Homework 1

- Problems marked with **P** are graded on progress, which means that they are graded subjectively on the perceived progress toward a solution, rather than **solely** on correctness.
- Problems marked with a **G** are group problems. A group of 1-3 students may work on these together and turn in one assignment for the entire group. Each member listed should have made real contributions. The group problems will be turned in separately on gradescope.

For bonus and most extra credit questions, we will not provide any insight during office hours or Piazza, and we do not guarantee anything about the difficulty of these questions. We strongly encourage you to typeset your solutions in LaTeX. If you collaborated with someone, you must state their name(s). You must write your own solution for all problems and may not look at any other student's write-up.

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0. If applicable, state the name(s) and uniqname(s) of your collaborator(s).

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1. *Extra credit*: (10) Typeset this **entire** assignment in LaTeX and draw a table with two columns that includes the *name* (i.e., "fraction") and an *example* of each of the following:
   - less than or equal to
   - superscript and subscript
   - fraction (using `\frac`)
   - set intersection
   - summation using sigma (Σ) notation

   Unlike most extra credit questions, we will help with this in office hours if asked though it will be a lower priority.

Solution:

name	example
less than or equal to	$\leq$
superscript and subscript	x_2^2
fraction	$\frac{1}{2}$
set intersection	$\cap$
summation using sigma notation	$\sum_{i=0}^{\infty} 3^i$

P 2. (20) For the following pairs of $f(n)$ and $g(n)$, is $f(n) = O(g(n))$? Justify your answer by applying the definition of big-O or by applying a limit argument.

Hint: You may find L'Hôpital's Rule useful.

(a) $f(n) = n + 2\log_2(n) + 3$, $g(n) = \frac{1}{3}n + 2$.

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n+2\log_2(n)+3}{\frac{1}{3}n+2}$
$= lim_{x\to\infty} \frac{1+\frac{2}{xln(2)}}{\frac{1}{3}}$ L'Hopital's Rule
$= lim_{x\to\infty} 3 + \frac{6}{xln(2)}$
$= 3$
Thus $f(n) = O(g(n))$

(b) $f(n) = n^2$, $g(n) = 2^n$

Solution: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2}{2^n}$
$= \lim_{n\to\infty} \frac{2n}{2^n ln(2)}$ L'Hopital's Rule
$= \lim_{n\to\infty} \frac{2}{2^n ln(2)^2}$ L'Hopital's Rule
$= \lim_{n\to\infty} \frac{0}{2^n ln(2)^3}$ L'Hopital's Rule
$= 0$
Thus $f(n) = O(g(n))$

(c) $f(n) = 3^n$, $g(n) = 2^n$

Solution: $lim_{n\to\infty} \frac{f(n)}{g(n)} = lim_{n\to\infty} \frac{3^n}{2^n}$
$= \lim_{n\to\infty} (\frac{3}{2})^n$
$= \infty$
Thus $f(n) \neq O(g(n))$

(d) $f(n) = \ln(n)$, $g(n) = \ln(\frac{n}{5})$

Solution: $lim_{n\to\infty} \frac{f(n)}{g(n)} = lim_{n\to\infty} \frac{ln(n)}{ln(\frac{n}{5})}$
$= lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}}$
$= lim_{n\to\infty} \frac{n}{n}$
$= 1$
Thus $f(n) = O(g(n))$

(e) $f(n) = (\log_2 n)^3$, $g(n) = 2^{\log_2 n}$

Solution: $lim_{n\to\infty} \frac{f(n)}{g(n)} = lim_{n\to\infty} \frac{(\log_2 n)^3}{2^{\log_2 n}}$
$= lim_{n\to\infty} \frac{\frac{3(\log n)^2}{\log(2)^3 n}}{1}$ L'Hopital's Rule
$= lim_{n\to\infty} \frac{\frac{6\log n}{ln(10)n}}{\log(2)^3}$ L'Hopital's Rule
$= lim_{n\to\infty} \frac{\frac{6}{ln(10)n}}{\log(2)^3 ln(10)}$ L'Hopital's Rule
$= lim_{n\to\infty} \frac{-6}{ln(10)x^2}$ L'Hopital's Rule

$= 0$
Thus $f(n) = O(g(n))$

G 3. (25) Prove the following claims by filling in the blanks provided.

Pay special attention to the exposition. Both proofs start from the assumptions and each make their way to the conclusion making sure that each step is complete before going to the next step and do not wander between claims. The proofs in (a) are proofs by cases, and the proof in (b) is a direct proof.

Another feature to notice and consider introducing in your mathematical writing is overviews of proofs. These are especially helpful for more complex proofs. The proof in (a) provides an overview of its structure which makes the proof easier to read by guiding the reader on how the counting will be done.

(a) Each face of a six-sided die is randomly painted red or blue with equal probability. What is the probability that there is at least one corner of the die where all three sides that make up that corner are of the same color? Clearly show your work.

There are two different approaches we can take: we can do a direct count of how many cubes have a 1-color corner or we can instead invert the problem to compute the number of cubes without a 1-color corner. Either way, it tends to be much easier to get an accurate count if we split it up by how many sides of a given color there are.

Direct:

6 red sides: only 1 cube and it has a 1-color corner.

5 red sides: ______ ways to pick the blue side. All cubes have a 1-color corner.

4 red sides: The 2 blue sides can either be next to each other or across from each other. If they are next to each other, ______________________________, while if they are across, ______________________________. To count the number of ways we can pick them making them adjacent, we pick a side (6 ways), pick an adjacent side (4 ways), and then note that we counted everything twice so we have ______ 1-color corner cubes.

3 red sides: After some reasoning, we can conclude that the 3 red sides will either be wrapped around a corner or stretched out in a line. If they are in a line, there is no 1-color corner, while if they are around a corner, there is. To count this, we can just pick which corner to wrap the red sides around, giving ______ options.

2 red sides: By symmetry this is the same as ______ red sides, ______ options.

1 red side: By symmetry this is the same as ______ red sides, ______ options.

<u>0 red sides:</u> By symmetry this is the same as _____ red sides, _____ options.

In total, this gives _____ cubes with a 1-color corner out of _____, so the probability is _____.

Indirect:

In order to guarantee there is no 1-color corner, we need a pair of opposite sides to be red and another pair of opposite sides to be blue. When there are only 0 or 1 of a given color, this is therefore impossible, so we need only consider the case of 4 and 2, 3 and 3, and 2 and 4.

<u>4 red sides</u>: In order to have the 2 blue sides be opposite each other, we pick one axis of the cube for them to be on. There are _____ ways to do this, so there are _____ cubes without a 1-color corner.

<u>3 red sides</u>: If we have 1 opposite pair of red sides and 1 opposite pair of blue sides, the other pair will be 1 red 1 blue, meaning all _____ can be told apart. To count this, we pick one pair for red (3), 1 pair for blue from what's left (2), and the remaining pair can be either BR or RB (2), so in total, there are _____ cubes without a 1-color corner.

<u>2 red sides</u>: By symmetry this is the same as _____ red sides, _____ cubes without a 1-color corner.

In total this gives _____ cubes without a 1-color corner out of _____, so the probability is _______.

Solution: Direct:

<u>6 red sides:</u> only 1 cube and it has a 1-color corner.

<u>5 red sides:</u> 6 ways to pick the blue side. All cubes have a 1-color corner.

<u>4 red sides:</u> The 2 blue sides can either be next to each other or across from each other. If they are next to each other, there will be a 1-color corner on the cube, while if they are across, there will not be a one color corner on the cube. To count the number of ways we can pick them making them adjacent, we pick a side (6 ways), pick an adjacent side (4 ways), and then note that we counted everything twice so we have 12 1-color corner cubes.

<u>3 red sides:</u> After some reasoning, we can conclude that the 3 red sides will either

be wrapped around a corner or stretched out in a line. If they are in a line, there is no 1-color corner, while if they are around a corner, there is. To count this, we can just pick which corner to wrap the red sides around, giving 8 options.

<u>2 red sides:</u> By symmetry this is the same as 4 red sides, 12 options.

<u>1 red side:</u> By symmetry this is the same as 5 red sides, 6 options.

<u>0 red sides:</u> By symmetry this is the same as 6 red sides, 1 options.

In total, this gives 46 cubes with a 1-color corner out of 64, so the probability is .719.

Indirect:

In order to guarantee there is no 1-color corner, we need a pair of opposite sides to be red and another pair of opposite sides to be blue. When there are only 0 or 1 of a given color, this is therefore impossible, so we need only consider the case of 4 and 2, 3 and 3, and 2 and 4.

<u>4 red sides</u>: In order to have the 2 blue sides be opposite each other, we pick one axis of the cube for them to be on. There are 3 ways to do this, so there are 3 cubes without a 1-color corner.

<u>3 red sides</u>: If we have 1 opposite pair of red sides and 1 opposite pair of blue sides, the other pair will be 1 red 1 blue, meaning all pairs of opposite sides can be told apart. To count this, we pick one pair for red (3), 1 pair for blue from what's left (2), and the remaining pair can be either BR or RB (2), so in total, there are 12 cubes without a 1-color corner.

<u>2 red sides</u>: By symmetry this is the same as 4 red sides, 3 cubes without a 1-color corner.

In total this gives 18 cubes without a 1-color corner out of 64, so the probability of no 1-color corners is .281. Therefore the probability of at least 1 1-color corner is .719

(b) Say we have five points (x_i, y_i), each distinct for $i = 1$ to 5 where each x and y value is an integer. Now say you draw a line connecting each pair of points. Prove that the midpoint of at least one of those lines has an (x, y) location where both x and y are integers.

Every integer is even or odd, so there are only 4 ordered pairs of even or odd: (even, even), (even, odd), (odd, even), (odd, odd). If we sort each point into these 4 categories by the parity of its x and y coordinates, we find that because we have more points than categories, there must be at least one category that has ______________. Given two points (x_i, y_i) and (x_j, y_j), the midpoint of the line between them is ______________. If x_i and x_j are both even or both odd, the x coordinate of the midpoint ______________ (and the same thing for the y coordinates). This means the pair of points in ______________ will have a midpoint that has integer coordinates.

Solution: Every integer is even or odd, so there are only 4 ordered pairs of even or odd: (even, even), (even, odd), (odd, even), (odd, odd). If we sort each point into these 4 categories by the parity of its x and y coordinates, we find that because we have more points than categories, there must be at least one category that has at least two points. Given two points (x_i, y_i) and (x_j, y_j), the midpoint of the line between them is $(\frac{x_j - x_i}{2}, \frac{y_j - y_i}{2})$. If x_i and x_j are both even or both odd, the x coordinate of the midpoint will be an even number divided by two which will be an integer (and the same thing for the y coordinates). This means the pair of points in the same category of whether their x and y coordinates are even or odd will have a midpoint that has integer coordinates.

P 4. (25) Do the following.

(a) Prove that $\sqrt{27}$ is irrational. (**Hint:** you may find it easiest to use a proof by contradiction.)

Solution: Proof by contradiction:
Assume $\sqrt{27}$ is rational, then by definition of a rational, $\sqrt{27} = \frac{a}{b}$ where a and b are integers that share no common factors greater than 1.
Then $27 = \frac{a^2}{b^2}$ and $27b^2 = a^2$
Since a^2 is divisible by 27 and 27 is prime then it must also be true that a is divisible by 27.
Then a = 27k for some integer k. Then $27b^2 = (27k)^2$ which is $27b^2 = 27^2k^2$. We then get $b^2 = 27k^2$. This shows b^2 is divisible by 27 and since 27 is prime b must also be divisible by 27. This makes a contradiction as we have shown a and b both to be divisible by 27 when they must not share any common factors greater than 1. Therefore by proof of contradiction the assumption of $\sqrt{27}$ being rational is false and we prove that $\sqrt{27}$ is irrational.

(b) Prove by induction that the number of n-character strings over the alphabet $\{a, b, c\}$ (i.e. strings of length exactly n, where each character is chosen from the set $\{a, b, c\}$) is 3^n.

Solution: Given the number of n character strings over the alphabet a,b,c is 3^n.
Base Case:
n=1
1-character strings from the alphabet a,b,c are "a","b","c". There are three of them.
$3^1 = 3$ therefore proving the base case.
Inductive Case:
Assume for an integer $k >= 1$ there will be 3^k possible k-character strings from the alphabet a,b,c.
If we were to add another character to find $k+1$ character strings we would have 3 options to choose from to add another character. This gives us $3^k * 3$ character strings when k=k+1.
$3^k * 3 = 3^{k+1}$, so therefore there is 3^{k+1} $k+1$-character strings. By proving $k+1$ is true we prove all values of n are true.
By proving both the base case and inductive step we prove that the number of n-character strings over the alphabet $\{a, b, c\}$ is 3^n.

(c) Suppose you want to unambiguously represent all of the elements of $\{0, 1, \ldots, n-1\}$ using ternary strings of a certain length k, i.e., strings with k characters from the set $\{a, b, c\}$. What's the smallest possible choice for k?

Solution: The set $\{0, 1, ..., n-1\}$ has n distinct elements so we will need n different strings to unambiguously represent all of the elements.
Given that there are 3^k unique strings possible from the alphabet when the strings are k-characters long. This is proven in part b of this question. We need at least n unique strings to represent all the elements so we know:
$n \leq 3^k = \log n \leq \log 3^k = \log n \leq k \log 3$
$= k \geq \frac{\log n}{\log 3}$
Therefore the smallest choice for k is $\lceil \frac{\log n}{\log 3} \rceil$

P 5. (10) Consider the following homework solver:

```
function SOLVE-HW(X):
    for i in range(sqrt(X)):
        print 'hey grader, give me ' + X + ' points '
    return X
```

Give an asymptotic complexity bound on the number of characters printed by the program as a function of the input X (X is an integer). Make this bound as tight as possible. Assume that X is printed in decimal (base-10) format.

Note: our providing this program does not constitute a guarantee of any particular lower bound on the number of points you should expect to receive if you choose to implement it and run it to solve your own homeworks.

Solution: The loop runs $\sqrt{X}$ times due to the for loop. Each time the loop runs 29 letter characters are printed and $\frac{X}{10}$ numerical characters are printed. Therefore since $30 + \frac{X}{10}$ characters are printed $\sqrt{X}$ times, the asymptotic complexity bound is $\Theta((30 + \frac{X}{10})\sqrt{X}) = \Theta(X\sqrt{X})$

6. (20) Euclidean algorithm

   (a) Use the Euclidean algorithm to compute the GCDs of the following pairs of integers. Show each step of the algorithm, and compute the value of the potential s_i at each stage. Verify that indeed $s_{i+1} \leq \frac{2}{3}s_i$. How many recursive calls does the algorithm invoke, excluding any in which only the order of the operands was swapped?

   You are encouraged to use a computer for this part.

   i. $(45, 162)$
   ii. $(183, 101)$
   iii. $(233, 144)$
   iv. $(1234, 3456)$

Solution: i. gcd(162,45) = gcd(45, 27) step 0, $s_0 = 72$ $\frac{2}{3}s_0 = 48$
gcd(45,27) = gcd(27,18) step 1, $s_1 = 45$ $\frac{2}{3}s_1 = 30$ $45 \leq 48$
gcd(27,18) = gcd(18,9) step 2, $s_2 = 27$ $\frac{2}{3}s_2 = 18$ $27 \leq 30$
gcd(18,9) = gcd(9,0) step 3, $s_3 = 9$ $\frac{2}{3}s_3 = 6$ $9 \leq 18$
gcd(9,0) = 9 step 4, $0 \leq 6$
In total, 4 recursive calls and the gcd(45,162)=9

ii. gcd(183,101) = gcd(101,82) step 0, $s_0 = 183$ $\frac{2}{3}s_0 = 122$
gcd(101,82) = gcd(82,19) step 1, $s_1 = 101$ $\frac{2}{3}s_1 = 67.333$ $101 \leq 122$
gcd(82,19) = gcd(19,6) step 2, $s_2 = 25$ $\frac{2}{3}s_2 = 16.6667$ $25 \leq 67.333$
gcd(19,6) = gcd(6,1) step 3, $s_3 = 7$ $\frac{2}{3}s_3 = 4.667$ $7 \leq 16.667$
gcd(6,1) = 1 step 4, $1 \leq 4.6667$
In total, 4 recursive calls and the gcd(183,101)=1

iii.gcd(233,144) = gcd(144,89) step 0, $s_0 = 233$ $\frac{2}{3}s_0 = 155.333$
gcd(144,89) = gcd(89,55) step 1, $s_1 = 144$ $\frac{2}{3}s_1 = 96$ $144 \leq 155.333$
gcd(89,55) = gcd(55,34) step 2, $s_2 = 89$ $\frac{2}{3}s_2 = 59.3333$ $89 \leq 96$
gcd(55,34) = gcd(34,21) step 3, $s_3 = 55$ $\frac{2}{3}s_3 = 36.6667$ $55 \leq 59.33$
gcd(34,21) = gcd(21,13) step 4, $s_4 = 34$ $\frac{2}{3}s_4 = 22.667$ $34 \leq 36.667$
gcd(21,13) = gcd(13,8) step 5, $s_5 = 21$ $\frac{2}{3}s_5 = 14$ $21 \leq 22.667$
gcd(13,8) = gcd(8,5) step 6, $s_6 = 13$ $\frac{2}{3}s_6 = 8.667$ $13 \leq 14$
gcd(8,5) = gcd(5,3) step 7, $s_7 = 8$ $\frac{2}{3}s_7 = 5.333$ $8 \leq 8.667$
gcd(5,3) = gcd(3,2) step 8, $s_8 = 5$ $\frac{2}{3}s_8 = 3.333$ $5 \leq 5.333$
gcd(3,2) = gcd(2,1) step 9, $s_9 = 3$ $\frac{2}{3}s_9 = 2$ $3 \leq 3.333$
gcd(2,1) = 1 step 10, $1 \leq 2$
In total, 10 recursive calls and the gcd(233,144)=1

iv.gcd(3456,1234) = gcd(1234,988) step 0, $s_0 = 3456$ $\frac{2}{3}s_0 = 2304$
gcd(1234,988) = gcd(988,246) step 1, $s_1 = 1234$ $\frac{2}{3}s_1 = 822.667$ $1234 \leq 2304$
gcd(988,246) = gcd(246,4) step 2, $s_2 = 250$ $\frac{2}{3}s_2 = 166.667$ $250 \leq 822.667$
gcd(246,4) = gcd(4,2) step 3, $s_3 = 6$ $\frac{2}{3}s_3 = 4$ $6 \leq 166.667$
gcd(4,2) = gcd(2,0) step 4, $s_4 = 2$ $\frac{2}{3}s_4 = 1.333$ $2 \leq 4$
gcd(2,0) = 2 step 5
In total, 5 recursive calls and the gcd(1234,3456)=2

P (b) Try to find pairs of inputs (x, y) such that the number of iterations of Euclid(x, y) is "large", that is, as close as possible to the upper bound of $\log_{3/2}(x + y)$ that we derived in lecture. Can you come up with a hypothesis about what kinds of inputs yield the worst-case running time?

Hint 1: Recall the inequalities used when showing the potential function for the Euclidean algorithm. To minimize the difference between the potential function on steps i and $i + 1$, what should q_i be?

Hint 2: Question 6(a)iii is a worst-case example. Can we apply our choice of q_i to derive a relationship between inputs in worst-case examples?

Solution: The worst case Euclidean algorithm occurs when there is the largest possible remainder at each call. This occurs when two numbers are consecutive in the Fibonacci sequence because all though the remainder for one specific step may not be as high as it possibly could be, the remainders of the following steps will continue to be high without lowering the remainders of the following steps. i.e. The value of 15 as a would produce the highest remainder of 7 for a value of 8 as b, but the next call would be 8 and 7 and the remainder for that would only be 1. Therefore pairs of inputs where the numbers are consecutive numbers in the Fibonacci sequence would produce the worst case scenario. Examples of these are (21,34),(55,89),(233,377).