Convergence of Gradient Descent: Smooth and Convex Settings

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Outline

1. Gradient Descent and Decent lemma for Smooth functions
    - GD for Non-Convex Smooth Functions
    - GD for Convex and Smooth function
    - GD for Smooth and Strongly Convex functions

2. Projection

Gradient Descent(GD) Formulation

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Let $f : \mathbb{R}^n \to \mathbb{R}$ *be* M*-smooth (but not necessarily convex).*

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2. In general for non-convex functions, this is the best one can establish for GD.
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Since $\bar{f} \le f(x^T)$, it follows

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\sum_{k=0}^{T-1}\left\|\nabla f(x^k)\right\|^2 \;\le\; 2M\big(f(x^0) - \bar{f}\big).
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- In practice, choose a large T and apply early stopping.

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- Requires differentiability; non-smooth cases call for subgradient methods.

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Since $\{f(x^k)\}$ is non-increasing due to Decent lemma and hence any term will be bounded by avarage quantity,

Proof Contd..

Summing over $k = 0, \ldots, T-1$ gives

$$\sum_{k=0}^{T-1} (f(x^{k+1}) - f(x^*)) \;\leq\; \tfrac{M}{2}(\|x^0 - x^*\|^2 - \|x^T - x^*\|^2) \;\leq\; \tfrac{M}{2}\|x^0 - x^*\|^2.$$

Since $\{f(x^k)\}$ is non-increasing due to Decent lemma and hence any term will be bounded by avarage quantity,

$$f(x^T) - f(x^*) \;\leq\; \frac{1}{T}\sum_{k=0}^{T-1}(f(x^k) - f(x^*)) \;\leq\; \frac{M}{2T}\|x^0 - x^*\|^2.$$

Case 3:M-Smooth and m-Strongly Convex

Lemma

If f is m-strongly convex and let x^* is a minima of f then

$$\|\nabla f(x)\|^2 \geq 2m(f(x) - f(x^*)) \tag{3}$$

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$$\Rightarrow y = x - \frac{1}{m} \nabla f(x).$$

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f(x^*) &\geq f(x) + \langle \nabla f(x), -\frac{1}{m}\nabla f(x) \rangle + \frac{m}{2}\left\| -\frac{1}{m}\nabla f(x) \right\|^2 \\
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$$\Rightarrow \left\|\nabla f(x)\right\|^2 \geq 2m\big(f(x) - f(x^*)\big).$$

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$$f(x^{k+1}) \leq f(x^k) - \frac{1}{2M} \left\|\nabla f(x^k)\right\|^2$$

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- Using previous lemma for m-strongly convex function ,we get

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Proof cond..

$$f(x^{k+1}) - f(x^*) \leq f(x^k) - \frac{m}{M}(f(x^k) - f(x^*)) - f(x^*)$$

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1. In this case, we can give a guarantee on $\|x^T - x^*\|$. The choice of step-size in that case would be $\alpha_k = \frac{2}{M+m}$
2. Left as exercise!

Convergence Rates of Gradient Descent

Assumptions	Rate	Guarantee

Convergence Rates of Gradient Descent

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f is M-smooth	$\min_{0\leq k<T} \|\nabla f(x_k)\|^2 \leq \frac{2M\left(f(x_0)-f^*\right)}{T}$	Best-iterate gradient norm

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f is M-smooth and convex	$f(x_T) - f^* \le \frac{M}{2T} \|x_0 - x^*\|^2$	Final-iterate function gap
f is M-smooth, m-strongly convex	$f(x_T) - f^* \le \left(1 - \frac{m}{M}\right)^T \left(f(x_0) - f^*\right)$	Exponential (linear) convergence

Let Here, $\kappa = M/m$, $R = \|x_0 - x^*\|$, and $\Delta_0 = f(x_0) - f^*$.

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Gradient Descent: Rates under Various Assumptions

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f is m-strongly convex, M-smooth	$\|x_T - x^*\| \leq \varepsilon$	$O\big(\kappa \ln(R/\varepsilon)\big)$

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f is nonconvex, M-smooth	$\min_{0 \le k < T} \|\nabla f(x_k)\| \le \varepsilon$	$O\Big(\frac{M\,\Delta_0}{\varepsilon^2}\Big)$

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Consider the following constrained optimization problem:

$$\min_{x \in \Omega} f(x) \tag{5}$$

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$$P_\Omega(x) := \arg\min_{z \in \Omega} \frac{1}{2}\|z - x\|^2 \tag{6}$$

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$$[P_\Omega(y)]_i = \max\{0, y_i\}, \quad \text{for each } i = 1, \ldots, d \tag{8}$$

Example 2:Box

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Exercise:Given a point $y \in \mathbb{R}^d$, determine the projection $P_\Omega(y)$ onto Ω.

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Contraction/Non-expansive Property of Projection:

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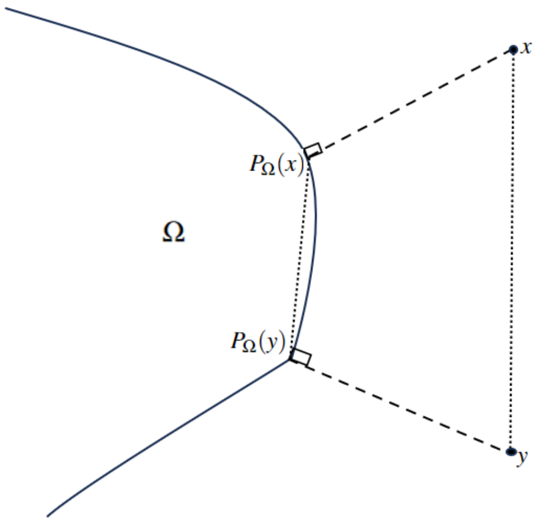
Contraction/Non-expansive Property of Projection:

$$\boxed{\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n} \tag{11}$$

Figure: Illustration of Contraction Property



Convergence of GD for Constrained Optimization

Projected Gradient Descent:

$$x^{k+1} = P_{\Omega}[x^k - \alpha_k \nabla f(x_k)]$$

1. Thanks to the Contraction property of Projection, the distance gets smaller after projection
2. The GD algorithm remains the same with the projection step
3. The convergence rates remain the same

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Thank You!