

HOW TO SOLVE AN LQ PROBLEM WITH TIME-VARYING TARGETS BY RICCATI'S THEORY

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In this short manual, we explain how to use Riccati's theory to solve an LQ control problem with targets. The related MATLAB code is downloadable freely.

We minimize over $u \in L^2(0, T)$ the functional

$$J(u) = \frac{1}{2} \left[\int_0^T \|u(t) - q(t)\|^2 dt + \beta \int_0^T \|C(x(t) - z(t))\|^2 dt + \gamma \|D(x(T) - z(T))\|^2 \right],$$

where:

$$(1) \quad \begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & t \in (0, T) \\ x(0) = x_0. \end{cases}$$

In the above control problem, $A \in \mathcal{M}_{n \times n}$, $B \in \mathcal{M}_{n \times m}$, $C \in \mathcal{M}_{r \times n}$ and $D \in \mathcal{M}_{r \times n}$. The control $u : [0, T] \rightarrow \mathbb{R}^m$, while the state $x : [0, T] \rightarrow \mathbb{R}^n$. The control target is $q \in C^1([0, T]; \mathbb{R}^m)$ and the state target is $z \in C^1([0, T]; \mathbb{R}^n)$. $\beta \geq 0$ and $\gamma \geq 0$ are positive parameters.

By the Direct Methods in the Calculus of Variations and strict convexity, the above problem admits an unique optimal control u^T . The corresponding optimal state is denoted by x^T .

1. DESCRIPTION OF THE ALGORITHM

We compute the optimal pair (u^T, x^T) by using the well-known Riccati's theory (see, for instance, [1, Lemma 2.6] and [2, section 4.3]).

Step 1 Computation of the solution to the Riccati Differential Equation

We determine the Riccati operator, satisfying the Riccati Differential Equation

$$\begin{cases} \mathcal{E}_t = \beta C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} & \forall t \in (0, +\infty) \\ \mathcal{E}(0) = \gamma D^*D. \end{cases}$$

We solve the above nonlinear ODE, by employing its linear representation, as suggested by [2, Proposition 4.3.5]. Namely:

- (1) we compute the matrix exponential $\exp(-t\text{Ham})$, where Ham is the Hamiltonian matrix

$$\text{Ham} := \begin{pmatrix} -A & -BB^* \\ -\beta C^*C & A^* \end{pmatrix};$$

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(2) we decompose the exponential in $n \times n$ -blocks

$$\exp(-t\text{Ham}) = \begin{pmatrix} R_1(t) & R_2(t) \\ R_3(t) & R_4(t) \end{pmatrix}.$$

(3) we obtain the Riccati operator

$$\mathcal{E}(t) = (R_3(t) + \gamma R_4(t) D^* D) (R_1(t) + \gamma R_2(t) D^* D)^{-1}.$$

Step 2 Computation of the remainder function

We take into account the targets computing the remainder function h^T

$$\begin{cases} -\frac{d}{dt}h^T + (A^* + \mathcal{E}(T-t)BB^*)h^T + \mathcal{E}(T-t)\left(\frac{dz}{dt} + Az - Bq\right) = 0 & t \in (0, T) \\ h^T(T) = 0. \end{cases}$$

Step 3 Computation of the optimal state

The optimal state solves the closed loop system

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = -BB^* (\mathcal{E}(T-t)(x^T(t) - z(t)) + h^T(t)) + Bq(t) & t \in (0, T) \\ x^T(0) = x_0. \end{cases}$$

Step 4 Computation of the optimal control

We compute the optimal control in a feedback form

$$u^T(t) = -B^* (\mathcal{E}(T-t)(x^T(t) - z) + h^T(t)) + q(t).$$

2. EXAMPLE

Take

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose $\beta = 26$, $\gamma = 0$, $x_0 = [1.4; 1.4]$, $q \equiv 0$, $z(t) = [\sin(t); \sin(t)]$ and $T = 10$. We obtain figures [1](#), [2](#) and [3](#).

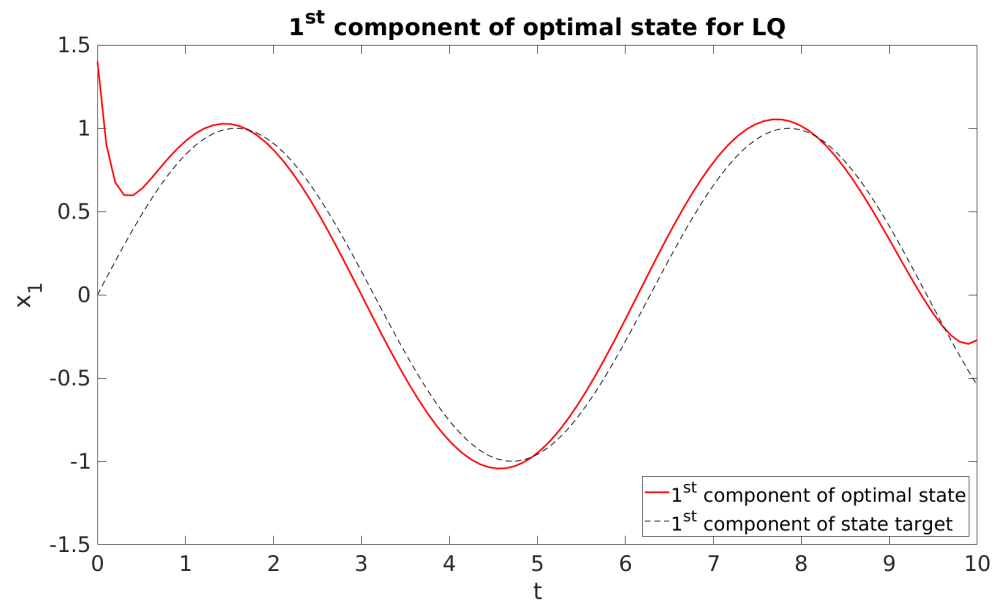


FIGURE 1. first component of the optimal state.

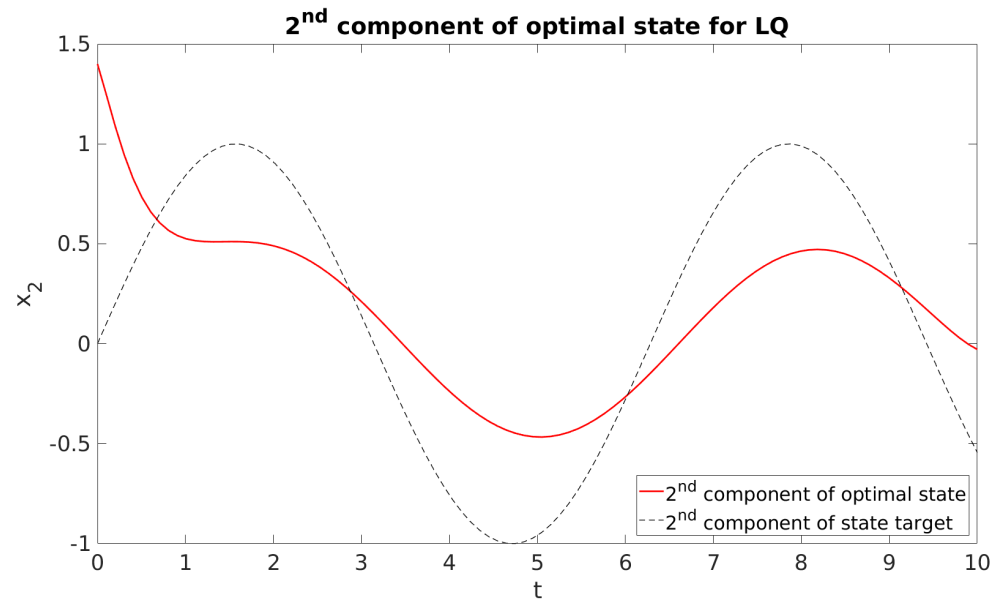


FIGURE 2. second component of the optimal state.

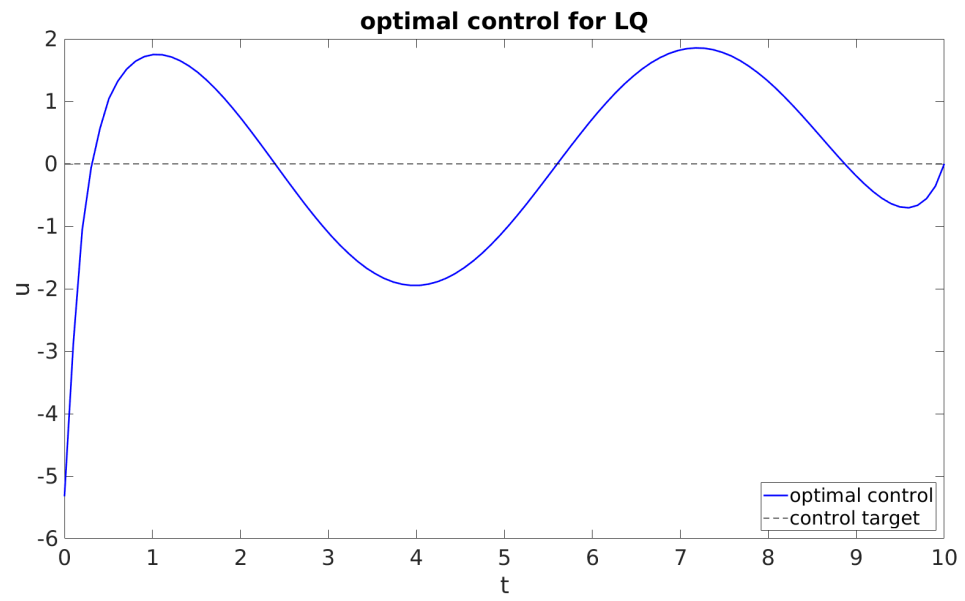


FIGURE 3. optimal control.

Since the parameter β is large enough and the control acts only on the first component of (1)

- the first component of the state is close to the target;
- the second component of the state is less close to the target;
- the control is far from its target.

The algorithm described in this guide can be employed to test the fulfillment of the turnpike property (see, for instance, [1] and [3]). In agreement with the theory, the turnpike effect is evident if:

- the targets are constants;
- (A, B) is controllable;
- (A, C) is observable, $\beta > 0$ and $\gamma = 0$;
- the time horizon T is large enough.

REFERENCES

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