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Dynamics of Mini Hydraulic Backhoe Excavator: A Lagrange-Euler (L-E) Approach

Bhaveshkumar P. Patel, J. M. Prajapati

Abstract—Excavators are high power machines used in the mining, agricultural and construction industry whose principal functions are digging (material removing), ground leveling and material transport operations. During the digging task there are certain unknown forces exerted by the bucket on the soil and the digging operation is repetitive in nature. Automation of the digging task can be performed by an automatically controlled excavator system, which is not only control the forces but also follow the planned digging trajectories. To develop such a controller for automated excavation, it is required to develop a dynamic model to describe the behavior of the control system during digging operation and motion of excavator with time. The presented work described a dynamic model needed for controller design and which is derived by applying Lagrange-Euler approach. The developed dynamic model is intended for further development of an automated excavation control system for light duty construction work and can be applied for heavy duty or all types of backhoe excavators.

Keywords—Backhoe excavator, controller, digging, excavation, trajectory.

I. INTRODUCTION

BACKHOE excavators are used primarily to excavate below the natural surface of the ground on which the machine rests. According to forestry, earthmoving, and excavator statistics program a backhoe excavator is defined as “A ride-on dual purpose self propelled wheeled machine for on and off road operation. One end with loader arms that can support a full width bucket or attachment and the other end incorporating a boom and arm combination capable of swinging half circle for the purpose of digging or attachment manipulation.” In other words a backhoe excavator is actually three pieces of construction equipment combined into one unit. These three pieces are a tractor, a loader, and a backhoe [4]. The third piece of the equipment of a backhoe also known as a backhoe excavator attachment is the area of research reported in this paper. Earthmoving machines, such as bulldozers, wheel loaders, excavators, scrapers and graders are commonly used in construction work. An excavator is a typical hydraulic heavy-duty human-operated machine used in general versatile construction operations, such as digging, ground leveling, carrying loads, dumping loads and straight traction. However, there are many tasks, such as hazard

environment (nuclear decomposition, earthquake, etc.) which is not suitable for human to work on site. The remotely controllable excavators are required to work in such environment [5].

However, operators who control hydraulic excavators must be trained for many years to do such work quickly and skillfully. A hydraulic backhoe excavator has three links: boom, dipper and bucket; and an operator have two arms. Thus, it is not easy for beginners to execute elaborate work that manipulates three links at the same time. Moreover, because the operators have to run work in various dangerous and dirty environments, the number of skillful operators is ever decreasing. For that reason, studying the automation of hydraulic excavators is necessary for improving productivity, efficiency, and safety [3].

For an autonomous operation it is very important to study the motion of the various links to operate the machine in planned digging trajectory as well as to control the same during excavation task, i.e. kinematics of the backhoe excavator. The various forces exerted during the digging operation can be understand and evaluated by developing a dynamics model for backhoe excavator.

II. PROBLEM FORMULATION

Due to severe working conditions, excavator parts are subjected to high loads. The excavator mechanism must work reliably under unpredictable working conditions. The excavation task is in cyclic nature. Poor strength properties of excavator parts like boom, arm and bucket limit life expectancy of the excavator. Dynamic modeling is an important step in the design of excavator parts. The dynamic behavior of the backhoe provides relationship between joint actuator torques and motion of links for simulation and design of control algorithms. During the work cycle of the backhoe, at some point of time it needs to be accelerated, move at varying speeds, and then decelerate. This time varying position and orientation of the backhoe is termed as the dynamic behavior of the backhoe. These time varying torques are applied at the pin joints between swing and boom link, between boom and arm link, and between arm and bucket link by the corresponding hydraulic actuators to balance out the internal and external forces. The internal forces are caused by motion (velocity, and acceleration) of links, inertia, Coriolis, centripetal and frictional forces. On the other hand load (a load vector of reaction forces due to bucket ground interaction, acting on the bucket) and gravitational forces are considered to be external forces in context of robotics. Moreover; the excavation operations can be made automated by an

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automatically controlled backhoe system. Such a system is able to perform autonomously a planned digging work and to quickly comply with interacting forces experienced during excavation. The development of such an automated control system is usually based on a dynamic model of the system that describes the motion with the time [1].

The dynamic model of the backhoe is useful in computation of the required torques for execution of a typical work cycle, which is vital information for design of links, joints, and actuators. Apart from this in designing the controller for the backhoe also requires the dynamic model to obtain the desired performance, because the controller directly depends on the accuracy of the dynamic model and control algorithms. The backhoe is a serial link manipulator (open kinematic chain) and represents a complex dynamic system, which can be modeled by systematically using the known physical laws of Lagrangian mechanics or Newtonian mechanics. Approaches such as Lagrange-Euler (L-E) which is energy based, and Newton-Euler (N-E) based on force balance, can be systematically applied to develop the backhoe equation of motion (EOM). The resulting EOM are a set of second order, coupled, nonlinear differential equations, consisting of inertia loading and coupling reaction forces between joints. The next section described research work reported by other researchers in the field of dynamics of backhoe excavator with their limitations and the scope of the further work to be carried out.

III. RELATED WORK

P. K. Vaha and M. J. Skibniewski have developed a dynamic model of a hydraulic excavator in the digging mode (for the three degrees of freedom only) in 1993. Vaha used the Newton-Euler approach (a force balance approach) for the development of the mathematical dynamic model. But few of his assumptions were not realistic in practice, and these are: For the determination of the mass moment of inertia tensor, and to make it less computationally complicated, he assumed (i) The axes of the link frame are aligned with the principle axes of the link for all links, to make the products of inertia I_{xy} , I_{yz} and I_{xz} to be zero, which is a valid assumption, (ii) The origin of frame of the link is made to coincide with the centre of mass or centroid of the link to make the last row and last column of the moment of inertia tensor to be zero row and zero column respectively, which is not true in actual practice because if the origin of the frame is translated to the centroid of the link, then whole previously derived kinematic relations will no longer be valid for the dynamic model. He also assumed that the gravity centre or the centre of mass of all links lie on the line joining the two end points of the links, which is also not true [8].

S. Singh has attempted to develop a dynamic model for the excavator in 1995. But as his main part of the research was to develop tactical plans for the robotic excavation; not to develop a dynamic model for the hydraulic excavator, his dynamic model was incomplete and only torques resulted from the gravitational forces were presented in an appendix of his Ph D thesis. So his dynamic model was incomplete in

context of the application of the model for controlling of the excavator [11].

A. J. Koivo et al. developed a complete dynamic model in 1996 using the same Newton-Euler approach for three degrees of freedom mechanism of a hydraulic excavator, by overcoming the shortcomings of Vaha's dynamic model, as he neglected the assumption of the centre of mass to be on the line joining the two end points of the link made by Vaha, and considered the real location of the centre of mass for each link. He also neglected the assumption that the origin of frame of the link is made to coincide with the centre of mass or centroid of the link as this was the assumption of Vaha's dynamic model, and considered the inertia tensor in general form (without neglecting the last row and last column). His dynamic model is so thorough that it can directly be applied as an input to the excavator controller to carry out the excavation operations in autonomous mode. He also gave a sound example of the utilization of his dynamic model into the PD (proportional plus derivative) controller for autonomous control of an excavator [1].

J. Frankel has presented only the torque equations for different joints, and presented just a procedure to develop a dynamic model for the backhoe excavator using Lagrange-Euler approach in 2004. But even in his procedure he did not include the load vector (a vector of torques on each link due to the interaction of the bucket with the ground while excavating) in his dynamic model procedure. So he just provided the procedure to find the dynamic model not the dynamic model based on Lagrange-Euler approach [7].

P. H. Yang et al. have carried out the dynamic model for three degrees of freedom mechanism of an excavator in 2008. But as the model was developed for the testing of his designed controller only, he did not consider the torques due to gravity loading on each link, and torques due to the load vector. He did not give any systematic procedure, and instead directly presented the dynamic model, and thus leaving the model with too many undefined terms [8].

To be concluded that the dynamic model of A. J. Koivo et al. was proved to be a thorough model over the period of time for many researchers to do the research work on the controller design for the excavator. It means for the researchers working in the area of designing new control algorithms for controlling the operations of the excavator, the dynamic model of A. J. Koivo proved to be a comprehensive input [1].

The development of such a comprehensive dynamic model for the backhoe excavator in digging mode (in three degrees of freedom only) using Lagrange-Euler approach and comparison of the same with A. J. Koivo's N-E dynamic model is another area of the research reported in this paper.

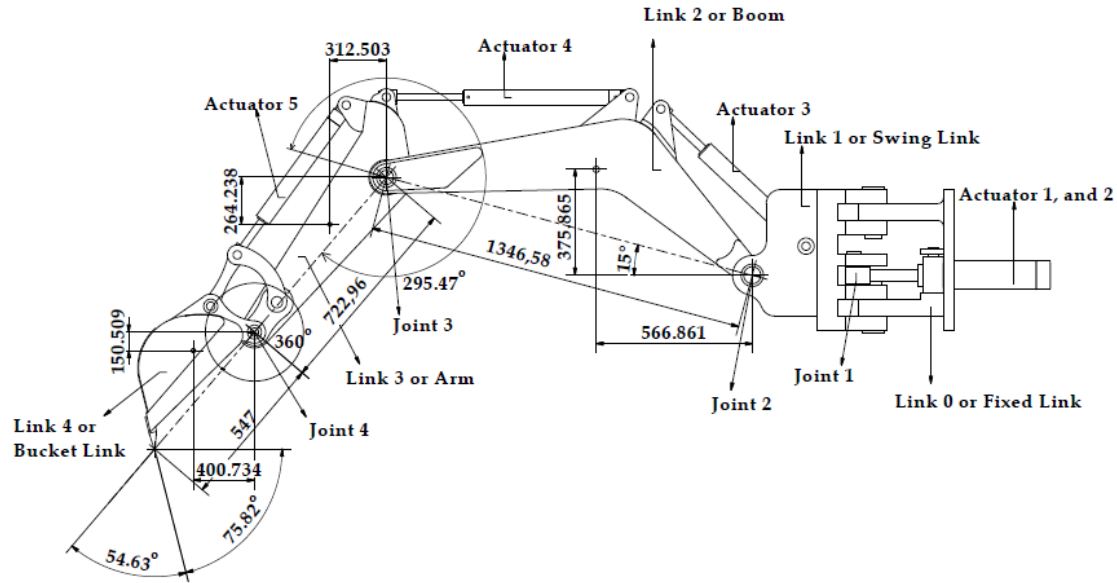


Fig. 1 Schematic view of a backhoe excavator attachment

IV. KINEMATICS OF BACKHOE EXCAVATOR

Kinematics is the science of motion which treats motion without regard to the forces that cause it. Within the science of kinematics one studies the position, velocity, acceleration, and all higher order derivatives of the position variables (with respect to time or any other variables) [6]. Fundamentally a backhoe excavator has five links starting from the fixed link or base link, swing link, boom link, arm link (dipper link), and bucket link. These links are connected to each other by joints, which allow revolute motion between connected links each of which exhibits just one degree of freedom. This leads to the four degree of freedom R-RRR configuration of the backhoe, where R stands for a revolute joint. Fig. 1 describes the schematic view of the backhoe excavator attachment. To analyze the motion of the backhoe excavator for performing a specific task, it becomes necessary to define a world coordinate system to describe the position and orientation of the bucket (collectively known as configuration of the bucket). A right-hand Cartesian coordinate system $X_w Y_w Z_w$ is chosen, and its origin is placed at an arbitrary point on the ground level in the workspace of the backhoe excavator. After assigning the world coordinate frame the local coordinate frames for all links are assigned by following the DH guideline for link frame assignment algorithm [10].

A backhoe excavator is designed to perform a task in the 3-D space. The bucket of the backhoe is required to follow a planned trajectory to carry out the digging task in the workspace. This requires control of position of each link (swing link, boom, arm, and the bucket) and joints of the backhoe to control both the position and orientation of the bucket. To program the bucket motion and the joint link motions, a mathematical model of the backhoe is required to refer all geometrical and/or time based properties of the motion. In other words kinematic model purely encodes the geometric relationship of the mechanism. Moreover; the

kinematic model gives relation between the position and orientation of the bucket and spatial positions of joint-links. A problem of describing the complete kinematic model for an autonomous operation of the backhoe excavator, and the static model of the backhoe considered for study. This study includes direct kinematic model (forward kinematic model), inverse kinematic model (Backward kinematic model), the differential motion of the backhoe refer to the differential motion velocity only, and static force model of the backhoe excavator but it is not a part of this paper, this paper focuses only on the dynamics of the backhoe excavator based on the Lagrange-Euler formulation.

V. LAGRANGE-EULER FORMULATION

A scalar function called Lagrange function or Lagrangian L_a is defined as the difference between the total kinetic energy K_E and the total potential energy P_E of a mechanical system.

$$L_a = K_E - P_E \quad (1)$$

The Lagrange-Euler dynamic formulation is based on a set of generalized coordinates to describe the system variables. In the generalized coordinates, generalized angular displacement θ (radian), is used as a joint variable or joint angle, and $\dot{\theta}$ (rad/sec) describes the angular velocity of the joint, and $\ddot{\theta}$ describes (rad/sec²) the angular acceleration of the joint. Similarly τ is a generalized torque required at the joint to produce the desired dynamics [9].

The dynamic model based on Lagrange-Euler formulation is obtained from the Lagrangian, as a set of equations,

$$\frac{d}{dt} \left(\frac{\partial L_a}{\partial \dot{\theta}_i} \right) - \left(\frac{\partial L_a}{\partial \theta_i} \right) = \tau_i \quad (2)$$

The left sides of the dynamic equation refer to (2) can be interpreted as sum of the torques due to kinetic and potential

energy present in the system. In the right hand side of the equation refer to (2) τ_i is the joint torque for joint i that is provided by the actuator ($i+1$) in our case. If $\tau_i = 0$, it means that joint i does not move and if τ_i is not equal to zero, it means the backhoe movement is modified by the actuator ($i + 1$) of joint i .

Now the kinetic energy of the differential mass dm_i of link i , for $i = 2, 3$, and 4 moving with velocity ${}^0v_i (= v_i)$ with respect to the base frame $\{0\}$ is,

$$dK_{E_i} = \frac{1}{2} dm_i (v_i)^2 \quad (3)$$

The trace operator (If A is a matrix of size $n \times n$, then trace operator $\text{Tr}(A) = \sum_{i=1}^n a_{ii}$) is used to obtain $(v_i)^2$ as,

$$v_i^2 = \text{Tr}(v_i v_i^T) \quad (4)$$

The total kinetic energy is then given by,

$$K_E = \frac{1}{2} \sum_{i=2}^4 \sum_{j=2}^i \sum_{k=2}^i \text{Tr}[({}^0T_{j-1} Q_j {}^{j-1}T_i) I_i ({}^0T_{k-1} Q_k {}^{k-1}T_i)^T] \dot{\theta}_j \dot{\theta}_k \quad (5)$$

Note that here limits are taken from $i = 2$ to $i = 4$, because the dynamic model presented is for the boom, arm and bucket linkage only (RRR configuration) and the swing motion is not considered. Where, ${}^0T_{j-1}$ is a transformation matrix of the size 4×4 of frame $\{j-1\}$ relative to the base frame $\{0\}$, O_j is a matrix when post multiplied with ${}^0T_{j-1}$, and pre multiplied with ${}^{j-1}T_i$ gives the partial derivative of the transformation matrix 0T_i with respect to the corresponding joint angle θ_i , and can be given by:

$$Q_j = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The mass of link contributes inertia forces during motion of the link. The mass properties, which reflect all the inertial loads with respect to rotations about the origin of frame of interest, are presented by a moment of inertia tensor I_i . The moment of inertia tensor I_i is a 4×4 symmetric matrix which characterizes the distribution of mass of a rigid body or link i , and it can be defined as:

$$I_i = \begin{bmatrix} \frac{1}{2}(-I_{xx_i} + I_{yy_i} + I_{zz_i}) & I_{xy} & I_{xz} & m_i \bar{x}_i \\ I_{xy} & \frac{1}{2}(I_{xx_i} - I_{yy_i} + I_{zz_i}) & I_{yz} & m_i \bar{y}_i \\ I_{xz} & I_{yz} & \frac{1}{2}(I_{xx_i} + I_{yy_i} - I_{zz_i}) & m_i \bar{z}_i \\ m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i & m_i \end{bmatrix} \quad (6)$$

where, I_{xx} , I_{yy} , I_{zz} are moment of inertias of a body with respect to X , Y and Z axes respectively, I_{xy} , I_{yz} , and I_{xz} are the products of inertia and are taken as zero by assuming that the axes of the reference frame are aligned with the principle axes of the body, m_i is the mass of link i , and ${}^i\bar{r}_i = [\bar{x}_i \ \bar{y}_i \ \bar{z}_i \ 1]^T$ is a homogeneous coordinate vector defining the position of the centre of mass or centroid of the link i from the origin of the

link i of O_i . So, new inertia tensor for our case will now be modified as:

$$I_i = \begin{bmatrix} \frac{1}{2}(-I_{xx_i} + I_{yy_i} + I_{zz_i}) & 0 & 0 & m_i \bar{x}_i \\ 0 & \frac{1}{2}(I_{xx_i} - I_{yy_i} + I_{zz_i}) & 0 & m_i \bar{y}_i \\ 0 & 0 & \frac{1}{2}(I_{xx_i} + I_{yy_i} - I_{zz_i}) & m_i \bar{z}_i \\ m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i & m_i \end{bmatrix} \quad (7)$$

The total potential energy of backhoe mechanism is sum of the potential energy of the links, i.e. boom, arm and bucket, then given by,

$$P_E = - \sum_{i=2}^4 m_i g^T {}^i\bar{r}_i \quad (8)$$

The negative sign indicates that the work is done on the system to raise link i against gravity. Where g is the vector of acceleration due to gravity (m/s^2):

$$g = [g_x \ g_y \ g_z \ 0] = [0 \ -9.81 \ 0 \ 0] \\ g^T = [g_x \ g_y \ g_z \ 0]^T = [0 \ -9.81 \ 0 \ 0]^T$$

The acceleration due to gravity g is the 4×1 gravity vector with respect to base frame $\{0\}$.

VI. EQUATIONS OF MOTION

The following procedure should be adopted to develop an L-E dynamic model [10].

The Lagrangian, $L_a = K_E - P_E$ obtained by refer to (5) and (8) is given by:

$$L_a = \frac{1}{2} \sum_{i=2}^4 \sum_{j=2}^i \sum_{k=2}^i \text{Tr}[({}^0T_{j-1} Q_j {}^{j-1}T_i) I_i ({}^0T_{k-1} Q_k {}^{k-1}T_i)^T] \dot{\theta}_j \dot{\theta}_k + \sum_{i=2}^4 m_i g^T {}^i\bar{r}_i \quad (9)$$

According to the Lagrange-Euler dynamic formulation, the generalized torque τ_i of actuator $i+1$ at joint i , to drive link i of the backhoe is given by (2) as:

$$\tau_i = \frac{d}{dt} \left(\frac{\partial L_a}{\partial \dot{\theta}_i} \right) - \left(\frac{\partial L_a}{\partial \theta_i} \right)$$

By substituting L_a and carrying out the differentiation, the generalized torque τ_i applied to link i for 3 degree of freedom backhoe is obtained [10]. The final EOM (dynamic model) is;

$$\tau_i = \sum_{j=2}^4 M_{ij} \ddot{\theta}_j + \sum_{j=2}^4 \sum_{k=2}^4 h_{ijk} \dot{\theta}_j \dot{\theta}_k + G_i + F_{load}, \quad \text{For } i = 2, 3, 4. \quad (10)$$

$$\text{Where, } M_{ij} = \sum_{p=\max(i,j)}^4 \text{Tr}[d_{pj} I_p d_{pi}^T] \quad (11)$$

$$h_{ijk} = \frac{1}{2} \left[\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{jk}}{\partial \theta_i} - \frac{\partial M_{ik}}{\partial \theta_j} \right] \quad (12)$$

$$G_i = \sum_{p=i}^4 m_p g^T d_{pi} \bar{r}_p \quad (13)$$

$$F_{load}(F_t, F_n) = \begin{bmatrix} a_2 [F_t \sin(\theta_2 - \rho) - F_n \cos(\theta_2 - \rho)] \\ a_3 [F_t \sin(\theta_{23} - \rho) - F_n \cos(\theta_{23} - \rho)] \\ a_4 [-F_t \sin \lambda + F_n \cos \lambda] \end{bmatrix} \quad (14)$$

$$d_{ij} = \begin{cases} {}^1T_{j-1}Q_j^{j-1}T_i & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases} \quad (15)$$

$$\frac{\partial d_{ij}}{\partial \theta_k} = \begin{cases} {}^1T_{j-1}Q_j^{j-1}T_{k-1}Q_k^{k-1}T_i & \text{for } i \geq k \geq j \\ {}^1T_{k-1}Q_k^{k-1}T_{j-1}Q_j^{j-1}T_i & \text{for } i \geq j \geq k \\ 0 & \text{for } i < j \text{ or } i < k \end{cases} \quad (16)$$

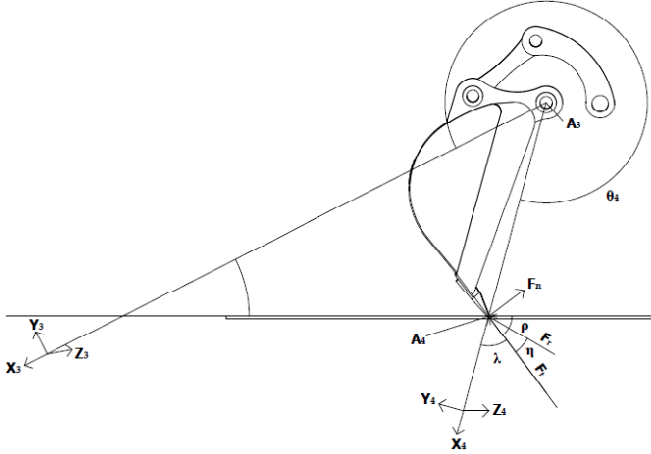


Fig. 2 Resistive force resolution

Equation (12) has been taken from [12]. Fig. 2 shows the resolution of the resistive force offered by the ground on the teeth of the bucket. This resistive force can be resolved into the tangential and normal directions on the bucket teeth as shown in Fig. 2, and for load vector refer to (14) can be found as given in [1].

Equation (10) is the dynamic model of the backhoe in generalized form and gives a set of 3 nonlinear, coupled, second order ordinary differential equations for 3 links of the three degree of freedom backhoe. These equations are the equations of motion or the dynamic equation of motions for the backhoe. M_{ij} represents an inertia matrix refer to (10). It is known as effective inertia when acceleration of joint i cause a torque at joint i , and coupling inertia when acceleration at joint j causes a torque at joint i .

The coefficient h_{ijk} represents the velocity induced reaction torque at joint i . $h_{ijk}\dot{\theta}_j^2$ represents the Centrifugal or Centripetal force acting at joint i due to velocity at joint j , and term $h_{ijk}\dot{\theta}_j\dot{\theta}_k$ represents the Coriolis force acting at joint i due to velocities at joint j and k . G_i is the gravity loading vector, and F_{load} is the load vector acting on the bucket teeth due to the forceful and sudden interaction of the bucket teeth with the ground and (14) can be determined from the geometry of the backhoe link mechanism and also it can be found as given in [1].

VII. ASSUMPTIONS OF THE PROPOSED DYNAMIC MODEL

The assumptions of the proposed dynamic model are as follows:

- The model has been presented in digging mode only, so for the proposed dynamic model only three degrees of freedom RRR (excluding the swing motion) are

considered (because while swinging, backhoe experiences an extra load and that is payload in the bucket that does not cause large amount of torques at different joints as compared to the digging task does, so the fourth degree of freedom is ignored).

- The frictional effects are neglected.
- The inertia tensor of the backhoe components is taken in a generalized form with no assumption like the frame axes are not coinciding with the axes of center of mass of the body, but it has been assumed that the link frame axes are coinciding with the principle axes of the body to cause the products of inertia to be zero in an inertia tensor.

VIII. BACKHOE L-E DYNAMIC MODEL

Now the dynamic model of the backhoe in digging mode is presented. The procedure used to carry out the final equations of motion is described in section VI. The transformation matrices ${}^{i-1}T_i$ from the direct kinematic model is directly utilized in equations of motion. The final form of the equations of motion (or dynamic model) for the backhoe can be given by:

$$\tau_i = M(\theta)\ddot{\theta} + H(\theta, \dot{\theta})\dot{\theta} + G(\theta) + F_{Load}(F_t, F_n) \quad (17)$$

$M(\theta)$ represents the 3×3 symmetric inertia matrix, $\ddot{\theta}$ represents the 3×1 joint acceleration matrix, $H(\theta, \dot{\theta})$ represents the 3×3 velocity induced torque matrix or the matrix of centripetal and Coriolis torques, $\dot{\theta}$ represents the 3×1 joint velocity matrix, $G(\theta)$ is the 3×1 gravity induced torque vector, and $F_{Load}(F_t, F_n)$ is the 3×1 load vector acting on the bucket teeth offered by the ground or soil, and is the function of tangential force F_t and normal force F_n acting on the bucket teeth refer to (17). Now the calculations of the elements of the matrices and vectors as described above are presented. Note that for each matrix or vector the sample calculation of the single element has been shown over here, other elements can be found out in the similar way.

A. Determination of the Elements of an Inertia Matrix

Using (11) all the nine elements of the symmetric inertia matrix can be determined as M_{ij} for joint $i = 2, 3$, and 4; and joint $j = 2, 3$, and 4. So total nine elements $M_{22}, M_{23}, M_{24}, M_{32} (= M_{23}), M_{33}, M_{34}, M_{42} (= M_{24}), M_{43} (= M_{34})$, and M_{44} have to be determined as a function of the joint angles θ_i for $i = 2, 3$, and 4. Let us first determine the element M_{22} with the use of (11) as follows:

$$M_{22} = \sum_{p=\max(2,2)}^4 \text{Tr}[d_{p2}I_p d_{p2}^T]$$

$$\therefore M_{22} = \text{Tr}[d_{22}I_2 d_{22}^T] + \text{Tr}[d_{32}I_3 d_{32}^T] + \text{Tr}[d_{42}I_4 d_{42}^T] \quad (18)$$

Now from (15) let us first determine the term $\text{Tr}[d_{22}I_2 d_{22}^T]$, in this term let us first determine d_{22} , as both i and j are equal ($i = 2$, and $j = 2$) the first condition of (15) can be applied to determine d_{22} as follows:

$$d_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore d_{22} = \begin{bmatrix} -S_2 & -C_2 & 0 & -a_2 S_2 \\ C_2 & -S_2 & 0 & a_2 C_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

d_{22}^T is the transposition of the matrix d_{22} .

The inertia tensor for $i = 2$ can be given as follows:

$$I_2 = \begin{bmatrix} \frac{1}{2}(-I_{xx2} + I_{yy2} + I_{zz2}) & 0 & 0 & m_2 \bar{x}_2 \\ 0 & \frac{1}{2}(I_{xx2} - I_{yy2} + I_{zz2}) & 0 & m_2 \bar{y}_2 \\ 0 & 0 & \frac{1}{2}(I_{xx2} + I_{yy2} - I_{zz2}) & m_2 \bar{z}_2 \\ m_2 \bar{x}_2 & m_2 \bar{y}_2 & m_2 \bar{z}_2 & m_2 \end{bmatrix}$$

$$\therefore \text{Tr}[d_{22} I_2 d_{22}^T] = I_{zz2} + m_2(2a_2 \bar{x}_2 + a_2^2) \quad (19)$$

Similarly, the other two terms $\text{Tr}[d_{32} I_3 d_{32}^T]$, and $\text{Tr}[d_{42} I_4 d_{42}^T]$ can be determined and given as follows:

$$\text{Tr}[d_{32} I_3 d_{32}^T] = I_{zz3} + m_3[\bar{x}_3(2a_3 + 2a_2 c_3) - \bar{y}_3(2a_2 s_3) + a_2^2 + a_3^2 + 2a_2 a_3 c_3] \quad (20)$$

$$\text{Tr}[d_{42} I_4 d_{42}^T] = I_{zz4} + m_4[\bar{x}_4(2a_4 + 2a_2 c_{34} + 2a_3 c_4) - \bar{y}_4(2a_2 s_{34} + 2a_3 s_4) + a_2^2 + a_3^2 + a_4^2 + 2a_2 a_3 c_3 + 2a_3 a_4 c_4 + 2a_2 a_4 c_{34}] \quad (21)$$

Equations (19), (20), and (21) when added together will form the first element of the inertia matrix M_{22} as follows:

$$M_{22} = I_{zz2} + I_{zz3} + I_{zz4} + m_2[(2\bar{x}_2 a_2) + (a_2)^2] + m_3[\bar{x}_3(2a_3 + 2a_2 c_3) - \bar{y}_3(2a_2 s_3) + a_2^2 + a_3^2 + 2a_2 a_3 c_3] + m_4[\bar{x}_4(2a_4 + 2a_2 c_{34} + 2a_3 c_4) - \bar{y}_4(2a_2 s_{34} + 2a_3 s_4) + a_2^2 + a_3^2 + a_4^2 + 2a_2 a_3 c_3 + 2a_3 a_4 c_4 + 2a_2 a_4 c_{34}] \quad (22)$$

So the final inertia matrix can be given by:

$$M(\theta) = \begin{bmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{bmatrix}$$

All the other elements are given in section VIII-F. In current section only the methodology to derive the inertia matrix elements have been explained with the illustration of the determination of the first element M_{22} .

B. Determination of the Elements of a Joint Acceleration Vector

The joint angle velocities $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ and $\dot{\theta}_4$ and respective angles θ_1 , θ_2 , θ_3 and θ_4 are derived based on the joints 1, 2, 3 and 4 but this derivation is included in kinematics of backhoe excavator not a part of this paper. The term $\ddot{\theta}$ represents the 3

x 1 joint acceleration vector, in which the acceleration of joint 2, $\ddot{\theta}_2$ can be determined by time derivative of the velocity of the joint 2, $\dot{\theta}_2$.

$$\dot{\theta}_2 = \left\{ \frac{V_{A_5 A_6} (A_5 A_6)}{-(A_1 A_5)(A_1 A_6) \sin(\pi - \gamma_1 - \gamma_2 - \theta_2)} \right\} \quad (23)$$

Therefore, the acceleration of joint 2 becomes,

$$\ddot{\theta}_2 = \frac{(A_5 A_6)}{(A_1 A_5)(A_1 A_6)} \left\{ \frac{\cos(\pi - \gamma_1 - \gamma_2 - \theta_2) \dot{\theta}_2 V_{A_5 A_6} - \sin(\pi - \gamma_1 - \gamma_2 - \theta_2) \alpha_{A_5 A_6}}{\sin^2(\pi - \gamma_1 - \gamma_2 - \theta_2)} \right\} \quad (24)$$

The time derivative of (25) yields the acceleration of the joint 3, $\ddot{\theta}_3$.

$$\dot{\theta}_3 = \left\{ \frac{V_{A_7 A_8} (A_7 A_8)}{-(A_2 A_7)(A_2 A_8) \sin(3\pi - \delta_1 - \delta_2 - \theta_3)} \right\} \quad (25)$$

Therefore, the acceleration of joint 3 becomes,

$$\ddot{\theta}_3 = \frac{(A_7 A_8)}{(A_2 A_7)(A_2 A_8)} \left\{ \frac{\cos(3\pi - \delta_1 - \delta_2 - \theta_3) \dot{\theta}_3 V_{A_7 A_8} - \sin(3\pi - \delta_1 - \delta_2 - \theta_3) \alpha_{A_7 A_8}}{\sin^2(3\pi - \delta_1 - \delta_2 - \theta_3)} \right\} \quad (26)$$

By differentiating (27) with respect to time yields the acceleration of the actuator 5 piston in terms of the angle acceleration $\ddot{\zeta}_1$, and then $\ddot{\zeta}_1$ will be determined in terms of the joint 4 angle acceleration $\ddot{\theta}_4$:

Therefore,

$$\dot{\zeta}_1 = \left\{ \frac{V_{A_9 A_{10}} (A_9 A_{10})}{-(A_9 A_{12})(A_{10} A_{12}) \sin(2\pi - \varepsilon_1 - \zeta_1)} \right\} \quad (27)$$

and

$$\ddot{\zeta}_1 = \frac{-(A_9 A_{10})}{(A_9 A_{12})(A_{10} A_{12})} \left\{ \frac{\cos(2\pi - \varepsilon_1 - \zeta_1) \dot{\zeta}_1 V_{A_9 A_{10}} - \sin(2\pi - \varepsilon_1 - \zeta_1) \alpha_{A_9 A_{10}}}{\sin^2(2\pi - \varepsilon_1 - \zeta_1)} \right\} \quad (28)$$

The following two relations are known to us from (29) and (30).

$$\zeta_1 + \zeta_2 = \theta_4 - \zeta_3 \quad (29)$$

$$\dot{\zeta}_2 = \left[\frac{(A_3 A_{12})(A_{10} A_{12}) \sin \zeta_1}{(A_3 A_{11})(A_{10} A_{11}) \sin \zeta_2} \right] \dot{\zeta}_1 \quad (30)$$

Now substituting (30) into (29) and rearranging the (29) yields,

$$\dot{\theta}_4 = \dot{\zeta}_1 \left[1 + \frac{(A_3 A_{12})(A_{10} A_{12}) \sin \zeta_1}{(A_3 A_{11})(A_{10} A_{11}) \sin \zeta_2} \right] + \dot{\zeta}_3 \quad (31)$$

The acceleration of joint 4 becomes,

$$\ddot{\theta}_4 = \dot{\zeta}_1 \left[\frac{(A_3 A_{12})(A_{10} A_{12})}{(A_3 A_{11})(A_{10} A_{11})} \left\{ \frac{\sin \zeta_2 \cos \zeta_1 \cdot \dot{\zeta}_1 - \cos \zeta_1 \sin \zeta_2 \cdot \dot{\zeta}_2}{\sin^2 \zeta_2} \right\} \right] + \dot{\zeta}_1 \left[1 + \frac{(A_3 A_{12})(A_{10} A_{12}) \sin \zeta_1}{(A_3 A_{11})(A_{10} A_{11}) \sin \zeta_2} \right] + \ddot{\zeta}_3 \quad (32)$$

C.Determination of the Elements of a Velocity Induced Torque Matrix

Using (12) all the nine elements of the velocity induced torque matrix can be determined as H_{ij} for joint $i = 2, 3$, and 4 ; and joint $j = 2, 3$, and 4 . So total nine elements $H_{22}, H_{23}, H_{24}, H_{32}, H_{33}, H_{34}, H_{42}, H_{43}$, and H_{44} have to be determined as a function of the joint angle velocities $\dot{\theta}_i$ for $i = 2, 3$, and 4 . The H_{ij} terms can be determined by the following equation as given in [12]:

$$H_{ij} = \sum_{k=2}^4 h_{ijk} \dot{\theta}_k \quad (33)$$

Let us first determine the element H_{22} with the use of (33), and (12) as follows:

$$H_{22} = h_{222} \dot{\theta}_2 + h_{223} \dot{\theta}_3 + h_{224} \dot{\theta}_4 \quad (34)$$

Now let us first determine the term, $h_{222} \dot{\theta}_2$ and for this h_{222} is to be determined using (12) as follows:

$$h_{222} = \frac{1}{2} \left[\frac{\partial M_{22}}{\partial \theta_2} + \frac{\partial M_{22}}{\partial \theta_2} - \frac{\partial M_{22}}{\partial \theta_2} \right] = \frac{1}{2} \left[\frac{\partial M_{22}}{\partial \theta_2} \right] = 0$$

This is because it is clear from (22) that the element M_{22} is not a function of the joint 2 angle θ_2 , thus partial derivative of the element M_{22} with respect to the joint 2 angle θ_2 will become zero. This leads to;

$$h_{222} \dot{\theta}_2 = (0) \dot{\theta}_2 \quad (35)$$

Now let us determine the second term $h_{223} \dot{\theta}_3$ of (34) and for this to be determined h_{223} can be given as follows:

$$h_{223} = \frac{1}{2} \left[\frac{\partial M_{22}}{\partial \theta_3} + \frac{\partial M_{23}}{\partial \theta_2} - \frac{\partial M_{23}}{\partial \theta_2} \right] = \frac{1}{2} \left[\frac{\partial M_{22}}{\partial \theta_3} \right]$$

$$\therefore h_{223} = -(m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) + m_4 \bar{x}_4(a_2 s_{34}) + m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34})) \dot{\theta}_3 \quad (36)$$

So, the term $h_{223} \dot{\theta}_3$ can be given as:

$$h_{223} \dot{\theta}_3 = -(m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) + m_4 \bar{x}_4(a_2 s_{34}) + m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34})) \dot{\theta}_3 \quad (37)$$

The last term of (34) can be given by:

$$h_{224} \dot{\theta}_4 = -(m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34})) \dot{\theta}_4 \quad (38)$$

The first element of the H_{ij} matrix, H_{22} can be given as follows:

$$H_{22} = (0) \dot{\theta}_2 - (m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) + m_4 \bar{x}_4(a_2 s_{34}) + m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34})) \dot{\theta}_3 - (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34})) \dot{\theta}_4 \quad (39)$$

So, the final velocity induced torque matrix or matrix of centripetal and Coriolis torques can be given as:

$$H(\theta, \dot{\theta}) = \begin{bmatrix} H_{22} & H_{23} & H_{24} \\ H_{32} & H_{33} & H_{34} \\ H_{42} & H_{43} & H_{44} \end{bmatrix} \quad (40)$$

All the other elements are given in section VIII-F. In this section only the methodology to derive the velocity induced torque matrix elements have been explained with the illustration of the determination of the first element H_{22} .

D.Determination of the Elements of a Gravity Torque Vector

The gravity torque acting at joint 2, 3 and 4 is the function of joint displacement or joint angle θ_i for joints $i = 2, 3$, and 4 , and can be collectively represented by a vector as follows:

$$G(\theta) = \begin{Bmatrix} G_2 \\ G_3 \\ G_4 \end{Bmatrix} \quad (41)$$

Using (13) let us first determine the first element of the gravity torque vector, G_2 as follows:

$$G_2 = -(m_2 g d_{22}^2 \bar{r}_2 + m_3 g d_{32}^3 \bar{r}_3 + m_4 g d_{42}^4 \bar{r}_4)$$

where, $g = [g_x \ g_y \ g_z \ 0] = [0 \ -9.81 \ 0 \ 0]$ (assuming the acceleration due to gravity is only acting in downward y direction), ${}^i \bar{r}_i = [\bar{x}_i \ \bar{y}_i \ \bar{z}_i \ 1]^T$ is a homogeneous coordinate vector defining the position of the centre of mass or centroid of the link i from the origin O_i of the link i . This leads to:

$$G_2 = -(m_2 g [c_2 \bar{x}_2 - s_2 \bar{y}_2 + a_2 c_2] + m_3 g [c_{23} \bar{x}_3 - s_{23} \bar{y}_3 + a_2 c_2 + a_3 c_{23}] + m_4 g [c_{234} \bar{x}_4 - s_{234} \bar{y}_4 + a_2 c_2 + a_3 c_{23} + a_4 c_{234}]) \quad (42)$$

All the other elements are given in section VIII-F. In this section only the methodology to derive the Gravity torque vector elements have been explained with the illustration of the determination of the first element G_2 .

E.Determination of the Elements of a Load Vector

The loading vector resulting from the interactive forces due to soil tool interaction and its elements are given in [1] and can be written as:

$$F_{Load}(F_t, F_n) = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

The elements F_2 , F_3 and F_4 of the loading vector also determined from the geometry of backhoe excavator by resolving the resistive forces offered by ground as the

tangential and normal components and can be written as per (14).

Therefore, the load vector,

$$F_{Load}(F_t, F_n) = \begin{bmatrix} a_2[F_t \sin(\theta_2 - \rho) - F_n \cos(\theta_2 - \rho)] \\ a_3[F_t \sin(\theta_{23} - \rho) - F_n \cos(\theta_{23} - \rho)] \\ a_4[-F_t \sin \lambda + F_n \cos \lambda] \end{bmatrix}$$

As shown in Fig. 2 the resistive force F_r is acting at the bucket teeth by making an angle η with the plane defined by the bottom plate of the bucket. While resolving F_r in tangential direction will give the value of the tangential force F_t , and resolving F_r in the normal direction will give the value of the normal force F_n as follows:

$$F_t = F_r \cos(\eta)$$

and,

$$F_n = F_r \sin(\eta)$$

But according to A. J. Koivo [1] these two forces can be given by: $F_t = F_r \cos(0.1)$, and $F_n = F_r \sin(0.1)$.

F. Determination of the Final Equations of Motions

The final form of the equations of motion (or dynamic model) for the backhoe can be given as follows:

$$\tau_i = M(\theta)\ddot{\theta} + H(\theta, \dot{\theta})\dot{\theta} + G(\theta) + F_{Load}(F_t, F_n) \quad (43)$$

where the inertia matrix and its elements are given by:

$$M(\theta) = \begin{bmatrix} M_{22} & M_{23} & M_{24} \\ M_{32} & M_{33} & M_{34} \\ M_{42} & M_{43} & M_{44} \end{bmatrix}$$

Where the elements of the inertia matrix are given as:

$$\begin{aligned} M_{22} = & I_{ZZ_2} + I_{ZZ_3} + I_{ZZ_4} + m_2[(2\bar{x}_2 a_2) + (a_2)^2] \\ & + m_3[\bar{x}_3(2a_3 + 2a_2 c_3) - \bar{y}_3(2a_2 s_3) + a_2^2 + a_3^2 \\ & + 2a_2 a_3 c_3] + m_4[\bar{x}_4(2a_4 + 2a_2 c_{34} + 2a_3 c_4) \\ & - \bar{y}_4(2a_2 s_{34} + 2a_3 s_4) + a_2^2 + a_3^2 + a_4^2 \\ & + 2a_2 a_3 c_3 + 2a_3 a_4 c_4 + 2a_2 a_4 c_{34}] \end{aligned}$$

$$\begin{aligned} M_{23} = M_{32} = & I_{ZZ_3} + I_{ZZ_4} \\ & + m_3[\bar{x}_3(2a_3 + a_2 c_3) - \bar{y}_3(a_2 s_3) + a_3^2 + a_2 a_3 c_3] \\ & + m_4[\bar{x}_4(2a_4 + a_2 c_{34} + 2a_3 c_4) \\ & - \bar{y}_4(a_2 s_{34} + 2a_3 s_4) + a_3^2 + a_4^2 + a_2 a_3 c_3 \\ & + 2a_3 a_4 c_4 + a_2 a_4 c_{34}] \end{aligned}$$

$$\begin{aligned} M_{24} = M_{42} = & I_{ZZ_4} + m_4[\bar{x}_4(2a_4 + a_2 c_{34} + a_3 c_4) - \bar{y}_4(a_2 s_{34} + a_3 s_4) \\ & + a_4^2 + a_3 a_4 c_4 + a_2 a_4 c_{34}] \end{aligned}$$

$$\begin{aligned} M_{33} = & I_{ZZ_3} + I_{ZZ_4} + m_3[\bar{x}_3(2a_3) + a_3^2] + m_4[\bar{x}_4(2a_4 + 2a_3 c_4) \\ & - \bar{y}_4(2a_3 s_4) + a_3^2 + a_4^2 + 2a_3 a_4 c_4] \end{aligned}$$

$$M_{34} = M_{43} = I_{ZZ_4} + m_4[\bar{x}_4(2a_4 + a_3 c_4) - \bar{y}_4(a_3 s_4) + a_4^2 + a_3 a_4 c_4]$$

and

$$M_{44} = I_{ZZ_4} + m_4[\bar{x}_4(2a_4) + a_4^2].$$

The joint acceleration column matrix can be written as follows and derived in Section VIII-B;

$$\ddot{\theta} = \begin{bmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix}$$

The velocity induced torques matrix or the matrix of centripetal and Coriolis torques and its elements are given by:

$$H(\theta, \dot{\theta}) = \begin{bmatrix} H_{22} & H_{23} & H_{24} \\ H_{32} & H_{33} & H_{34} \\ H_{42} & H_{43} & H_{44} \end{bmatrix}$$

where the elements of the velocity induced torques are given as:

$$\begin{aligned} H_{22} = & (0)(\dot{\theta}_2) - (m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) \\ & + m_4 \bar{x}_4(a_2 s_{34}) + m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 \\ & + a_2 a_4 s_{34}))(\dot{\theta}_3) \\ & - (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) \\ & + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34}))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{23} = & -(m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) + m_4 \bar{x}_4(a_2 s_{34}) \\ & + m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34}))(\dot{\theta}_2) \\ & - (m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) \\ & + m_4 \bar{x}_4(a_2 s_{34}) + m_4 \bar{y}_4(a_2 c_{34}) \\ & + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34}))(\dot{\theta}_3) \\ & - (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) \\ & + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34}))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{24} = & -(m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) + m_4(a_3 a_4 s_4 \\ & + a_2 a_4 s_{34}))(\dot{\theta}_2) - (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) \\ & + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34}))(\dot{\theta}_3) - (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) \\ & + m_4(a_3 a_4 s_4 + a_2 a_4 s_{34}))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{32} = & (m_3 \bar{x}_3(a_2 s_3) + m_3 \bar{y}_3(a_2 c_3) + m_3(a_2 a_3 s_3) + m_4 \bar{x}_4(a_2 s_{34}) + \\ & m_4 \bar{y}_4(a_2 c_{34}) + m_4(a_2 a_3 s_3 + a_2 a_4 s_{34}))(\dot{\theta}_2) + 0(\dot{\theta}_3) - (m_4 \bar{x}_4(a_3 s_4) + \\ & m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{33} = & 0(\dot{\theta}_2) + 0(\dot{\theta}_3) - (m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{34} = & -(m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_2) - \\ & (m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_3) - (m_4 \bar{x}_4(a_3 s_4) + \\ & m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{42} = & (m_4 \bar{x}_4(a_2 s_{34} + a_3 s_4) + m_4 \bar{y}_4(a_2 c_{34} + a_3 c_4) + m_4(a_3 a_4 s_4 \\ & + a_2 a_4 s_{34}))(\dot{\theta}_2) \\ & + (m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_3) \\ & + 0(\dot{\theta}_4) \end{aligned}$$

$$\begin{aligned} H_{43} = & (m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_2) \\ & + (m_4 \bar{x}_4(a_3 s_4) + m_4 \bar{y}_4(a_3 c_4) + m_4(a_3 a_4 s_4))(\dot{\theta}_3) \\ & + 0(\dot{\theta}_4) \end{aligned}$$

and

$$H_{44} = 0(\dot{\theta}_2) + 0(\dot{\theta}_3) + 0(\dot{\theta}_4).$$

The joint velocity column matrix can be given by:

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix}$$

The gravity torque vector and its elements are given by:

$$G(\theta) = \begin{Bmatrix} G_2 \\ G_3 \\ G_4 \end{Bmatrix}$$

where the elements of the gravity torque vector are given as:

$$G_2 = -(m_2g[c_{23}\bar{x}_2 - s_{23}\bar{y}_2 + a_2c_2] + m_3g[c_{23}\bar{x}_3 - s_{23}\bar{y}_3 + a_2c_2 + a_3c_{23}] + m_4g[c_{234}\bar{x}_4 - s_{234}\bar{y}_4 + a_2c_2 + a_3c_{23} + a_4c_{234}])$$

$$G_3 = -(m_3g[c_{23}\bar{x}_3 - s_{23}\bar{y}_3 + a_3c_{23}] + m_4g[c_{234}\bar{x}_4 - s_{234}\bar{y}_4 + a_3c_{23} + a_4c_{234}])$$

and

$$G_4 = -(m_4g[c_{234}\bar{x}_4 - s_{234}\bar{y}_4 + a_4c_{234}]).$$

The loading vector resulting from the interactive forces due to soil tool interaction and its elements are given in [1] and written as follows:

$$F_{Load}(F_t, F_n) = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

where the elements of the loading vector are given as:

$$F_2 = a_2[F_t \sin(\theta_2 - \rho) - F_n \cos(\theta_2 - \rho)],$$

$$F_3 = a_3[F_t \sin(\theta_{23} - \rho) - F_n \cos(\theta_{23} - \rho)],$$

$$\text{and } F_4 = a_4[-F_t \sin\lambda + F_n \cos\lambda].$$

IX. RESULTS AND DISCUSSION

The proposed dynamic model has been derived by Lagrange-Euler (L-E) formulation for the three degrees of freedom of the backhoe excavator. The proposed dynamic model determines the required joint torques for the given set of trajectory points, joint angle vector θ , joint speed vector $\dot{\theta}$, joint acceleration vector $\ddot{\theta}$. This proposed dynamic model will be useful to the autonomous controlling problem of the backhoe excavator. The MATLAB codes for the proposed dynamic model and for A. J. Koivo's dynamic model [1] are developed. Parameters used for both the models are listed in Table I and as well as shown in Fig. 1. To obtain the joint velocity and acceleration the quadratic nonlinear equation is consider for trajectory planning. The total angle moved by boom, arm and bucket are 117.64° , 120.46° and 157.52° respectively and time required to move these angles from start position to end position are 9.5 seconds, 10.3 seconds and 8.48 seconds respectively. For the validation purpose here we have considered the maximum breakout force condition.

To reach at the maximum breakout force position, the boom, arm and bucket moves the angles of 44.13° , 107.69° and 137.31° respectively from its starting position and it required the time of 3.97 seconds, 7.54 seconds and 6.8

seconds respectively. Based on the MATLAB codes the joint velocities and acceleration are obtained. This leads to the joint parameters θ_i , $\dot{\theta}_i$, and $\ddot{\theta}_i$ for $i = 2, 3$, and 4 as shown in Table I.

Here, the joint torques calculated for the maximum breakout force condition of 7626 N [2], and the parameters as listed in Table I are used in the MATLAB codes for both the proposed dynamic model and A. J. Koivo's dynamic model [1], will give the following joint torque vectors.

The calculated torque vector for the proposed model:

$$\tau_i = \begin{Bmatrix} \tau_2 \\ \tau_3 \\ \tau_4 \end{Bmatrix} = \begin{Bmatrix} -10518 \\ -5346 \\ -3788 \end{Bmatrix}$$

The calculated torque vector for Koivo's model:

$$\tau_i = \begin{Bmatrix} \tau_2 \\ \tau_3 \\ \tau_4 \end{Bmatrix} = \begin{Bmatrix} -9578.5 \\ -4980.3 \\ -3698.6 \end{Bmatrix}$$

TABLE I
PARAMETERS USED IN MATLAB CODES FOR DYNAMIC MODEL

Description	Symbol	Value	Unit
Moment of inertia of link 2 (boom), link 3 (arm), and link (4) bucket with respect to z axis of the frame {2}, {3}, and {4} respectively	I_{zz2} I_{zz3} I_{zz4}	8.809856 5.875598 2.343296	$\text{Kg}\cdot\text{m}^2$
Mass of link 2 (boom), link 3 (arm), and link 4 (bucket)	m_2 m_3 m_4	51.664 32.450 22.007	Kg
The distance of centre of mass of boom from the origin of frame {2} in X_2 , Y_2 , and Z_2 directions	x_2 y_2 z_2	0.566861 0.375865 0.000000	m
The distance of centre of mass of arm from the origin of frame {3} in X_3 , Y_3 , and Z_3 directions	x_3 y_3 z_3	0.312503 0.264238 0.000000	m
The distance of centre of mass of bucket from the origin of frame {4} in X_4 , Y_4 , and Z_4 directions	x_4 y_4 z_4	0.400734 0.150509 0.000000	m
Acceleration due to the gravity of the earth	g	9.81	m/s^2
Link lengths of boom, arm and bucket	a_2 a_3 a_4	1.34658 0.72296 0.547	m
Joint angles for joint 2, joint 3, and joint 4	θ_2 θ_3 θ_4	0.2618 5.1569 6.2832	radian
Joint angle velocities for joint 2, joint 3, and joint 4	$\dot{\theta}_2$ $\dot{\theta}_3$ $\dot{\theta}_4$	0.3156 0.2402 0.3442	rad/sec
Joint angle accelerations for joint 2, joint 3, and joint 4	$\ddot{\theta}_2$ $\ddot{\theta}_3$ $\ddot{\theta}_4$	0.0223 -0.0552 -0.1543	rad/sec^2
The resistive force	F_r	7626	N
The digging angle	ρ	75.82	degree
The angle between the bucket bottom plate plane and the positive X_4 axis	λ $= \theta_{234}$ $= 3\pi - \rho$	54.63	degree

It can be seen that the highest torque is acting at the joint 2, and lowest at joint 4 for both the models. The difference between the two models is 8.93 % for τ_2 , 6.84 % for τ_3 and 2.36 % for τ_4 . The difference of torque at joint 2 of τ_2 is higher compare to others because the of the geometrical difference in excavator model taken for study by A. J. Koivo et al. as given in [1], in which the front end of the boom

cylinders are attached at both side of the boom (link 2) and rear end of the boom supported by link 1 from bottom, whereas in our case the front end of the boom cylinder is hinged on top surface of the boom and rear end hinged on swing link at upper side of the boom as shown in Fig. 1. The differences in results are very less and acceptable.

X.CONCLUSION

A complete dynamic model of the backhoe in digging mode is presented using L-E approach. The proposed dynamic model can be used as the basis for automating the digging operation of the backhoe. This can be accomplished by designing the controller so that the entire system can be operated in autonomous mode. The approach presented can equally be applied to the operations of all type of backhoe excavators. The proposed dynamic model of the backhoe is useful in computation of the required torques for execution of a typical work cycle, which can be used for design of backhoe mechanism and helpful for autonomous application of excavator. Apart from this in designing the controller for the backhoe, this dynamic model can be utilized to obtain the desired performance, because the controller directly depends on the accuracy of the dynamic model and control algorithms. MATLAB codes are developed for our presented dynamic model which is based on L-E formulation and dynamic model presented by A. J. Koivo et al. [1] which is based on N-E formulation and compared them for same parameters. The results show the acceptable difference in values of torque. The variation in the torque is due to only the geometrical differences in the backhoe excavator models.

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