A Nambu-Routh Representation of Mechanics

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Abstract: Basic formulations in the Nambu-Routh mechanics are presented. The generalized Legendre transformation are proposed to formulate Nambu-Routh functions. Exterior differential forms are involved to enhance the convenience of derivation process.

Keywords: Nambu-Routh mechanics; generalized Legendre transformation

1. Introduction

Lagrangian and Hamiltonian mechanics, cornerstone approaches in analytical mechanics, offer distinct yet complementary perspectives on dynamical systems. Lagrangian mechanics (LM), formulated by Joseph-Louis Lagrange in 1788, simplifies system analysis through generalized coordinates and the Lagrangian, encapsulating kinetic and potential energies, which facilitates the study of constrained systems. By contrast, Hamiltonian mechanics (HM), developed by William Rowan Hamilton in 1833, introduces the phase space, leveraging the Hamiltonian—related to the Lagrangian via Legendre transformation—and first-order differential equations to explore system evolution. HM excels in identifying symmetries, conservation laws, and conserved quantities, making it particularly advantageous for systems with high symmetry and in areas like celestial mechanics and quantum mechanics precursors. Both frameworks have profound implications, guiding advancements in physics and underscoring the versatility and elegance of these mathematical formulations.

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Beyond LM and HM, Edward John Routh developed the Routhian mechanics (RM), which is a hybrid formulation of LM and HM. RM offers substantial advantages when addressing problems involving cyclic coordinates. Nambu (1973) generalized the classical Hamiltonian mechanics into a novel framework that incorporates multiple Hamiltonians, a theory now famously recognized as Nambu mechanics (NM). Ogawa and Sagae (2000) investigated Nambu mechanics in the Lagrangian formalism and elucidated the NM in terms of exterior differential form, Nambu bracket, and generalized Legendre transformation.

For the purpose of analysis, Nambu mechanics and its Lagrangian formalism are briefly elucidated in Section 2. In Section 3, we show a generalized Legendre transformation designed for Nambu-Routh functions and formulate Nambu-Routh mechanics for the cases with three and multiple Nambu-Routh momenta.

2. Nambu Mechanics

In classical analytical mechanics, the Hamilton's equations are in the form of

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial \mathcal{H}}{\partial \mathbf{p}} \\ \dot{\mathbf{p}} = -\dfrac{\partial \mathcal{H}}{\partial \mathbf{q}} \end{cases} \tag{1}$$

where $\mathbf{q}$ and $\mathbf{p}$ are dual variables, which represent canonical coordinates and momenta respectively. For brevity, the summation symbol for the case with plural degrees of freedom is also omitted in the following, unless it is necessary to avoid any potential confusion. Given a function $f(\mathbf{q},\mathbf{p})$, the temporal derivative of which may be written in terms of Poisson bracket:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f,\mathcal{H}\} = \frac{\partial(f,\mathcal{H})}{\partial(\mathbf{q},\mathbf{p})} \tag{2}$$

Where $\{\#_1,\#_2\}$ is the Poisson bracket; $\partial(\#_1,\#_2)/\partial(\#_{1'},\#_{2'})$ represents the Jacobian.

Nambu (1973) ingeniously generalized the aforementioned form to cases that consist of more variables (e.g., triple variables) as:

$$\frac{\mathrm{d}f(\mathbf{q},\mathbf{p},\mathbf{r})}{\mathrm{d}t}=[f,\mathcal{H}_1,\mathcal{H}_2]=\frac{\partial(f,\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{p},\mathbf{r})} \tag{3}$$

$$\begin{cases}\dot{\mathbf{q}}=\dfrac{\partial(\mathbf{q},\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{p},\mathbf{r})}=\dfrac{\partial(\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{p},\mathbf{r})}\\ \dot{\mathbf{p}}=\dfrac{\partial(\mathbf{p},\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{p},\mathbf{r})}=-\dfrac{\partial(\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{r})}\\ \dot{\mathbf{r}}=\dfrac{\partial(\mathbf{r},\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{p},\mathbf{r})}=\dfrac{\partial(\mathcal{H}_1,\mathcal{H}_2)}{\partial(\mathbf{q},\mathbf{p})}\end{cases} \tag{4}$$

where $[\#_1,\#_2,\#_3]$ is the Nambu bracket.

As for the exterior differential forms for Hamiltonian and Nambu mechanics, we first define differential 1-form $\Omega^{(1)}$ and differential 2-form $\Omega^{(2)}$ respectively as (Ogawa and Sagae, 2000)

$$\begin{aligned}\Omega^{(1)}&=\mathbf{p}\mathrm{d}\mathbf{q}-\mathcal{H}\mathrm{d}t\\ \Omega^{(2)}&=\mathbf{q}\mathrm{d}\mathbf{p}\wedge\mathrm{d}\mathbf{r}-\mathcal{H}_1\mathrm{d}\mathcal{H}_2\wedge\mathrm{d}t\end{aligned} \tag{5}$$

where the differential 1-form $\Omega^{(1)}$ is also recognized as the Poincaré-Cartan integral invariant. Concerning the Pfaffian equations, we will arrive at the same results as Eq. (1) and Eq. (4) by calculating $\mathrm{d}\Omega^{(1)}$ and $\mathrm{d}\Omega^{(2)}$, respectively.

3. Nambu-Routh Mechanics

For the case of three Nambu-Routh momenta, the Lagrangians are $\mathcal{L}_k\left(\mathbf{q},\boldsymbol{\xi},\dot{\mathbf{q}},\dot{\boldsymbol{\xi}}\right)\equiv\mathcal{L}_k\left(q_i,\xi_j,\dot{q}_i,\dot{\xi}_j\right)$ $(k=1,2;i=1,\cdots,n;j=1,\cdots,m)$, then

$$\mathrm{d}\mathcal{L}_k=\frac{\partial\mathcal{L}_k}{\partial\mathbf{q}}\mathrm{d}\mathbf{q}+\frac{\partial\mathcal{L}_k}{\partial\dot{\mathbf{q}}}\mathrm{d}\dot{\mathbf{q}}+\frac{\partial\mathcal{L}_k}{\partial\boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi}+\frac{\partial\mathcal{L}_k}{\partial\dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}} \tag{6}$$

Since the first canonical momenta $\mathbf{p}$ and second canonical momenta $\mathbf{r}$ together with their derivatives are defined as

$$
\begin{cases}
\mathbf{p} = \dfrac{\partial \mathcal{L}_1}{\partial \dot{\mathbf{q}}} \\
\dot{\mathbf{p}} = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}_1}{\partial \dot{\mathbf{q}}} = \dfrac{\partial \mathcal{L}_1}{\partial \mathbf{q}} \\
\mathbf{r} = \dfrac{\partial \mathcal{L}_2}{\partial \dot{\mathbf{q}}} \\
\dot{\mathbf{r}} = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}_2}{\partial \dot{\mathbf{q}}} = \dfrac{\partial \mathcal{L}_2}{\partial \mathbf{q}}
\end{cases} \tag{7}
$$

Substituting Eq. (7) into Eq. (6), we have

$$
\begin{cases}
\mathrm{d}\mathcal{L}_1 = \dot{\mathbf{p}}\mathrm{d}\mathbf{q} + \mathbf{p}\mathrm{d}\dot{\mathbf{q}} + \dfrac{\partial \mathcal{L}_1}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \dfrac{\partial \mathcal{L}_1}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}} \\
\mathrm{d}\mathcal{L}_2 = \dot{\mathbf{r}}\mathrm{d}\mathbf{q} + \mathbf{r}\mathrm{d}\dot{\mathbf{q}} + \dfrac{\partial \mathcal{L}_2}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \dfrac{\partial \mathcal{L}_2}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}}
\end{cases} \tag{8}
$$

Considering

$$
\begin{cases}
\mathbf{p}\mathrm{d}\dot{\mathbf{q}} = \mathrm{d}\left(\dot{\mathbf{q}}\mathbf{p}\right) - \dot{\mathbf{q}}\mathrm{d}\mathbf{p} \\
\mathbf{r}\mathrm{d}\dot{\mathbf{q}} = \mathrm{d}\left(\dot{\mathbf{q}}\mathbf{r}\right) - \dot{\mathbf{q}}\mathrm{d}\mathbf{r}
\end{cases} \tag{9}
$$

then Eq. (8) can be rewritten as

$$
\begin{cases}
\mathrm{d}\left(\mathcal{L}_1 - \mathbf{p}\dot{\mathbf{q}}\right) = \dot{\mathbf{p}}\mathrm{d}\mathbf{q} - \dot{\mathbf{q}}\mathrm{d}\mathbf{p} + \dfrac{\partial \mathcal{L}_1}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \dfrac{\partial \mathcal{L}_1}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}} \\
\mathrm{d}\left(\mathcal{L}_2 - \mathbf{r}\dot{\mathbf{q}}\right) = \dot{\mathbf{r}}\mathrm{d}\mathbf{q} - \dot{\mathbf{q}}\mathrm{d}\mathbf{r} + \dfrac{\partial \mathcal{L}_2}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \dfrac{\partial \mathcal{L}_2}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}}
\end{cases} \tag{10}
$$

The Nambu-Routh functions are expanded as

$$
\mathrm{d}\mathcal{R}_k = \frac{\partial \mathcal{R}_k}{\partial \mathbf{q}}\mathrm{d}\mathbf{q} + \frac{\partial \mathcal{R}_k}{\partial \mathbf{p}}\mathrm{d}\mathbf{p} + \frac{\partial \mathcal{R}_k}{\partial \mathbf{r}}\mathrm{d}\mathbf{r} + \frac{\partial \mathcal{R}_k}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \frac{\partial \mathcal{R}_k}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}} \tag{11}
$$

which lead to

$$
\begin{aligned}
\mathrm{d}\mathcal{R}_1 \wedge \mathrm{d}\mathcal{R}_2 &= \left(\frac{\partial \mathcal{R}_1}{\partial \mathbf{q}}\mathrm{d}\mathbf{q} + \frac{\partial \mathcal{R}_1}{\partial \mathbf{p}}\mathrm{d}\mathbf{p} + \frac{\partial \mathcal{R}_1}{\partial \mathbf{r}}\mathrm{d}\mathbf{r} + \frac{\partial \mathcal{R}_1}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \frac{\partial \mathcal{R}_1}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}}\right) \\
&\quad \wedge \left(\frac{\partial \mathcal{R}_2}{\partial \mathbf{q}}\mathrm{d}\mathbf{q} + \frac{\partial \mathcal{R}_2}{\partial \mathbf{p}}\mathrm{d}\mathbf{p} + \frac{\partial \mathcal{R}_2}{\partial \mathbf{r}}\mathrm{d}\mathbf{r} + \frac{\partial \mathcal{R}_2}{\partial \boldsymbol{\xi}}\mathrm{d}\boldsymbol{\xi} + \frac{\partial \mathcal{R}_2}{\partial \dot{\boldsymbol{\xi}}}\mathrm{d}\dot{\boldsymbol{\xi}}\right) \\
&= \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \mathbf{r}\right)}\mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{r} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{r}\right)}\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{r} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{p}\right)}\mathrm{d}\mathbf{q} \wedge \mathrm{d}\mathbf{p} \\
&\quad + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \boldsymbol{\xi}\right)}\mathrm{d}\mathbf{q} \wedge \mathrm{d}\boldsymbol{\xi} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \boldsymbol{\xi}\right)}\mathrm{d}\mathbf{p} \wedge \mathrm{d}\boldsymbol{\xi} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \boldsymbol{\xi}\right)}\mathrm{d}\mathbf{r} \wedge \mathrm{d}\boldsymbol{\xi} \\
&\quad + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \dot{\boldsymbol{\xi}}\right)}\mathrm{d}\mathbf{q} \wedge \mathrm{d}\dot{\boldsymbol{\xi}} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \dot{\boldsymbol{\xi}}\right)}\mathrm{d}\mathbf{p} \wedge \mathrm{d}\dot{\boldsymbol{\xi}} + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \dot{\boldsymbol{\xi}}\right)}\mathrm{d}\mathbf{r} \wedge \mathrm{d}\dot{\boldsymbol{\xi}} \\
&\quad + \frac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\right)}\mathrm{d}\boldsymbol{\xi} \wedge \mathrm{d}\dot{\boldsymbol{\xi}}
\end{aligned} \tag{12}
$$

With the generalized Legendre transformation,

$$\mathrm{d}\mathcal{R}_1 \wedge \mathrm{d}\mathcal{R}_2 = \frac{1}{\dot{\mathbf{q}}^{\odot}} \mathrm{d}\left(\mathbf{p}\dot{\mathbf{q}} - \mathcal{L}_1\right) \wedge \mathrm{d}\left(\mathbf{r}\dot{\mathbf{q}} - \mathcal{L}_2\right) \tag{13}$$

we will arrive at

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \mathbf{r}\right)}; \ \dot{\mathbf{p}} = -\dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{r}\right)}; \ \dot{\mathbf{r}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{p}\right)} \\ \dfrac{\dot{\mathbf{p}}}{\dot{\mathbf{q}}} \dfrac{\partial \mathcal{L}_2}{\partial \boldsymbol{\xi}} - \dfrac{\dot{\mathbf{r}}}{\dot{\mathbf{q}}} \dfrac{\partial \mathcal{L}_1}{\partial \boldsymbol{\xi}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \boldsymbol{\xi}\right)}; \ -\dfrac{\partial \mathcal{L}_2}{\partial \boldsymbol{\xi}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \boldsymbol{\xi}\right)}; \ \dfrac{\partial \mathcal{L}_1}{\partial \boldsymbol{\xi}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \boldsymbol{\xi}\right)} \\ \dfrac{\dot{\mathbf{p}}}{\dot{\mathbf{q}}} \dfrac{\partial \mathcal{L}_2}{\partial \dot{\boldsymbol{\xi}}} - \dfrac{\dot{\mathbf{r}}}{\dot{\mathbf{q}}} \dfrac{\partial \mathcal{L}_1}{\partial \dot{\boldsymbol{\xi}}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \dot{\boldsymbol{\xi}}\right)}; \ -\dfrac{\partial \mathcal{L}_2}{\partial \dot{\boldsymbol{\xi}}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \dot{\boldsymbol{\xi}}\right)}; \ \dfrac{\partial \mathcal{L}_1}{\partial \dot{\boldsymbol{\xi}}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \dot{\boldsymbol{\xi}}\right)} \\ \dfrac{1}{\dot{\mathbf{q}}} \dfrac{\partial\left(\mathcal{L}_1, \mathcal{L}_2\right)}{\partial\left(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\right)} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}\right)} \end{cases} \tag{14}$$

by comparing the two sides of Eq. (13), where $\mathbf{ab}/\mathbf{c}^{\odot} = \sum_i a_i b_i / c_i$. We notice that $\mathcal{L}_k$ fulfill Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}_k}{\partial \boldsymbol{\xi}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}_k}{\partial \dot{\boldsymbol{\xi}}} = 0 \tag{15}$$

through which the Nambu-Routh equations are obtained as

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \mathbf{r}\right)} \\ \dot{\mathbf{p}} = -\dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{r}\right)} \\ \dot{\mathbf{r}} = \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{q}, \mathbf{p}\right)} \\ \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \boldsymbol{\xi}\right)} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{p}, \dot{\boldsymbol{\xi}}\right)} = 0 \\ \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \boldsymbol{\xi}\right)} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial\left(\mathcal{R}_1, \mathcal{R}_2\right)}{\partial\left(\mathbf{r}, \dot{\boldsymbol{\xi}}\right)} = 0 \end{cases} \tag{16}$$

Besides, we can also prove that the rest equations shown in Eq. (14) are identities, as stated in Appendix.

The results in Eq. (16) may be generalized to the cases involving multiple Nambu-Routh momenta. Assuming that there exist $s-1$ Lagrangians (i.e., $\mathcal{L}_1\left(\mathbf{q}, \boldsymbol{\xi}, \dot{\mathbf{q}}, \dot{\boldsymbol{\xi}}\right), \cdots, \mathcal{L}_{s-1}\left(\mathbf{q}, \boldsymbol{\xi}, \dot{\mathbf{q}}, \dot{\boldsymbol{\xi}}\right)$) and the principle of least action holds for each Lagrangian $\mathcal{L}_k\left(\mathbf{q}, \boldsymbol{\xi}, \dot{\mathbf{q}}, \dot{\boldsymbol{\xi}}\right)$. The corresponding momenta are defined as ${}_k\mathbf{p} \equiv \partial \mathcal{L}_k / \partial \dot{\mathbf{q}}$, which lead to the generalized Legendre transformation containing $s-1$ Nambu-Routh functions:

$$\mathrm{d}\mathcal{R}_1 \wedge \cdots \wedge \mathrm{d}\mathcal{R}_{s-1} = \frac{1}{\left(\dot{\mathbf{q}}^{\odot}\right)^{s-2}} \mathrm{d}\left({}_1\mathbf{p}\dot{\mathbf{q}} - \mathcal{L}_1\right) \wedge \cdots \wedge \mathrm{d}\left({}_{s-1}\mathbf{p}\dot{\mathbf{q}} - \mathcal{L}_{s-1}\right) \quad (s \geq 2) \tag{17}$$

The Nambu-Routh equations are derived accordingly as

$$\begin{cases} \dot{\mathbf{q}} = \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_1\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}\right)} \\ \vdots \\ {}_k\dot{\mathbf{p}} = (-1)^k \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left(\mathbf{q}, \cdots, {}_{k-1}\mathbf{p}, {}_{k+1}\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}\right)} \\ \vdots \\ {}_{s-1}\dot{\mathbf{p}} = (-1)^{s-1} \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left(\mathbf{q}, \cdots, {}_{s-2}\mathbf{p}\right)} \end{cases}$$
$$\begin{cases} \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_1\mathbf{p}, \cdots, {}_{s-2}\mathbf{p}, \boldsymbol{\xi}\right)} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_1\mathbf{p}, \cdots, {}_{s-2}\mathbf{p}, \dot{\boldsymbol{\xi}}\right)} = 0 \\ \vdots \\ \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_1\mathbf{p}, \cdots, {}_{k-1}\mathbf{p}, {}_{k+1}\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}, \boldsymbol{\xi}\right)} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_1\mathbf{p}, \cdots, {}_{k-1}\mathbf{p}, {}_{k+1}\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}, \dot{\boldsymbol{\xi}}\right)} = 0 \\ \vdots \\ \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_2\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}, \boldsymbol{\xi}\right)} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial\left(\mathcal{R}_1, \cdots, \mathcal{R}_{s-1}\right)}{\partial\left({}_2\mathbf{p}, \cdots, {}_{s-1}\mathbf{p}, \dot{\boldsymbol{\xi}}\right)} = 0 \end{cases} \tag{18}$$

4. Conclusions

In this paper, we derived basic formulations of Nambu-Routh mechanics. With the generalized Legendre transformation, we arrive at Nambu-Routh equations and several identities.

Appendix

We can prove that

$$\left(\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)-\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}-\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}\right)$$
$$\equiv\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}-\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}\right)$$

for $\dot{\mathbf{p}}\dfrac{\partial\mathcal{L}_2}{\partial\boldsymbol{\xi}}-\dot{\mathbf{r}}\dfrac{\partial\mathcal{L}_1}{\partial\boldsymbol{\xi}}=\dot{\mathbf{q}}\dfrac{\partial(\mathcal{R}_1,\mathcal{R}_2)}{\partial(\mathbf{q},\boldsymbol{\xi})}$;

$$\left(\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)-\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}-\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}\right)$$
$$\equiv\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{q}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}-\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{q}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}\right)$$

for $\dot{\mathbf{p}}\dfrac{\partial\mathcal{L}_2}{\partial\boldsymbol{\xi}}-\dot{\mathbf{r}}\dfrac{\partial\mathcal{L}_1}{\partial\boldsymbol{\xi}}=\dot{\mathbf{q}}\dfrac{\partial(\mathcal{R}_1,\mathcal{R}_2)}{\partial(\mathbf{q},\dot{\boldsymbol{\xi}})}$; and

$$\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}-\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}\right)-\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}-\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}\right)$$
$$\equiv\left(\frac{\partial\mathcal{R}_1}{\partial\mathbf{p}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{r}}-\frac{\partial\mathcal{R}_1}{\partial\mathbf{r}}\frac{\partial\mathcal{R}_2}{\partial\mathbf{p}}\right)\left(\frac{\partial\mathcal{R}_1}{\partial\boldsymbol{\xi}}\frac{\partial\mathcal{R}_2}{\partial\dot{\boldsymbol{\xi}}}-\frac{\partial\mathcal{R}_1}{\partial\dot{\boldsymbol{\xi}}}\frac{\partial\mathcal{R}_2}{\partial\boldsymbol{\xi}}\right)$$

for $\dfrac{\partial(\mathcal{L}_1,\mathcal{L}_2)}{\partial(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})}=\dot{\mathbf{q}}\dfrac{\partial(\mathcal{R}_1,\mathcal{R}_2)}{\partial(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})}$.

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