Project Euler.net

Problem 1:

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.

Problem 2:

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.

Problem 3:

The prime factors of 13195 are 5, 7, 13 and 29.

What is the largest prime factor of the number 600851475143 ?

Problem 4:

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is 9009 = 91 × 99.

Find the largest palindrome made from the product of two 3-digit numbers.

Problem 5:

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?

Problem 6:

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + ... + 10^2 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + ... + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is 3025 − 385 = 2640.

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.

Problem 7:

By listing the first six prime numbers: 2, 3, 5, 7, 11, and 13, we can see that the 6th prime is 13.

What is the 10 001st prime number?

Problem 8:

Find the greatest product of five consecutive digits in the 1000-digit number.

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
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Problem 9:

A Pythagorean triplet is a set of three natural numbers, $a < b < c$, for which,

$$a^2 + b^2 = c^2$$

For example, $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

There exists exactly one Pythagorean triplet for which $a + b + c = 1000$.
Find the product abc.

Problem 10:

The sum of the primes below 10 is 2 + 3 + 5 + 7 = 17.

Find the sum of all the primes below two million.

Problem 11:

In the 20×20 grid below, four numbers along a diagonal line have been marked in red.

```
08 02 22 97 38 15 00 40 00 75 04 05 07 78 52 12 50 77 91 08
49 49 99 40 17 81 18 57 60 87 17 40 98 43 69 48 04 56 62 00
81 49 31 73 55 79 14 29 93 71 40 67 53 88 30 03 49 13 36 65
52 70 95 23 04 60 11 42 69 24 68 56 01 32 56 71 37 02 36 91
22 31 16 71 51 67 63 89 41 92 36 54 22 40 40 28 66 33 13 80
24 47 32 60 99 03 45 02 44 75 33 53 78 36 84 20 35 17 12 50
32 98 81 28 64 23 67 10 26 38 40 67 59 54 70 66 18 38 64 70
67 26 20 68 02 62 12 20 95 63 94 39 63 08 40 91 66 49 94 21
24 55 58 05 66 73 99 26 97 17 78 78 96 83 14 88 34 89 63 72
21 36 23 09 75 00 76 44 20 45 35 14 00 61 33 97 34 31 33 95
78 17 53 28 22 75 31 67 15 94 03 80 04 62 16 14 09 53 56 92
16 39 05 42 96 35 31 47 55 58 88 24 00 17 54 24 36 29 85 57
86 56 00 48 35 71 89 07 05 44 44 37 44 60 21 58 51 54 17 58
19 80 81 68 05 94 47 69 28 73 92 13 86 52 17 77 04 89 55 40
04 52 08 83 97 35 99 16 07 97 57 32 16 26 26 79 33 27 98 66
88 36 68 87 57 62 20 72 03 46 33 67 46 55 12 32 63 93 53 69
04 42 16 73 38 25 39 11 24 94 72 18 08 46 29 32 40 62 76 36
20 69 36 41 72 30 23 88 34 62 99 69 82 67 59 85 74 04 36 16
20 73 35 29 78 31 90 01 74 31 49 71 48 86 81 16 23 57 05 54
01 70 54 71 83 51 54 69 16 92 33 48 61 43 52 01 89 19 67 48
```

The product of these numbers is 26 × 63 × 78 × 14 = 1788696.

What is the greatest product of four adjacent numbers in the same direction (up, down, left, right, or diagonally) in the 20×20 grid?

Problem 12:

The sequence of triangle numbers is generated by adding the natural numbers. So the 7^{th} triangle number would be 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28. The first ten terms would be:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Let us list the factors of the first seven triangle numbers:

1: 1
3: 1,3
6: 1,2,3,6
10: 1,2,5,10
15: 1,3,5,15
21: 1,3,7,21
28: 1,2,4,7,14,28

We can see that 28 is the first triangle number to have over five divisors.

What is the value of the first triangle number to have over five hundred divisors?

Problem 13:

Work out the first ten digits of the sum of the following one-hundred 50-digit numbers.

```
37107287533902102798797998220837590246510135740250
46376937677490009712648124896970078050417018260538
74324986199524741059474233309513058123726617309629
91942213363574161572522430563301811072406154908250
23067588207539346171171980310421047513778063246676
89261670696623633820136378418383684178734361726757
28112879812849979408065481931592621691275889832738
44274228917432520321923589422876796487670272189318
47451445736001306439091167216856844588711603153276
70386486105843025439939619828917593665686757934951
62176457141856560629502157223196586755079324193331
64906352462741904929101432445813822663347944758178
92575867718337217661963751590579239728245598838407
58203565325359399008402633568948830189458628227828
80181199384826282014278194139940567587151170094390
35398664372827112653829987240784473053190104293586
86515506006295864861532075273371959191420517255829
71693888707715466499115593487603532921714970056938
54370070576826684624621495650076471787294438377604
53282654108756828443191190634694037855217779295145
36123272525000296071075082563815656710885258350721
45876576172410976447339110607218265236877223636045
17423706905851860660448207621209813287860733969412
81142660418086830619328460811191061556940512689692
51934325451728388641918047049293215058642563049483
62467221648435076201727918039944693004732956340691
15732444386908125794514089057706229429197107928209
55037687525678773091862540744969844508330393682126
18336384825330154686196124348767681297534375946515
80386287592878490201521685554828717201219257766954
78182833757993103614740356856449095527097864797581
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16726320100436897842553539920931837441497806860984
48403098129077791799088218795327364475675590848030
87086987551392711854517078544161852424320693150332
59959406895756536782107074926966537676326235447210
69793950679652694742597709739166693763042633987085
41052684708299085211399427365734116182760315001271
65378607361501080857009149939512557028198746004375
35829035317434717326932123578154982629742552737307
94953759765105305946966067683156574377167401875275
88902802571733229619176668713819931811048770190271
25267680276078003013678680992525463401061632866526
36270218540497705585629946580636237993140746255962
24074486908231174977792365466257246923322810917141
91430288197103288597806669760892938638285025333403
34413065578016127815921815005561868836468420090470
23053081172816430487623791969842487255036638784583
11487696932154902810424020138335124462181441773470
63783299490636259666498587618221225225512486764533
67720186971698544312419572409913959008952310058822
95548255300263520781532296796249481641953868218774
76085327132285723110424803456124867697064507995236
37774242535411291684276865538926205024910326572967
23701913275725675285653248258265463092207058596522
29798860272258331913126375147341994889534765745501
18495701454879288984856827726077713721403798879715
38298203783031473527721580348144513491373226651381
34829543829199918180278916522431027392251122869539
40957953066405232632538044100059654939159879593635
29746152185502371307642255121183693803580388584903
41698116222072977186158236678424689157993532961922
62467957194401269043877107275048102390895523597457
23189706772547915061505504953922979530901129967519
86188088225875314529584099251203829009407770775672
11306739708304724483816533873502340845647058077308
82959174767140363198008187129011875491310547126581
97623331044818386269515456334926366572897563400500
42846280183517070527831839425882145521227251250327
55121603546981200581762165212827652751691296897789
32238195734329339946437501907836945765883352399886
75506164965184775180738168837861091527357929701337
62177842752192623401942399639168044983993173312731
32924185707147349566916674687634660915035914677504
99518671430235219628894890102423325116913619626622
73267460800591547471830798392868535206946944540724
76841822524674417161514036427982273348055556214818
97142617910342598647204516893989422179826088076852
87783646182799346313767754307809363333018982642090
10848802521674670883215120185883543223812876952786
71329612474782464538636993009049310363619763878039
62184073572399794223406235393808339651327408011116
66627891981488087797941876876144230030984490851411
60661826293682836764744779239180335110989069790714
85786944089552990653640447425576083659976645795096
66024396409905389607120198219976047599490197230297
64913982680032973156037120041377903785566085089252
16730939319872750275468906903707539413042652315011
94809377245048795150954100921645863754710598436791
78639167021187492431995700641917969777599028300699
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15368713711936614952811305876380278410754449733078
40789923115535562561142322423255033685442488917353
44889911501440648020369068063960672322193204149535
41503128880339536053299340368006977710650566631954
81234880673210146739058568557934581403627822703280
82616570773948327592232845941706525094512325230608
22918802058777319719839450180888072429661980811197
77158542502016545090413245809786882778948721859617
72107838435069186155435662884062257473692284509516
20849603980134001723930671666823555245252804609722
53503534226472524250874054075591789781264330331690
```

Problem 14:

The following iterative sequence is defined for the set of positive integers:

$n \rightarrow n/2$ (n is even)
$n \rightarrow 3n + 1$ (n is odd)

Using the rule above and starting with 13, we generate the following sequence:

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

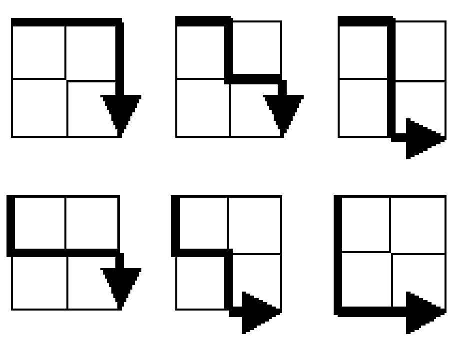
It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

NOTE: Once the chain starts the terms are allowed to go above one million.

Problem 15:

Starting in the top left corner of a 2×2 grid, there are 6 routes (without backtracking) to the bottom right corner.



How many routes are there through a 20×20 grid?

Problem 16:

$2^{15} = 32768$ and the sum of its digits is $3 + 2 + 7 + 6 + 8 = 26$.

What is the sum of the digits of the number 2^{1000}?

Problem 17:

If the numbers 1 to 5 are written out in words: one, two, three, four, five, then there are 3 + 3 + 5 + 4 + 4 = 19 letters used in total.

If all the numbers from 1 to 1000 (one thousand) inclusive were written out in words, how many letters would be used?

NOTE: Do not count spaces or hyphens. For example, 342 (three hundred and forty-two) contains 23 letters and 115 (one hundred and fifteen) contains 20 letters. The use of "and" when writing out numbers is in compliance with British usage.

Problem 18:

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
   3
  7 4
 2 4 6
8 5 9 3
```

That is, 3 + 7 + 4 + 9 = 23.

Find the maximum total from top to bottom of the triangle below:

```
                            75
                          95 64
                        17 47 82
                      18 35 87 10
                    20 04 82 47 65
                  19 01 23 75 03 34
                88 02 77 73 07 63 67
              99 65 04 28 06 16 70 92
            41 41 26 56 83 40 80 70 33
          41 48 72 33 47 32 37 16 94 29
        53 71 44 65 25 43 91 52 97 51 14
      70 11 33 28 77 73 17 78 39 68 17 57
    91 71 52 38 17 14 91 43 58 50 27 29 48
  63 66 04 68 89 53 67 30 73 16 69 87 40 31
04 62 98 27 23 09 70 98 73 93 38 53 60 04 23
```

NOTE: As there are only 16384 routes, it is possible to solve this problem by trying every route. However, Problem 67, is the same challenge with a triangle containing one-hundred rows; it cannot be solved by brute force, and requires a clever method! ;o)

Problem 19:

You are given the following information, but you may prefer to do some research for yourself.

- 1 Jan 1900 was a Monday.
- Thirty days has September,
  April, June and November.
  All the rest have thirty-one,
  Saving February alone,
  Which has twenty-eight, rain or shine.
  And on leap years, twenty-nine.
- A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400.

How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?

Problem 20:

$n!$ means $n \times (n - 1) \times ... \times 3 \times 2 \times 1$

For example, $10! = 10 \times 9 \times ... \times 3 \times 2 \times 1 = 3628800$,
and the sum of the digits in the number 10! is $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$.

Find the sum of the digits in the number 100!

Problem 21:

Let $d(n)$ be defined as the sum of proper divisors of n (numbers less than n which divide evenly into n).
If $d(a) = b$ and $d(b) = a$, where $a \neq b$, then a and b are an amicable pair and each of a and b are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore d(220) = 284. The proper divisors of 284 are 1, 2, 4, 71 and 142; so d(284) = 220.

Evaluate the sum of all the amicable numbers under 10000.

Problem 22:

Using names.txt (right click and 'Save Link/Target As...'), a 46K text file containing over five-thousand first names, begin by sorting it into alphabetical order. Then working out the alphabetical value for each name, multiply this value by its alphabetical position in the list to obtain a name score.

For example, when the list is sorted into alphabetical order, COLIN, which is worth 3 + 15 + 12 + 9 + 14 = 53, is the 938th name in the list. So, COLIN would obtain a score of 938 × 53 = 49714.

What is the total of all the name scores in the file?

Problem 23:

A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be 1 + 2 + 4 + 7 + 14 = 28, which means that 28 is a perfect number.

A number n is called deficient if the sum of its proper divisors is less than n and it is called abundant if this sum exceeds n.

As 12 is the smallest abundant number, 1 + 2 + 3 + 4 + 6 = 16, the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.

Find the sum of all the positive integers which cannot be written as the sum of two abundant numbers.

Problem 24:

A permutation is an ordered arrangement of objects. For example, 3124 is one possible permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are:

012   021   102   120   201   210

What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9?

Problem 25:

The Fibonacci sequence is defined by the recurrence relation:

$F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$.

Hence the first 12 terms will be:

$F_1 = 1$
$F_2 = 1$

$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$
$F_9 = 34$
$F_{10} = 55$
$F_{11} = 89$
$F_{12} = 144$

The 12th term, F_{12}, is the first term to contain three digits.

What is the first term in the Fibonacci sequence to contain 1000 digits?

Problem 26:

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$^1/_2$ = 0.5
$^1/_3$ = 0.(3)
$^1/_4$ = 0.25
$^1/_5$ = 0.2
$^1/_6$ = 0.1(6)
$^1/_7$ = 0.(142857)
$^1/_8$ = 0.125
$^1/_9$ = 0.(1)
$^1/_{10}$ = 0.1

Where 0.1(6) means 0.166666..., and has a 1-digit recurring cycle. It can be seen that $^1/_7$ has a 6-digit recurring cycle.

Find the value of $d < 1000$ for which $^1/_d$ contains the longest recurring cycle in its decimal fraction part.

Problem 27:

Euler published the remarkable quadratic formula:

$$n^2 + n + 41$$

It turns out that the formula will produce 40 primes for the consecutive values *n* = 0 to 39. However, when *n* = 40, $40^2 + 40 + 41 = 40(40 + 1) + 41$ is divisible by 41, and certainly when *n* = 41, $41^2 + 41 + 41$ is clearly divisible by 41.

Using computers, the incredible formula $n^2 - 79n + 1601$ was discovered, which produces 80 primes for the consecutive values *n* = 0 to 79. The product of the coefficients, −79 and 1601, is −126479.

Considering quadratics of the form:

> $n^2 + an + b$, where $|a| < 1000$ and $|b| < 1000$
>
> where $|n|$ is the modulus/absolute value of *n*
> e.g. $|11| = 11$ and $|-4| = 4$

Find the product of the coefficients, *a* and *b*, for the quadratic expression that produces the maximum number of primes for consecutive values of *n*, starting with *n* = 0.

Problem 28:

Starting with the number 1 and moving to the right in a clockwise direction a 5 by 5 spiral is formed as follows:

```
21 22 23 24 25
20  7  8  9 10
19  6  1  2 11
18  5  4  3 12
17 16 15 14 13
```

It can be verified that the sum of the numbers on the diagonals is 101.

What is the sum of the numbers on the diagonals in a 1001 by 1001 spiral formed in the same way?

Problem 29:

Consider all integer combinations of a^b for $2 \le a \le 5$ and $2 \le b \le 5$:

> $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$
> $3^2=9$, $3^3=27$, $3^4=81$, $3^5=243$
> $4^2=16$, $4^3=64$, $4^4=256$, $4^5=1024$
> $5^2=25$, $5^3=125$, $5^4=625$, $5^5=3125$

If they are then placed in numerical order, with any repeats removed, we get the following sequence of 15 distinct terms:

4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125

How many distinct terms are in the sequence generated by a^b for $2 \le a \le 100$ and $2 \le b \le 100$?

Problem 30:

Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits:

$1634 = 1^4 + 6^4 + 3^4 + 4^4$
$8208 = 8^4 + 2^4 + 0^4 + 8^4$
$9474 = 9^4 + 4^4 + 7^4 + 4^4$

As $1 = 1^4$ is not a sum it is not included.

The sum of these numbers is 1634 + 8208 + 9474 = 19316.

Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.

Problem 31:

In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation:

1p, 2p, 5p, 10p, 20p, 50p, £1 (100p) and £2 (200p).

It is possible to make £2 in the following way:

1×£1 + 1×50p + 2×20p + 1×5p + 1×2p + 3×1p

How many different ways can £2 be made using any number of coins?

Problem 32:

We shall say that an n-digit number is pandigital if it makes use of all the digits 1 to n exactly once; for example, the 5-digit number, 15234, is 1 through 5 pandigital.

The product 7254 is unusual, as the identity, $39 \times 186 = 7254$, containing multiplicand, multiplier, and product is 1 through 9 pandigital.

Find the sum of all products whose multiplicand/multiplier/product identity can be written as a 1 through 9 pandigital.

HINT: Some products can be obtained in more than one way so be sure to only include it once in your sum.

Problem 33:

The fraction $^{49}/_{98}$ is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that $^{49}/_{98} = {}^{4}/_{8}$, which is correct, is obtained by cancelling the 9s.

We shall consider fractions like, $^{30}/_{50} = {}^{3}/_{5}$, to be trivial examples.

There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator.

If the product of these four fractions is given in its lowest common terms, find the value of the denominator.

Problem 34:

145 is a curious number, as 1! + 4! + 5! = 1 + 24 + 120 = 145.

Find the sum of all numbers which are equal to the sum of the factorial of their digits.

Note: as 1! = 1 and 2! = 2 are not sums they are not included.

Problem 35:

The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime.

There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular primes are there below one million?

Problem 36:

The decimal number, 585 = 1001001001_2 (binary), is palindromic in both bases.

Find the sum of all numbers, less than one million, which are palindromic in base 10 and base 2.

(Please note that the palindromic number, in either base, may not include leading zeros.)

Problem 37:

The number 3797 has an interesting property. Being prime itself, it is possible to continuously remove digits from left to right, and remain prime at each stage: 3797, 797, 97, and 7. Similarly we can work from right to left: 3797, 379, 37, and 3.

Find the sum of the only eleven primes that are both truncatable from left to right and right to left.

NOTE: 2, 3, 5, and 7 are not considered to be truncatable primes.

Problem 38:

Take the number 192 and multiply it by each of 1, 2, and 3:

$192 \times 1 = 192$
$192 \times 2 = 384$
$192 \times 3 = 576$

By concatenating each product we get the 1 to 9 pandigital, 192384576. We will call 192384576 the concatenated product of 192 and (1,2,3)

The same can be achieved by starting with 9 and multiplying by 1, 2, 3, 4, and 5, giving the pandigital, 918273645, which is the concatenated product of 9 and (1,2,3,4,5).

What is the largest 1 to 9 pandigital 9-digit number that can be formed as the concatenated product of an integer with (1,2, ... , n) where $n > 1$?

Problem 39:

If p is the perimeter of a right angle triangle with integral length sides, {a,b,c}, there are exactly three solutions for $p = 120$.

{20,48,52}, {24,45,51}, {30,40,50}

For which value of $p \leq 1000$, is the number of solutions maximised?

Problem 40:

An irrational decimal fraction is created by concatenating the positive integers:

0.123456789101112131415161718192021...

It can be seen that the 12^{th} digit of the fractional part is 1.

If d_n represents the n^{th} digit of the fractional part, find the value of the following expression.

$$d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10000} \times d_{100000} \times d_{1000000}$$

Problem 41:

We shall say that an n-digit number is pandigital if it makes use of all the digits 1 to n exactly once. For example, 2143 is a 4-digit pandigital and is also prime.

What is the largest n-digit pandigital prime that exists?

Problem 42:

The n^{th} term of the sequence of triangle numbers is given by, $t_n = ½n(n+1)$; so the first ten triangle numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

By converting each letter in a word to a number corresponding to its alphabetical position and adding these values we form a word value. For example, the word value for SKY is 19 + 11 + 25 = 55 = t_{10}. If the word value is a triangle number then we shall call the word a triangle word.

Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, how many are triangle words?

Problem 43:

The number, 1406357289, is a 0 to 9 pandigital number because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property.

Let d_1 be the 1^{st} digit, d_2 be the 2^{nd} digit, and so on. In this way, we note the following:

- $d_2d_3d_4$=406 is divisible by 2
- $d_3d_4d_5$=063 is divisible by 3
- $d_4d_5d_6$=635 is divisible by 5
- $d_5d_6d_7$=357 is divisible by 7
- $d_6d_7d_8$=572 is divisible by 11
- $d_7d_8d_9$=728 is divisible by 13
- $d_8d_9d_{10}$=289 is divisible by 17

Find the sum of all 0 to 9 pandigital numbers with this property.

Problem 44:

Pentagonal numbers are generated by the formula, $P_n=n(3n-1)/2$. The first ten pentagonal numbers are:

1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ...

It can be seen that $P_4 + P_7 = 22 + 70 = 92 = P_8$. However, their difference, 70 − 22 = 48, is not pentagonal.

Find the pair of pentagonal numbers, P_j and P_k, for which their sum and difference is pentagonal and D = $|P_k - P_j|$ is minimised; what is the value of D?

Problem 45:

Triangle, pentagonal, and hexagonal numbers are generated by the following formulae:

Triangle	$T_n=n(n+1)/2$	1, 3, 6, 10, 15, ...
Pentagonal	$P_n=n(3n-1)/2$	1, 5, 12, 22, 35, ...
Hexagonal	$H_n=n(2n-1)$	1, 6, 15, 28, 45, ...

It can be verified that $T_{285} = P_{165} = H_{143} = 40755$.

Find the next triangle number that is also pentagonal and hexagonal.

Problem 46:

It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square.

$9 = 7 + 2\times1^2$
$15 = 7 + 2\times2^2$
$21 = 3 + 2\times3^2$
$25 = 7 + 2\times3^2$
$27 = 19 + 2\times2^2$
$33 = 31 + 2\times1^2$

It turns out that the conjecture was false.

What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?

Problem 47:

The first two consecutive numbers to have two distinct prime factors are:

$14 = 2 \times 7$
$15 = 3 \times 5$

The first three consecutive numbers to have three distinct prime factors are:

$644 = 2^2 \times 7 \times 23$
$645 = 3 \times 5 \times 43$
$646 = 2 \times 17 \times 19$.

Find the first four consecutive integers to have four distinct primes factors. What is the first of these numbers?

Problem 48:

The series, $1^1 + 2^2 + 3^3 + ... + 10^{10} = 10405071317$.

Find the last ten digits of the series, $1^1 + 2^2 + 3^3 + ... + 1000^{1000}$.

Problem 49:

The arithmetic sequence, 1487, 4817, 8147, in which each of the terms increases by 3330, is unusual in two ways: (i) each of the three terms are prime, and, (ii) each of the 4-digit numbers are permutations of one another.

There are no arithmetic sequences made up of three 1-, 2-, or 3-digit primes, exhibiting this property, but there is one other 4-digit increasing sequence.

What 12-digit number do you form by concatenating the three terms in this sequence?

Problem 50:

The prime 41, can be written as the sum of six consecutive primes:

$$41 = 2 + 3 + 5 + 7 + 11 + 13$$

This is the longest sum of consecutive primes that adds to a prime below one-hundred.

The longest sum of consecutive primes below one-thousand that adds to a prime, contains 21 terms, and is equal to 953.

Which prime, below one-million, can be written as the sum of the most consecutive primes?

Problem 51:

By replacing the 1st digit of *3, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3rd and 4th digits of 56**3 with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

Problem 52:

It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order.

Find the smallest positive integer, *x*, such that 2*x*, 3*x*, 4*x*, 5*x*, and 6*x*, contain the same digits.

Problem 53:

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation, ${}^5C_3 = 10$.

In general,

$${}^nC_r = \frac{n!}{r!(n-r)!}$$,where $r \le n$, $n! = n\times(n-1)\times...\times3\times2\times1$, and $0! = 1$.

It is not until $n = 23$, that a value exceeds one-million: ${}^{23}C_{10} = 1144066$.

How many, not necessarily distinct, values of ${}^nC_r$, for $1 \le n \le 100$, are greater than one-million?

Problem 54:

In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way:

- **High Card**: Highest value card.
- **One Pair**: Two cards of the same value.
- **Two Pairs**: Two different pairs.
- **Three of a Kind**: Three cards of the same value.
- **Straight**: All cards are consecutive values.
- **Flush**: All cards of the same suit.
- **Full House**: Three of a kind and a pair.
- **Four of a Kind**: Four cards of the same value.
- **Straight Flush**: All cards are consecutive values of same suit.
- **Royal Flush**: Ten, Jack, Queen, King, Ace, in same suit.

The cards are valued in the order:
2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair of eights beats a pair of fives (see example 1 below). But if two ranks tie, for example, both players have a pair of queens, then highest cards in each hand are compared (see example 4 below); if the highest cards tie then the next highest cards are compared, and so on.

Consider the following five hands dealt to two players:

Hand	Player 1	Player 2	Winner
1	5H 5C 6S 7S KD Pair of Fives	2C 3S 8S 8D TD Pair of Eights	Player 2
2	5D 8C 9S JS AC Highest card Ace	2C 5C 7D 8S QH Highest card Queen	Player 1
3	2D 9C AS AH AC Three Aces	3D 6D 7D TD QD Flush with Diamonds	Player 2
4	4D 6S 9H QH QC Pair of Queens Highest card Nine	3D 6D 7H QD QS Pair of Queens Highest card Seven	Player 1
5	2H 2D 4C 4D 4S Full House With Three Fours	3C 3D 3S 9S 9D Full House with Three Threes	Player 1

The file, poker.txt, contains one-thousand random hands dealt to two players. Each line of the file contains ten cards (separated by a single space): the first five are Player 1's cards and the last five are Player 2's cards. You can assume that all hands are valid (no invalid characters or repeated cards), each player's hand is in no specific order, and in each hand there is a clear winner.

How many hands does Player 1 win?

Problem 55:

If we take 47, reverse and add, 47 + 74 = 121, which is palindromic.

Not all numbers produce palindromes so quickly. For example,

349 + 943 = 1292,
1292 + 2921 = 4213
4213 + 3124 = 7337

That is, 349 took three iterations to arrive at a palindrome.

Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a palindrome. In fact, 10677 is the first number to be shown to require over fifty iterations before producing a palindrome: 4668731596684224866951378664 (53 iterations, 28-digits).

Surprisingly, there are palindromic numbers that are themselves Lychrel numbers; the first example is 4994.

How many Lychrel numbers are there below ten-thousand?

NOTE: Wording was modified slightly on 24 April 2007 to emphasise the theoretical nature of Lychrel numbers.

Problem 56:

A googol (10^{100}) is a massive number: one followed by one-hundred zeros; 100^{100} is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form, a^b, where $a, b < 100$, what is the maximum digital sum?

Problem 57:

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$$\sqrt{2} = 1 + 1/(2 + 1/(2 + 1/(2 + ...))) = 1.414213...$$

By expanding this for the first four iterations, we get:

1 + 1/2 = 3/2 = 1.5
1 + 1/(2 + 1/2) = 7/5 = 1.4
1 + 1/(2 + 1/(2 + 1/2)) = 17/12 = 1.41666...
1 + 1/(2 + 1/(2 + 1/(2 + 1/2))) = 41/29 = 1.41379...

The next three expansions are 99/70, 239/169, and 577/408, but the eighth expansion, 1393/985, is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than denominator?

Problem 58:

Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed.

37	36	35	34	33	32	**31**
38	**17**	16	15	14	**13**	30
39	18	**5**	4	**3**	12	29
40	19	6	1	2	11	28
41	20	**7**	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of 8/13 ≈ 62%.

If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

Problem 59:

Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (*) = 42, and lowercase k = 107.

A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example, 65 XOR 42 = 107, then 107 XOR 42 = 65.

For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message.

Unfortunately, this method is impractical for most users, so the modified method is to use a password as a key. If the password is shorter than the message, which is likely, the key is repeated cyclically throughout the message. The balance for this method is using a sufficiently long password key for security, but short enough to be memorable.

Your task has been made easy, as the encryption key consists of three lower case characters. Using cipher1.txt (right click and 'Save Link/Target As...'), a file containing the encrypted ASCII codes, and the knowledge that the plain text must contain common English words, decrypt the message and find the sum of the ASCII values in the original text.

Problem 60:

The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property.

Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.

Problem 61:

Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae:

Triangle	$P_{3,n}=n(n+1)/2$	1, 3, 6, 10, 15, ...
Square	$P_{4,n}=n^2$	1, 4, 9, 16, 25, ...
Pentagonal	$P_{5,n}=n(3n-1)/2$	1, 5, 12, 22, 35, ...

Hexagonal	$P_{6,n}=n(2n-1)$	1, 6, 15, 28, 45, ...
Heptagonal	$P_{7,n}=n(5n-3)/2$	1, 7, 18, 34, 55, ...
Octagonal	$P_{8,n}=n(3n-2)$	1, 8, 21, 40, 65, ...

The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties.

1. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first).
2. Each polygonal type: triangle ($P_{3,127}$=8128), square ($P_{4,91}$=8281), and pentagonal ($P_{5,44}$=2882), is represented by a different number in the set.
3. This is the only set of 4-digit numbers with this property.

Find the sum of the only ordered set of six cyclic 4-digit numbers for which each polygonal type: triangle, square, pentagonal, hexagonal, heptagonal, and octagonal, is represented by a different number in the set.

Problem 62:

The cube, 41063625 (345^3), can be permuted to produce two other cubes: 56623104 (384^3) and 66430125 (405^3). In fact, 41063625 is the smallest cube which has exactly three permutations of its digits which are also cube.

Find the smallest cube for which exactly five permutations of its digits are cube.

Problem 63:

The 5-digit number, $16807=7^5$, is also a fifth power. Similarly, the 9-digit number, $134217728=8^9$, is a ninth power.

How many n-digit positive integers exist which are also an nth power?

Problem 64:

All square roots are periodic when written as continued fractions and can be written in the form:

$$\sqrt{N} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

For example, let us consider $\sqrt{23}$:

$$\sqrt{23} = 4 + \sqrt{23} — 4 = 4 + \frac{1}{} = 4 + \frac{1}{}$$

$$\frac{1}{\sqrt{23}-4} \qquad 1+\frac{\sqrt{23}-3}{7}$$

If we continue we would get the following expansion:

$$\sqrt{23} = 4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{8 + \ldots}}}}$$

The process can be summarised as follows:

$$a_0 = 4, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7}$$
$$a_1 = 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2}$$
$$a_2 = 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7}$$
$$a_3 = 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4$$
$$a_4 = 8, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7}$$
$$a_5 = 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2}$$
$$a_6 = 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7}$$
$$a_7 = 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4$$

It can be seen that the sequence is repeating. For conciseness, we use the notation $\sqrt{23}$ = [4;(1,3,1,8)], to indicate that the block (1,3,1,8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

$\sqrt{2}$=[1;(2)], period=1
$\sqrt{3}$=[1;(1,2)], period=2
$\sqrt{5}$=[2;(4)], period=1
$\sqrt{6}$=[2;(2,4)], period=2
$\sqrt{7}$=[2;(1,1,1,4)], period=4
$\sqrt{8}$=[2;(1,4)], period=2
$\sqrt{10}$=[3;(6)], period=1
$\sqrt{11}$=[3;(3,6)], period=2
$\sqrt{12}$= [3;(2,6)], period=2
$\sqrt{13}$=[3;(1,1,1,1,6)], period=5

Exactly four continued fractions, for $N \leq 13$, have an odd period.

How many continued fractions for $N \leq 10000$ have an odd period?

Problem 65:

The square root of 2 can be written as an infinite continued fraction.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}$$

The infinite continued fraction can be written, $\sqrt{2}$ = [1;(2)], (2) indicates that 2 repeats *ad infinitum*. In a similar way, $\sqrt{23}$ = [4;(1,3,1,8)].

It turns out that the sequence of partial values of continued fractions for square roots provide the best rational approximations. Let us consider the convergents for $\sqrt{2}$.

$$1 + \frac{1}{2} = 3/2$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2}} = 7/5$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 17/12$$

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = 41/29$$

Hence the sequence of the first ten convergents for $\sqrt{2}$ are:

1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1393/985, 3363/2378, ...

What is most surprising is that the important mathematical constant,
e = [2; 1,2,1, 1,4,1, 1,6,1 , ... , 1,2*k*,1, ...].

The first ten terms in the sequence of convergents for *e* are:

2, 3, 8/3, 11/4, 19/7, 87/32, 106/39, 193/71, 1264/465, 1457/536, ...

The sum of digits in the numerator of the 10^{th} convergent is 1+4+5+7=17.

Find the sum of digits in the numerator of the 100^{th} convergent of the continued fraction for *e*.

Problem 66:

Consider quadratic Diophantine equations of the form:

$$x^2 - Dy^2 = 1$$

For example, when D=13, the minimal solution in *x* is $649^2 - 13 \times 180^2 = 1$.

It can be assumed that there are no solutions in positive integers when D is square.

By finding minimal solutions in *x* for D = {2, 3, 5, 6, 7}, we obtain the following:

$3^2 - 2 \times 2^2 = 1$
$2^2 - 3 \times 1^2 = 1$
$\mathbf{9}^2 - 5 \times 4^2 = 1$
$5^2 - 6 \times 2^2 = 1$
$8^2 - 7 \times 3^2 = 1$

Hence, by considering minimal solutions in *x* for D ≤ 7, the largest *x* is obtained when D=5.

Find the value of D ≤ 1000 in minimal solutions of *x* for which the largest value of *x* is obtained.

Problem 67:

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

3
7 4
2 **4** 6
8 5 **9** 3

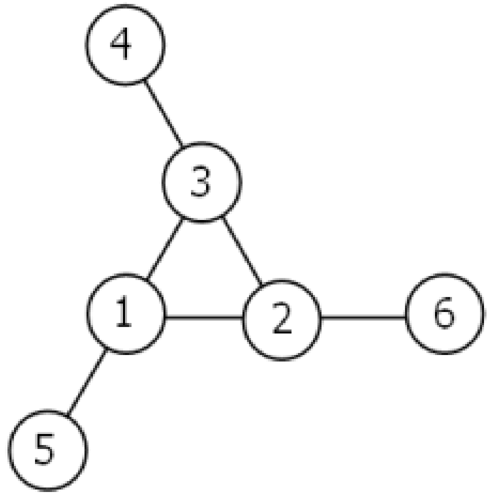
That is, 3 + 7 + 4 + 9 = 23.

Find the maximum total from top to bottom in triangle.txt (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows.

NOTE: This is a much more difficult version of Problem 18. It is not possible to try every route to solve this problem, as there are 2^{99} altogether! If you could check one trillion (10^{12}) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

Problem 68:

Consider the following "magic" 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine.



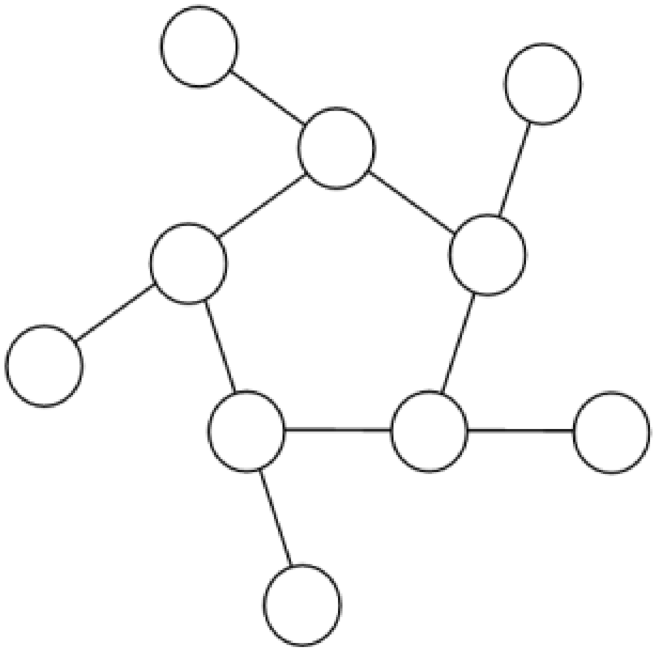
Working **clockwise**, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3.

It is possible to complete the ring with four different totals: 9, 10, 11, and 12. There are eight solutions in total.

Total	**Solution Set**
9	4,2,3; 5,3,1; 6,1,2
9	4,3,2; 6,2,1; 5,1,3
10	2,3,5; 4,5,1; 6,1,3
10	2,5,3; 6,3,1; 4,1,5
11	1,4,6; 3,6,2; 5,2,4
11	1,6,4; 5,4,2; 3,2,6
12	1,5,6; 2,6,4; 3,4,5
12	1,6,5; 3,5,4; 2,4,6

By concatenating each group it is possible to form 9-digit strings; the maximum string for a 3-gon ring is 432621513.

Using the numbers 1 to 10, and depending on arrangements, it is possible to form 16- and 17-digit strings. What is the maximum **16-digit** string for a "magic" 5-gon ring?



Problem 69:

Euler's Totient function, φ(*n*) [sometimes called the phi function], is used to determine the number of numbers less than *n* which are relatively prime to *n*. For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, φ(9)=6.

n	**Relatively Prime**	**φ(*n*)**	***n*/φ(*n*)**
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that *n*=6 produces a maximum *n*/φ(*n*) for *n* ≤ 10.

Find the value of *n* ≤ 1,000,000 for which *n*/φ(*n*) is a maximum.

Problem 70:

Euler's Totient function, φ(*n*) [sometimes called the phi function], is used to determine the number of

positive numbers less than or equal to n which are relatively prime to n. For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine, φ(9)=6.
The number 1 is considered to be relatively prime to every positive number, so φ(1)=1.

Interestingly, φ(87109)=79180, and it can be seen that 87109 is a permutation of 79180.

Find the value of n, $1 < n < 10^7$, for which $\varphi(n)$ is a permutation of n and the ratio $n/\varphi(n)$ produces a minimum.

Problem 71:

Consider the fraction, *n/d*, where *n* and *d* are positive integers. If $n<d$ and HCF(*n,d*)=1, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, **2/5**, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8

It can be seen that 2/5 is the fraction immediately to the left of 3/7.

By listing the set of reduced proper fractions for $d \leq 1{,}000{,}000$ in ascending order of size, find the numerator of the fraction immediately to the left of 3/7.

Problem 72:

Consider the fraction, *n/d*, where *n* and *d* are positive integers. If $n<d$ and HCF(*n,d*)=1, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8

It can be seen that there are 21 elements in this set.

How many elements would be contained in the set of reduced proper fractions for $d \leq 1{,}000{,}000$?

Problem 73:

Consider the fraction, *n/d*, where *n* and *d* are positive integers. If $n<d$ and HCF(*n,d*)=1, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \leq 8$ in ascending order of size, we get:

1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, **3/8, 2/5, 3/7**, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8

It can be seen that there are 3 fractions between 1/3 and 1/2.

How many fractions lie between 1/3 and 1/2 in the sorted set of reduced proper fractions for $d \leq 12,000$?

Problem 74:

The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145:

$1! + 4! + 5! = 1 + 24 + 120 = 145$

Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist:

169 → 363601 → 1454 → 169
871 → 45361 → 871
872 → 45362 → 872

It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example,

69 → 363600 → 1454 → 169 → 363601 (→ 1454)
78 → 45360 → 871 → 45361 (→ 871)
540 → 145 (→ 145)

Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms.

How many chains, with a starting number below one million, contain exactly sixty non-repeating terms?

Problem 75:

It turns out that 12 cm is the smallest length of wire that can be bent to form an integer sided right angle triangle in exactly one way, but there are many more examples.

12 cm: (3,4,5)
24 cm: (6,8,10)
30 cm: (5,12,13)
36 cm: (9,12,15)
40 cm: (8,15,17)
48 cm: (12,16,20)

In contrast, some lengths of wire, like 20 cm, cannot be bent to form an integer sided right angle triangle, and other lengths allow more than one solution to be found; for example, using 120 cm it is possible to form exactly three different integer sided right angle triangles.

120 cm: (30,40,50), (20,48,52), (24,45,51)

Given that L is the length of the wire, for how many values of $L \leq 1,500,000$ can exactly one integer sided right angle triangle be formed?

Problem 76:

It is possible to write five as a sum in exactly six different ways:

4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

How many different ways can one hundred be written as a sum of at least two positive integers?

Problem 77:

It is possible to write ten as the sum of primes in exactly five different ways:

7 + 3
5 + 5
5 + 3 + 2
3 + 3 + 2 + 2
2 + 2 + 2 + 2 + 2

What is the first value which can be written as the sum of primes in over five thousand different ways?

Problem 78:

Let p(*n*) represent the number of different ways in which *n* coins can be separated into piles. For example, five coins can separated into piles in exactly seven different ways, so p(5)=7.

OOOOO

OOOO O

OOO OO

OOO O O

OO OO O

OO O O O

O O O O O

Find the least value of *n* for which p(*n*) is divisible by one million.

Problem 79:

A common security method used for online banking is to ask the user for three random characters from a passcode. For example, if the passcode was 531278, they may ask for the 2nd, 3rd, and 5th characters; the expected reply would be: 317.

The text file, keylog.txt, contains fifty successful login attempts.

Given that the three characters are always asked for in order, analyse the file so as to determine the shortest possible secret passcode of unknown length.

Problem 80:

It is well known that if the square root of a natural number is not an integer, then it is irrational. The decimal expansion of such square roots is infinite without any repeating pattern at all.

The square root of two is 1.41421356237309504880..., and the digital sum of the first one hundred decimal digits is 475.

For the first one hundred natural numbers, find the total of the digital sums of the first one hundred decimal digits for all the irrational square roots.

Problem 81:

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by **only moving to the right and down**, is indicated in bold red and is equal to 2427.

$$\begin{pmatrix} \mathbf{131} & 673 & 234 & 103 & 18 \\ \mathbf{201} & \mathbf{96} & \mathbf{342} & 965 & 150 \\ 630 & 803 & \mathbf{746} & \mathbf{422} & 111 \\ 537 & 699 & 497 & \mathbf{121} & 956 \\ 805 & 732 & 524 & \mathbf{37} & \mathbf{331} \end{pmatrix}$$

Find the minimal path sum, in matrix.txt (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by only moving right and down.

Problem 82:

NOTE: This problem is a more challenging version of Problem 81.

The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994.

$$\begin{pmatrix} 131 & 673 & \mathbf{234} & \mathbf{103} & \mathbf{18} \\ \mathbf{201} & \mathbf{96} & \mathbf{342} & 965 & 150 \\ 630 & 803 & 746 & 422 & 111 \\ 537 & 699 & 497 & 121 & 956 \\ 805 & 732 & 524 & 37 & 331 \end{pmatrix}$$

Find the minimal path sum, in matrix.txt (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the left column to the right column.

Problem 83:

NOTE: This problem is a significantly more challenging version of Problem 81.

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297.

$$\begin{pmatrix} \mathbf{131} & 673 & \mathbf{234} & \mathbf{103} & \mathbf{18} \\ \mathbf{201} & \mathbf{96} & \mathbf{342} & 965 & \mathbf{150} \\ 630 & 803 & 746 & \mathbf{422} & \mathbf{111} \\ 537 & 699 & 497 & \mathbf{121} & 956 \\ 805 & 732 & 524 & \mathbf{37} & \mathbf{331} \end{pmatrix}$$

Find the minimal path sum, in matrix.txt (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by moving left, right, up, and down.

Problem 84:

In the game, *Monopoly*, the standard board is set up in the following way:

GO	A1	CC1	A2	T1	R1	B1	CH1	B2	B3	JAIL
H2										C1
T2										U1
H1										C2
CH3										C3
R4										R2
G3										D1
CC3										CC2

G2										D2
G1										D3
G2J	F3	U2	F2	F1	R3	E3	E2	CH2	E1	FP

A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution.

In addition to G2J, and one card from each of CC and CH, that orders the player to go directly to jail, if a player rolls three consecutive doubles, they do not advance the result of their 3rd roll. Instead they proceed directly to jail.

At the beginning of the game, the CC and CH cards are shuffled. When a player lands on CC or CH they take a card from the top of the respective pile and, after following the instructions, it is returned to the bottom of the pile. There are sixteen cards in each pile, but for the purpose of this problem we are only concerned with cards that order a movement; any instruction not concerned with movement will be ignored and the player will remain on the CC/CH square.

- Community Chest (2/16 cards):
  1. Advance to GO
  2. Go to JAIL
- Chance (10/16 cards):
  1. Advance to GO
  2. Go to JAIL
  3. Go to C1
  4. Go to E3
  5. Go to H2
  6. Go to R1
  7. Go to next R (railway company)
  8. Go to next R
  9. Go to next U (utility company)
  10. Go back 3 squares.

The heart of this problem concerns the likelihood of visiting a particular square. That is, the probability of finishing at that square after a roll. For this reason it should be clear that, with the exception of G2J for which the probability of finishing on it is zero, the CH squares will have the lowest probabilities, as 5/8 request a movement to another square, and it is the final square that the player finishes at on each roll that we are interested in. We shall make no distinction between "Just Visiting" and being sent to JAIL, and we shall also ignore the rule about requiring a double to "get out of jail", assuming that they pay to get out on their next turn.

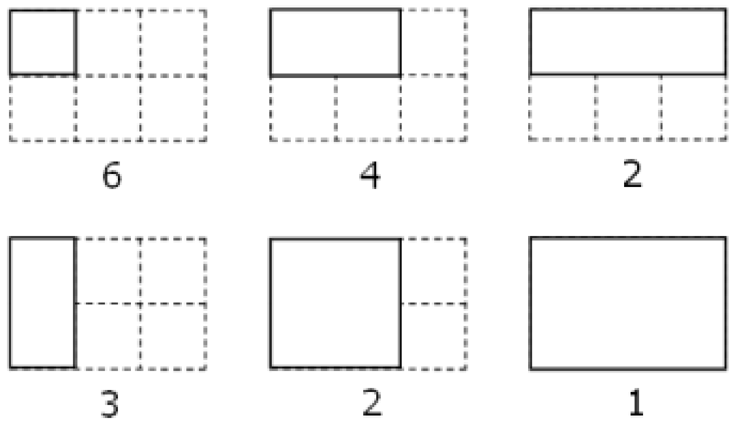
By starting at GO and numbering the squares sequentially from 00 to 39 we can concatenate these two-digit numbers to produce strings that correspond with sets of squares.

Statistically it can be shown that the three most popular squares, in order, are JAIL (6.24%) = Square 10, E3 (3.18%) = Square 24, and GO (3.09%) = Square 00. So these three most popular squares can be listed with the six-digit modal string: 102400.

If, instead of using two 6-sided dice, two 4-sided dice are used, find the six-digit modal string.

Problem 85:

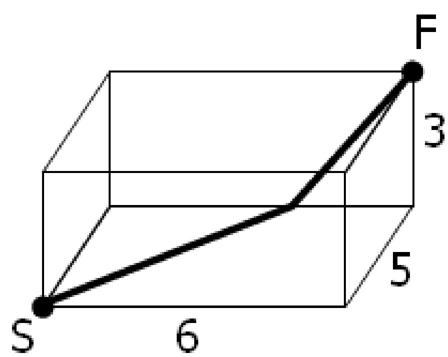
By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles:



Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

Problem 86:

A spider, S, sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly, F, sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from S to F is 10 and the path is shown on the diagram.



However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length.

By considering all cuboid rooms with integer dimensions, up to a maximum size of M by M by M, there are exactly 2060 cuboids for which the shortest route has integer length when M=100, and this is the least value of M for which the number of solutions first exceeds two thousand; the number of solutions is 1975 when M=99.

Find the least value of M such that the number of solutions first exceeds one million.

Problem 87:

The smallest number expressible as the sum of a prime square, prime cube, and prime fourth power is 28. In fact, there are exactly four numbers below fifty that can be expressed in such a way:

$28 = 2^2 + 2^3 + 2^4$
$33 = 3^2 + 2^3 + 2^4$
$49 = 5^2 + 2^3 + 2^4$
$47 = 2^2 + 3^3 + 2^4$

How many numbers below fifty million can be expressed as the sum of a prime square, prime cube, and prime fourth power?

Problem 88:

A natural number, N, that can be written as the sum and product of a given set of at least two natural numbers, $\{a_1, a_2, ... , a_k\}$ is called a product-sum number: $N = a_1 + a_2 + ... + a_k = a_1 \times a_2 \times ... \times a_k$.

For example, $6 = 1 + 2 + 3 = 1 \times 2 \times 3$.

For a given set of size, k, we shall call the smallest N with this property a minimal product-sum number. The minimal product-sum numbers for sets of size, k = 2, 3, 4, 5, and 6 are as follows.

k=2: $4 = 2 \times 2 = 2 + 2$
k=3: $6 = 1 \times 2 \times 3 = 1 + 2 + 3$
k=4: $8 = 1 \times 1 \times 2 \times 4 = 1 + 1 + 2 + 4$
k=5: $8 = 1 \times 1 \times 2 \times 2 \times 2 = 1 + 1 + 2 + 2 + 2$
k=6: $12 = 1 \times 1 \times 1 \times 1 \times 2 \times 6 = 1 + 1 + 1 + 1 + 2 + 6$

Hence for $2 \leq k \leq 6$, the sum of all the minimal product-sum numbers is 4+6+8+12 = 30; note that 8 is only counted once in the sum.

In fact, as the complete set of minimal product-sum numbers for $2 \leq k \leq 12$ is {4, 6, 8, 12, 15, 16}, the sum is 61.

What is the sum of all the minimal product-sum numbers for $2 \leq k \leq 12000$?

Problem 89:

The rules for writing Roman numerals allow for many ways of writing each number (see About Roman Numerals...). However, there is always a "best" way of writing a particular number.

For example, the following represent all of the legitimate ways of writing the number sixteen:

```
IIIIIIIIIIIIIIII
VIIIIIIIIII
```

VVIIIIII
XIIIIII
VVVI
XVI

The last example being considered the most efficient, as it uses the least number of numerals.

The 11K text file, roman.txt (right click and 'Save Link/Target As...'), contains one thousand numbers written in valid, but not necessarily minimal, Roman numerals; that is, they are arranged in descending units and obey the subtractive pair rule (see About Roman Numerals... for the definitive rules for this problem).

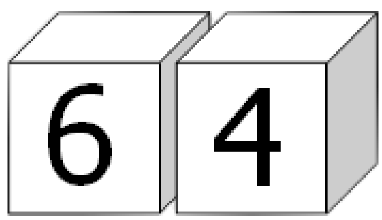
Find the number of characters saved by writing each of these in their minimal form.

Note: You can assume that all the Roman numerals in the file contain no more than four consecutive identical units.

Problem 90:

Each of the six faces on a cube has a different digit (0 to 9) written on it; the same is done to a second cube. By placing the two cubes side-by-side in different positions we can form a variety of 2-digit numbers.

For example, the square number 64 could be formed:



In fact, by carefully choosing the digits on both cubes it is possible to display all of the square numbers below one-hundred: 01, 04, 09, 16, 25, 36, 49, 64, and 81.

For example, one way this can be achieved is by placing {0, 5, 6, 7, 8, 9} on one cube and {1, 2, 3, 4, 8, 9} on the other cube.

However, for this problem we shall allow the 6 or 9 to be turned upside-down so that an arrangement like {0, 5, 6, 7, 8, 9} and {1, 2, 3, 4, 6, 7} allows for all nine square numbers to be displayed; otherwise it would be impossible to obtain 09.

In determining a distinct arrangement we are interested in the digits on each cube, not the order.

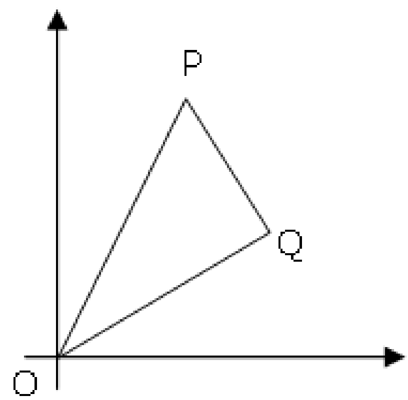
{1, 2, 3, 4, 5, 6} is equivalent to {3, 6, 4, 1, 2, 5}
{1, 2, 3, 4, 5, 6} is distinct from {1, 2, 3, 4, 5, 9}

But because we are allowing 6 and 9 to be reversed, the two distinct sets in the last example both represent the extended set {1, 2, 3, 4, 5, 6, 9} for the purpose of forming 2-digit numbers.

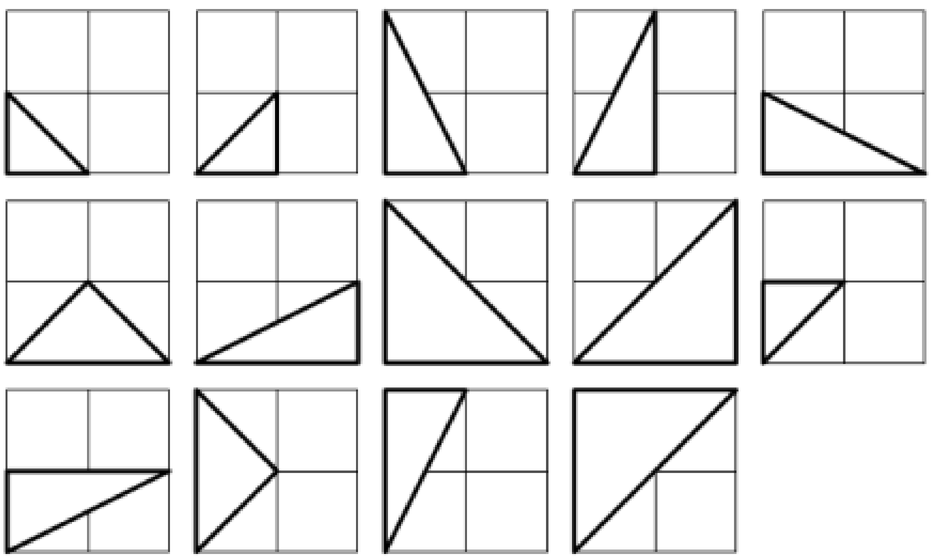
How many distinct arrangements of the two cubes allow for all of the square numbers to be displayed?

Problem 91:

The points P (x_1, y_1) and Q (x_2, y_2) are plotted at integer co-ordinates and are joined to the origin, O(0,0), to form ΔOPQ.



There are exactly fourteen triangles containing a right angle that can be formed when each co-ordinate lies between 0 and 2 inclusive; that is,
$0 \le x_1, y_1, x_2, y_2 \le 2$.



Given that $0 \le x_1, y_1, x_2, y_2 \le 50$, how many right triangles can be formed?

Problem 92:

A number chain is created by continuously adding the square of the digits in a number to form a new number until it has been seen before.

For example,

44 → 32 → 13 → 10 → **1** → **1**
85 → **89** → 145 → 42 → 20 → 4 → 16 → 37 → 58 → **89**

Therefore any chain that arrives at 1 or 89 will become stuck in an endless loop. What is most amazing is that EVERY starting number will eventually arrive at 1 or 89.

How many starting numbers below ten million will arrive at 89?

Problem 93:

By using each of the digits from the set, {1, 2, 3, 4}, exactly once, and making use of the four arithmetic operations (+, −, *, /) and brackets/parentheses, it is possible to form different positive integer targets.

For example,

```
8 = (4 * (1 + 3)) / 2
14 = 4 * (3 + 1 / 2)
19 = 4 * (2 + 3) − 1
36 = 3 * 4 * (2 + 1)
```

Note that concatenations of the digits, like 12 + 34, are not allowed.

Using the set, {1, 2, 3, 4}, it is possible to obtain thirty-one different target numbers of which 36 is the maximum, and each of the numbers 1 to 28 can be obtained before encountering the first non-expressible number.

Find the set of four distinct digits, $a < b < c < d$, for which the longest set of consecutive positive integers, 1 to n, can be obtained, giving your answer as a string: *abcd*.

Problem 94:

It is easily proved that no equilateral triangle exists with integral length sides and integral area. However, the *almost equilateral triangle* 5-5-6 has an area of 12 square units.

We shall define an *almost equilateral triangle* to be a triangle for which two sides are equal and the third differs by no more than one unit.

Find the sum of the perimeters of all *almost equilateral triangles* with integral side lengths and area and whose perimeters do not exceed one billion (1,000,000,000).

Problem 95:

The proper divisors of a number are all the divisors excluding the number itself. For example, the proper divisors of 28 are 1, 2, 4, 7, and 14. As the sum of these divisors is equal to 28, we call it a perfect number.

Interestingly the sum of the proper divisors of 220 is 284 and the sum of the proper divisors of 284 is 220, forming a chain of two numbers. For this reason, 220 and 284 are called an amicable pair.

Perhaps less well known are longer chains. For example, starting with 12496, we form a chain of five numbers:

$$12496 \rightarrow 14288 \rightarrow 15472 \rightarrow 14536 \rightarrow 14264 \ (\rightarrow 12496 \rightarrow \ldots)$$

Since this chain returns to its starting point, it is called an amicable chain.

Find the smallest member of the longest amicable chain with no element exceeding one million.

Problem 96:

Su Doku (Japanese meaning *number place*) is the name given to a popular puzzle concept. Its origin is unclear, but credit must be attributed to Leonhard Euler who invented a similar, and much more difficult, puzzle idea called Latin Squares. The objective of Su Doku puzzles, however, is to replace the blanks (or zeros) in a 9 by 9 grid in such that each row, column, and 3 by 3 box contains each of the digits 1 to 9. Below is an example of a typical starting puzzle grid and its solution grid.

0	0	3	0	2	0	6	0	0
9	0	0	3	0	5	0	0	1
0	0	1	8	0	6	4	0	0
0	0	8	1	0	2	9	0	0
7	0	0	0	0	0	0	0	8
0	0	6	7	0	8	2	0	0
0	0	2	6	0	9	5	0	0
8	0	0	2	0	3	0	0	9
0	0	5	0	1	0	3	0	0

4	8	3	9	2	1	6	5	7
9	6	7	3	4	5	8	2	1
2	5	1	8	7	6	4	9	3
5	4	8	1	3	2	9	7	6
7	2	9	5	6	4	1	3	8
1	3	6	7	9	8	2	4	5
3	7	2	6	8	9	5	1	4
8	1	4	2	5	3	7	6	9
6	9	5	4	1	7	3	8	2

A well constructed Su Doku puzzle has a unique solution and can be solved by logic, although it may be necessary to employ "guess and test" methods in order to eliminate options (there is much contested opinion over this). The complexity of the search determines the difficulty of the puzzle; the example above is considered *easy* because it can be solved by straight forward direct deduction.

The 6K text file, sudoku.txt (right click and 'Save Link/Target As...'), contains fifty different Su Doku puzzles ranging in difficulty, but all with unique solutions (the first puzzle in the file is the example above).

By solving all fifty puzzles find the sum of the 3-digit numbers found in the top left corner of each solution grid; for example, 483 is the 3-digit number found in the top left corner of the solution grid above.

Problem 97:

The first known prime found to exceed one million digits was discovered in 1999, and is a Mersenne prime of the form $2^{6972593}-1$; it contains exactly 2,098,960 digits. Subsequently other Mersenne primes, of the form 2^p-1, have been found which contain more digits.

However, in 2004 there was found a massive non-Mersenne prime which contains 2,357,207 digits: $28433 \times 2^{7830457}+1$.

Find the last ten digits of this prime number.

Problem 98:

By replacing each of the letters in the word CARE with 1, 2, 9, and 6 respectively, we form a square number: $1296 = 36^2$. What is remarkable is that, by using the same digital substitutions, the anagram, RACE, also forms a square number: $9216 = 96^2$. We shall call CARE (and RACE) a square anagram word pair and specify further that leading zeroes are not permitted, neither may a different letter have the same digital value as another letter.

Using words.txt (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, find all the square anagram word pairs (a palindromic word is NOT considered to be an anagram of itself).

What is the largest square number formed by any member of such a pair?

NOTE: All anagrams formed must be contained in the given text file.

Problem 99:

Comparing two numbers written in index form like 2^{11} and 3^7 is not difficult, as any calculator would confirm that $2^{11} = 2048 < 3^7 = 2187$.

However, confirming that $632382^{518061} > 519432^{525806}$ would be much more difficult, as both numbers contain over three million digits.

Using base_exp.txt (right click and 'Save Link/Target As...'), a 22K text file containing one thousand lines with a base/exponent pair on each line, determine which line number has the greatest numerical value.

NOTE: The first two lines in the file represent the numbers in the example given above.

Problem 100:

If a box contains twenty-one coloured discs, composed of fifteen blue discs and six red discs, and two discs were taken at random, it can be seen that the probability of taking two blue discs, $P(BB) = (15/21) \times (14/20) = 1/2$.

The next such arrangement, for which there is exactly 50% chance of taking two blue discs at random, is a box containing eighty-five blue discs and thirty-five red discs.

By finding the first arrangement to contain over 10^{12} = 1,000,000,000,000 discs in total, determine the number of blue discs that the box would contain.

Problem 101:

If we are presented with the first k terms of a sequence it is impossible to say with certainty the value of the next term, as there are infinitely many polynomial functions that can model the sequence.

As an example, let us consider the sequence of cube numbers. This is defined by the generating function, $u_n = n^3$: 1, 8, 27, 64, 125, 216, ...

Suppose we were only given the first two terms of this sequence. Working on the principle that "simple is best" we should assume a linear relationship and predict the next term to be 15 (common difference 7). Even if we were presented with the first three terms, by the same principle of simplicity, a quadratic relationship should be assumed.

We shall define OP(k, n) to be the n^{th} term of the optimum polynomial generating function for the first k terms of a sequence. It should be clear that OP(k, n) will accurately generate the terms of the sequence for $n \le k$, and potentially the *first incorrect term* (FIT) will be OP(k, k+1); in which case we shall call it a *bad OP* (BOP).

As a basis, if we were only given the first term of sequence, it would be most sensible to assume constancy; that is, for $n \ge 2$, OP(1, n) = u_1.

Hence we obtain the following OPs for the cubic sequence:

OP(1, n) = 1	1, **1**, 1, 1, ...
OP(2, n) = $7n-6$	1, 8, **15**, ...
OP(3, n) = $6n^2-11n+6$	1, 8, 27, **58**, ...
OP(4, n) = n^3	1, 8, 27, 64, 125, ...

Clearly no BOPs exist for $k \ge 4$.

By considering the sum of FITs generated by the BOPs (indicated in **red** above), we obtain 1 + 15 + 58 = 74.

Consider the following tenth degree polynomial generating function:

$$u_n = 1 - n + n^2 - n^3 + n^4 - n^5 + n^6 - n^7 + n^8 - n^9 + n^{10}$$

Find the sum of FITs for the BOPs.

Problem 102:

Three distinct points are plotted at random on a Cartesian plane, for which $-1000 \leq x, y \leq 1000$, such that a triangle is formed.

Consider the following two triangles:

A(-340,495), B(-153,-910), C(835,-947)

X(-175,41), Y(-421,-714), Z(574,-645)

It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not.

Using triangles.txt (right click and 'Save Link/Target As...'), a 27K text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin.

NOTE: The first two examples in the file represent the triangles in the example given above.

Problem 103:

Let S(A) represent the sum of elements in set A of size n. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. $S(B) \neq S(C)$; that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then $S(B) > S(C)$.

If S(A) is minimised for a given n, we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

$n = 1$: {1}
$n = 2$: {1, 2}
$n = 3$: {2, 3, 4}
$n = 4$: {3, 5, 6, 7}
$n = 5$: {6, 9, 11, 12, 13}

It *seems* that for a given optimum set, $A = \{a_1, a_2, ... , a_n\}$, the next optimum set is of the form $B = \{b, a_1+b, a_2+b, ... ,a_n+b\}$, where b is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for $n = 6$ to be A = {11, 17, 20, 22, 23, 24}, with S(A) = 117. However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for $n = 6$ is A = {11, 18, 19, 20, 22, 25}, with S(A) = 115 and corresponding set string: 111819202225.

Given that A is an optimum special sum set for $n = 7$, find its set string.

NOTE: This problem is related to problems 105 and 106.

Problem 104:

The Fibonacci sequence is defined by the recurrence relation:

$F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1$ and $F_2 = 1$.

It turns out that F_{541}, which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And F_{2749}, which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital.

Given that F_k is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find k.

Problem 105:

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

For example, {81, 88, 75, 42, 87, 84, 86, 65} is not a special sum set because 65 + 87 + 88 = 75 + 81 + 84, whereas {157, 150, 164, 119, 79, 159, 161, 139, 158} satisfies both rules for all possible subset pair combinations and S(A) = 1286.

Using sets.txt (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets, A_1, A_2, ..., A_k, and find the value of $S(A_1) + S(A_2) + ... + S(A_k)$.

NOTE: This problem is related to problems 103 and 106.

Problem 106:

Let S(A) represent the sum of elements in set A of size *n*. We shall call it a special sum set if for any two non-empty disjoint subsets, B and C, the following properties are true:

i. S(B) ≠ S(C); that is, sums of subsets cannot be equal.
ii. If B contains more elements than C then S(B) > S(C).

For this problem we shall assume that a given set contains *n* strictly increasing elements and it already satisfies the second rule.

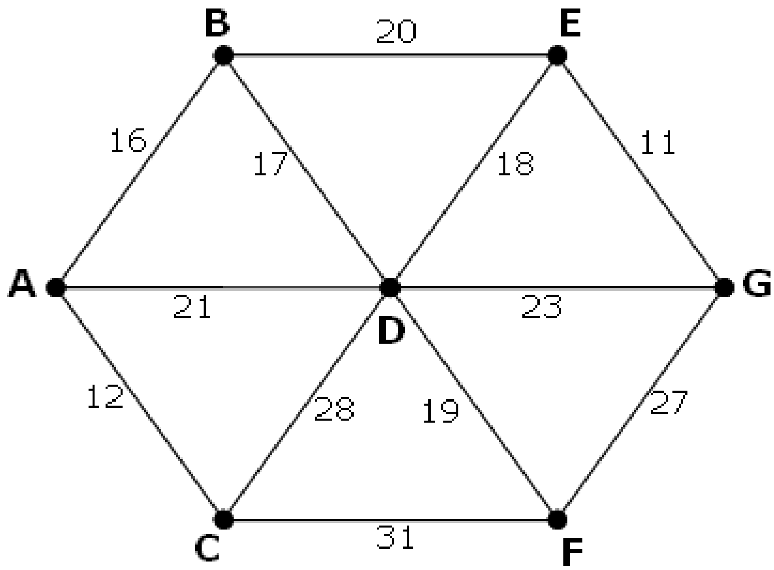
Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which *n* = 4, only 1 of these pairs need to be tested for equality (first rule). Similarly, when *n* = 7, only 70 out of the 966 subset pairs need to be tested.

For *n* = 12, how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to problems 103 and 105.

Problem 107:

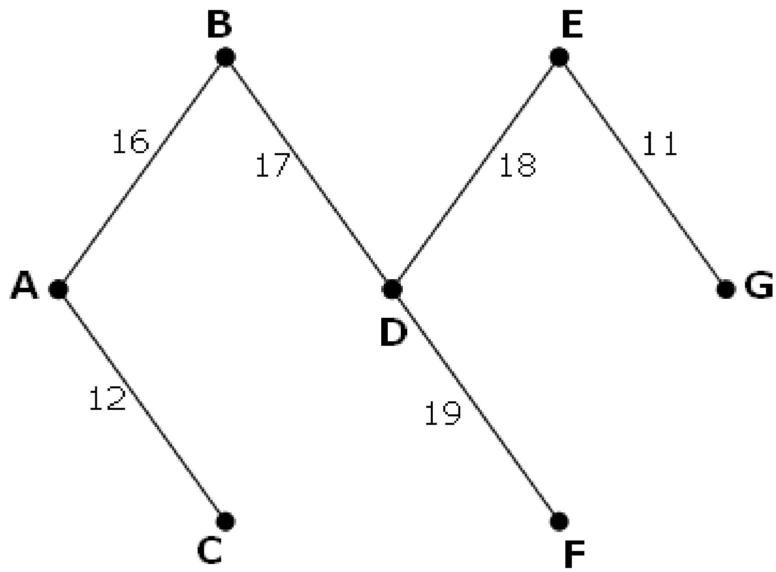
The following undirected network consists of seven vertices and twelve edges with a total weight of 243.



The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of 243 − 93 = 150 from the original network.



Using network.txt (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.

Problem 108:

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For $n = 4$ there are exactly three distinct solutions:

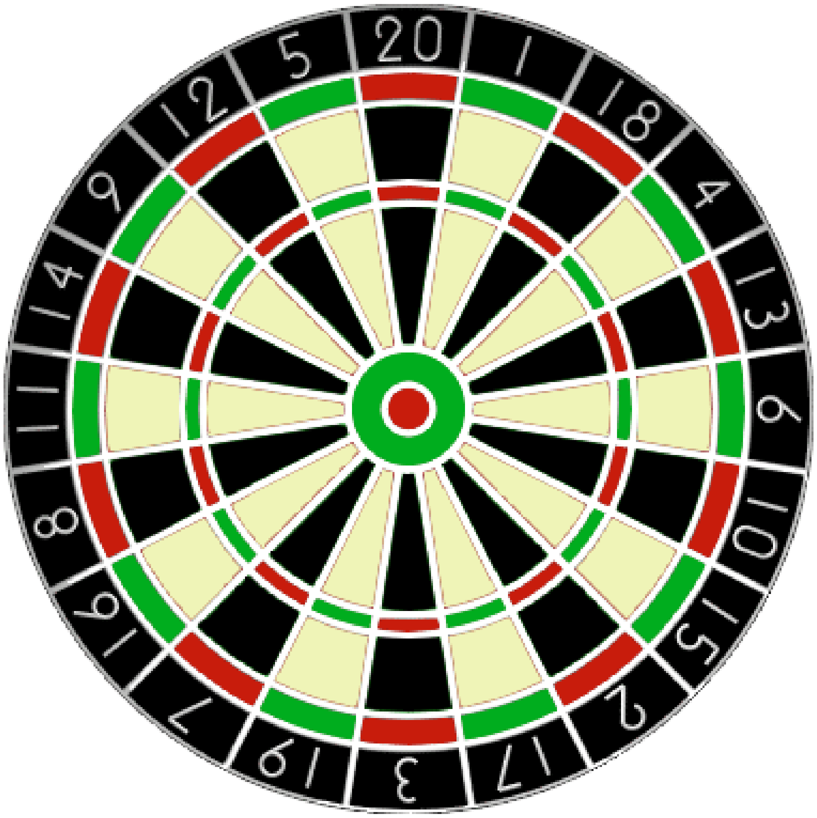
$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$$
$$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

What is the least value of n for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of problem 110; it is strongly advised that you solve this one first.

Problem 109:

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.



The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	

D2	D1	
S4	D1	
S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

Problem 110:

In the following equation x, y, and n are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when n = 1260 there are 113 distinct solutions and this is the least value of n for which the total number of distinct solutions exceeds one hundred.

What is the least value of n for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of problem 108 and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

Problem 111:

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that M(n, d) represents the maximum number of repeated digits for an n-digit prime where d is the repeated digit, N(n, d) represents the number of such primes, and S(n, d) represents the sum of these primes.

So M(4, 1) = 3 is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are N(4, 1) = 9 such primes, and the sum of these primes is S(4, 1) = 22275. It turns out that for d = 0, it is only possible to have M(4, 0) = 2 repeated digits, but there are N(4, 0) = 13 such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, *d*	**M(4, *d*)**	**N(4, *d*)**	**S(4, *d*)**
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214
4	3	2	8888
5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For d = 0 to 9, the sum of all S(4, d) is 273700.

Find the sum of all S(10, d).

Problem 112:

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50% is 538.

Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%.

Find the least number for which the proportion of bouncy numbers is exactly 99%.

Problem 113:

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

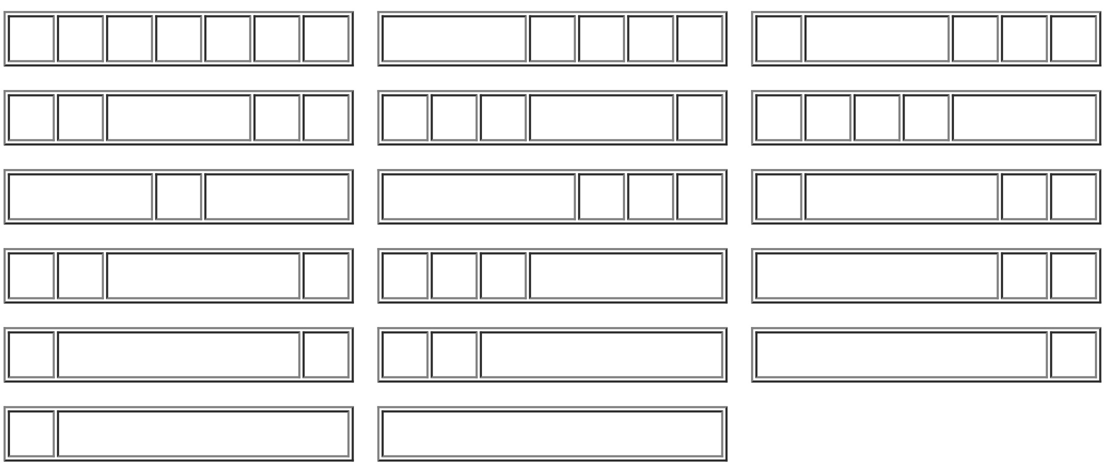
We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

As n increases, the proportion of bouncy numbers below n increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below 10^{10}.

How many numbers below a googol (10^{100}) are not bouncy?

Problem 114:

A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. There are exactly seventeen ways of doing this.



How many ways can a row measuring fifty units in length be filled?

NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), black (1), and red (4).

Problem 115:

NOTE: This is a more difficult version of problem 114.

A row measuring *n* units in length has red blocks with a minimum length of *m* units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square.

Let the fill-count function, F(*m*, *n*), represent the number of ways that a row can be filled.

For example, F(3, 29) = 673135 and F(3, 30) = 1089155.

That is, for *m* = 3, it can be seen that *n* = 30 is the smallest value for which the fill-count function first exceeds one million.

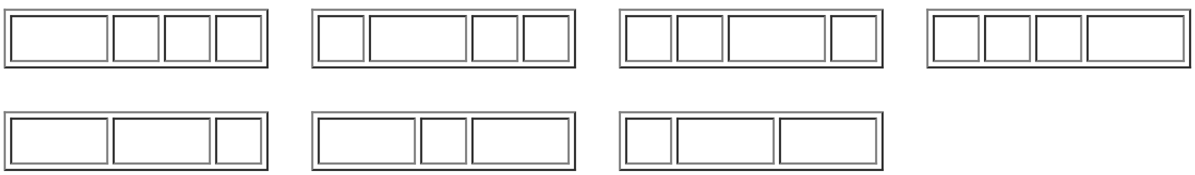
In the same way, for *m* = 10, it can be verified that F(10, 56) = 880711 and F(10, 57) = 1148904, so *n* = 57 is the least value for which the fill-count function first exceeds one million.

For *m* = 50, find the least value of *n* for which the fill-count function first exceeds one million.

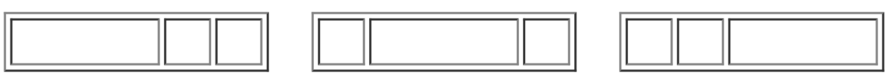
Problem 116:

A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

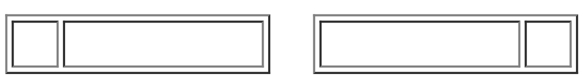
If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.



And if blue tiles are chosen there are two ways.



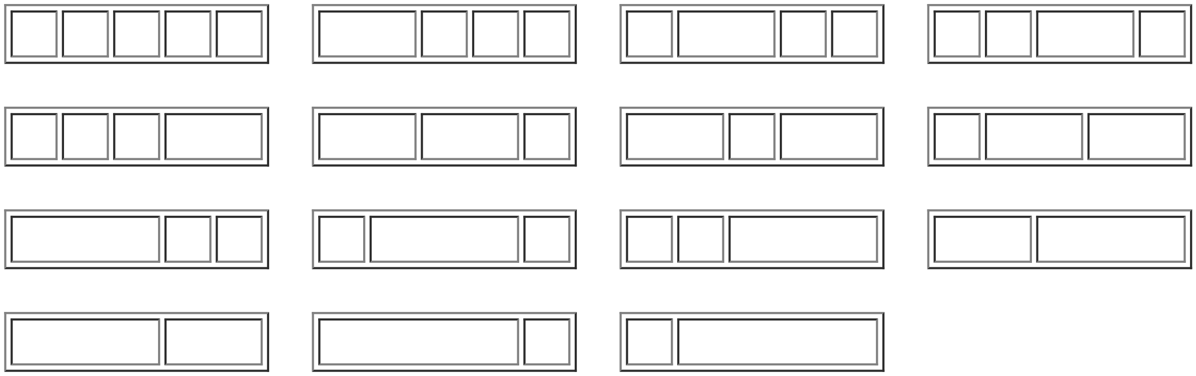
Assuming that colours cannot be mixed there are 7 + 3 + 2 = 12 ways of replacing the black tiles in a row measuring five units in length.

How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to problem 117.

Problem 117:

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to problem 116.

Problem 118:

Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set {2,5,47,89,631}, all of the elements belonging to it are prime.

How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?

Problem 119:

The number 512 is interesting because it is equal to the sum of its digits raised to some power: 5 + 1 + 2 = 8, and $8^3 = 512$. Another example of a number with this property is $614656 = 28^4$.

We shall define a_n to be the nth term of this sequence and insist that a number must contain at least two digits to have a sum.

You are given that $a_2 = 512$ and $a_{10} = 614656$.

Find a_{30}.

Problem 120:

Let r be the remainder when $(a-1)^n + (a+1)^n$ is divided by a^2.

For example, if $a = 7$ and $n = 3$, then $r = 42$: $6^3 + 8^3 = 728 \equiv 42 \bmod 49$. And as n varies, so too will r, but for $a = 7$ it turns out that $r_{max} = 42$.

For $3 \leq a \leq 1000$, find $\sum r_{max}$.

Problem 121:

A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random.

The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game.

If the game is played for four turns, the probability of a player winning is exactly 11/120, and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9.

Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.

Problem 122:

The most naive way of computing n^{15} requires fourteen multiplications:

$$n \times n \times \ldots \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$n \times n = n^2$$
$$n^2 \times n^2 = n^4$$
$$n^4 \times n^4 = n^8$$
$$n^8 \times n^4 = n^{12}$$
$$n^{12} \times n^2 = n^{14}$$
$$n^{14} \times n = n^{15}$$

However it is yet possible to compute it in only five multiplications:

$$n \times n = n^2$$
$$n^2 \times n = n^3$$
$$n^3 \times n^3 = n^6$$
$$n^6 \times n^6 = n^{12}$$
$$n^{12} \times n^3 = n^{15}$$

We shall define m(k) to be the minimum number of multiplications to compute n^k; for example m(15) = 5.

For $1 \leq k \leq 200$, find Σ m(k).

Problem 123:

Let p_n be the nth prime: 2, 3, 5, 7, 11, ..., and let r be the remainder when $(p_n-1)^n + (p_n+1)^n$ is divided by p_n^2.

For example, when $n = 3$, $p_3 = 5$, and $4^3 + 6^3 = 280 \equiv 5 \bmod 25$.

The least value of n for which the remainder first exceeds 10^9 is 7037.

Find the least value of n for which the remainder first exceeds 10^{10}.

Problem 124:

The radical of n, rad(n), is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) = $2 \times 3 \times 7 = 42$.

If we calculate rad(n) for $1 \leq n \leq 10$, then sort them on rad(n), and sorting on n if the radical values are equal, we get:

Unsorted			**Sorted**	
n	**rad(*n*)**	***n***	**rad(*n*)**	**k**
1	1	1	1	1
2	2	2	2	2
3	3	4	2	3
4	2	8	2	4
5	5	3	3	5
6	6	9	3	6
7	7	5	5	7
8	2	6	6	8
9	3	7	7	9
10	10	10	10	10

Let E(k) be the kth element in the sorted n column; for example, E(4) = 8 and E(6) = 9.

If rad(n) is sorted for $1 \leq n \leq 100000$, find E(10000).

Problem 125:

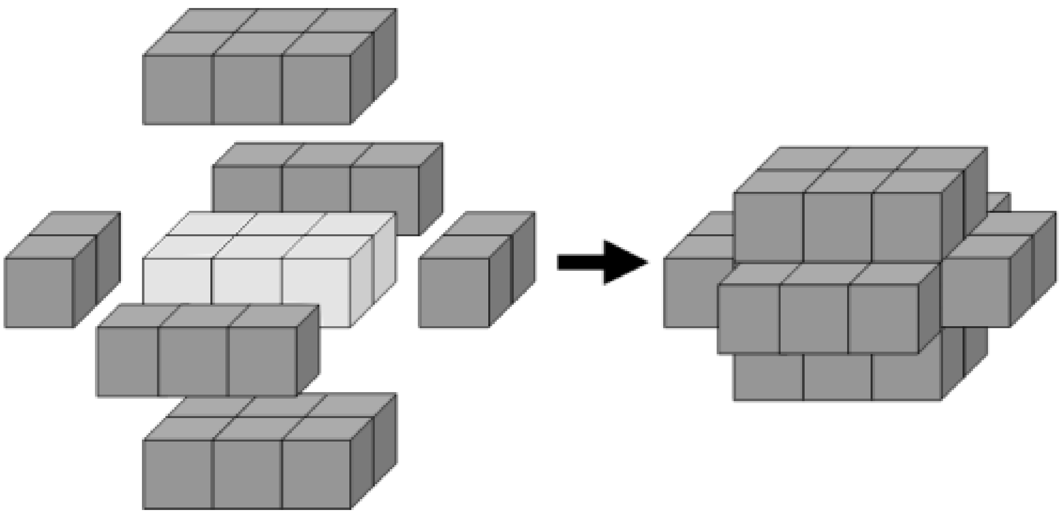
The palindromic number 595 is interesting because it can be written as the sum of consecutive squares: $6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$.

There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that $1 = 0^2 + 1^2$ has not been included as this problem is concerned with the squares of positive integers.

Find the sum of all the numbers less than 10^8 that are both palindromic and can be written as the sum of consecutive squares.

Problem 126:

The minimum number of cubes to cover every visible face on a cuboid measuring 3 x 2 x 1 is twenty-two.



If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring 5 x 1 x 1 also requires twenty-two cubes; similarly the first layer on cuboids measuring 5 x 3 x 1, 7 x 2 x 1, and 11 x 1 x 1 all contain forty-six cubes.

We shall define C(n) to represent the number of cuboids that contain n cubes in one of its layers. So C(22) = 2, C(46) = 4, C(78) = 5, and C(118) = 8.

It turns out that 154 is the least value of n for which C(n) = 10.

Find the least value of n for which C(n) = 1000.

Problem 127:

The radical of n, rad(n), is the product of distinct prime factors of n. For example, $504 = 2^3 \times 3^2 \times 7$, so rad(504) = $2 \times 3 \times 7 = 42$.

We shall define the triplet of positive integers (a, b, c) to be an abc-hit if:

1. GCD(a, b) = GCD(a, c) = GCD(b, c) = 1
2. $a < b$
3. $a + b = c$
4. rad(abc) $< c$

For example, (5, 27, 32) is an abc-hit, because:

1. GCD(5, 27) = GCD(5, 32) = GCD(27, 32) = 1
2. 5 < 27
3. 5 + 27 = 32
4. rad(4320) = 30 < 32

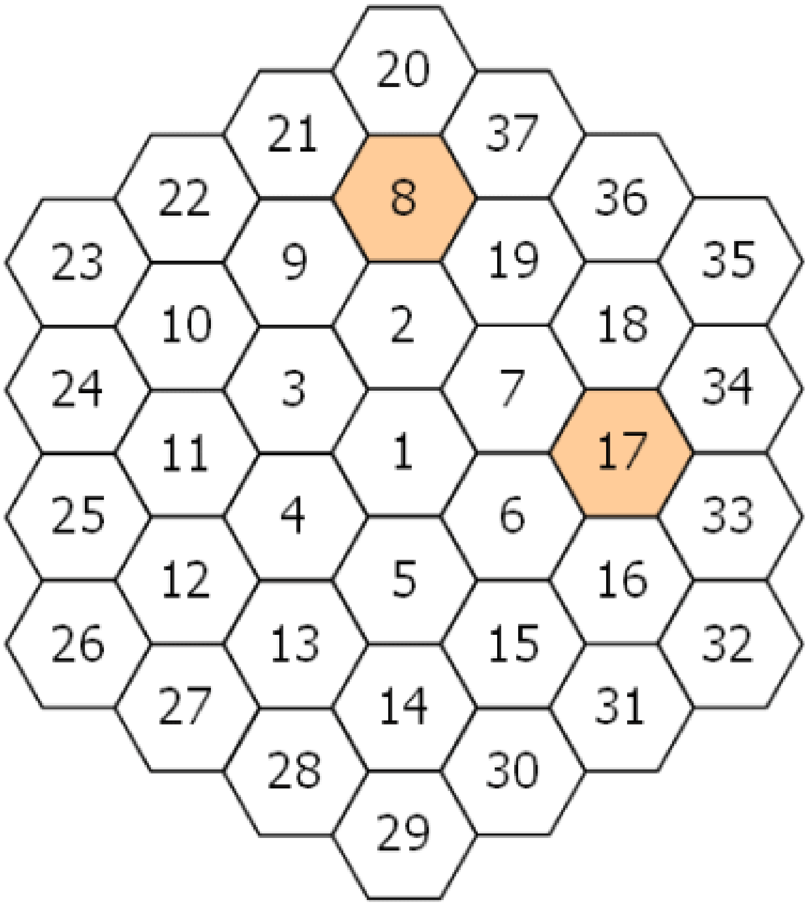
It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for $c < 1000$, with $\sum c = 12523$.

Find $\sum c$ for $c < 120000$.

Problem 128:

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile *n* and each its six neighbours we shall define PD(*n*) to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So PD(8) = 3.

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence PD(17) = 2.

It can be shown that the maximum value of PD(*n*) is 3.

If all of the tiles for which PD(*n*) = 3 are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.

Problem 129:

A number consisting entirely of ones is called a repunit. We shall define R(*k*) to be a repunit of length *k*; for example, R(6) = 111111.

Given that *n* is a positive integer and GCD(*n*, 10) = 1, it can be shown that there always exists a value, *k*, for which R(*k*) is divisible by *n*, and let A(*n*) be the least such value of *k*; for example, A(7) = 6 and A(41) = 5.

The least value of *n* for which A(*n*) first exceeds ten is 17.

Find the least value of *n* for which A(*n*) first exceeds one-million.

Problem 130:

A number consisting entirely of ones is called a repunit. We shall define R(*k*) to be a repunit of length *k*; for example, R(6) = 111111.

Given that *n* is a positive integer and GCD(*n*, 10) = 1, it can be shown that there always exists a value, *k*, for which R(*k*) is divisible by *n*, and let A(*n*) be the least such value of *k*; for example, A(7) = 6 and A(41) = 5.

You are given that for all primes, $p > 5$, that $p - 1$ is divisible by A(*p*). For example, when *p* = 41, A(41) = 5, and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of *n* for which
GCD(*n*, 10) = 1 and $n - 1$ is divisible by A(*n*).

Problem 131:

There are some prime values, *p*, for which there exists a positive integer, *n*, such that the expression $n^3 + n^2p$ is a perfect cube.

For example, when *p* = 19, $8^3 + 8^2 \times 19 = 12^3$.

What is perhaps most surprising is that for each prime with this property the value of *n* is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?

Problem 132:

A number consisting entirely of ones is called a repunit. We shall define R(*k*) to be a repunit of length *k*.

For example, R(10) = 1111111111 = 11×41×271×9091, and the sum of these prime factors is 9414.

Find the sum of the first forty prime factors of R(10^9).

Problem 133:

A number consisting entirely of ones is called a repunit. We shall define R(k) to be a repunit of length k; for example, R(6) = 111111.

Let us consider repunits of the form $R(10^n)$.

Although R(10), R(100), or R(1000) are not divisible by 17, R(10000) is divisible by 17. Yet there is no value of n for which $R(10^n)$ will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of $R(10^n)$.

Find the sum of all the primes below one-hundred thousand that will never be a factor of $R(10^n)$.

Problem 134:

Consider the consecutive primes $p_1 = 19$ and $p_2 = 23$. It can be verified that 1219 is the smallest number such that the last digits are formed by p_1 whilst also being divisible by p_2.

In fact, with the exception of $p_1 = 3$ and $p_2 = 5$, for every pair of consecutive primes, $p_2 > p_1$, there exist values of n for which the last digits are formed by p_1 and n is divisible by p_2. Let S be the smallest of these values of n.

Find $\sum S$ for every pair of consecutive primes with $5 \leq p_1 \leq 1000000$.

Problem 135:

Given the positive integers, x, y, and z, are consecutive terms of an arithmetic progression, the least value of the positive integer, n, for which the equation, $x^2 - y^2 - z^2 = n$, has exactly two solutions is $n = 27$:

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27$$

It turns out that $n = 1155$ is the least value which has exactly ten solutions.

How many values of n less than one million have exactly ten distinct solutions?

Problem 136:

The positive integers, x, y, and z, are consecutive terms of an arithmetic progression. Given that n is a positive integer, the equation, $x^2 - y^2 - z^2 = n$, has exactly one solution when $n = 20$:

$$13^2 - 10^2 - 7^2 = 20$$

In fact there are twenty-five values of n below one hundred for which the equation has a unique solution.

How many values of n less than fifty million have exactly one solution?

Problem 137:

Consider the infinite polynomial series $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + ...$, where F_k is the kth term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... ; that is, $F_k = F_{k-1} + F_{k-2}$, $F_1 = 1$ and $F_2 = 1$.

For this problem we shall be interested in values of x for which $A_F(x)$ is a positive integer.

Surprisingly $A_F(1/2) = (1/2).1 + (1/2)^2.1 + (1/2)^3.2 + (1/2)^4.3 + (1/2)^5.5 + ...$
$= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + ...$
$= 2$

The corresponding values of x for the first five natural numbers are shown below.

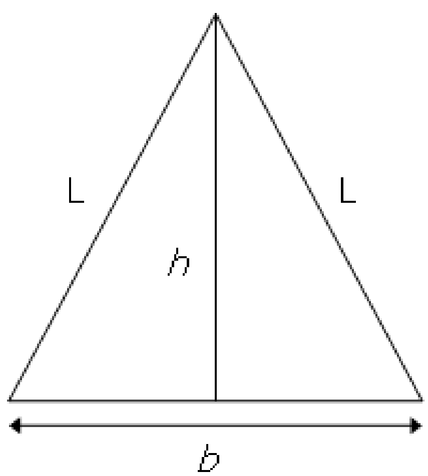
x	$A_F(x)$
$\sqrt{2}-1$	1
1/2	2
$(\sqrt{13}-2)/3$	3
$(\sqrt{89}-5)/8$	4
$(\sqrt{34}-3)/5$	5

We shall call $A_F(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

Problem 138:

Consider the isosceles triangle with base length, $b = 16$, and legs, $L = 17$.



By using the Pythagorean theorem it can be seen that the height of the triangle, $h = \sqrt{(17^2 - 8^2)} = 15$, which is one less than the base length.

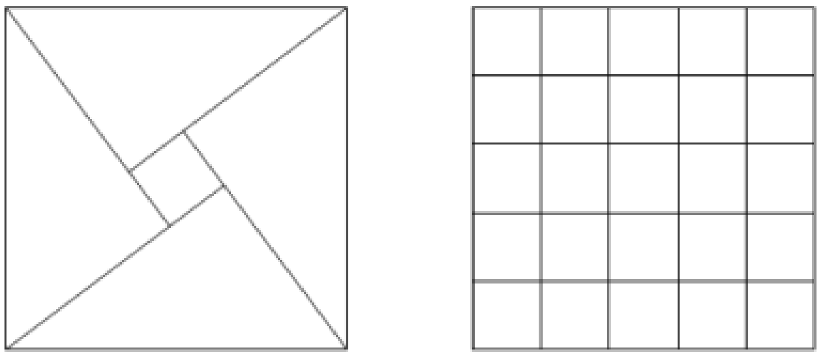
With $b = 272$ and $L = 305$, we get $h = 273$, which is one more than the base length, and this is the second smallest isosceles triangle with the property that $h = b \pm 1$.

Find Σ L for the twelve smallest isosceles triangles for which $h = b \pm 1$ and b, L are positive integers.

Problem 139:

Let (a, b, c) represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length c.

For example, (3, 4, 5) triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.



However, if (5, 12, 13) triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?

Problem 140:

Consider the infinite polynomial series $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \ldots$, where G_k is the kth term of the second order recurrence relation $G_k = G_{k-1} + G_{k-2}$, $G_1 = 1$ and $G_2 = 4$; that is, 1, 4, 5, 9, 14, 23, ...

For this problem we shall be concerned with values of x for which $A_G(x)$ is a positive integer.

The corresponding values of x for the first five natural numbers are shown below.

x	$A_G(x)$
$(\sqrt{5}-1)/4$	1
2/5	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
1/2	5

We shall call $A_G(x)$ a golden nugget if x is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

Problem 141:

A positive integer, n, is divided by d and the quotient and remainder are q and r respectively. In addition d, q, and r are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio 3/2).
We will call such numbers, n, progressive.

Some progressive numbers, such as 9 and 10404 = 102^2, happen to also be perfect squares.
The sum of all progressive perfect squares below one hundred thousand is 124657.

Find the sum of all progressive perfect squares below one trillion (10^{12}).

Problem 142:

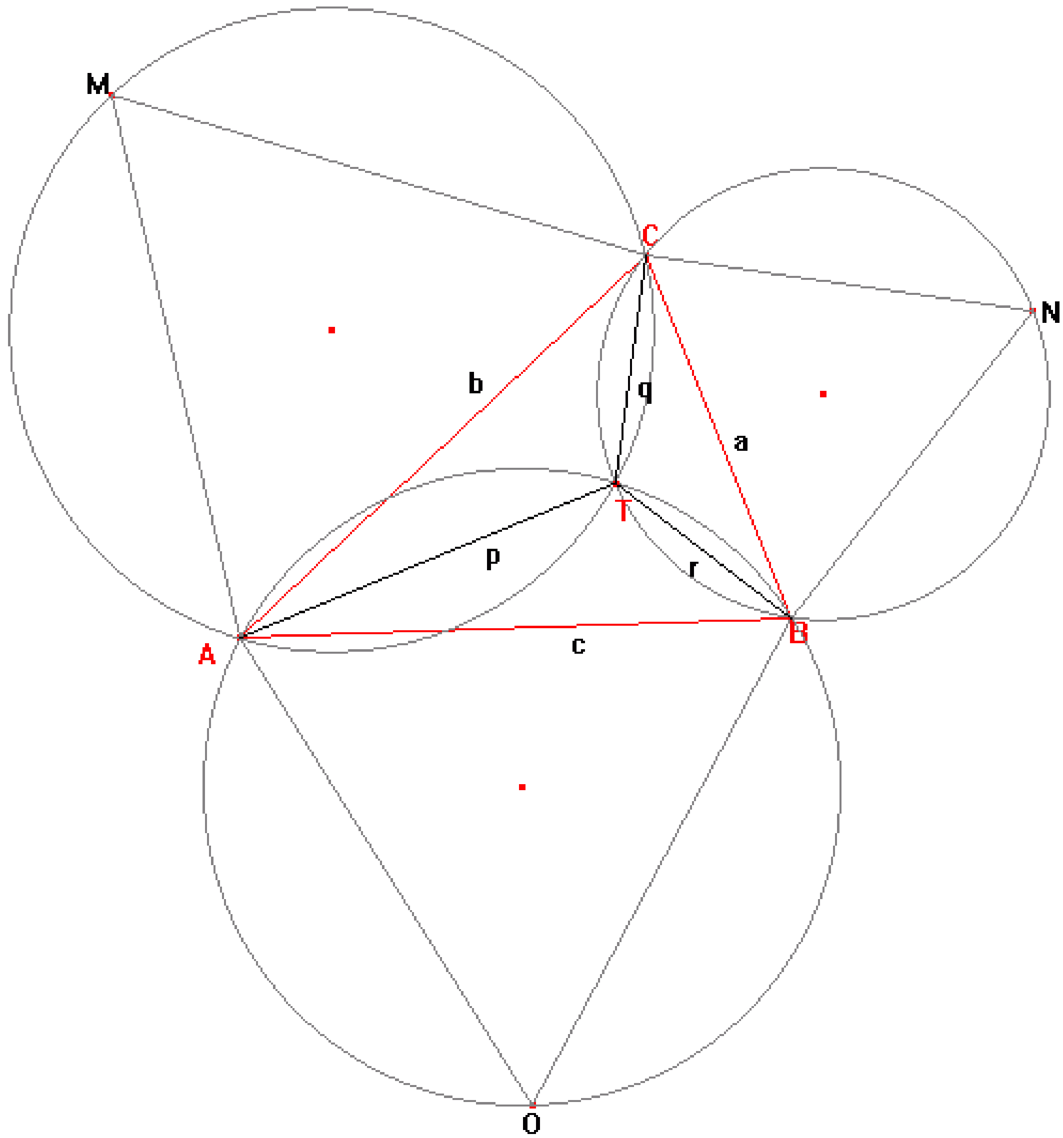
Find the smallest $x + y + z$ with integers $x > y > z > 0$ such that $x + y$, $x - y$, $x + z$, $x - z$, $y + z$, $y - z$ are all perfect squares.

Problem 143:

Let ABC be a triangle with all interior angles being less than 120 degrees. Let X be any point inside the triangle and let XA = p, XB = q, and XC = r.

Fermat challenged Torricelli to find the position of X such that p + q + r was minimised.

Torricelli was able to prove that if equilateral triangles AOB, BNC and AMC are constructed on each side of triangle ABC, the circumscribed circles of AOB, BNC, and AMC will intersect at a single point, T, inside the triangle. Moreover he proved that T, called the Torricelli/Fermat point, minimises p + q + r. Even more remarkable, it can be shown that when the sum is minimised, AN = BM = CO = p + q + r and that AN, BM and CO also intersect at T.



If the sum is minimised and a, b, c, p, q and r are all positive integers we shall call triangle ABC a Torricelli triangle. For example, a = 399, b = 455, c = 511 is an example of a Torricelli triangle, with p + q + r = 784.

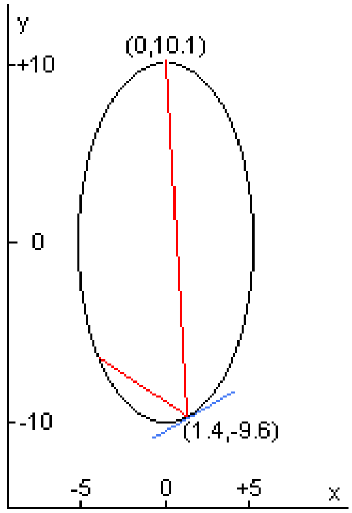
Find the sum of all distinct values of p + q + r ≤ 120000 for Torricelli triangles.

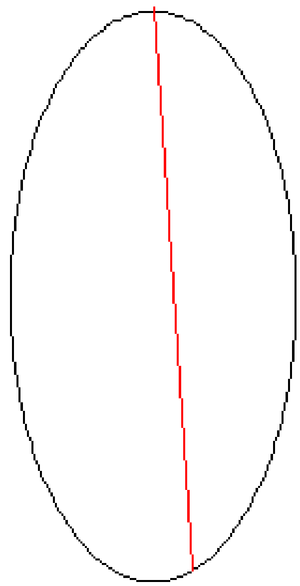
Problem 144:

In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out.

The specific white cell we will be considering is an ellipse with the equation $4x^2 + y^2 = 100$

The section corresponding to $-0.01 \leq x \leq +0.01$ at the top is missing, allowing the light to enter and exit through the hole.





The light beam in this problem starts at the point (0.0,10.1) just outside the white cell, and the beam first impacts the mirror at (1.4,-9.6).

Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence.

In the figure on the left, the red line shows the first two points of contact between the laser beam and the wall of the white cell; the blue line shows the line tangent to the ellipse at the point of incidence of the first bounce.

The slope m of the tangent line at any point (x,y) of the given ellipse is: $m = -4x/y$

The normal line is perpendicular to this tangent line at the point of incidence.

The animation on the right shows the first 10 reflections of the beam.

How many times does the beam hit the internal surface of the white cell before exiting?

Problem 145:

Some positive integers n have the property that the sum [n + reverse(n)] consists entirely of odd (decimal) digits. For instance, 36 + 63 = 99 and 409 + 904 = 1313. We will call such numbers *reversible*; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either n or reverse(n).

There are 120 reversible numbers below one-thousand.

How many reversible numbers are there below one-billion (10^9)?

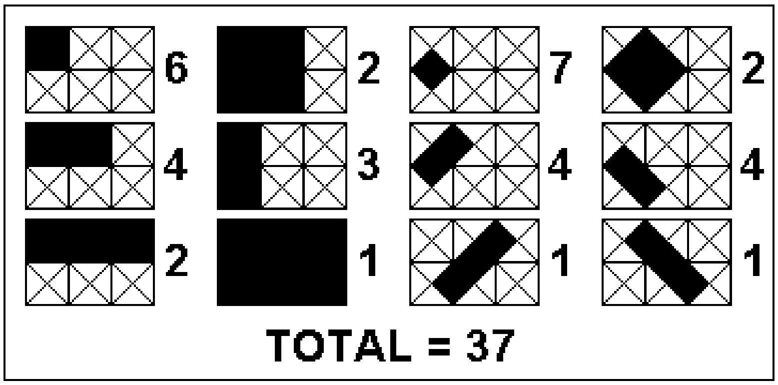
Problem 146:

The smallest positive integer n for which the numbers n^2+1, n^2+3, n^2+7, n^2+9, n^2+13, and n^2+27 are consecutive primes is 10. The sum of all such integers n below one-million is 1242490.

What is the sum of all such integers n below 150 million?

Problem 147:

In a 3x2 cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch.



There are 5 grids smaller than 3x2, vertical and horizontal dimensions being important, i.e. 1x1, 2x1, 3x1, 1x2 and 2x2. If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids:

1x1: 1
2x1: 4
3x1: 8
1x2: 4
2x2: 18

Adding those to the 37 of the 3x2 grid, a total of 72 different rectangles could be situated within 3x2 and smaller grids.

How many different rectangles could be situated within 47x43 and smaller grids?

Problem 148:

We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are *not* divisible by 7.

Find the number of entries which are *not* divisible by 7 in the first one billion (10^9) rows of Pascal's triangle.

Problem 149:

Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1).

−2	5	3	2
9	−6	5	1
3	2	7	3
−1	8	−4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

For $1 \leq k \leq 55$, $s_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000) − 500000.
For $56 \leq k \leq 4000000$, $s_k = [s_{k-24} + s_{k-55} + 1000000]$ (modulo 1000000) − 500000.

Thus, $s_{10} = -393027$ and $s_{100} = 86613$.

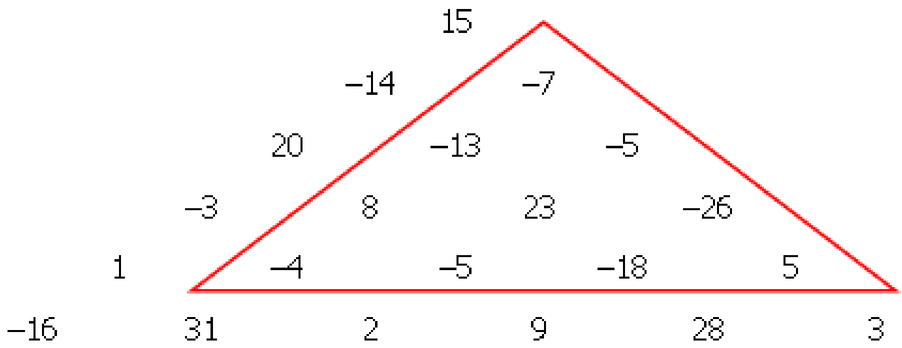
The terms of s are then arranged in a 2000×2000 table, using the first 2000 numbers to fill the first row (sequentially), the next 2000 numbers to fill the second row, and so on.

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal or anti-diagonal).

Problem 150:

In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible.

In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of −42.



We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers s_k in the range $\pm 2^{19}$, using a type of random number generator (known as a Linear Congruential Generator) as follows:

> t := 0
> for k = 1 up to k = 500500:
> t := (615949*t + 797807) modulo 2^{20}
> $s_k := t - 2^{19}$

Thus: s_1 = 273519, s_2 = −153582, s_3 = 450905 etc

Our triangular array is then formed using the pseudo-random numbers thus:

$$\begin{array}{c} s_1 \\ s_2\ \ s_3 \\ s_4\ \ s_5\ \ s_6 \\ s_7\ \ s_8\ \ s_9\ \ s_{10} \\ \ldots \end{array}$$

Sub-triangles can start at any element of the array and extend down as far as we like (taking-in the two elements directly below it from the next row, the three elements directly below from the row after that, and so on).
The "sum of a sub-triangle" is defined as the sum of all the elements it contains.
Find the smallest possible sub-triangle sum.

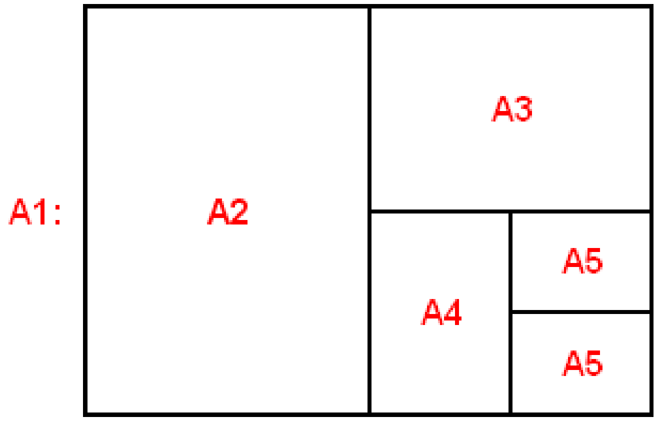
Problem 151:

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.



At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

Problem 152:

There are several ways to write the number 1/2 as a sum of inverse squares using *distinct* integers.

For instance, the numbers {2,3,4,5,7,12,15,20,28,35} can be used:

$$\frac{1}{2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{28^2} + \frac{1}{35^2}$$

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being: {2,3,4,6,7,9,10,20,28,35,36,45} and {2,3,4,6,7,9,12,15,28,30,35,36,45}.

How many ways are there to write the number 1/2 as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

Problem 153:

As we all know the equation x^2=-1 has no solutions for real x.
If we however introduce the imaginary number i this equation has two solutions: $x=i$ and $x=-i$.
If we go a step further the equation (x-3)^2=-4 has two complex solutions: x=3+2i and x=3-2i.
x=3+2i and x=3-2i are called each others' complex conjugate.
Numbers of the form $a+bi$ are called complex numbers.
In general $a+bi$ and $a-bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a+bi$ such that both a and b are integers.
The regular integers are also Gaussian integers (with $b=0$).
To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."
A Gaussian integer is called a divisor of a rational integer n if the result is also a Gaussian integer.

If for example we divide 5 by $1+2i$ we can simplify $\frac{5}{1+2i}$ in the following manner:

Multiply numerator and denominator by the complex conjugate of $1+2i$: $1-2i$.

The result is $\frac{5}{1+2i} = \frac{5}{1+2i}\frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$.

So $1+2i$ is a divisor of 5.

Note that $1+i$ is not a divisor of 5 because $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$.

Note also that if the Gaussian Integer $(a+bi)$ is a divisor of a rational integer n, then its complex conjugate $(a-bi)$ is also a divisor of n.

In fact, 5 has six divisors such that the real part is positive: $\{1, 1+2i, 1-2i, 2+i, 2-i, 5\}$.
The following is a table of all of the divisors for the first five positive rational integers:

n	Gaussian integer divisors with positive real part	Sum s(n) of these divisors
1	1	1
2	1, $1+i$, $1-i$, 2	5
3	1, 3	4
4	1, $1+i$, $1-i$, 2, $2+2i$, $2-2i$,4	13
5	1, $1+2i$, $1-2i$, $2+i$, $2-i$, 5	12

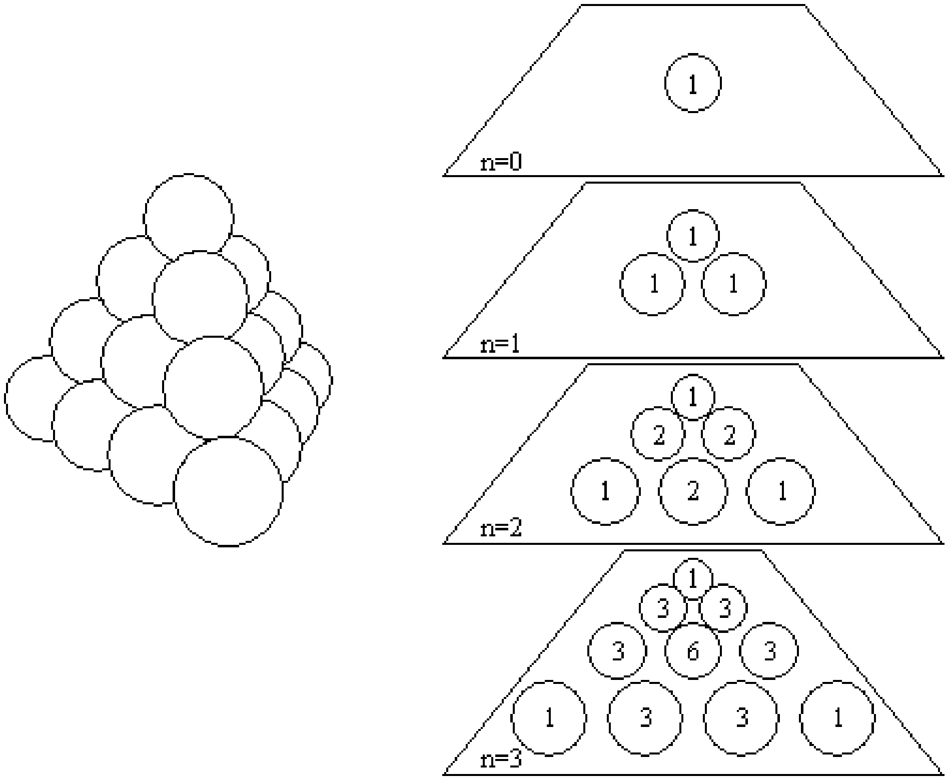
For divisors with positive real parts, then, we have: $\sum_{n=1}^{5} s(n) = 35$.

For $1 \leq n \leq 10^5$, $\sum s(n)=17924657155$.

What is $\sum s(n)$ for $1 \leq n \leq 10^8$?

Problem 154:

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.



Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

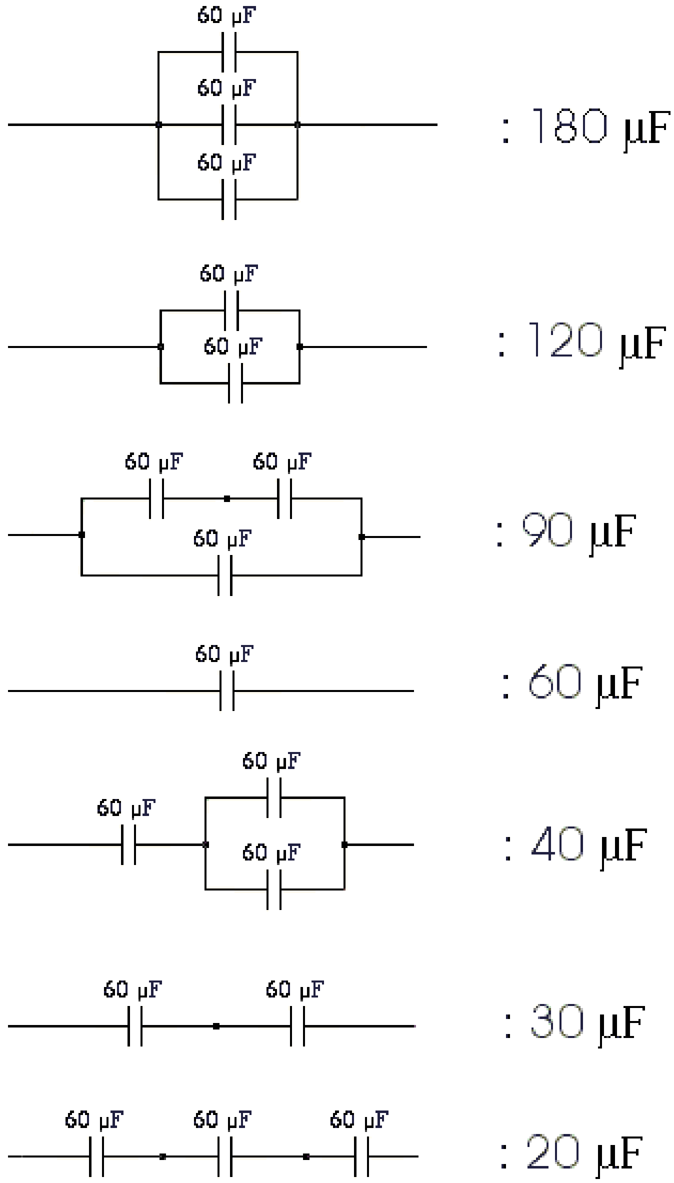
The result is *Pascal's pyramid* and the numbers at each level n are the coefficients of the trinomial expansion $(x + y + z)^n$.

How many coefficients in the expansion of $(x + y + z)^{200000}$ are multiples of 10^{12}?

Problem 155:

An electric circuit uses exclusively identical capacitors of the same value C.
The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to n identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to n=3 capacitors of 60 μF each, we can obtain the following 7 distinct total capacitance values:



If we denote by $D(n)$ the number of distinct total capacitance values we can obtain when using up to n equal-valued capacitors and the simple procedure described above, we have: $D(1)=1$, $D(2)=3$, $D(3)=7$ ...

Find $D(18)$.

Reminder : When connecting capacitors C_1, C_2 etc in parallel, the total capacitance is $C_T = C_1 + C_2 + \ldots$, whereas when connecting them in series, the overall capacitance is given by: $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$

Problem 156:

Starting from zero the natural numbers are written down in base 10 like this:
0 1 2 3 4 5 6 7 8 9 10 11 12....

Consider the digit d=1. After we write down each number n, we will update the number of ones that have occurred and call this number $f(n,1)$. The first values for $f(n,1)$, then, are as follows:

n	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	2
11	4
12	5

Note that $f(n,1)$ never equals 3.
So the first two solutions of the equation $f(n,1)=n$ are $n=0$ and $n=1$. The next solution is $n=199981$.

In the same manner the function $f(n,d)$ gives the total number of digits d that have been written down after the number n has been written.
In fact, for every digit $d \neq 0$, 0 is the first solution of the equation $f(n,d)=n$.

Let $s(d)$ be the sum of all the solutions for which $f(n,d)=n$.
You are given that $s(1)=22786974071$.

Find $\sum s(d)$ for $1 \leq d \leq 9$.

Note: if, for some n, $f(n,d)=n$ for more than one value of d this value of n is counted again for every value of d for which $f(n,d)=n$.

Problem 157:

Consider the diophantine equation $^1/_a+^1/_b=^p/_{10^n}$ with a, b, p, n positive integers and $a \leq b$.
For $n=1$ this equation has 20 solutions that are listed below:

$^1/_1+^1/_1=^{20}/_{10}$	$^1/_1+^1/_2=^{15}/_{10}$	$^1/_1+^1/_5=^{12}/_{10}$	$^1/_1+^1/_{10}=^{11}/_{10}$	$^1/_2+^1/_2=^{10}/_{10}$
$^1/_2+^1/_5=^7/_{10}$	$^1/_2+^1/_{10}=^6/_{10}$	$^1/_3+^1/_6=^5/_{10}$	$^1/_3+^1/_{15}=^4/_{10}$	$^1/_4+^1/_4=^5/_{10}$
$^1/_4+^1/_{20}=^3/_{10}$	$^1/_5+^1/_5=^4/_{10}$	$^1/_5+^1/_{10}=^3/_{10}$	$^1/_6+^1/_{30}=^2/_{10}$	$^1/_{10}+^1/_{10}=^2/_{10}$
$^1/_{11}+^1/_{110}=^1/_{10}$	$^1/_{12}+^1/_{60}=^1/_{10}$	$^1/_{14}+^1/_{35}=^1/_{10}$	$^1/_{15}+^1/_{30}=^1/_{10}$	$^1/_{20}+^1/_{20}=^1/_{10}$

How many solutions has this equation for $1 \leq n \leq 9$?

Problem 158:

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.
Examples are 'abc', 'hat' and 'zyx'.
When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.
For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.
In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of $n \leq 26$ different characters from the alphabet.
For every n, p(n) is the number of strings of length n for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of p(n)?

Problem 159:

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

24 = 2x2x2x3
24 = 2x3x4
24 = 2x2x6
24 = 4x6
24 = 3x8
24 = 2x12
24 = 24

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number. The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
2x2x2x3	9
2x3x4	9
2x2x6	10
4x6	10
3x8	11

2x12	5
24	6

The maximum Digital Root Sum of 24 is 11.
The function mdrs(n) gives the maximum Digital Root Sum of n. So mdrs(24)=11.
Find $\sum$mdrs(n) for $1 < n < 1{,}000{,}000$.

Problem 160:

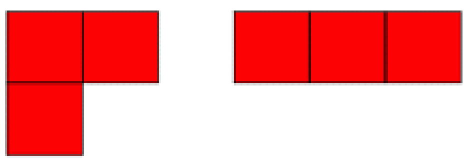
For any N, let f(N) be the last five digits before the trailing zeroes in N!.
For example,

> 9! = 362880 so f(9)=36288
> 10! = 3628800 so f(10)=36288
> 20! = 2432902008176640000 so f(20)=17664

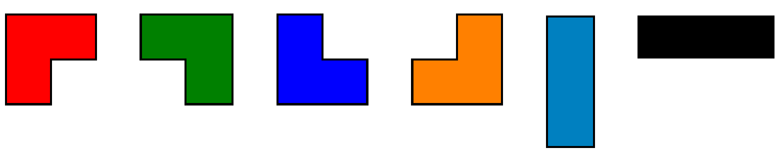
Find f(1,000,000,000,000)

Problem 161:

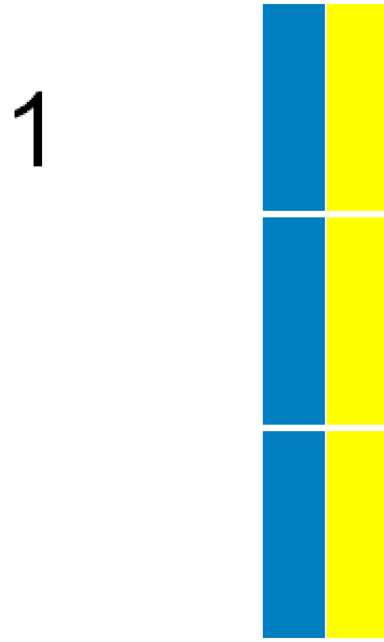
A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:



If all possible orientations are taken into account there are six:



Any n by m grid for which nxm is divisible by 3 can be tiled with triominoes.
If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

Problem 162:

In the hexadecimal number system numbers are represented using 16 different digits:

0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

The hexadecimal number AF when written in the decimal number system equals 10x16+15=175.

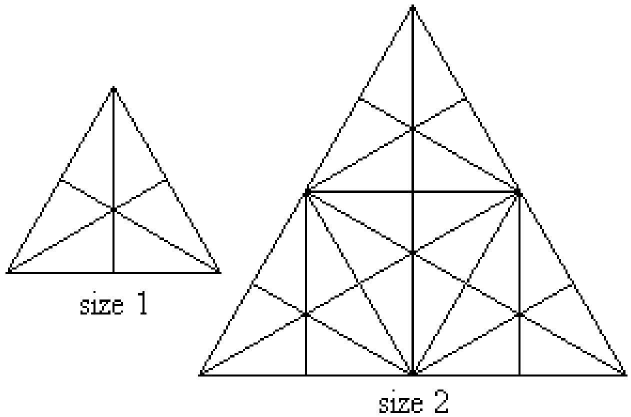
In the 3-digit hexadecimal numbers 10A, 1A0, A10, and A01 the digits 0,1 and A are all present.
Like numbers written in base ten we write hexadecimal numbers without leading zeroes.

How many hexadecimal numbers containing at most sixteen hexadecimal digits exist with all of the digits 0,1, and A present at least once?
Give your answer as a hexadecimal number.

(A,B,C,D,E and F in upper case, without any leading or trailing code that marks the number as hexadecimal and without leading zeroes , e.g. 1A3F and not: 1a3f and not 0x1a3f and not $1A3F and not #1A3F and not 0000001A3F)

Problem 163:

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.



Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain n^2 *size 1* triangle building blocks.

If we denote T(n) as the number of triangles present in a triangle of *size* n, then

T(1) = 16
T(2) = 104

Find T(36).

Problem 164:

How many 20 digit numbers n (without any leading zero) exist such that no three consecutive digits of n have a sum greater than 9?

Problem 165:

A segment is uniquely defined by its two endpoints.
By considering two line segments in plane geometry there are three possibilities:
the segments have zero points, one point, or infinitely many points in common.

Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments.
We will call a common point T of two segments L_1 and L_2 a true intersection point of L_1 and L_2 if T is the only common point of L_1 and L_2 and T is an interior point of both segments.

Consider the three segments L_1, L_2, and L_3:

L_1: (27, 44) to (12, 32)
L_2: (46, 53) to (17, 62)
L_3: (46, 70) to (22, 40)

It can be verified that line segments L_2 and L_3 have a true intersection point. We note that as the one of the end points of L_3: (22,40) lies on L_1 this is not considered to be a true point of intersection. L_1 and L_2 have no common point. So among the three line segments, we find one true intersection point.

Now let us do the same for 5000 line segments. To this end, we generate 20000 numbers using the so-called "Blum Blum Shub" pseudo-random number generator.

$s_0 = 290797$

$s_{n+1} = s_n \times s_n$ (modulo 50515093)

$t_n = s_n$ (modulo 500)

To create each line segment, we use four consecutive numbers t_n. That is, the first line segment is given by:

(t_1, t_2) to (t_3, t_4)

The first four numbers computed according to the above generator should be: 27, 144, 12 and 232. The first segment would thus be (27,144) to (12,232).

How many distinct true intersection points are found among the 5000 line segments?

Problem 166:

A 4x4 grid is filled with digits d, $0 \leq d \leq 9$.

It can be seen that in the grid

6 3 3 0
5 0 4 3
0 7 1 4
1 2 4 5

the sum of each row and each column has the value 12. Moreover the sum of each diagonal is also 12.

In how many ways can you fill a 4x4 grid with the digits d, $0 \leq d \leq 9$ so that each row, each column, and both diagonals have the same sum?

Problem 167:

For two positive integers a and b, the Ulam sequence U(a,b) is defined by $U(a,b)_1 = a$, $U(a,b)_2 = b$ and for k > 2, $U(a,b)_k$ is the smallest integer greater than $U(a,b)_{(k-1)}$ which can be written in exactly one way as the sum of two distinct previous members of U(a,b).

For example, the sequence U(1,2) begins with
1, 2, 3 = 1 + 2, 4 = 1 + 3, 6 = 2 + 4, 8 = 2 + 6, 11 = 3 + 8;
5 does not belong to it because 5 = 1 + 4 = 2 + 3 has two representations as the sum of two previous members, likewise 7 = 1 + 6 = 3 + 4.

Find $\sum U(2,2n+1)_k$ for $2 \leq n \leq 10$, where $k = 10^{11}$.

Problem 168:

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285.
It can be verified that $714285=5\times142857$.
This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers n, $10 < n < 10^{100}$, that have this property.

Problem 169:

Define f(0)=1 and f(n) to be the number of different ways n can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example, f(10)=5 since there are five different ways to express 10:

1 + 1 + 8
1 + 1 + 4 + 4
1 + 1 + 2 + 2 + 4
2 + 4 + 4
2 + 8

What is f(10^{25})?

Problem 170:

Take the number 6 and multiply it by each of 1273 and 9854:

$6 \times 1273 = 7638$
$6 \times 9854 = 59124$

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

Problem 171:

For a positive integer n, let f(n) be the sum of the squares of the digits (in base 10) of n, e.g.

$f(3) = 3^2 = 9$,
$f(25) = 2^2 + 5^2 = 4 + 25 = 29$,
$f(442) = 4^2 + 4^2 + 2^2 = 16 + 16 + 4 = 36$

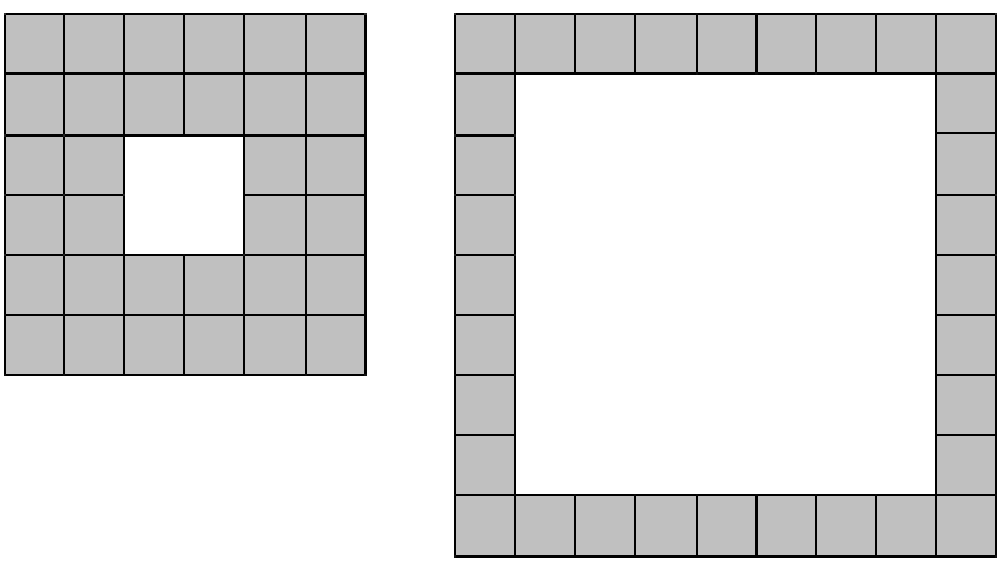
Find the last nine digits of the sum of all n, $0 < n < 10^{20}$, such that f(n) is a perfect square.

Problem 172:

How many 18-digit numbers n (without leading zeros) are there such that no digit occurs more than three times in n?

Problem 173:

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae:



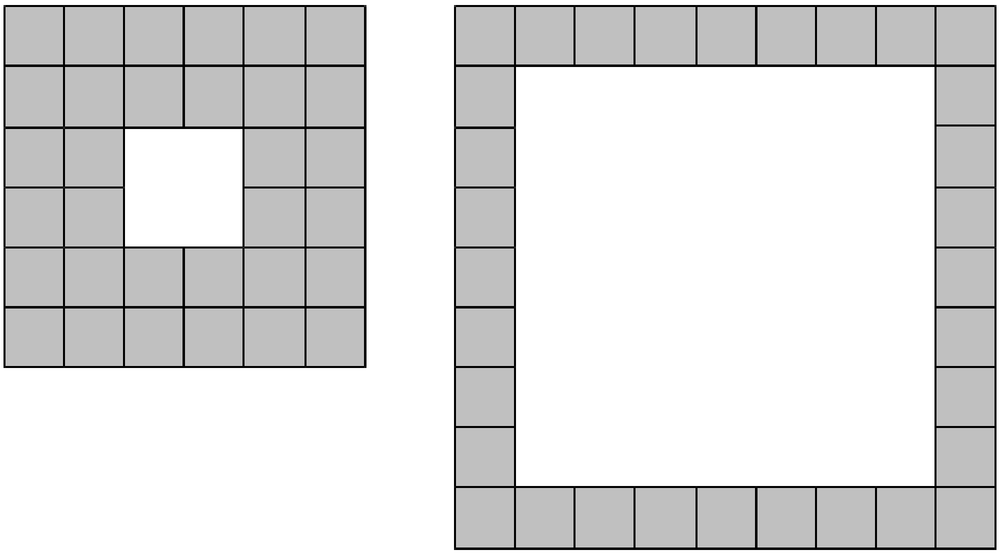
With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae.

Using up to one million tiles how many different square laminae can be formed?

Problem 174:

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry.

Given eight tiles it is possible to form a lamina in only one way: 3x3 square with a 1x1 hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae.



If t represents the number of tiles used, we shall say that t = 8 is type L(1) and t = 32 is type L(2).

Let N(n) be the number of $t \leq$ 1000000 such that t is type L(n); for example, N(15) = 832.

What is $\sum$ N(n) for 1 $\leq n \leq$ 10?

Problem 175:

Define f(0)=1 and f(n) to be the number of ways to write n as a sum of powers of 2 where no power occurs more than twice.

For example, f(10)=5 since there are five different ways to express 10:
10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1

It can be shown that for every fraction p/q (p>0, q>0) there exists at least one integer n such that f(n)/f(n-1)=p/q.

For instance, the smallest n for which f(n)/f(n-1)=13/17 is 241.
The binary expansion of 241 is 11110001.

Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest n for which
$f(n)/f(n\text{-}1)=123456789/987654321$.

Give your answer as comma separated integers, without any whitespaces.

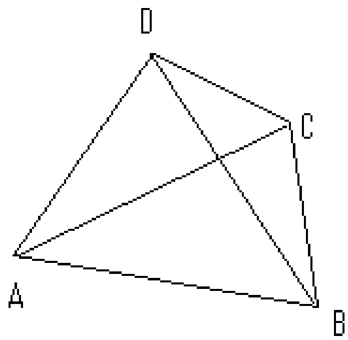
Problem 176:

The four rectangular triangles with sides (9,12,15), (12,16,20), (5,12,13) and (12,35,37) all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided rectangular triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided rectangular triangles.

Problem 177:

Let ABCD be a convex quadrilateral, with diagonals AC and BD. At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.



For example, at vertex A, the two angles are CAD, CAB.

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are 45°. Another example is given by DAC = 20°, BAC = 60°, ABD = 50°, CBD = 30°, BCA = 40°, DCA = 30°, CDB = 80°, ADB = 50°.

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of 10^{-9} of an integer value.

Problem 178:

Consider the number 45656.
It can be seen that each pair of consecutive digits of 45656 has a difference of one.
A number for which every pair of consecutive digits has a difference of one is called a step number.
A pandigital number contains every decimal digit from 0 to 9 at least once.
How many pandigital step numbers less than 10^{40} are there?

Problem 179:

Find the number of integers $1 < n < 10^7$, for which n and $n + 1$ have the same number of positive divisors. For example, 14 has the positive divisors 1, 2, 7, 14 while 15 has 1, 3, 5, 15.

Problem 180:

For any integer n, consider the three functions

$$f_{1,n}(x,y,z) = x^{n+1} + y^{n+1} - z^{n+1}$$
$$f_{2,n}(x,y,z) = (xy + yz + zx)*(x^{n-1} + y^{n-1} - z^{n-1})$$
$$f_{3,n}(x,y,z) = xyz*(x^{n-2} + y^{n-2} - z^{n-2})$$

and their combination

$$f_n(x,y,z) = f_{1,n}(x,y,z) + f_{2,n}(x,y,z) - f_{3,n}(x,y,z)$$

We call (x,y,z) a golden triple of order k if x, y, and z are all rational numbers of the form a / b with $0 < a < b \le k$ and there is (at least) one integer n, so that $f_n(x,y,z) = 0$.

Let $s(x,y,z) = x + y + z$.
Let $t = u / v$ be the sum of all distinct $s(x,y,z)$ for all golden triples (x,y,z) of order 35.
All the $s(x,y,z)$ and t must be in reduced form.

Find $u + v$.

Problem 181:

Having three black objects B and one white object W they can be grouped in 7 ways like this:

(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?

Problem 182:

The RSA encryption is based on the following procedure:

Generate two distinct primes p and q.
Compute $n=pq$ and φ=(p-1)(q-1).
Find an integer e, $1<e<φ$, such that gcd(e,φ)=1.

A message in this system is a number in the interval [0,n-1].
A text to be encrypted is then somehow converted to messages (numbers in the interval [0,n-1]).
To encrypt the text, for each message, m, $c=m^e$ mod n is calculated.

To decrypt the text, the following procedure is needed: calculate d such that ed=1 mod φ, then for each encrypted message, c, calculate $m=c^d$ mod n.

There exist values of e and m such that m^e mod $n=m$.
We call messages m for which m^e mod $n=m$ unconcealed messages.

An issue when choosing e is that there should not be too many unconcealed messages.
For instance, let p=19 and q=37.
Then n=19*37=703 and φ=18*36=648.
If we choose e=181, then, although gcd(181,648)=1 it turns out that all possible messages m ($0 \le m \le n$-1) are unconcealed when calculating m^e mod n.
For any valid choice of e there exist some unconcealed messages.
It's important that the number of unconcealed messages is at a minimum.

Choose p=1009 and q=3643.
Find the sum of all values of e, $1<e<φ$(1009,3643) and gcd(e,φ)=1, so that the number of unconcealed messages for this value of e is at a minimum.

Problem 183:

Let N be a positive integer and let N be split into k equal parts, r = N/k, so that N = $r + r + ... + r$.
Let P be the product of these parts, P = $r \times r \times ... \times r = r^k$.

For example, if 11 is split into five equal parts, 11 = 2.2 + 2.2 + 2.2 + 2.2 + 2.2, then P = 2.2^5 = 51.53632.

Let M(N) = P_{max} for a given value of N.

It turns out that the maximum for N = 11 is found by splitting eleven into four equal parts which leads to $P_{max} = (11/4)^4$; that is, M(11) = 14641/256 = 57.19140625, which is a terminating decimal.

However, for N = 8 the maximum is achieved by splitting it into three equal parts, so M(8) = 512/27, which is a non-terminating decimal.

Let D(N) = N if M(N) is a non-terminating decimal and D(N) = -N if M(N) is a terminating decimal.

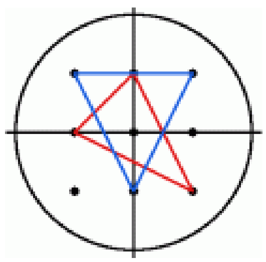
For example, ΣD(N) for 5 $\leq$ N $\leq$ 100 is 2438.

Find ΣD(N) for 5 $\leq$ N $\leq$ 10000.

Problem 184:

Consider the set I_r of points (x,y) with integer co-ordinates in the interior of the circle with radius r, centered at the origin, i.e. $x^2 + y^2 < r^2$.

For a radius of 2, I_2 contains the nine points (0,0), (1,0), (1,1), (0,1), (-1,1), (-1,0), (-1,-1), (0,-1) and (1,-1). There are eight triangles having all three vertices in I_2 which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation.



For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in I_3 and for I_5 the number is 10600.

How many triangles are there containing the origin in the interior and having all three vertices in I_{105}?

Problem 185:

The game Number Mind is a variant of the well known game Master Mind.

Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place.

For instance, given the following guesses for a 5-digit secret sequence,

90342 ;2 correct
70794 ;0 correct
39458 ;2 correct
34109 ;1 correct
51545 ;2 correct
12531 ;1 correct

The correct sequence 39542 is unique.

Based on the following guesses,

5616185650518293 ;2 correct
3847439647293047 ;1 correct
5855462940810587 ;3 correct
9742855507068353 ;3 correct
4296849643607543 ;3 correct
3174248439465858 ;1 correct
4513559094146117 ;2 correct
7890971548908067 ;3 correct
8157356344118483 ;1 correct
2615250744386899 ;2 correct
8690095851526254 ;3 correct
6375711915077050 ;1 correct
6913859173121360 ;1 correct
6442889055042768 ;2 correct
2321386104303845 ;0 correct
2326509471271448 ;2 correct
5251583379644322 ;2 correct
1748270476758276 ;3 correct
4895722652190306 ;1 correct
3041631117224635 ;3 correct
1841236454324589 ;3 correct
2659862637316867 ;2 correct

Find the unique 16-digit secret sequence.

Problem 186:

Here are the records from a busy telephone system with one million users:

RecNr	Caller	Called
1	200007	100053
2	600183	500439
3	600863	701497
...	...	...

The telephone number of the caller and the called number in record n are Caller(n) = S_{2n-1} and Called(n) = S_{2n} where $S_{1,2,3,...}$ come from the "Lagged Fibonacci Generator":

For $1 \leq k \leq 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \leq k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

If Caller(n) = Called(n) then the user is assumed to have misdialled and the call fails; otherwise the call is successful.

From the start of the records, we say that any pair of users X and Y are friends if X calls Y or vice-versa. Similarly, X is a friend of a friend of Z if X is a friend of Y and Y is a friend of Z; and so on for longer chains.

The Prime Minister's phone number is 524287. After how many successful calls, not counting misdials, will 99% of the users (including the PM) be a friend, or a friend of a friend etc., of the Prime Minister?

Problem 187:

A composite is a number containing at least two prime factors. For example, $15 = 3 \times 5$; $9 = 3 \times 3$; $12 = 2 \times 2 \times 3$.

There are ten composites below thirty containing precisely two, not necessarily distinct, prime factors: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26.

How many composite integers, $n < 10^8$, have precisely two, not necessarily distinct, prime factors?

Problem 188:

The *hyperexponentiation* or *tetration* of a number a by a positive integer b, denoted by a↑↑b or $^b a$, is recursively defined by:

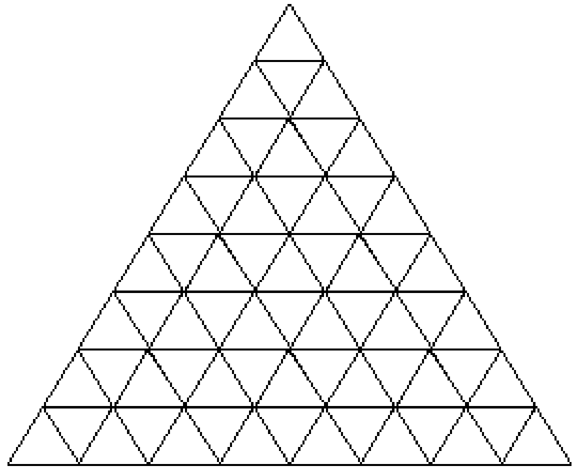
a↑↑1 = a,
a↑↑(k+1) = $a^{(a\uparrow\uparrow k)}$.

Thus we have e.g. 3↑↑2 = 3^3 = 27, hence 3↑↑3 = 3^{27} = 7625597484987 and 3↑↑4 is roughly $10^{3.6383346400240996*10\text{^}12}$.

Find the last 8 digits of 1777↑↑1855.

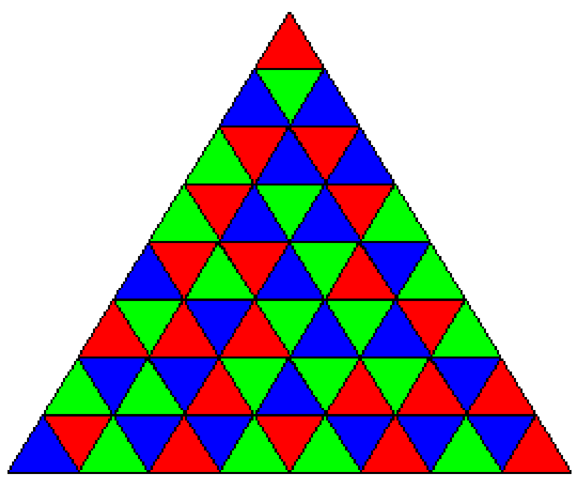
Problem 189:

Consider the following configuration of 64 triangles:



We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge.
Note: if they only share a vertex, then they are not neighbours.

For example, here is a valid colouring of the above grid:



A colouring C' which is obtained from a colouring C by rotation or reflection is considered *distinct* from C unless the two are identical.

How many distinct valid colourings are there for the above configuration?

Problem 190:

Let $S_m = (x_1, x_2, ..., x_m)$ be the m-tuple of positive real numbers with $x_1 + x_2 + ... + x_m = m$ for which $P_m = x_1 * x_2^2 * ... * x_m^m$ is maximised.

For example, it can be verified that $[P_{10}] = 4112$ ([ ] is the integer part function).

Find $\Sigma[P_m]$ for $2 \leq m \leq 15$.

Problem 191:

A particular school offers cash rewards to children with good attendance and punctuality. If they are absent for three consecutive days or late on more than one occasion then they forfeit their prize.

During an n-day period a trinary string is formed for each child consisting of L's (late), O's (on time), and A's (absent).

Although there are eighty-one trinary strings for a 4-day period that can be formed, exactly forty-three strings would lead to a prize:

```
OOOO OOOA OOOL OOAO OOAA OOAL OOLO OOLA OAOO OAOA
OAOL OAAO OAAL OALO OALA OLOO OLOA OLAO OLAA AOOO
AOOA AOOL AOAO AOAA AOAL AOLO AOLA AAOO AAOA AAOL
AALO AALA ALOO ALOA ALAO ALAA LOOO LOOA LOAO LOAA
LAOO LAOA LAAO
```

How many "prize" strings exist over a 30-day period?

Problem 192:

Let x be a real number.
A *best approximation* to x for the *denominator bound* d is a rational number r/s *in reduced form*, with $s \leq d$, such that any rational number which is closer to x than r/s has a denominator larger than d:

$$|p/q-x| < |r/s-x| \Rightarrow q > d$$

For example, the best approximation to $\sqrt{13}$ for the denominator bound 20 is 18/5 and the best approximation to $\sqrt{13}$ for the denominator bound 30 is 101/28.

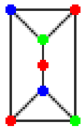
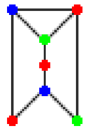
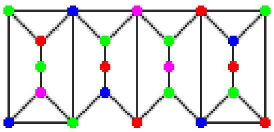
Find the sum of all denominators of the best approximations to $\sqrt{n}$ for the denominator bound 10^{12}, where n is not a perfect square and $1 < n \leq 100000$.

Problem 193:

A positive integer n is called squarefree, if no square of a prime divides n, thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12.

How many squarefree numbers are there below 2^{50}?

Problem 194:

Consider graphs built with the units A:  and B: , where the units are glued along the vertical edges as in the graph .

A configuration of type (a,b,c) is a graph thus built of a units A and b units B, where the graph's vertices are coloured using up to c colours, so that no two adjacent vertices have the same colour.
The compound graph above is an example of a configuration of type (2,2,6), in fact of type (2,2,c) for all $c \geq 4$.

Let N(a,b,c) be the number of configurations of type (a,b,c).
For example, N(1,0,3) = 24, N(0,2,4) = 92928 and N(2,2,3) = 20736.

Find the last 8 digits of N(25,75,1984).

Problem 195:

Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle.
Let r be the radius of the inscribed circle of such a 60-degree triangle.

There are 1234 60-degree triangles for which $r \leq 100$.
Let T(n) be the number of 60-degree triangles for which $r \leq n$, so
T(100) = 1234, T(1000) = 22767, and T(10000) = 359912.

Find T(1053779).

Problem 196:

Build a triangle from all positive integers in the following way:

1
2 3
4 5 6

```
 7  8  9 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45
46 47 48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63 64 65 66
. . .
```

Each positive integer has up to eight neighbours in the triangle.

A set of three primes is called a *prime triplet* if one of the three primes has the other two as neighbours in the triangle.

For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet.

If row 8 is considered, it contains two primes which are elements of some prime triplet, i.e. 29 and 31.
If row 9 is considered, it contains only one prime which is an element of some prime triplet: 37.

Define S(n) as the sum of the primes in row n which are elements of any prime triplet.
Then S(8)=60 and S(9)=37.

You are given that S(10000)=950007619.

Find S(5678027) + S(7208785).

Problem 197:

Given is the function $f(x) = \lfloor 2^{30.403243784-x^2} \rfloor \times 10^{-9}$ ($\lfloor\ \rfloor$ is the floor-function),
the sequence u_n is defined by $u_0 = -1$ and $u_{n+1} = f(u_n)$.

Find $u_n + u_{n+1}$ for $n = 10^{12}$.
Give your answer with 9 digits after the decimal point.

Problem 198:

A best approximation to a real number x for the denominator bound d is a rational number r/s (in reduced form) with $s \leq d$, so that any rational number p/q which is closer to x than r/s has $q > d$.

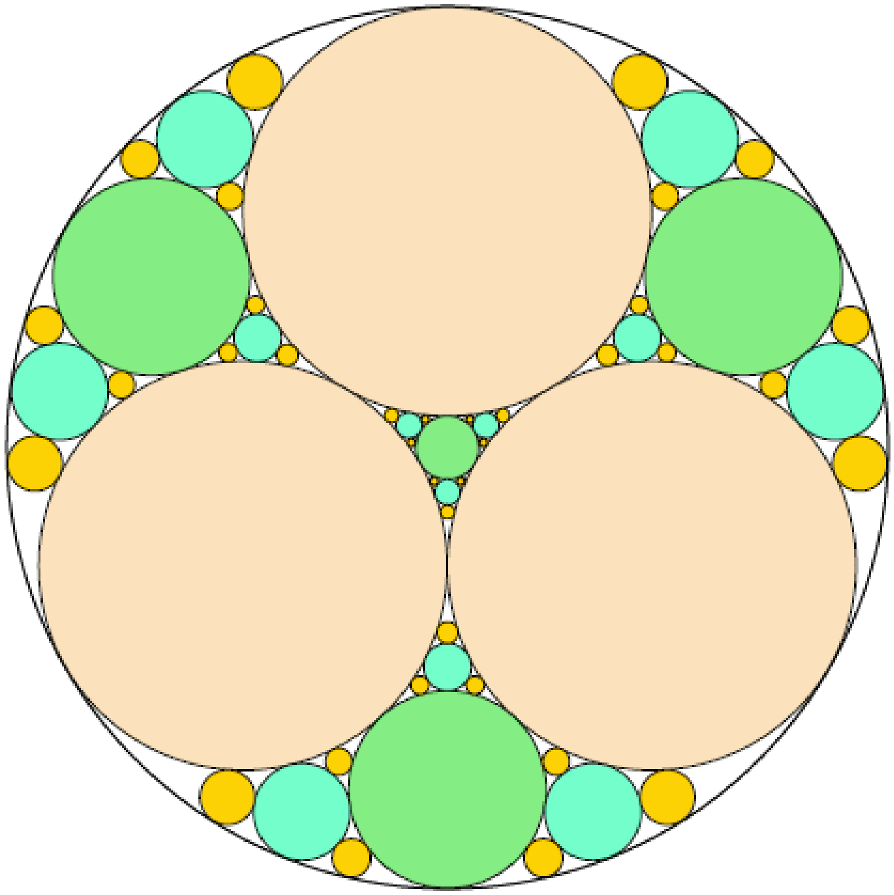
Usually the best approximation to a real number is uniquely determined for all denominator bounds. However, there are some exceptions, e.g. 9/40 has the two best approximations 1/4 and 1/5 for the denominator bound 6. We shall call a real number x *ambiguous*, if there is at least one denominator bound for which x possesses two best approximations. Clearly, an ambiguous number is necessarily rational.

How many ambiguous numbers $x = p/q$, $0 < x < 1/100$, are there whose denominator q does not exceed

10^8?

Problem 199:

Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles.



At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places.

What fraction of the area is not covered by circles after 10 iterations?
Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 200:

We shall define a sqube to be a number of the form, p^2q^3, where p and q are distinct primes.
For example, 200 = $5^2 2^3$ or 120072949 = $23^2 61^3$.

The first five squbes are 72, 108, 200, 392, and 500.

Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof. The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008.

Find the 200th prime-proof sqube containing the contiguous sub-string "200".

Problem 201:

For any set A of numbers, let sum(A) be the sum of the elements of A.
Consider the set B = {1,3,6,8,10,11}.
There are 20 subsets of B containing three elements, and their sums are:

sum({1,3,6}) = 10,
sum({1,3,8}) = 12,
sum({1,3,10}) = 14,
sum({1,3,11}) = 15,
sum({1,6,8}) = 15,
sum({1,6,10}) = 17,
sum({1,6,11}) = 18,
sum({1,8,10}) = 19,
sum({1,8,11}) = 20,
sum({1,10,11}) = 22,
sum({3,6,8}) = 17,
sum({3,6,10}) = 19,
sum({3,6,11}) = 20,
sum({3,8,10}) = 21,
sum({3,8,11}) = 22,
sum({3,10,11}) = 24,
sum({6,8,10}) = 24,
sum({6,8,11}) = 25,
sum({6,10,11}) = 27,
sum({8,10,11}) = 29.

Some of these sums occur more than once, others are unique.
For a set A, let U(A,k) be the set of unique sums of k-element subsets of A, in our example we find U(B,3) = {10,12,14,18,21,25,27,29} and sum(U(B,3)) = 156.

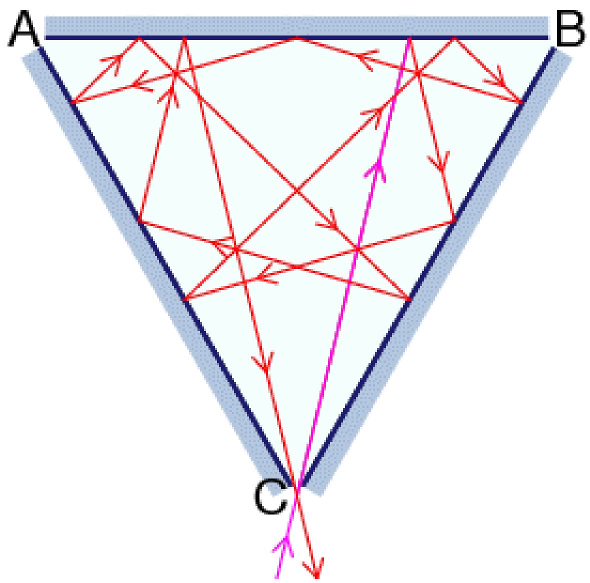
Now consider the 100-element set S = {1^2, 2^2, ... , 100^2}.
S has 100891344545564193334812497256 50-element subsets.

Determine the sum of all integers which are the sum of exactly one of the 50-element subsets of S, i.e. find sum(U(S,50)).

Problem 202:

Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass.

Label the vertices A, B and C. There are 2 ways in which a laser beam may enter vertex C, bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that.



There are 80840 ways in which a laser beam may enter vertex C, bounce off 1000001 surfaces, then exit through the same vertex.

In how many ways can a laser beam enter at vertex C, bounce off 12017639147 surfaces, then exit through the same vertex?

Problem 203:

The binomial coefficients ^nC_k can be arranged in triangular form, Pascal's triangle, like this:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
....

It can be seen that the first eight rows of Pascal's triangle contain twelve distinct numbers:
1, 2, 3, 4, 5, 6, 7, 10, 15, 20, 21 and 35.

A positive integer n is called squarefree if no square of a prime divides n. Of the twelve distinct numbers in the first eight rows of Pascal's triangle, all except 4 and 20 are squarefree. The sum of the distinct squarefree numbers in the first eight rows is 105.

Find the sum of the distinct squarefree numbers in the first 51 rows of Pascal's triangle.

Problem 204:

A Hamming number is a positive number which has no prime factor larger than 5.
So the first few Hamming numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15.
There are 1105 Hamming numbers not exceeding 10^8.

We will call a positive number a generalised Hamming number of type n, if it has no prime factor larger than n.
Hence the Hamming numbers are the generalised Hamming numbers of type 5.

How many generalised Hamming numbers of type 100 are there which don't exceed 10^9?

Problem 205:

Peter has nine four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4.
Colin has six six-sided (cubic) dice, each with faces numbered 1, 2, 3, 4, 5, 6.

Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal.

What is the probability that Pyramidal Pete beats Cubic Colin? Give your answer rounded to seven decimal places in the form 0.abcdefg

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Problem 206:

Find the unique positive integer whose square has the form 1_2_3_4_5_6_7_8_9_0,
where each “_” is a single digit.

Problem 207:

For some positive integers k, there exists an integer partition of the form $4^t = 2^t + k$,
where 4^t, 2^t, and k are all positive integers and t is a real number.

The first two such partitions are $4^1 = 2^1 + 2$ and $4^{1.5849625...} = 2^{1.5849625...} + 6$.

Partitions where t is also an integer are called *perfect*.
For any $m \geq 1$ let P(m) be the proportion of such partitions that are perfect with $k \leq m$.
Thus P(6) = 1/2.

In the following table are listed some values of P(m)

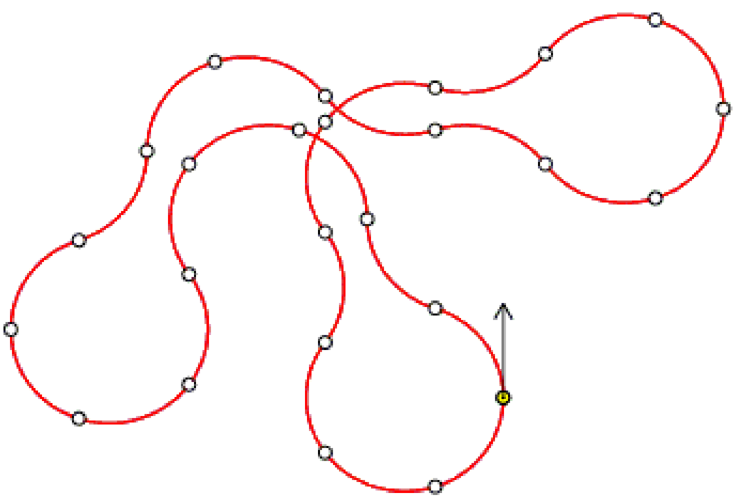
P(5) = 1/1
P(10) = 1/2
P(15) = 2/3
P(20) = 1/2
P(25) = 1/2
P(30) = 2/5
...
P(180) = 1/4
P(185) = 3/13

Find the smallest m for which P(m) $<$ 1/12345

Problem 208:

A robot moves in a series of one-fifth circular arcs (72°), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot.

One of 70932 possible closed paths of 25 arcs starting northward is



Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position?
(Any arc may be traversed multiple times.)

Problem 209:

A k-input *binary truth table* is a map from k input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are:

x	y	x AND y
0	0	0
0	1	0
1	0	0
1	1	1

x	y	x XOR y
0	0	0
0	1	1
1	0	1
1	1	0

How many 6-input binary truth tables, τ, satisfy the formula

$$\tau(a, b, c, d, e, f) \text{ AND } \tau(b, c, d, e, f, a \text{ XOR } (b \text{ AND } c)) = 0$$

for all 6-bit inputs (a, b, c, d, e, f)?

Problem 210:

Consider the set S(r) of points (x,y) with integer coordinates satisfying $|x| + |y| \leq r$.
Let O be the point (0,0) and C the point (r/4,r/4).
Let N(r) be the number of points B in S(r), so that the triangle OBC has an obtuse angle, i.e. the largest angle α satisfies 90°<α<180°.
So, for example, N(4)=24 and N(8)=100.

What is N(1,000,000,000)?

Problem 211:

For a positive integer n, let $\sigma_2(n)$ be the sum of the squares of its divisors. For example,

$$\sigma_2(10) = 1 + 4 + 25 + 100 = 130.$$

Find the sum of all n, $0 < n < 64{,}000{,}000$ such that $\sigma_2(n)$ is a perfect square.

Problem 212:

An *axis-aligned cuboid*, specified by parameters { (x_0,y_0,z_0), (dx,dy,dz) }, consists of all points (X,Y,Z) such that $x_0 \leq X \leq x_0+dx$, $y_0 \leq Y \leq y_0+dy$ and $z_0 \leq Z \leq z_0+dz$. The volume of the cuboid is the product, $dx \times dy \times dz$. The *combined volume* of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap.

Let $C_1,\ldots,C_{50000}$ be a collection of 50000 axis-aligned cuboids such that C_n has parameters

$x_0 = S_{6n-5}$ modulo 10000
$y_0 = S_{6n-4}$ modulo 10000
$z_0 = S_{6n-3}$ modulo 10000

dx = 1 + (S_{6n-2} modulo 399)
dy = 1 + (S_{6n-1} modulo 399)
dz = 1 + (S_{6n} modulo 399)

where $S_1,...,S_{300000}$ come from the "Lagged Fibonacci Generator":

For $1 \le k \le 55$, $S_k = [100003 - 200003k + 300007k^3]$ (modulo 1000000)
For $56 \le k$, $S_k = [S_{k-24} + S_{k-55}]$ (modulo 1000000)

Thus, C_1 has parameters {(7,53,183),(94,369,56)}, C_2 has parameters {(2383,3563,5079),(42,212,344)}, and so on.

The combined volume of the first 100 cuboids, $C_1,...,C_{100}$, is 723581599.

What is the combined volume of all 50000 cuboids, $C_1,...,C_{50000}$?

Problem 213:

A 30×30 grid of squares contains 900 fleas, initially one flea per square.
When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners).

What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.

Problem 214:

Let φ be Euler's totient function, i.e. for a natural number n, φ(n) is the number of k, $1 \le k \le n$, for which gcd(k,n) = 1.

By iterating φ, each positive integer generates a decreasing chain of numbers ending in 1.
E.g. if we start with 5 the sequence 5,4,2,1 is generated.
Here is a listing of all chains with length 4:

5,4,2,1
7,6,2,1
8,4,2,1
9,6,2,1
10,4,2,1
12,4,2,1
14,6,2,1
18,6,2,1

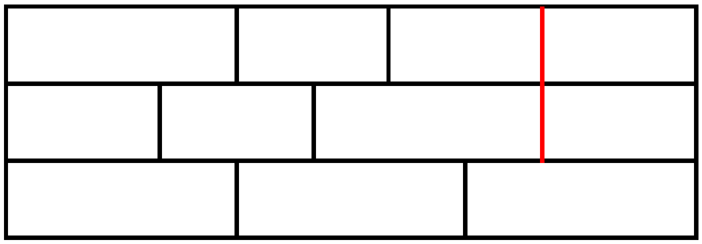
Only two of these chains start with a prime, their sum is 12.

What is the sum of all primes less than 40000000 which generate a chain of length 25?

Problem 215:

Consider the problem of building a wall out of 2×1 and 3×1 bricks (horizontal×vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack".

For example, the following 9×3 wall is not acceptable due to the running crack shown in red:



There are eight ways of forming a crack-free 9×3 wall, written W(9,3) = 8.

Calculate W(32,10).

Problem 216:

Consider numbers $t(n)$ of the form $t(n) = 2n^2-1$ with $n > 1$.
The first such numbers are 7, 17, 31, 49, 71, 97, 127 and 161.
It turns out that only 49 = 7*7 and 161 = 7*23 are not prime.
For $n \leq$ 10000 there are 2202 numbers $t(n)$ that are prime.

How many numbers $t(n)$ are prime for $n \leq$ 50,000,000 ?

Problem 217:

A positive integer with k (decimal) digits is called balanced if its first $\lceil k/2 \rceil$ digits sum to the same value as its last $\lceil k/2 \rceil$ digits, where $\lceil x \rceil$, pronounced *ceiling* of x, is the smallest integer $\geq x$, thus $\lceil \pi \rceil = 4$ and $\lceil 5 \rceil = 5$.

So, for example, all palindromes are balanced, as is 13722.

Let T(n) be the sum of all balanced numbers less than 10^n.
Thus: T(1) = 45, T(2) = 540 and T(5) = 334795890.

Find T(47) mod 3^{15}

Problem 218:

Consider the right angled triangle with sides a=7, b=24 and c=25. The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28.
Moreover it is a primitive right angled triangle as gcd(a,b)=1 and gcd(b,c)=1.
Also c is a perfect square.

We will call a right angled triangle perfect if
-it is a primitive right angled triangle
-its hypotenuse is a perfect square

We will call a right angled triangle super-perfect if
-it is a perfect right angled triangle and
-its area is a multiple of the perfect numbers 6 and 28.

How many perfect right-angled triangles with $c \le 10^{16}$ exist that are not super-perfect?

Problem 219:

Let **A** and **B** be bit strings (sequences of 0's and 1's).
If **A** is equal to the leftmost length(**A**) bits of **B**, then **A** is said to be a *prefix* of **B**.
For example, 00110 is a prefix of 001101001, but not of 00111 or 100110.

A *prefix-free code of size* n is a collection of n distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6:

0000, 0001, 001, 01, 10, 11

Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'.
Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question.
In short, we write Cost(6) = 35.

What is Cost(10^9) ?

Problem 220:

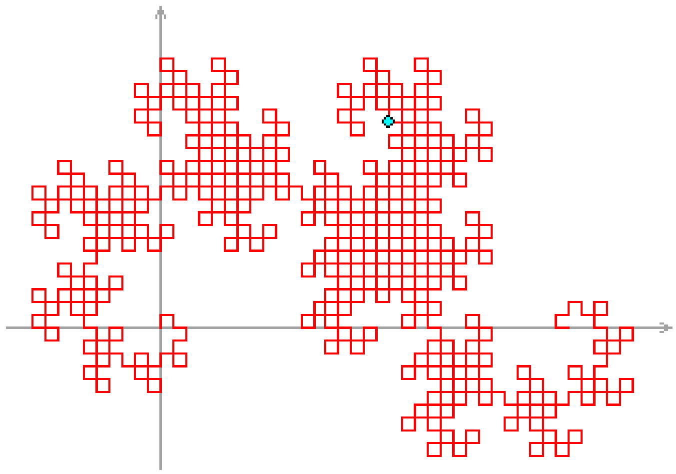
Let $\boldsymbol{D}_0$ be the two-letter string "Fa". For $n \ge 1$, derive $\boldsymbol{D}_n$ from $\boldsymbol{D}_{n-1}$ by the string-rewriting rules:

"a" → "aRbFR"
"b" → "LFaLb"

Thus, $\boldsymbol{D}_0$ = "Fa", $\boldsymbol{D}_1$ = "FaRbFR", $\boldsymbol{D}_2$ = "FaRbFRRLFaLbFR", and so on.

These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1).

Then D_n is an exotic drawing known as the *Heighway Dragon* of order n. For example, D_{10} is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps.



What is the position of the cursor after 10^{12} steps in D_{50} ?
Give your answer in the form *x*,*y* with no spaces.

Problem 221:

We shall call a positive integer A an "Alexandrian integer", if there exist integers p, q, r such that:

$$A = p \cdot q \cdot r \quad \text{and} \quad \frac{1}{A} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

For example, 630 is an Alexandrian integer ($p = 5$, $q = -7$, $r = -18$). In fact, 630 is the 6^{th} Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630.

Find the 150000^{th} Alexandrian integer.

Problem 222:

What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm?

Give your answer in micrometres (10^{-6} m) rounded to the nearest integer.

Problem 223:

Let us call an integer sided triangle with sides $a \leq b \leq c$ *barely acute* if the sides satisfy

$a^2 + b^2 = c^2 + 1$.

How many barely acute triangles are there with perimeter $\leq$ 25,000,000?

Problem 224:

Let us call an integer sided triangle with sides $a \leq b \leq c$ *barely obtuse* if the sides satisfy
$a^2 + b^2 = c^2 - 1$.

How many barely obtuse triangles are there with perimeter $\leq$ 75,000,000?

Problem 225:

The sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201 ...
is defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$.

It can be shown that 27 does not divide any terms of this sequence.
In fact, 27 is the first odd number with this property.

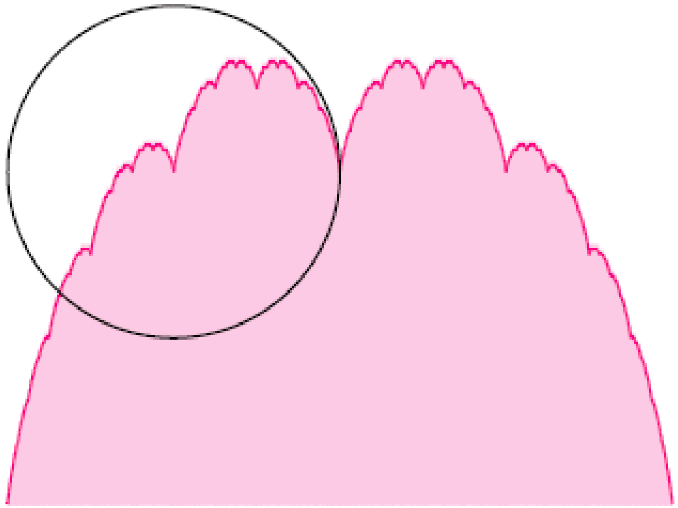
Find the 124^{th} odd number that does not divide any terms of the above sequence.

Problem 226:

The *blancmange curve* is the set of points (x,y) such that $0 \leq x \leq 1$ and $y = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$,

where $s(x)$ = the distance from x to the nearest integer.

The area under the blancmange curve is equal to ½, shown in pink in the diagram below.



Let C be the circle with centre (¼,½) and radius ¼, shown in black in the diagram.

What area under the blancmange curve is enclosed by C?
Give your answer rounded to eight decimal places in the form *0.abcdefgh*

Problem 227:

"*The Chase*" is a game played with two dice and an even number of players.

The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it.
If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn.
The game ends when one player has both dice after they have been rolled and passed; that player has then lost.

In a game with 100 players, what is the expected number of turns the game lasts?

Give your answer rounded to ten significant digits.

Problem 228:

Let S_n be the regular n-sided polygon – or *shape* – whose vertices v_k ($k = 1,2,\ldots,n$) have coordinates:

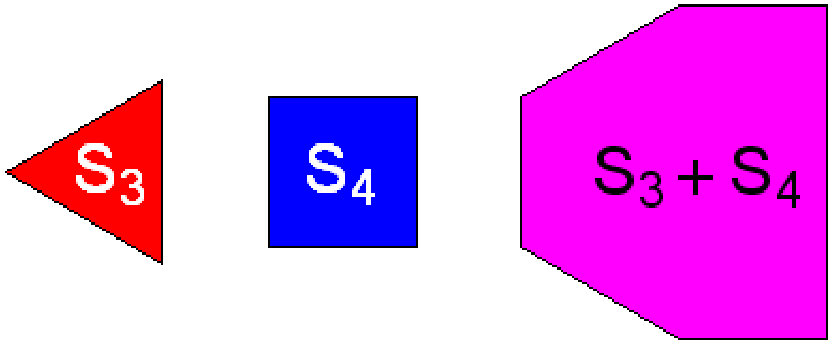
$$x_k = \cos\left({}^{2k-1}/_{n} \times 180° \right)$$

$$y_k = \sin\left({}^{2k-1}/_{n} \times 180° \right)$$

Each S_n is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior.

The *Minkowski sum*, $S+T$, of two shapes S and T is the result of adding every point in S to every point in T, where point addition is performed coordinate-wise: $(u, v) + (x, y) = (u+x, v+y)$.

For example, the sum of S_3 and S_4 is the six-sided shape shown in pink below:



How many sides does $S_{1864} + S_{1865} + \ldots + S_{1909}$ have?

Problem 229:

Consider the number 3600. It is very special, because

$$3600 = 48^2 + \quad 36^2$$

$$3600 = 20^2 + 2\times40^2$$

$$3600 = 30^2 + 3\times30^2$$

$$3600 = 45^2 + 7\times15^2$$

Similarly, we find that $88201 = 99^2 + 280^2 = 287^2 + 2\times54^2 = 283^2 + 3\times52^2 = 197^2 + 7\times84^2$.

In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers n which admit representations of all of the following four types:

$$n = a_1^2 + \quad b_1^2$$

$$n = a_2^2 + 2\ b_2^2$$

$$n = a_3^2 + 3\ b_3^2$$

$$n = a_7^2 + 7\ b_7^2,$$

where the a_k and b_k are positive integers.

There are 75373 such numbers that do not exceed 10^7.
How many such numbers are there that do not exceed 2×10^9?

Problem 230:

For any two strings of digits, A and B, we define $F_{A,B}$ to be the sequence (A,B,AB,BAB,ABBAB,...) in which each term is the concatenation of the previous two.

Further, we define $D_{A,B}(n)$ to be the n^{th} digit in the first term of $F_{A,B}$ that contains at least n digits.

Example:

Let A=1415926535, B=8979323846. We wish to find $D_{A,B}(35)$, say.

The first few terms of $F_{A,B}$ are:
1415926535
8979323846
14159265358979323846
897932384614159265358979323846
14159265358979323846897932384614159265358979323846

Then $D_{A,B}(35)$ is the 35^{th} digit in the fifth term, which is 9.

Now we use for A the first 100 digits of π behind the decimal point:

14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679

and for B the next hundred digits:

82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196 .

Find $\sum_{n=0,1,...,17} 10^n\times D_{A,B}((127+19n)\times7^n)$.

Problem 231:

The binomial coefficient ${}^{10}C_3 = 120$.
$120 = 2^3 \times 3 \times 5 = 2 \times 2 \times 2 \times 3 \times 5$, and 2 + 2 + 2 + 3 + 5 = 14.
So the sum of the terms in the prime factorisation of ${}^{10}C_3$ is 14.

Find the sum of the terms in the prime factorisation of ${}^{20000000}C_{15000000}$.

Problem 232:

Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer T and tosses the coin T times: if it comes up all Heads, she scores 2^{T-1} points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points.

On each turn Player 2 selects the number, T, of coin tosses that maximises the probability of her winning.

What is the probability that Player 2 wins?

Give your answer rounded to eight decimal places in the form 0.*abcdefgh* .

Problem 233:

Let $f(N)$ be the number of points with integer coordinates that are on a circle passing through (0,0), (N,0), (0,N), and (N,N).

It can be shown that $f(10000) = 36$.

What is the sum of all positive integers $N \le 10^{11}$ such that $f(N) = 420$?

Problem 234:

For an integer $n \ge 4$, we define the *lower prime square root* of n, denoted by lps(n), as the largest prime $\le \sqrt{n}$ and the *upper prime square root* of n, ups(n), as the smallest prime $\ge \sqrt{n}$.

So, for example, lps(4) = 2 = ups(4), lps(1000) = 31, ups(1000) = 37.
Let us call an integer $n \ge 4$ *semidivisible*, if one of lps(n) and ups(n) divides n, but not both.

The sum of the semidivisible numbers not exceeding 15 is 30, the numbers are 8, 10 and 12.
15 is not semidivisible because it is a multiple of both lps(15) = 3 and ups(15) = 5.
As a further example, the sum of the 92 semidivisible numbers up to 1000 is 34825.

What is the sum of all semidivisible numbers not exceeding 999966663333 ?

Problem 235:

Given is the arithmetic-geometric sequence $u(k) = (900-3k)r^{k-1}$.
Let $s(n) = \Sigma_{k=1...n} u(k)$.

Find the value of r for which s(5000) = -600,000,000,000.

Give your answer rounded to 12 places behind the decimal point.

Problem 236:

Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market:

Product	'A'	'B'
Beluga Caviar	5248	640
Christmas Cake	1312	1888
Gammon Joint	2624	3776
Vintage Port	5760	3776
Champagne Truffles	3936	5664

Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage - *i.e.* products gone bad.

The suppliers compare their performance using two types of statistic:

- The five *per-product spoilage rates* for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn.
- The *overall spoilage rate* for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that supplier.

To their surprise, the suppliers found that each of the five per-product spoilage rates was worse (higher) for 'B' than for 'A' by the same factor (ratio of spoilage rates), $m>1$; and yet, paradoxically, the overall spoilage rate was worse for 'A' than for 'B', also by a factor of m.

There are thirty-five $m>1$ for which this surprising result could have occurred, the smallest of which is 1476/1475.

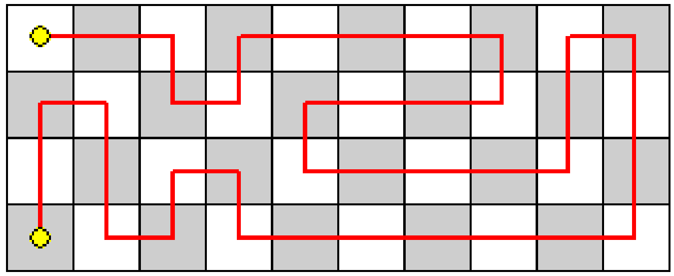
What's the largest possible value of m?
Give your answer as a fraction reduced to its lowest terms, in the form u/v.

Problem 237:

Let $T(n)$ be the number of tours over a $4 \times n$ playing board such that:

- The tour starts in the top left corner.
- The tour consists of moves that are up, down, left, or right one square.
- The tour visits each square exactly once.
- The tour ends in the bottom left corner.

The diagram shows one tour over a 4×10 board:



T(10) is 2329. What is T(10^{12}) modulo 10^8?

Problem 238:

Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator:

$$s_0 = 14025256$$
$$s_{n+1} = s_n^2 \bmod 20300713$$

Concatenate these numbers $s_0 s_1 s_2 \ldots$ to create a string w of infinite length.
Then, w = 14025256741014958470038053646…

For a positive integer k, if no substring of w exists with a sum of digits equal to k, $p(k)$ is defined to be zero. If at least one substring of w exists with a sum of digits equal to k, we define $p(k) = z$, where z is the starting position of the earliest such substring.

For instance:

The substrings 1, 14, 1402, …
with respective sums of digits equal to 1, 5, 7, …
start at position **1**, hence $p(1) = p(5) = p(7) = \ldots =$ **1**.

The substrings 4, 402, 4025, …
with respective sums of digits equal to 4, 6, 11, …
start at position **2**, hence $p(4) = p(6) = p(11) = \ldots =$ **2**.

The substrings 02, 0252, …
with respective sums of digits equal to 2, 9, …
start at position **3**, hence $p(2) = p(9) = \ldots =$ **3**.

Note that substring 025 starting at position **3**, has a sum of digits equal to 7, but there was an earlier substring (starting at position **1**) with a sum of digits equal to 7, so $p(7) = 1$, *not* 3.

We can verify that, for $0 < k \leq 10^3$, $\sum p(k) = 4742$.

Find $\sum p(k)$, for $0 < k \leq 2 \cdot 10^{15}$.

Problem 239:

A set of disks numbered 1 through 100 are placed in a line in random order.

What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions?
(Any number of non-prime disks may also be found in or out of their natural positions.)

Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

Problem 240:

There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are:

D_1,D_2,D_3,D_4,D_5 = 4,3,6,3,5
D_1,D_2,D_3,D_4,D_5 = 4,3,3,5,6
D_1,D_2,D_3,D_4,D_5 = 3,3,3,6,6
D_1,D_2,D_3,D_4,D_5 = 6,6,3,3,3

In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?

Problem 241:

For a positive integer n, let $\sigma(n)$ be the sum of all divisors of n, so e.g. $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

A perfect number, as you probably know, is a number with $\sigma(n) = 2n$.

Let us define the **perfection quotient** of a positive integer as $p(n) = \dfrac{\sigma(n)}{n}$.

Find the sum of all positive integers $n \le 10^{18}$ for which $p(n)$ has the form $k + \frac{1}{2}$, where k is an integer.

Problem 242:

Given the set $\{1,2,...,n\}$, we define $f(n,k)$ as the number of its k-element subsets with an odd sum of elements. For example, $f(5,3) = 4$, since the set $\{1,2,3,4,5\}$ has four 3-element subsets having an odd sum of elements, i.e.: $\{1,2,4\}$, $\{1,3,5\}$, $\{2,3,4\}$ and $\{2,4,5\}$.

When all three values n, k and $f(n,k)$ are odd, we say that they make an *odd-triplet* $[n,k,f(n,k)]$.

There are exactly five odd-triplets with $n \leq 10$, namely:
[1,1,f(1,1) = 1], [5,1,f(5,1) = 3], [5,5,f(5,5) = 1], [9,1,f(9,1) = 5] and [9,9,f(9,9) = 1].

How many odd-triplets are there with $n \leq 10^{12}$?

Problem 243:

A positive fraction whose numerator is less than its denominator is called a proper fraction.
For any denominator, d, there will be $d-1$ proper fractions; for example, with $d = 12$:
$^1/_{12}$, $^2/_{12}$, $^3/_{12}$, $^4/_{12}$, $^5/_{12}$, $^6/_{12}$, $^7/_{12}$, $^8/_{12}$, $^9/_{12}$, $^{10}/_{12}$, $^{11}/_{12}$.

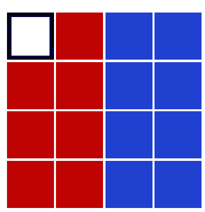
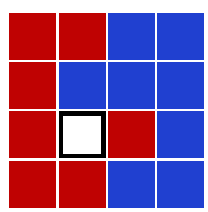
We shall call a fraction that cannot be cancelled down a *resilient fraction*.
Furthermore we shall define the *resilience* of a denominator, $R(d)$, to be the ratio of its proper fractions that are resilient; for example, $R(12) = {}^4/_{11}$.
In fact, $d = 12$ is the smallest denominator having a resilience $R(d) < {}^4/_{10}$.

Find the smallest denominator d, having a resilience $R(d) < {}^{15499}/_{94744}$.

Problem 244:

You probably know the game *Fifteen Puzzle*. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles.

A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (**S**), by the sequence **LULUR** we reach the configuration (**E**):

(**S**)  , (**E**) 

For each path, its checksum is calculated by (pseudocode):

checksum = 0
checksum = (checksum $\times$ 243 + m_1) mod 100 000 007
checksum = (checksum $\times$ 243 + m_2) mod 100 000 007
…
checksum = (checksum $\times$ 243 + m_n) mod 100 000 007

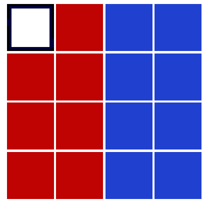
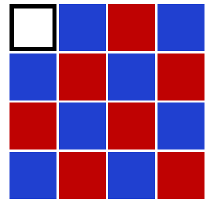
where m_k is the ASCII value of the k^{th} letter in the move sequence and the ASCII values for the moves are:

L	76
R	82
U	85

D	68

For the sequence **LULUR** given above, the checksum would be 19761398.

Now, starting from configuration (**S**), find all shortest ways to reach configuration (**T**).

(**S**)  , (**T**) 

What is the sum of all checksums for the paths having the minimal length?

Problem 245:

We shall call a fraction that cannot be cancelled down a resilient fraction.
Furthermore we shall define the resilience of a denominator, $R(d)$, to be the ratio of its proper fractions that are resilient; for example, $R(12) = {}^4\!/_{11}$.

The resilience of a number $d > 1$ is then $\dfrac{\varphi(d)}{d - 1}$, where φ is Euler's totient function.

We further define the **coresilience** of a number $n > 1$ as $C(n) = \dfrac{n - \varphi(n)}{n - 1}$.

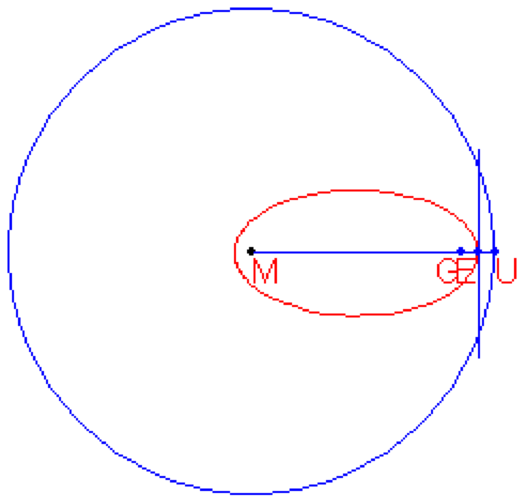
The coresilience of a prime p is $C(p) = \dfrac{1}{p - 1}$.

Find the sum of all **composite** integers $1 < n \le 2\times 10^{11}$, for which $C(n)$ is a unit fraction.

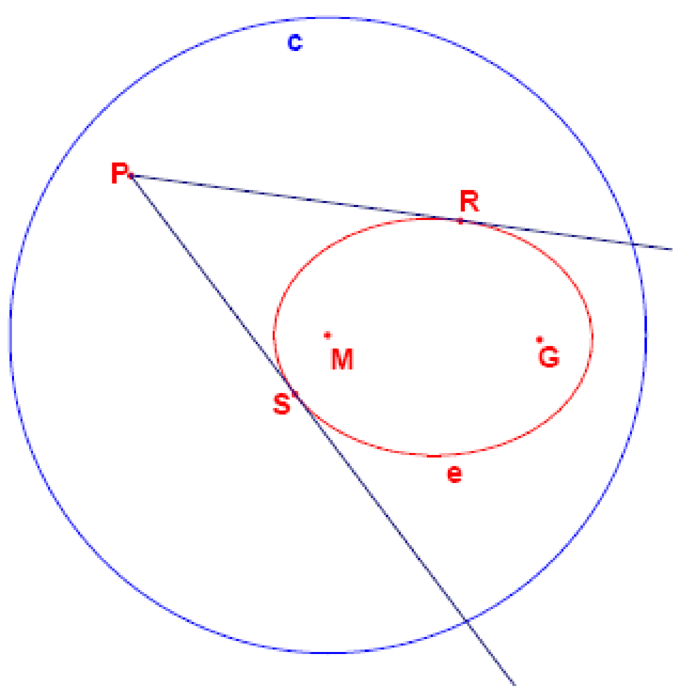
Problem 246:

A definition for an ellipse is:
Given a circle c with centre M and radius r and a point G such that d(G,M)<r, the locus of the points that are equidistant from c and G form an ellipse.

The construction of the points of the ellipse is shown below.



Given are the points M(-2000,1500) and G(8000,1500).
Given is also the circle c with centre M and radius 15000.
The locus of the points that are equidistant from G and c form an ellipse e.
From a point P outside e the two tangents t_1 and t_2 to the ellipse are drawn.
Let the points where t_1 and t_2 touch the ellipse be R and S.

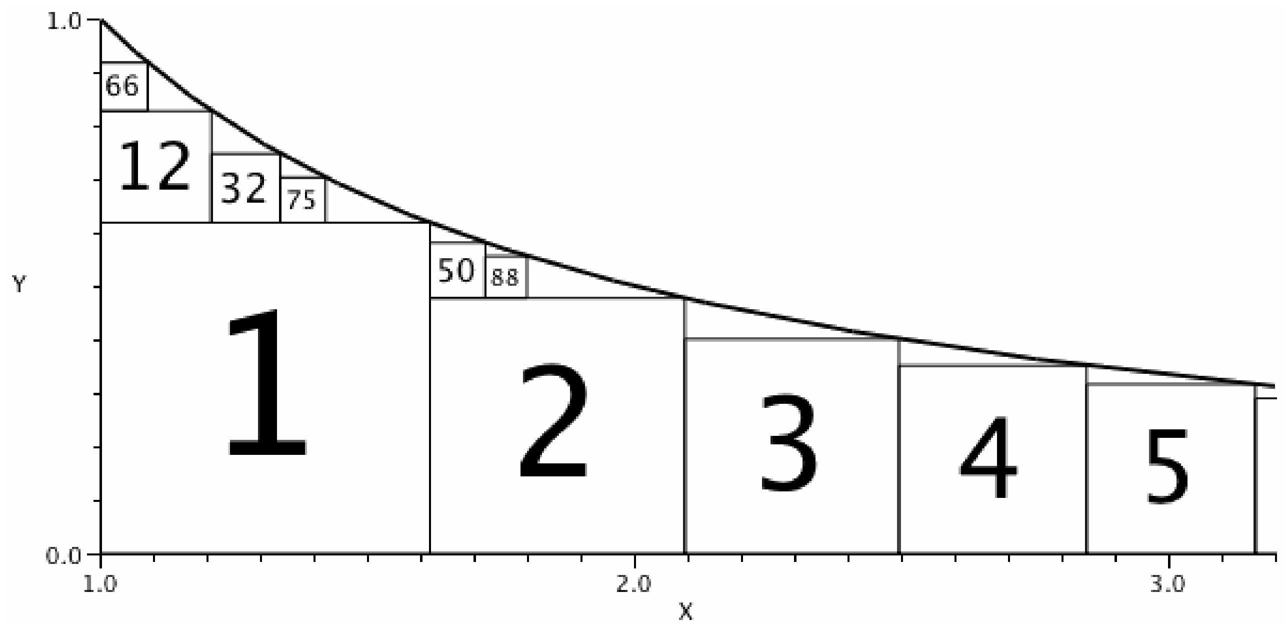


For how many lattice points P is angle RPS greater than 45 degrees?

Problem 247:

Consider the region constrained by $1 \le x$ and $0 \le y \le {}^1/_x$.

Let S_1 be the largest square that can fit under the curve.
Let S_2 be the largest square that fits in the remaining area, and so on.
Let the *index* of S_n be the pair (left, below) indicating the number of squares to the left of S_n and the number of squares below S_n.



The diagram shows some such squares labelled by number.
S_2 has one square to its left and none below, so the index of S_2 is (1,0).
It can be seen that the index of S_{32} is (1,1) as is the index of S_{50}.
50 is the largest n for which the index of S_n is (1,1).

What is the largest n for which the index of S_n is (3,3)?

Problem 248:

The first number n for which φ(n)=13! is 6227180929.

Find the 150,000th such number.

Problem 249:

Let S = {2, 3, 5, ..., 4999} be the set of prime numbers less than 5000.

Find the number of subsets of S, the sum of whose elements is a prime number.

Enter the rightmost 16 digits as your answer.

Problem 250:

Find the number of non-empty subsets of $\{1^1, 2^2, 3^3, ..., 250250^{250250}\}$, the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

Problem 251:

A triplet of positive integers (a,b,c) is called a Cardano Triplet if it satisfies the condition:

$$\sqrt[3]{a+b\sqrt{c}}+\sqrt[3]{a-b\sqrt{c}}=1$$

For example, (2,1,5) is a Cardano Triplet.

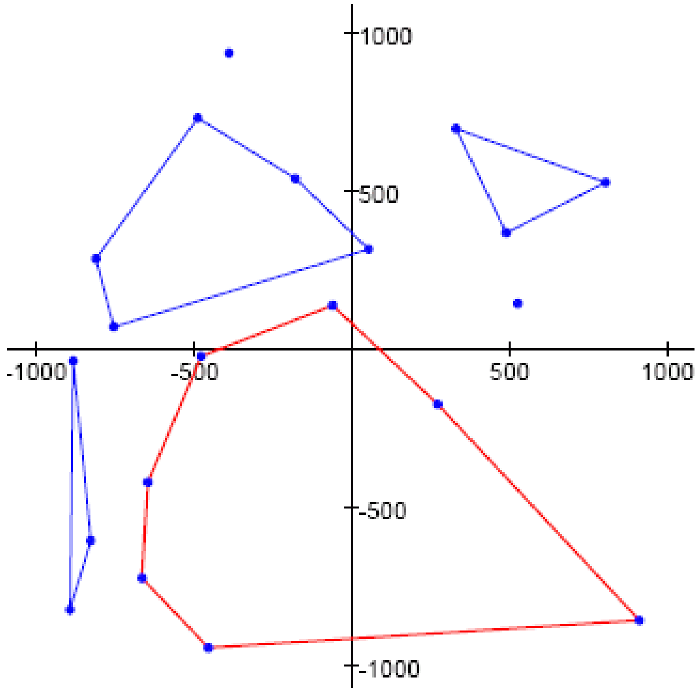
There exist 149 Cardano Triplets for which $a+b+c \le 1000$.

Find how many Cardano Triplets exist such that $a+b+c \le 110{,}000{,}000$.

Problem 252:

Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon).

As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points.



For our example, we used the first 20 points (T_{2k-1}, T_{2k}), for $k = 1,2,\ldots,20$, produced with the pseudo-random number generator:

$$S_0 = 290797$$
$$S_{n+1} = S_n^2 \bmod 50515093$$
$$T_n = (S_n \bmod 2000) - 1000$$

i.e. (527, 144), (−488, 732), (−454, −947), …

What is the maximum area for a convex hole on the set containing the first 500 points in the pseudo-random sequence?
Specify your answer including one digit after the decimal point.

Problem 253:

A small child has a “number caterpillar” consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order.

Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order.
As the caterpillar is built up in this way, it forms distinct segments that gradually merge together.
The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed).

For example:

Piece Placed	Segments So Far

12	1
4	2
29	3
6	4
34	5
5	4
35	4
…	…

Let M be the maximum number of segments encountered during a random tidy-up of the caterpillar.
For a caterpillar of ten pieces, the number of possibilities for each M is

M	**Possibilities**
1	512
2	250912
3	1815264
4	1418112
5	144000

so the most likely value of M is 3 and the average value is $^{385643}/_{113400}$ = 3.400732, rounded to six decimal places.

The most likely value of M for a forty-piece caterpillar is 11; but what is the average value of M?

Give your answer rounded to six decimal places.

Problem 254:

Define f(n) as the sum of the factorials of the digits of n. For example, f(342) = 3! + 4! + 2! = 32.

Define sf(n) as the sum of the digits of f(n). So sf(342) = 3 + 2 = 5.

Define g(i) to be the smallest positive integer n such that sf(n) = i. Though sf(342) is 5, sf(25) is also 5, and it can be verified that g(5) is 25.

Define sg(i) as the sum of the digits of g(i). So sg(5) = 2 + 5 = 7.

Further, it can be verified that g(20) is 267 and $\sum$ sg(i) for 1 $\leq i \leq$ 20 is 156.

What is $\sum$ sg(i) for 1 $\leq i \leq$ 150?

Problem 255:

We define the *rounded-square-root* of a positive integer n as the square root of n rounded to the nearest integer.

The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of n:

Let d be the number of digits of the number n.
If d is odd, set $x_0 = 2\times10^{(d-1)/2}$.
If d is even, set $x_0 = 7\times10^{(d-2)/2}$.
Repeat:

$$x_{k+1} = \left\lfloor \frac{x_k + \lceil n/x_k \rceil}{2} \right\rfloor$$

until $x_{k+1} = x_k$.

As an example, let us find the rounded-square-root of $n = 4321$.
n has 4 digits, so $x_0 = 7\times10^{(4-2)/2} = 70$.

$$x_1 = \left\lfloor \frac{70 + \lceil 4321/70 \rceil}{2} \right\rfloor = 66.$$

$$x_2 = \left\lfloor \frac{66 + \lceil 4321/66 \rceil}{2} \right\rfloor = 66.$$

Since $x_2 = x_1$, we stop here.
So, after just two iterations, we have found that the rounded-square-root of 4321 is 66 (the actual square root is 65.7343137…).

The number of iterations required when using this method is surprisingly low.
For example, we can find the rounded-square-root of a 5-digit integer ($10{,}000 \le n \le 99{,}999$) with an average of 3.2102888889 iterations (the average value was rounded to 10 decimal places).

Using the procedure described above, what is the average number of iterations required to find the rounded-square-root of a 14-digit number ($10^{13} \le n < 10^{14}$)?
Give your answer rounded to 10 decimal places.

Note: The symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ represent the floor function and ceiling function respectively.

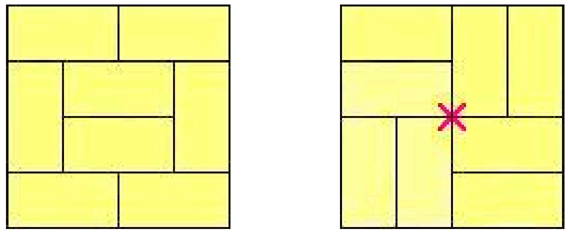
Problem 256:

Tatami are rectangular mats, used to completely cover the floor of a room, without overlap.

Assuming that the only type of available tatami has dimensions 1×2, there are obviously some limitations for the shape and size of the rooms that can be covered.

For this problem, we consider only rectangular rooms with integer dimensions a, b and even size $s = a \cdot b$.
We use the term 'size' to denote the floor surface area of the room, and — without loss of generality — we add the condition $a \leq b$.

There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet.
For example, consider the two arrangements below for a 4×4 room:



The arrangement on the left is acceptable, whereas the one on the right is not: a red "**X**" in the middle, marks the point where four tatami meet.

Because of this rule, certain even-sized rooms cannot be covered with tatami: we call them tatami-free rooms.
Further, we define $T(s)$ as the number of tatami-free rooms of size s.

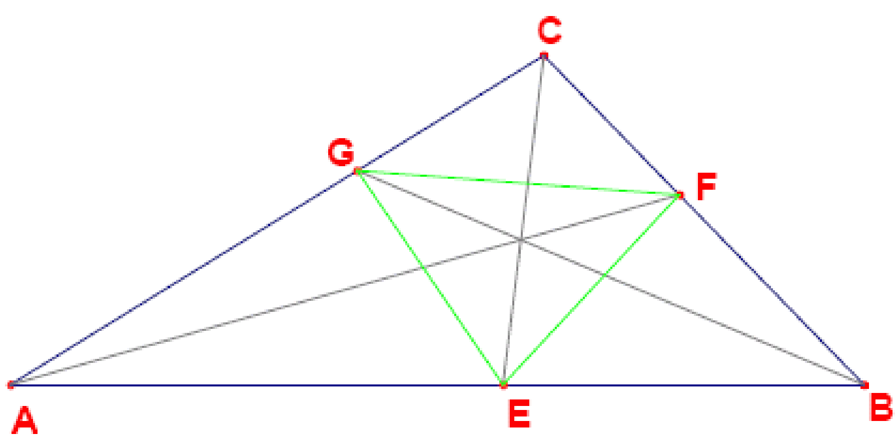
The smallest tatami-free room has size $s = 70$ and dimensions 7×10.
All the other rooms of size $s = 70$ can be covered with tatami; they are: 1×70, 2×35 and 5×14.
Hence, $T(70) = 1$.

Similarly, we can verify that $T(1320) = 5$ because there are exactly 5 tatami-free rooms of size $s = 1320$:
20×66, 22×60, 24×55, 30×44 and 33×40.
In fact, $s = 1320$ is the smallest room-size s for which $T(s) = 5$.

Find the smallest room-size s for which $T(s) = 200$.

Problem 257:

Given is an integer sided triangle ABC with sides a ≤ b ≤ c. (AB = c, BC = a and AC = b).
The angular bisectors of the triangle intersect the sides at points E, F and G (see picture below).



The segments EF, EG and FG partition the triangle ABC into four smaller triangles: AEG, BFE, CGF and EFG.
It can be proven that for each of these four triangles the ratio area(ABC)/area(subtriangle) is rational. However, there exist triangles for which some or all of these ratios are integral.

How many triangles ABC with perimeter≤100,000,000 exist so that the ratio area(ABC)/area(AEG) is integral?

Problem 258:

A sequence is defined as:

- g_k = 1, for 0 ≤ k ≤ 1999
- $g_k = g_{k\text{-}2000} + g_{k\text{-}1999}$, for k ≥ 2000.

Find g_k mod 20092010 for $k = 10^{18}$.

Problem 259:

A positive integer will be called *reachable* if it can result from an arithmetic expression obeying the following rules:

- Uses the digits 1 through 9, in that order and exactly once each.
- Any successive digits can be concatenated (for example, using the digits 2, 3 and 4 we obtain the number 234).
- Only the four usual binary arithmetic operations (addition, subtraction, multiplication and division) are allowed.
- Each operation can be used any number of times, or not at all.
- Unary minus is not allowed.
- Any number of (possibly nested) parentheses may be used to define the order of operations.

For example, 42 is reachable, since (1/23) * ((4*5)-6) * (78-9) = 42.

What is the sum of all positive reachable integers?

Problem 260:

A game is played with three piles of stones and two players.
At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles.

In other words, the player chooses some N>0 and removes:

- N stones from any single pile; or
- N stones from each of any two piles (2N total); or

- N stones from each of the three piles (3N total).

The player taking the last stone(s) wins the game.

A *winning configuration* is one where the first player can force a win.
For example, (0,0,13), (0,11,11) and (5,5,5) are winning configurations because the first player can immediately remove all stones.

A *losing configuration* is one where the second player can force a win, no matter what the first player does.
For example, (0,1,2) and (1,3,3) are losing configurations: any legal move leaves a winning configuration for the second player.

Consider all losing configurations (x_i,y_i,z_i) where $x_i \le y_i \le z_i \le 100$.
We can verify that $\Sigma(x_i+y_i+z_i) = 173895$ for these.

Find $\Sigma(x_i+y_i+z_i)$ where (x_i,y_i,z_i) ranges over the losing configurations
with $x_i \le y_i \le z_i \le 1000$.

Problem 261:

Let us call a positive integer k a *square-pivot*, if there is a pair of integers $m > 0$ and $n \ge k$, such that the sum of the $(m+1)$ consecutive squares up to k equals the sum of the m consecutive squares from $(n+1)$ on:

$$(k-m)^2 + ... + k^2 = (n+1)^2 + ... + (n+m)^2.$$

Some small square-pivots are

- **4**: $3^2 + \mathbf{4}^2 = 5^2$
- **21**: $20^2 + \mathbf{21}^2 = 29^2$
- **24**: $21^2 + 22^2 + 23^2 + \mathbf{24}^2 = 25^2 + 26^2 + 27^2$
- **110**: $108^2 + 109^2 + \mathbf{110}^2 = 133^2 + 134^2$

Find the sum of all **distinct** square-pivots $\le 10^{10}$.

Problem 262:

The following equation represents the *continuous* topography of a mountainous region, giving the elevation h at any point (x,y):

$$h = \left(5000 - \frac{x^2 + y^2 + xy}{200} + \frac{25(x+y)}{2}\right) \cdot e^{-\left|\frac{x^2+y^2}{1000000} - \frac{3(x+y)}{2000} + \frac{7}{10}\right|}$$

A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by

$0 \le x, y \le 1600$.

Because of the intervening mountains, it first rises straight up to a point A', having elevation f. Then, while remaining at the same elevation f, it flies around any obstacles until it arrives at a point B' directly above B.

First, determine f_{min} which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area.
Then, find the length of the shortest path between A' and B', while flying at that constant elevation f_{min}.

Give that length as your answer, rounded to three decimal places.

Note: For convenience, the elevation function shown above is repeated below, in a form suitable for most programming languages:
h=(5000-0.005*(x*x+y*y+x*y)+12.5*(x+y)) * exp(-abs(0.000001*(x*x+y*y)-0.0015*(x+y)+0.7))

Problem 263:

Consider the number 6. The divisors of 6 are: 1,2,3 and 6.
Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6:
1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6.
A number n is called a practical number if every number from 1 up to and including n can be expressed as a sum of distinct divisors of n.

A pair of consecutive prime numbers with a difference of six is called a sexy pair (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29).

We may occasionally find a triple-pair, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair.

We shall call a number n such that :

- (n-9, n-3), (n-3,n+3), (n+3, n+9) form a triple-pair, and
- the numbers n-8, n-4, n, n+4 and n+8 are all practical,

an engineers’ paradise.

Find the sum of the first four engineers’ paradises.

Problem 264:

Consider all the triangles having:

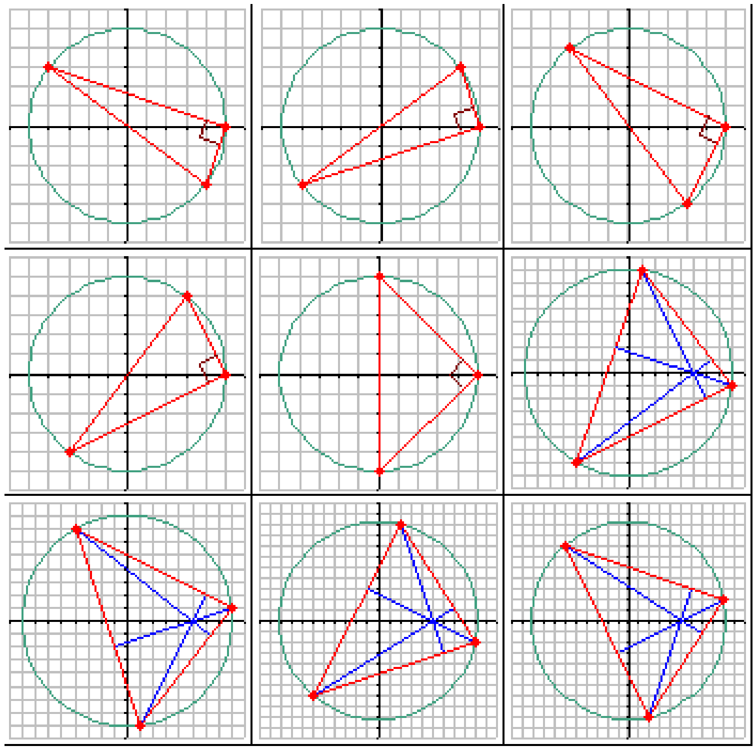
- All their vertices on lattice points.
- Circumcentre at the origin O.
- Orthocentre at the point H(5, 0).

There are nine such triangles having a perimeter ≤ 50.
Listed and shown in ascending order of their perimeter, they are:

A(-4, 3), B(5, 0), C(4, -3)
A(4, 3), B(5, 0), C(-4, -3)
A(-3, 4), B(5, 0), C(3, -4)

A(3, 4), B(5, 0), C(-3, -4)
A(0, 5), B(5, 0), C(0, -5)
A(1, 8), B(8, -1), C(-4, -7)

A(8, 1), B(1, -8), C(-4, 7)
A(2, 9), B(9, -2), C(-6, -7)
A(9, 2), B(2, -9), C(-6, 7)



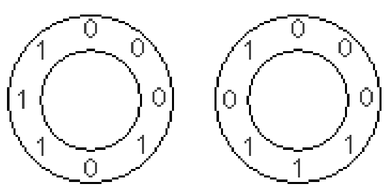
The sum of their perimeters, rounded to four decimal places, is 291.0089.

Find all such triangles with a perimeter ≤ 10^5.
Enter as your answer the sum of their perimeters rounded to four decimal places.

Problem 265:

2^N binary digits can be placed in a circle so that all the N-digit clockwise subsequences are distinct.

For N=3, two such circular arrangements are possible, ignoring rotations:



For the first arrangement, the 3-digit subsequences, in clockwise order, are:
000, 001, 010, 101, 011, 111, 110 and 100.

Each circular arrangement can be encoded as a number by concatenating the binary digits starting with the subsequence of all zeros as the most significant bits and proceeding clockwise. The two arrangements for N=3 are thus represented as 23 and 29:

$$00010111_2 = 23$$
$$00011101_2 = 29$$

Calling S(N) the sum of the unique numeric representations, we can see that S(3) = 23 + 29 = 52.

Find S(5).

Problem 266:

The divisors of 12 are: 1,2,3,4,6 and 12.
The largest divisor of 12 that does not exceed the square root of 12 is 3.
We shall call the largest divisor of an integer n that does not exceed the square root of n the pseudo square root (PSR) of n.
It can be seen that PSR(3102)=47.

Let p be the product of the primes below 190.
Find PSR(p) mod 10^{16}.

Problem 267:

You are given a unique investment opportunity.

Starting with £1 of capital, you can choose a fixed proportion, f, of your capital to bet on a fair coin toss repeatedly for 1000 tosses.

Your return is double your bet for heads and you lose your bet for tails.

For example, if $f = 1/4$, for the first toss you bet £0.25, and if heads comes up you win £0.5 and so then have £1.5. You then bet £0.375 and if the second toss is tails, you have £1.125.

Choosing f to maximize your chances of having at least £1,000,000,000 after 1,000 flips, what is the chance that you become a billionaire?

All computations are assumed to be exact (no rounding), but give your answer rounded to 12 digits behind the decimal point in the form 0.abcdefghijkl.

Problem 268:

It can be verified that there are 23 positive integers less than 1000 that are divisible by at least four distinct primes less than 100.

Find how many positive integers less than 10^{16} are divisible by at least four distinct primes less than 100.

Problem 269:

A root or zero of a polynomial $P(x)$ is a solution to the equation $P(x) = 0$.
Define P_n as the polynomial whose coefficients are the digits of n.
For example, $P_{5703}(x) = 5x^3 + 7x^2 + 3$.

We can see that:

- $P_n(0)$ is the last digit of n,
- $P_n(1)$ is the sum of the digits of n,
- $P_n(10)$ is n itself.

Define $Z(k)$ as the number of positive integers, n, not exceeding k for which the polynomial P_n has at least one integer root.

It can be verified that Z(100 000) is 14696.

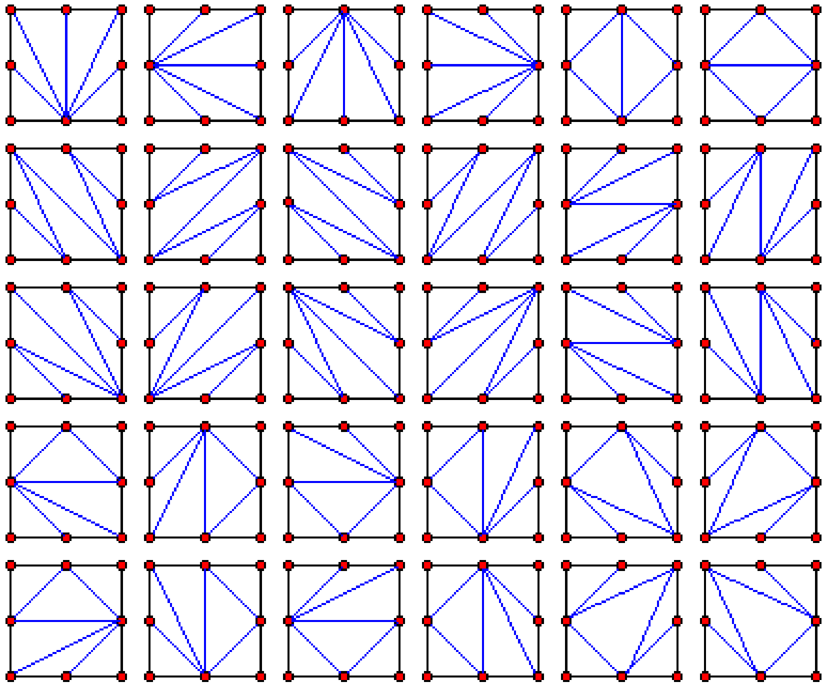
What is $Z(10^{16})$?

Problem 270:

A square piece of paper with integer dimensions N×N is placed with a corner at the origin and two of its sides along the x- and y-axes. Then, we cut it up respecting the following rules:

- We only make straight cuts between two points lying on different sides of the square, and having integer coordinates.
- Two cuts cannot cross, but several cuts can meet at the same border point.
- Proceed until no more legal cuts can be made.

Counting any reflections or rotations as distinct, we call C(N) the number of ways to cut an N×N square.
For example, C(1) = 2 and C(2) = 30 (shown below).



What is C(30) mod 10^8 ?

Problem 271:

For a positive number n, define S(n) as the sum of the integers x, for which $1<x<n$ and $x^3 \equiv 1 \bmod n$.

When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, S(91)=9+16+22+29+53+74+79+81=363.

Find S(13082761331670030).

Problem 272:

For a positive number n, define C(n) as the number of the integers x, for which $1<x<n$ and $x^3 \equiv 1 \bmod n$.

When n=91, there are 8 possible values for x, namely : 9, 16, 22, 29, 53, 74, 79, 81.
Thus, C(91)=8.

Find the sum of the positive numbers $n \le 10^{11}$ for which C(n)=242.

Problem 273:

Consider equations of the form: $a^2 + b^2 = N$, $0 \le a \le b$, a, b and N integer.

For N=65 there are two solutions:

a=1, b=8 and a=4, b=7.

We call S(N) the sum of the values of a of all solutions of $a^2 + b^2 = N$, $0 \le a \le b$, a, b and N integer.

Thus S(65) = 1 + 4 = 5.

Find $\sum$S(N), for all squarefree N only divisible by primes of the form $4k+1$ with $4k+1 < 150$.

Problem 274:

For each integer $p > 1$ coprime to 10 there is a positive *divisibility multiplier* $m < p$ which preserves divisibility by p for the following function on any positive integer, n:

$f(n)$ = (all but the last digit of n) + (the last digit of n) * m

That is, if m is the divisibility multiplier for p, then $f(n)$ is divisible by p if and only if n is divisible by p.

(When n is much larger than p, $f(n)$ will be less than n and repeated application of f provides a multiplicative divisibility test for p.)

For example, the divisibility multiplier for 113 is 34.

f(76275) = 7627 + 5 * 34 = 7797 : 76275 and 7797 are both divisible by 113
f(12345) = 1234 + 5 * 34 = 1404 : 12345 and 1404 are both not divisible by 113

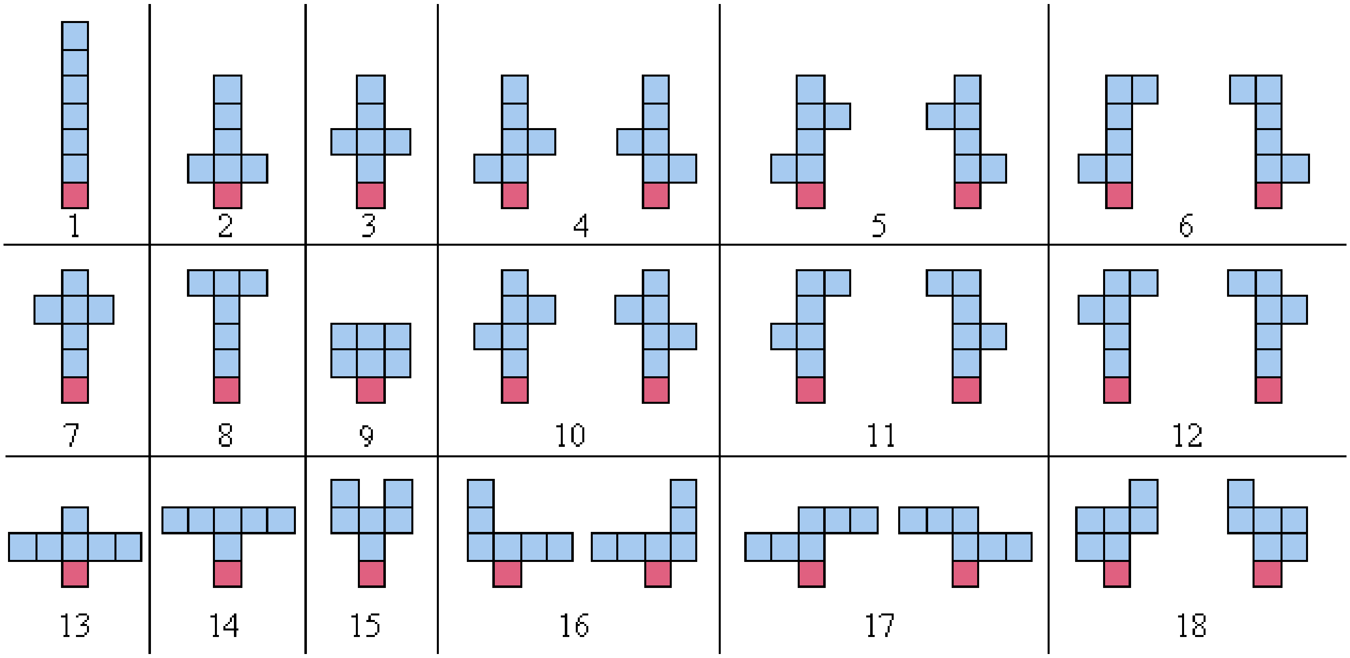
The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517.
What is the sum of the divisibility multipliers for the primes that are coprime to 10 and less than 10^7?

Problem 275:

Let us define a *balanced sculpture* of order n as follows:

- A polyomino made up of n+1 tiles known as the *blocks* (n tiles)
  and the *plinth* (remaining tile);
- the plinth has its centre at position (x = 0, y = 0);
- the blocks have y-coordinates greater than zero (so the plinth is the unique lowest tile);
- the centre of mass of all the blocks, combined, has x-coordinate equal to zero.

When counting the sculptures, any arrangements which are simply reflections about the y-axis, are <u>not</u> counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the y-axis) is counted as one sculpture:



There are 964 balanced sculptures of order 10 and 360505 of order 15.
How many balanced sculptures are there of order 18?

Problem 276:

Consider the triangles with integer sides a, b and c with a $\leq$ b $\leq$ c.
An integer sided triangle (a,b,c) is called primitive if gcd(a,b,c)=1.
How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

Problem 277:

A modified Collatz sequence of integers is obtained from a starting value a_1 in the following way:

$a_{n+1} = a_n/3$ if a_n is divisible by 3. We shall denote this as a large downward step, "D".

$a_{n+1} = (4a_n + 2)/3$ if a_n divided by 3 gives a remainder of 1. We shall denote this as an upward step, "U".

$a_{n+1} = (2a_n - 1)/3$ if a_n divided by 3 gives a remainder of 2. We shall denote this as a small downward step, "d".

The sequence terminates when some $a_n = 1$.

Given any integer, we can list out the sequence of steps.
For instance if a_1=231, then the sequence $\{a_n\}$={231,77,51,17,11,7,10,14,9,3,1} corresponds to the steps "DdDddUUdDD".

Of course, there are other sequences that begin with that same sequence "DdDddUUdDD....".
For instance, if a_1=1004064, then the sequence is DdDddUUdDDDdUDUUUdDdUUDDDUdDD.
In fact, 1004064 is the smallest possible $a_1 > 10^6$ that begins with the sequence DdDddUUdDD.

What is the smallest $a_1 > 10^{15}$ that begins with the sequence
"UDDDUdddDDUDDddDdDddDDUDDdUUDd"?

Problem 278:

Given the values of integers $1 < a_1 < a_2 <... < a_n$, consider the linear combination
$q_1a_1 + q_2a_2 + ... + q_na_n = b$, using only integer values $q_k \geq 0$.

Note that for a given set of a_k, it may be that not all values of b are possible.
For instance, if $a_1 = 5$ and $a_2 = 7$, there are no $q_1 \geq 0$ and $q_2 \geq 0$ such that b could be
1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18 or 23.
In fact, 23 is the largest impossible value of b for $a_1 = 5$ and $a_2 = 7$.
We therefore call $f(5, 7) = 23$.
Similarly, it can be shown that $f(6, 10, 15)=29$ and $f(14, 22, 77) = 195$.

Find $\sum f(p*q,p*r,q*r)$, where p, q and r are prime numbers and $p < q < r < 5000$.

Problem 279:

How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed 10^8?

Problem 280:

A laborious ant walks randomly on a 5x5 grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position.

At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches.

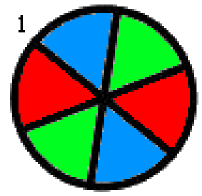
What's the expected number of steps until all seeds have been dropped in the top row?
Give your answer rounded to 6 decimal places.

Problem 281:

You are given a pizza (perfect circle) that has been cut into $m \cdot n$ equal pieces and you want to have exactly one topping on each slice.

Let $f(m,n)$ denote the number of ways you can have toppings on the pizza with m different toppings ($m \geq 2$), using each topping on exactly n slices ($n \geq 1$).
Reflections are considered distinct, rotations are not.

Thus, for instance, $f(2,1) = 1$, $f(2,2) = f(3,1) = 2$ and $f(3,2) = 16$.
$f(3,2)$ is shown below:



Find the sum of all $f(m,n)$ such that $f(m,n) \leq 10^{15}$.

Problem 282:

For non-negative integers m, n, the Ackermann function $A(m, n)$ is defined as follows:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

For example $A(1, 0) = 2$, $A(2, 2) = 7$ and $A(3, 4) = 125$.

Find $\sum_{n=0}^{6} A(n, n)$ and give your answer mod 14^8.

Problem 283:

Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1.
Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2.

Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.

Problem 284:

The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits: 376^2 = 141376. Let's call a number with this property a steady square.

Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square: $c37^2$ = aa0c37, and the sum of its digits is c+3+7=18 in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system.

For $1 \leq n \leq 9$, the sum of the digits of all the n-digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0's are not allowed.

Find the sum of the digits of all the n-digit steady squares in the base 14 numbering system for $1 \leq n \leq 10000$ (decimal) and give your answer in the base 14 system using lower case letters where necessary.

Problem 285:

Albert chooses a positive integer k, then two real numbers a, b are randomly chosen in the interval [0,1] with uniform distribution.
The square root of the sum $(k \cdot a+1)^2 + (k \cdot b+1)^2$ is then computed and rounded to the nearest integer. If the result is equal to k, he scores k points; otherwise he scores nothing.

For example, if $k = 6$, $a = 0.2$ and $b = 0.85$, then $(k \cdot a+1)^2 + (k \cdot b+1)^2 = 42.05$.
The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6.
This is equal to k, so he scores 6 points.

It can be shown that if he plays 10 turns with $k = 1$, $k = 2$, ..., $k = 10$, the expected value of his total score, rounded to five decimal places, is 10.20914.

If he plays 10^5 turns with $k = 1$, $k = 2$, $k = 3$, ..., $k = 10^5$, what is the expected value of his total score, rounded to five decimal places?

Problem 286:

Barbara is a mathematician and a basketball player. She has found that the probability of scoring a point when shooting from a distance x is exactly $(1 - {}^{x}/_{q})$, where q is a real constant greater than 50.

During each practice run, she takes shots from distances $x = 1$, $x = 2$, ..., $x = 50$ and, according to her records, she has precisely a 2 % chance to score a total of exactly 20 points.

Find q and give your answer rounded to 10 decimal places.

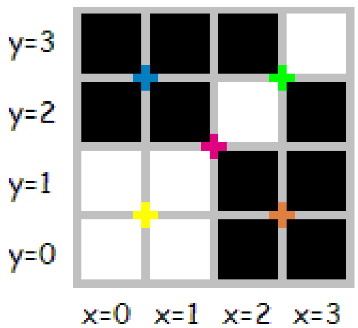
Problem 287:

The quadtree encoding allows us to describe a $2^N \times 2^N$ black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this:

- the first bit deals with the complete $2^N \times 2^N$ region;
- "0" denotes a split:

  the current $2^n \times 2^n$ region is divided into 4 sub-regions of dimension $2^{n-1} \times 2^{n-1}$,
  the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order;
- "10" indicates that the current region contains only black pixels;
- "11" indicates that the current region contains only white pixels.

Consider the following 4×4 image (colored marks denote places where a split can occur):



This image can be described by several sequences, for example : "001010101001011111011010101010", of length 30, or
"0100101111101110", of length 16, which is the minimal sequence for this image.

For a positive integer N, define D_N as the $2^N \times 2^N$ image with the following coloring scheme:

- the pixel with coordinates $x = 0$, $y = 0$ corresponds to the bottom left pixel,
- if $(x - 2^{N-1})^2 + (y - 2^{N-1})^2 \leq 2^{2N-2}$ then the pixel is black,
- otherwise the pixel is white.

What is the length of the minimal sequence describing D_{24} ?

Problem 288:

For any prime p the number $N(p,q)$ is defined by $N(p,q) = \sum_{n=0 \text{ to } q} T_n * p^n$
with T_n generated by the following random number generator:

$S_0 = 290797$

$S_{n+1} = S_n^2 \bmod 50515093$

$T_n = S_n \bmod p$

Let Nfac(p,q) be the factorial of N(p,q).
Let NF(p,q) be the number of factors p in Nfac(p,q).

You are given that NF(3,10000) mod 3^{20}=624955285.

Find NF(61,10^7) mod 61^{10}

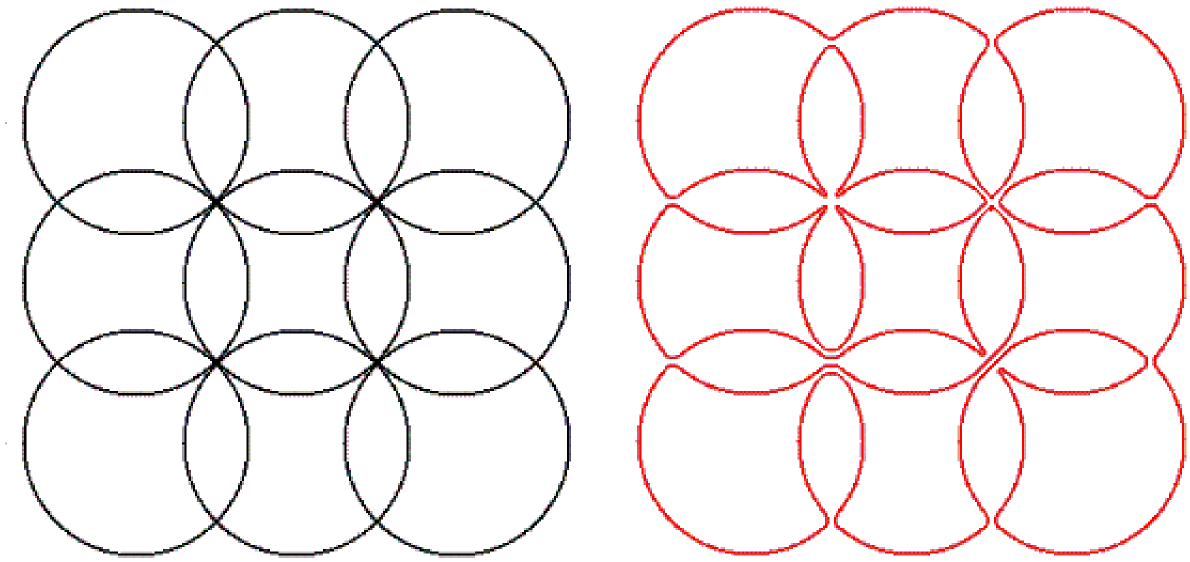
Problem 289:

Let C(x,y) be a circle passing through the points (x, y), (x, y+1), (x+1, y) and (x+1, y+1).

For positive integers m and n, let E(m,n) be a configuration which consists of the $m \cdot n$ circles:
{ C(x,y): $0 \le x < m$, $0 \le y < n$, x and y are integers }

An Eulerian cycle on E(m,n) is a closed path that passes through each arc exactly once.
Many such paths are possible on E(m,n), but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself.

The image below shows E(3,3) and an example of an Eulerian non-crossing path.



Let L(m,n) be the number of Eulerian non-crossing paths on E(m,n).
For example, L(1,2) = 2, L(2,2) = 37 and L(3,3) = 104290.

Find L(6,10) mod 10^{10}.

Problem 290:

How many integers $0 \le n < 10^{18}$ have the property that the sum of the digits of n equals the sum of digits of $137n$?

Problem 291:

A prime number p is called a Panaitopol prime if $p = \frac{x^4 - y^4}{x^3 + y^3}$ for some positive integers x and y.

Find how many Panaitopol primes are less than 5×10^{15}.

Problem 292:

We shall define a *pythagorean polygon* to be a **convex polygon** with the following properties:

- there are at least three vertices,
- no three vertices are aligned,
- each vertex has **integer coordinates**,
- each edge has **integer length**.

For a given integer n, define P(n) as the number of distinct pythagorean polygons for which the perimeter is $\leq n$.
Pythagorean polygons should be considered distinct as long as none is a translation of another.

You are given that P(4) = 1, P(30) = 3655 and P(60) = 891045.
Find P(120).

Problem 293:

An even positive integer N will be called admissible, if it is a power of 2 or its distinct prime factors are consecutive primes.
The first twelve admissible numbers are 2,4,6,8,12,16,18,24,30,32,36,48.

If N is admissible, the smallest integer M $>$ 1 such that N+M is prime, will be called the pseudo-Fortunate number for N.

For example, N=630 is admissible since it is even and its distinct prime factors are the consecutive primes 2,3,5 and 7.
The next prime number after 631 is 641; hence, the pseudo-Fortunate number for 630 is M=11.
It can also be seen that the pseudo-Fortunate number for 16 is 3.

Find the sum of all distinct pseudo-Fortunate numbers for admissible numbers N less than 10^9.

Problem 294:

For a positive integer k, define d(k) as the sum of the digits of k in its usual decimal representation. Thus d(42) = 4+2 = 6.

For a positive integer n, define S(n) as the number of positive integers k < 10^n with the following properties :

- k is divisible by 23 and
- d(k) = 23.

You are given that S(9) = 263626 and S(42) = 6377168878570056.

Find S(11^{12}) and give your answer mod 10^9.

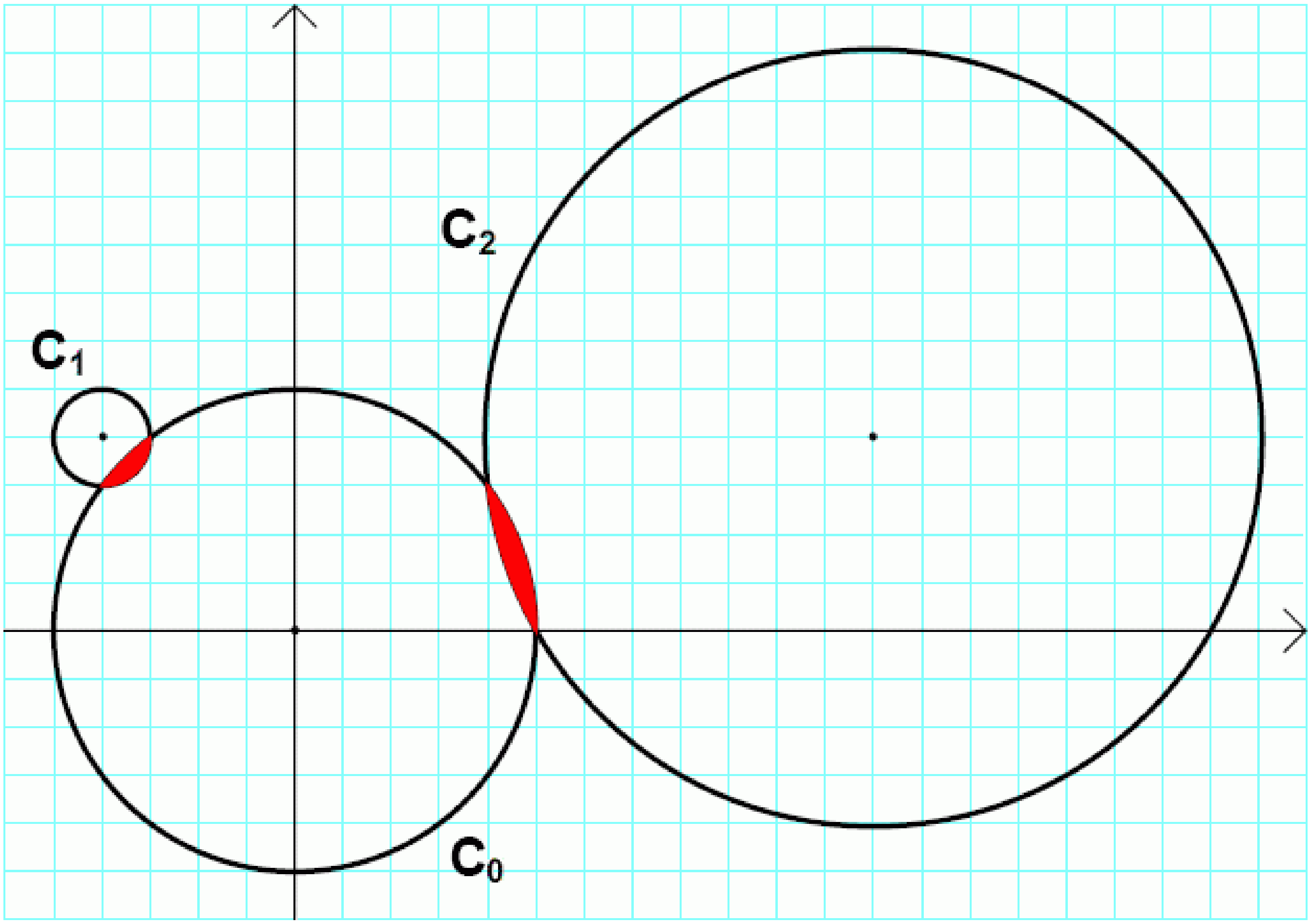
Problem 295:

We call the convex area enclosed by two circles a *lenticular hole* if:

- The centres of both circles are on lattice points.
- The two circles intersect at two distinct lattice points.
- The interior of the convex area enclosed by both circles does not contain any lattice points.

Consider the circles:
C_0: $x^2+y^2=25$
C_1: $(x+4)^2+(y-4)^2=1$
C_2: $(x-12)^2+(y-4)^2=65$

The circles C_0, C_1 and C_2 are drawn in the picture below.



C_0 and C_1 form a lenticular hole, as well as C_0 and C_2.

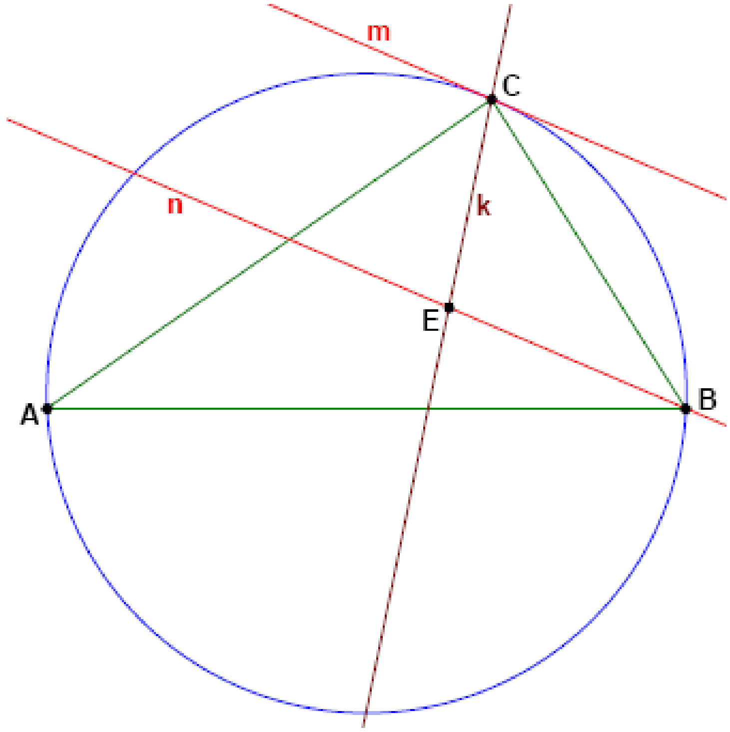
We call an ordered pair of positive real numbers (r_1, r_2) a *lenticular pair* if there exist two circles with radii r_1 and r_2 that form a lenticular hole. We can verify that (1, 5) and (5, $\sqrt{65}$) are the lenticular pairs of the example above.

Let L(N) be the number of **distinct** lenticular pairs (r_1, r_2) for which $0 < r_1 \le r_2 \le N$.
We can verify that L(10) = 30 and L(100) = 3442.

Find L(100 000).

Problem 296:

Given is an integer sided triangle ABC with $BC \le AC \le AB$.
k is the angular bisector of angle ACB.
m is the tangent at C to the circumscribed circle of ABC.
n is a line parallel to m through B.
The intersection of n and k is called E.



How many triangles ABC with a perimeter not exceeding 100 000 exist such that BE has integral length?

Problem 297:

Each new term in the Fibonacci sequence is generated by adding the previous two terms.
Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example, 100 = 3 + 8 + 89.
Such a sum is called the **Zeckendorf representation** of the number.

For any integer $n>0$, let $z(n)$ be the number of terms in the Zeckendorf representation of n.
Thus, $z(5) = 1$, $z(14) = 2$, $z(100) = 3$ etc.
Also, for $0<n<10^6$, $\sum z(n) = 7894453$.

Find $\sum z(n)$ for $0<n<10^{17}$.

Problem 298:

Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full.

Both players start with empty memories. Both players always add new missed numbers to their memory but

use a different strategy in deciding which number to remove:
Larry's strategy is to remove the number that hasn't been called in the longest time.
Robin's strategy is to remove the number that's been in the memory the longest time.

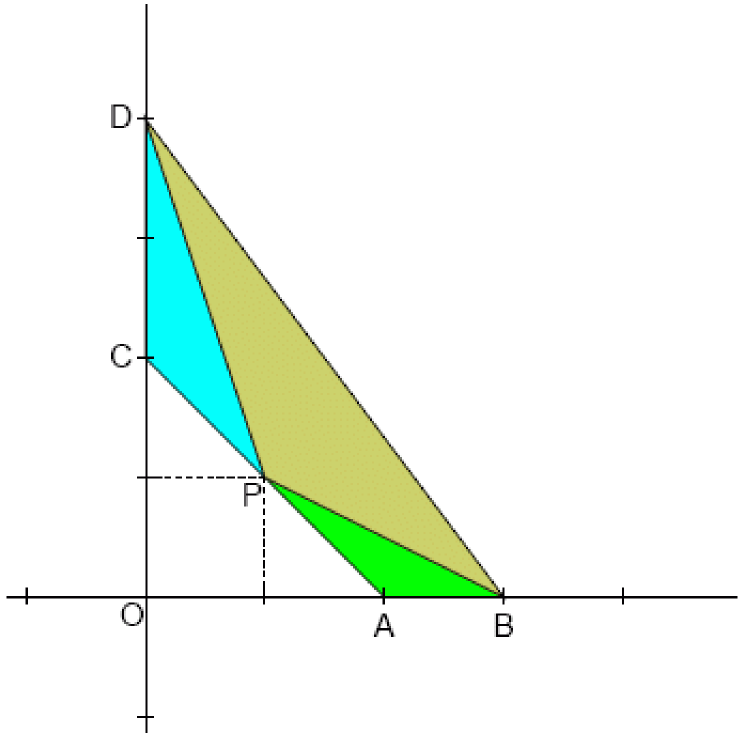
Example game:

Turn	Called number	Larry's memory	Larry's score	Robin's memory	Robin's score
1	1	1	0	1	0
2	2	1,2	0	1,2	0
3	4	1,2,4	0	1,2,4	0
4	6	1,2,4,6	0	1,2,4,6	0
5	1	1,2,4,6	1	1,2,4,6	1
6	8	1,2,4,6,8	1	1,2,4,6,8	1
7	10	1,4,6,8,10	1	2,4,6,8,10	1
8	2	1,2,6,8,10	1	2,4,6,8,10	2
9	4	1,2,4,8,10	1	2,4,6,8,10	3
10	1	1,2,4,8,10	2	1,4,6,8,10	3

Denoting Larry's score by L and Robin's score by R, what is the expected value of $|L-R|$ after 50 turns? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

Problem 299:

Four points with integer coordinates are selected:
A(a, 0), B(b, 0), C(0, c) and D(0, d), with $0 < a < b$ and $0 < c < d$.
Point P, also with integer coordinates, is chosen on the line AC so that the three triangles ABP, CDP and BDP are all similar.



It is easy to prove that the three triangles can be similar, only if $a=c$.

So, given that $a=c$, we are looking for triplets (a,b,d) such that at least one point P (with integer coordinates) exists on AC, making the three triangles ABP, CDP and BDP all similar.

For example, if (a,b,d)=(2,3,4), it can be easily verified that point P(1,1) satisfies the above condition. Note that the triplets (2,3,4) and (2,4,3) are considered as distinct, although point P(1,1) is common for both.

If $b+d < 100$, there are 92 distinct triplets (a,b,d) such that point P exists.
If $b+d < 100\,000$, there are 320471 distinct triplets (a,b,d) such that point P exists.

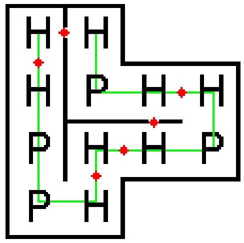
If $b+d < 100\,000\,000$, how many distinct triplets (a,b,d) are there such that point P exists?

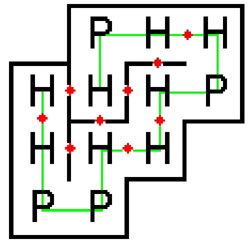
Problem 300:

In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HHPPHHHPHHPH.
For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are 2^n distinct proteins consisting of n elements.

When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous.
As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside.
Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions.

The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).





The folding on the left has only six H-H contact points, thus it would never occur naturally.
On the other hand, the folding on the right has nine H-H contact points, which is optimal for this string.

Assuming that H and P elements are equally likely to occur in any position along the string, the average number of H-H contact points in an optimal folding of a random protein string of length 8 turns out to be $850 / 2^8 = 3.3203125$.

What is the average number of H-H contact points in an optimal folding of a random protein string of length 15?
Give your answer using as many decimal places as necessary for an exact result.

Problem 301:

Nim is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain.

We'll consider the three-heap normal-play version of Nim, which works as follows:
- At the start of the game there are three heaps of stones.
- On his turn the player removes any positive number of stones from any single heap.
- The first player unable to move (because no stones remain) loses.

If (n_1,n_2,n_3) indicates a Nim position consisting of heaps of size n_1, n_2 and n_3 then there is a simple function $X(n_1,n_2,n_3)$ — that you may look up or attempt to deduce for yourself — that returns:

- zero if, with perfect strategy, the player about to move will eventually lose; or
- non-zero if, with perfect strategy, the player about to move will eventually win.

For example $X(1,2,3) = 0$ because, no matter what the current player does, his opponent can respond with a move that leaves two heaps of equal size, at which point every move by the current player can be mirrored by his opponent until no stones remain; so the current player loses. To illustrate:
- current player moves to (1,2,1)
- opponent moves to (1,0,1)
- current player moves to (0,0,1)
- opponent moves to (0,0,0), and so wins.

For how many positive integers $n \leq 2^{30}$ does $X(n,2n,3n) = 0$?

Problem 302:

A positive integer n is **powerful** if p^2 is a divisor of n for every prime factor p in n.

A positive integer n is a **perfect power** if n can be expressed as a power of another positive integer.

A positive integer n is an **Achilles number** if n is powerful but not a perfect power. For example, 864 and 1800 are Achilles numbers: $864 = 2^5 \cdot 3^3$ and $1800 = 2^3 \cdot 3^2 \cdot 5^2$.

We shall call a positive integer S a *Strong Achilles number* if both S and $\varphi(S)$ are Achilles numbers.[1]
For example, 864 is a Strong Achilles number: $\varphi(864) = 288 = 2^5 \cdot 3^2$. However, 1800 isn't a Strong Achilles number because: $\varphi(1800) = 480 = 2^5 \cdot 3^1 \cdot 5^1$.

There are 7 Strong Achilles numbers below 10^4 and 656 below 10^8.

How many Strong Achilles numbers are there below 10^{18}?

[1] φ denotes **Euler's totient function**.

Problem 303:

For a positive integer n, define $f(n)$ as the least positive multiple of n that, written in base 10, uses only digits $\leq$ 2.

Thus $f(2)=2$, $f(3)=12$, $f(7)=21$, $f(42)=210$, $f(89)=1121222$.

Also, $\sum_{n=1}^{100} \frac{f(n)}{n} = 11363107$.

Find $\sum_{n=1}^{10000} \frac{f(n)}{n}$.

Problem 304:

For any positive integer n the function next_prime(n) returns the smallest prime p such that $p > n$.

The sequence $a(n)$ is defined by:
$a(1)=\text{next\_prime}(10^{14})$ and $a(n)=\text{next\_prime}(a(n-1))$ for $n > 1$.

The fibonacci sequence f(n) is defined by: f(0)=0, f(1)=1 and f(n)=f(n-1)+f(n-2) for $n>1$.

The sequence b(n) is defined as f(a(n)).

Find $\sum$b(n) for $1 \le n \le 100\,000$. Give your answer mod 1234567891011.

Problem 305:

Let's call S the (infinite) string that is made by concatenating the consecutive positive integers (starting from 1) written down in base 10.
Thus, S = 1234567891011121314151617181920212223242...

It's easy to see that any number will show up an infinite number of times in S.

Let's call f(n) the starting position of the n^{th} occurrence of n in S.
For example, f(1)=1, f(5)=81, f(12)=271 and f(7780)=111111365.

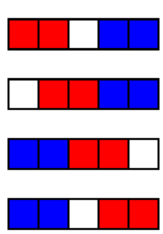
Find $\sum f(3^k)$ for $1 \le k \le 13$.

Problem 306:

The following game is a classic example of Combinatorial Game Theory:

Two players start with a strip of n white squares and they take alternate turns.
On each turn, a player picks two contiguous white squares and paints them black.
The first player who cannot make a move loses.

- If n = 1, there are no valid moves, so the first player loses automatically.
- If n = 2, there is only one valid move, after which the second player loses.
- If n = 3, there are two valid moves, but both leave a situation where the second player loses.
- If n = 4, there are three valid moves for the first player; she can win the game by painting the two middle squares.
- If n = 5, there are four valid moves for the first player (shown below in red); but no matter what she does, the second player (blue) wins.



So, for $1 \le n \le 5$, there are 3 values of n for which the first player can force a win.
Similarly, for $1 \le n \le 50$, there are 40 values of n for which the first player can force a win.

For 1 ≤ n ≤ 1 000 000, how many values of n are there for which the first player can force a win?

Problem 307:

k defects are randomly distributed amongst n integrated-circuit chips produced by a factory (any number of defects may be found on a chip and each defect is independent of the other defects).

Let p(k,n) represent the probability that there is a chip with at least 3 defects.
For instance p(3,7) ≈ 0.0204081633.

Find p(20 000, 1 000 000) and give your answer rounded to 10 decimal places in the form 0.abcdefghij

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Problem 308:

A program written in the programming language Fractran consists of a list of fractions.

The internal state of the Fractran Virtual Machine is a positive integer, which is initially set to a seed value. Each iteration of a Fractran program multiplies the state integer by the first fraction in the list which will leave it an integer.

For example, one of the Fractran programs that John Horton Conway wrote for prime-generation consists of the following 14 fractions:

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}.$$

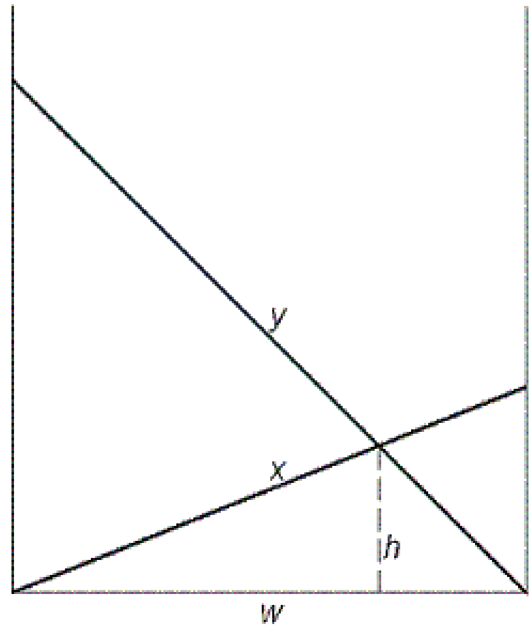
Starting with the seed integer 2, successive iterations of the program produce the sequence:
15, 825, 725, 1925, 2275, 425, ..., 68, **4**, 30, ..., 136, **8**, 60, ..., 544, **32**, 240, ...

The powers of 2 that appear in this sequence are 2^2, 2^3, 2^5, ...
It can be shown that *all* the powers of 2 in this sequence have prime exponents and that *all* the primes appear as exponents of powers of 2, in proper order!

If someone uses the above Fractran program to solve Project Euler Problem 7 (find the 10001st prime), how many iterations would be needed until the program produces $2^{\text{10001st prime}}$?

Problem 309:

In the classic "Crossing Ladders" problem, we are given the lengths x and y of two ladders resting on the opposite walls of a narrow, level street. We are also given the height h above the street where the two ladders cross and we are asked to find the width of the street (w).



Here, we are only concerned with instances where all four variables are positive integers.
For example, if $x = 70$, $y = 119$ and $h = 30$, we can calculate that $w = 56$.

In fact, for integer values x, y, h and $0 < x < y < 200$, there are only five triplets (x,y,h) producing integer solutions for w:
(70, 119, 30), (74, 182, 21), (87, 105, 35), (100, 116, 35) and (119, 175, 40).

For integer values x, y, h and $0 < x < y < 1\,000\,000$, how many triplets (x,y,h) produce integer solutions for w?

Problem 310:

Alice and Bob play the game Nim Square.
Nim Square is just like ordinary three-heap normal play Nim, but the players may only remove a square number of stones from a heap.
The number of stones in the three heaps is represented by the ordered triple (a,b,c).
If $0 \le a \le b \le c \le 29$ then the number of losing positions for the next player is 1160.

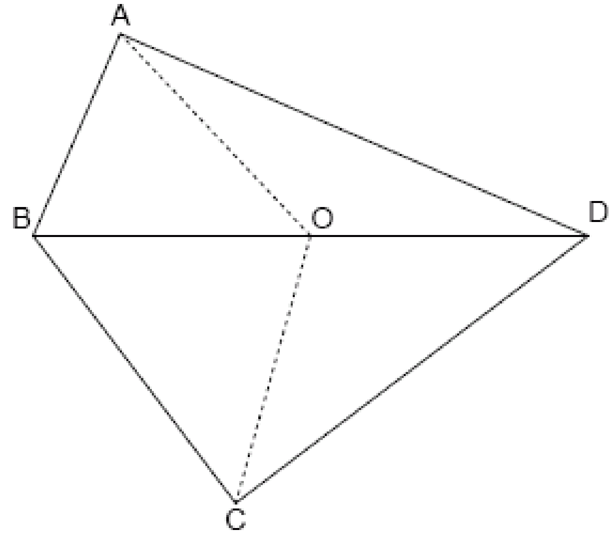
Find the number of losing positions for the next player if $0 \le a \le b \le c \le 100\,000$.

Problem 311:

ABCD is a convex, integer sided quadrilateral with $1 \le AB < BC < CD < AD$.
BD has integer length. O is the midpoint of BD. AO has integer length.
We'll call ABCD a *biclinic integral quadrilateral* if $AO = CO \le BO = DO$.

For example, the following quadrilateral is a biclinic integral quadrilateral:

AB = 19, BC = 29, CD = 37, AD = 43, BD = 48 and AO = CO = 23.

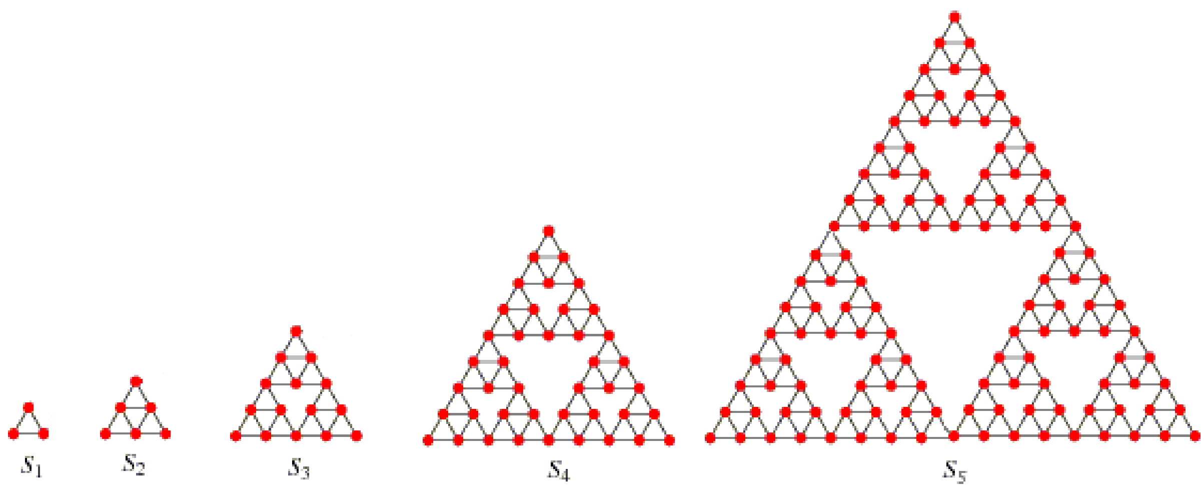


Let B(N) be the number of distinct biclinic integral quadrilaterals ABCD that satisfy $AB^2+BC^2+CD^2+AD^2 \leq N$.
We can verify that B(10 000) = 49 and B(1 000 000) = 38239.

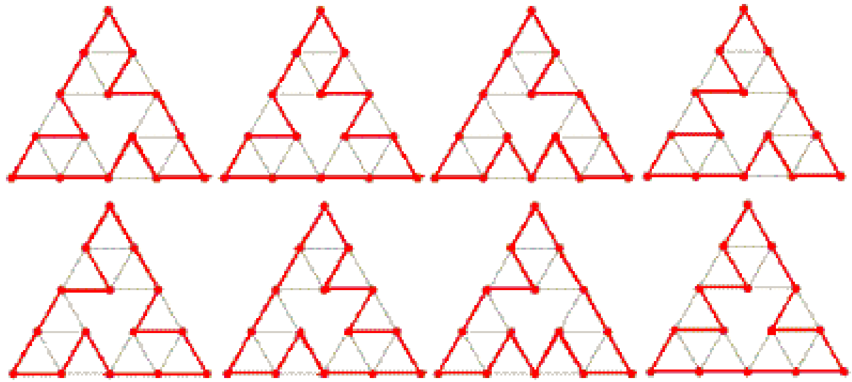
Find B(10 000 000 000).

Problem 312:

- A **Sierpiński graph** of order-1 (S_1) is an equilateral triangle.
- S_{n+1} is obtained from S_n by positioning three copies of S_n so that every pair of copies has one common corner.



Let C(n) be the number of cycles that pass exactly once through all the vertices of S_n.
For example, C(3) = 8 because eight such cycles can be drawn on S_3, as shown below:

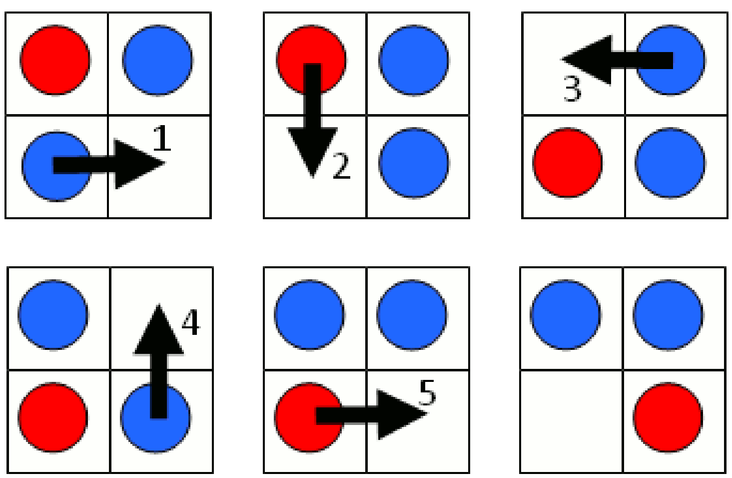


It can also be verified that :
C(1) = C(2) = 1
C(5) = 71328803586048
C(10 000) mod 10^8 = 37652224
C(10 000) mod 13^8 = 617720485

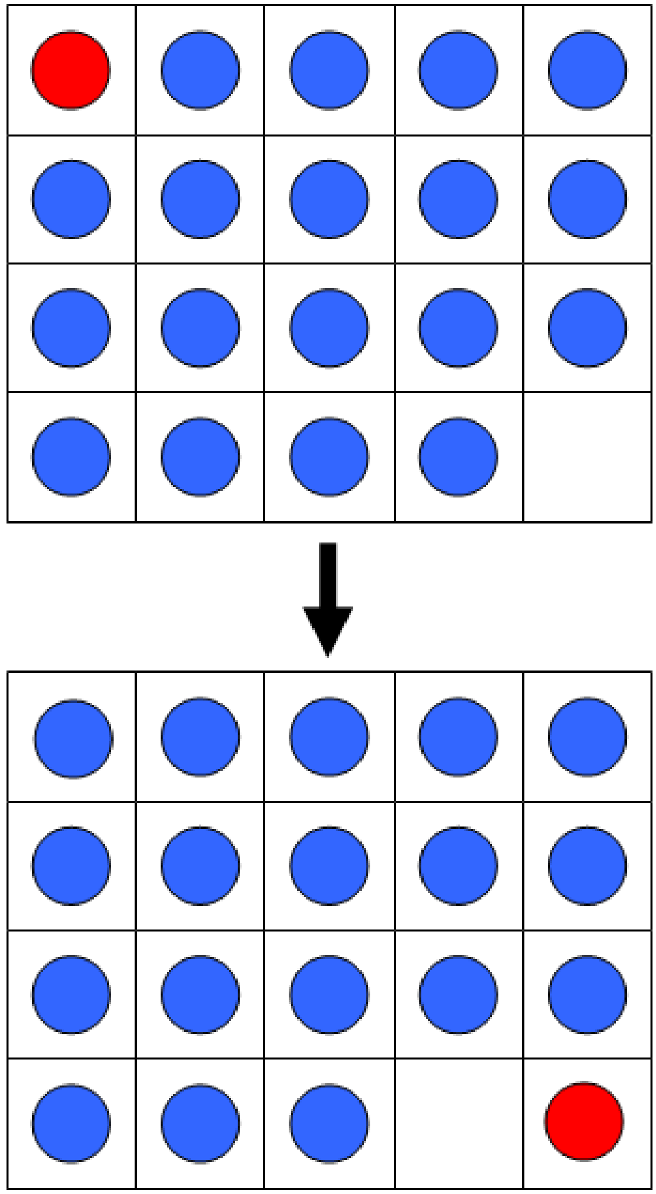
Find C(C(C(10 000))) mod 13^8.

Problem 313:

In a sliding game a counter may slide horizontally or vertically into an empty space. The objective of the game is to move the red counter from the top left corner of a grid to the bottom right corner; the space always starts in the bottom right corner. For example, the following sequence of pictures show how the game can be completed in five moves on a 2 by 2 grid.



Let S(m,n) represent the minimum number of moves to complete the game on an m by n grid. For example, it can be verified that S(5,4) = 25.



There are exactly 5482 grids for which $S(m,n) = p^2$, where $p < 100$ is prime.

How many grids does $S(m,n) = p^2$, where $p < 10^6$ is prime?

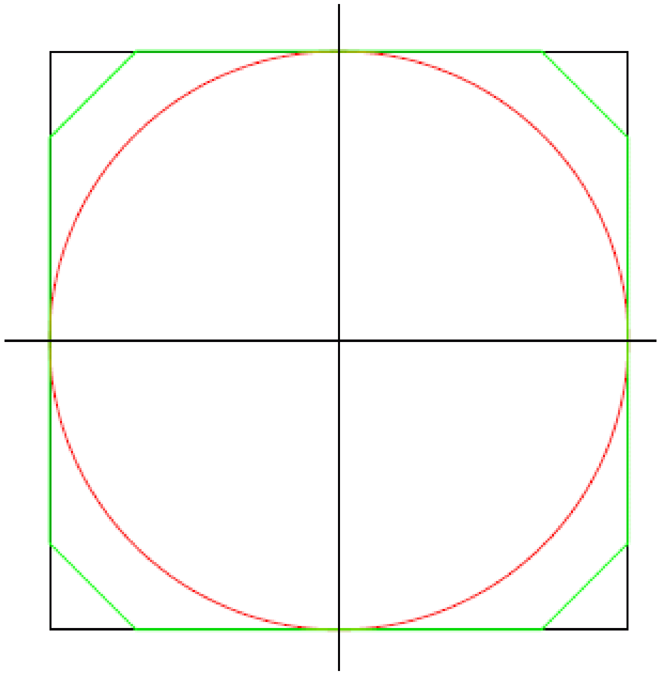
Problem 314:

The moon has been opened up, and land can be obtained for free, but there is a catch. You have to build a wall around the land that you stake out, and building a wall on the moon is expensive. Every country has been allotted a 500 m by 500 m square area, but they will possess only that area which they wall in. 251001 posts have been placed in a rectangular grid with 1 meter spacing. The wall must be a closed series of straight lines, each line running from post to post.

The bigger countries of course have built a 2000 m wall enclosing the entire 250 000 m^2 area. The Duchy of Grand Fenwick, has a tighter budget, and has asked you (their Royal Programmer) to compute what shape would get best maximum enclosed-area/wall-length ratio.

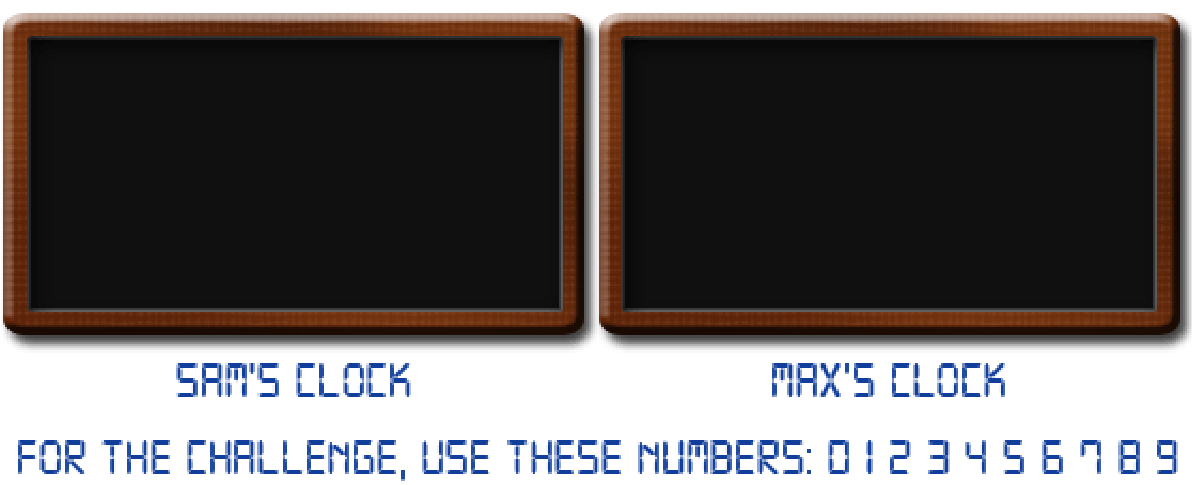
You have done some preliminary calculations on a sheet of paper. For a 2000 meter wall enclosing the 250 000 m^2 area the enclosed-area/wall-length ratio is 125.
Although not allowed , but to get an idea if this is anything better: if you place a circle inside the square area touching the four sides the area will be equal to $\pi*250^2$ m^2 and the perimeter will be $\pi*500$ m, so the enclosed-area/wall-length ratio will also be 125.

However, if you cut off from the square four triangles with sides 75 m, 75 m and $75\sqrt{2}$ m the total area becomes 238750 m^2 and the perimeter becomes $1400+300\sqrt{2}$ m. So this gives an enclosed-area/wall-length ratio of 130.87, which is significantly better.



Find the maximum enclosed-area/wall-length ratio.
Give your answer rounded to 8 places behind the decimal point in the form abc.defghijk.

Problem 315:



Sam and Max are asked to transform two digital clocks into two "digital root" clocks.
A digital root clock is a digital clock that calculates digital roots step by step.

When a clock is fed a number, it will show it and then it will start the calculation, showing all the intermediate values until it gets to the result.
For example, if the clock is fed the number 137, it will show: "**137**" → "**11**" → "**2**" and then it will go black, waiting for the next number.

Every digital number consists of some light segments: three horizontal (top, middle, bottom) and four vertical (top-left, top-right, bottom-left, bottom-right).
Number "**1**" is made of vertical top-right and bottom-right, number "**4**" is made by middle horizontal and vertical top-left, top-right and bottom-right. Number "**8**" lights them all.

The clocks consume energy only when segments are turned on/off.
To turn on a "**2**" will cost 5 transitions, while a "**7**" will cost only 4 transitions.

Sam and Max built two different clocks.

Sam's clock is fed e.g. number 137: the clock shows "**137**", then the panel is turned off, then the next number ("**11**") is turned on, then the panel is turned off again and finally the last number ("**2**") is turned on and, after some time, off.
For the example, with number 137, Sam's clock requires:

"**137**" : $(2 + 5 + 4) \times 2 = 22$ transitions ("**137**" on/off).
"**11**" : $(2 + 2) \times 2 = 8$ transitions ("**11**" on/off).
"**2**" : $(5) \times 2 = 10$ transitions ("**2**" on/off).
For a grand total of 40 transitions.

Max's clock works differently. Instead of turning off the whole panel, it is smart enough to turn off only those segments that won't be needed for the next number.
For number 137, Max's clock requires:

"**137**" : 2 + 5 + 4 = 11 transitions ("**137**" on)
7 transitions (to turn off the segments that are not needed for number "**11**").
"**11**" : 0 transitions (number "**11**" is already turned on correctly)
3 transitions (to turn off the first "**1**" and the bottom part of the second "**1**"; the top part is common with number "**2**").
"**2**" : 4 tansitions (to turn on the remaining segments in order to get a "**2**")
5 transitions (to turn off number "**2**").
For a grand total of 30 transitions.

Of course, Max's clock consumes less power than Sam's one.
The two clocks are fed all the prime numbers between $A = 10^7$ and $B = 2\times10^7$.
Find the difference between the total number of transitions needed by Sam's clock and that needed by Max's one.

Problem 316:

Let $p = p_1\, p_2\, p_3$ ... be an infinite sequence of random digits, selected from {0,1,2,3,4,5,6,7,8,9} with equal probability.
It can be seen that p corresponds to the real number $0.p_1\, p_2\, p_3$ ....
It can also be seen that choosing a random real number from the interval [0,1) is equivalent to choosing an infinite sequence of random digits selected from {0,1,2,3,4,5,6,7,8,9} with equal probability.

For any positive integer n with d decimal digits, let k be the smallest index such that
$p_k, p_{k+1}, \ldots p_{k+d-1}$ are the decimal digits of n, in the same order.
Also, let $g(n)$ be the expected value of k; it can be proven that $g(n)$ is always finite and, interestingly, always an integer number.

For example, if n = 535, then
for p = 31415926**535**897...., we get k = 9
for p = 355287143650049560000490848764084685**35**4..., we get k = 36
etc and we find that $g(535) = 1008$.

Given that $\sum_{n=2}^{999} g\left(\left\lfloor \frac{10^6}{n} \right\rfloor\right) = 27280188$, find $\sum_{n=2}^{999999} g\left(\left\lfloor \frac{10^{16}}{n} \right\rfloor\right)$

Note: $\lfloor x \rfloor$ represents the floor function.

Problem 317:

A firecracker explodes at a height of 100 m above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of 20 m/s.

We assume that the fragments move without air resistance, in a uniform gravitational field with g=9.81 m/s^2.

Find the volume (in m^3) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

Problem 318:

Consider the real number $\sqrt{2}+\sqrt{3}$.
When we calculate the even powers of $\sqrt{2}+\sqrt{3}$ we get:
$(\sqrt{2}+\sqrt{3})^2 = 9.898979485566356...$
$(\sqrt{2}+\sqrt{3})^4 = 97.98979485566356...$
$(\sqrt{2}+\sqrt{3})^6 = 969.998969071069263...$
$(\sqrt{2}+\sqrt{3})^8 = 9601.99989585502907...$
$(\sqrt{2}+\sqrt{3})^{10} = 95049.999989479221...$
$(\sqrt{2}+\sqrt{3})^{12} = 940897.9999989371855...$
$(\sqrt{2}+\sqrt{3})^{14} = 9313929.99999989263$

$(\sqrt{2}+\sqrt{3})^{16} = 92198401.99999998915...$

It looks like that the number of consecutive nines at the beginning of the fractional part of these powers is non-decreasing.
In fact it can be proven that the fractional part of $(\sqrt{2}+\sqrt{3})^{2n}$ approaches 1 for large n.

Consider all real numbers of the form $\sqrt{p}+\sqrt{q}$ with p and q positive integers and $p<q$, such that the fractional part of $(\sqrt{p}+\sqrt{q})^{2n}$ approaches 1 for large n.

Let C(p,q,n) be the number of consecutive nines at the beginning of the fractional part of $(\sqrt{p}+\sqrt{q})^{2n}$.

Let N(p,q) be the minimal value of n such that C(p,q,n) $\geq$ 2011.

Find $\sum$N(p,q) for p+q $\leq$ 2011.

Problem 319:

Let $x_1, x_2,..., x_n$ be a sequence of length n such that:

- $x_1 = 2$
- for all $1 < i \leq n : x_{i-1} < x_i$
- for all i and j with $1 \leq i, j \leq n : (x_i)^j < (x_j + 1)^i$

There are only five such sequences of length 2, namely: {2,4}, {2,5}, {2,6}, {2,7} and {2,8}.
There are 293 such sequences of length 5; three examples are given below:
{2,5,11,25,55}, {2,6,14,36,88}, {2,8,22,64,181}.

Let $t(n)$ denote the number of such sequences of length n.
You are given that $t(10) = 86195$ and $t(20) = 5227991891$.

Find $t(10^{10})$ and give your answer modulo 10^9.

Problem 320:

Let N(i) be the smallest integer n such that $n!$ is divisible by $(i!)^{1234567890}$

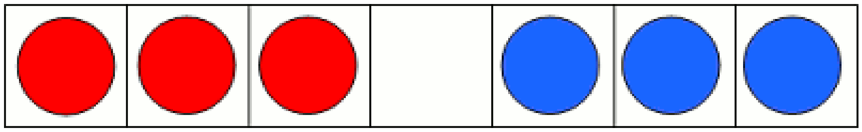
Let S(u)=$\sum$N(i) for $10 \leq i \leq u$.

S(1000)=614538266565663.

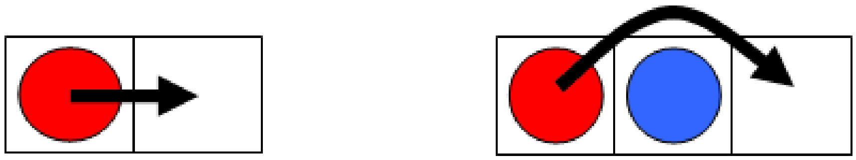
Find S(1 000 000) mod 10^{18}.

Problem 321:

A horizontal row comprising of $2n + 1$ squares has n red counters placed at one end and n blue counters at the other end, being separated by a single empty square in the centre. For example, when $n = 3$.



A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the square next to that counter is unoccupied.



Let M(n) represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left.

It can be verified M(3) = 15, which also happens to be a triangle number.

If we create a sequence based on the values of n for which M(n) is a triangle number then the first five terms would be:
1, 3, 10, 22, and 63, and their sum would be 99.

Find the sum of the first forty terms of this sequence.

Problem 322:

Let T(m, n) be the number of the binomial coefficients ${}^iC_n$ that are divisible by 10 for $n \le i < m$(i, m and n are positive integers).
You are given that T(10^9, 10^7-10) = 989697000.

Find T(10^{18}, 10^{12}-10).

Problem 323:

Let $y_0, y_1, y_2,$... be a sequence of random unsigned 32 bit integers
(i.e. $0 \le y_i < 2^{32}$, every value equally likely).

For the sequence x_i the following recursion is given:

- $x_0 = 0$ and

- $x_i = x_{i-1} \mid y_{i-1}$, for $i > 0$. ($\mid$ is the bitwise-OR operator)

It can be seen that eventually there will be an index N such that $x_i = 2^{32} - 1$ (a bit-pattern of all ones) for all $i \geq$ N.

Find the expected value of N.
Give your answer rounded to 10 digits after the decimal point.

Problem 324:

Let $f(n)$ represent the number of ways one can fill a $3 \times 3 \times n$ tower with blocks of $2 \times 1 \times 1$.
You're allowed to rotate the blocks in any way you like; however, rotations, reflections etc of the tower itself are counted as distinct.

For example (with q = 100000007) :
$f(2)$ = 229,
$f(4)$ = 117805,
$f(10)$ mod q = 96149360,
$f(10^3)$ mod q = 24806056,
$f(10^6)$ mod q = 30808124.

Find $f(10^{10000})$ mod 100000007.

Problem 325:

A game is played with two piles of stones and two players. At her turn, a player removes a number of stones from the larger pile. The number of stones she removes must be a positive multiple of the number of stones in the smaller pile.

E.g., let the ordered pair(6,14) describe a configuration with 6 stones in the smaller pile and 14 stones in the larger pile, then the first player can remove 6 or 12 stones from the larger pile.

The player taking all the stones from a pile wins the game.

A *winning configuration* is one where the first player can force a win. For example, (1,5), (2,6) and (3,12) are winning configurations because the first player can immediately remove all stones in the second pile.

A *losing configuration* is one where the second player can force a win, no matter what the first player does. For example, (2,3) and (3,4) are losing configurations: any legal move leaves a winning configuration for the second player.

Define S(N) as the sum of (x_i+y_i) for all losing configurations (x_i,y_i), $0 < x_i < y_i \leq N$. We can verify that S(10) = 211 and S(10^4) = 230312207313.

Find S(10^{16}) mod 7^{10}.

Problem 326:

Let a_n be a sequence recursively defined by: $a_1 = 1, \quad a_n = \left(\sum_{k=1}^{n-1} k \cdot a_k \right) \bmod n$.

So the first 10 elements of a_n are: 1,1,0,3,0,3,5,4,1,9.

Let $f(N,M)$ represent the number of pairs (p,q) such that:

$$1 \le p \le q \le N \quad \text{and} \quad \left(\sum_{i=p}^{q} a_i \right) \bmod M = 0$$

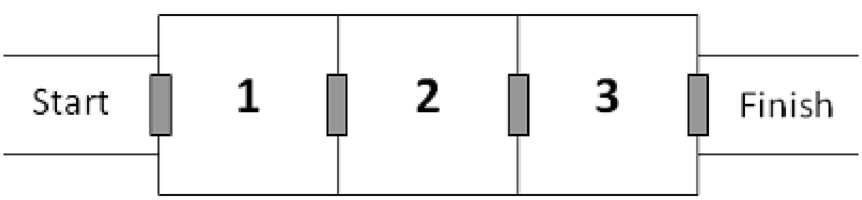
It can be seen that $f(10,10)$=4 with the pairs (3,3), (5,5), (7,9) and (9,10).

You are also given that $f(10^4,10^3)$=97158.

Find $f(10^{12},10^6)$.

Problem 327:

A series of three rooms are connected to each other by automatic doors.



Each door is operated by a security card. Once you enter a room the door automatically closes and that security card cannot be used again. A machine at the start will dispense an unlimited number of cards, but each room (including the starting room) contains scanners and if they detect that you are holding more than three security cards or if they detect an unattended security card on the floor, then all the doors will become permanently locked. However, each room contains a box where you may safely store any number of security cards for use at a later stage.

If you simply tried to travel through the rooms one at a time then as you entered room 3 you would have used all three cards and would be trapped in that room forever!

However, if you make use of the storage boxes, then escape is possible. For example, you could enter room 1 using your first card, place one card in the storage box, and use your third card to exit the room back to the start. Then after collecting three more cards from the dispensing machine you could use one to enter room 1 and collect the card you placed in the box a moment ago. You now have three cards again and will be able to travel through the remaining three doors. This method allows you to travel through all three rooms using six security cards in total.

It is possible to travel through six rooms using a total of 123 security cards while carrying a maximum of 3 cards.

Let C be the maximum number of cards which can be carried at any time.

Let R be the number of rooms to travel through.

Let M(C,R) be the minimum number of cards required from the dispensing machine to travel through R rooms carrying up to a maximum of C cards at any time.

For example, M(3,6)=123 and M(4,6)=23.
And, ΣM(C,6)=146 for $3 \leq C \leq 4$.

You are given that ΣM(C,10)=10382 for $3 \leq C \leq 10$.

Find ΣM(C,30) for $3 \leq C \leq 40$.

Problem 328:

We are trying to find a hidden number selected from the set of integers {1, 2, ..., n} by asking questions. Each number (question) we ask, has a cost equal to the number asked and we get one of three possible answers:

- "Your guess is lower than the hidden number", or
- "Yes, that's it!", or
- "Your guess is higher than the hidden number".

Given the value of n, an *optimal strategy* minimizes the total cost (i.e. the sum of all the questions asked) for the worst possible case. E.g.

If n=3, the best we can do is obviously to ask the number "**2**". The answer will immediately lead us to find the hidden number (at a total cost = 2).

If n=8, we might decide to use a "binary search" type of strategy: Our first question would be "**4**" and if the hidden number is higher than 4 we will need one or two additional questions.
Let our second question be "**6**". If the hidden number is still higher than 6, we will need a third question in order to discriminate between 7 and 8.
Thus, our third question will be "**7**" and the total cost for this worst-case scenario will be 4+6+7=**17**.

We can improve considerably the worst-case cost for n=8, by asking "**5**" as our first question.
If we are told that the hidden number is higher than 5, our second question will be "**7**", then we'll know for certain what the hidden number is (for a total cost of 5+7=**12**).
If we are told that the hidden number is lower than 5, our second question will be "**3**" and if the hidden number is lower than 3 our third question will be "**1**", giving a total cost of 5+3+1=**9**.

Since **12**>**9**, the worst-case cost for this strategy is **12**. That's better than what we achieved previously with the "binary search" strategy; it is also better than or equal to any other strategy.
So, in fact, we have just described an optimal strategy for n=8.

Let C(n) be the worst-case cost achieved by an optimal strategy for n, as described above.
Thus C(1) = 0, C(2) = 1, C(3) = 2 and C(8) = 12.
Similarly, C(100) = 400 and
