthogonality relation is valid when $\sigma = k$, then for $\sigma = k + 1$, the symplectic inner products are

$$\begin{cases} \left\langle \boldsymbol{\Phi}_i^{(\rho)}, (\mathcal{H} + \beta\mathcal{B})\boldsymbol{\Psi}_j^{(\theta)} \right\rangle = \left\langle \boldsymbol{\Phi}_i^{(\rho)}, \mu_j \boldsymbol{\Psi}_j^{(\theta)} \right\rangle + \left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta-1)} \right\rangle = \mu_j \left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle, \\ \left\langle \boldsymbol{\Psi}_j^{(\theta)}, \mathcal{H}\boldsymbol{\Phi}_i^{(\rho)} \right\rangle = \left\langle \boldsymbol{\Psi}_j^{(\theta)}, \mu_i \boldsymbol{\Phi}_i^{(\rho)} \right\rangle + \left\langle \boldsymbol{\Psi}_j^{(\theta)}, \boldsymbol{\Phi}_i^{(\rho-1)} \right\rangle = \mu_i \left\langle \boldsymbol{\Psi}_j^{(\theta)}, \boldsymbol{\Phi}_i^{(\rho)} \right\rangle = -\mu_i \left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle, \end{cases} \tag{6}$$

according to

$$\begin{cases} \mathcal{H}\boldsymbol{\Phi}_i^{(\rho)} = \mu_i \boldsymbol{\Phi}_i^{(\rho)} + \boldsymbol{\Phi}_i^{(\rho-1)}, \\ (\mathcal{H} + \beta\mathcal{B})\boldsymbol{\Psi}_j^{(\theta)} = \mu_j \boldsymbol{\Psi}_j^{(\theta)} + \boldsymbol{\Psi}_j^{(\theta-1)}. \end{cases} \tag{7}$$

Considering $\left\langle \boldsymbol{\Phi}_i^{(\rho)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\boldsymbol{\Psi}_j^{(\theta)} \right\rangle = \left\langle \boldsymbol{\Psi}_j^{(\theta)}, \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}_i^{(\rho)} \right\rangle - \alpha\left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle$, we arrive at $(\mu_i+\mu_j+\alpha)\left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle = 0$, which leads to $\left\langle \boldsymbol{\Phi}_i^{(\rho)}, \boldsymbol{\Psi}_j^{(\theta)} \right\rangle = 0$.

□

Proposition. 3. *Let μ_i and $-\mu_i-\alpha$ be the symplectic adjoint eigenvalues of the operators $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}}$ with multiplicity ω, respectively. The corresponding adjoint symplectic orthogonal sets are $\{\boldsymbol{\Phi}_i^{(0)}, \boldsymbol{\Phi}_i^{(1)}, \boldsymbol{\Phi}_i^{(2)}, \cdots, \boldsymbol{\Phi}_i^{(\omega-1)}\}$ and $\{\boldsymbol{\Psi}_j^{(0)}, \boldsymbol{\Psi}_j^{(1)}, \boldsymbol{\Psi}_j^{(2)}, \cdots, \boldsymbol{\Psi}_j^{(\omega-1)}\}$ satisfying*

$$\left\langle \boldsymbol{\Phi}^{(i)}, \boldsymbol{\Psi}^{(j)} \right\rangle = \begin{cases} (-1)^i \Upsilon \neq 0, & (i+j=\omega-1). \\ 0, & (i+j \neq \omega-1). \end{cases} \tag{8}$$

Proof. We still take mathematical induction: for $i=0$, when $j \le \omega-2$,

$$\begin{cases} \left\langle \boldsymbol{\Phi}^{(0)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\boldsymbol{\Psi}^{(j+1)} \right\rangle = -(\mu+\alpha)\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle + \left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j)} \right\rangle, \\ \left\langle \boldsymbol{\Psi}^{(j+1)}, \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}^{(0)} \right\rangle = \mu\left\langle \boldsymbol{\Psi}^{(j+1)}, \boldsymbol{\Phi}^{(0)} \right\rangle = -\mu\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle. \end{cases} \tag{9}$$

Since $\left\langle \boldsymbol{\Phi}^{(0)}, (\boldsymbol{\mathcal{H}}+\beta\boldsymbol{\mathcal{B}})\boldsymbol{\Psi}^{(j+1)} \right\rangle = \left\langle \boldsymbol{\Psi}^{(j+1)}, \boldsymbol{\mathcal{H}}\boldsymbol{\Phi}^{(0)} \right\rangle - \alpha\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle$, then $\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j)} \right\rangle = 0$. Therefore, $\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(j)} \right\rangle = \Upsilon \neq 0$ for $j=\omega-1$. Otherwise, $\boldsymbol{\Phi}^{(0)} \equiv 0$, which is contradictory.

If we further assume this property holds when $i=k$, then for $i=k+1$, the set $\{\boldsymbol{\Phi}^{(0)}, \cdots, \boldsymbol{\Phi}^{(k)}, \tilde{\boldsymbol{\Phi}}^{(k+1)}, \cdots, \tilde{\boldsymbol{\Phi}}^{(\omega-1)}\}$ still fulfill the eigen equation of eigenvalue μ and $\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(\omega-1)} \right\rangle = \left\langle \boldsymbol{\Phi}^{(k+1)}, \boldsymbol{\Psi}^{(\omega-1)} \right\rangle + r\left\langle \boldsymbol{\Phi}^{(0)}, \boldsymbol{\Psi}^{(\omega-1)} \right\rangle = 0$, where $\tilde{\boldsymbol{\Phi}}^{(k+1+n)} = \boldsymbol{\Phi}^{(k+1+n)} + r\boldsymbol{\Phi}^{(n)}$, $(n=0,1,\cdots,\omega-k-2)$, and $r = -\frac{1}{\Upsilon}\left\langle \boldsymbol{\Phi}^{(k+1)}, \boldsymbol{\Psi}^{(\omega-1)} \right\rangle$.

For $j<\omega-1$,

$$\begin{cases}\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, (\mathcal{H}+\beta\mathcal{B})\boldsymbol{\Psi}^{(j+1)} \right\rangle = -(\mu+\alpha)\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle + \left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j)} \right\rangle, \\ \left\langle \boldsymbol{\Psi}^{(j+1)}, \mathcal{H}\tilde{\boldsymbol{\Phi}}^{(k+1)} \right\rangle = \mu\left\langle \boldsymbol{\Psi}^{(j+1)}, \tilde{\boldsymbol{\Phi}}^{(k+1)} \right\rangle + \left\langle \boldsymbol{\Psi}^{(j+1)}, \boldsymbol{\Phi}^{(k)} \right\rangle = -\mu\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle - \left\langle \boldsymbol{\Phi}^{(k)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle. \end{cases} \tag{10}$$

Considering $\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, (\mathcal{H}+\beta\mathcal{B})\boldsymbol{\Psi}^{(j+1)} \right\rangle = \left\langle \boldsymbol{\Psi}^{(j+1)}, \mathcal{H}\tilde{\boldsymbol{\Phi}}^{(k+1)} \right\rangle - \alpha\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle$, then we have $\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j)} \right\rangle = -\left\langle \boldsymbol{\Phi}^{(k)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle$. When $i=k$, Eq. (8) is valid, then for $i=k+1$,

$$\left\langle \tilde{\boldsymbol{\Phi}}^{(k+1)}, \boldsymbol{\Psi}^{(j)} \right\rangle = -\left\langle \boldsymbol{\Phi}^{(k)}, \boldsymbol{\Psi}^{(j+1)} \right\rangle = \begin{cases} (-1)^{(k+1)}\Upsilon \neq 0, & (k+1+j=\omega-1). \\ 0, & (k+1+j\neq\omega-1). \end{cases} \tag{11}$$

$\square$