GEOMETRY

Through Algebra

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Contents

Chapter 1

Triangle

Consider a triangle with vertices and midpoints

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1}$$

$$\mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \mathbf{E} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix}, \mathbf{F} = \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} \tag{1.2}$$

1.1. Perpendicular Bisector

1.1.1. The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B} + \mathbf{C}}{2}\right)(\mathbf{B} - \mathbf{C}) = 0 \tag{1.1.1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution:

On substituting the values,

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-7\\-8\end{pmatrix}, \mathbf{B}-\mathbf{C} = \begin{pmatrix}3\\0\end{pmatrix} \tag{1.1.1.2}$$

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-2\\-9\end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix}2\\-1\end{pmatrix} \tag{1.1.1.3}$$

$$\frac{\mathbf{C}+\mathbf{A}}{\mathbf{2}} = \frac{1}{2}\begin{pmatrix}-5\\-9\end{pmatrix}, \mathbf{C}-\mathbf{A} = \begin{pmatrix}-5\\1\end{pmatrix} \tag{1.1.1.4}$$

$$\tag{1.1.1.5}$$

yielding

$$(\mathbf{B}-\mathbf{C})^\top \left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix}3 & 0\end{pmatrix}\begin{pmatrix}\frac{7}{2}\\-4\end{pmatrix} = \frac{-21}{2} \tag{1.1.1.6}$$

$$(\mathbf{A}-\mathbf{B})^\top \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix}2 & -1\end{pmatrix}\begin{pmatrix}-1\\\frac{-9}{2}\end{pmatrix} = \frac{5}{2} \tag{1.1.1.7}$$

$$(\mathbf{C}-\mathbf{A})^\top \left(\frac{\mathbf{C}+\mathbf{A}}{2}\right) = \begin{pmatrix}-5 & 1\end{pmatrix}\begin{pmatrix}\frac{-5}{2}\\\frac{-9}{2}\end{pmatrix} = 8 \tag{1.1.1.8}$$

Thus, the perpendicular bisectors are obtained from as

$$BC: \quad \begin{pmatrix}3 & 0\end{pmatrix}\mathbf{x} = \frac{-21}{2} \tag{1.1.1.9}$$

$$CA: \quad \begin{pmatrix}2 & -1\end{pmatrix}\mathbf{x} = \frac{5}{2} \tag{1.1.1.10}$$

$$AB: \quad \begin{pmatrix}-5 & 1\end{pmatrix}\mathbf{x} = 8 \tag{1.1.1.11}$$

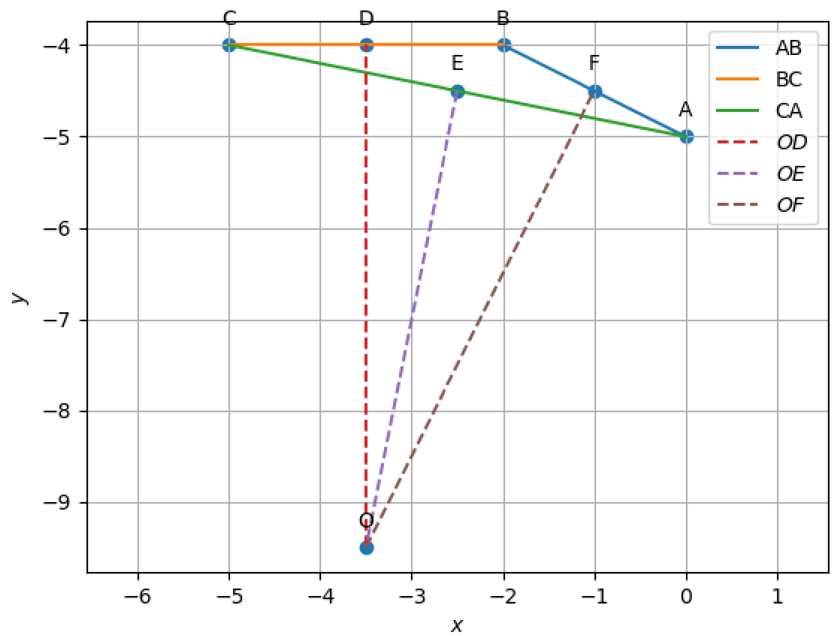

Figure 1.1: Perpendicular Bisectors of $\triangle ABC$

1.1.2. Find the intersection $\mathbf{O}$ of the perpendicular bisectors of AB and AC.

Solution:

Given vector equation of perpendicular bisector of $\mathbf{A} - \mathbf{B}$ is

$$(\mathbf{A} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2}\right) = 0 \tag{1.1.2.1}$$

where,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} -2 \\ -9 \end{pmatrix} \tag{1.1.2.2}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \tag{1.1.2.3}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} 2 & -1 \end{pmatrix} \tag{1.1.2.4}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{F}$ is

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \left(\mathbf{x} - \frac{1}{2}\begin{pmatrix} -2 \\ -9 \end{pmatrix}\right) = 0 \tag{1.1.2.5}$$

$$\implies \begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = \frac{1}{2}\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} -2 \\ -9 \end{pmatrix} \tag{1.1.2.6}$$

$$\begin{pmatrix} 2 & -1 \end{pmatrix} \mathbf{x} = \frac{5}{2} \tag{1.1.2.7}$$

Vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{C}$ is

$$(\mathbf{A}-\mathbf{C})^\top \left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{C}}{2}\right) = 0 \tag{1.1.2.8}$$

where,

$$\mathbf{A}+\mathbf{C}=\begin{pmatrix}-5\\-9\end{pmatrix} \tag{1.1.2.9}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}5\\-1\end{pmatrix} \tag{1.1.2.10}$$

$$\implies (\mathbf{A}-\mathbf{C})^\top=\begin{pmatrix}5 & -1\end{pmatrix} \tag{1.1.2.11}$$

$\therefore$ The vector equatio of $\mathbf{O}-\mathbf{E}$ is

$$\begin{pmatrix}5 & -1\end{pmatrix}\left(\mathbf{x}-\frac{1}{2}\begin{pmatrix}-5\\-9\end{pmatrix}\right)=0 \tag{1.1.2.12}$$

$$\implies \begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x}=\frac{1}{2}\begin{pmatrix}5 & -1\end{pmatrix}\begin{pmatrix}-5\\-9\end{pmatrix} \tag{1.1.2.13}$$

$$\begin{pmatrix}5 & -1\end{pmatrix}\mathbf{x}=-8 \tag{1.1.2.14}$$

Thus,

$$\begin{pmatrix} 2 & -1 & \frac{5}{2} \\ 5 & -1 & -8 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R1}{2}} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-5}{4} \\ 5 & 1 & -8 \end{pmatrix} \tag{1.1.2.15}$$

$$\xleftrightarrow{R_2 \leftarrow R2 - 5R1} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-5}{4} \\ 0 & \frac{3}{2} & \frac{-57}{4} \end{pmatrix} \tag{1.1.2.16}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{2R_2}{3}} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-5}{4} \\ 0 & 1 & \frac{-19}{2} \end{pmatrix} \tag{1.1.2.17}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{R_2}{2}} \begin{pmatrix} 1 & 0 & \frac{-7}{2} \\ 0 & 1 & \frac{-19}{2} \end{pmatrix} \tag{1.1.2.18}$$

Therefore, the point of intersection of perpendicular bisectors of $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}-\mathbf{C}$ is $\mathbf{O} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix}$

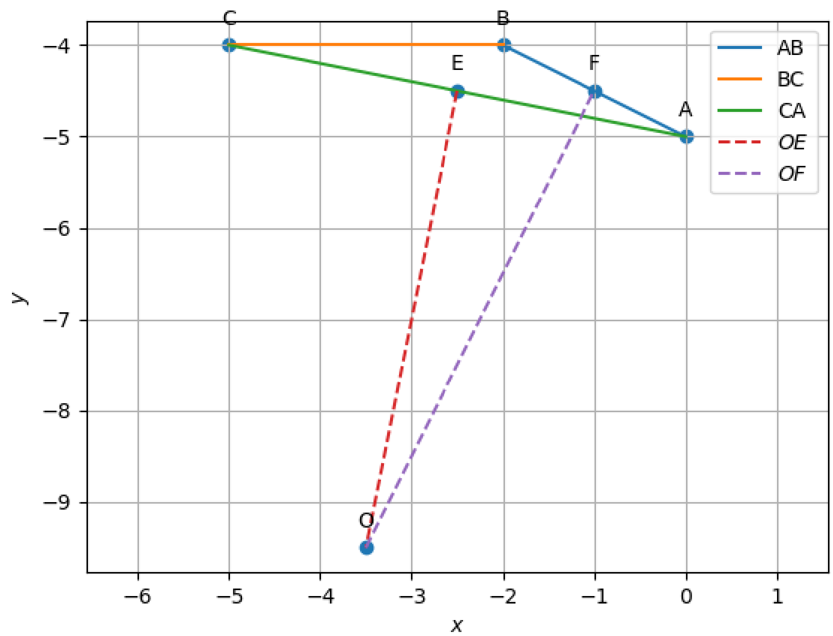

Figure 1.2: Perpendicular Bisectors $\mathbf{OE}, \mathbf{OF}$of $\mathbf{AC}, \mathbf{AB}$

1.1.3. Verify that $\mathbf{O}$ satisfies (1.1.1.1). $\mathbf{O}$ is known as the circumcentre.

Solution:

From the previous question we get,

$$\mathbf{O} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix} \tag{1.1.3.1}$$

when substituted in the above equation,

$$= \left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right).(\mathbf{B}-\mathbf{C}) \tag{1.1.3.2}$$

$$= \left(\frac{1}{2}\begin{pmatrix}-7\\-19\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-7\\-8\end{pmatrix}\right)^{\top}\begin{pmatrix}3\\0\end{pmatrix} \tag{1.1.3.3}$$

$$= \frac{1}{2}\begin{pmatrix}0 & -11\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix} \tag{1.1.3.4}$$

$$= 0 \tag{1.1.3.5}$$

It is hence proved that **O** satisfies the equation (1.1.1.1)

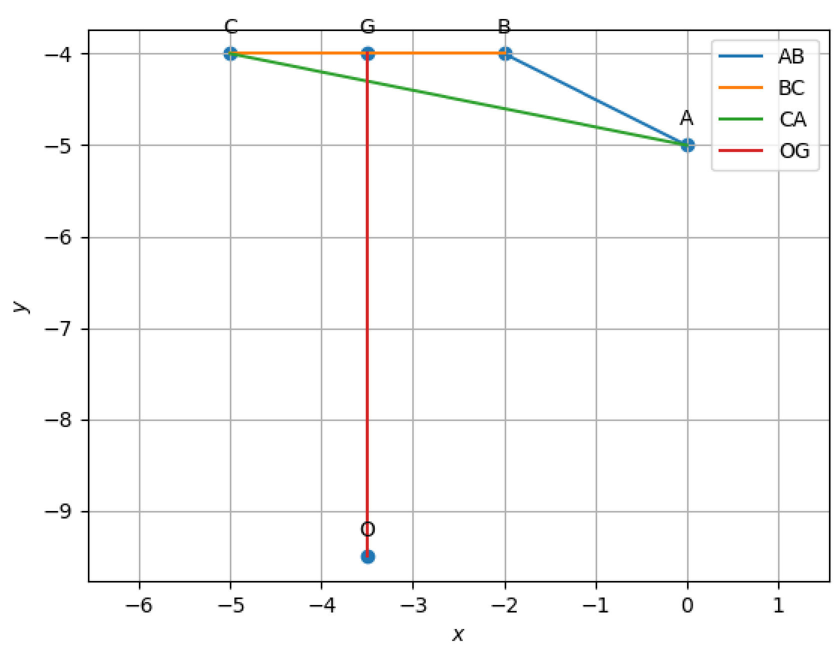

Figure 1.3: Circumcenter(**OG**) of $\triangle ABC$

1.1.4. Verify that

$$OA = OB = OC \tag{1.1.4.1}$$

Solution:

Given,

$$\mathbf{O} - \mathbf{A} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.4.2}$$

$$= \begin{pmatrix} \frac{-7}{2} \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.4.3}$$

$$\mathbf{O} - \mathbf{B} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.4.4}$$

$$= \begin{pmatrix} \frac{-3}{2} \\ \frac{-11}{2} \end{pmatrix} \tag{1.1.4.5}$$

$$\mathbf{O} - \mathbf{A} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.4.6}$$

$$= \begin{pmatrix} \frac{3}{2} \\ \frac{-11}{2} \end{pmatrix} \tag{1.1.4.7}$$

By substituting the above values

(a)

$$\mathbf{OA} = \sqrt{(\mathbf{O} - \mathbf{A})^\top (\mathbf{O} - \mathbf{A})} \tag{1.1.4.8}$$

$$= \sqrt{\begin{pmatrix} \frac{-7}{2} & \frac{-9}{2} \end{pmatrix} \begin{pmatrix} \frac{-7}{2} \\ \frac{-9}{2} \end{pmatrix}} \tag{1.1.4.9}$$

$$= \sqrt{\frac{-7}{2}^2 + \frac{-9}{2}^2} \tag{1.1.4.10}$$

$$= \frac{\sqrt{130}}{2} \tag{1.1.4.11}$$

(b)

$$\mathbf{OB} = \sqrt{(\mathbf{O} - \mathbf{B})^\top (\mathbf{O} - \mathbf{B})} \tag{1.1.4.12}$$

$$= \sqrt{\begin{pmatrix} \frac{-3}{2} & \frac{-11}{2} \end{pmatrix} \begin{pmatrix} \frac{-3}{2} \\ \frac{-11}{2} \end{pmatrix}} \tag{1.1.4.13}$$

$$= \sqrt{\frac{-3}{2}^2 + \frac{-11}{2}^2} \tag{1.1.4.14}$$

$$= \frac{\sqrt{130}}{2} \tag{1.1.4.15}$$

(c)

$$OC = \sqrt{(\mathbf{O}-\mathbf{C})^\top(\mathbf{O}-\mathbf{C})} \tag{1.1.4.16}$$
$$= \sqrt{\begin{pmatrix} \frac{3}{2} & \frac{-11}{2} \end{pmatrix}\begin{pmatrix} \frac{3}{2} \\ \frac{-11}{2} \end{pmatrix}} \tag{1.1.4.17}$$
$$= \sqrt{\frac{3}{2}^2 + \frac{-11}{2}^2} \tag{1.1.4.18}$$
$$= \frac{\sqrt{130}}{2} \tag{1.1.4.19}$$

From above,

$$\mathbf{OA} = \mathbf{OB} = \mathbf{OC} \tag{1.1.4.20}$$

1.1.5. Draw the circle with centre at **O** and radius

$$\mathbf{R} = \mathbf{OA} \tag{1.1.5.1}$$

This is known as the circumradius.

Solution:

$$\mathbf{O} = \frac{1}{2}\begin{pmatrix} -7 \\ -19 \end{pmatrix} \tag{1.1.5.2}$$

Now we will calculate the radius,

$$R = OA \tag{1.1.5.3}$$

$$= \|\mathbf{A} - \mathbf{O}\| \tag{1.1.5.4}$$

$$= \left\| \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -7 \\ -19 \end{pmatrix} \right\| \tag{1.1.5.5}$$

$$= \left\| \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \right\| \tag{1.1.5.6}$$

$$= \frac{\sqrt{130}}{2} \tag{1.1.5.7}$$

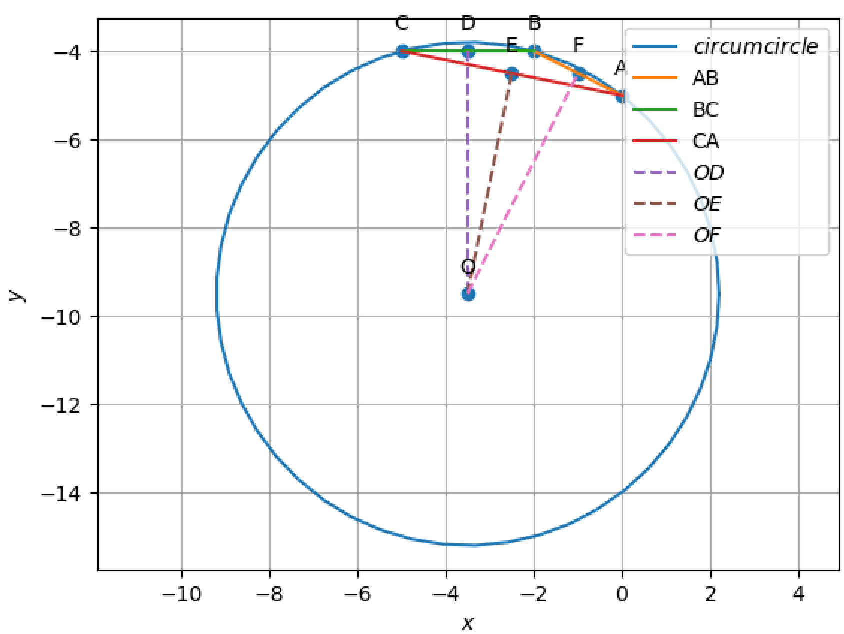

Figure 1.4: Circumcircle of $\triangle ABC$ with center $\mathbf{O}$

1.1.6. Verify that

$$\angle BOC = 2\angle BAC. \tag{1.1.6.1}$$

Solution:

We have a point $\mathbf{O} = \begin{pmatrix} \frac{-7}{2} \\ \frac{-19}{2} \end{pmatrix}$ which is intersection point of the perpendicular bisectors of AB and AC and is circumcentre of the triangle made by points A,B and C.

(a) To find the value of $\angle BOC$:

$$\mathbf{B} - \mathbf{O} = \begin{pmatrix} \frac{3}{2} \\ \frac{11}{2} \end{pmatrix} \tag{1.1.6.2}$$

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{-3}{2} \\ \frac{11}{2} \end{pmatrix} \tag{1.1.6.3}$$

calculating the norm of $\mathbf{B} - \mathbf{O}$ and $\mathbf{C} - \mathbf{O}$,we get:

$$\|\mathbf{B} - \mathbf{O}\| = \frac{\sqrt{130}}{2} \tag{1.1.6.4}$$

$$\|\mathbf{C} - \mathbf{O}\| = \frac{\sqrt{130}}{2} \tag{1.1.6.5}$$

by doing matrix multiplication, we get:

$$(\mathbf{B} - \mathbf{O})^\top (\mathbf{C} - \mathbf{O}) = 28 \tag{1.1.6.6}$$

to calcuate the $\angle BOC$:

$$\cos BOC = \frac{\left(\mathbf{B}-\mathbf{O}\right)^{\top}\left(\mathbf{C}-\mathbf{O}\right)}{\left\|\mathbf{B}-\mathbf{O}\right\|\left\|\mathbf{C}-\mathbf{O}\right\|} \tag{1.1.6.7}$$

$$= \frac{28}{\frac{\sqrt{130}}{2}\times\frac{\sqrt{130}}{2}} \tag{1.1.6.8}$$

$$= \frac{112}{130} \tag{1.1.6.9}$$

$$\implies \angle BOC = \cos^{-1}\left(\frac{112}{130}\right) \tag{1.1.6.10}$$

$$= 30.5^{\circ} \tag{1.1.6.11}$$

Taking the reflex of above angle we get

$$\angle BOC = 360^{\circ} - 30.5^{\circ} \qquad = 329.5^{\circ} \tag{1.1.6.12}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B}-\mathbf{A} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.6.13}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} 5 \\ -1 \end{pmatrix} \tag{1.1.6.14}$$

calculating the norm of $\mathbf{B}-\mathbf{A}$ and $\mathbf{C}-\mathbf{A}$,we get:

$$\left\|\mathbf{B}-\mathbf{A}\right\| = \sqrt{5} \tag{1.1.6.15}$$

$$\left\|\mathbf{C}-\mathbf{A}\right\| = \sqrt{26} \tag{1.1.6.16}$$

by doing matrix multiplication, we get:

$$(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{A}) = -11 \tag{1.1.6.17}$$

to calcuate the $\angle BAC$:

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^{\top}(\mathbf{C}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{A}\|} \tag{1.1.6.18}$$

$$= \frac{-11}{\sqrt{5}\times\sqrt{26}} \tag{1.1.6.19}$$

$$= \frac{-11}{\sqrt{130}} \tag{1.1.6.20}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{-11}{\sqrt{130}}\right) \tag{1.1.6.21}$$

$$= 164.75^{\circ} \tag{1.1.6.22}$$

from equation (1.1.6.22):

$$2\times\angle BAC = 329.5^{\circ} \tag{1.1.6.23}$$

On comparing equation (1.1.6.12) and equation (1.1.6.23):

$$\angle BOC = 2\times\angle BAC \tag{1.1.6.24}$$

Hence,verified.

1.1.7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.1.7.1}$$

Find θ if

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{1.1.7.2}$$

Solution:

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{-3}{2} \\ \frac{11}{2} \end{pmatrix} \tag{1.1.7.3}$$

$$\mathbf{A} - \mathbf{O} = \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.1.7.4}$$

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.1.7.5}$$

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{1.1.7.6}$$

Now from (1.1.6.30)

$$\begin{pmatrix} \frac{-3}{2} \\ \frac{11}{2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{7}{2} \\ \frac{9}{2} \end{pmatrix} \tag{1.1.7.7}$$

solving using matrix multiplication,we get

$$\begin{pmatrix} \frac{-3}{2} \\ \frac{11}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2}\cos\theta - \frac{9}{2}\sin\theta \\ \frac{7}{2}\sin\theta + \frac{9}{2}\cos\theta \end{pmatrix} \tag{1.1.7.8}$$

Comparing on Both sides ,we get

$$7\cos\theta - 9\sin\theta = -3 \tag{1.1.7.9}$$

$$7\sin\theta + 9\cos\theta = 11 \tag{1.1.7.10}$$

On solving equations (1.1.6.33) and (1.1.6.34)

$$\cos\theta = \frac{3}{5} \tag{1.1.7.11}$$

$$\sin\theta = \frac{4}{5} \tag{1.1.7.12}$$

$$\theta = \cos^{-1}\frac{3}{5} \tag{1.1.7.13}$$

$$= 53.13 \tag{1.1.7.14}$$

$$\therefore \theta = 53.13 \tag{1.1.7.15}$$