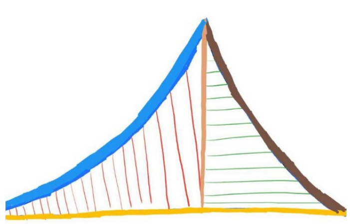
GEOMETRY

Through Algebra

Chakali Suresh



Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1}$$

1.1. Altitude

1.1.1. $\mathbf{D}_1$ is a point on BC such that

$$AD_1 \perp BC \tag{1.1.1.1}$$

and AD_1 is defined to be the altitude. Find the normal vector of AD_1.

Solution:

Given

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \tag{1.1.1.2}$$

$$\mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \tag{1.1.1.3}$$

$$\mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.1.4}$$

Normal vector of $AD1$ is orthogonal to $AD1$ and hence parallel to BC.Direction vector $\mathbf{m_{BC}}$

$$= \mathbf{C} - \mathbf{B} \tag{1.1.1.5}$$

$$= \begin{pmatrix} -5 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.1.6}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.1.7}$$

$$\text{Normal vector of } \mathbf{AD1} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.1.8}$$

1.1.2. Find the equation of AD_1.

Solution:

The normal vector of

$$\implies \mathbf{n} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.2.1}$$

The equation of AD_1 is

$$\mathbf{n}^\top(\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.2.2}$$

$$\mathbf{n}^\top(\mathbf{x}) = \mathbf{n}^\top(\mathbf{A}) \tag{1.1.2.3}$$

$$\implies \begin{pmatrix} -3 & 0 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.2.4}$$

$$\begin{pmatrix} 0 & 3 \end{pmatrix} \mathbf{x} = 0 \tag{1.1.2.5}$$

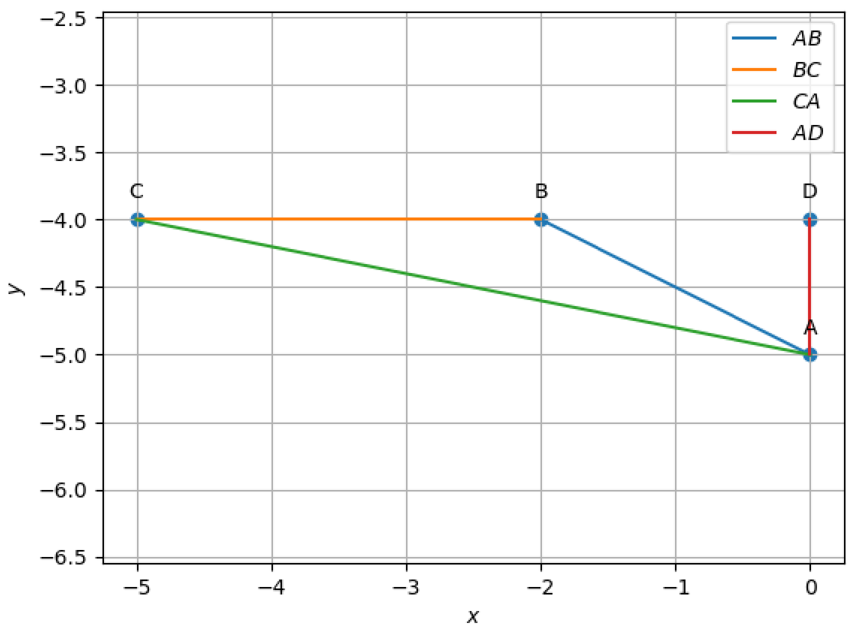


Figure 1.1: Altitude AD_1

1.1.3. Find the equations of the altitudes BE_1 and CF_1 to the sides AC and AB respectively.

Solution:

The normal equation of CF_1 is

$$\mathbf{n} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{C}) = 0 \tag{1.1.3.2}$$

$$\mathbf{n}^\top (\mathbf{x}) = \mathbf{n}^\top (\mathbf{C}) \tag{1.1.3.3}$$

$$\implies \begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -2 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.3.4}$$

$$\implies \begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.1.3.5}$$

The normal equation of BE_1 is

$$\mathbf{n} = \begin{pmatrix} -5 \\ 1 \end{pmatrix} \tag{1.1.3.6}$$

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{B}) = 0 \tag{1.1.3.7}$$

$$\mathbf{n}^\top (\mathbf{x}) = \mathbf{n}^\top (\mathbf{B}) \tag{1.1.3.8}$$

$$\implies \begin{pmatrix} -5 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -5 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.3.9}$$

$$\implies \begin{pmatrix} -5 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.1.3.10}$$

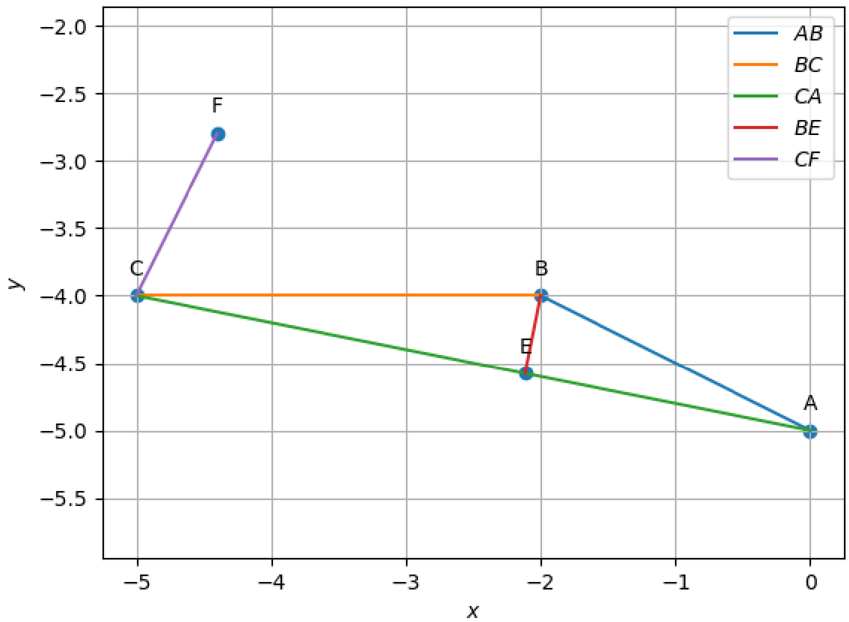


Figure 1.2: Altitudes BE_1 and CF_1

1.1.4. Find the intersection $\mathbf{H}$ of BE_1 and CF_1.

Solution:

Equation of BE_1

$$\begin{pmatrix} -5 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.1.4.1}$$

Equation of CF_1

$$\begin{pmatrix} -2 & 1 \end{pmatrix} \mathbf{x} = 6 \tag{1.1.4.2}$$

Therefore, we need to solve the following equation to get **H**:

$$\begin{pmatrix} -5 & 1 \\ -2 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \tag{1.1.4.3}$$

Solving the above equation by Gauss-Jordan method

$$\begin{pmatrix} -5 & 1 & 6 \\ -2 & 1 & 6 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{-5}} \begin{pmatrix} 1 & \frac{-1}{5} & \frac{-6}{5} \\ -2 & 1 & 6 \end{pmatrix} \tag{1.1.4.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & \frac{-1}{5} & \frac{-6}{5} \\ 0 & \frac{3}{5} & \frac{18}{5} \end{pmatrix} \tag{1.1.4.5}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{5R_2}{3}} \begin{pmatrix} 1 & \frac{-1}{5} & \frac{-6}{5} \\ 0 & 1 & 6 \end{pmatrix} \tag{1.1.4.6}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{R_2}{5}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \end{pmatrix} \tag{1.1.4.7}$$

Therefore point of intersection **H** is

$$= \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.1.4.8}$$

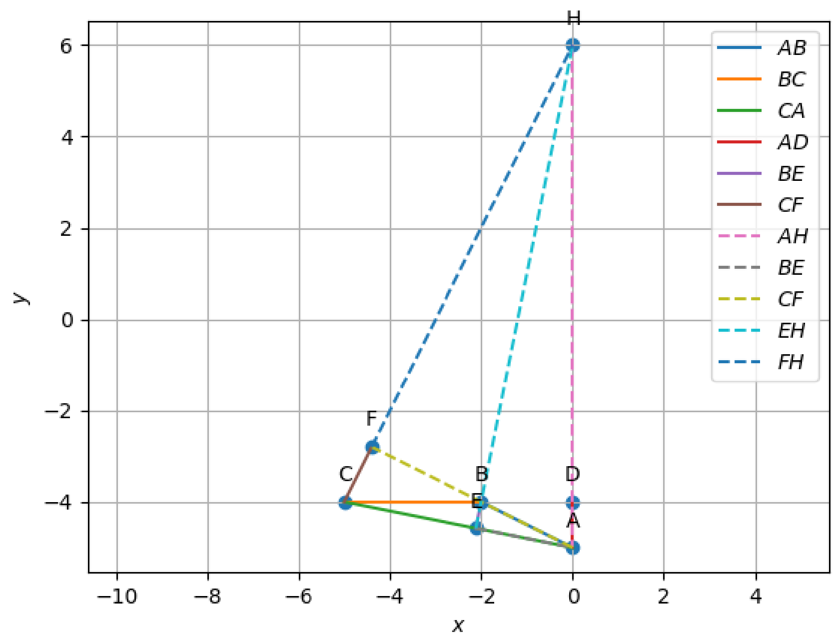


Figure 1.3: Intersection point $\mathbf{H}$ of altitudes BE_1 and CF_1

1.1.5. Verify that

$$\left(\mathbf{A}-\mathbf{H}\right)^{\top}\left(\mathbf{B}-\mathbf{C}\right)=0 \tag{1.1.5.1}$$

Solution:

Given

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.5.2}$$

$$\mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \mathbf{H} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.1.5.3}$$

To solve the equation

$$\mathbf{A} - \mathbf{H} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{1.1.5.4}$$

$$= \begin{pmatrix} 0 \\ -11 \end{pmatrix} \tag{1.1.5.5}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.5.6}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.1.5.7}$$

$$\implies (\mathbf{A} - \mathbf{H})^\top (\mathbf{B} - \mathbf{C}) = \begin{pmatrix} 0 & -11 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{1.1.5.8}$$

$$= 0 \tag{1.1.5.9}$$

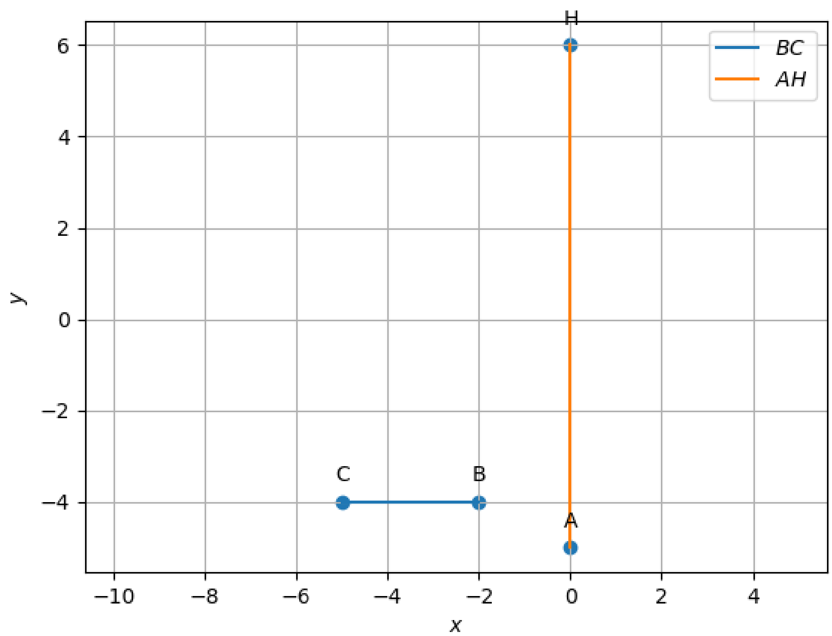


Figure 1.4: Plot of points A, B, C and H