Contents

1 Triangle **1**

1.1 Matrix 1

1.1.1 Vectors 2

1.1.2 Median 6

1.1.3 Altitude 13

1.1.4 Perpendicular Bisector 15

1.1.5 Angle Bisector 17

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}, \tag{1.1}$$

1.1. Matrix

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \tag{1.3}$$

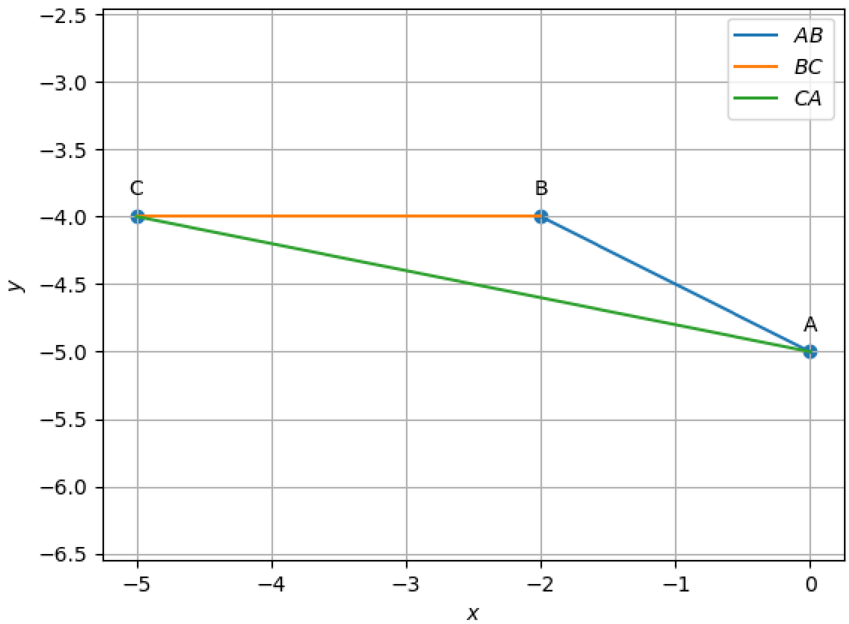

Figure 1.1: $\triangle ABC$

1.1.1. Vectors

1.1.1.1. Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.1.1.1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.1.1.1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.1.1.1.3}$$

$$= \begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.1.1.1.4}$$

Using Matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 2 & 3 & -5 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.1.1.1.5}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row.

1.1.1.2. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1.1.2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{1.1.1.2.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.1.1.2.3}$$

Using matrix multiplication

$$\mathbf{N} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -5 \end{pmatrix} \tag{1.1.1.2.4}$$

1.1.1.3. Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{1.1.1.3.1}$$

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -5 \\ -1 & 0 & 1 \end{pmatrix} \tag{1.1.1.3.2}$$

Using matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 5 & 6 & -11 \\ 6 & 9 & -15 \\ -11 & -15 & 26 \end{pmatrix} \tag{1.1.1.3.3}$$

$$\mathbf{d} = \sqrt{\operatorname{diag}\left(\begin{pmatrix} 5 & 6 & -11 \\ 6 & 9 & -15 \\ -11 & -15 & 26 \end{pmatrix}\right)} \tag{1.1.1.3.4}$$

$$= \begin{pmatrix} \sqrt{5} & 3 & \sqrt{26} \end{pmatrix} \tag{1.1.1.3.5}$$

1.1.1.4. Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \operatorname{diag}\left\{\left(\mathbf{N}^\top\mathbf{P}\right)\right\} \tag{1.1.1.4.1}$$

$$\mathbf{N}^\top\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & -5 \end{pmatrix}\begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \tag{1.1.1.4.2}$$

$$\tag{1.1.1.4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} -10 & -10 & -13 \\ -15 & -12 & -12 \\ 25 & 22 & 25 \end{pmatrix} \tag{1.1.1.4.4}$$

$$\mathbf{c} = \text{diag}\left(\begin{pmatrix} -10 & -10 & -13 \\ -15 & -12 & -12 \\ 25 & 22 & 25 \end{pmatrix}\right) \tag{1.1.1.4.5}$$

$$= \begin{pmatrix} -10 & -12 & 25 \end{pmatrix} \tag{1.1.1.4.6}$$

1.1.2. Median

1.1.2.1. Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.1.2.1.1}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.1.2.1.2}$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{-7}{2} & \frac{-5}{2} & -1 \\ -4 & \frac{-9}{2} & \frac{-9}{2} \end{pmatrix} \tag{1.1.2.1.3}$$

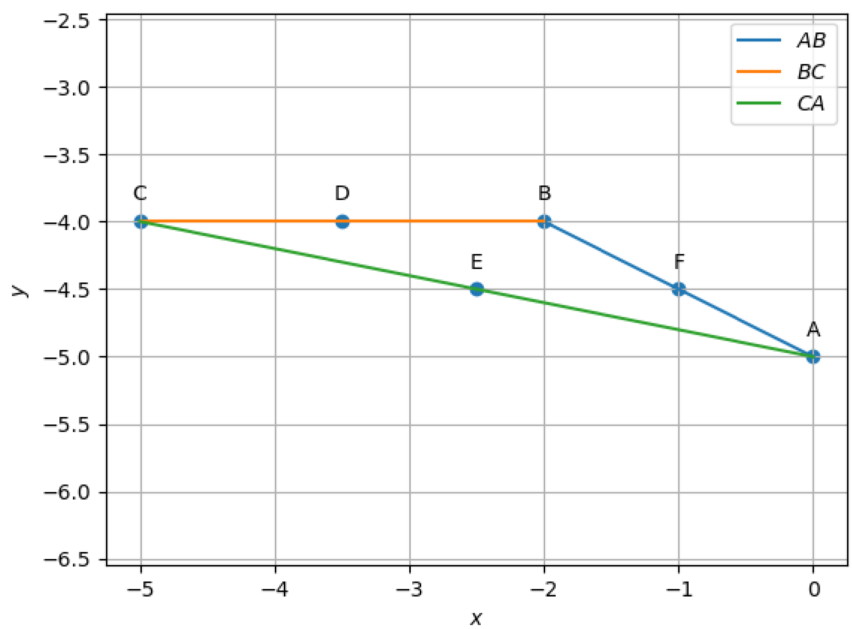

Figure 1.2: Midpoints of $\triangle ABC$

1.1.2.2. Obtain the median direction matrix.

Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{1.1.2.2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{1.1.2.2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.1.2.2.3}$$

$$= \begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{1.1.2.2.4}$$

Using matrix multiplication

$$\mathbf{M}_1 = \begin{pmatrix} \frac{7}{2} & \frac{1}{2} & -4 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.1.2.2.5}$$

1.1.2.3. Obtain the median normal matrix.

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1.2.3.1}$$

the normal matrix is obtained as

$$\mathbf{N}_1 = \mathbf{R}\mathbf{M}_1 \tag{1.1.2.3.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{7}{2} & \frac{1}{2} & -4 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.1.2.3.3}$$

$$\mathbf{N}_1 = \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{7}{2} & \frac{1}{2} & -4 \end{pmatrix} \tag{1.1.2.3.4}$$

1.1.2.4. Obtian the median equation constants.

$$\mathbf{c}_1 = \text{diag}\left(\left(\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right)\right) \tag{1.1.2.4.1}$$

$$\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 1 & \frac{7}{2} \\ \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & -4 \end{pmatrix} \begin{pmatrix} \frac{-7}{2} & \frac{-5}{2} & -1 \\ -4 & \frac{-9}{2} & \frac{-9}{2} \end{pmatrix} \tag{1.1.2.4.2}$$

$$\tag{1.1.2.4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} \frac{-35}{2} & \frac{-73}{4} & \frac{-67}{4} \\ \frac{-1}{2} & -1 & \frac{-7}{4} \\ \frac{71}{4} & \frac{77}{4} & \frac{37}{2} \end{pmatrix} \tag{1.1.2.4.4}$$

$$\mathbf{c}_1 = \text{diag}\left(\begin{pmatrix} \frac{-35}{2} & \frac{-73}{4} & \frac{-67}{4} \\ \frac{-1}{2} & -1 & \frac{-7}{4} \\ \frac{71}{4} & \frac{77}{4} & \frac{37}{2} \end{pmatrix}\right) \tag{1.1.2.4.5}$$

$$\mathbf{c}_1 = \begin{pmatrix} \frac{-35}{2} & -1 & \frac{37}{2} \end{pmatrix} \tag{1.1.2.4.6}$$

1.1.2.5. Obtain the centroid by finding the intersection of the medians.

Solution:

$$\left(\mathbf{N}_1^\top \;\middle|\; \mathbf{c}^\top\right) = \left(\begin{array}{cc|c} 1 & \frac{-5}{2} & \frac{19}{2} \\ 1 & \frac{13}{2} & \frac{-59}{2} \\ -2 & -4 & 20 \end{array}\right) \tag{1.1.2.5.1}$$

Using Gauss-Elimination method:

$$\left(\begin{array}{cc|c} 1 & \frac{-5}{2} & \frac{19}{2} \\ 1 & \frac{13}{2} & \frac{-59}{2} \\ -2 & -4 & 20 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} 0 & \frac{-5}{2} & \frac{19}{2} \\ 0 & 9 & -39 \\ -2 & -4 & 20 \end{array}\right) \tag{1.1.2.5.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_1} \left(\begin{array}{cc|c} 1 & \frac{-5}{2} & \frac{19}{2} \\ 0 & 9 & -39 \\ 0 & -9 & 39 \end{array}\right) \tag{1.1.2.5.3}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{9} R_2} \left(\begin{array}{cc|c} 1 & \frac{-5}{2} & \frac{19}{2} \\ 0 & 1 & \frac{-13}{3} \\ 0 & -9 & 39 \end{array}\right) \tag{1.1.2.5.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{5}{2} R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-4}{3} \\ 0 & 1 & \frac{-13}{3} \\ 0 & -9 & 39 \end{array}\right) \tag{1.1.2.5.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 9R_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-4}{3} \\ 0 & 1 & \frac{-13}{3} \\ 0 & 0 & 0 \end{array}\right) \tag{1.1.2.5.6}$$

$$\text{Therefore } \mathbf{G} = \begin{pmatrix} \frac{-4}{3} \\ \frac{-13}{3} \end{pmatrix} \tag{1.1.2.5.7}$$

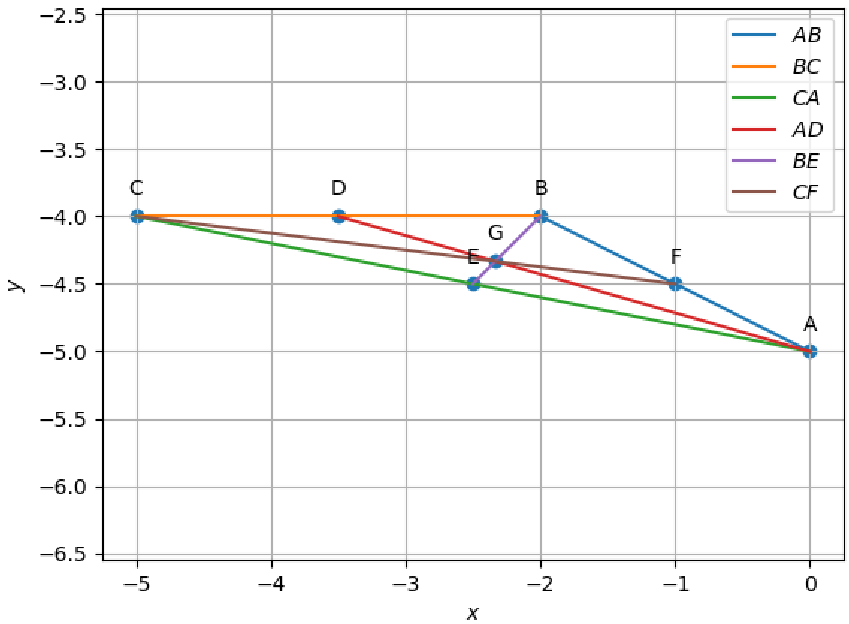

Figure 1.3: Centroid of $\triangle ABC$

1.1.3. Altitude

1.1.3.1. Find the normal matrix for the altitudes

Solution: The desired matrix is

$$
\begin{aligned}
\mathbf{M}_2 &= \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} && (1.1.3.1.1) \\
&= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} && (1.1.3.1.2) \\
&= \begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} && (1.1.3.1.3)
\end{aligned}
$$

Using Matrix multiplication

$$
\mathbf{M}_2 = \begin{pmatrix} 3 & -5 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.1.3.1.4}
$$

1.1.3.2. Find the constants vector for the altitudes.

Solution: The desired vector is

$$
\mathbf{c}_2 = \text{diag}\left\{ \left(\mathbf{M}^\top \mathbf{P} \right) \right\} \tag{1.1.3.2.1}
$$

$$
\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} 3 & 0 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & -2 & -5 \\ -5 & -4 & -4 \end{pmatrix} \tag{1.1.3.2.2}
$$

$$
\tag{1.1.3.2.3}
$$

Using matrix multiplication

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} 0 & -6 & -15 \\ -5 & 6 & 21 \\ -5 & 0 & -6 \end{pmatrix} \tag{1.1.3.2.4}$$

$$\mathbf{c}_2 = \text{diag}\left(\begin{pmatrix} 0 & -6 & -15 \\ -5 & 6 & 21 \\ -5 & 0 & -6 \end{pmatrix}\right) \tag{1.1.3.2.5}$$

$$\mathbf{c}_2 = \begin{pmatrix} 0 & 6 & -6 \end{pmatrix} \tag{1.1.3.2.6}$$

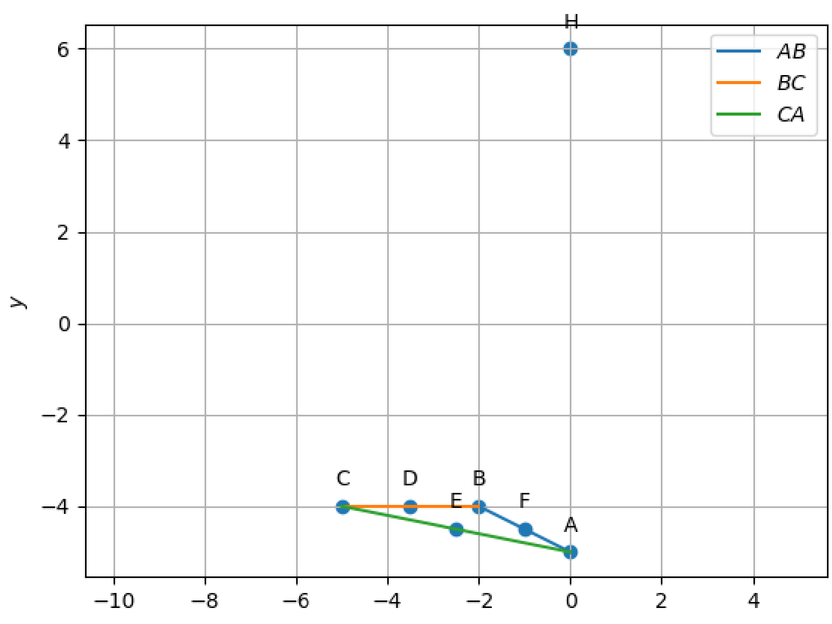

Figure 1.4: Orthocentre of $\triangle ABC$

1.1.4. Perpendicular Bisector

1.1.4.1. Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} 3 & -5 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{1.1.4.1.1}$$

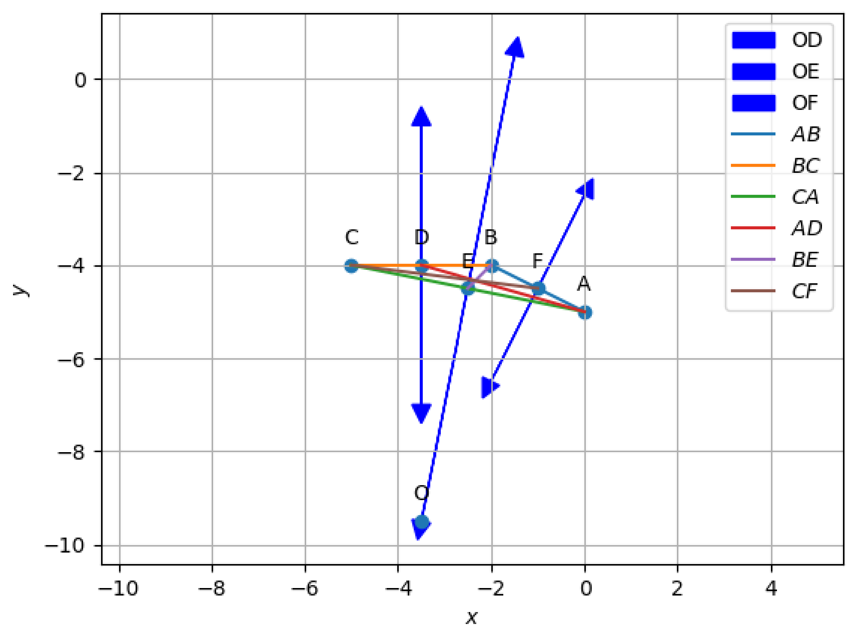

Figure 1.5: Perpendicular bisectors of $\triangle ABC$

1.1.4.2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.1.4.2.1}$$

Solution:

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right\} \tag{1.1.4.2.2}$$

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -5 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{-7}{2} & \frac{-5}{2} & -1 \\ -4 & \frac{-9}{2} & \frac{-9}{2} \end{pmatrix} \tag{1.1.4.2.3}$$

$$\tag{1.1.4.2.4}$$

Using matrix multiplication

$$\mathbf{M}_2^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{-21}{2} & \frac{-15}{2} & -3 \\ \frac{27}{2} & 8 & \frac{1}{2} \\ -3 & \frac{-1}{2} & \frac{-5}{2} \end{pmatrix} \tag{1.1.4.2.5}$$

$$\mathbf{c}_3 = \text{diag}\left(\begin{pmatrix} \frac{-21}{2} & \frac{-15}{2} & -3 \\ \frac{27}{2} & 8 & \frac{1}{2} \\ -3 & \frac{-1}{2} & \frac{-5}{2} \end{pmatrix}\right) \tag{1.1.4.2.6}$$

$$\mathbf{c}_3 = \begin{pmatrix} \frac{-21}{2} & 8 & \frac{5}{2} \end{pmatrix} \tag{1.1.4.2.7}$$

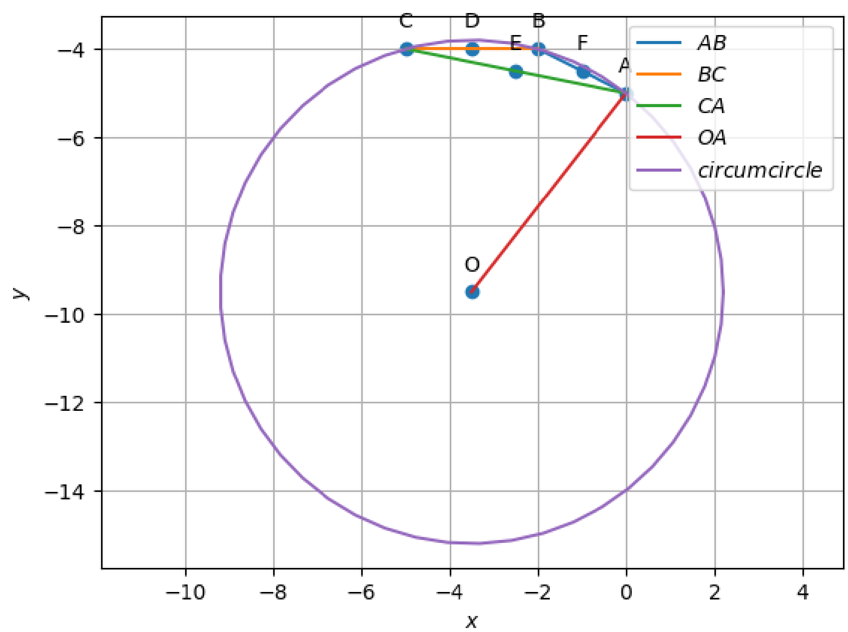

Figure 1.6: Circumcentre and Circumcircle of $\triangle ABC$

1.1.5. Angle Bisector

1.1.5.1. Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} & \frac{m\mathbf{C}+n\mathbf{B}}{m+n} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} \tag{1.1.5.1.1}$$

$$\begin{pmatrix}\mathbf{p} & \mathbf{m} & \mathbf{n}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbf{a} & \mathbf{b} & \mathbf{c}\end{pmatrix}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix} \tag{1.1.5.1.2}$$

$$= \frac{1}{2}\begin{pmatrix}\sqrt{5} & 3 & \sqrt{26}\end{pmatrix}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix} \tag{1.1.5.1.3}$$

Using matrix multiplication

$$\begin{pmatrix}\mathbf{p} & \mathbf{m} & \mathbf{n}\end{pmatrix} = \begin{pmatrix}2.9314757685 & 2.1675437455 & 0.0685242315\end{pmatrix} \tag{1.1.5.1.4}$$

$$= \begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\frac{n}{b} & \frac{m}{c} & 0\\ 0 & \frac{p}{c} & \frac{n}{a}\\ \frac{p}{b} & 0 & \frac{m}{a}\end{pmatrix} \tag{1.1.5.1.5}$$

$$= \begin{pmatrix}0 & -2 & -5\\ -5 & -4 & -4\end{pmatrix}\begin{pmatrix}\frac{0.0685242315}{\sqrt{26}} & \frac{2.1675437455}{\sqrt{5}} & 0\\ 0 & \frac{2.9314757685}{\sqrt{5}} & \frac{0.0685242315}{3}\\ \frac{2.9314757685}{\sqrt{26}} & 0 & \frac{2.1675437455}{3}\end{pmatrix} \tag{1.1.5.1.6}$$

Using matrix multiplication We get the points of contact

$$= \begin{pmatrix}-2.874548489127433 & -2.62199163724261 & -3.6582557313\\ -2.366832328963313 & -10.090758435829073 & -2.9814239706\end{pmatrix} \tag{1.1.5.1.7}$$

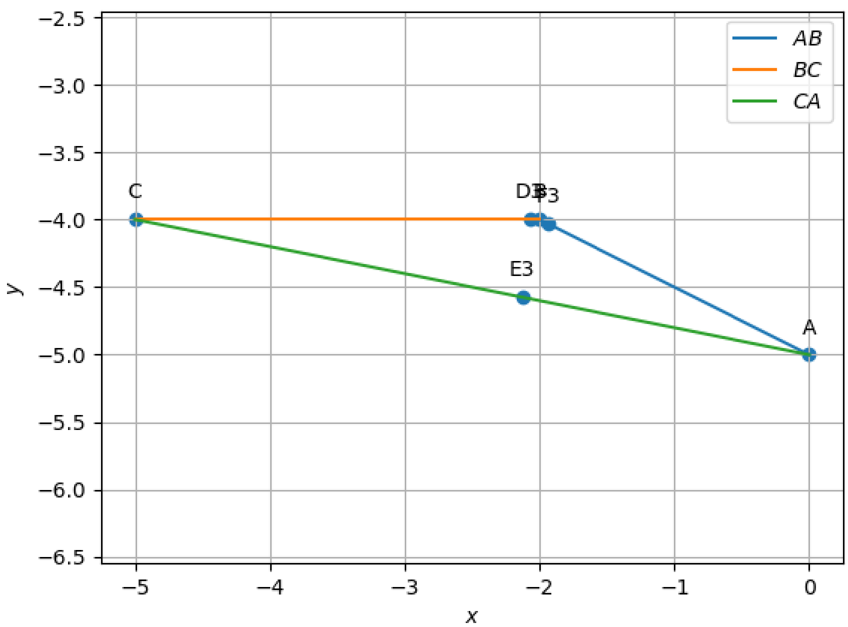

Figure 1.7: Contact points of Incircle of $\triangle ABC$

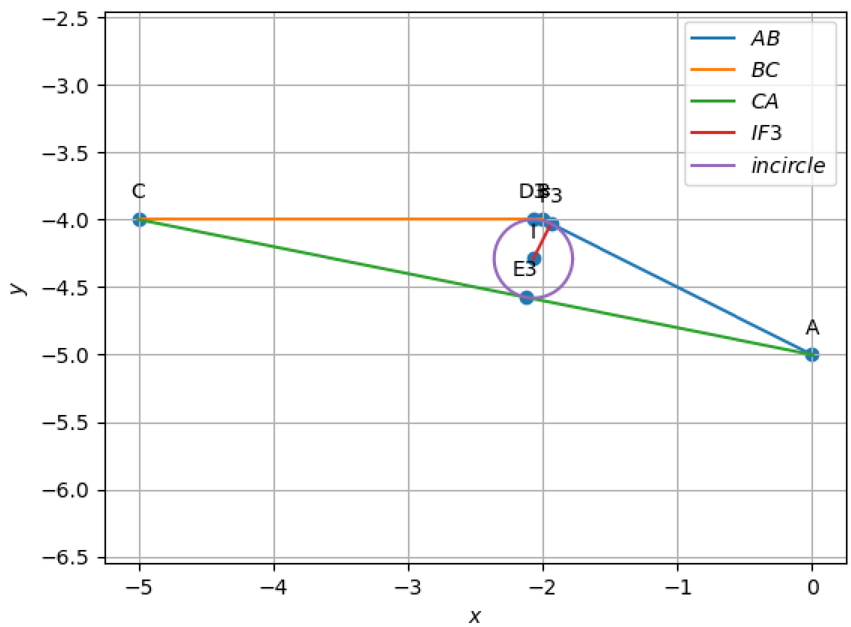

Figure 1.8: Incentre and Incircle of $\triangle ABC$