Optimization

Reference:

Logic Synthesis and Verification Algorithms by GD Hachtel and F Somenzi

Contents

- Logic Optimization Techniques
- Branch Method
- Petrick Method

Quine Mcluskey Method /Tabular Method

- First founded by Quine and later improved by Mcluskey.
- Combining Theorem is repeatedly used.

  $XY+XY'=X$

Problem

- $f(a,b,c,d)=\Sigma m(0,2,5,6,7,8,10,12,13,14,15)$

$F(a,b,c,d)=\Sigma m(0000,0010,0101,0110,0111,1000,1010,1100,1101,1110,1111)$

0	0 0 0 0
2	0 0 1 0
8	1 0 0 0
5	0 1 0 1
6	0 1 1 0
10	1 0 1 0
12	1 1 0 0
7	0 1 1 1
13	1 1 0 1
14	1 1 1 0
15	1 1 1 1

(0,2)	**0 0 - 0**
(0,8)	**- 0 0 0**
(2,6)	**0 - 1 0**
(2,10)	**- 0 1 0**
(8,10)	**1 0 - 0**
(8,12)	**1 - 0 0**
(5,7)	**0 1 - 0**
(5,13)	**- 1 0 1**
(6,7)	**0 1 1 -**
(6,14)	**- 1 1 0**
(10,14)	**1 - 1 0**
(12,13)	**1 1 0 -**
(12,14)	**1 1 - 0**
(7,15)	**- 1 1 1**
(13,15)	**1 1 - 1**
(14,15)	**1 1 1 -**

(0,2)	0 0 - 0
(0,8)	- 0 0 0
(2,6)	0 - 1 0
(2,10)	- 0 1 0
(8,10)	1 0 - 0
(8,12)	1 - 0 0
(5,7)	0 1 - 0
(5,13)	- 1 0 1
(6,7)	0 1 1 -
(6,14)	- 1 1 0
(10,14)	1 - 1 0
(12,13)	1 1 0 -
(12,14)	1 1 - 0
(7,15)	- 1 1 1
(13,15)	1 1 - 1
(14,15)	1 1 1 -

(0,2,8,10)	- 0 - 0
Duplicate Entry	
(2,6,10,14)	- - 1 0
Duplicate Entry	
(8,10,12,14)	1 - - 0
Duplicate Entry	
(5,7,13,15)	- 1 - 1
Duplicate Entry	
(6,7,14,15)	- 1 1 -
Duplicate Entry	
(12,13,14,15)	1 1 - -
Duplicate Entry	

0	0 0 0 0 √
2	0 0 1 0 √
8	1 0 0 0 √
5	0 1 0 1 √
6	0 1 1 0 √
10	1 0 1 0 √
12	1 1 0 0 √
7	0 1 1 1 √
13	1 1 0 1 √
14	1 1 1 0 √
15	1 1 1 1 √

(0,2)	0 0 - 0 √
(0,8)	- 0 0 0 √
(2,6)	0 - 1 0 √
(2,10)	- 0 1 0 √
(8,10)	1 0 - 0 √
(8,12)	1 - 0 0 √
(5,7)	0 1 - 0 √
(5,13)	- 1 0 1 √
(6,7)	0 1 1 - √
(6,14)	- 1 1 0 √
(10,14)	1 - 1 0 √
(12,13)	1 1 0 - √
(12,14)	1 1 - 0 √
(7,15)	- 1 1 1 √
(13,15)	1 1 - 1 √
(14,15)	1 1 1 - √

(0,2,8,10)	- 0 - 0
(2,6,10,14)	- - 1 0
(8,10,12,14)	1 - - 0
(5,7,13,15)	- 1 - 1
(6,7,14,15)	- 1 1 -
(12,13,14,15)	1 1 - -

	a b c d
(0,2,8,10)	**- 0 - 0**
(2,6,10,14)	**- - 1 0**
(8,10,12,14)	**1 - - 0**
(5,7,13,15)	**- 1 - 1**
(6,7,14,15)	**- 1 1 -**
(12,13,14,15)	**1 1 - -**

f(a,b,c,d)=**(0,2,8,10)+(2,6,10,14)+(8,10,12,14)+(5,7,13,15)+(6,7,14,15)+(12,13,14,15)**

f(a,b,c,d)=**b'd'+cd'+ad'+bd+bc+ab**

These are the Prime Implicants.

Prime Implicant Chart

- The prime implicant chart displays pictorially the covering relationships between the prime implicants and the minterms of the function.
- Consists of array of U columns and V arrays.

Procedure

- Place prime implicants in the column and minterm in row.
- Place a ‘ * ‘ in all the in all the intersections of minterms and the implicants that cover them.
- If there is any column with a single “ * “ the corresponding prime implicant is an essential prime implicant, since that minterm is covered only by this prime implicant.

$$F(A,B,C,D) = \Sigma m(0,2,5,6,7,8,10,12,13,14,15)$$

	$B'D'$	CD'	BD	BC	AD'	AB
	(0,2,8,10)	(2,6,10,14)	(5,7,13,15)	(6,7,14,15)	(8,10,12,14)	(12,13,14,15)
0	X					
2	X	X				
5			X			
6		X		X		
7			X	X		
8	X				X	
10	X	X			X	
12					X	X
13			X			X
14		X		X	X	X
15			X	X		X

Remove Primary Essential prime Implicants

Primary EPI

- These are implicants which will appear in any solution.
- A row which is covered by only 1 PI is called a distinguished row.
- The PI which covers it is EPI.

	$B'D'$(*)	CD'	BD(*)	BC	AD'	AB
	(0,2,8,10)	(2,6,10,14)	(5,7,13,15)	(6,7,14,15)	(8,10,12,14)	(12,13,14,15)
(ʊ) 0	X					
2	X	X				
(ʊ) 5			X			
6		X		X		
7			X	X		
8	X				X	
10	X	X			X	
12					X	X
13			X			X
14		X		X	X	X
15			X	X		X

* indicates an essential prime implicant

ʊ indicates a distinguished row, i.e. a row covered by only 1 prime implicant

- So Primary EPI are

  B’D’ and BD

  This term will appear in any minimal solution.

Row Dominance

	CD'	BC	AD'	AB
	(2,6,10,14)	(6,7,14,15)	(8,10,12,14)	(12,13,14,15)
6	X	X		
12			X	X
~~14~~	~~X~~	~~X~~	~~X~~	~~X~~

Row 14 dominates row 6 as well as 12.

Dominating Row can always be removed.

Column Dominance

	CD'	BC	AD'	AB
	(2,6,10,14)	(6,7,14,15)	(8,10,12,14)	(12,13,14,15)
6	X	X		
12			X	X

CD’ and BC both dominate each other such columns are said as Co-dominate.

Remove any one of the column.

Remove Secondary essential Prime Implicant

Secondary EPI

- A row which is covered by only one prime implicant is called a distinguished row
- The prime implicant which covers it is a (secondary) Essential Prime Implicant.

	CD'(**)	AD'(**)
	(2,6,10,14)	(8,10,12,14)
(▫)6	X	
(▫)12		X

** indicates a secondary essential prime implicant

▫ indicates a distinguished row

Result

- No other rows to be covered ,so no further steps required. Therefore minimum-cost solution consists of the primary and secondary essential prime implicants.
- So F= B’D’ + BD + CD’+ AD’

Example 2

$F(A, B, C, D) = \Sigma m(0, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$

	$A'D'$	$B'D'$	$C'D'$	$A'C$	$B'C$	$A'B$	BC'	AB'	AC'
0	X	X	X						
2	X	X		X	X				
3				X	X				
4	X		X			X	X		
5						X	X		
6	X			X		X			
7				X		X			
8		X	X					X	X
9								X	X
10		X			X			X	
11					X			X	
12			X				X		X
13							X		X

Reduce Prime Implicant Table

Remove Primary EPI

- There are no primary Essential prime Implicants each row is covered by atleast two products.

Row Dominance

	A'D'	B'D'	C'D'	A'C	B'C	A'B	BC'	AB'	AC'
0	X	X	X						
2	X	X		X	X				
3				X	X				
4	X		X			X	X		
5						X	X		
6	X			X		X			
7				X		X			
8		X	X					X	X
9								X	X
10		X			X			X	
11					X			X	
12			X				X		X
13							X		X

Dominating Rows are Removed.

Row Dominance

	A'D'	B'D'	C'D'	A'C	B'C	A'B	BC'	AB'	AC'
0	X	X	X						
~~2~~	~~X~~	~~X~~		~~X~~	~~X~~				
3				X	X				
~~4~~	~~X~~		~~X~~			~~X~~	~~X~~		
5						X	X		
~~6~~	~~X~~			~~X~~		~~X~~			
7				X		X			
~~8~~		~~X~~	~~X~~					~~X~~	~~X~~
9								X	X
~~10~~		~~X~~			~~X~~			~~X~~	
11					X			X	
~~12~~			~~X~~				~~X~~		~~X~~
13							X		X

Column Dominance

	A'D'	B'D'	C'D'	A'C	B'C	A'B	BC'	AB'	AC'
0	X	X	X						
3				X	X				
5						X	X		
7				X		X			
9								X	X
11					X			X	
13							X		X

- A'D' & B'D' and C'D' each dominate one another.
- We can remove any two of them
- Then Remove Secondary Essential EPI

	A'D'	B'D'	C'D'	A'C	B'C	A'B	BC'	AB'	AC'
0	X	X	X						
3				X	X				
5						X	X		
7				X		X			
9								X	X
11					X			X	
13							X		X

Column A'D' B'D''' and C'D' each dominate open another. Remove any two of them.

Remove Secondary EPI

	A'D'(**)	A'C	B'C	A'B	BC'	AB'	AC'
(o)6	X						
3		X	X				
5				X	X		
7		X		X			
9						X	X
11			X			X	
13					X		X

** indicates a secondary essential prime implicant
o indicates a distinguished row

Column A'D' is a secondary EPI; it is removed from the table.

Solve Prime Implicant Table

	A'C	B'C	A'B	BC'	AB'	AC'
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

No further **Row Dominance** is possible

Also no further **Column Dominance** is possible

- No additional Secondary EPI
- Cyclic Covering Problem
- Solution can be obtained by
  - Petrick Method
  - Branch Method

Petricks Method

- Increased Number of PI and complexity may lead to trial and error method.

- Method to determine all SOP solutions from PI Chart.

- Petricks method more systematic.

Steps of Petricks Method

1. Reduce the PI Chart by eliminating the EPI.
2. Label the columns of reduced PI chart as P1,P2,…….Pn
3. Form a logic function P which is true when all the rows are covered. P consists of POS form.
4. Reduce P to a minimum SOP by applying

   (X+Y)(X+Z)=X+YZ and X+XY=X
5. Each term represent a solution ,that is a set which covers all the minterms.

   Find the terms with minimum number of variables.

   Each of these terms represent a solution with a minimum number of PI.
6. For each term count the number of literals in each PI and find the total number of literals.

   Choose the term or terms correspond to minimum number of literals.

Petrick’s Method

- Lets Apply petrick method for the problem shown below

	A'C	B'C	A'B	BC'	AB'	AC'
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

Petrick’s Method

- Problem is cyclic
- No column or row dominance is possible.
- In Petrick method a Boolean Expression P is formed which describes all possible solutions of the table.
- P1=A’C P2=B’C P3=A’B P4=BC’ P5=AB’ P6=AC’

Petrick's Method

	P1	P2	P3	P4	P5	P6
	$A'C$	$B'C$	$A'B$	BC'	AB'	AC'
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

Petrick’s Method

- For the problem stated

  P=(P1+P2)(P3+P4)(P1+P3)(P5+P6)(P2+P5)(P4+P6)

  If P=1 each disjunctive clauses are satisfied. And all rows are covered.

  - In the expression (P1+P2) covers row 3.So product P1 or P2 must be included in the solution.
  - Similarly for other’s=>(P3+P4) covers row 5.P3 or P4 should be included.
  - These sums are ANDed together, since requirements are satisfied.

- $P=\underline{(P1+P2)}\underline{(P3+P4)}(\underline{(P1+P3)}\underline{(P5+P6)}\underline{(P2+P5)}\underline{(P4+P6)}$

- Use $(X+Y)(X+Z)=X+YZ$ and multiply out.

- $P=(P1+P2P3)(P4+P3P6)(P5+P2P6)$

- $P=(P1P4+P2P3P4+P1P3P6+P2P3P6)(P5+P2P6)$

- $P=P1P4P5+P2P3P4P5+P1P3P5P6+P2P3P5P6+P1P2P4P6+P2P3P4P6+P1P2P3P6+P2P3P6$

- Apply $X+XY=X$
- $P=P1P4P5+P2P3P4P5+P1P3P5P6+P2P3P5P6+P1P2P4P6+P2P3P4P6+\underline{\cancel{P1P2P3P6}}+\underline{P2P3P6}$
- $P=P1P4P5+P2P3P4P5+P1P3P5P6+P2P3P5P6+P1P2P4P6+\underline{\cancel{P2P3P4P6}}+\underline{P2P3P6}$
- $P=P1P4P5+P2P3P4P5+P1P3P5P6+P2P3P5P6+P1P2P4P6+P2P3P6$

Petrick Method

- Since P is a Boolean expression, it can be multiplied by out into SOP form.

P= P1P4P5 + P1P3P5P6 + P2P3P4P5 +
P2P3P5P6 + P1P2P4P6 + P1P2P3P6 +
P2P3P4P6 + P2P3P6

Petrick Method

- So minimal solution are

  F=A’D’+A’C+BC’+AB’

  F=A’D’+B’C+A’B+AC’

Here A’D’ is the secondary EPI which was found earlier in the problem.

Let Us Verify This with branch method

	A'C	B'C	A'B	BC'	AB'	AC'
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

Select A'C as Arbitary

	A'C	B'C	A'B	BC'	AB'	AC'
3	X	X				
5			X	X		
7	X		X			
9					X	X
11		X			X	
13				X		X

A'C is one of the solution in the minimal.

Column Dominance

	$B'C$	$A'B$	BC'	AB'	AC'
5		X	X		
9				X	X
11	X			X	
13			X		X

	BC'	AB'	AC'
5	X		
9		X	X
11		X	
13	X		X

Row 5 is a distinguished Row

So **BC'** is a solution of minimal expression

	AB'	AC'
9	X	X
11	X	

- AB’ Dominates AC’
- Dominated Column can be removed

AB’ is one of the solution to the minimal solution

Minimal Solution

F=A’D’+A’C+BC’+AB’

We can see that the minimal solution obtained from Petrick’s Method is same as the branch method.

Branch Method

Branch Method

Cyclic Prime Implicant Chart

	(0,1)	(1,5)	(5,7)	(7,15)	(14,15)	(10,14)	(8,10)	(0,8)
0	*							*
1	*	*						
5		*	*					
7			*	*				
8							*	*
10						*	*	
14					*	*		
15				*	*			

Branch Method

- To find the minimal expression
  - Make an arbitrary selection of one of the row and then apply the reduction technique.

- For the problem given let us select row 0 arbitrarily and remove.

Cyclic Prime Implicant Chart

	(0,1)	(1,5)	(5,7)	(7,15)	(14,15)	(10,14)	(8,10)	(0,8)
0	*							*
1	*	*						
5		*	*					
7			*	*				
8							*	*
10						*	*	
14					*	*		
15				*	*			

(0,1) will be a solution of any minimal solution.

Reduced Chart after deleted Column (0,1)

	(1,5)	(5,7)	(7,15)	(14,15)	(10,14)	(8,10)	(0,8)
5	*	*					
7		*	*				
8						*	*
10					*	*	
14				*	*		
15			*	*			

	(5,7)	(7,15)	(14,15)	(10,14)	(8,10)
5	*				
7	*	*			
8					*
10				*	*
14			*	*	
15		*	*		

So (5,7) and (8,10) are one of the minimal solution.

	(7,15)	(14,15)	(10,14)
14		*	*
15	*	*	

(14,15) dominates (7,15) and (10,14)

We are left out with (14,15) PI

So (14,15) must be a solution to minimal solution.

F=(0,1)+(5,7)+(8,10)+(14,15)

Same procedure is repeated by removing (0,8) instead of (0,1).

We get

F=(1,5)+(7,15)+(10,15)+(0,8)

Since both the expressions for F have same number of literals, both are minimal.

Assignment

- Find a minimal SOP

1. $F(v,w,x,y,z)=\sum(0,4,12,16,19,24,27,28,29,31)$
2. $F(a,b,c,d,e) = \sum(2,3,7,10,12,15,27) + \sum d(5,18,19,21,23)$
3. Find $f(a,b,c,d)=\Sigma m(2,3,7,9,11,13) + \sum d(1,10,15)$
4. $Y = \sum(0, 3, 6) + d(2, 5)$