Convexity, Gradient Descent, Convergence Guarantees

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Outline

1. Convexity and Smoothness

2. Strong Convexity and Examples

3. Gradient Descent and Decent lemma for Smooth functions

Convex set

Convex set

Definition (Convex Set)

A set $\mathcal{X} \subseteq \mathbb{R}^n$ is *convex* if for any points $x, y \in \mathcal{X}$ and any scalar $\lambda \in [0, 1]$, the combination

$$\lambda x + (1 - \lambda)\, y \ \in \ \mathcal{X}.$$

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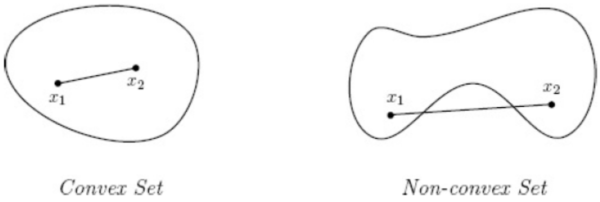

Figure: Convex and Non-convex set.

Convex Function

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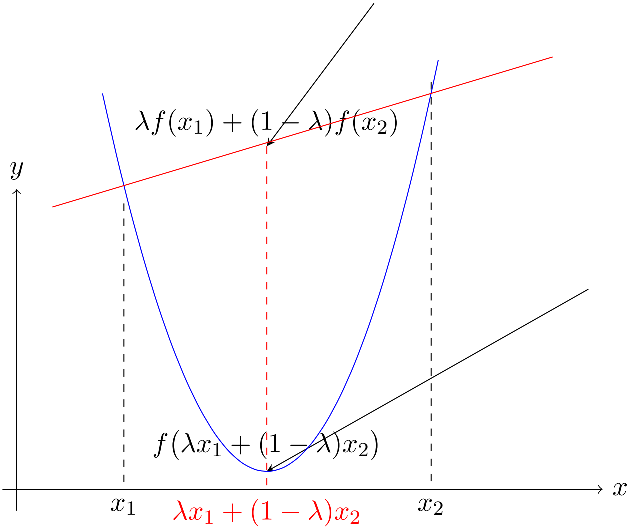
Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a convex set. A function $f : \mathcal{X} \to \mathbb{R}$ is called *convex* if, for all $x, y \in \mathcal{X}$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda) y) \ \leq \ \lambda f(x) + (1 - \lambda) f(y).$$

Illustration of Convex function



First order Characterization of Convex Functions

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Theorem

Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a convex set and $f : \mathcal{X} \to \mathbb{R}$ be differentiable. Then f is convex if and only if for all $x, y \in \mathcal{X}$,

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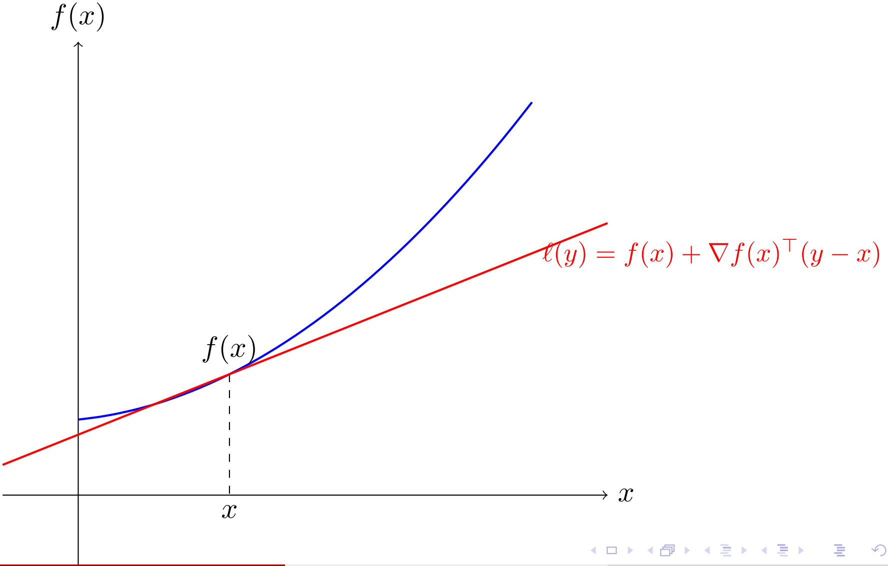
$$f(y) \ \geq \ f(x) \ + \ \nabla f(x)^\top (y - x).$$

Proof: Exercise!

Figure of First order charecterization of Convexity



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$$\nabla^2 f(x) \succeq 0$$

This means the Hessian matrix $\nabla^2 f(x)$ is Positive Semi definite (meaning that the eigen values are non-negative) **Examples?**

If $d = 1$, implies $f''(x) \geq 0$ for all x

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Proof of Theorem: Exercise!

Local and Global Minima of a Function

Definition

Let $f : \mathcal{X} \to \mathbb{R}$. A point $x^* \in \mathcal{X}$ is

- **Local minimum** if there exists an open neighborhood U of x^* such that

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Question: Prove that, for convex functions, all local minima are global minima.

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Suppose $f : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$||\nabla f(x) - \nabla f(y)||_2 \leq M||x - y||_2$$

Where $M > 0$, then we call f as M-smooth/ M-Gradient Lipschitz.

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Hence Quadratic objective function is M-smooth.

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4. $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{M}\|\nabla f(x) - \nabla f(y)\|^2$
5. f *is twice differentiable then:*

$$-M\mathbb{I} \preccurlyeq \nabla^2 f \preccurlyeq M\mathbb{I}$$

where $M\mathbb{I} - \nabla^2 f$ *is P.S.D. matrix.*

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For $(2) \Rightarrow (3)$, we use the Interpolation method where $G: \ [0,1] \rightarrow \mathbb{R}$,

$$G(t) = f(x + t(y - x)) - f(x) - \langle \nabla f(x), t(y - x) \rangle$$

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$$G(1) - G(0) = \int_0^1 G'(t) dt$$

Proof Contd..

$$\text{LHS of } (3) = \int_0^1 \langle \nabla f(x + t(y-x)), (y-x) \rangle - 0 - \langle \nabla f(x), y - x \rangle \, dt$$

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For $(3) \Rightarrow (5)$, we utilize the Taylor Series Expansion.

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For $(4) \Rightarrow (1)$, we utilize the Cauchy Schwartz Inequality.

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Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a nonempty convex set. A function $f : \mathcal{C} \to \mathbb{R}$ is called m-*strongly convex* if there exists $m > 0$ such that for all $x, y \in \mathcal{C}$ and every $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \;\le\; \lambda f(x) \;+\; (1-\lambda) f(y) \;-\; \frac{m}{2} \lambda(1-\lambda) \|x - y\|^2.$$

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Remark

- **Convexity:** Ensures for any $x, y \in \mathcal{C}$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \;\leq\; \lambda f(x) + (1-\lambda) f(y).$$

The graph lies below the chord connecting $(x, f(x))$ and $(y, f(y))$.

Remarks

- m-**Strong Convexity:** Strengthens convexity by quantifying the gap with a quadratic term:

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- **Interpretation:** Strong convexity not only bounds f below its secant, but guarantees a minimum separation (proportional to $\|x-y\|^2$) between the function's graph and that chord (see below Figure).

Figure: Interpretation of m-strongly convex Function

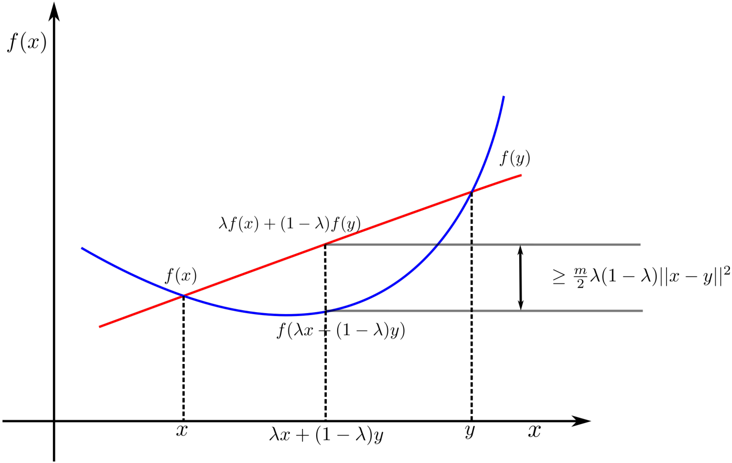

Figure: $m-$strongly convex function f

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Theorem

Let $\mathcal{C} \subseteq \mathbb{R}^d$ be a nonempty convex set and $f : \mathcal{C} \to \mathbb{R}$ be continuously differentiable. Then the following are equivalent:

a. *f is m-strongly convex.*

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b. $\|\nabla f(x) - \nabla f(y)\| \;\geq\; m\,\|x - y\|$ *for all $x, y \in \mathcal{C}$.*

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- (b) $\|\nabla f(x) - \nabla f(y)\| \;\geq\; m\,\|x - y\|$ *for all* $x, y \in \mathcal{C}$.
- (c) $(\nabla f(x) - \nabla f(y))^\top (x - y) \;\geq\; m\,\|x - y\|^2$ *for all* $x, y \in \mathcal{C}$.

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c. $(\nabla f(x) - \nabla f(y))^\top (x - y) \ \geq \ m\,\|x - y\|^2$ *for all* $x, y \in \mathcal{C}$.
d. $f(y) - (f(x) + \nabla f(x)^\top (y - x)) \ \geq \ \frac{m}{2}\,\|x - y\|^2$ *for all* $x, y \in \mathcal{C}$.

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- (e) $(\nabla f(x) - \nabla f(y))^\top (x - y) \leq \frac{1}{m}\|\nabla f(x) - \nabla f(y)\|^2$ *for all* $x, y \in \mathcal{C}$.

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- (e) $(\nabla f(x) - \nabla f(y))^\top (x - y) \;\leq\; \frac{1}{m}\,\|\nabla f(x) - \nabla f(y)\|^2$ *for all* $x, y \in \mathcal{C}$.
- (f) *If* f *is twice differentiable, then* $\nabla^2 f(x) \succeq mI$ *for all* $x \in \mathcal{C}$.

Example 1: Quadratic Function

Let $f : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2} x^\top Q\, x \;-\; b^\top x,$$

where $Q \in \mathbb{R}^{d \times d}$ is symmetric positive definite and $b \in \mathbb{R}^d$.

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- Since $Q \succeq \lambda_{\min}(Q)\, I$, It follows

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- Then

$$\|\nabla f(x) - \nabla f(y)\| = \|Q\,(x - y)\|$$

- Since $Q \succeq \lambda_{\min}(Q)\, I$, It follows

$$\|\nabla f(x) - \nabla f(y)\| \;\geq\; \lambda_{\min}(Q)\, \|x - y\|.$$

Example 1: Quadratic Function

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Thus By Above Theorem , f is $\lambda_{\min}(Q)$-strongly convex.

Example 2: Log–Sum–Exp Function

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$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \log(1 + e^x).$$

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- **Convex but not strongly convex on $[0,\infty)$:** $\inf_{x \geq 0} \nabla^2 f(x) = 0$, so no $m > 0$ exists with $\nabla^2 f(x) \geq m$ for all $x \geq 0$.

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- **Smoothness:** $\nabla^2 f(x) \leq \nabla^2 f(0) = \frac{1}{4}$ for all $x \geq 0$, hence f is M-smooth with

$$M = \frac{1}{4}.$$

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so $0 \leq f''(x) \leq 12$ for all $x \in [-1, 1]$.
- Hence f is convex and M-smooth with $M = 12$.
- However, since $f''(0) = 0$, there is no $m > 0$ such that $f''(x) \geq m$ on $[-1, 1]$; thus f is not strongly convex.

Example 4: Invex Function (Non-Convex but Smooth)

- Define

$$f : [-\tfrac{1}{2}, \infty) \times [1, \infty) \to \mathbb{R}, \qquad f(x, y) = y\,(x^2 - 1)^2.$$

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$$\nabla^2 f(x,y)=\begin{pmatrix}4y\,(3x^2-1) & 4x\,(x^2-1)\\ 4x\,(x^2-1) & 0\end{pmatrix}.$$

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- $f \in C^2$ with continuous Hessian $\Rightarrow$ on any bounded subset of $[-\frac{1}{2},\infty) \times [1,\infty)$, f is M-smooth.

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Thank You!