GEOMETRY

Through Algebra

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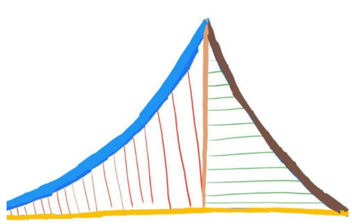

Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1}$$

1.1. Median

1.1.1. if $\mathbf{D}$ divides BC in the ratio of $k : 1$,

$$\mathbf{D} = \frac{k\mathbf{C} + \mathbf{B}}{K + 1} \tag{1.1.1.1}$$

Find the mid points $\mathbf{D}, \mathbf{E}, \mathbf{F}$ of the sides AB, BC and CA respectively.

Solution:

Since $\mathbf{D}$ is the midpoint of BC

$$\text{Let} k = 1 \tag{1.1.1.2}$$

$$\therefore \text{we get} \tag{1.1.1.3}$$

$$\implies \mathbf{D} = \frac{\mathbf{C} + \mathbf{B}}{2} \tag{1.1.1.4}$$

$$= \frac{1}{2} \times \begin{pmatrix} -5 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.1.5}$$

$$= \frac{1}{2} \begin{pmatrix} -7 \\ -8 \end{pmatrix} \tag{1.1.1.6}$$

Similarly,

$$\implies \mathbf{E} = \frac{\mathbf{A} + \mathbf{C}}{2} \tag{1.1.1.7}$$

$$= \frac{1}{2} \begin{pmatrix} -5 \\ -9 \end{pmatrix} \tag{1.1.1.8}$$

$$\implies \mathbf{F} = \frac{\mathbf{A} + \mathbf{B}}{2} \tag{1.1.1.9}$$

$$= \frac{1}{2} \begin{pmatrix} -2 \\ -9 \end{pmatrix} \tag{1.1.1.10}$$

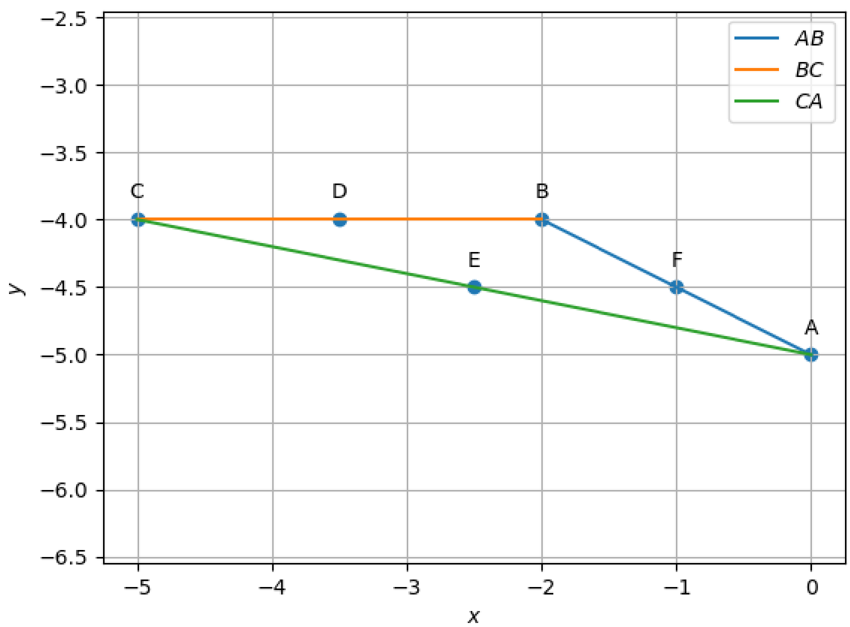

Figure 1.1: Triangle **ABC** with midpoints **D**, **E**, **F**

1.1.2. Find the equations of AD, BE and CF. **Solution:** :

Given,

$$\mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \tag{1.1.2.1}$$

$$\mathbf{E} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.2.2}$$

$$\mathbf{F} = \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.2.3}$$

(a) The normal equation for the median AD is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.2.4}$$

$$\therefore \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.2.5}$$

$$\mathbf{m} = \mathbf{D} - \mathbf{A} \tag{1.1.2.6}$$

$$= \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \end{pmatrix} = \begin{pmatrix} \frac{-7}{2} \\ 1 \end{pmatrix} \tag{1.1.2.7}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.8}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-7}{2} \\ 1 \end{pmatrix} \tag{1.1.2.9}$$

$$= \begin{pmatrix} 1 \\ \frac{7}{2} \end{pmatrix} \tag{1.1.2.10}$$

$$\therefore \mathbf{n}^\top = \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \tag{1.1.2.11}$$

Hence the normal equation of median AD is

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.2.12}$$

$$\implies \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.2.13}$$

$$\begin{pmatrix} 1 & \frac{7}{2} \end{pmatrix} \mathbf{x} = -\frac{35}{2} \tag{1.1.2.14}$$

$$\begin{pmatrix} 2 & 7 \end{pmatrix} \mathbf{x} = -35 \tag{1.1.2.15}$$

(b) The normal equation for the median BE is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B} \right) = 0 \tag{1.1.2.16}$$

$$\therefore \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{1.1.2.17}$$

$$\mathbf{m} = \mathbf{E} - \mathbf{B} \tag{1.1.2.18}$$

$$= \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix} - \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.2.19}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.20}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.2.21}$$

$$= \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{1.1.2.22}$$

$$\therefore \mathbf{n}^\top = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.1.2.23}$$

Hence the normal equation of median BE is

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{1.1.2.24}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ -4 \end{pmatrix} \tag{1.1.2.25}$$

$$\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = -1 \tag{1.1.2.26}$$

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -2 \tag{1.1.2.27}$$

(c) The normal equation for the median CF is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{1.1.2.28}$$

$$\therefore \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{1.1.2.29}$$

$$\mathbf{m} = \mathbf{F} - \mathbf{C} = \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} - \begin{pmatrix} -5 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.2.30}$$

$$\implies \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.2.31}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.2.32}$$

$$= \begin{pmatrix} \frac{-1}{2} \\ -4 \end{pmatrix} \tag{1.1.2.33}$$

$$\therefore \mathbf{n}^\top = \begin{pmatrix} \frac{-1}{2} & -4 \end{pmatrix} \tag{1.1.2.34}$$

Hence the normal equation of median CF is

$$\mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{1.1.2.35}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & -4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} \frac{-1}{2} & -4 \end{pmatrix} \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.2.36}$$

$$\begin{pmatrix} \frac{-1}{2} & -4 \end{pmatrix} \mathbf{x} = \frac{37}{2} \tag{1.1.2.37}$$

$$\begin{pmatrix} -1 & -8 \end{pmatrix} \mathbf{x} = 37 \tag{1.1.2.38}$$

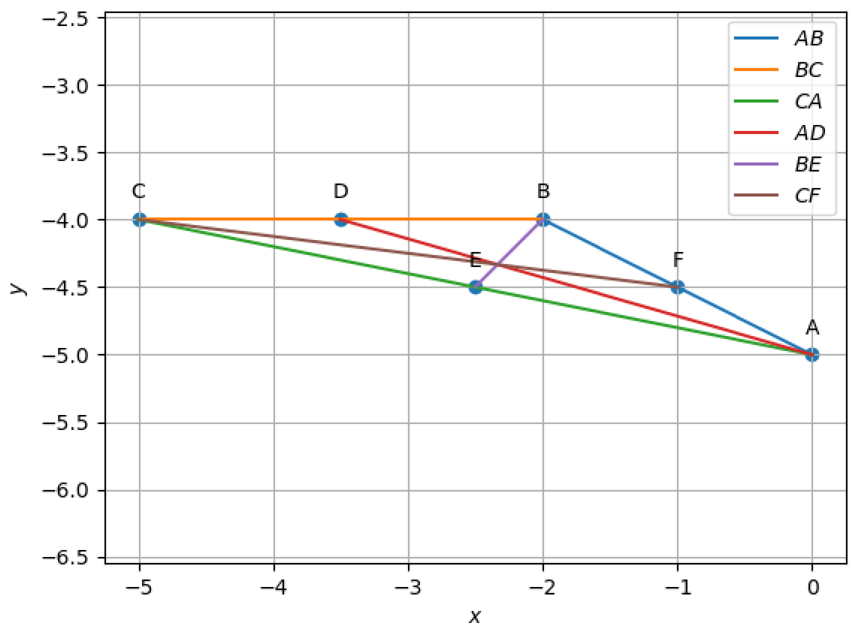

Figure 1.2: Triangle ABC with medians AD, BE, CF

1.1.3. Find the intersection $\mathbf{G}$ of BE and CF.

Solution:

The equations of BE is,

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = -2 \tag{1.1.3.1}$$

The equations of CF is,

$$\begin{pmatrix} -1 & -8 \end{pmatrix} \mathbf{x} = 37 \tag{1.1.3.2}$$

From (1.1.3.1) and (1.1.3.2) the augmented matrix is

$$\begin{pmatrix} -1 & 1 & -2 \\ -1 & -8 & 37 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 \times -1} \begin{pmatrix} 1 & -1 & 2 \\ -1 & -8 & 37 \end{pmatrix} \tag{1.1.3.3}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -9 & 39 \end{pmatrix} \tag{1.1.3.4}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 \times -1/9} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & \frac{-13}{3} \end{pmatrix} \tag{1.1.3.5}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & \frac{-7}{3} \\ 0 & 1 & \frac{-13}{3} \end{pmatrix} \tag{1.1.3.6}$$

using Gauss elimination. Therefore,

$$\mathbf{G} = \begin{pmatrix} \frac{-7}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.1.3.7}$$

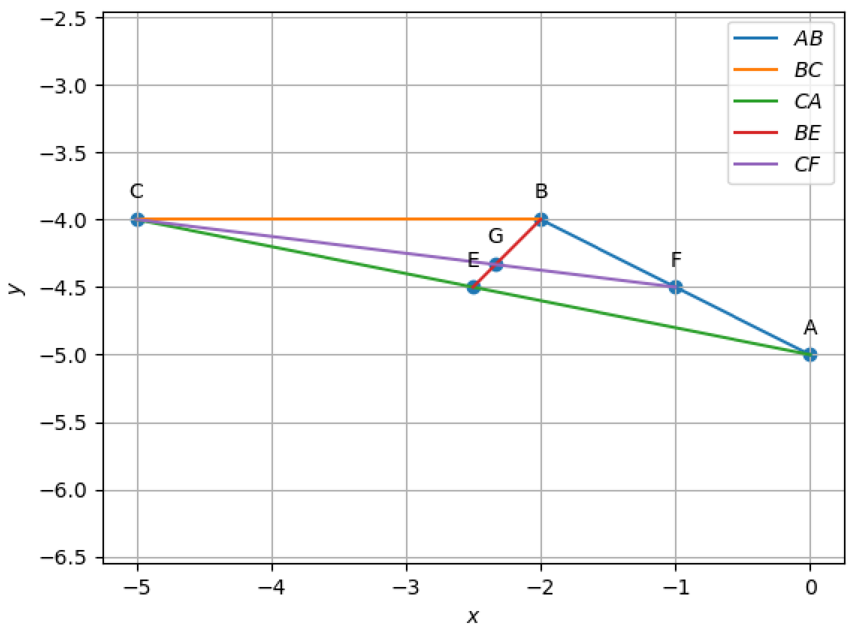

Figure 1.3: Centroid($\mathbf{G}$) of triangle ABC

1.1.4. Verify that

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.1}$$

Solution:

In order to verify the above equation we first need to find $\mathbf{G}$.$\mathbf{G}$ is the intersection of BE and CF,Using the value of $\mathbf{G}$ from (1.2.3).

$$\mathbf{G} = \begin{pmatrix} \frac{-7}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.1.4.2}$$

Also,We know that $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}$ and $\mathbf{F}$ are midpoints of BC, CA and AB respectively from (1.2.1).

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.4.3}$$

$$\mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix}, \mathbf{E} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix}, \mathbf{F} = \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.4.4}$$

(a) Calculating the ratio of BG and GE,

$$\mathbf{G} - \mathbf{B} = \begin{pmatrix} \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix} \tag{1.1.4.5}$$

$$\mathbf{E} - \mathbf{G} = \begin{pmatrix} \frac{-1}{6} \\ \frac{-1}{6} \end{pmatrix} \tag{1.1.4.6}$$

$$\|\mathbf{G} - \mathbf{B}\| = \sqrt{\left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^2} = \frac{\sqrt{2}}{3} \tag{1.1.4.7}$$

$$\|\mathbf{E} - \mathbf{G}\| = \sqrt{\left(\frac{-1}{6}\right)^2 + \left(\frac{-1}{6}\right)^2} = \frac{\sqrt{2}}{6} \tag{1.1.4.8}$$

$$\frac{BG}{GE} = \frac{\|\mathbf{G} - \mathbf{B}\|}{\|\mathbf{E} - \mathbf{G}\|} = \frac{\frac{\sqrt{2}}{3}}{\frac{\sqrt{2}}{6}} = 2 \tag{1.1.4.9}$$

(b) Calculating the ratio of CG and GF,

$$\mathbf{G}-\mathbf{C}=\begin{pmatrix}\frac{8}{3}\\\frac{-1}{3}\end{pmatrix} \tag{1.1.4.10}$$

$$\mathbf{F}-\mathbf{G}=\begin{pmatrix}\frac{4}{3}\\\frac{-1}{6}\end{pmatrix} \tag{1.1.4.11}$$

$$\|\mathbf{G}-\mathbf{C}\|=\sqrt{\left(\frac{8}{3}\right)^2+\left(\frac{-1}{3}\right)^2} \qquad =\frac{\sqrt{65}}{3} \tag{1.1.4.12}$$

$$\|\mathbf{F}-\mathbf{G}\|=\sqrt{\left(\frac{4}{3}\right)^2+\left(\frac{-1}{6}\right)^2} \qquad =\frac{\sqrt{65}}{6} \tag{1.1.4.13}$$

$$\frac{CG}{GF}=\frac{\|\mathbf{G}-\mathbf{C}\|}{\|\mathbf{F}-\mathbf{G}\|} \qquad =\frac{\frac{\sqrt{65}}{3}}{\frac{\sqrt{65}}{6}}=2 \tag{1.1.4.14}$$

(c) Calculating the ratio of AG and GD,

$$\mathbf{G}-\mathbf{A}=\begin{pmatrix}\frac{-7}{3}\\\frac{2}{3}\end{pmatrix} \tag{1.1.4.15}$$

$$\mathbf{D}-\mathbf{G}=\begin{pmatrix}\frac{-7}{6}\\\frac{1}{3}\end{pmatrix} \tag{1.1.4.16}$$

$$\|\mathbf{G}-\mathbf{A}\|=\sqrt{\left(\frac{-7}{3}\right)^2+\left(\frac{2}{3}\right)^2} \qquad =\frac{\sqrt{53}}{3} \tag{1.1.4.17}$$

$$\|\mathbf{D}-\mathbf{G}\|=\sqrt{\left(\frac{-7}{6}\right)^2+\left(\frac{1}{3}\right)^2} \qquad =\frac{\sqrt{53}}{6} \tag{1.1.4.18}$$

$$\frac{AG}{GD}=\frac{\|\mathbf{G}-\mathbf{A}\|}{\|\mathbf{D}-\mathbf{G}\|} \qquad =\frac{\frac{\sqrt{53}}{3}}{\frac{\sqrt{53}}{6}}=2 \tag{1.1.4.19}$$

From (1.1.4.9), (1.1.4.14), (1.1.4.19)

$$\frac{BG}{GE} = \frac{CG}{GF} = \frac{AG}{GD} = 2 \tag{1.1.4.20}$$

1.1.5. Show that $\mathbf{A}, \mathbf{G}$ and $\mathbf{D}$ are collinear.

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \tag{1.1.5.1}$$

We need to show that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear. From Problem 1.2.3 We know that, The point $\mathbf{G}$ is

$$\mathbf{G} = \begin{pmatrix} \frac{-7}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.1.5.2}$$

And from Problem 1.2.1 We know that, The point $\mathbf{D}$ is

$$\mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ 4 \end{pmatrix} \tag{1.1.5.3}$$

In Problem 1.1.3, There is a theorem/law mentioned i.e.,

Points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are defined to be collinear if

$$\text{rank} \begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{D} & \mathbf{G} \end{pmatrix} = 2 \tag{1.1.5.4}$$

Using the above law/Theorem Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{-7}{2} & \frac{-7}{3} \\ -5 & 4 & \frac{-13}{3} \end{pmatrix} \tag{1.1.5.5}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{-7}{2} & \frac{-7}{3} \\ -5 & 4 & \frac{-13}{3} \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 5R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \frac{-7}{2} & \frac{-7}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \tag{1.1.5.6}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{-7}{2} R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} \end{pmatrix} \tag{1.1.5.7}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.5.8}$$

Rank of above matrix is 2.

Hence, we proved that that points $\mathbf{A}, \mathbf{D}, \mathbf{G}$ are collinear.

1.1.6. Verify that

$$\mathbf{G} = \frac{\mathbf{A} + \mathbf{B} + \mathbf{C}}{3} \tag{1.1.6.1}$$

$\mathbf{G}$ is known as the centroid of $\triangle ABC$.

Solution:

$$\mathbf{G} = \frac{\mathbf{A}+\mathbf{B}+\mathbf{C}}{3} \tag{1.1.6.2}$$

let us first evaluate the R.H.S of the equation

$$\begin{aligned}\mathbf{G} &= \frac{\begin{pmatrix}0\\-5\end{pmatrix}+\begin{pmatrix}-2\\-4\end{pmatrix}+\begin{pmatrix}-5\\-4\end{pmatrix}}{3}\\ &= \begin{pmatrix}\frac{0-2-5}{3}\\ \frac{-5-4-4}{3}\end{pmatrix}\\ &= \begin{pmatrix}\frac{-7}{3}\\ \frac{-13}{3}\end{pmatrix}\end{aligned} \tag{1.1.6.3}$$

Hence verified.

1.1.7. Verify that

$$\mathbf{A}-\mathbf{F} = \mathbf{E}-\mathbf{D} \tag{1.1.7.1}$$

The quadrilateral $AFDE$ is defined to be a parallelogram.

Solution:

Given that,

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} -5 \\ -4 \end{pmatrix} \tag{1.1.7.2}$$

$$\mathbf{D} = \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.7.3}$$

Evaluating the L.H.S of the equation

$$\mathbf{A} - \mathbf{F} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -1 \\ \frac{-9}{2} \end{pmatrix} \tag{1.1.7.4}$$

$$= \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.7.5}$$

Evaluating the R.H.S of the equation

$$\mathbf{E} - \mathbf{D} = \begin{pmatrix} \frac{-5}{2} \\ \frac{-9}{2} \end{pmatrix} - \begin{pmatrix} \frac{-7}{2} \\ -4 \end{pmatrix} \tag{1.1.7.6}$$

$$= \begin{pmatrix} -1 \\ \frac{-1}{2} \end{pmatrix} \tag{1.1.7.7}$$

Hence verified that, R.H.S = L.H.S

$$\mathbf{A} - \mathbf{F} = \mathbf{E} - \mathbf{D} \tag{1.1.7.8}$$

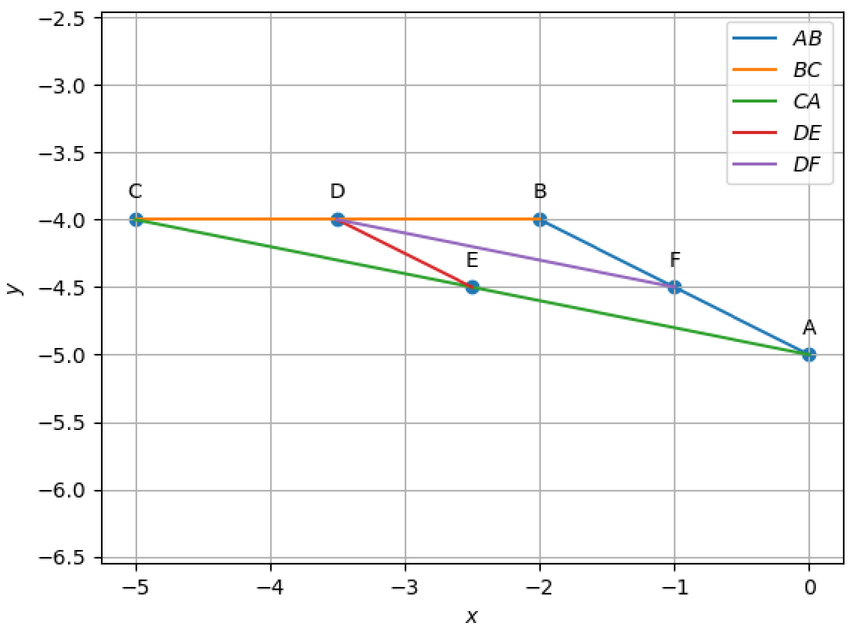

Figure 1.4: $AFDE$ form a parallelogram in triangle ABC