- Problems marked with **P** are graded on progress, which means that they are graded subjectively on the perceived progress toward a solution, rather than **solely** on correctness.
- Problems marked with a **G** are group problems. A group of 1-3 students may work on these together and turn in one assignment for the entire group. Each member listed should have made real contributions. The group problems will be turned in separately on gradescope.

For bonus and most extra credit questions, we will not provide any insight during office hours or Piazza, and we do not guarantee anything about the difficulty of these questions. We strongly encourage you to typeset your solutions in LaTeX. If you collaborated with someone, you must state their name(s). You must write your own solution for all problems and may not look at any other student's write-up.

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0. If applicable, state the name(s) and uniqname(s) of your collaborator(s).

Solution: Collaborator: Hannah Dunn - hrdunn

1. *Extra credit*: (10) Typeset this **entire** assignment in LaTeX and draw a table with two columns that includes the *name* (i.e., "fraction") and an *example* of each of the following:
   - less than or equal to
   - superscript and subscript
   - fraction (using `\frac`)
   - set intersection
   - summation using sigma (Σ) notation

   Unlike most extra credit questions, we will help with this in office hours if asked though it will be a lower priority.

Solution:

Column 1: Name	Column 2: Symbol
less than or equal to	$\leq$
superscript and subscript	x^y $and\, x_y$
fraction	$\frac{a}{b}$
set intersection	$\cap$
summation using sigma	$\sum_{b=1}^{\infty} 2^b$

P 2. (20) For the following pairs of $f(n)$ and $g(n)$, is $f(n) = O(g(n))$? Justify your answer by applying the definition of big-O or by applying a limit argument.

Hint: You may find L'Hôpital's Rule useful.

(a) $f(n) = n + 2\log_2(n) + 3$, $g(n) = \frac{1}{3}n + 2$.

Solution: We know $\log_2(n) \leq n$
$2log_2(n) \leq 2n$
$n + 2n = 3n$
$Suppose\, C = 9, n = 1:$
$3n \leq \frac{1}{3}$(n) + 2

3n $\leq 3n + 18$
$Hence,$ f(n) = O(g(n))

(b) $f(n) = n^2$, $g(n) = 2^n$

Solution: n$^2:\ 0, 1, 4, 9, 16, 25, 36, ...$
$2^n:\ 1, 2, 4, 8, 16, 32, 64, ...$
$\lim_{x\to\infty}(\frac{n^2}{2^n}) = 0$
This shows n$^2 = o(2^n)$
$Little - o\, implies\, Big - O$
$Therefore, n^2 = O(2^n) -> f(n) = O(g(n))$

(c) $f(n) = 3^n$, $g(n) = 2^n$

Solution: $2^n:\ 1, 2, 4, 8, 16, 32, 64, ...$
$3^n:\ 1, 3, 9, 27, 81, 243, 729, ...$
$\lim_{n\to\infty}(\frac{2^n}{3^n}) = 0$
This shows g(n) = o(f(n))
Therefore, it is impossible for f(n) = O(g(n))

(d) $f(n) = \ln(n)$, $g(n) = \ln(\frac{n}{5})$

Solution: $\lim_{n\to\infty}(\frac{ln(n)}{ln(n/5)})$
L'Hopital's Rule:

$\lim_{n\to\infty}(\frac{d/dn(ln(n))}{d/dn(ln(n/5))})$

d/dn (ln(n/5)) = 5/n * d/dn(n/5) = 5/n * 1/5 = 1/n

$\lim_{n\to\infty}(\frac{1/n}{1/n}) = 1$

Since $f(n)$ and $g(n)$ grow at the same rate, by definition, $f(n) = O(g(n))$.

(e) $f(n) = (\log_2 n)^3$, $g(n) = 2^{\log_2 n}$

Solution: Suppose $a = log_2(n)$
$(log_2(n))^3 = a^3$, $2^{log_2(n)} = 2^a$
$\lim_{a\to\infty}(\frac{a^3}{2^a})$

L'Hopitals's Rule:

$\lim_{a\to\infty}(\frac{d/da(a^3)}{d/dn(2^a)}) = (\frac{3a^2}{2^a ln(a)})$

Using L'Hopital's Rule 2 more times, you get:
$\lim_{a\to\infty}(\frac{6}{2^a ln^3(a)}) = 0$
Therefore, $f(a) = O(g(a))$ and consequently $f(n) = O(g(n))$

G 3. (25) Prove the following claims by filling in the blanks provided.

Pay special attention to the exposition. Both proofs start from the assumptions and each make their way to the conclusion making sure that each step is complete before going to the next step and do not wander between claims. The proofs in (a) are proofs by cases, and the proof in (b) is a direct proof.

Another feature to notice and consider introducing in your mathematical writing is overviews of proofs. These are especially helpful for more complex proofs. The proof in (a) provides an overview of its structure which makes the proof easier to read by guiding the reader on how the counting will be done.

(a) Each face of a six-sided die is randomly painted red or blue with equal probability. What is the probability that there is at least one corner of the die where all three sides that make up that corner are of the same color? Clearly show your work.

There are two different approaches we can take: we can do a direct count of how many cubes have a 1-color corner or we can instead invert the problem to compute the number of cubes without a 1-color corner. Either way, it tends to be much easier to get an accurate count if we split it up by how many sides of a given color there are.

Direct:

<u>6 red sides:</u> only 1 cube and it has a 1-color corner.

<u>5 red sides:</u> ______ ways to pick the blue side. All cubes have a 1-color corner.

<u>4 red sides:</u> The 2 blue sides can either be next to each other or across from each other. If they are next to each other, ______________________________, while if they are across, ______________________________. To count the number of ways we can pick them making them adjacent, we pick a side (6 ways), pick an adjacent side (4 ways), and then note that we counted everything twice so we have ______ 1-color corner cubes.

<u>3 red sides:</u> After some reasoning, we can conclude that the 3 red sides will either be wrapped around a corner or stretched out in a line. If they are in a line, there is no 1-color corner, while if they are around a corner, there is. To count this, we can just pick which corner to wrap the red sides around, giving _____ options.

<u>2 red sides:</u> By symmetry this is the same as _____ red sides, _____ options.

<u>1 red side:</u> By symmetry this is the same as _____ red sides, _____ options.

<u>0 red sides:</u> By symmetry this is the same as _____ red sides, _____ options.

In total, this gives _____ cubes with a 1-color corner out of _____, so the probability is _____.

Indirect:

In order to guarantee there is no 1-color corner, we need a pair of opposite sides to be red and another pair of opposite sides to be blue. When there are only 0 or 1 of a given color, this is therefore impossible, so we need only consider the case of 4 and 2, 3 and 3, and 2 and 4.

<u>4 red sides</u>: In order to have the 2 blue sides be opposite each other, we pick one axis of the cube for them to be on. There are ____________________ ways to do this, so there are ____________________ cubes without a 1-color corner.

<u>3 red sides</u>: If we have 1 opposite pair of red sides and 1 opposite pair of blue sides, the other pair will be 1 red 1 blue, meaning all ____________________ can be told apart. To count this, we pick one pair for red (3), ____________________ pair for blue from what's left (2), and the remaining pair can be either BR or RB (2), so in total, there are ____________________ cubes without a 1-color corner.

<u>2 red sides</u>: By symmetry this is the same as ____________________ red sides, ____________________ cubes without a 1-color corner.

In total this gives ____________________ cubes without a 1-color corner out of ____________________, so the probability is ____________________

Solution: 6 red sides: only 1 cube and it has a 1-color corner.
5 red sides: 6 ways to pick the blue side. All cubes have a 1-color corner.
4 red sides: The 2 blue sides can either be next to each other or across from each other. If they are next to each other, there will be 1 red corner, while if they are across, none of the corners will have a 1-color corner. To count the number of ways we can pick them making them adjacent, we pick a side (6 ways), pick an adjacent side (4 ways), and then note that we counted everything twice so we have 12 1-color corner cubes.
3 red sides: After some reasoning, we can conclude that the 3 red sides will either be wrapped around a corner or stretched out in a line. If they are in a line, there is no 1-color corner, while if they are around a corner, there is. To count this, we can just pick which corner to wrap the red sides around, giving 8 options.
2 red sides: By symmetry this is the same as 4 red sides, 12 options.
1 red side: By symmetry this is the same as 5 red sides, 6 options.
0 red sides: By symmetry this is the same as 6 red sides, 1 options.
In total, this gives 6 cubes with a 1-color corner out of 46, so the probability is $46/64 = 0.7188$.

In order to guarantee there is no 1-color corner, we need a pair of opposite sides to be red and another pair of opposite sides to be blue. When there are only 0 or 1 of a given color, this is therefore impossible, so we need only consider the case of 4 and 2, 3 and 3, and 2 and 4.
4 red sides: In order to have the 2 blue sides be opposite each other, we pick one axis of the cube for them to be on. There are 3 ways to do this, so there are 3 cubes without a 1-color corner.
3 red sides: If we have 1 opposite pair of red sides and 1 opposite pair of blue sides, the other pair will be 1 red 1 blue, meaning all pairs of opposite sides can be told apart. To count this, we pick one pair for red (3), one pair for blue from what's left (2), and the remaining pair can be either BR or RB (2), so in total, there are 12 cubes without a 1-color corner.
2 red sides: By symmetry this is the same as 4 red sides, 3 cubes without a 1-color corner.
In total this gives 18 cubes without a 1-color corner out of 64, so the probability is $18/64 = 0.2813$

(b) Say we have five points (x_i, y_i), each distinct for $i = 1$ to 5 where each x and y value is an integer. Now say you draw a line connecting each pair of points. Prove that the midpoint of at least one of those lines has an (x, y) location where both x and y are integers.

Every integer is even or odd, so there are only 4 ordered pairs of even or odd: (even, even), (even, odd), (odd, even), (odd, odd). If we sort each point into these 4 categories by the parity of its x and y coordinates, we find that because we have more points than categories, there must be at least one category that has ________________________. Given two points (x_i, y_i) and (x_j, y_j), the midpoint of the line between them is ____________________________.

If x_i and x_j are both even or both odd, the x coordinate of the midpoint ______________________ (and the same thing for the y coordinates). This means the pair of points in ______________________ will have a midpoint that has integer coordinates.

Solution: Every integer is even or odd, so there are only 4 ordered pairs of even or odd: (even, even), (even, odd), (odd, even), (odd, odd). If we sort each point into these 4 categories by the parity of its x and y coordinates, we find that because we have more points than categories, there must be at least one category that has 2 points. Given two points (x_i, y_i) and (x_j, y_j), the midpoint of the line between them is $(\frac{(x_j - x_i)}{2}, \frac{(y_j - y_i)}{2})$. If x_i and x_j are both even or both odd, the x coordinate of the midpoint will be an even integer and consequently the midpoint is even/2 which will also be an even integer (and the same thing for the y coordinates). This means the pair of points in the same category will have a midpoint that has integer coordinates.

P 4. (25) Do the following.

(a) Prove that $\sqrt{27}$ is irrational. (**Hint:** you may find it easiest to use a proof by contradiction.)

Solution: Proof By Contradiction
Suppose $\sqrt{27}$ is rational
Then it can be represented by $\frac{a}{b}$ where a and b are integers and co-prime

$$27 = \frac{a^2}{b^2}$$
$$27b^2 = a^2$$

This implies a^2 is divisible by 27 and consequently a is also divisible by 27. Then, let $a = 27c$ where c is some arbitrary integer

$$27b^2 = (27c)^2$$
$$b^2 = 27c^2$$

This implies b^2 is divisible by 27 and consequently b is also divisible by 27.
Since both a and b are divisible by 27, then they are not co-prime. This is a contradiction. Hence, the original statement is true.

(b) Prove by induction that the number of n-character strings over the alphabet $\{a, b, c\}$ (i.e. strings of length exactly n, where each character is chosen from the set $\{a, b, c\}$) is 3^n.

Solution:
Base Case: $n = 1$
3 options a, b, c

$$3^1 = 3$$

Hence, it is true for n = 1
Inductive Hypothesis: Suppose $P(k)$ is true for some positive integer k
Inductive Step: We want to show P(k+1) is also true
$P(k + 1) = P(k) * 3$
Suppose we have a string of length k, Using the inductive hypothesis, we know the number of character strings is 3^k.
Now, if he have $k + 1$ strings, there will be 3 options a, b, c for the $k + 1$th string
Therefore, we will have $3^{k+1} = 3^k * 3$ number of strings for size $k + 1$

$$3^k(3) = 3^k(3)$$

By mathematical induction, P(n) is true for all n ≥ 1

(c) Suppose you want to unambiguously represent all of the elements of $\{0, 1, \ldots, n - 1\}$ using ternary strings of a certain length k, i.e., strings with k characters from the set $\{a, b, c\}$. What's the smallest possible choice for k?

Solution: From part (b), we know that the number of distinct k-character strings we can make from the set $\{a, b, c\}$ is 3^k
0 to $n - 1$ strings $= n$ strings
Therefore,

$$n \leq 3^k$$

Hence, the minimum value of k would be

$$log_3(n) \leq k$$

P 5. (10) Consider the following homework solver:

```
function SOLVE_HW(X):
    for i in range(sqrt(X)):
        print 'hey grader, give me ' + X + ' points '
    return X
```

Give an asymptotic complexity bound on the number of characters printed by the program as a function of the input X (X is an integer). Make this bound as tight as possible. Assume that X is printed in decimal (base-10) format.

Note: our providing this program does not constitute a guarantee of any particular lower bound on the number of points you should expect to receive if you choose to implement it and run it to solve your own homeworks.

Solution: We know

$$log(x) < x, \forall x > 0$$

So, by substituting $\sqrt{x}$ into x, we get

$$log(\sqrt{x}) < \sqrt{x}$$

$$2log(\sqrt{x}) < 2\sqrt{x}$$

$$log(x) < 2\sqrt{x}$$

As x approaches infinity, $log(x) < \sqrt(x)$
Hence, $log(x) = O(\sqrt{x})$
Therefore, the asymptotic complexity bound of the number of characters printed is $log(x)$

6. (20) Euclidean algorithm

(a) Use the Euclidean algorithm to compute the GCDs of the following pairs of integers. Show each step of the algorithm, and compute the value of the potential s_i at each stage. Verify that indeed $s_{i+1} \leq \frac{2}{3}s_i$. How many recursive calls does the algorithm invoke, excluding any in which only the order of the operands was swapped?

You are encouraged to use a computer for this part.

i. $(45, 162)$
ii. $(183, 101)$
iii. $(233, 144)$
iv. $(1234, 3456)$

Solution: (i) $(45, 162)$

$$162 = 3 * 45 + 27, 162(2/3) = 108 \geq 45$$

$$45 = 1 * 27 + 18, 45(2/3) = 36 \geq 27$$

$$27 = 1 * 18 + 9, 27(2/3) = 18 \geq 18$$

$$18 = 2 * 9 + 0, 18(2/3) = 12 \geq 9$$

Thus, the GCD of 45 and 162 is 9
Number of iterations = 4

(ii) $(183, 101)$

$$183 = 1 * 101 + 82, 183(2/3) = 122 \geq 101$$

$$101 = 1 * 82 + 19, 101(2/3) = 67.33 \geq 82$$

$$82 = 4 * 19 + 6, 82(2/3) = 54.67 \geq 19$$

$$19 = 3 * 6 + 1, 19(2/3) = 12.67 \geq 6$$

$$6 = 6 * 1 + 0, 6(2/3) = 4 \geq 1$$

Thus, the GCD of 183 and 101 is 6
Number of iterations = 5

(iii) $(233, 144)$

$$233 = 1 * 144 + 89, 233(2/3) = 155.33 \geq 144$$
$$144 = 1 * 89 + 55, 144(2/3) = 96 \geq 89$$
$$89 = 1 * 55 + 34, 89(2/3) = 59.33 \geq 55$$
$$55 = 1 * 34 + 21, 55(2/3) = 36.67 \geq 34$$
$$34 = 1 * 21 + 13, 34(2/3) = 22.67 \geq 21$$
$$21 = 1 * 13 + 8, 21(2/3) = 14 \geq 13$$
$$13 = 1 * 8 + 5, 13(2/3) = 8.67 \geq 8$$
$$8 = 1 * 5 + 3, 8(2/3) = 5.33 \geq 5$$
$$5 = 1 * 3 + 2, 5(2/3) = 3.33 \geq 3$$
$$3 = 1 * 2 + 1, 3(2/3) = 2 \geq 2$$
$$2 = 2 * 1 + 0, 2(2/3) = 1.33 \geq 1$$

Thus, the GCD of 233 and 144 is 1
Number of iterations = 11

(iv) $(1234, 3456)$

$$3456 = 2 * 1234 + 988, 3456(2/3) = 2304 \geq 1234$$
$$1234 = 1 * 988 + 246, 1234(2/3) = 822.67 \leq 988$$
$$988 = 4 * 246 + 4, 988(2/3) = 658.67 \geq 246$$
$$246 = 61 * 4 + 2, 246(2/3) = 164 \geq 4$$
$$4 = 2 * 2 + 0, 4(2/3) = 2.67 \geq 2$$

Thus, the GCD of 1234 and 3456 is 2
Number of iterations = 5

P (b) Try to find pairs of inputs (x, y) such that the number of iterations of Euclid(x, y) is "large", that is, as close as possible to the upper bound of $\log_{3/2}(x + y)$ that we derived in lecture. Can you come up with a hypothesis about what kinds of inputs yield the worst-case running time?

Hint 1: Recall the inequalities used when showing the potential function for the Euclidean algorithm. To minimize the difference between the potential function on steps i and $i + 1$, what should q_i be?

Hint 2: Question 6(a)iii is a worst-case example. Can we apply our choice of q_i to derive a relationship between inputs in worst-case examples?

Solution: Based on the different pairs of numbers used in part (a), numbers within the Fibonacci Sequence seem to yield the worst-case running time. For example, the pair (233, 144) had 11 iterations and the GCD was 1.
The $s_i * (2/3)$ values were also the closest to the s_{i+1} in the Fibonacci sequence in part (iii), which is another indicator of the number of iterations being high.
Therefore, if we were to using pairs of Fibonacci Numbers like $(8, 5)$ where there would be 5 iterations. This is pretty close to the upper bound $log_{3/2}(8 + 5) = 6.326$.
Or the pair $(13, 21)$ where there would 6 iterations, with the upper bound being $log_{3/2}(21 + 13) = 8.697$
The GCD for the pairs of consecutive Fibonacci numbers is 1, therefore, these kinds of inputs will have the worst-case running time.