Practice Set

Course Code: CS6007 Instructor: Dr. Avishek Ghosh

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Instructions:

- This practice set covers basic optimization concepts.
- Attempt all questions for a better understanding of optimization foundations.

Q1. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality,

$$C = \left\{ x \in \mathbb{R}^n \;\middle|\; x^\top A x + b^\top x + c \leq 0 \right\},$$

where $A \in \mathbb{S}^n$ (the set of symmetric $n \times n$ matrices), $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that C is convex if $A \succeq 0$ (i.e., A is positive semidefinite).

(b) Let $H = \{x \in \mathbb{R}^n \mid g^\top x + h = 0\}$ be a hyperplane with $g \neq 0$. Show that the intersection $C \cap H$ is convex if $A + \lambda g g^\top \succeq 0$ for some $\lambda \in \mathbb{R}$.

Q2. Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is convex and M-smooth. Show that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{M} \|\nabla f(x) - \nabla f(y)\|^2$$

Hint: Use the Descent Lemma on the following two functions:

(a) $h_x(z) = f(z) - \langle \nabla f(x), z \rangle$

(b) $h_y(z) = f(z) - \langle \nabla f(y), z \rangle$

Q3. Let f be a (M, m) smooth and strongly convex function ($M > m$) on $\mathbb{R}^d$. Then, show that

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{mM}{m + M} \|x - y\|^2 + \frac{1}{M + m} \|\nabla f(x) - \nabla f(y)\|^2$$

Hint: Use the result from Question 2.

Q4. [Gradient Descent for Polyak Lojasiewicz (PL) functions]

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is M-smooth and possibly non-convex. However, it satisfies the PL condition (i.e., $\|\nabla f(x)\|^2 \geq 2m[f(x) - f(x^*)]$, for all x where $m > 0$ and x^* is any minimizer of f). Show that running Gradient Descent with initial point x^0 for T steps with step size $\alpha = 1/M$ for such functions yields

$$f(x^T) - f(x^*) \leq \left(1 - \frac{m}{M}\right)^T [f(x^0) - f(x^*)].$$

Q5. Let V be a vector space. A norm on V is a function $p : V \to [0, \infty)$ satisfying the following properties:

- $p(\lambda v) = |\lambda|\, p(v)$ for all $v \in V$, $\lambda \in \mathbb{R}$
- $p(u + v) \leq p(u) + p(v)$ for all $u, v \in V$ (Triangle inequality)
- $p(v) = 0$ if and only if $v = 0$

Now suppose $v \in V$, $r > 0$, and let p be a norm. Show that the set

$$B := \{x \in V \mid p(x - v) < r\}$$

is convex.

Q6. A function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is called **monotone** if for all $x, y \in \mathrm{dom}(\psi)$,

$$(\psi(x) - \psi(y))^\top (x - y) \geq 0.$$

(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone.

(b) Is the converse true? That is, is every monotone mapping the gradient of a convex function? Justify your answer.

Q7. Let $C \subseteq \mathbb{R}^n$ be the solution set of a quadratic inequality given by:

$$C = \left\{x \in \mathbb{R}^n \;\middle|\; x^\top A x + b^\top x + c \leq 0\right\},$$

where $A \in \mathbb{S}^n$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

(a) Show that the set C is convex if $A \succeq 0$, i.e., A is positive semidefinite.

(b) Does the converse hold true? Justify your answer with a counterexample or reasoning.

End of Practice Set