HOW TO SOLVE AN LQ PROBLEM WITH TIME-VARYING TARGETS BY RICCATI'S THEORY

Dario Pighin* - June 28, 2018

Departamento de Matemáticas, Universidad Autónoma de Madrid
28049 Madrid, Spain

In this short manual, we explain how to use Riccati's theory to solve an LQ control problem with targets. The related MATLAB code is downloadable freely.

We minimize over $u \in L^2(0,T)$ the functional

$$J(u) = \frac{1}{2}\left[\int_0^T \|u(t)-q(t)\|^2 dt + \beta\int_0^T \|C(x(t)-z(t))\|^2 dt + \gamma\|D(x(T)-z(T))\|^2\right],$$

where:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = Bu(t) & t \in (0,T) \\ x(0) = x_0. \end{cases} \tag{1}$$

In the above control problem, $A \in \mathcal{M}_{n\times n}$, $B \in \mathcal{M}_{n\times m}$, $C \in \mathcal{M}_{r\times n}$ and $D \in \mathcal{M}_{r\times n}$. The control $u : [0,T] \longrightarrow \mathbb{R}^m$, while the state $x : [0,T] \longrightarrow \mathbb{R}^n$. The control target is $q \in C^1([0,T];\mathbb{R}^m)$ and the state target is $z \in C^1([0,T];\mathbb{R}^n)$. $\beta \geq 0$ and $\gamma \geq 0$ are positive parameters.

By the Direct Methods in the Calculus of Variations and strict convexity, the above problem admits an unique optimal control u^T. The corresponding optimal state is denoted by x^T.

1. Description of the algorithm

We compute the optimal pair (u^T, x^T) by using the well-known Riccati's theory (see, for instance, [1, Lemma 2.6] and [2, section 4.3]).

Step 1 **Computation of the solution to the Riccati Differential Equation**
We determine the Riccati operator, satisfying the Riccati Differential Equation

$$\begin{cases} \mathcal{E}_t = \beta C^*C - (\mathcal{E}A + A^*\mathcal{E}) - \mathcal{E}BB^*\mathcal{E} & \forall t \in (0,+\infty) \\ \mathcal{E}(0) = \gamma D^*D. \end{cases}$$

We solve the above nonlinear ODE, by employing its linear representation, as suggested by [2, Proposition 4.3.5]. Namely:

(1) we compute the matrix exponential $\exp(-t\mathrm{Ham})$, where Ham is the Hamiltonian matrix

$$\mathrm{Ham} := \begin{pmatrix} -A & -BB^* \\ -\beta C^*C & A^* \end{pmatrix};$$

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(2) we decompose the exponential in $n \times n$-blocks

$$\exp(-t\mathrm{Ham}) = \begin{pmatrix} R_1(t) & R_2(t) \\ R_3(t) & R_4(t) \end{pmatrix}.$$

(3) we obtain the Riccati operator

$$\mathcal{E}(t) = \left(R_3(t) + \gamma R_4(t) D^* D\right)\left(R_1(t) + \gamma R_2(t) D^* D\right)^{-1}.$$

Step 2 **Computation of the remainder function**
We take into account the targets computing the remainder function h^T

$$\begin{cases} -\frac{d}{dt}h^T + (A^* + \mathcal{E}(T-t)BB^*)h^T + \mathcal{E}(T-t)\left(\frac{dz}{dt} + Az - Bq\right) = 0 & t \in (0,T) \\ h^T(T) = 0. \end{cases}$$

Step 3 **Computation of the optimal state**
The optimal state solves the closed loop system

$$\begin{cases} \frac{d}{dt}x^T(t) + Ax^T(t) = -BB^*\left(\mathcal{E}(T-t)(x^T(t) - z(t)) + h^T(t)\right) + Bq(t) & t \in (0,T) \\ x^T(0) = x_0. \end{cases}$$

Step 4 **Computation of the optimal control**
We compute the optimal control in a feedback form

$$u^T(t) = -B^*\left(\mathcal{E}(T-t)(x^T(t) - z) + h^T(t)\right) + q(t).$$

2. Example

Take

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad D := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Choose $\beta = 26$, $\gamma = 0$, $x_0 = [1.4; 1.4]$, $q \equiv 0$, $z(t) = [\sin(t); \sin(t)]$ and $T = 10$. We obtain figures 1, 2 and 3.

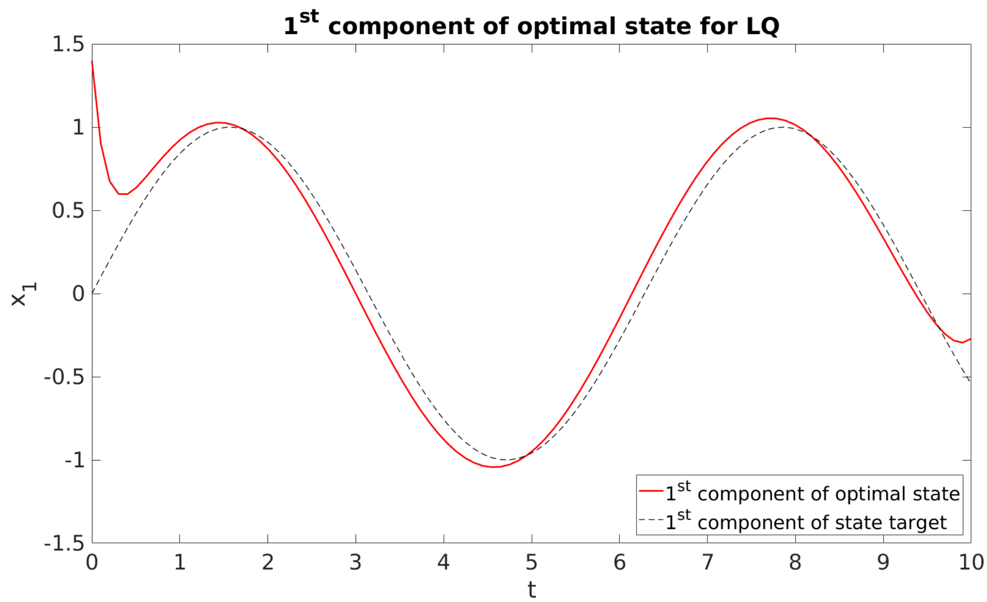

FIGURE 1. first component of the optimal state.

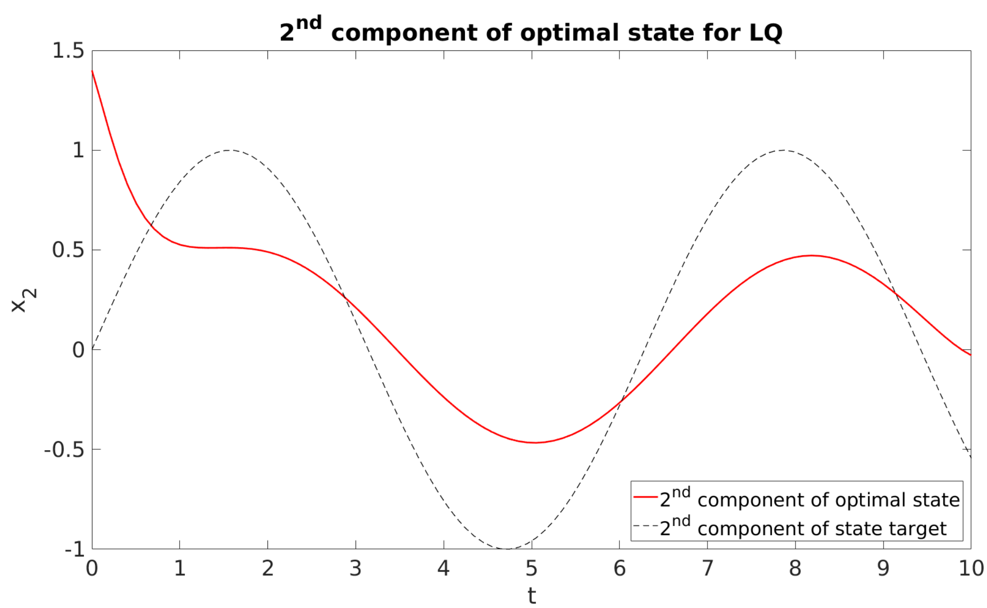

FIGURE 2. second component of the optimal state.

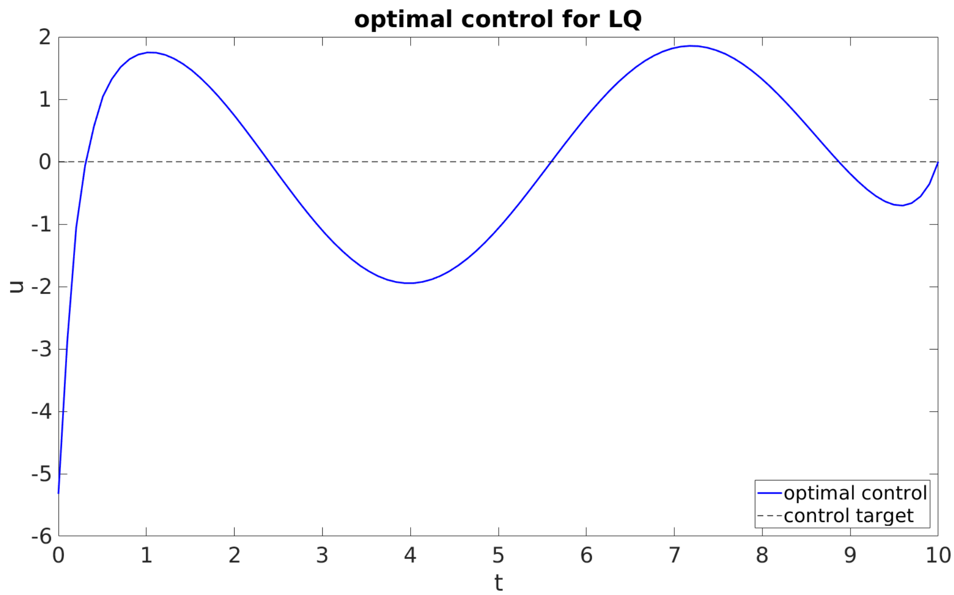

FIGURE 3. optimal control.

Since the parameter β is large enough and the control acts only on the first component of (1)

- the first component of the state is close to the target;
- the second component of the state is less close to the target;
- the control is far from its target.

The algorithm described in this guide can be employed to test the fulfillment of the turnpike property (see, for instance, [1] and [3]). In agreement with the theory, the turnpike effect is evident if:

- the targets are constants;
- (A, B) is controllable;
- (A, C) is observable, $\beta > 0$ and $\gamma = 0$;
- the time horizon T is large enough.

References

[1] A. Porretta and E. Zuazua, *Long time versus steady state optimal control*, SIAM J. Control Optim., 51 (2013), pp. 4242–4273. available online: https://epubs.siam.org/doi/pdf/10.1137/130907239.

[2] E. Trélat, *Contrôle optimal: théorie & applications*, Mathématiques Concrètes., Vuibert, Paris, 2005. available online: https://www.ljll.math.upmc.fr/trelat/fichiers/livreopt2.pdf.

[3] E. Trélat and E. Zuazua, *The turnpike property in finite-dimensional nonlinear optimal control*, Journal of Differential Equations, 258 (2015), pp. 81–114. available online: https://arxiv.org/abs/1402.3263.

E-mail address: `dario.pighin@uam.es`