GEOMETRY
Through Algebra

Chakali Suresh

Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1}$$

1.1. Vectors

1.1.1. the direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Find the direction vectors of AB, BC and CA.

Solution:

(a) The Direction vector of AB is

$$= \mathbf{B} - \mathbf{A} \qquad (1.1.1.2)$$

$$= \begin{pmatrix} -2 - (0) \\ -4 - (-5) \end{pmatrix} \qquad (1.1.1.3)$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad (1.1.1.4)$$

(b) The Direction vector of BC

$$= \mathbf{C} - \mathbf{B} \qquad (1.1.1.5)$$

$$= \begin{pmatrix} -5 - (-2) \\ -4 - (-4) \end{pmatrix} \qquad (1.1.1.6)$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \qquad (1.1.1.7)$$

(c) The Direction vector of CA

$$= \mathbf{A} - \mathbf{C} \qquad (1.1.1.8)$$

$$= \begin{pmatrix} 0 - (-5) \\ -5 - (-4) \end{pmatrix} \qquad (1.1.1.9)$$

$$= \begin{pmatrix} 5 \\ -1 \end{pmatrix} \qquad (1.1.1.10)$$

1.1.2. The length of side AB, BC and AC is

Solution: Given,

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \tag{1.1.2.1}$$

$$\mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \tag{1.1.2.2}$$

$$\mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.2.3}$$

Now solving for AB,

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{(\mathbf{A} - \mathbf{B})^{\top}(\mathbf{A} - \mathbf{B})} \tag{1.1.2.4}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.1.2.5}$$

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{\begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}} \tag{1.1.2.6}$$

$$= \sqrt{(2)^2 + (-1)^2} \tag{1.1.2.7}$$

$$\implies \|\mathbf{A} - \mathbf{B}\| = \sqrt{5} \tag{1.1.2.8}$$

Now solving for BC,

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{(\mathbf{B} - \mathbf{C})^{\top}(\mathbf{B} - \mathbf{C})} \qquad (1.1.2.9)$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad (1.1.2.10)$$

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{\begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}} \qquad (1.1.2.11)$$

$$= \sqrt{(3)^2 + (0)^2} \qquad (1.1.2.12)$$

$$\implies \|\mathbf{B} - \mathbf{C}\| = \sqrt{9} \qquad (1.1.2.13)$$

Now solving for AC,

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{(\mathbf{A} - \mathbf{C})^{\top}(\mathbf{A} - \mathbf{C})} \qquad (1.1.2.14)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \qquad (1.1.2.15)$$

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{\begin{pmatrix} -5 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix}} \qquad (1.1.2.16)$$

$$= \sqrt{(-5)^2 + (1)^2} \qquad (1.1.2.17)$$

$$\implies \|\mathbf{A} - \mathbf{C}\| = \sqrt{26} \qquad (1.1.2.18)$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be colliner if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \qquad (1.1.3.1)$$

Solution:

Given that,

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \qquad (1.1.3.2)$$

Given that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} < 3 \qquad (1.1.3.3)$$

Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \qquad (1.1.3.4)$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \xleftarrow{R_3 \leftarrow R_3 + (5+R_2)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & -1 & -4 \end{pmatrix} \qquad (1.1.3.5)$$

$$\xleftarrow{R_3 \leftarrow -2R_3 + R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 3 \end{pmatrix} \qquad (1.1.3.6)$$

There are no zero rows. So,

$$\operatorname{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \qquad (1.1.3.7)$$

Hence, from (1.1.3.3) the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear.
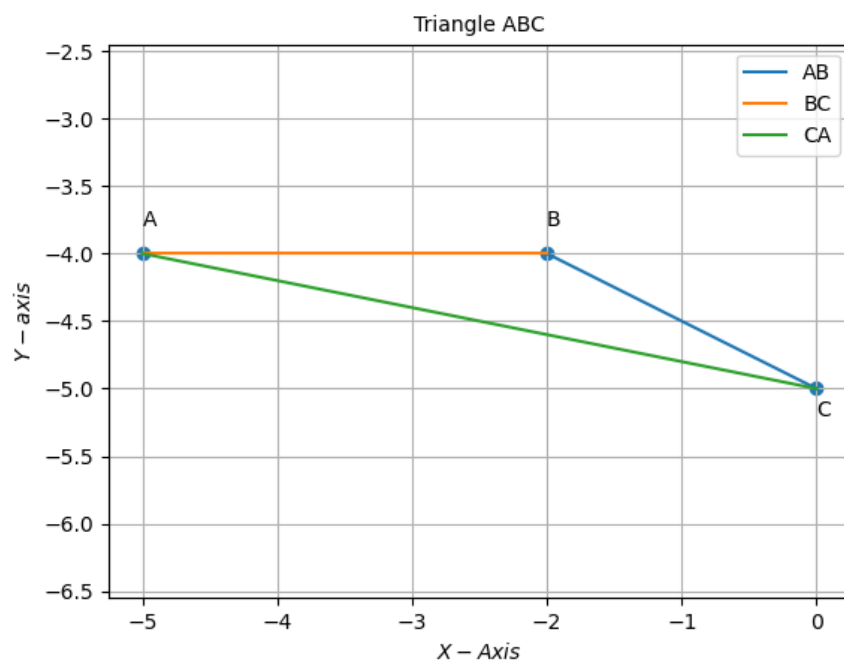
Figure 1.1: $\mathbf{A}, \mathbf{B}, \mathbf{C}$ plot

From Fig. 1.1, We can see that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear .

1.1.4. The parametric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

7

is the direction vector of AB. Find the parametric equations of AB, BC and CA.

Solution:

The parametric equation for AB is given by

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.3}$$

$$\text{where, } \mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.4}$$

$$= \begin{pmatrix} -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.4.5}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.4.6}$$

Hence we get,

$$\mathbf{AB} : \mathbf{x} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} + k \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.4.7}$$

Similarly,

$$\mathbf{BC} : \mathbf{x} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} + k \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.4.8}$$

$$\mathbf{CA} : \mathbf{x} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} + k \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{1.1.4.9}$$

1.1.5. The normal form of the equation of $\mathbf{AB}$ is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A} \right) = 0 \qquad (1.1.5.1)$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top \left(\mathbf{B} - \mathbf{A} \right) = 0 \qquad (1.1.5.2)$$

or,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \qquad (1.1.5.3)$$

then find the normal form of the equations of $\mathbf{AB}$ $\mathbf{BC}$ and $\mathbf{CA}$

Solution: :

The normal equation for the side AB is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A} \right) = 0 \qquad (1.1.5.4)$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \qquad (1.1.5.5)$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given.

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \qquad (1.1.5.6)$$

9

Here $\mathbf{m} = \mathbf{B} - \mathbf{A}$ for side $\mathbf{AB}$

$$\implies \mathbf{m} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.5.7}$$

$$= \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.5.8}$$

Now as we have obtained vector $\mathbf{m}$.we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{1.1.5.9}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^\top = \begin{pmatrix} 1 & 2 \end{pmatrix} \tag{1.1.5.10}$$

Hence the normal equation of side AB is

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.5.11}$$

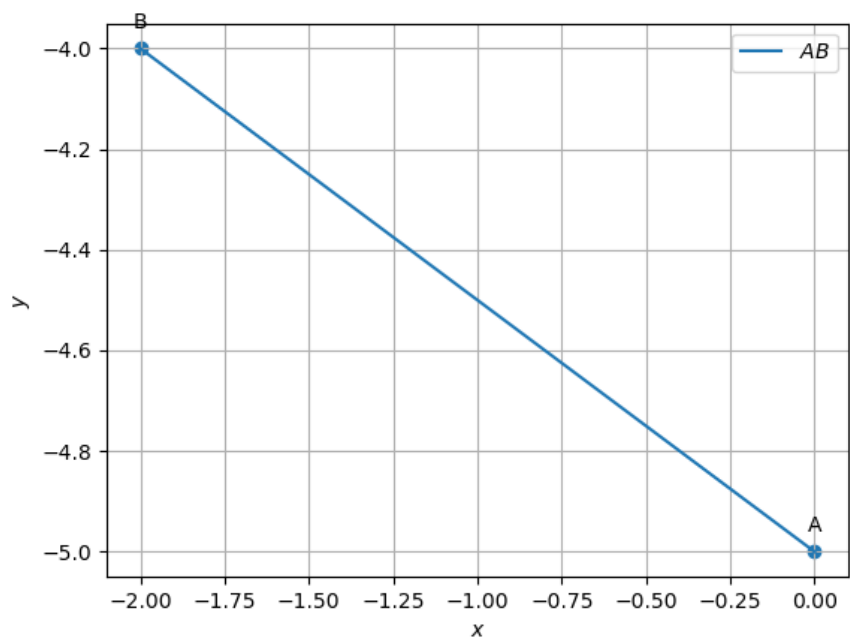$$\implies \begin{pmatrix} 1 & 2 \end{pmatrix} \mathbf{x} = -10 \tag{1.1.5.12}$$

Figure 1.2: The line **AB** plotted

Similarly

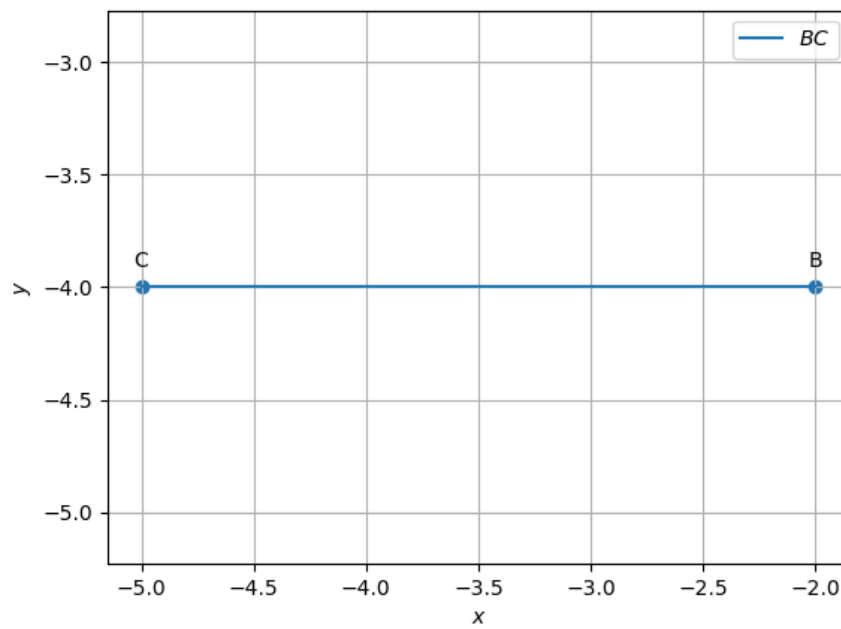$$\implies \mathbf{BC} : \begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = -12 \qquad (1.1.5.13)$$

Figure 1.3: The line **BC** plotted

$$\implies \mathbf{CA} : \begin{pmatrix} -1 & -5 \end{pmatrix} \mathbf{x} = 25 \qquad (1.1.5.14)$$
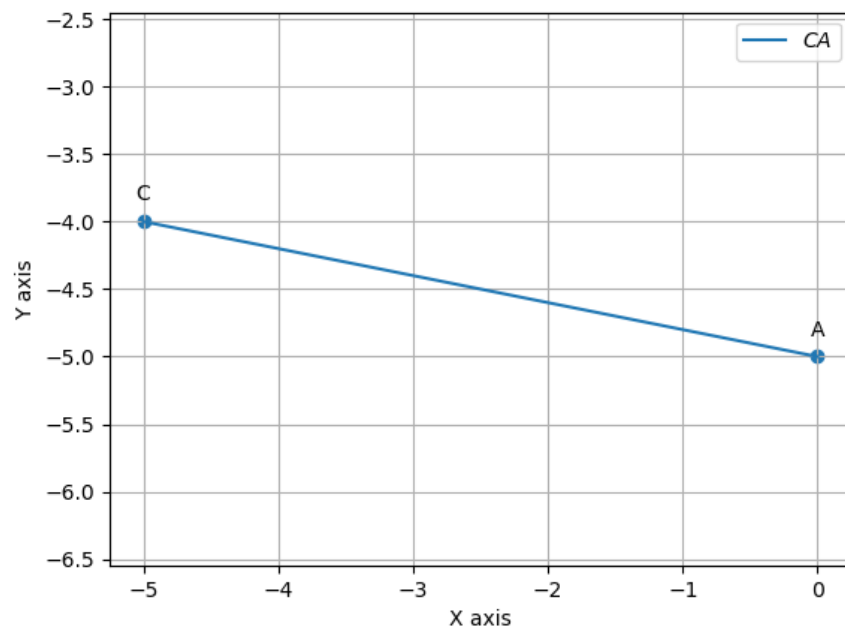
Figure 1.4: The line **CA** plotted

1.1.6. Find the area of the $\triangle ABC$

Solution:

Given,

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} ; \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} ; \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.6.1}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$(1.1.6.2)$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$(1.1.6.3)$$

$$\therefore (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 2 & -1 \\ 5 & -1 \end{vmatrix} \qquad (1.1.6.4)$$

$$= 2 \times -1 - 5 \times (-1) \qquad (1.1.6.5)$$

$$= -2 + 5 \qquad (1.1.6.6)$$

$$= 3 \qquad (1.1.6.7)$$

$$\implies \frac{1}{2} \|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| = \frac{1}{2} \|3\| = \frac{3}{2} \qquad (1.1.6.8)$$

1.1.7. Find the angles A, B, C if

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})^\top \mathbf{C} - \mathbf{A}}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{C} - \mathbf{A}\|} \qquad (1.1.7.1)$$

Solution:

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

14

(a) Finding the value of angle A

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.7.2}$$

and

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \tag{1.1.7.3}$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{5} \tag{1.1.7.4}$$

$$\|\mathbf{C} - \mathbf{A}\| = \sqrt{26} \tag{1.1.7.5}$$

and by doing matrix multiplication we get,

$$(\mathbf{B} - \mathbf{A})^{\top}(\mathbf{C} - \mathbf{A}) = \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix} = 11 \tag{1.1.7.6}$$

So, we get

$$\cos A = \frac{11}{\sqrt{5}\sqrt{26}} \tag{1.1.7.7}$$

$$= \frac{11}{\sqrt{130}} \tag{1.1.7.8}$$

$$\implies A = \cos^{-1}\frac{11}{\sqrt{130}} \tag{1.1.7.9}$$

(b) Finding the value of angle B

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.7.10}$$

and

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.1.7.11}$$

also calculating the values of norms

$$\|\mathbf{C} - \mathbf{B}\| = \sqrt{9} = 3 \tag{1.1.7.12}$$

$$\|\mathbf{A} - \mathbf{B}\| = \sqrt{5} \tag{1.1.7.13}$$

and by doing matrix multiplication we get,

$$(\mathbf{C} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B}) = \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -6 \tag{1.1.7.14}$$

So, we get

$$\cos B = \frac{-6}{3\sqrt{5}} \tag{1.1.7.15}$$

$$= \frac{-2}{\sqrt{5}} \tag{1.1.7.16}$$

$$\implies B = \cos^{-1} \frac{-2}{\sqrt{5}} \tag{1.1.7.17}$$

(c) Finding the value of angle C

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{1.1.7.18}$$

and

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.1.7.19}$$

also calculating the values of norms

$$\|\mathbf{A} - \mathbf{C}\| = \sqrt{26} \tag{1.1.7.20}$$

$$\|\mathbf{B} - \mathbf{C}\| = \sqrt{9} = 3 \tag{1.1.7.21}$$

and by doing matrix multiplication we get,

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.1.7.22}$$

$$= 15$$

so

$$\cos C = \frac{15}{3\sqrt{26}} \tag{1.1.7.23}$$

$$= \frac{5}{\sqrt{26}} \tag{1.1.7.24}$$

$$\implies C = \cos^{-1} \frac{5}{\sqrt{26}} \tag{1.1.7.25}$$

17