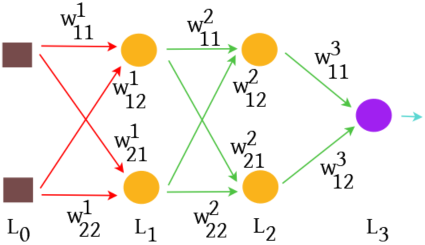
Deep Learning - Theory and Practice

IE 643
Lecture 6

Aug 14, 2025.

1. Recap
   - MLP-Data Perspective
2. Optimization Concepts
   - Gradient Descent
   - Stochastic Gradient Descent
   - Mini-batch SGD
3. Sample-wise Gradient Computation
   - MLP for prediction tasks

Multi Layer Perceptron - Data Perspective



- **Input:** Training Data $D = \{(x^s, y^s)\}_{s=1}^{S}$.
- For each sample x^s the prediction $\hat{y}^s = \text{MLP}(x^s)$.
- **Error:** $e^s = E(y^s, \hat{y}^s)$.
- **Aim:** To minimize $\sum_{s=1}^{S} e^s$.

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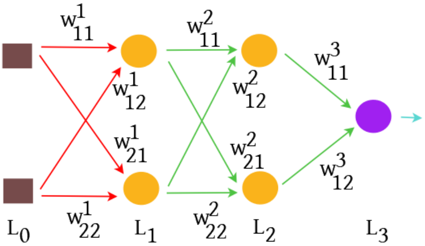
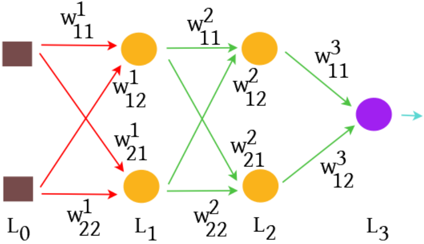


Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s$$

Multi Layer Perceptron - Data Perspective

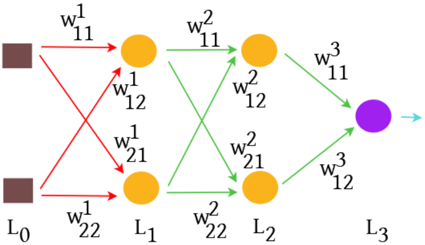


Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s = \sum_{s=1}^{S} E(y^s, \hat{y}^s)$$

Multi Layer Perceptron - Data Perspective



Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s = \sum_{s=1}^{S} E(y^s, \hat{y}^s) = \sum_{s=1}^{S} E(y^s, \text{MLP}(x^s))$$

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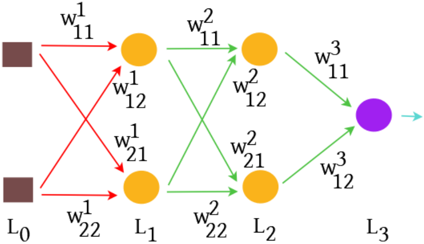


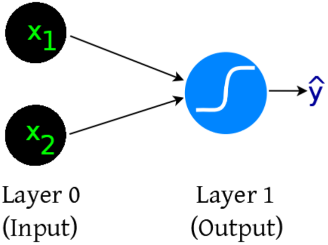
Optimization perspective

- Given training data $D = \{(x^s, y^s)\}_{s=1}^{S}$,

$$\min \sum_{s=1}^{S} e^s = \sum_{s=1}^{S} E(y^s, \hat{y}^s) = \sum_{s=1}^{S} E(y^s, \mathrm{MLP}(x^s))$$

- Note: The minimization is over the weights of the MLP $W^1, \ldots, W^L$, where L denotes number of layers in MLP.

MLP - Data Perspective: A Simple Example



$$\hat{y} = \sigma\left(w_{11}^{1}x_1 + w_{12}^{1}x_2\right) = \frac{1}{1 + \exp\left(-[w_{11}^{1}x_1 + w_{12}^{1}x_2]\right)}$$

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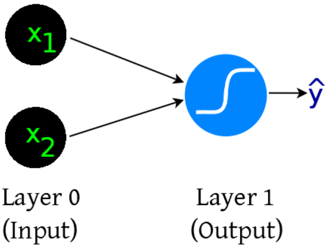
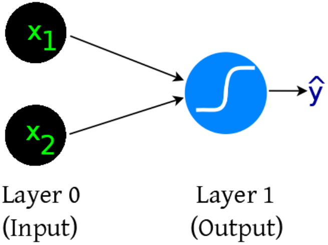


$$\hat{y} = \sigma(w^1_{11}x_1 + w^1_{12}x_2) = \frac{1}{1 + \exp\left(-[w^1_{11}x_1 + w^1_{12}x_2]\right)}$$

Property of 0-1 sigmoid $\sigma : \mathbb{R} \to [0, 1]$

- σ is continuous
- σ is monotonic
- $\sigma(z) \to \begin{cases} 0 & \text{if } z \to -\infty \\ 1 & \text{if } z \to +\infty \end{cases}$

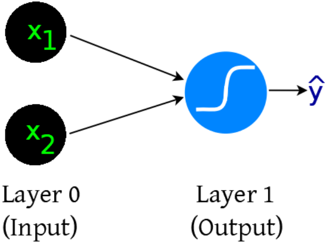
MLP - Data Perspective: A Simple Example



- Let

$$
\begin{aligned}
D = \{&(x^1 = (-3, -3), y^1 = 1),\\
&(x^2 = (-2, -2), y^2 = 1),\\
&(x^3 = (4, 4), y^3 = 0),\\
&(x^4 = (2, -5), y^4 = 0)\}.
\end{aligned}
$$

MLP - Data Perspective: A Simple Example



x_1	x_2	y	$\hat{y} = \sigma(w_{11}^1 x_1 + w_{12}^1 x_2)$
-3	-3	1	$\sigma(-3w_{11}^1 - 3w_{12}^1)$
-2	-2	1	$\sigma(-2w_{11}^1 - 2w_{12}^1)$
4	4	0	$\sigma(4w_{11}^1 + 4w_{12}^1)$
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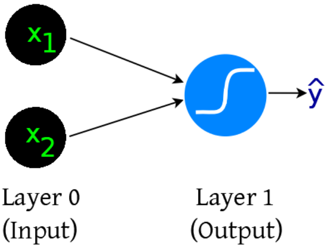

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- **Assume:** $\mathrm{Err}(y, \hat{y}) = (y - \hat{y})^2$.

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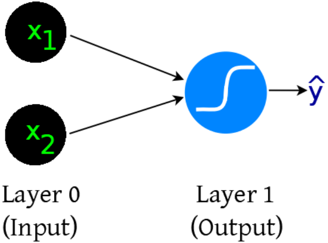

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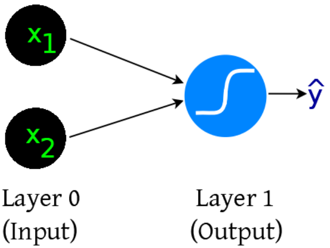
- **Assume:** $\mathrm{Err}(y, \hat{y}) = (y - \hat{y})^2$.
- Popularly called the squared error.

MLP - Data Perspective: A Simple Example



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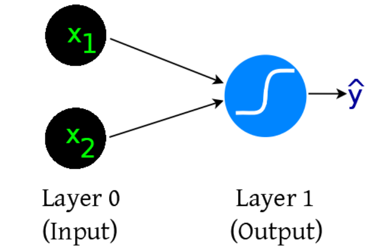
- Total error (or loss):

$$E = \sum_{i=1}^{4} e^i = \sum_{i=1}^{4} \text{Err}(y^i, \hat{y}^i)$$

MLP - Data Perspective: A Simple Example



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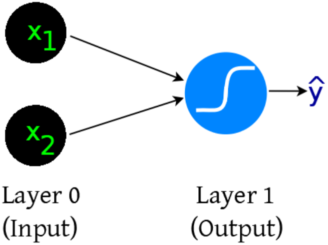
- Total error (or loss):

$$E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

MLP - Data Perspective: A Simple Example



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- Aim: To minimize the total error (or loss), which is

$$\min_{w_{11}^1, w_{12}^1} E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

MLP - Data Perspective: A Simple Example

Visualizing the loss surface:

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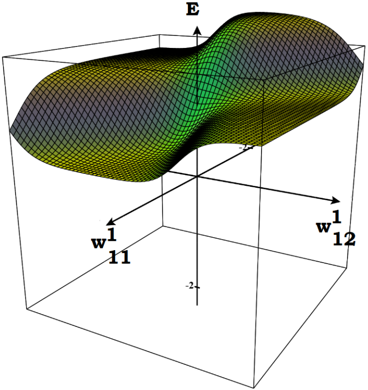

$$E = \sum_{i=1}^{4} \left(y^i - \frac{1}{1 + \exp\left(-\left[w_{11}^1 x_1^i + w_{12}^1 x_2^i\right]\right)} \right)^2$$

Optimization Concepts

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x)$$

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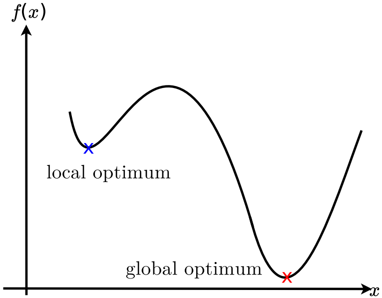
- f is called objective function and $\mathcal{C}$ is called feasible set.
- Let $f^* = \min_{x \in \mathcal{C}} f(x)$ denote the **optimal objective function value**.
- **Optimal Solution Set** $S^* = \{x \in \mathcal{C} : f(x) = f^*\}$.
- Let us denote by x^* an optimal solution in S^*.

General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x) \tag{OP}$$



General Optimization Problem

$$\min_{x \in \mathcal{C}} f(x) \tag{OP}$$

Local Optimal Solution

A solution z to (OP) is called local optimal solution if $f(z) \leq f(\hat{z})$, $\forall \hat{z} \in \mathcal{N}(z, \epsilon)$ for some $\epsilon > 0$.

Note: $\mathcal{N}(z, \epsilon)$ denotes suitable $\epsilon-$neighborhood of z.

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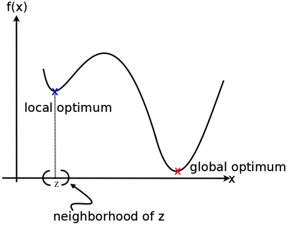

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$\epsilon-$ Neighborhood of $z \in \mathcal{C}$

$$\mathcal{N}(z, \epsilon) = \{u \in \mathcal{C} : \text{dist}(z, u) \leq \epsilon\}.$$

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Global Optimal Solution

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