Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \tag{1.1}$$

1.1. Angle Bisector

1.1.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \tag{1.1.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Question 1.1.2.

Solution: From Question 1.1.2

$$a = 3 \tag{1.1.1.2}$$

$$b = \sqrt{26} \tag{1.1.1.3}$$

$$c = \sqrt{5} \tag{1.1.1.4}$$

From the given information,

$$a = m + n, \tag{1.1.1.5}$$

$$b = n + p, \tag{1.1.1.6}$$

$$c = m + p \tag{1.1.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.8}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.9}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.1.1.10}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.11}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.12}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.1.1.13}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{5}+\sqrt{26}-3}{2} \\ m &= \frac{a+c-b}{2} = \frac{3+\sqrt{5}-\sqrt{26}}{2} \\ n &= \frac{a+b-c}{2} = \frac{3+\sqrt{26}-\sqrt{5}}{2} \end{aligned} \tag{1.1.1.14}$$

on solving above equations we get

$$p = 2.167543746 \tag{1.1.1.15}$$

$$m = 0.068524232 \tag{1.1.1.16}$$

$$n = 2.931475768 \tag{1.1.1.17}$$

1.1.2. Using section formula, find $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$.

Solution: Given

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n},\ \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p},\ \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.1.2.1}$$

Here

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \tag{1.1.2.2}$$

$$p = 2.167543746,\ m = 0.068524232,\ n = 2.931475768 \tag{1.1.2.3}$$

On substituting (1.1.2.2) and (1.1.2.3) in (**??**) We get

$$\mathbf{D}_3 = \frac{0.068524232\begin{pmatrix}-5\\-4\end{pmatrix} + 2.931475768\begin{pmatrix}-2\\-4\end{pmatrix}}{0.068524232 + 2.931475768} \tag{1.1.2.4}$$

$$\mathbf{E}_3 = \frac{2.931475768\begin{pmatrix}0\\-5\end{pmatrix} + 2.167543746\begin{pmatrix}-5\\-4\end{pmatrix}}{2.931475768 + 2.167543746} \tag{1.1.2.5}$$

$$\mathbf{F}_3 = \frac{2.167543746\begin{pmatrix}-2\\-4\end{pmatrix} + 0.068524232\begin{pmatrix}0\\-5\end{pmatrix}}{2.167543746 + 0.068524232} \tag{1.1.2.6}$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix}-2.06852423\\-4\end{pmatrix} \tag{1.1.2.7}$$

$$\mathbf{E}_3 = \begin{pmatrix}-2.12545151\\-4.57490969\end{pmatrix} \tag{1.1.2.8}$$

$$\mathbf{F}_3 = \begin{pmatrix}-1.93871006\\-4.03064497\end{pmatrix} \tag{1.1.2.9}$$

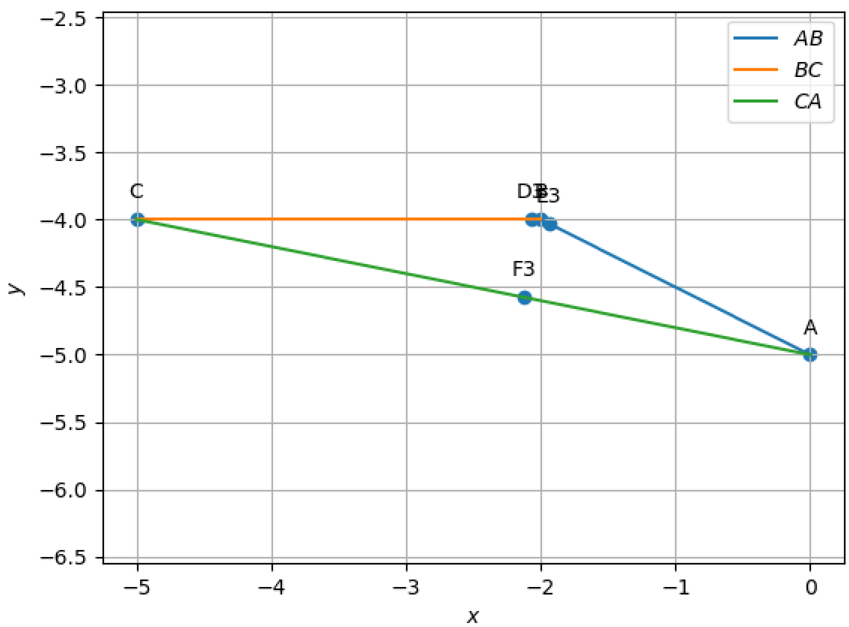

Figure 1.1: Points $D3, E3, F3$

1.1.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

Solution: Given

$$\mathbf{D}_3 = \begin{pmatrix} -2.06852423 \\ -4 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} -2.12545151 \\ -4.57490969 \end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} -1.93871006 \\ -4.03064497 \end{pmatrix} \tag{1.1.3.3}$$

(a) For circumcentre

Vector equation of $\mathbf{D}-\mathbf{E}$ is

$$(\mathbf{D}_3-\mathbf{E}_3)^\top\left(\mathbf{x}-\frac{\mathbf{D}_3+\mathbf{E}_3}{2}\right)=0 \tag{1.1.3.4}$$

$$(\mathbf{D}_3-\mathbf{F}_3)^\top\left(\mathbf{x}-\frac{\mathbf{D}_3+\mathbf{F}_3}{2}\right)=0 \tag{1.1.3.5}$$

on Substituting the values of $\mathbf{D}_3,\mathbf{E}_3,\mathbf{F}_3$ and solving We get,

$$\begin{pmatrix}0.05692728 & 0.57490969\end{pmatrix}\mathbf{x}=-2.5427151 \tag{1.1.3.6}$$

$$\begin{pmatrix}-0.1291417 & 0.03064497\end{pmatrix}\mathbf{x}=0.1370484596 \tag{1.1.3.7}$$

Thus on solving (1.1.3.6) and (1.1.3.7) using gauss elimination We get

$$\begin{pmatrix}0.05692728 & 0.057490969 & -2.5427151\\ -0.1291417 & 0.03064497 & 0.1370484596\end{pmatrix} \tag{1.1.3.8}$$

$$\therefore \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mathbf{x}=\begin{pmatrix}-2.06852423\\ -4.29027329\end{pmatrix} \tag{1.1.3.9}$$

$$\implies \mathbf{x}=\begin{pmatrix}-2.06852423\\ -4.29027329\end{pmatrix} \tag{1.1.3.10}$$

(b) The circiumradius is obtained from $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} -2.06852423 \\ -4.29027329 \end{pmatrix} \tag{1.1.3.11}$$

$$\mathbf{D}_3 = \begin{pmatrix} -2.06852423 \\ -4 \end{pmatrix} \tag{1.1.3.12}$$

$$\mathbf{I} - \mathbf{D_3} = \begin{pmatrix} 0 \\ -0.29027329 \end{pmatrix} \tag{1.1.3.13}$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \tag{1.1.3.14}$$

$$r = 0.29027329 \tag{1.1.3.15}$$

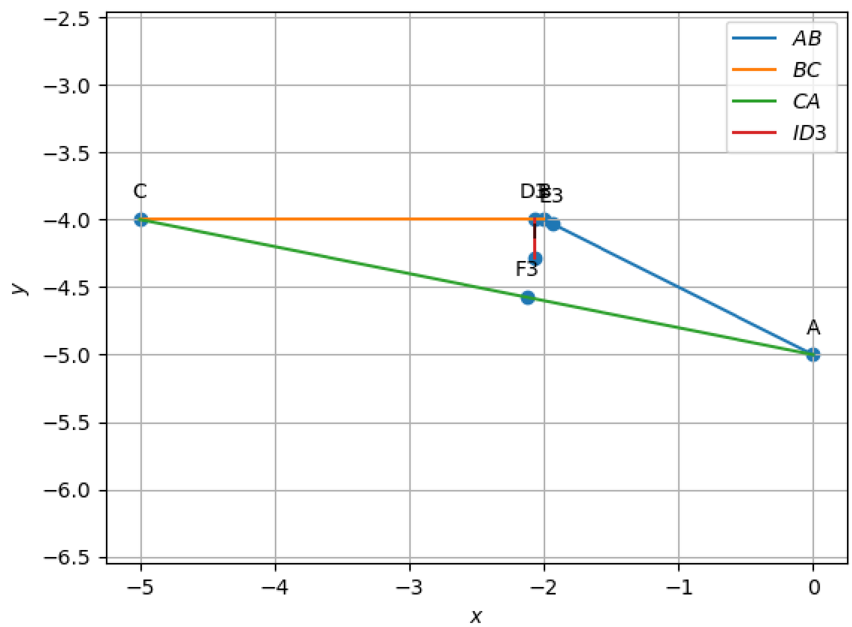

Figure 1.2: Incentre and Inradius of $\triangle ABC$

1.1.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution:

$$\mathbf{D}_3 = \begin{pmatrix} -2.06852423 \\ -4 \end{pmatrix} \tag{1.1.4.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} -2.12545151 \\ -4.57490969 \end{pmatrix} \tag{1.1.4.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} -1.93871006 \\ -4.03064497 \end{pmatrix} \tag{1.1.4.3}$$

$$\text{Incentre} \tag{1.1.4.4}$$

$$\mathbf{I} = \begin{pmatrix} -2.06852423 \\ -4.29027329 \end{pmatrix} \tag{1.1.4.5}$$

$$\text{Radius} \tag{1.1.4.6}$$

$$\mathbf{r} = 0.29027329 \tag{1.1.4.7}$$

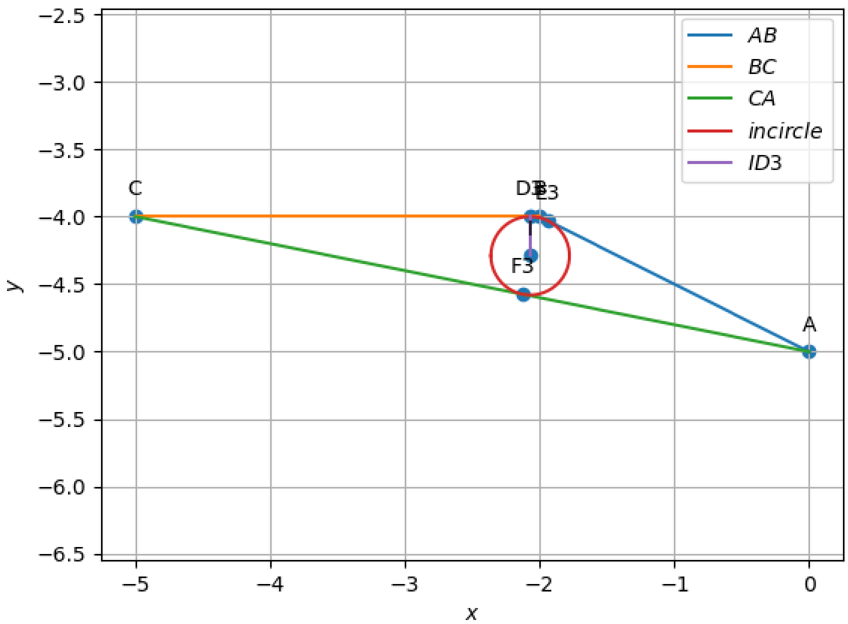

Figure 1.3: Incircle of $\triangle ABC$

1.1.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \tag{1.1.5.1}$$

AI is the bisector of $\angle A$.

Solution:

$$\cos \angle BAI \triangleq \frac{(\mathbf{B}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.2}$$

$$\cos \angle CAI \triangleq \frac{(\mathbf{C}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.3}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{I}$,

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.5.4}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \tag{1.1.5.5}$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} -2.06852423 \\ 0.70972671 \end{pmatrix} \tag{1.1.5.6}$$

also calculating the values of norms

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{5} \tag{1.1.5.7}$$

$$\|\mathbf{C} - \mathbf{A}\| = \sqrt{26} \tag{1.1.5.8}$$

$$\|\mathbf{I} - \mathbf{A}\| = 2.1868938 \tag{1.1.5.9}$$

$$\tag{1.1.5.10}$$

(a) for $\angle BAI$:

On substtuting the values in (1.1.5.2) ,We get

$$\cos \angle BAI \triangleq \frac{\begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} -2.06852423 \\ 0.70972671 \end{pmatrix}}{\sqrt{5} \times 2.1868938} \tag{1.1.5.11}$$

$$\tag{1.1.5.12}$$

On solving

$$\angle BAI = 7.62755^\circ \tag{1.1.5.13}$$

(b) for $\angle CAI$:

On substtuting the values in (1.1.5.2) ,We get

$$\cos \angle CAI \triangleq \frac{\begin{pmatrix} -5 & 1 \end{pmatrix} \begin{pmatrix} -2.06852423 \\ 0.70972671 \end{pmatrix}}{\sqrt{26} \times 2.1868938} \tag{1.1.5.14}$$

$$\tag{1.1.5.15}$$

On solving

$$\angle CAI = 7.62755^\circ \tag{1.1.5.16}$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

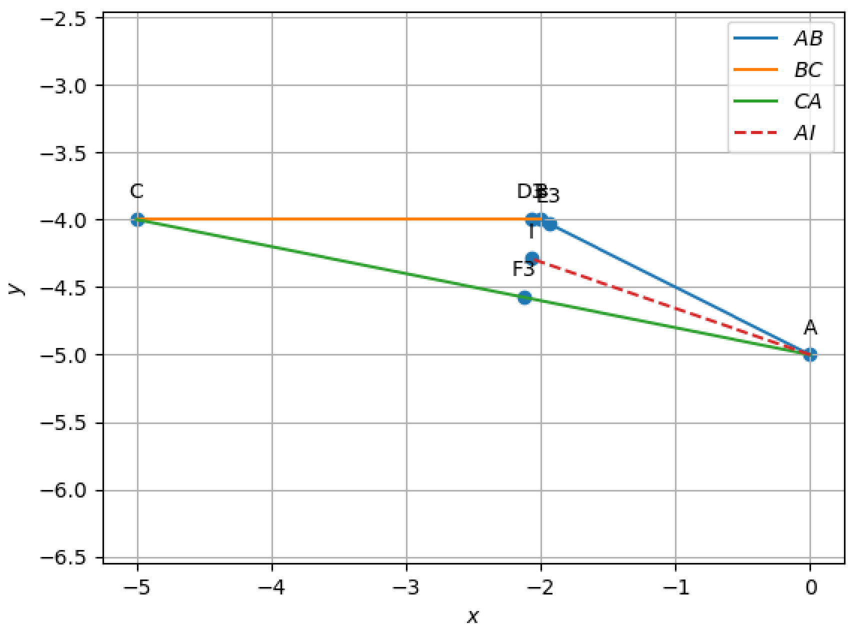

Figure 1.4: Angular bisector AI

1.1.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

Solution:

(a) To prove BI is an angular bisector of $\angle B$

$$\cos\angle ABI \triangleq \frac{(\mathbf{A}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.1.6.1}$$

$$\cos\angle CBI \triangleq \frac{(\mathbf{C}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{C}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.1.6.2}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}2\\-1\end{pmatrix} \tag{1.1.6.3}$$

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}-3\\0\end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{I}-\mathbf{B}=\begin{pmatrix}-0.06852423\\-0.29027329\end{pmatrix} \tag{1.1.6.5}$$

also calculating the values of norms

$$\|\mathbf{A}-\mathbf{B}\|=\sqrt{36} \qquad =\sqrt{5} \tag{1.1.6.6}$$

$$\|\mathbf{C}-\mathbf{B}\|=3 \tag{1.1.6.7}$$

$$\|\mathbf{I}-\mathbf{B}\|=0.2982518281 \tag{1.1.6.8}$$

$$\tag{1.1.6.9}$$

i. for $\angle ABI$:

On substtuting the values in (1.1.6.1) ,We get

$$\cos\angle ABI \triangleq \frac{\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}-0.06852423\\-0.29027329\end{pmatrix}}{\sqrt{5}\times 0.2982518281} \tag{1.1.6.10}$$

$$\tag{1.1.6.11}$$

On solving

$$\angle ABI = 76.71746973^\circ \tag{1.1.6.12}$$

ii. for $\angle CBI$:

On substtuting the values in (1.1.6.1) ,We get

$$\cos \angle CBI \triangleq \frac{\begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} -0.06852423 \\ -0.29027329 \end{pmatrix}}{3 \times 0.2982518281} \tag{1.1.6.13}$$

$$\tag{1.1.6.14}$$

On solving

$$\angle CBI = 76.71746973^\circ \tag{1.1.6.15}$$

Therefore $\angle ABI = \angle CBI$. and BI is the bisector of $\angle B$.

(b) To prove CI is an angular bisector of $\angle C$

$$\cos \angle BCI \triangleq \frac{(\mathbf{B} - \mathbf{C})\top(\mathbf{I} - \mathbf{C})}{\|\mathbf{B} - \mathbf{C}\| \, \|\mathbf{I} - \mathbf{C}\|} \tag{1.1.6.16}$$

$$\cos \angle ACI \triangleq \frac{(\mathbf{A} - \mathbf{C})\top(\mathbf{I} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{I} - \mathbf{C}\|} \tag{1.1.6.17}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}3\\0\end{pmatrix} \tag{1.1.6.18}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}5\\-1\end{pmatrix} \tag{1.1.6.19}$$

$$\mathbf{I}-\mathbf{C}=\begin{pmatrix}2.93147577\\0.29027329\end{pmatrix} \tag{1.1.6.20}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{C}\|=3 \tag{1.1.6.21}$$

$$\|\mathbf{A}-\mathbf{C}\|=\sqrt{26} \tag{1.1.6.22}$$

$$\|\mathbf{I}-\mathbf{C}\|=2.945812074 \tag{1.1.6.23}$$

$$\tag{1.1.6.24}$$

i. for $\angle BCI$:

On substtuting the values in (1.1.6.16) ,We get

$$\cos\angle BCI \triangleq \frac{\begin{pmatrix}3 & 0\end{pmatrix}\begin{pmatrix}2.93147577\\0.29027329\end{pmatrix}}{3\times 2.945812074} \tag{1.1.6.25}$$

$$\tag{1.1.6.26}$$

On solving

$$\angle BCI = 5.655156179^\circ \tag{1.1.6.27}$$

ii. for $\angle ACI$:

On substtuting the values in (1.1.6.16) ,We get

$$\cos \angle ACI \triangleq \frac{\begin{pmatrix} 5 & -1 \end{pmatrix} \begin{pmatrix} 2.93147577 \\ 0.29027329 \end{pmatrix}}{\sqrt{26} \times 2.945812074} \tag{1.1.6.28}$$

(1.1.6.29)

On solving

$$\angle ACI = 5.655156179^\circ \tag{1.1.6.30}$$

Therefore $\angle BCI = \angle ACI$. and CI is the bisector of $\angle C$.

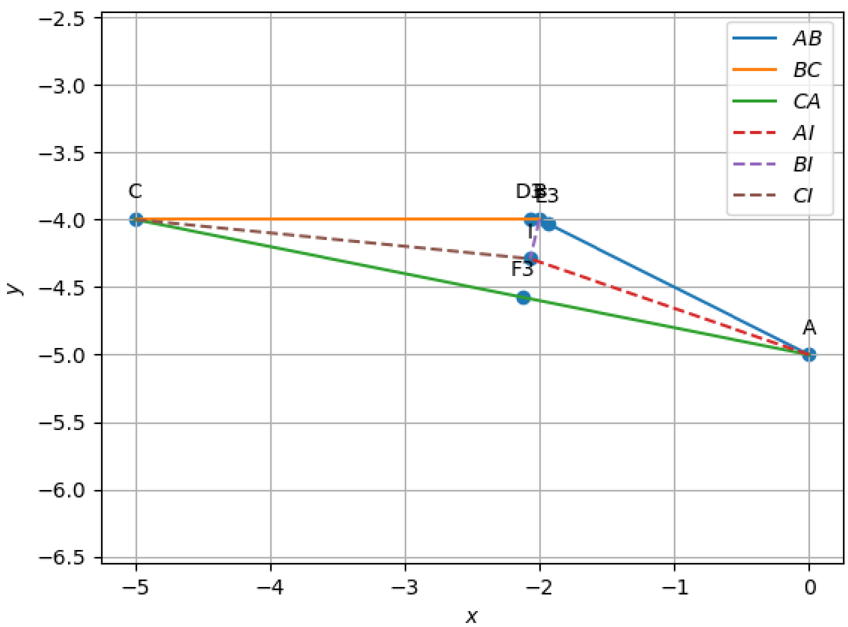

Figure 1.5: Angular bisectors BI, CI