3D Symplectic Expansion in Spherical Coordinates

Lizichen Chen[a]

Abstract: This note is an extension of a report on constructing 3D symplectic expansion in spherical coordinates.

Keywords: 3D symplectic expansion; spherical coordinates

1. Basic formulations

There are some limitations in the application of the symplectic form in 3D (Chen and Chen, 2025a), nevertheless we give the solution in spherical coordinates. For an artifact in spherical coordinates, governing equations for linear deformation are:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi r}}{\partial \varphi}+\frac{1}{r}\left(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}+\sigma_{r\theta}\cot\theta\right)=0\\
&\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\theta}}{\partial \varphi}+\frac{1}{r}\left[\left(\sigma_{\theta\theta}-\sigma_{\varphi\varphi}\right)\cot\theta+3\sigma_{r\theta}\right]=0\\
&\frac{\partial \sigma_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\varphi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi}+\frac{1}{r}\left(3\sigma_{r\varphi}+2\sigma_{\theta\varphi}\cot\theta\right)=0
\end{aligned}\tag{1}$$

$$\begin{aligned}
\sigma_{rr}&=\left(\lambda+2G\right)\frac{\partial u_r}{\partial r}+\lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+\lambda\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)\\
\sigma_{\theta\theta}&=\lambda\frac{\partial u_r}{\partial r}+\left(\lambda+2G\right)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+\lambda\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)\\
\sigma_{\varphi\varphi}&=\lambda\frac{\partial u_r}{\partial r}+\lambda\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+\left(\lambda+2G\right)\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)\\
\sigma_{\theta\varphi}&=G\left[\frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta}-u_\varphi\cot\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi}\right]\\
\sigma_{r\varphi}&=G\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right)\\
\sigma_{r\theta}&=G\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)
\end{aligned}\tag{2}$$

[a] Department of Engineering Mechanics, Zhejiang University, Hangzhou 310027, P.R. China

and the Lagrange function is

$$\begin{aligned}\mathcal{L}=\frac{1}{2}r^2\sin\theta\Bigg\{&\left(\lambda+2G\right)\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)^2\right]\\&+2\lambda\left[\left(\frac{\partial u_r}{\partial r}\right)\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)+\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right)\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)\right.\\&\left.+\left(\frac{\partial u_r}{\partial r}\right)\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_\theta}{r}\cot\theta+\frac{u_r}{r}\right)\right]+G\left[\frac{1}{r}\left(\frac{\partial u_\varphi}{\partial\theta}-u_\varphi\cot\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right]^2\\&+G\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r}\right]^2+G\left[\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right]^2\Bigg\}\end{aligned}\tag{3}$$

where λ and G are the first and second Lamé constant, respectively. If we take $\mathbf{q}=\left[u_r,u_\theta,u_\varphi\right]^{\mathrm{T}}$, then the dual variables are derived through

$$\mathbf{p}=\frac{\partial\mathcal{L}}{\partial\left(\partial\mathbf{q}/\partial\ln r\right)}=\left[r\sin\theta\sigma_{rr},r\sin\theta\sigma_{r\theta},r\sin\theta\sigma_{r\varphi}\right]^{\mathrm{T}}\tag{4}$$

Therefore, we have

$$\begin{aligned}\frac{\partial u_r}{\partial\ln r}&=-\frac{\lambda}{\lambda+2G}\left(\frac{\partial u_\theta}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+u_\theta\cot\theta+2u_r\right)+\frac{r\sin\theta\sigma_{rr}}{\left(\lambda+2G\right)\sin\theta}\\\frac{\partial u_\theta}{\partial\ln r}&=-\frac{\partial u_r}{\partial\theta}+u_\theta+\frac{r\sin\theta\sigma_{r\theta}}{G\sin\theta}\\\frac{\partial u_\varphi}{\partial\ln r}&=-\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi}+u_\varphi+\frac{r\sin\theta\sigma_{r\varphi}}{G\sin\theta}\\\frac{\partial r\sin\theta\sigma_{rr}}{\partial\ln r}&=-\frac{\partial r\sin\theta\sigma_{r\theta}}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial r\sin\theta\sigma_{r\varphi}}{\partial\varphi}\\&\quad+2G\frac{3\lambda+2G}{\lambda+2G}\left(\sin\theta\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_\varphi}{\partial\varphi}+u_\theta\cos\theta+2u_r\sin\theta\right)+\frac{\lambda-2G}{\lambda+2G}r\sin\theta\sigma_{rr}\\\frac{\partial r\sin\theta\sigma_{r\theta}}{\partial\ln r}&=-2G\frac{3\lambda+2G}{\lambda+2G}\sin\theta\frac{\partial u_r}{\partial\theta}-\left(G\frac{3\lambda+2G}{\lambda+2G}\frac{\partial^2}{\partial\theta\partial\varphi}-G\frac{5\lambda+6G}{\lambda+2G}\cot\theta\frac{\partial}{\partial\varphi}\right)u_\varphi\\&\quad-\left(4G\frac{\lambda+G}{\lambda+2G}\sin\theta\frac{\partial^2}{\partial\theta^2}+4G\frac{\lambda+G}{\lambda+2G}\cos\theta\frac{\partial}{\partial\theta}+G\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}-2G\frac{\lambda}{\lambda+2G}\sin\theta-4G\frac{\lambda+G}{\lambda+2G}\cos\theta\cot\theta\right)u_\theta\\&\quad-\left(\frac{\lambda}{\lambda+2G}\frac{\partial}{\partial\theta}-\frac{\lambda}{\lambda+2G}\cot\theta\right)r\sin\theta\sigma_{rr}-2r\sin\theta\sigma_{r\theta}\\\frac{\partial r\sin\theta\sigma_{r\varphi}}{\partial\ln r}&=-2G\frac{3\lambda+2G}{\lambda+2G}\frac{\partial u_r}{\partial\varphi}-\left(G\frac{3\lambda+2G}{\lambda+2G}\frac{\partial^2}{\partial\theta\partial\varphi}+G\frac{5\lambda+6G}{\lambda+2G}\cot\theta\frac{\partial}{\partial\varphi}\right)u_\theta-\frac{\lambda}{\lambda+2G}\frac{1}{\sin\theta}\frac{\partial r\sin\theta\sigma_{rr}}{\partial\varphi}-2r\sin\theta\sigma_{r\varphi}\\&\quad-\left(G\sin\theta\frac{\partial^2}{\partial\theta^2}+4G\frac{\lambda+G}{\lambda+2G}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}+G\cos\theta\frac{\partial}{\partial\theta}+G\sin\theta-G\cos\theta\cot\theta\right)u_\varphi\end{aligned}\tag{5}$$

together with the supplementary equations:

$$\begin{cases} r\sin\theta\sigma_{\theta\theta} = 4G\dfrac{\lambda+G}{\lambda+2G}\sin\theta\left(\dfrac{\partial u_\theta}{\partial\theta}+u_r\right)+2G\dfrac{\lambda}{\lambda+2G}\left(\dfrac{\partial u_\varphi}{\partial\varphi}+u_\theta\cos\theta+u_r\sin\theta\right)+\dfrac{\lambda}{\lambda+2G}r\sin\theta\sigma_{rr} \\ r\sin\theta\sigma_{\varphi\varphi} = 2G\dfrac{\lambda}{\lambda+2G}\sin\theta\left(\dfrac{\partial u_\theta}{\partial\theta}+u_r\right)+4G\dfrac{\lambda+G}{\lambda+2G}\left(\dfrac{\partial u_\varphi}{\partial\varphi}+u_\theta\cos\theta+u_r\sin\theta\right)+\dfrac{\lambda}{\lambda+2G}r\sin\theta\sigma_{rr} \\ r\sin\theta\sigma_{\theta\varphi} = G\left(\sin\theta\dfrac{\partial u_\varphi}{\partial\theta}-u_\varphi\cos\theta+\dfrac{\partial u_\theta}{\partial\varphi}\right) \end{cases} \tag{6}$$

We may rewrite Eq. (5) in matrix form as

$$\frac{\partial}{\partial\ln r}\boldsymbol{f} = \boldsymbol{\mathcal{H}}\boldsymbol{f} \tag{7}$$

where $\boldsymbol{f}=\left[\mathbf{q},\mathbf{p}\right]^{\mathrm{T}}$, and $\boldsymbol{\mathcal{H}}$ is detailed in Appendix A. Under homogeneous lateral boundary conditions at $\theta=0$ and $\theta=\pi$, we can prove a unit shifted Hamiltonian transformation:

$$\left\langle \boldsymbol{f}^{\alpha},\left(\boldsymbol{\mathcal{H}}+\mathbf{I}_6\right)\boldsymbol{f}^{\beta}\right\rangle = \left\langle \boldsymbol{f}^{\beta},\boldsymbol{\mathcal{H}}\boldsymbol{f}^{\alpha}\right\rangle \tag{8}$$

where the superscript α or β denotes a specified state vector, $\mathbf{I}_n$ is an nth-order identity matrix, and symplectic inner product is defined as:

$$\left\langle \boldsymbol{f}^{\alpha},\boldsymbol{f}^{\beta}\right\rangle = \int_0^{2\pi}\int_0^{\pi}\left(\boldsymbol{f}^{\alpha}\right)^{\mathrm{T}}\mathbf{J}\boldsymbol{f}^{\beta}\mathrm{d}\theta\mathrm{d}\varphi \tag{9}$$

where $\mathbf{J}$ is a unit symplectic matrix. If we separate the variables in state vector $\boldsymbol{f}$, i.e., $\boldsymbol{f}(r,\theta,\varphi)=\boldsymbol{\Phi}(\theta,\varphi)\xi(r)$, the eigen equation is derived as $\boldsymbol{\mathcal{H}}\boldsymbol{\Phi}=\mu\boldsymbol{\Phi}$, and we also obtain $\xi(r)=\mathrm{e}^{\mu\ln r}=r^{\mu}$. It is noteworthy that the unit shifted Hamiltonian transformation in Eq. (8) indicates that the symplectic adjoint eigenvalue of μ is $-\mu-1$.

2. Special and general eigen-solutions

According to the uniqueness theorem in theory of elasticity, we may take Papkovich-Neuber type solution

$$\mathbf{q}=\mathbf{B}-\frac{\lambda+G}{2\left(\lambda+2G\right)}\nabla\left(\mathbf{r}\cdot\mathbf{B}+B_{0}\right) \tag{10}$$

Without loss of generality, B_0 is set to be zero, then Eq. (10) is in the form of

$$\begin{cases} u_{r}=B_{r}-\dfrac{\lambda+G}{2(\lambda+2G)}\dfrac{\partial}{\partial r}\left(rB_{r}\right) \\ u_{\theta}=B_{\theta}-\dfrac{\lambda+G}{2(\lambda+2G)}\dfrac{1}{r}\dfrac{\partial}{\partial\theta}\left(rB_{r}\right) \\ u_{\varphi}=B_{\varphi}-\dfrac{\lambda+G}{2(\lambda+2G)}\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\varphi}\left(rB_{r}\right) \end{cases} \tag{11}$$

where $\mathbf{B}$ fulfills $\nabla^2\mathbf{B}=0$, and $\nabla^2\equiv\frac{1}{r^2}\frac{\partial}{\partial r}(r^r\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$. Derivatives of unit vectors in spherical coordinates are tabulated in Appendix A. Considering $\mathbf{B}(r,\theta,0)=\mathbf{B}(r,\theta,2\pi)$, we may further expand $\mathbf{B}$ as

$$\left(B_{r},B_{\theta},B_{\varphi}\right)=\sum_{m}\left(\Theta_{r}^{\{m\}},\Theta_{\theta}^{\{m\}},\mathrm{i}\Theta_{\varphi}^{\{m\}}\right)\mathrm{e}^{\mu\ln r}\mathrm{e}^{\mathrm{i}m\varphi} \tag{12}$$

where $\Theta_{r}^{\{m\}}=\sin\theta\Theta_{\varphi}^{\{m\}}$ and $\Theta_{\theta}^{\{m\}}=\cos\theta\Theta_{\varphi}^{\{m\}}$, since the convergence of solutions at $\theta=0$ and π should be satisfied, which also serves as the boundary conditions of governing equation:

$$\frac{\partial^2\Theta_{\varphi}^{\{m\}}}{\partial\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\partial\Theta_{\varphi}^{\{m\}}}{\partial\theta}+\mu\left(\mu+1\right)\Theta_{\varphi}^{\{m\}}-\frac{\left(m-1\right)^2}{\sin^2\theta}\Theta_{\varphi}^{\{m\}}=0 \tag{13}$$

which is an associated Legendre equation. The solution to Eq. (13) fulfills

$$P_{\mu}^{+(m-1)}(\cos\theta)=P_{-\mu-1}^{+(m-1)}(\cos\theta),\quad P_{\mu}^{-(m-1)}(\cos\theta)=P_{-\mu-1}^{-(m-1)}(\cos\theta) \tag{14}$$

which are in agreement with the relations between symplectic adjoint eigenvalues.

As for special eigenvalues (i.e., 0 and -1), the eigenvectors can be deduced through substituting the eigenvalues in Eq. (11), which constitute several Jordan chains, e.g., $\boldsymbol{\Phi}_{0,1}^{(0)}=\left[\sin\theta\cos\varphi,\cos\theta\cos\varphi,-\sin\varphi,0,0,0\right]^{\mathrm{T}}$, $\boldsymbol{\Phi}_{0,2}^{(0)}=\left[\sin\theta\sin\varphi,\cos\theta\sin\varphi,\cos\varphi,0,0,0\right]^{\mathrm{T}}$, and $\boldsymbol{\Phi}_{0,3}^{(0)}=\left[\cos\theta,\sin\theta,0,0,0,0\right]^{\mathrm{T}}$. Finally, the eigen-solutions can be obtained accordingly and form the Saint-Venant solution.

Appendix A

$$
\boldsymbol{\mathcal{H}} = \left[\begin{array}{ccc|ccc}
-2\dfrac{\lambda}{\lambda+2G} & -\dfrac{\lambda}{\lambda+2G}\left(\dfrac{\partial}{\partial\theta}+\cot\theta\right) & -\dfrac{\lambda}{\lambda+2G}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi} & \dfrac{1}{\left(\lambda+2G\right)\sin\theta} & 0 & 0 \\
-\dfrac{\partial}{\partial\theta} & 1 & 0 & 0 & \dfrac{1}{G\sin\theta} & 0 \\
-\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi} & 0 & 1 & 0 & 0 & \dfrac{1}{G\sin\theta} \\
\hline
4G\dfrac{3\lambda+2G}{\lambda+2G}\sin\theta & 2G\dfrac{3\lambda+2G}{\lambda+2G}\sin\theta\left(\dfrac{\partial}{\partial\theta}+\cot\theta\right) & 2G\dfrac{3\lambda+2G}{\lambda+2G}\dfrac{\partial}{\partial\varphi} & \dfrac{\lambda-2G}{\lambda+2G} & -\dfrac{\partial}{\partial\theta} & -\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi} \\
-2G\dfrac{3\lambda+2G}{\lambda+2G}\sin\theta\dfrac{\partial}{\partial\theta} & \mathcal{D}_1 & -G\left(\dfrac{3\lambda+2G}{\lambda+2G}\dfrac{\partial^2}{\partial\theta\partial\varphi}-\dfrac{5\lambda+6G}{\lambda+2G}\cot\theta\dfrac{\partial}{\partial\varphi}\right) & -\dfrac{\lambda}{\lambda+2G}\left(\dfrac{\partial}{\partial\theta}-\cot\theta\right) & -2 & 0 \\
-2G\dfrac{3\lambda+2G}{\lambda+2G}\dfrac{\partial}{\partial\varphi} & -G\left(\dfrac{3\lambda+2G}{\lambda+2G}\dfrac{\partial^2}{\partial\theta\partial\varphi}+\dfrac{5\lambda+6G}{\lambda+2G}\cot\theta\dfrac{\partial}{\partial\varphi}\right) & \mathcal{D}_2 & -\dfrac{\lambda}{\lambda+2G}\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\varphi} & 0 & -2
\end{array}\right]
$$

where

$$
\mathcal{D}_1 \equiv -\left(4G\frac{\lambda+G}{\lambda+2G}\sin\theta\frac{\partial^2}{\partial\theta^2}+4G\frac{\lambda+G}{\lambda+2G}\cos\theta\frac{\partial}{\partial\theta}+G\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}-2G\frac{\lambda}{\lambda+2G}\sin\theta-4G\frac{\lambda+G}{\lambda+2G}\cos\theta\cot\theta\right)
$$

$$
\mathcal{D}_2 \equiv -\left(G\sin\theta\frac{\partial^2}{\partial\theta^2}+4G\frac{\lambda+G}{\lambda+2G}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}+G\cos\theta\frac{\partial}{\partial\theta}+G\sin\theta-G\cos\theta\cot\theta\right)
$$

It is important to note that the lower-left block of $\boldsymbol{\mathcal{H}}$ is adjoint symmetric under the definition of adjoint transpose (Chen et al., 2025b).

Table 1. Derivatives of unit vectors

	$\boldsymbol{e}_r$	$\boldsymbol{e}_\theta$	$\boldsymbol{e}_\varphi$
∂_r	0	0	0
∂_θ	$\boldsymbol{e}_\theta$	$-\boldsymbol{e}_r$	0
∂_φ	$\boldsymbol{e}_\varphi \sin\theta$	$\boldsymbol{e}_\varphi \cos\theta$	$-\boldsymbol{e}_r \sin\theta - \boldsymbol{e}_\theta \cos\theta$

Appendix B

To derive the dual variables in sub-symplectic space representation, we first introduce the Lagrange function

$$
\begin{aligned}
\mathcal{L}_\mu = \frac{1}{2}\sin\theta \Bigg\{ & (\lambda + 2G)\left[\mu^2 u_r^2 + \left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_\theta \cot\theta + u_r\right)^2\right] \\
& + 2\lambda\left[\mu u_r \left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) + \mu u_r \left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_\theta \cot\theta + u_r\right) + \left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)\left(\frac{1}{\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + u_\theta \cot\theta + u_r\right)\right] \\
& + G\left[\left(\frac{\partial u_\varphi}{\partial \theta} - u_\varphi \cot\theta + \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \varphi}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \varphi} + \mu u_\varphi - u_\varphi\right)^2 + \left(\mu u_\theta + \frac{\partial u_r}{\partial \theta} - u_\theta\right)^2\right]\Bigg\}
\end{aligned}
$$

And

$$\tilde{\mathbf{p}} = \frac{\partial \mathcal{L}_\mu}{\partial\left(\partial \mathbf{q}/\partial\varphi\right)} = \begin{pmatrix} G\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_r}{\partial\varphi} + \mu u_\varphi - u_\varphi\right) \\ G\left(\dfrac{\partial u_\varphi}{\partial\theta} - u_\varphi \cot\theta + \dfrac{1}{\sin\theta}\dfrac{\partial u_\theta}{\partial\varphi}\right) \\ \left(\lambda+2G\right)\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_\varphi}{\partial\varphi} + u_\theta\cot\theta + u_r\right) + \lambda\mu u_r + \lambda\left(\dfrac{\partial u_\theta}{\partial\theta} + u_r\right) \end{pmatrix} = \begin{pmatrix} r\sigma_{r\varphi} \\ r\sigma_{\theta\varphi} \\ r\sigma_{\varphi\varphi} \end{pmatrix}$$

$$\begin{aligned} \mathcal{H}_\mu = {} & r\sigma_{r\varphi}\left(-\mu\sin\theta u_\varphi + \sin\theta u_\varphi + \sin\theta\frac{r\sigma_{r\varphi}}{G}\right) + r\sigma_{\theta\varphi}\left(-\sin\theta\frac{\partial u_\varphi}{\partial\theta} + \cos\theta u_\varphi + \sin\theta\frac{r\sigma_{\theta\varphi}}{G}\right) \\ & + r\sigma_{\varphi\varphi}\left[-\frac{\lambda}{\lambda+2G}\mu\sin\theta u_r - \frac{\lambda}{\lambda+2G}\sin\theta\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right) + \sin\theta\frac{r\sigma_{\varphi\varphi}}{\lambda+2G} - u_\theta\cos\theta - u_r\sin\theta\right] - \mathcal{L}_\mu \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial\varphi}\begin{pmatrix}\mathbf{q}\\ \tilde{\mathbf{p}}\end{pmatrix} = \begin{pmatrix} \dfrac{\partial\mathcal{H}_\mu}{\partial\tilde{\mathbf{p}}} \\ -\dfrac{\partial\mathcal{H}_\mu}{\partial\mathbf{q}} \end{pmatrix} = \left[\begin{array}{ccc|ccc} 0 & 0 & -\sin\theta\left(\mu-1\right) & \dfrac{\sin\theta}{G} & 0 & 0 \\ 0 & 0 & -\sin\theta\dfrac{\partial}{\partial\theta} + \cos\theta & 0 & \dfrac{\sin\theta}{G} & 0 \\ -\sin\theta\left(\dfrac{\lambda}{\lambda+2G}\mu + 2\dfrac{\lambda+G}{\lambda+2G}\right) & -\sin\theta\dfrac{\lambda}{\lambda+2G}\dfrac{\partial}{\partial\theta} - \cos\theta & 0 & 0 & 0 & \dfrac{\sin\theta}{\lambda+2G} \\ \hline \tilde{\mathcal{D}}_1 & \tilde{\mathcal{D}}_2 & 0 & 0 & 0 & \sin\theta\left[1 + \dfrac{\lambda}{\lambda+2G}\left(\mu+1\right)\right] \\ \tilde{\mathcal{D}}_3 & \tilde{\mathcal{D}}_4 & 0 & 0 & 0 & -\dfrac{\lambda}{\lambda+2G}\dfrac{\partial}{\partial\theta}\left(\sin\theta\,\cdot\right) + \cos\theta \\ 0 & 0 & 0 & \sin\theta\left(\mu-1\right) & -\dfrac{\partial}{\partial\theta}\left(\sin\theta\,\cdot\right) - \cos\theta & 0 \end{array}\right]\begin{pmatrix}\mathbf{q}\\ \tilde{\mathbf{p}}\end{pmatrix}$$

where

$$
\begin{aligned}
\tilde{\mathcal{D}}_1 &\equiv \frac{4G}{\lambda+2G}\left[\left(\lambda+G\right)\mu^2+\lambda\mu+\left(\lambda+G\right)\right]\sin\theta-G\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\\
\tilde{\mathcal{D}}_2 &\equiv \frac{2G}{\lambda+2G}\left[2\left(\lambda+G\right)+\lambda\mu\right]\frac{\partial}{\partial\theta}-G\left(\mu-1\right)\frac{\partial}{\partial\theta}\left(\sin\theta\,\cdot\right)\\
\tilde{\mathcal{D}}_3 &\equiv \left(G\frac{3\lambda+2G}{\lambda+2G}\mu-\lambda+3G\right)\frac{\partial}{\partial\theta}\left(\sin\theta\,\cdot\right)-G\left(\mu-1\right)\cos\theta\\
\tilde{\mathcal{D}}_4 &\equiv -\left(\lambda+2G\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+G\sin\theta\left(\mu-1\right)^2
\end{aligned}
$$

References

Chen, L.Z.C. and Chen, W.Q., 2025a. Symplectic contact analysis of a finite-sized horizontally graded magneto-electro-elastic plane. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 481, 2306, 20240591, https://doi.org/10.1098/rspa.2024.0591.

Chen, L.Z.C., Lim, C.W., Chen, W.Q., 2025b. Symplectic solution for size-dependent contact analysis. *Nano Micro Mechanics Review*, http://doi.org/10.1142/S3082805825500074.