On the projection onto the set of isotropic power spectra

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Preliminaries

- The Fourier transform of a bi-dimensional function in cartesian coordinates $\boldsymbol{\nu} = (\nu_1, \nu_2)$:

$$\hat{f}(\nu_1, \nu_2) = \int_{\mathbb{R}^2} f(x_1, x_2).e^{-i2\pi \mathbf{x}\cdot\boldsymbol{\nu}} dx_1 dx_2,$$

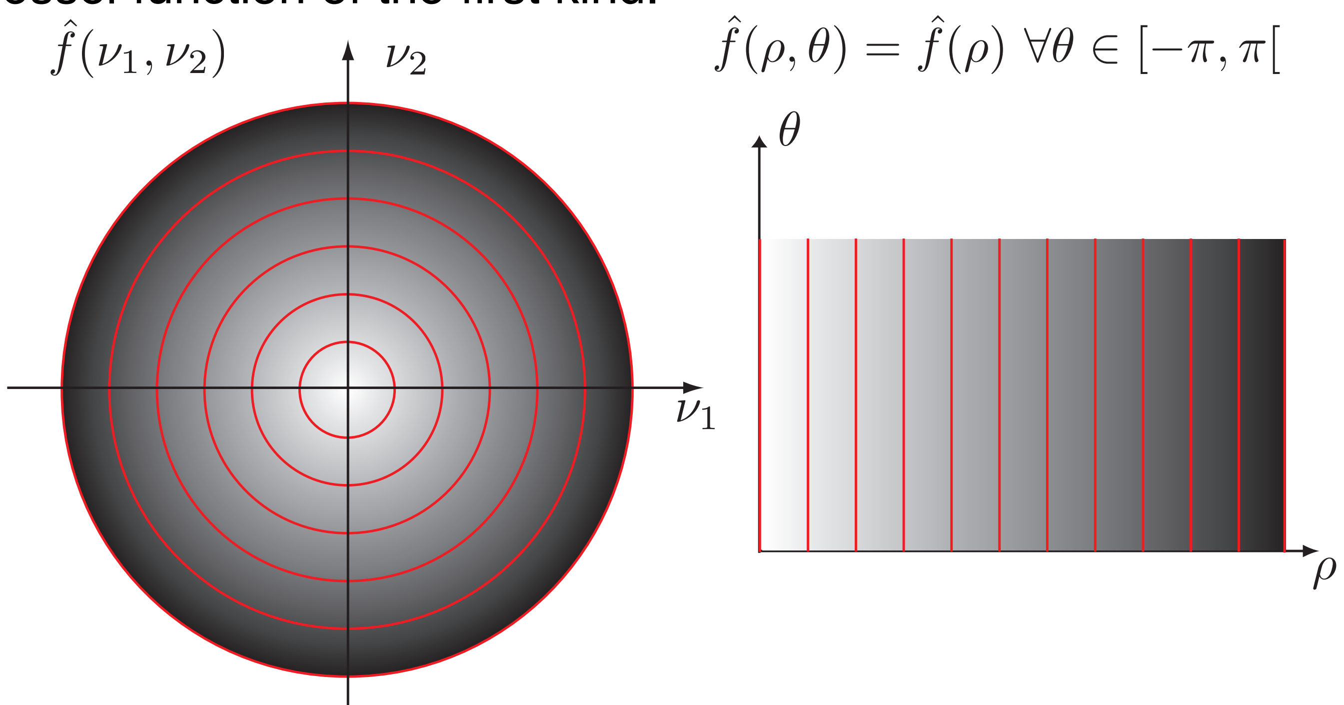
- or in polar coordinates (ρ, θ)

$$\hat{f}(\rho, \theta) = \int_0^{+\infty} \int_0^{2\pi} f(r, \varphi).e^{-i2\pi\rho r\cos(\theta-\varphi)} r dr d\varphi.$$

- If f is circularly (or radially) symmetric, then so is its Fourier transform:

$$\hat{f}(\rho, \theta) = \int_0^{+\infty} f(r) J_0(\rho r) r dr, \ \forall\theta \in [-\pi, \pi[,$$

- This is the zeroth order Hankel transform of the one-dimensional function $f(r)$, where $J_0(z)$ is a zeroth order Bessel function of the first kind.



Main definition

Definition 1 *Let's consider a bi-dimensional random field* $Y(s)$ *defined on* $\mathbb{R}^2$ *(continuous domain) with an appropriate probability space. Let* g *be the action of the rotation group* $SO(2)$ *(special orthogonal group).*

(i) The random process Y *is said to be* isotropic *if its distribution is invariant under the action of* g*. That is*

$$Y \stackrel{d}{=} Y \circ g.$$

(ii) If Y *is a Gaussian stationary random field (SGRF), then it is completely defined by its power-density spectrum (PSD) (its sufficient natural statistic).* Y *will be said* isotropic *if its PSD* $\hat{s}_y(\nu_1, \nu_2)$ *(hence its autocorrelation function, ACF) is circularly symmetric*

$$\begin{aligned} \mathbb{E}\left[Y \circ g(u) \cdot Y \circ g(u+s)\right] &= \mathrm{ACF}_{y \circ g}(s) = \mathrm{ACF}_y(s), \\ \hat{s}_{y \circ g(\nu_1, \nu_2)} &= \hat{s}_y(\nu_1, \nu_2) \iff \hat{s}_z(\rho, \theta) = \hat{s}_z(\rho)\ \forall \theta \in [-\pi, \pi[\ . \end{aligned}$$

Structural constraints on the PSD

Straightforward projection: stochastic programming

- Y is a (cyclo-)SGRV. The Karhunen-Loève basis of Y is the DFT.
- The samples of the DFT $\hat{Y}(\nu_1, \nu_2)$ are a collection of independent *heteroscedastic* complex Gaussian RV whose variance function is the PSD $\hat{s}_y$.
- The set of isotropic SGRFs is the closed set

$$\mathscr{S}_{\text{iso}} := \{Z \big| \hat{s}_z(\rho, \theta) = \hat{s}_z(\rho)\ \forall \theta \in [-\pi, \pi[\}.$$

- The naive projector of Y onto this set is the solution to the optimization problem

$$\min_{Z \in \mathscr{S}_{\text{iso}}} \|Y - Z\|_F^2 .$$

- Main difficulties:
  - Constraint set involves the expectation operator $\Rightarrow$ *stochastic programming*.
  - The constraint set is not convex.
  - Involves a generally uknown deterministic function $\hat{s}_z$ associated to the random process Z.
  - Can be relaxed it somehow by parametrizing $\hat{s}_z$ and optimizing with respect to these hyperparameters.

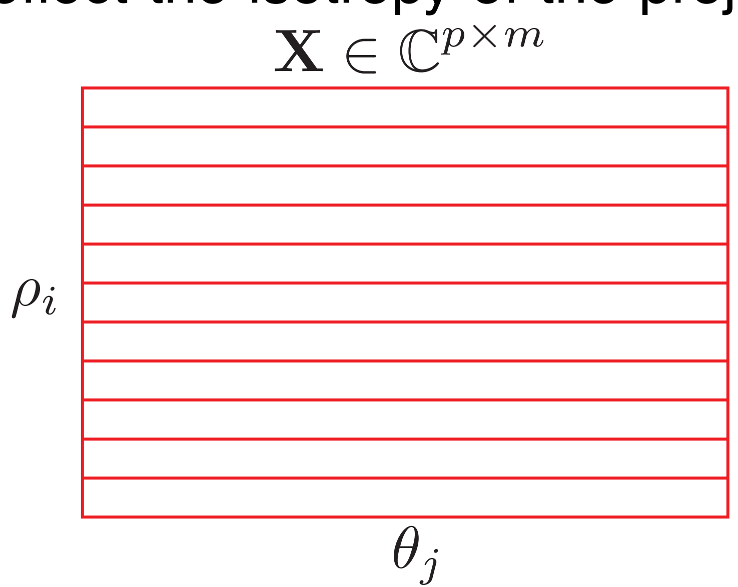
Structural constraints on the PSD

Projection as a matrix design problem

- The zero-mean SGRF Y defined on the square $[0, n-1]^2$. $\mathbf{X} \in \mathscr{M}_{p\times m}(\mathbb{C})$ the matrix storing the polar DFT of Y.
- The columns of $\mathbf{X}$ are the radial samples of the polar DFT, and its rows the angular samples.
- If $Y \in \mathscr{S}_{\text{iso}}$, the samples in each row i of $\mathbf{X}$ are iid complex $\mathcal{N}(0, \hat{s}_y(\rho_i))$.
- We then consider the optimization problem with a spectral constraint on $\mathbf{Z}\mathbf{Z}^*$,

$$\min_{\mathbf{Z}} \|\mathbf{X} - \mathbf{Z}\|_F^2 \text{ subject to } \mathbf{Z}\mathbf{Z}^* = \operatorname{diag}\left(\alpha \left(\sigma_i^2\right)_{1\le i\le p}\right),$$

- Off-diagonal elements in the constraint vanish because of independence.
- The choice of all $\sigma_i > 0$ must reflect the isotropy of the projected observation, and $\alpha > 0$ is a real constant to be optimized.



Structural constraints on the PSD

Projection as a matrix design problem

Theorem 1 *Let $p \leq m$, and $\mathbf{\Sigma} \in \mathscr{M}_{p\times m}(\mathbb{R})$ is such that $\mathbf{\Sigma} = \mathrm{diag}(\sqrt{\alpha}\sigma_i)$. The solution $\mathbf{Z}$ to matrix design problem is*

$$\mathbf{Z} = \mathbf{\Sigma}\mathbf{B}\mathbf{A}^*,$$

where $\mathbf{A}$ and $\mathbf{B}$ are the left and right matrices in the SVD of $\mathbf{X}^*\mathbf{\Sigma}$. The solution is uniquely determined if $\mathbf{X}^*\mathbf{\Sigma}$ is nonsingular. Let $\sigma_i = \sum_{j=1}^{m} |\mathbf{X}_{i,j}|$. The best α for which the projected process is isotropic almost surely is $1/m$. In this case, the equality spectral constraint is sharp.

Structural constraints on the PSD

Projection with a deterministic constraint

- Formulate the projection problem as finding the nearest $\mathbf{Z}$ to $\mathbf{X}$ such that its columns deviate the least possible from some pre-specified vector $\boldsymbol{\mu} \in (\mathbb{R}^+)^p$ (independent of the angle index):

$$\min_{\mathbf{Z} \in \mathcal{C}_\varepsilon} \|\mathbf{X} - \mathbf{Z}\|_F^2 \,,$$

$$\mathcal{C}_\varepsilon = \{\mathbf{Z} \in \mathscr{M}_{p \times m}(\mathbb{C}) : ||\mathbf{Z}_{i,j}| - \boldsymbol{\mu}_i| \leq \varepsilon, \ (i,j) \in \{1, \ldots, p\} \times \{1, \ldots, m\}\} \,.$$

- In fact, the most crucial part is to find or estimate the vector $\boldsymbol{\mu}$: to be related to $\hat{s}(\rho)$ for an isotropic process.

- In practice, when the process is not isotropic, the robust median estimator can be used.

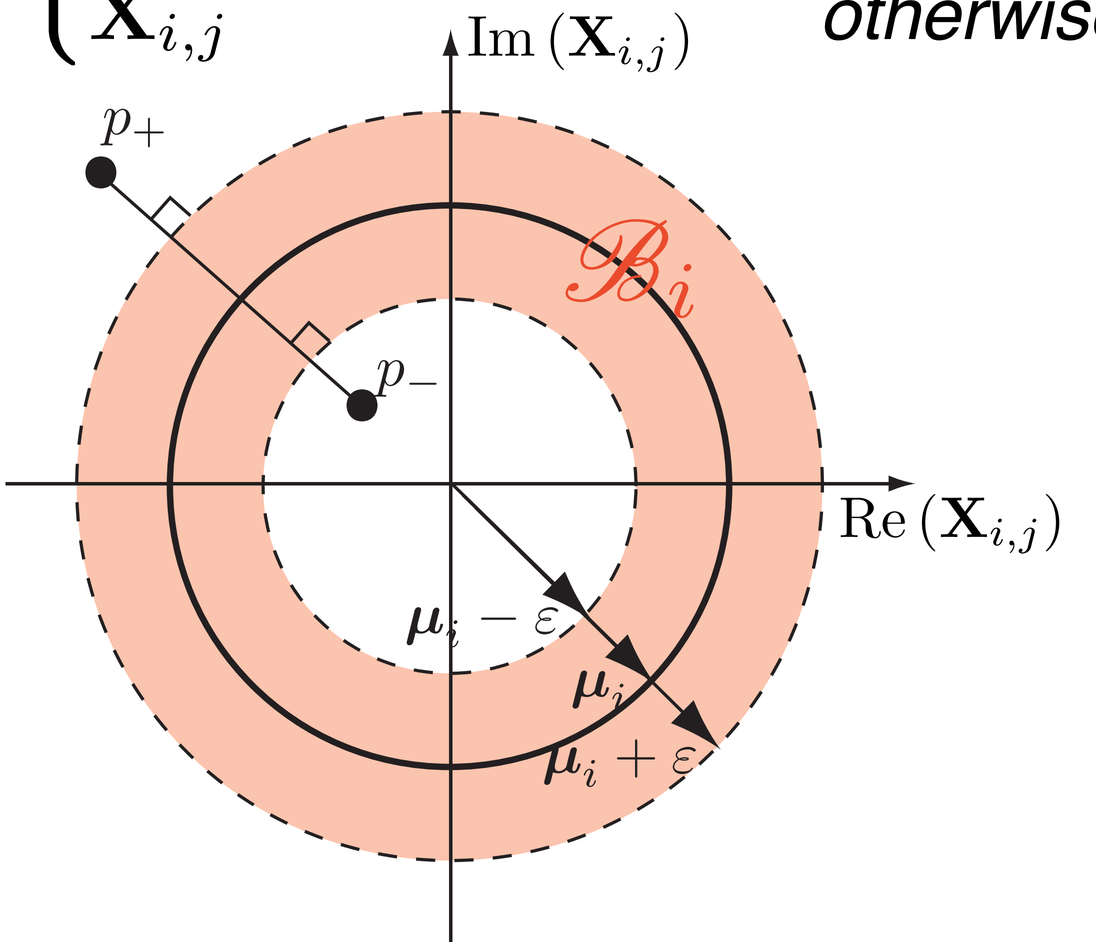
Structural constraints on the PSD

Projection with a deterministic constraint

- Only some constraints are convex, but $\mathcal{C}_\varepsilon = \bigtimes_{i,j} \mathscr{B}_i$.

Proposition 1 *The projector onto the closed partially convex set $\mathcal{C}_\varepsilon$ is unique*

$$\mathcal{P}_{\mathcal{C}_\varepsilon}(\mathbf{X}_{i,j}) = \begin{cases} (\boldsymbol{\mu}_i + \varepsilon)\mathbf{X}_{i,j} / |\mathbf{X}_{i,j}| & \textit{if } |\mathbf{X}_{i,j}| > \boldsymbol{\mu}_i + \varepsilon \ , \\ (\boldsymbol{\mu}_i - \varepsilon)\mathbf{X}_{i,j} / |\mathbf{X}_{i,j}| & \textit{if } |\mathbf{X}_{i,j}| < \boldsymbol{\mu}_i - \varepsilon \ , \\ \mathbf{X}_{i,j} & \textit{otherwise.} \end{cases}$$



Structural constraints on the PSD

On the choice of the constraint radius

- Devise a statistically meaningful choice of the constraint radius ε.
- First, investigate the null (isotropy) distribution of $|\mathbf{Z}_{i,j}| - \boldsymbol{\mu}_i, \forall (i,j)$, e.g. when $\boldsymbol{\mu}_i = \left(m^{-1} \sum_{j=1}^{m} |\mathbf{X}_{i,j}|^2\right)^{1/2}$.
- Classical PSD estimate results for SGRF

$$2 \left|\mathbf{Z}_{i,j}\right|^2 \sim \hat{s}_z(\rho_i)\chi^2(2) \quad \text{and} \quad 2m\boldsymbol{\mu}_i^2 \sim \hat{s}_z(\rho_i)\chi^2(2m), \quad 1 \leq i \leq p\,.$$

- By the Fisher asymptotic formula,

$$|\mathbf{Z}_{i,j}| \xrightarrow{d} \sqrt{\hat{s}_z(\rho_i)}\mathcal{N}(\sqrt{3/4}, 1/4) \quad \text{and} \quad \boldsymbol{\mu}_i \xrightarrow{d} \sqrt{\hat{s}_z(\rho_i)}\mathcal{N}(\sqrt{1 - 1/(4m)}, 1/(4m)), \quad 1 \leq i \leq p\,.$$

- Ignoring the obvious dependency between $|\mathbf{Z}_{i,j}|$ and $\boldsymbol{\mu}_i$,

$$|\mathbf{Z}_{i,j}| - \boldsymbol{\mu}_i \xrightarrow{d} \sqrt{\hat{s}_z(\rho_i)}\mathcal{N}(\sqrt{3/4} - \sqrt{1 - 1/(4m)}, (1 + 1/m)/4)\,.$$

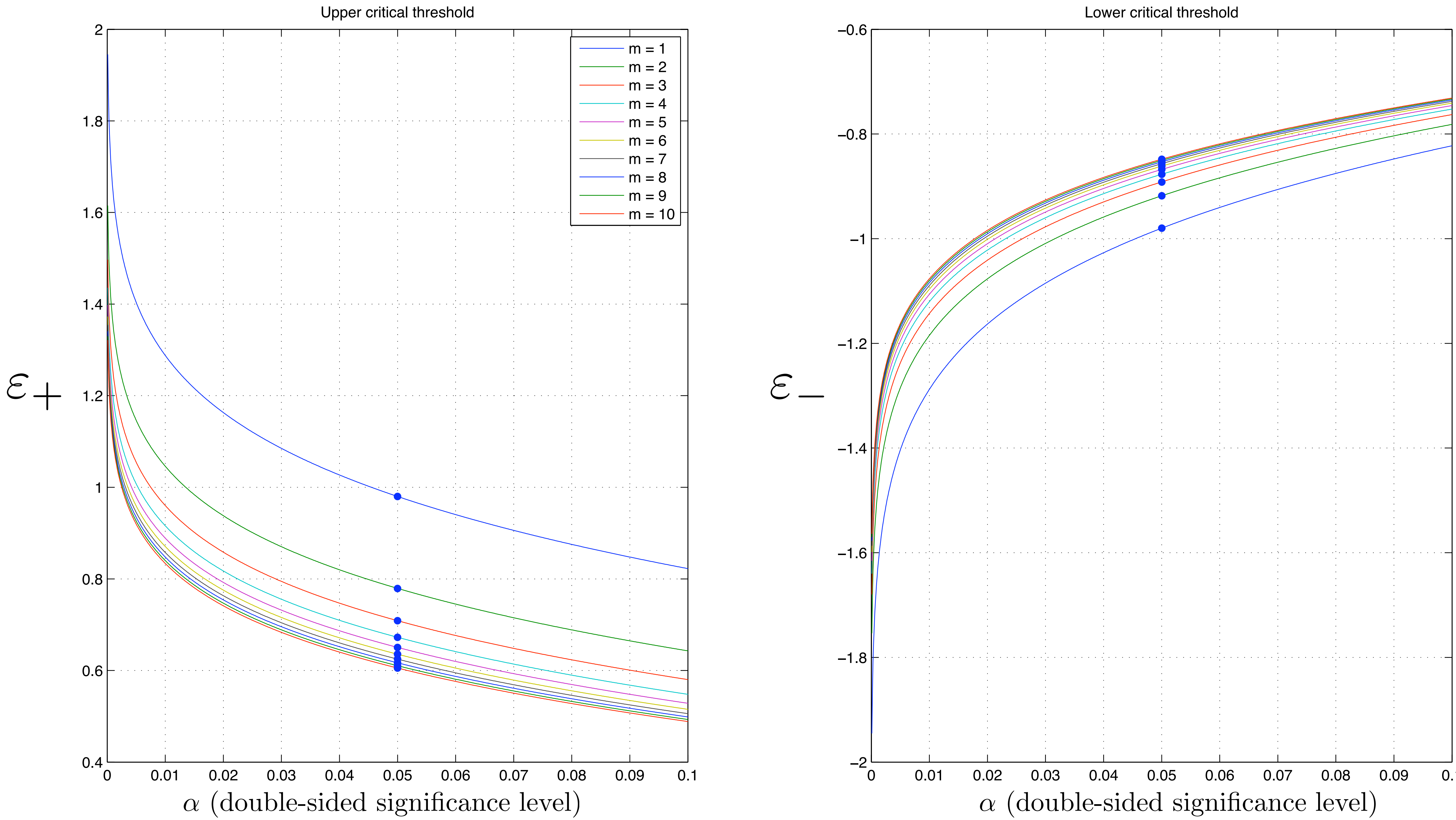
- The upper and lower critical thresholds at the double-sided significance level $0 \leq \alpha \leq 1$ are

$$\varepsilon_{\pm} = \sqrt{\hat{s}_z(\rho_i)}\left(\sqrt{3/4} - \sqrt{1 - 1/(4m)} \pm \sqrt{(1 + 1/m)/4}\Phi^{-1}(1 - \alpha/2)\right)\,,$$

- The thresholds are not symmetric $\Rightarrow$ different upper and lower values of ε should be used.

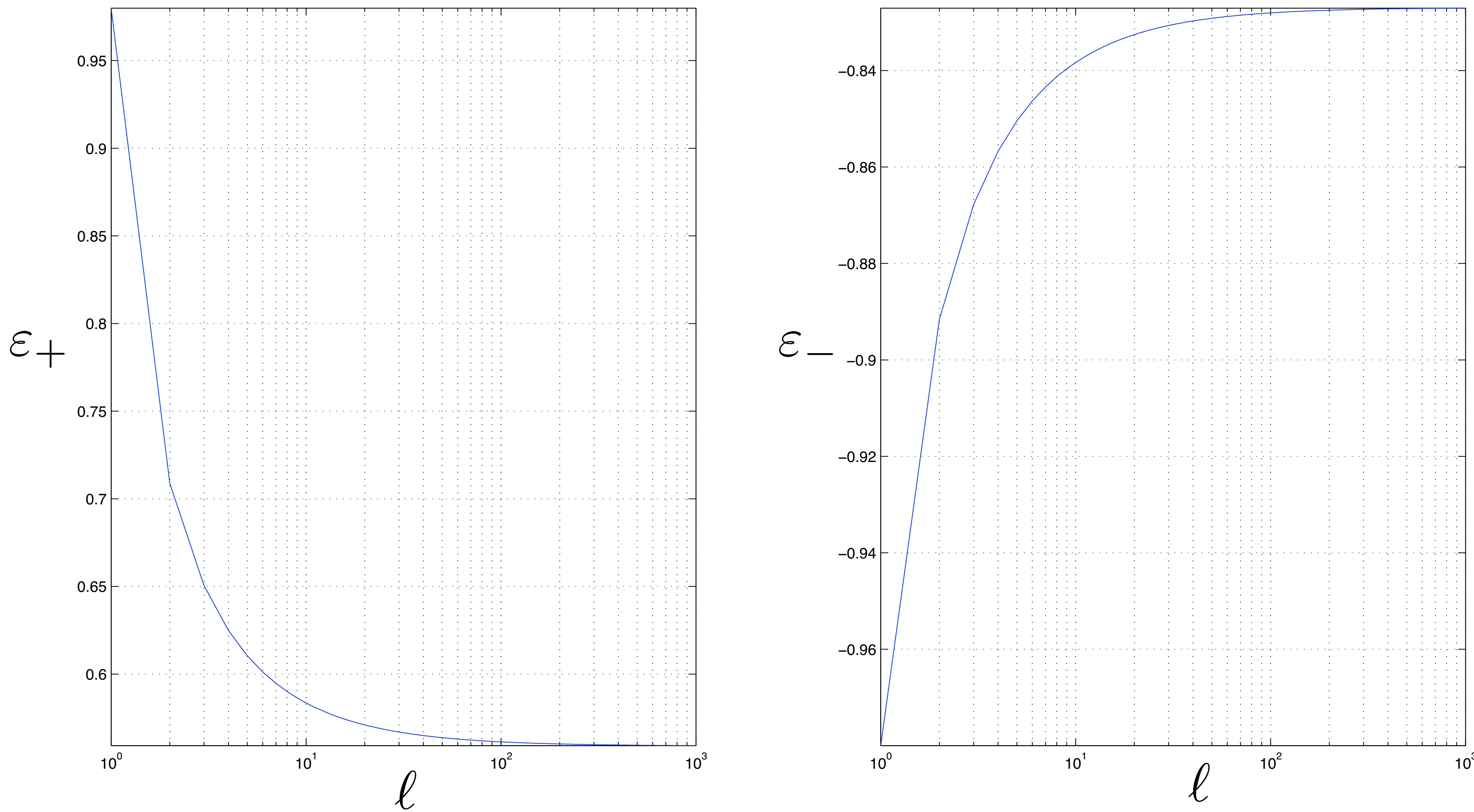
Structural constraints on the PSD

On the choice of the constraint radius



Structural constraints on the PSD

On the choice of the constraint radius



Application to random processes on the sphere, normalized critical thresholds for $\alpha = 0.05$ vs ℓ such that $m = 2\ell - 1$. For ℓ large enough (typically ≥ 100), the two thresholds attain the limit values $\sqrt{3/4} - 1 + 1.96/\sqrt{8} = 0.559$ and $\sqrt{3/4} - 1 - 1.96/\sqrt{8} = -0.8269$.

Application to CMB inpainting on the sphere

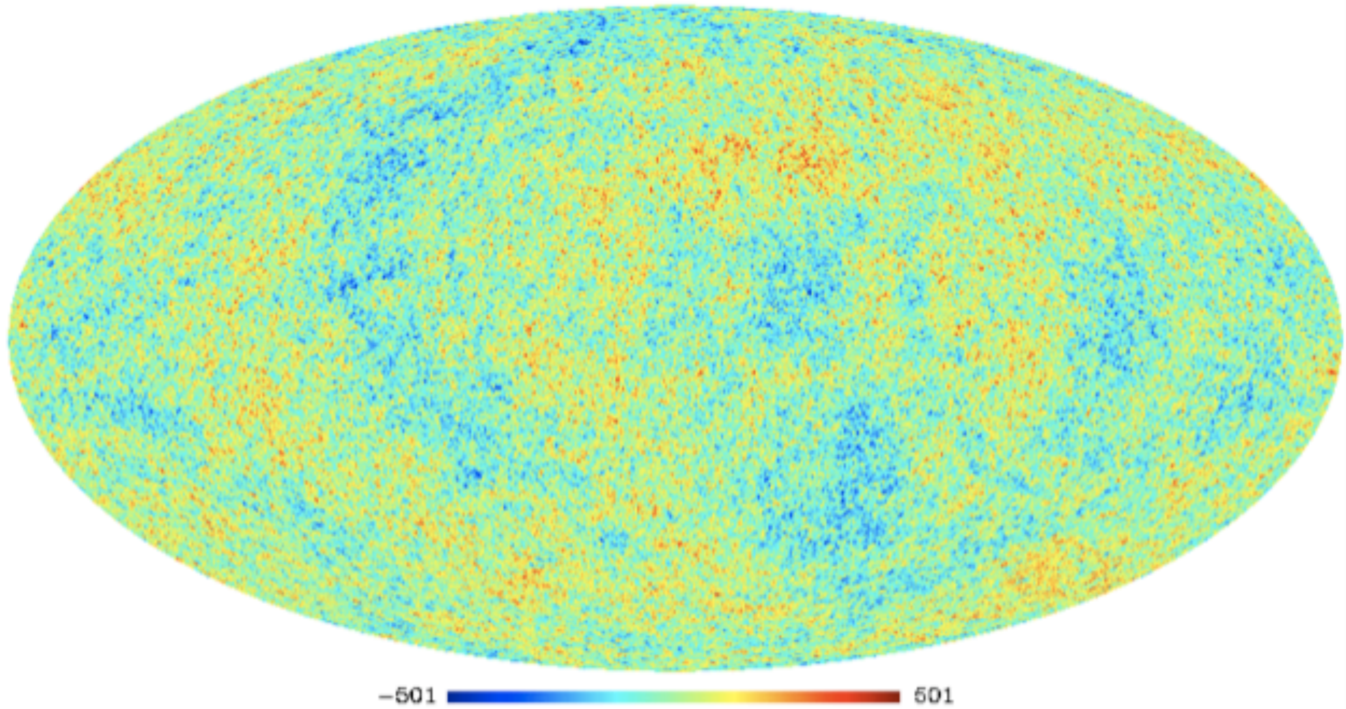
- Solve the hard *nonconvex* feasibility problem

$$\min_{Z} \|Y - Z\|_F^2 \text{ subject to } Z \in \mathcal{C}_\varepsilon \cap \{\mathbf{Z} | \mathbf{M}Y = \mathbf{M}Z\} \ .$$

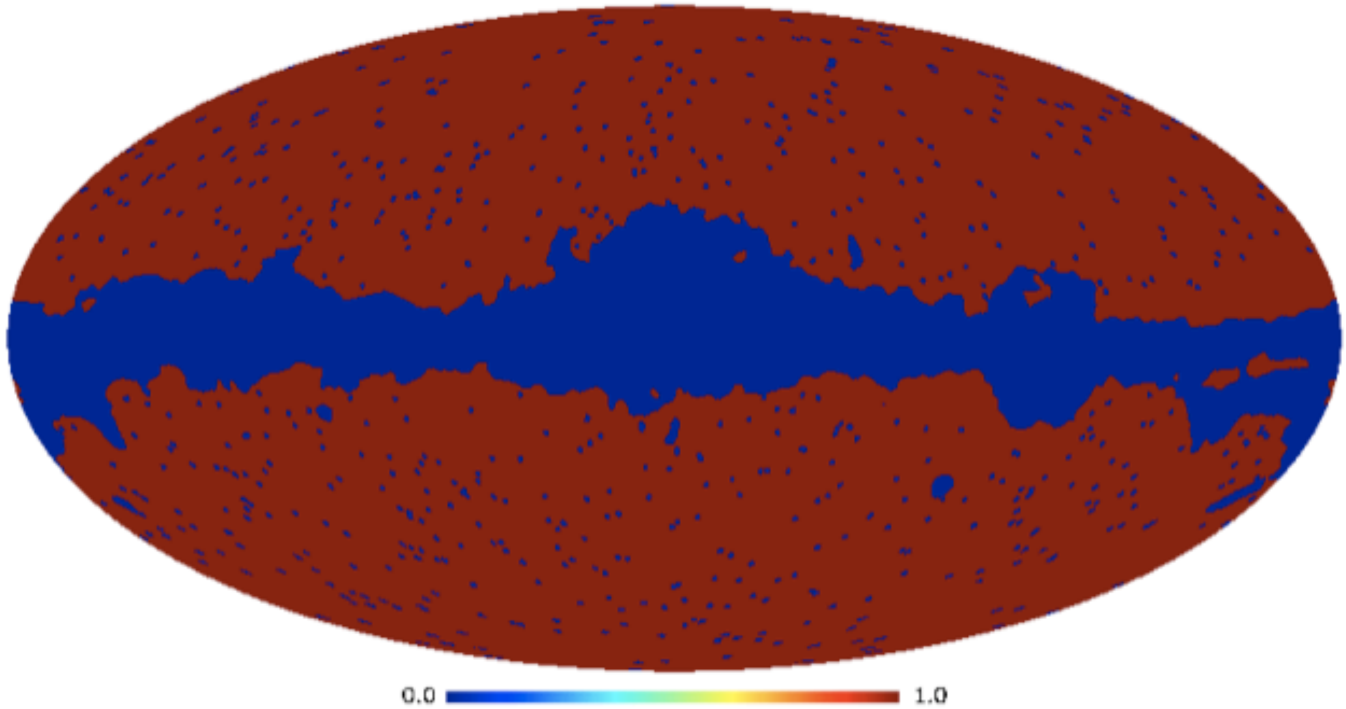
- The feasible set is nonempty since the constraint set is non-empty and closed; the CMB is in it.
- Apply alternating projection with a good starting point.
- Convergence proof can be inspired by arguments from [LewisMalick07].

Preliminary results

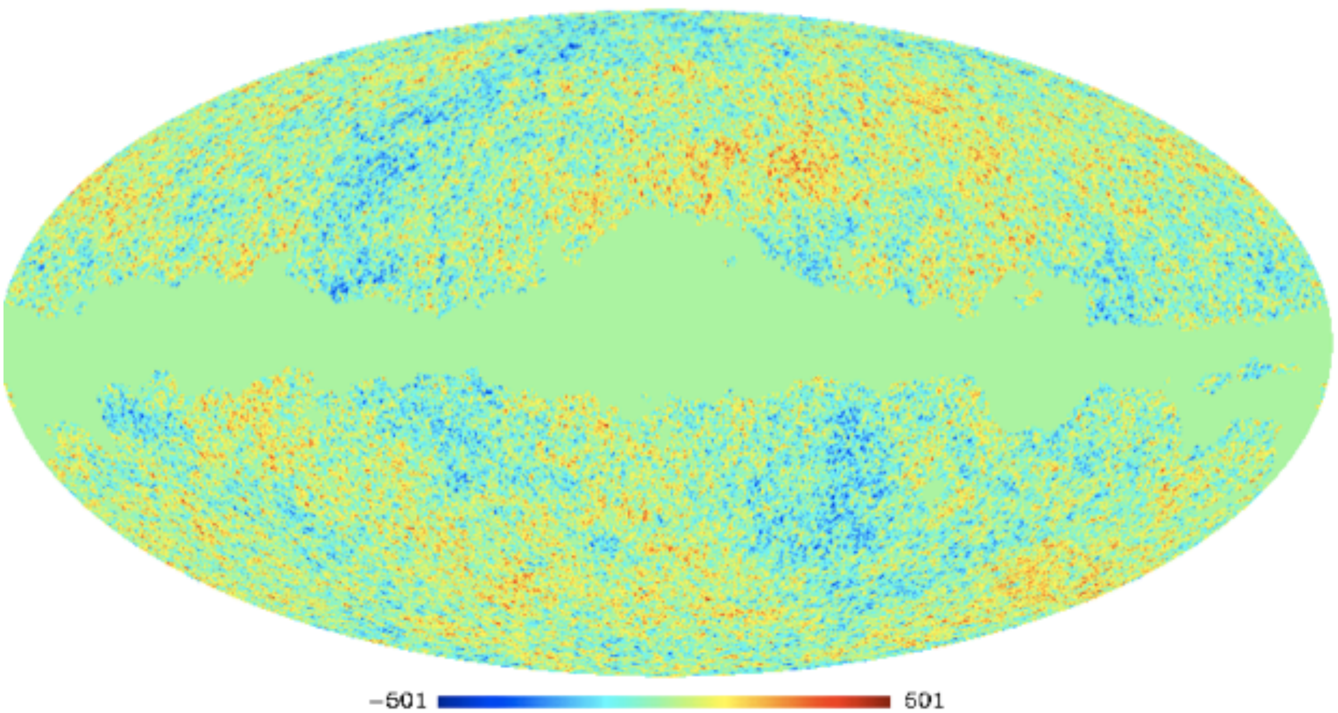
Ground truth simulated CMB map



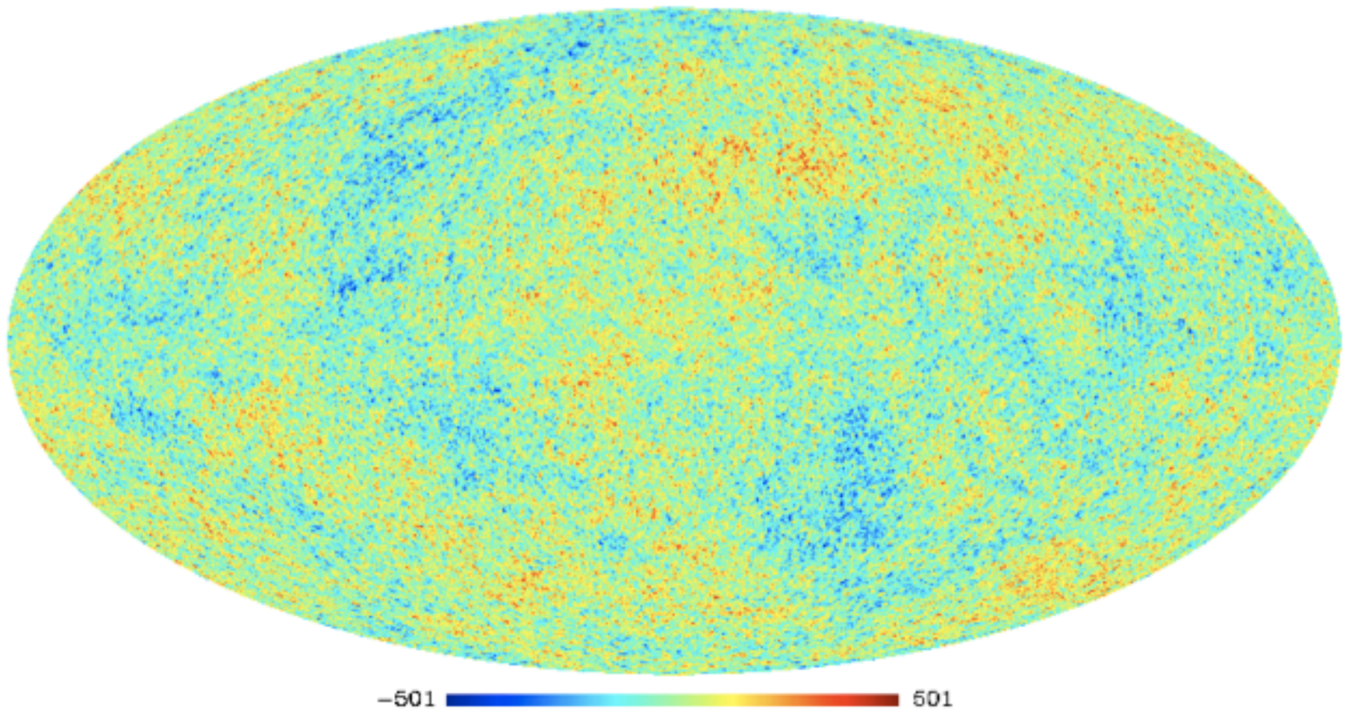
Mask map



Masked CMB map

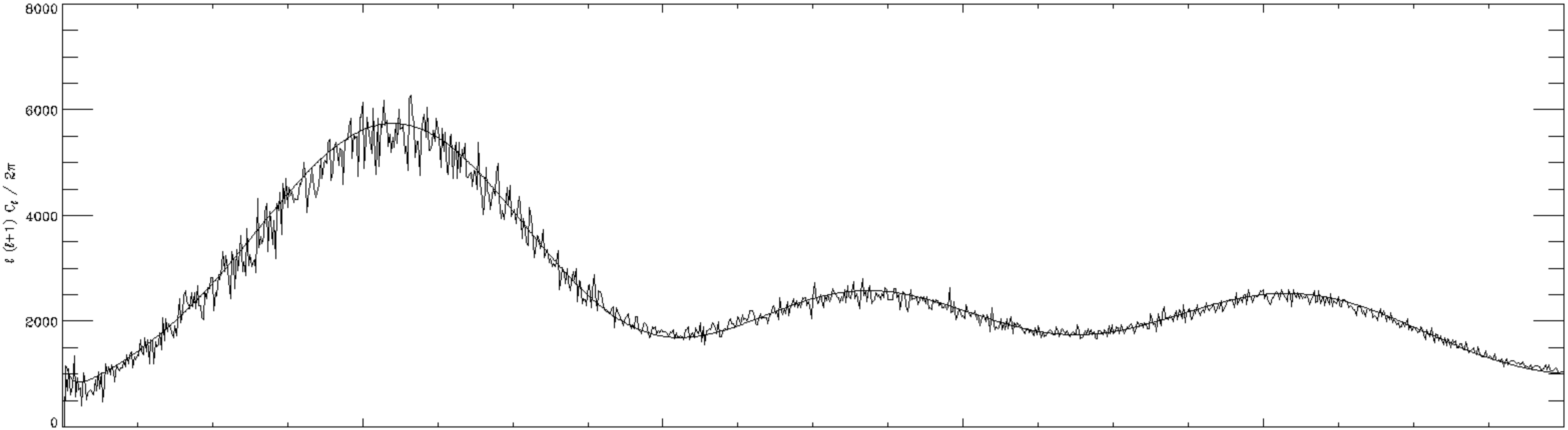


Inpainted using the isotropy-constraint



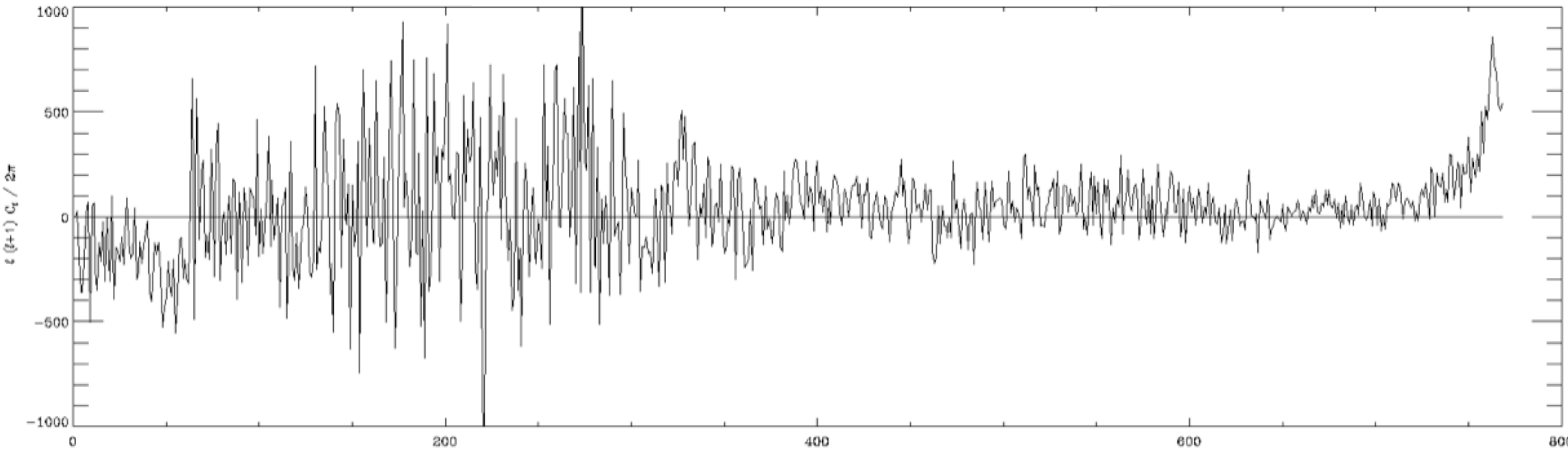
Preliminary results

CMB estimated and original PSD



Preliminary results

CMB PSD error using MCA inpainting



CMB PSD error using isotropy-constrained inpainting