Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \tag{1.1}$$

1.1. Angle Bisector

1.1.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \tag{1.1.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Question 1.1.2.

Solution: From Question 1.1.2

$$a = \sqrt{26} \tag{1.1.1.2}$$

$$b = \sqrt{10} \tag{1.1.1.3}$$

$$c = \sqrt{68} \tag{1.1.1.4}$$

From the given information,

$$a = m + n, \tag{1.1.1.5}$$

$$b = n + p, \tag{1.1.1.6}$$

$$c = m + p \tag{1.1.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.8}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.1.1.9}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.1.1.10}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.11}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.1.1.12}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.1.1.13}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{68}+\sqrt{10}-\sqrt{26}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{26}+\sqrt{68}-\sqrt{10}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{26}+\sqrt{10}-\sqrt{68}}{2} \end{aligned} \tag{1.1.1.14}$$

on solving above equations we get

$$p = 3.154734699 \tag{1.1.1.15}$$

$$m = 5.091476552 \tag{1.1.1.16}$$

$$n = 0.007542961263 \tag{1.1.1.17}$$

1.1.2. Using section formula, find $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$.

Solution: Given

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \; \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \; \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.1.2.1}$$

Here

$$\mathbf{A} = \begin{pmatrix} -6 \\ -5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}, \tag{1.1.2.2}$$

$$p = 3.154734699, \; m = 5.091476552, \; n = 0.007542961263 \tag{1.1.2.3}$$

On substituting (1.1.2.2) and (1.1.2.3) in (**??**) We get

$$\mathbf{D}_3 = \frac{5.091476552 \begin{pmatrix} 1 \\ -5 \end{pmatrix} + 0.007542961263 \begin{pmatrix} -6 \\ 1 \end{pmatrix}}{5.091476552 + 0.007542961263} \tag{1.1.2.4}$$

$$\mathbf{E}_3 = \frac{0.007542961263 \begin{pmatrix} -6 \\ -5 \end{pmatrix} + 3.154734699 \begin{pmatrix} 1 \\ -5 \end{pmatrix}}{0.007542961263 + 3.154734699} \tag{1.1.2.5}$$

$$\mathbf{F}_3 = \frac{3.154734699 \begin{pmatrix} -6 \\ 1 \end{pmatrix} + 0.007542961263 \begin{pmatrix} -6 \\ -5 \end{pmatrix}}{3.154734699 + 0.007542961263} \tag{1.1.2.6}$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix} -2.8796283 \\ -1.67460431 \end{pmatrix} \tag{1.1.2.7}$$

$$\mathbf{E}_3 = \begin{pmatrix} -6 \\ -3.10977223 \end{pmatrix} \tag{1.1.2.8}$$

$$\mathbf{F}_3 = \begin{pmatrix} -4.10977223 \\ -5 \end{pmatrix} \tag{1.1.2.9}$$

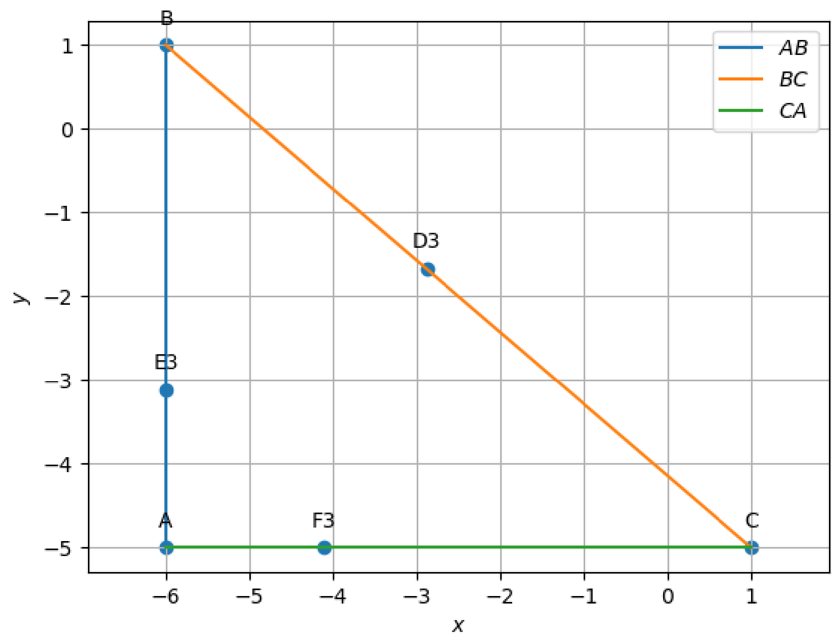

Figure 1.1: Points $D3, E3, F3$

1.1.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the incentre and inradius of $\triangle ABC$.

Solution: Given

$$\mathbf{D}_3 = \begin{pmatrix} -2.8796283 \\ -1.67460431 \end{pmatrix} \tag{1.1.3.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} -6 \\ -3.10977223 \end{pmatrix} \tag{1.1.3.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} -4.10977223 \\ -5 \end{pmatrix} \tag{1.1.3.3}$$

(a) For circumcentre

Vector equation of $\mathbf{D} - \mathbf{E}$ is

$$(\mathbf{D}_3 - \mathbf{E}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{E}_3}{2}\right) = 0 \tag{1.1.3.4}$$

$$(\mathbf{D}_3 - \mathbf{F}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{F}_3}{2}\right) = 0 \tag{1.1.3.5}$$

on Substituting the values of $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ and solving We get,

$$\begin{pmatrix} 3.1203717 & 1.43516792 \end{pmatrix} \mathbf{x} = -17.28706229 \tag{1.1.3.6}$$

$$\begin{pmatrix} 1.23014393 & 3.32539569 \end{pmatrix} \mathbf{x} = -15.39683453 \tag{1.1.3.7}$$

Thus on solving (1.1.3.6) and (1.1.3.7) using gauss elimination We get

$$\begin{pmatrix} 3.1203717 & 1.43516792 & -17.28706229 \\ 1.23014393 & 3.32539569 & -15.39683453 \end{pmatrix} \tag{1.1.3.8}$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -4.10977222 \\ -3.10977223 \end{pmatrix} \tag{1.1.3.9}$$

$$\implies \mathbf{x} = \begin{pmatrix} -4.10977222 \\ -3.10977223 \end{pmatrix} \tag{1.1.3.10}$$

(b) The circiumradius is obtained from $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} -4.10977223 \\ -3.10977223 \end{pmatrix} \tag{1.1.3.11}$$

$$\mathbf{D}_3 = \begin{pmatrix} -2.8796283 \\ -1.67460431 \end{pmatrix} \tag{1.1.3.12}$$

$$\mathbf{I} - \mathbf{D_3} = \begin{pmatrix} -1.23014393 \\ -1.43516792 \end{pmatrix} \tag{1.1.3.13}$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \tag{1.1.3.14}$$

$$r = 1.89022777 \tag{1.1.3.15}$$

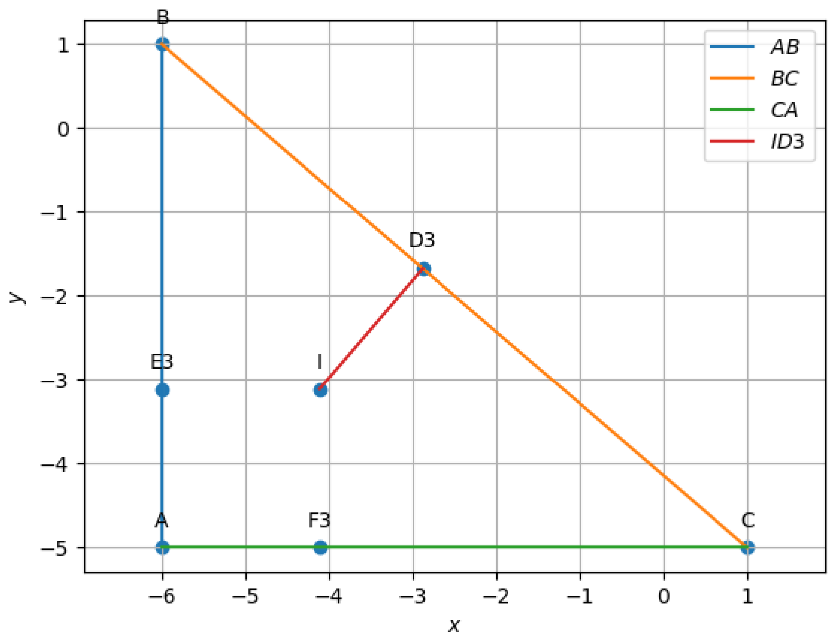

Figure 1.2: Incentre and Inradius of $\triangle ABC$

1.1.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution:

$$\mathbf{D}_3 = \begin{pmatrix} -2.8796283 \\ -1.67460431 \end{pmatrix} \tag{1.1.4.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} -6 \\ -3.10977223 \end{pmatrix} \tag{1.1.4.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} -4.10977223 \\ -5 \end{pmatrix} \tag{1.1.4.3}$$

$$\text{Incentre} \tag{1.1.4.4}$$

$$\mathbf{I} = \begin{pmatrix} -4.10977223 \\ -3.10977223 \end{pmatrix} \tag{1.1.4.5}$$

$$\text{Radius} \tag{1.1.4.6}$$

$$\mathbf{r} = 1.89022777 \tag{1.1.4.7}$$

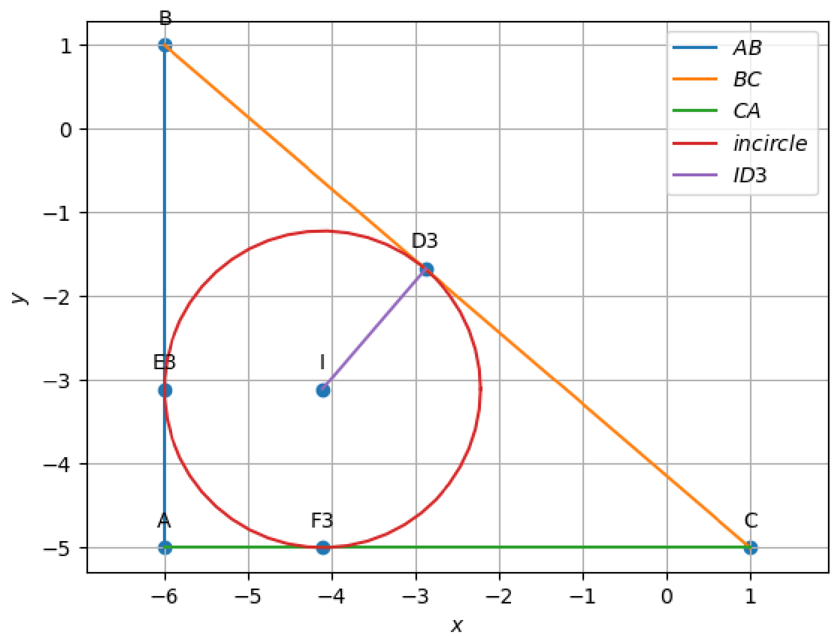

Figure 1.3: Incircle of $\triangle ABC$

1.1.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \tag{1.1.5.1}$$

AI is the bisector of $\angle A$.

Solution:

$$\cos \angle BAI \triangleq \frac{(\mathbf{B}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.2}$$

$$\cos \angle CAI \triangleq \frac{(\mathbf{C}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\| \, \|\mathbf{I}-\mathbf{A}\|} \tag{1.1.5.3}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{I}$,

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}0\\6\end{pmatrix} \tag{1.1.5.4}$$

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}7\\0\end{pmatrix} \tag{1.1.5.5}$$

$$\mathbf{I}-\mathbf{A}=\begin{pmatrix}1.89022777\\1.89022777\end{pmatrix} \tag{1.1.5.6}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\|=6 \tag{1.1.5.7}$$

$$\|\mathbf{C}-\mathbf{A}\|=7 \tag{1.1.5.8}$$

$$\|\mathbf{I}-\mathbf{A}\|=2.673185748 \tag{1.1.5.9}$$

$$\tag{1.1.5.10}$$

(a) for $\angle BAI$:

On substtuting the values in (1.1.5.2) ,We get

$$\cos\angle BAI \triangleq \frac{\begin{pmatrix}0 & 6\end{pmatrix}\begin{pmatrix}1.89022777\\1.89022777\end{pmatrix}}{6\times 2.673185748} \tag{1.1.5.11}$$

$$\tag{1.1.5.12}$$

On solving

$$\angle BAI = 44.999^\circ \tag{1.1.5.13}$$

(b) for $\angle CAI$:

On substtuting the values in (1.1.5.2) ,We get

$$\cos \angle CAI \triangleq \frac{\begin{pmatrix} 7 & 0 \end{pmatrix} \begin{pmatrix} 1.89022777 \\ 1.89022777 \end{pmatrix}}{7 \times 2.673185748} \tag{1.1.5.14}$$

(1.1.5.15)

On solving

$$\angle CAI = 44.999^\circ \tag{1.1.5.16}$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

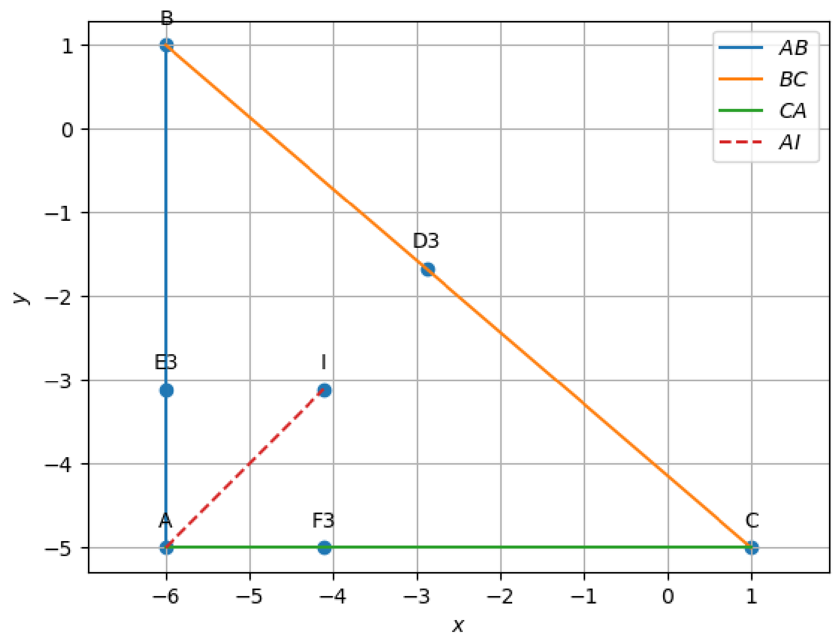

Figure 1.4: Angular bisector AI

1.1.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

Solution:

(a) To prove BI is an angular bisector of $\angle B$

$$\cos \angle ABI \triangleq \frac{(\mathbf{A} - \mathbf{B})\top(\mathbf{I} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{I} - \mathbf{B}\|} \tag{1.1.6.1}$$

$$\cos \angle CBI \triangleq \frac{(\mathbf{C} - \mathbf{B})\top(\mathbf{I} - \mathbf{B})}{\|\mathbf{C} - \mathbf{B}\| \, \|\mathbf{I} - \mathbf{B}\|} \tag{1.1.6.2}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}0\\-6\end{pmatrix} \tag{1.1.6.3}$$

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}7\\-6\end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{I}-\mathbf{B}=\begin{pmatrix}1.89022777\\-4.10977223\end{pmatrix} \tag{1.1.6.5}$$

also calculating the values of norms

$$\|\mathbf{A}-\mathbf{B}\|=6 \tag{1.1.6.6}$$

$$\|\mathbf{C}-\mathbf{B}\|=\sqrt{85} \tag{1.1.6.7}$$

$$\|\mathbf{I}-\mathbf{B}\|=4.523625626 \tag{1.1.6.8}$$

$$\tag{1.1.6.9}$$

i. for $\angle ABI$:

On substtuting the values in (1.1.6.1) ,We get

$$\cos\angle ABI \triangleq \frac{\begin{pmatrix}0 & -6\end{pmatrix}\begin{pmatrix}1.89022777\\-4.10977223\end{pmatrix}}{6\times 4.523625626} \tag{1.1.6.10}$$

$$\tag{1.1.6.11}$$

On solving

$$\angle ABI = 24.69935265^\circ \tag{1.1.6.12}$$

ii. for $\angle CBI$:

On substtuting the values in (1.1.6.1) ,We get

$$\cos \angle CBI \triangleq \frac{\begin{pmatrix} 7 & -6 \end{pmatrix} \begin{pmatrix} 1.89022777 \\ -4.10977223 \end{pmatrix}}{\sqrt{85} \times 4.523625626} \tag{1.1.6.13}$$

$$\tag{1.1.6.14}$$

On solving

$$\angle CBI = 24.6993527^\circ \tag{1.1.6.15}$$

Therefore $\angle ABI = \angle CBI$. and BI is the bisector of $\angle B$.

(b) To prove CI is an angular bisector of $\angle C$

$$\cos \angle BCI \triangleq \frac{(\mathbf{B} - \mathbf{C})\top(\mathbf{I} - \mathbf{C})}{\|\mathbf{B} - \mathbf{C}\| \, \|\mathbf{I} - \mathbf{C}\|} \tag{1.1.6.16}$$

$$\cos \angle ACI \triangleq \frac{(\mathbf{A} - \mathbf{C})\top(\mathbf{I} - \mathbf{C})}{\|\mathbf{A} - \mathbf{B}\| \, \|\mathbf{I} - \mathbf{C}\|} \tag{1.1.6.17}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}-7\\6\end{pmatrix} \tag{1.1.6.18}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-7\\0\end{pmatrix} \tag{1.1.6.19}$$

$$\mathbf{I}-\mathbf{C}=\begin{pmatrix}-5.10977223\\1.89022777\end{pmatrix} \tag{1.1.6.20}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{C}\|=\sqrt{85} \tag{1.1.6.21}$$

$$\|\mathbf{A}-\mathbf{C}\|=7 \tag{1.1.6.22}$$

$$\|\mathbf{I}-\mathbf{C}\|=5.448186236 \tag{1.1.6.23}$$

$$\tag{1.1.6.24}$$

i. for $\angle BCI$:

On substtuting the values in (1.1.6.16) ,We get

$$\cos\angle BCI \triangleq \frac{\begin{pmatrix}-7 & 6\end{pmatrix}\begin{pmatrix}-5.10977223\\1.89022777\end{pmatrix}}{\sqrt{85}\times 5.448186236} \tag{1.1.6.25}$$

$$\tag{1.1.6.26}$$

On solving

$$\angle BCI = 20.30064734^\circ \tag{1.1.6.27}$$

ii. for $\angle ACI$:

On substtuting the values in (1.1.6.16) ,We get

$$\cos\angle ACI \triangleq \frac{\begin{pmatrix} -7 & 0 \end{pmatrix}\begin{pmatrix} -5.10977223 \\ 1.89022777 \end{pmatrix}}{7 \times 5.448186236} \tag{1.1.6.28}$$

$$\tag{1.1.6.29}$$

On solving

$$\angle ACI = 20.30064729^\circ \tag{1.1.6.30}$$

Therefore $\angle BCI = \angle ACI$. and CI is the bisector of $\angle C$.

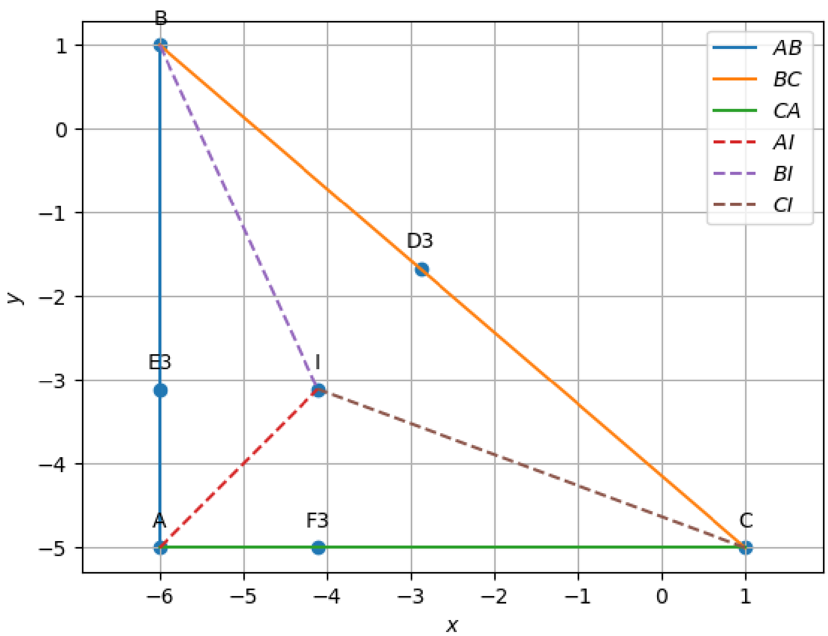

Figure 1.5: Angular bisectors BI, CI