Topological Insulators and Topological Phase Transitions

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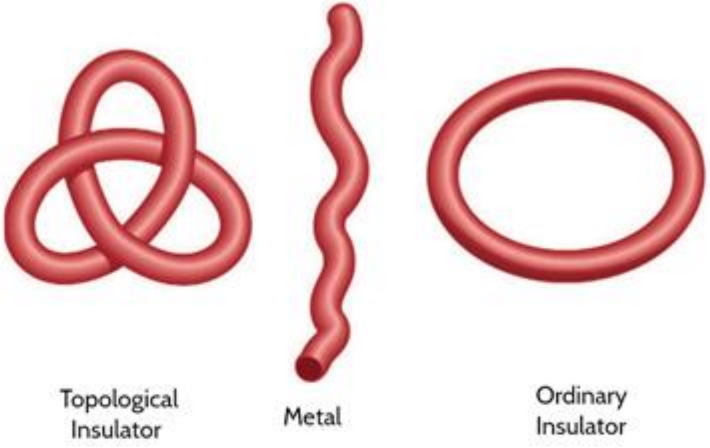
What are topological insulators?

- An insulator that always has a metallic boundary when placed next to an ordinary insulator
- Origin of metallic boundaries - topological invariants that cannot change as long as material is insulating



The Role of Symmetry

- Symmetry in **momentum** - $H(k)$
- Chiral Symmetry (S)

  $S\,H(k)\,S^{-1} = -H(k)$

- Particle-Hole Symmetry (Charge Conjugation) Symmetry (C)

  $C\,H(k)\,C^{-1} = -H(k)$

- Inversion (Time-Reversal) Symmetry (T)

  $T\,H(k)\,T^{-1} = H(k)$

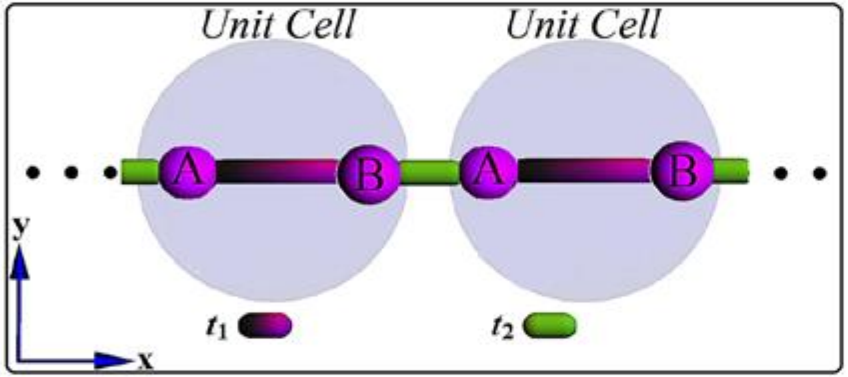
- 10 discrete symmetry classes

Symmetry Class	Time reversal symmetry	Particle hole symmetry	Chiral symmetry
A	No	No	No
AIII	No	No	Yes
AI	Yes, $T^2 = 1$	No	No
BDI	Yes, $T^2 = 1$	Yes, $C^2 = 1$	Yes
D	No	Yes, $C^2 = 1$	No
DIII	Yes, $T^2 = -1$	Yes, $C^2 = 1$	Yes
AII	Yes, $T^2 = -1$	No	No
CII	Yes, $T^2 = -1$	Yes, $C^2 = -1$	Yes
C	No	Yes, $C^2 = -1$	No
CI	Yes, $T^2 = 1$	Yes, $C^2 = -1$	Yes

Atomic Picture

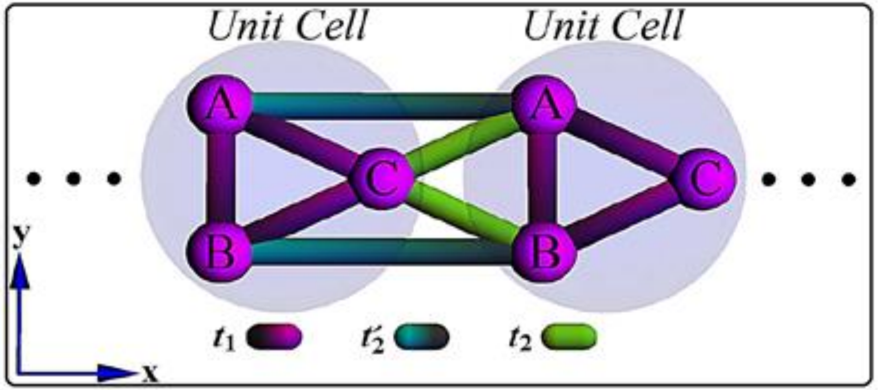
SSH Model (1D)

$$H = \sum_{n=1}^{N} t_1(A_n^\dagger B_n) + \sum_{n=1}^{N} t_2(B_n^\dagger A_{n+1})$$



JH Model (quasi 1D)

$$H = \sum_{n=1}^{N} t_1(A_n^\dagger B_n + B_n^\dagger C_n + A_n^\dagger C_n) + \sum_{n=1}^{N} t_2'(A_n^\dagger A_{n+1} + B_n^\dagger B_{n+1})$$
$$+ \sum_{n=1}^{N} t_2(C_n^\dagger A_{n+1} + C_n^\dagger B_{n+1})$$



Hamiltonian $H(k)$

SSH Model (1D)

$$\mathcal{H}(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{ik} \\ t_1 + t_2 e^{-ik} & 0 \end{pmatrix}$$

$$E^{\pm} = \pm\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 cos(k))}$$

Also, note that

$$\mathcal{H}(k) = (t_1 + t_2 cos(k))\sigma_x + (t_2 sin(k))\sigma_y$$

where σ_x and σ_y are Pauli matrices

$$\mathcal{H}(k) = d_x\sigma_x + d_y\sigma_y$$

JH Model (quasi 1D)

$$\mathcal{H}(k) = \begin{pmatrix} 2t_2' cos(k) & t_1 & t_1 + t_2 e^{ik} \\ t_1 & 2t_2' cos(k) & t_1 + t_2 e^{ik} \\ t_1 + t_2 e^{-ik} & t_1 + t_2 e^{-ik} & 0 \end{pmatrix}$$

$$E^0 = -2t_1 + \eta$$

$$E^{\pm} = \frac{1}{2}(\eta \pm \sqrt{\eta^2 + 2(t_1^2 + 4t_2'^2 + 8t_1 t_2 cos(k))}$$

where $\eta = t_1 + 2t_2' cos(k)$

Band closing and reopening at k= (0 , π)

SSH Model

$$E^{+} = E^{-} \text{ when}$$

$$t_1 = t_2$$

Topological Phase Transition is possible when coupling is same.

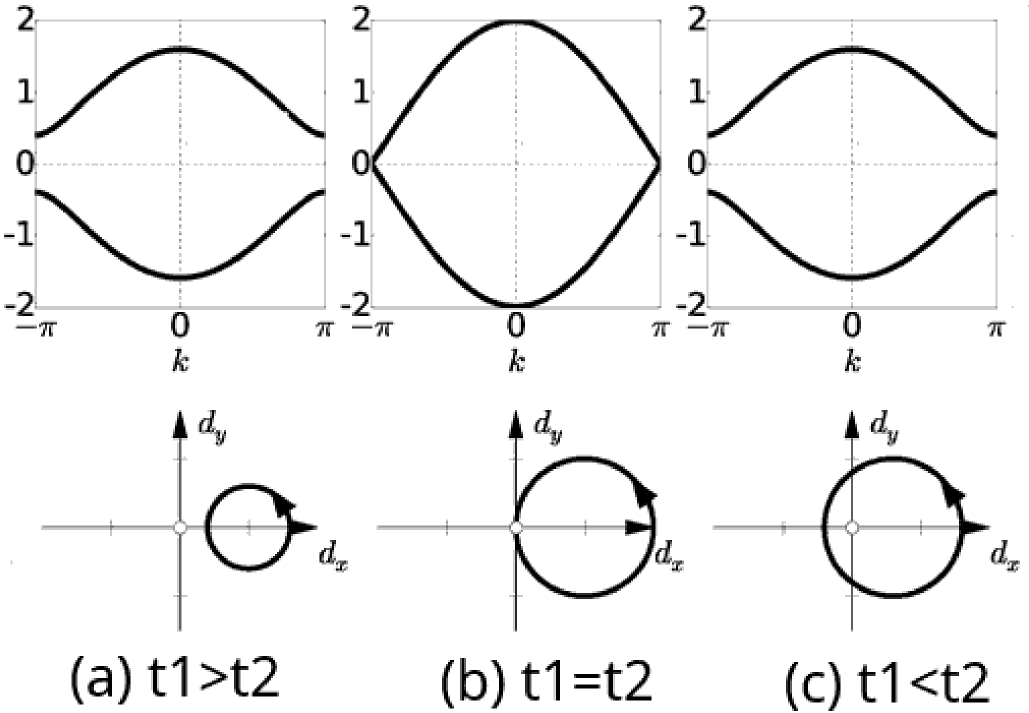
JH Model

$$E^{+} = E^{-} \text{ when}$$

$$t_1 = e^{ik}\frac{2}{9}\left(t_2' + 4t_2 \pm \frac{\sqrt{-(-2t_2' + t_2)^2}}{2}\right)$$

$$t_2' = \frac{t_2}{2} \text{ and } t_1 = e^{ik}t_2$$

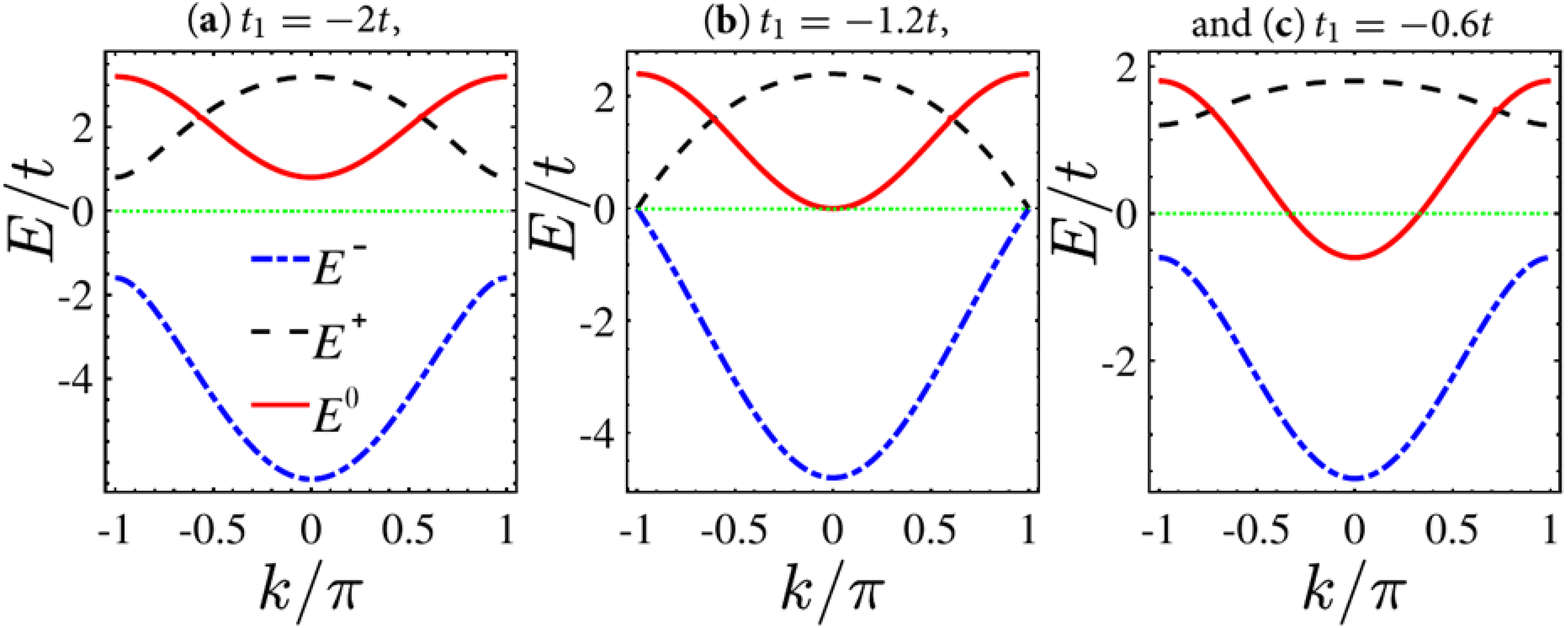
Transition in the SSH Model



Winding number

- Number of times the path encloses the origin
- Number of edge states
  - Zero for trivial states
  - Non-zero in topological states with edge modes

Phase Transition in the JH Model



Here, $t_2' = t_2/2 = -0.6t$.

Hidden symmetry in the JH Model

$H(k)$ has exchange symmetry :

$$\Upsilon \psi_k = \psi'_k = \begin{pmatrix} B_k \\ A_k \\ C_k \end{pmatrix}$$

So it can be block-diagonalised in the basis of the operator Υ, under the transformation $\mathscr{U}^{-1}\mathscr{H}(k)\mathscr{U} = \mathscr{H}^{BD}(k)$

with

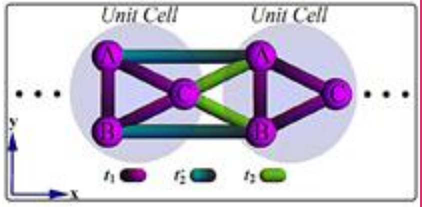
$$\mathscr{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

$$\mathscr{H}^{BD}(k) = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

$$h_1 = -2t_1 + \eta,$$

$$h_2 = \begin{pmatrix} \eta & \sqrt{2}(t_1 + t_2 e^{ik}) \\ \sqrt{2}(t_1 + t_2 e^{-ik}) & 0 \end{pmatrix}$$

Generalised SSH Hamiltonian



Invariants for the JH Model

- Definition of an integer topological invariant : $\mathbb{Z} = |n_0 - n_\pi|$

  where n_0 and n_π are the number of negative parities at $k_s = 0$ and $k_s = \pi$, .

- Analytical expression of topological invariant $\mathbb{Z}$ for the subsystem h_2 is

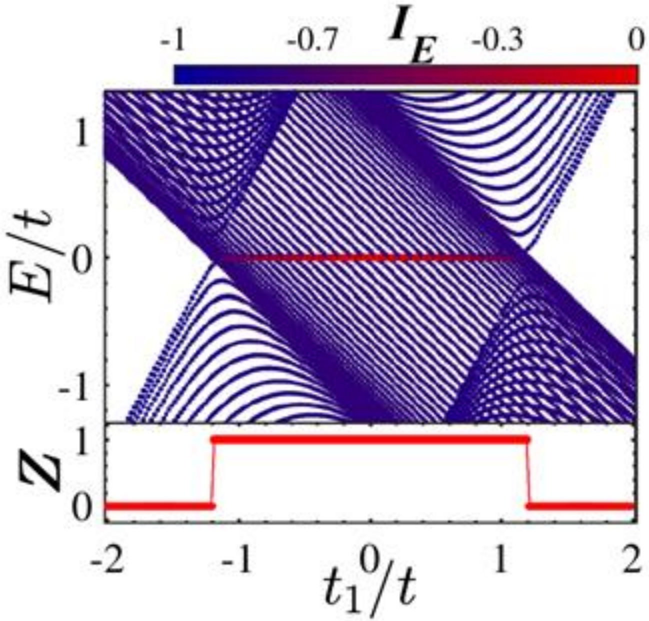
$$\mathbb{Z} = \begin{cases} 0, \text{ if } sgn(\eta(0)) = sgn(\eta(\pi)) \\ 1, \text{ if } sgn(\eta(0)) \neq sgn(\eta(\pi)) \end{cases},$$

- Definition of inverse participation ratio (IPR)

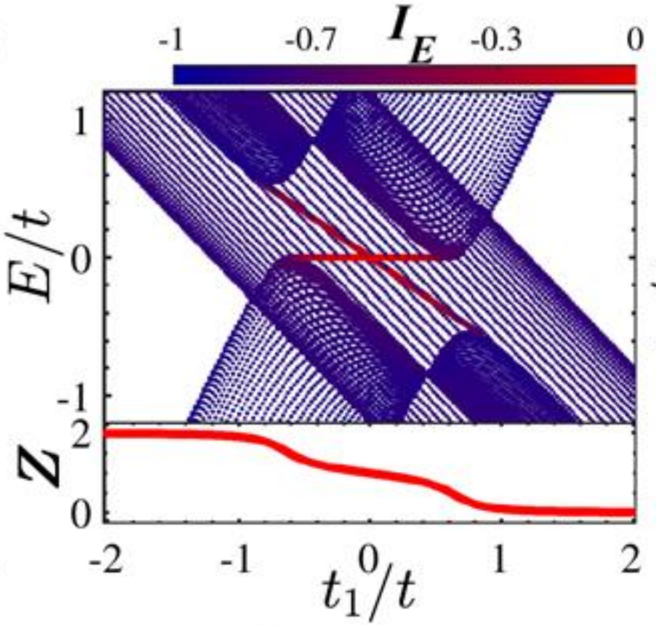
$$I_E = \frac{Ln \sum_j |\psi_E(j)|^4}{Ln 3N}.$$

  - $I_E = 0$ for a localised eigenstate
  - $I_E = -1$ for an extended eigenstate

Numerical Results for JH Model

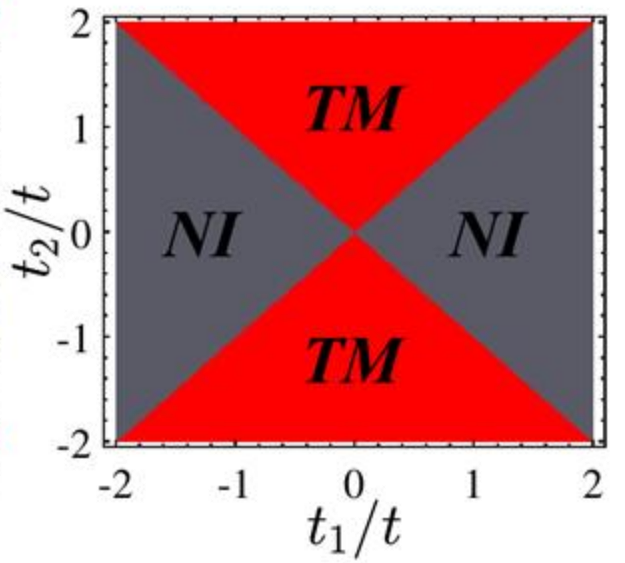


(a) $t_2' = t_2/2$
$t_2' = -0.6t$



(b) $t_2' \neq t_2/2 = -0.3.$
$t_2' = -0.6t$

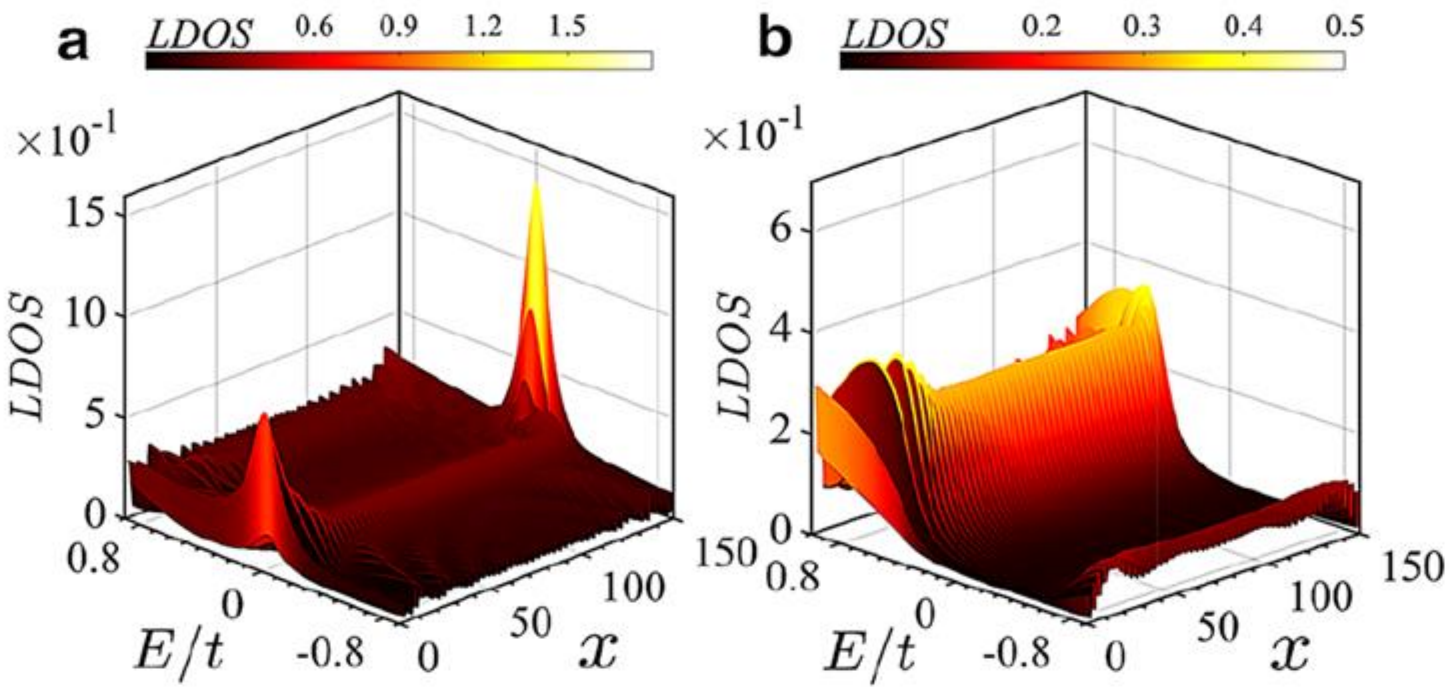
Topological phase diagram



Numerical Results for JH Model

Topological Metal Phase

Normal Insulator

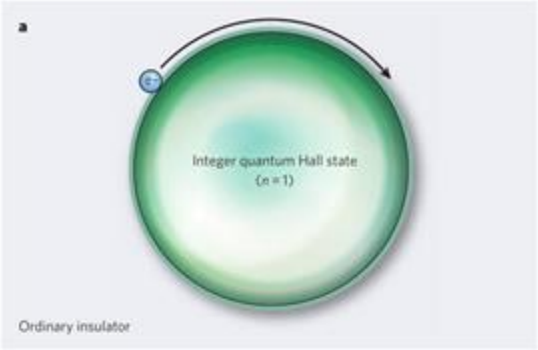


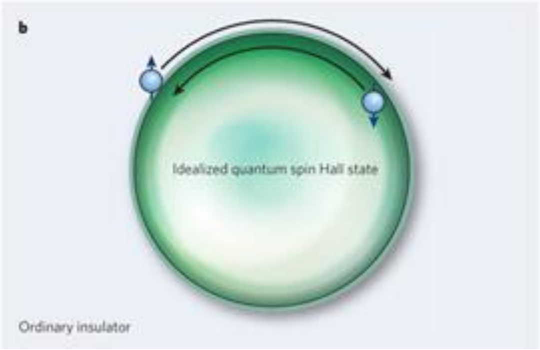
Possible experimental realisations of the JH Model

- Edge states created in the absence of chiral or particle-hole symmetry and *protected* by the hidden inversion (time-reversal) symmetry
- Experimentally, model can be realized by coupled acoustic resonators, topo-electrical circuits, optical lattices, photonic crystals and mechanical systems
- Can also simulate using cold atoms, to reveal the topological features employing density and momentum-distribution measurements
- LDOS can be probed using spatially resolved radio-frequency spectroscopy
- Can distinguish between topologically trivial and nontrivial edge states, using edge state transport in topological states of matter

2D topological order

- Quantum Hall effect and Quantum spin Hall effect
- Theoretical Advances by Kane and Mele - topological invariant
- Prediction by Bernevig, Hughes and Zhang that 2D topological insulators with quantized charge conductance would be realised in (Hg,Cd)Te quantum wells

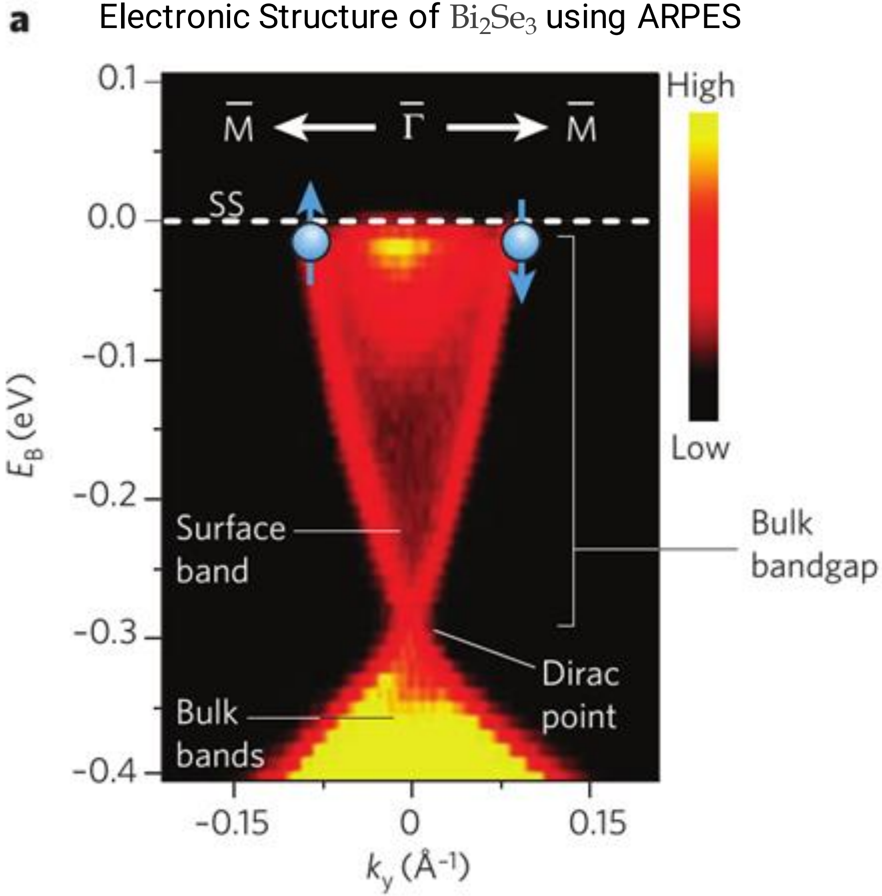




3D topological insulators

- ‘Weak’ and ‘Strong’ topological insulators
- The first topological insulator : alloy Bi_xSb_{1-x} using angle-resolved photoemission spectroscopy (ARPES) experiment
- ‘Next-generation’ topological insulators : Bi_2Se_3 and Bi_2Te_3

Electronic Structure of Bi_2Se_3 using ARPES



Applications and Future Scope

- Spin torque device for magnetic memory applications
    - Heterostructure - topological insulator + ferromagnet
    - Switching of ferromagnet by passing current in TI surface
- Magnetoelectric effect and Axion dynamics
    - Applied electric field generates a magnetic dipole and vice versa
    - Advantage of multiferroic materials - increase in speed and reproducibility without fatigue
- Emergent particles - Majorana fermions
    - Direct observation
    - Obey special kind of non-abelian quantum statistics
    - Topological quantum computers well protected from error

The Course - NE240

- Symmetry
  - Symmetry in Real space
  - Group Theory
  - 32 point and 230 space Groups
- Thermodynamics and Stat. Mech.
  - Atomic picture
  - Origin of various properties
- Phase transitions
  - Change in space group
  - Landau theory
  - Order Parameters
- Disorders
- Measurement

The Presentation

- Symmetry
  - Symmetry in k-space
  - Topology
  - 10 discrete symmetry classes
- Band Theory
  - SSH lattice and quasi-1D (paper)
  - Spin-orbital coupling, Magnetic field
- Phase transitions
  - Change in topological invariant
  - ~~Landau theory~~
  - Topological Invariant - Winding number
- Disorders
- Measurement

Thank You