HOW TO SOLVE AN LQ PROBLEM WITH TIME-VARYING TARGETS BY RICCATI'S THEORY

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In this short manual, we explain how to use Riccati's theory to solve an LQ control problem with targets. The related MATLAB code is downloadable freely.

We minimize over $u \in L^2(0,T)$ the functional

$$J(u) = \frac{1}{2}\left[\int_0^T \|u(t)-q(t)\|^2 dt + \beta\int_0^T \|C(x(t)-z(t))\|^2 dt + \gamma\|D(x(T)-z(T))\|^2\right],$$

where:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & t\in(0,T) \\ x(0) = x_0. \end{cases} \tag{1}$$

In the above control problem, $A \in \mathcal{M}_{n\times n}$, $B \in \mathcal{M}_{n\times m}$, $C \in \mathcal{M}_{r\times n}$ and $D \in \mathcal{M}_{r\times n}$. The control $u : [0,T] \longrightarrow \mathbb{R}^m$, while the state $x : [0,T] \longrightarrow \mathbb{R}^n$. The control target is $q \in C^1([0,T];\mathbb{R}^m)$ and the state target is $z \in C^1([0,T];\mathbb{R}^n)$. $\beta \geq 0$ and $\gamma \geq 0$ are positive parameters.

By the Direct Methods in the Calculus of Variations and strict convexity, the above problem admits an unique optimal control u^T. The corresponding optimal state is denoted by x^T.

1. Description of the algorithm

We compute the optimal pair (u^T, x^T) by using the well-known Riccati's theory (see, for instance, [1, Lemma 2.6] and [2, section 4.3]).

Step 1 **Computation of the solution to the Riccati Differential Equation**

We determine the Riccati operator, satisfying the Riccati Differential Equation

$$\begin{cases} \mathcal{E}_t = \beta C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} & \forall t\in(0,+\infty) \\ \mathcal{E}(0) = \gamma D^*D. \end{cases}$$

We solve the above nonlinear ODE, by employing its linear representation, as suggested by [2, Proposition 4.3.5]. Namely:

(1) we compute the matrix exponential $\exp(-t\mathrm{Ham})$, where Ham is the Hamiltonian matrix

$$\mathrm{Ham} := \begin{pmatrix} -A & -BB^* \\ -\beta C^*C & A^* \end{pmatrix};$$

(2) we decompose the exponential in $n\times n$-blocks

$$\exp(-t\mathrm{Ham}) = \begin{pmatrix} R_1(t) & R_2(t) \\ R_3(t) & R_4(t) \end{pmatrix}.$$

(3) we obtain the Riccati operator

$$\mathcal{E}(t) = (R_3(t) + \gamma R_4(t)D^*D)\left(R_1(t) + \gamma R_2(t)D^*D\right)^{-1}.$$

Step 2 **Computation of the remainder function**
We take into account the targets computing the remainder function h^T

$$\begin{cases} -\frac{d}{dt}h^T + (A^* + \mathcal{E}(T-t)BB^*)h^T + \mathcal{E}(T-t)\left(\frac{dz}{dt} + Az - Bq\right) = 0 & t \in (0,T) \\ h^T(T) = 0. \end{cases}$$

Step 3 **Computation of the optimal state**
The optimal state solves the closed loop system

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = -BB^*\left(\mathcal{E}(T-t)(x^T(t) - z(t)) + h^T(t)\right) + Bq(t) & t \in (0,T) \\ x^T(0) = x_0. \end{cases}$$

Step 4 **Computation of the optimal control**
We compute the optimal control in a feedback form

$$u^T(t) = -B^*\left(\mathcal{E}(T-t)(x^T(t) - z) + h^T(t)\right) + q(t).$$

2. Example

Take

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose $\beta = 26$, $\gamma = 0$, $x_0 = [1.4; 1.4]$, $q \equiv 0$, $z(t) = [\sin(t); \sin(t)]$ and $T = 10$. We obtain figures 1, 2 and 3.

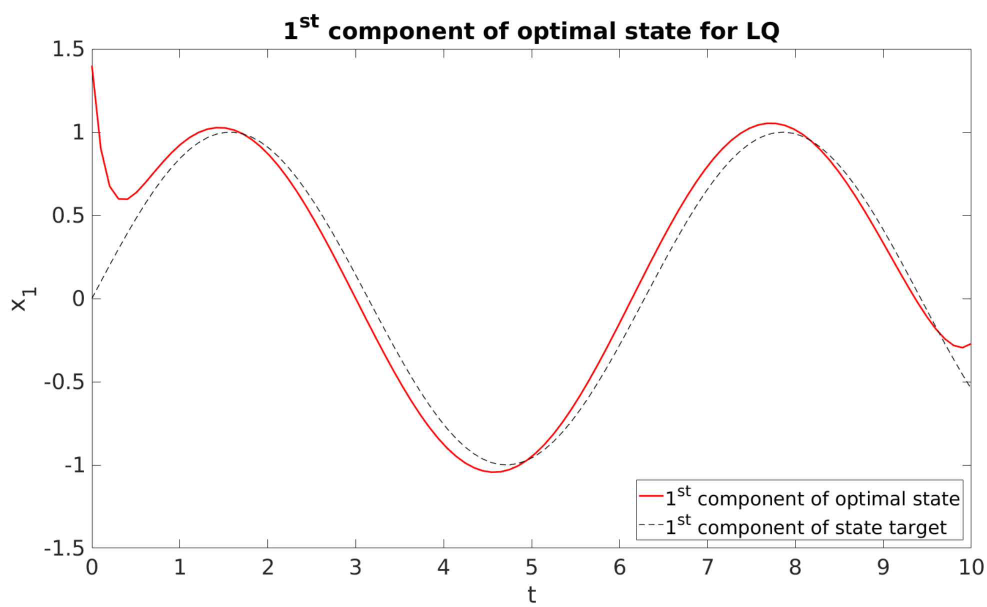

Figure 1. first component of the optimal state.

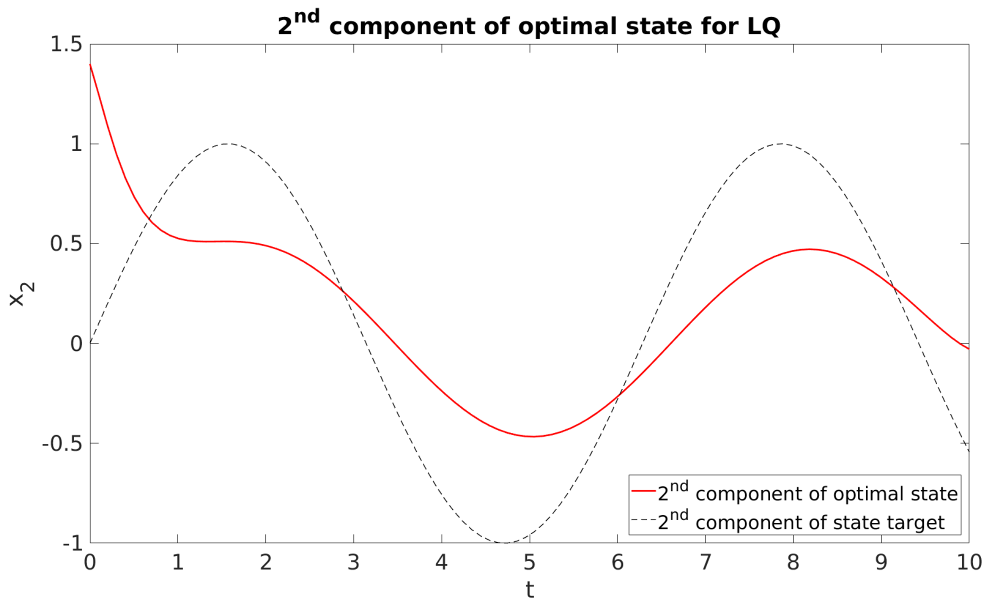

FIGURE 2. second component of the optimal state.

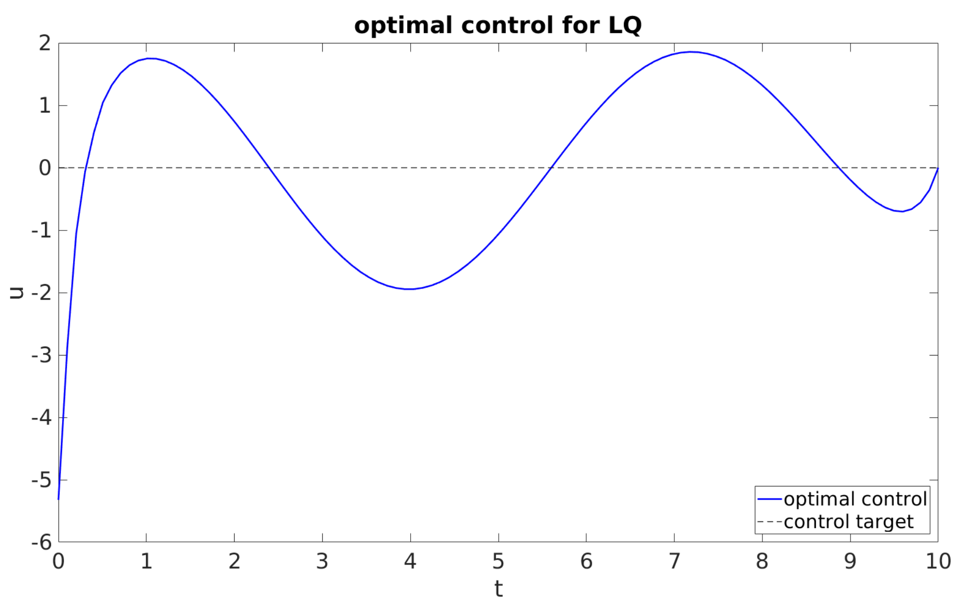

FIGURE 3. optimal control.

Since the parameter β is large enough and the control acts only on the first component of (1)

- the first component of the state is close to the target;
- the second component of the state is less close to the target;
- the control is far from its target.

The algorithm described in this guide can be employed to test the fulfillment of the turnpike property (see, for instance, [1] and [3]). In agreement with the theory, the turnpike effect is evident if:

- the targets are constants;
- (A, B) is controllable;
- (A, C) is observable, $\beta > 0$ and $\gamma = 0$;
- the time horizon T is large enough.

References

[1] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim., 51 (2013), pp. 4242–4273. available online: https://epubs.siam.org/doi/pdf/10.1137/130907239.

[2] E. Trélat, *Contrôle optimal: théorie & applications*, Mathématiques Concrètes., Vuibert, Paris, 2005. available online: https://www.ljll.math.upmc.fr/trelat/fichiers/livreopt2.pdf.

[3] E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations, 258 (2015), pp. 81–114. available online: https://arxiv.org/abs/1402.3263.