Financial Mathematics
Black-Scholes Model and Option Replication: Solutions

Solutions

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1 Introduction

This document provides detailed solutions to the problems related to the Black-Scholes model and option replication strategies. We will derive the necessary mathematical formulations, prove key results, and provide the theoretical foundation for the numerical implementations.

2 Black-Scholes Model and Option Pricing

2.1 Derivation of the Black-Scholes Model

We begin by considering a financial market with two assets:

- A risk-free asset (bond) with price process $(S_t^0)_{t\geq 0}$ following:

$$dS_t^0 = rS_t^0 dt \tag{1}$$

  where $r > 0$ is the constant risk-free interest rate.

- A risky asset (stock) with price process $(S_t)_{t\geq 0}$ following:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P \tag{2}$$

  where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility, and W_t^P is a standard Brownian motion under the physical probability measure P.

Let's first solve the stochastic differential equation (SDE) for the risk-free asset. The SDE for the bond price is:

$$dS_t^0 = rS_t^0 dt \tag{3}$$

This is a deterministic ordinary differential equation that can be solved by separation of variables:

$$\frac{dS_t^0}{S_t^0} = rdt \tag{4}$$

$$\int_{S_0^0}^{S_t^0} \frac{dS}{S} = \int_0^t rds \tag{5}$$

$$\ln(S_t^0) - \ln(S_0^0) = rt \tag{6}$$

$$\ln\left(\frac{S_t^0}{S_0^0}\right) = rt \tag{7}$$

$$\frac{S_t^0}{S_0^0} = e^{rt} \tag{8}$$

$$S_t^0 = S_0^0 e^{rt} \tag{9}$$

For the risky asset, we need to solve the SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P \tag{10}$$

This is a geometric Brownian motion. To solve it, we apply Itô's lemma to the function $f(S_t) = \ln(S_t)$. By Itô's formula:

$$\begin{aligned} d(\ln(S_t)) &= \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} d\langle S\rangle_t && (11)\\ &= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t^P) - \frac{1}{2}\frac{1}{S_t^2}\sigma^2 S_t^2 dt && (12)\\ &= \mu dt + \sigma dW_t^P - \frac{1}{2}\sigma^2 dt && (13)\\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t^P && (14) \end{aligned}$$

Integrating from 0 to t:

$$\ln(S_t) - \ln(S_0) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t^P \tag{15}$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t^P \tag{16}$$

$$\frac{S_t}{S_0} = \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t^P\right] \tag{17}$$

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t^P\right] \tag{18}$$

This gives us the explicit solution for the stock price process under the physical measure P. The stock price follows a log-normal distribution, with:

$$\ln(S_t) \sim \mathcal{N}\left(\ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma^2 t\right) \tag{19}$$

These two assets form the basis of the Black-Scholes model, which we will use to derive the option pricing formula.

2.2 Risk-Neutral Pricing

Let's define a European option with payoff $\Phi(S_T)$ at maturity $T > 0$. According to the risk-neutral pricing principle, the fair price of this option at time $t \in [0, T]$ is given by:

$$V_t = e^{-r(T-t)} E^Q[\Phi(S_T)|S_t] \tag{20}$$

where Q is the risk-neutral probability measure.

The risk-neutral pricing principle is a fundamental concept in mathematical finance that allows us to price derivatives without needing to know the market's risk preferences. This principle is based on the absence of arbitrage and the concept of risk-neutral valuation.

To understand why this principle holds, let's consider a portfolio that replicates the option payoff at maturity. If we can construct a self-financing portfolio $(P_t)_{t\in[0,T]}$ such that $P_T = \Phi(S_T)$, then by the law of one price, we must have $V_t = P_t$ for all $t \in [0, T]$ to avoid arbitrage.

Let's assume that such a replicating portfolio exists and is given by:

$$P_t = \delta_t S_t + \delta_t^0 S_t^0 \tag{21}$$

where δ_t is the number of shares of the stock and δ_t^0 is the number of units of the bond at time t.

The self-financing condition implies that:

$$dP_t = \delta_t dS_t + \delta_t^0 dS_t^0 \tag{22}$$

Under the physical measure P, we have:

$$\begin{align} dP_t &= \delta_t(\mu S_t dt + \sigma S_t dW_t^P) + \delta_t^0(r S_t^0 dt) \tag{23} \\ &= (\delta_t \mu S_t + \delta_t^0 r S_t^0)dt + \delta_t \sigma S_t dW_t^P \tag{24} \end{align}$$

The key insight of risk-neutral pricing is that we can change the probability measure from P to a risk-neutral measure Q under which the discounted price processes of all tradable assets are martingales. This means that under Q, the expected return of all assets is the risk-free rate r.

Under the risk-neutral measure Q, the stock price process follows:

$$dS_t = r S_t dt + \sigma S_t dW_t^Q \tag{25}$$

where W_t^Q is a standard Brownian motion under Q.

Now, the discounted portfolio value $\tilde{P}_t = e^{-rt} P_t$ should be a martingale under Q. This means:

$$E^Q[\tilde{P}_T|\mathcal{F}_t] = \tilde{P}_t \tag{26}$$

Since $P_T = \Phi(S_T)$, we have:

$$\begin{align} \tilde{P}_t &= E^Q[\tilde{P}_T|\mathcal{F}_t] \tag{27} \\ &= E^Q[e^{-rT}\Phi(S_T)|\mathcal{F}_t] \tag{28} \end{align}$$

Converting back to the undiscounted value:

$$\begin{align} P_t &= e^{rt}\tilde{P}_t \tag{29} \\ &= e^{rt} E^Q[e^{-rT}\Phi(S_T)|\mathcal{F}_t] \tag{30} \\ &= e^{-r(T-t)} E^Q[\Phi(S_T)|\mathcal{F}_t] \tag{31} \end{align}$$

Since the stock price process is Markovian, the conditional expectation given $\mathcal{F}_t$ is the same as the conditional expectation given S_t:

$$P_t = e^{-r(T-t)} E^Q[\Phi(S_T)|S_t] \tag{32}$$

Therefore, the option price $V_t = P_t$ is given by:

$$V_t = e^{-r(T-t)} E^Q[\Phi(S_T)|S_t] \tag{33}$$

This is the risk-neutral pricing formula, which allows us to price options without needing to know the market's risk preferences or the true drift μ of the stock.

2.3 Change of Measure

To move from the physical measure P to the risk-neutral measure Q, we need to apply Girsanov's theorem. Under the risk-neutral measure Q, the stock price process follows:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q \tag{34}$$

where W_t^Q is a standard Brownian motion under Q.

Girsanov's theorem provides a way to change the probability measure for stochastic processes. It tells us how the drift of a stochastic process changes when we change the probability measure, while the diffusion coefficient remains the same.

Let's start with the stock price process under the physical measure P:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P \tag{35}$$

We want to find a new measure Q such that under Q, the drift of the stock price process is the risk-free rate r. According to Girsanov's theorem, we can define a new Brownian motion W_t^Q under Q as:

$$W_t^Q = W_t^P + \theta t \tag{36}$$

where θ is a constant to be determined.

Substituting this into the original SDE:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^P && (37) \\ &= \mu S_t dt + \sigma S_t (dW_t^Q - \theta dt) && (38) \\ &= (\mu - \sigma\theta) S_t dt + \sigma S_t dW_t^Q && (39) \end{aligned}$$

For this to match the desired form $dS_t = rS_t dt + \sigma S_t dW_t^Q$, we need:

$$\mu - \sigma\theta = r \tag{40}$$

$$\theta = \frac{\mu - r}{\sigma} \tag{41}$$

This parameter θ is known as the market price of risk.

Now, Girsanov's theorem states that the Radon-Nikodym derivative that defines the change of measure from P to Q is:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\theta W_t^P - \frac{1}{2}\theta^2 t\right) \tag{42}$$

Substituting $\theta = \frac{\mu - r}{\sigma}$:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\frac{\mu - r}{\sigma} W_t^P - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right) \tag{43}$$

This Radon-Nikodym derivative is a martingale under P, which means:

$$E^P\left[\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}\right] = 1 \tag{44}$$

We can verify this by noting that $\exp\left(-\theta W_t^P - \frac{1}{2}\theta^2 t\right)$ is the exponential martingale associated with the process $-\theta W_t^P$.

The Radon-Nikodym derivative allows us to compute expectations under Q using expectations under P:

$$E^Q[X] = E^P\left[X \cdot \frac{dQ}{dP}\right] \tag{45}$$

for any random variable X.

In particular, for the stock price process, we can verify that under Q, the discounted stock price $\tilde{S}_t = e^{-rt} S_t$ is a martingale:

$$\begin{aligned}
d\tilde{S}_t &= d(e^{-rt} S_t) && (46)\\
&= -re^{-rt} S_t dt + e^{-rt} dS_t && (47)\\
&= -re^{-rt} S_t dt + e^{-rt}(r S_t dt + \sigma S_t dW_t^Q) && (48)\\
&= e^{-rt} \sigma S_t dW_t^Q && (49)
\end{aligned}$$

Since the drift term is zero, $\tilde{S}_t$ is a martingale under Q, which confirms that Q is indeed a risk-neutral measure.

2.4 Justification of $V_t = v(t, S_t)$

We need to justify that the option price V_t can be expressed as a function of time t and the current stock price S_t, i.e., $V_t = v(t, S_t)$ for some function v.

Under the risk-neutral measure Q, the stock price S_t follows a Markov process. The option price at time t is:

$$V_t = e^{-r(T-t)} E^Q[\Phi(S_T)|S_t] \tag{50}$$

To understand why S_t is a Markov process, let's recall the solution to the SDE under the risk-neutral measure Q:

$$S_t = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^Q\right] \tag{51}$$

For any $0 \leq u < t$, we can write:

$$\begin{aligned}
S_t &= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^Q\right] && (52)\\
&= S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right) u + \sigma W_u^Q\right] \cdot \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(t-u) + \sigma(W_t^Q - W_u^Q)\right] && (53)\\
&= S_u \cdot \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(t-u) + \sigma(W_t^Q - W_u^Q)\right] && (54)
\end{aligned}$$

Since $W_t^Q - W_u^Q$ is independent of $\mathcal{F}_u$ (the information available up to time u), the future value S_t depends on the past only through the current value S_u. This is precisely the Markov property.

Now, let's define the function $v(t,s)$ as:

$$v(t,s) = e^{-r(T-t)} E^Q[\Phi(S_T)|S_t = s] \tag{55}$$

This function gives the option price at time t when the stock price is s. We want to show that $V_t = v(t, S_t)$.

By the definition of conditional expectation:

$$V_t = e^{-r(T-t)} E^Q[\Phi(S_T)|\mathcal{F}_t] \tag{56}$$

Since S_t is a Markov process, the conditional expectation given $\mathcal{F}_t$ is the same as the conditional expectation given S_t:

$$E^Q[\Phi(S_T)|\mathcal{F}_t] = E^Q[\Phi(S_T)|S_t] \tag{57}$$

This is because the future evolution of the stock price depends on the past only through the current value S_t.

Therefore:

$$\begin{aligned} V_t &= e^{-r(T-t)} E^Q[\Phi(S_T)|\mathcal{F}_t] && (58)\\ &= e^{-r(T-t)} E^Q[\Phi(S_T)|S_t] && (59)\\ &= v(t, S_t) && (60) \end{aligned}$$

This confirms that the option price V_t can be expressed as a function $v(t, S_t)$ of time t and the current stock price S_t.

Furthermore, we can show that $v(t,s)$ satisfies the Feynman-Kac representation:

$$v(t,s) = e^{-r(T-t)} E^Q[\Phi(S_T^{t,s})] \tag{61}$$

where $S_T^{t,s}$ is the value at time T of the stock price process starting from s at time t.

Explicitly, we have:

$$S_T^{t,s} = s \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T^Q - W_t^Q)\right] \tag{62}$$

This representation will be useful when we derive the Black-Scholes partial differential equation (PDE) in the next section.

2.5 Derivation of the Black-Scholes PDE

Assuming that $v \in C^{1,2}([0,T) \times \mathbb{R}_+^*, \mathbb{R}) \cap C^0([0,T] \times \mathbb{R}_+^*, \mathbb{R})$, we can apply Itô's formula to $v(t, S_t)$ to derive the Black-Scholes partial differential equation (PDE).

Itô's formula for a function $f(t, X_t)$ of time and a stochastic process X_t is:

$$df(t, X_t) = \partial_t f(t, X_t)dt + \partial_x f(t, X_t)dX_t + \frac{1}{2}\partial_{xx}^2 f(t, X_t)d\langle X\rangle_t \tag{63}$$

In our case, we apply this to $f(t, X_t) = v(t, S_t)$, where S_t follows the SDE under the risk-neutral measure Q:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q \tag{64}$$

The quadratic variation of S_t is:

$$d\langle S\rangle_t = \sigma^2 S_t^2 dt \tag{65}$$

This follows from the fact that $d\langle W^Q\rangle_t = dt$ and the rules of stochastic calculus.

Now, applying Itô's formula to $v(t, S_t)$:

$$dv(t, S_t) = \partial_t v(t, S_t)dt + \partial_s v(t, S_t)dS_t + \frac{1}{2}\partial_{ss}^2 v(t, S_t)d\langle S\rangle_t \tag{66}$$

Substituting the expressions for dS_t and $d\langle S\rangle_t$:

$$\begin{aligned} dv(t, S_t) &= \partial_t v(t, S_t)dt + \partial_s v(t, S_t)(rS_t dt + \sigma S_t dW_t^Q) + \frac{1}{2}\partial_{ss}^2 v(t, S_t)\sigma^2 S_t^2 dt && (67) \\ &= \partial_t v(t, S_t)dt + rS_t\partial_s v(t, S_t)dt + \sigma S_t \partial_s v(t, S_t)dW_t^Q + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 v(t, S_t)dt && (68) \\ &= \left(\partial_t v(t, S_t) + rS_t\partial_s v(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 v(t, S_t)\right) dt + \sigma S_t \partial_s v(t, S_t)dW_t^Q && (69) \end{aligned}$$

On the other hand, we know that $V_t = v(t, S_t)$ is the price of a tradable asset. In an arbitrage-free market, any tradable asset must earn the risk-free rate of return under the risk-neutral measure. This means:

$$dV_t = rV_t dt + \text{martingale term} \tag{70}$$

The martingale term has zero drift and represents the risk associated with the asset. Since $V_t = v(t, S_t)$, we have:

$$dv(t, S_t) = rv(t, S_t)dt + \text{martingale term} \tag{71}$$

Comparing this with our expression for $dv(t, S_t)$ from Itô's formula, we can identify the martingale term as $\sigma S_t \partial_s v(t, S_t)dW_t^Q$. For the drift terms to match, we must have:

$$\partial_t v(t, S_t) + rS_t\partial_s v(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \partial_{ss}^2 v(t, S_t) = rv(t, S_t) \tag{72}$$

Rearranging to standard form:

$$\partial_t v(t, s) + rs\partial_s v(t, s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss}^2 v(t, s) - rv(t, s) = 0 \tag{73}$$

This is the Black-Scholes PDE. It is a second-order parabolic partial differential equation that the option price function $v(t, s)$ must satisfy.

To complete the specification of the problem, we need to add boundary conditions. For a European call option with payoff $\Phi(S_T) = \max(S_T - K, 0)$, where K is the strike price, the boundary conditions are:

$$\begin{aligned} v(T, s) &= \max(s - K, 0) \quad \text{for all } s > 0 \quad \text{(terminal condition)} && (74) \\ v(t, 0) &= 0 \quad \text{for all } t \in [0, T] \quad \text{(boundary condition at } s = 0) && (75) \\ \lim_{s\to\infty} v(t, s) &= s \quad \text{for all } t \in [0, T] \quad \text{(asymptotic behavior as } s \to \infty) && (76) \end{aligned}$$

For a European put option with payoff $\Phi(S_T) = \max(K - S_T, 0)$, the boundary conditions are:

$$\begin{aligned} v(T, s) &= \max(K - s, 0) \quad \text{for all } s > 0 \quad \text{(terminal condition)} && (77) \\ v(t, 0) &= Ke^{-r(T-t)} \quad \text{for all } t \in [0, T] \quad \text{(boundary condition at } s = 0) && (78) \\ \lim_{s\to\infty} v(t, s) &= 0 \quad \text{for all } t \in [0, T] \quad \text{(asymptotic behavior as } s \to \infty) && (79) \end{aligned}$$

The Black-Scholes PDE, together with the appropriate boundary conditions, uniquely determines the option price function $v(t, s)$.

2.6 Black-Scholes Formula

The Black-Scholes formula gives the price of European call and put options in closed form. For a European call option with strike price K and maturity T, the price at time $t \in [0, T]$ when the stock price is S_t is:

$$C(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{80}$$

where:

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{81}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{82}$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \tag{83}$$

For a European put option with the same parameters, the price is:

$$P(t, S_t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{84}$$

These formulas can be derived by solving the Black-Scholes PDE with the appropriate boundary conditions, or by directly computing the risk-neutral expectation of the option payoff.

Let's derive the formula for a European call option by computing the risk-neutral expectation:

$$C(t, S_t) = e^{-r(T-t)} E^Q[\max(S_T - K, 0)|S_t] \tag{85}$$

$$= e^{-r(T-t)} E^Q[\max(S_T - K, 0)|S_t] \tag{86}$$

Under the risk-neutral measure Q, the stock price at maturity is:

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T^Q - W_t^Q)\right] \tag{87}$$

Let's denote $Z = W_T^Q - W_t^Q$, which is normally distributed with mean 0 and variance $T - t$. Then:

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma Z\right] \tag{88}$$

The call option will be in-the-money (i.e., $S_T > K$) if and only if:

$$S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma Z\right] > K \tag{89}$$

$$\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma Z > \ln\left(\frac{K}{S_t}\right) \tag{90}$$

$$Z > \frac{\ln\left(\frac{K}{S_t}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma} \tag{91}$$

$$Z > -d_2 \tag{92}$$

Now, we can compute the expected payoff:

$$E^Q[\max(S_T - K, 0)|S_t] = E^Q[S_T \mathbf{1}_{\{S_T > K\}}|S_t] - KE^Q[\mathbf{1}_{\{S_T > K\}}|S_t] \tag{93}$$
$$= E^Q[S_T \mathbf{1}_{\{Z > -d_2\}}|S_t] - KP^Q(Z > -d_2) \tag{94}$$

For the first term, we have:

$$E^Q[S_T \mathbf{1}_{\{Z > -d_2\}}|S_t] = E^Q\left[S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma Z\right] \mathbf{1}_{\{Z > -d_2\}}\right] \tag{95}$$
$$= S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t)\right] E^Q\left[\exp\left[\sigma Z\right] \mathbf{1}_{\{Z > -d_2\}}\right] \tag{96}$$

To compute this expectation, we use a change of measure technique. Define a new measure $\tilde{Q}$ by:

$$\frac{d\tilde{Q}}{dQ} = \exp\left[\sigma Z - \frac{1}{2}\sigma^2(T-t)\right] \tag{97}$$

Under $\tilde{Q}$, the random variable Z is normally distributed with mean $\sigma(T-t)$ and variance $T-t$. We can write:

$$E^Q\left[\exp\left[\sigma Z\right] \mathbf{1}_{\{Z > -d_2\}}\right] = E^Q\left[\frac{d\tilde{Q}}{dQ} \exp\left[\frac{1}{2}\sigma^2(T-t)\right] \mathbf{1}_{\{Z > -d_2\}}\right] \tag{98}$$
$$= \exp\left[\frac{1}{2}\sigma^2(T-t)\right] E^Q\left[\frac{d\tilde{Q}}{dQ}\mathbf{1}_{\{Z > -d_2\}}\right] \tag{99}$$
$$= \exp\left[\frac{1}{2}\sigma^2(T-t)\right] E^{\tilde{Q}}\left[\mathbf{1}_{\{Z > -d_2\}}\right] \tag{100}$$
$$= \exp\left[\frac{1}{2}\sigma^2(T-t)\right] P^{\tilde{Q}}(Z > -d_2) \tag{101}$$

Under $\tilde{Q}$, we have $Z = \sigma(T-t) + \tilde{Z}$, where $\tilde{Z}$ is a standard normal random variable. Therefore:

$$P^{\tilde{Q}}(Z > -d_2) = P^{\tilde{Q}}(\sigma(T-t) + \tilde{Z} > -d_2) \tag{102}$$
$$= P^{\tilde{Q}}\left(\tilde{Z} > -d_2 - \sigma(T-t)\right) \tag{103}$$
$$= P^{\tilde{Q}}\left(\tilde{Z} > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} - \sigma(T-t)\right) \tag{104}$$
$$= P^{\tilde{Q}}\left(\tilde{Z} > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \tag{105}$$
$$= P^{\tilde{Q}}\left(\tilde{Z} > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{106}$$
$$= P^{\tilde{Q}}(\tilde{Z} > -d_1) \tag{107}$$
$$= P^{\tilde{Q}}(\tilde{Z} < d_1) \tag{108}$$
$$= N(d_1) \tag{109}$$

Therefore:

$$E^Q[S_T \mathbf{1}_{\{Z>-d_2\}}|S_t] = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t)\right] \exp\left[\frac{1}{2}\sigma^2(T-t)\right] N(d_1) \tag{110}$$

$$= S_t \exp\left[r(T-t)\right] N(d_1) \tag{111}$$

For the second term, we have:

$$P^Q(Z > -d_2) = P^Q\left(Z > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{112}$$

$$= P^Q\left(\frac{Z}{\sqrt{T-t}} > -\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma(T-t)}\right) \tag{113}$$

$$= P^Q\left(\frac{Z}{\sqrt{T-t}} < \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{114}$$

$$= P^Q\left(\frac{Z}{\sqrt{T-t}} < d_2\right) \tag{115}$$

$$= N(d_2) \tag{116}$$

Putting everything together:

$$C(t, S_t) = e^{-r(T-t)} E^Q[\max(S_T - K, 0)|S_t] \tag{117}$$

$$= e^{-r(T-t)}[E^Q[S_T \mathbf{1}_{\{S_T>K\}}|S_t] - KP^Q(S_T > K)] \tag{118}$$

$$= e^{-r(T-t)}[S_t \exp\left[r(T-t)\right] N(d_1) - KN(d_2)] \tag{119}$$

$$= S_t N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{120}$$

This is the Black-Scholes formula for a European call option. The formula for a European put option can be derived similarly, or by using the put-call parity relation:

$$P(t, S_t) = C(t, S_t) - S_t + Ke^{-r(T-t)} \tag{121}$$

3 Option Replication

3.1 Delta Hedging

Delta hedging is a key strategy for replicating options. The delta of an option is the sensitivity of the option price to changes in the underlying asset price:

$$\Delta = \frac{\partial v}{\partial s}(t, S_t) \tag{122}$$

For a European call option, the delta is:

$$\Delta_{\text{call}} = N(d_1) \tag{123}$$

For a European put option, the delta is:

$$\Delta_{\text{put}} = N(d_1) - 1 = -N(-d_1) \tag{124}$$

The delta represents the number of shares of the underlying asset that should be held in the replicating portfolio. The remaining wealth is invested in the risk-free asset.

Let's denote the value of the replicating portfolio at time t as P_t. The portfolio consists of Δ_t shares of the stock and the remaining wealth invested in the risk-free asset:

$$P_t = \Delta_t S_t + \beta_t S_t^0 \tag{125}$$

where β_t is the number of units of the risk-free asset.

For the portfolio to replicate the option, we need $P_t = V_t$ for all $t \in [0, T]$. This gives:

$$\beta_t = \frac{V_t - \Delta_t S_t}{S_t^0} \tag{126}$$

The self-financing condition requires that changes in the portfolio value come only from changes in the asset prices, not from injecting or withdrawing funds:

$$dP_t = \Delta_t dS_t + \beta_t dS_t^0 \tag{127}$$

In a continuous-time setting, the delta is continuously updated, and the portfolio is rebalanced accordingly. However, in practice, rebalancing occurs at discrete time points, which introduces replication error.

3.2 Discrete Replication

In practice, option replication is performed at discrete time points. Let's consider a partition of the time interval $[0, T]$ into N equal subintervals:

$$0 = t_0 < t_1 < \ldots < t_N = T \tag{128}$$

with $t_k = k \cdot \frac{T}{N}$ for $k = 0, 1, \ldots, N$.

At each time point t_k, we compute the delta $\Delta_{t_k} = \frac{\partial v}{\partial s}(t_k, S_{t_k})$ and rebalance the portfolio accordingly. The portfolio value at time t_{k+1} is:

$$P_{t_{k+1}} = \Delta_{t_k} S_{t_{k+1}} + \beta_{t_k} S_{t_{k+1}}^0 \tag{129}$$

The replication error at maturity is the difference between the portfolio value and the option payoff:

$$\epsilon_N = P_T - \Phi(S_T) \tag{130}$$

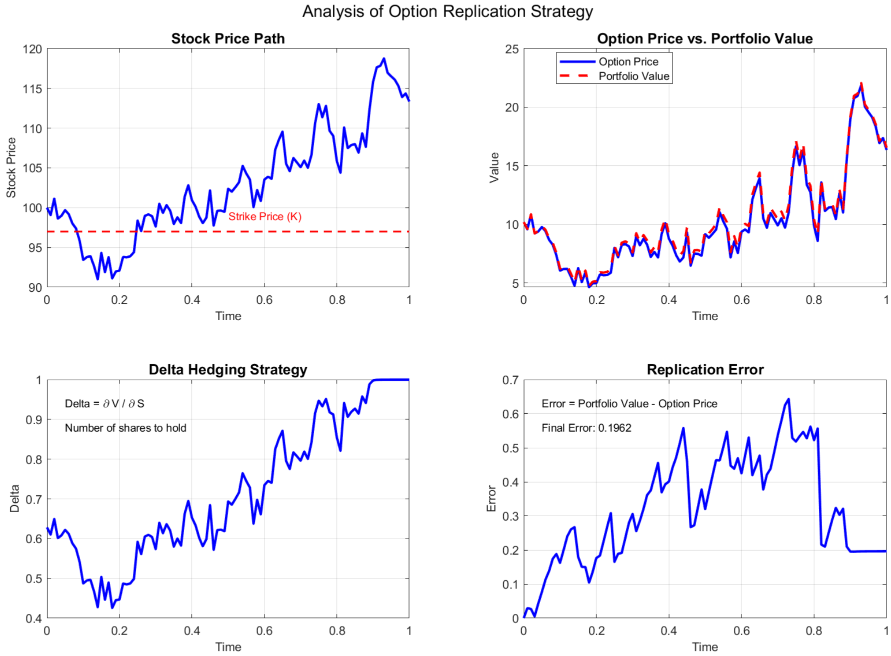

Figure 1: Analysis of Option Replication Strategy. The top-left panel shows the stock price path with the strike price indicated by the horizontal line. The top-right panel compares the option price with the replicating portfolio value. The bottom-left panel shows the delta hedging strategy (number of shares to hold). The bottom-right panel displays the replication error over time, with a final error of 0.1962.

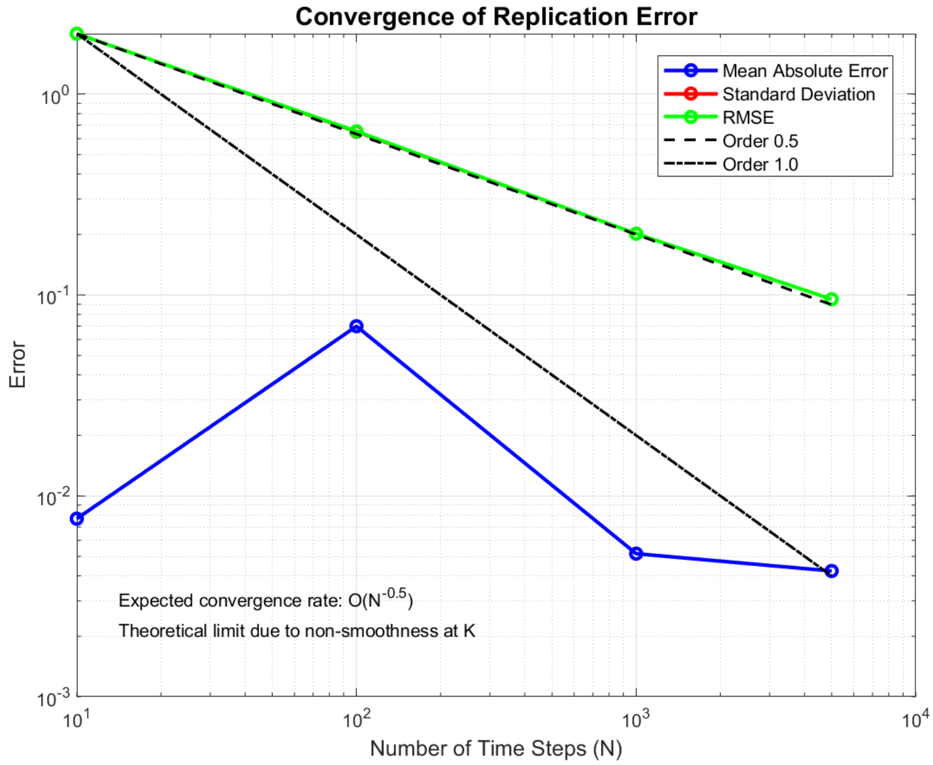

Figure 2: Convergence of Replication Error. The graph shows how the replication error decreases as the number of time steps increases. The standard deviation and RMSE follow a convergence rate of approximately $O(N^{-0.5})$, which is consistent with the theoretical limit due to the non-smoothness of the payoff at the strike price.

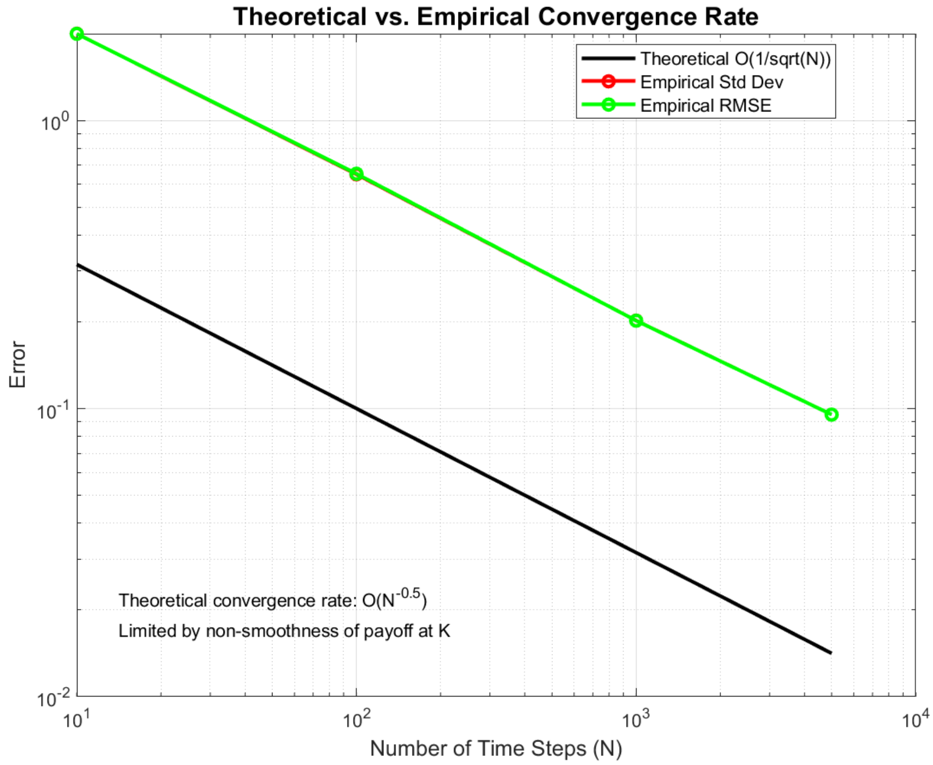

Figure 3: Theoretical vs. Empirical Convergence Rate. The graph compares the theoretical convergence rate of $O(N^{-0.5})$ with the empirically observed convergence rates for the standard deviation and RMSE of the replication error. The empirical rates closely follow the theoretical prediction, confirming that the convergence is limited by the non-smoothness of the payoff at the strike price.

3.3 Distribution of Replication Error

The distribution of the replication error depends on various factors, including the number of rebalancing points, the volatility of the underlying asset, and the moneyness of the option.

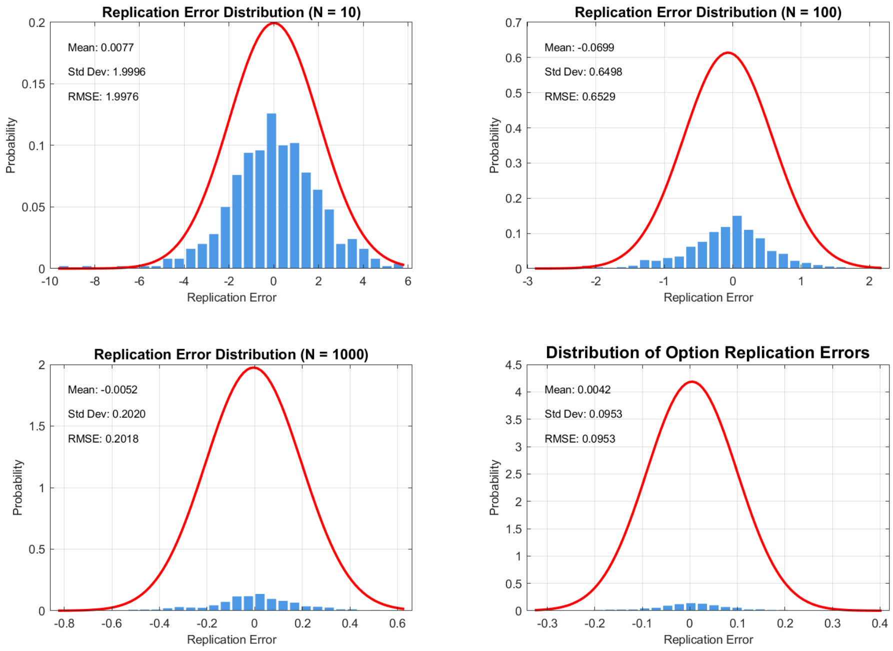

Figure 4: Distribution of Replication Error for different numbers of time steps. As the number of time steps increases from 10 to 1000, the distribution of replication errors becomes narrower, with the standard deviation decreasing from 1.9996 to 0.2020. This confirms the convergence of the replication strategy as the number of rebalancing points increases.

3.4 Impact of Volatility

The volatility of the underlying asset has a significant impact on the replication error. Higher volatility leads to larger price movements between rebalancing points, which increases the replication error.

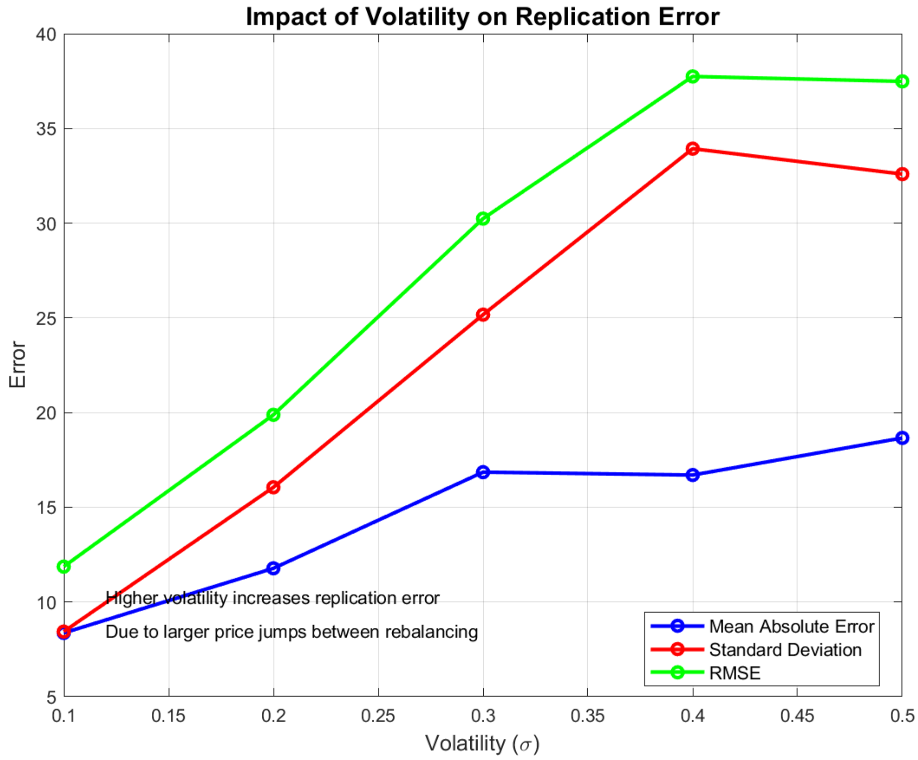

Figure 5: Impact of Volatility on Replication Error. The graph shows how the replication error increases with higher volatility. This is due to larger price jumps between rebalancing points, which makes the delta hedging strategy less effective. The mean absolute error, standard deviation, and RMSE all increase as volatility increases.

3.5 Impact of Moneyness

The moneyness of the option, defined as the ratio of the strike price to the initial stock price (K/S_0), also affects the replication error. Options that are at-the-money (i.e., $K/S_0 \approx 1$) tend to have higher replication errors.

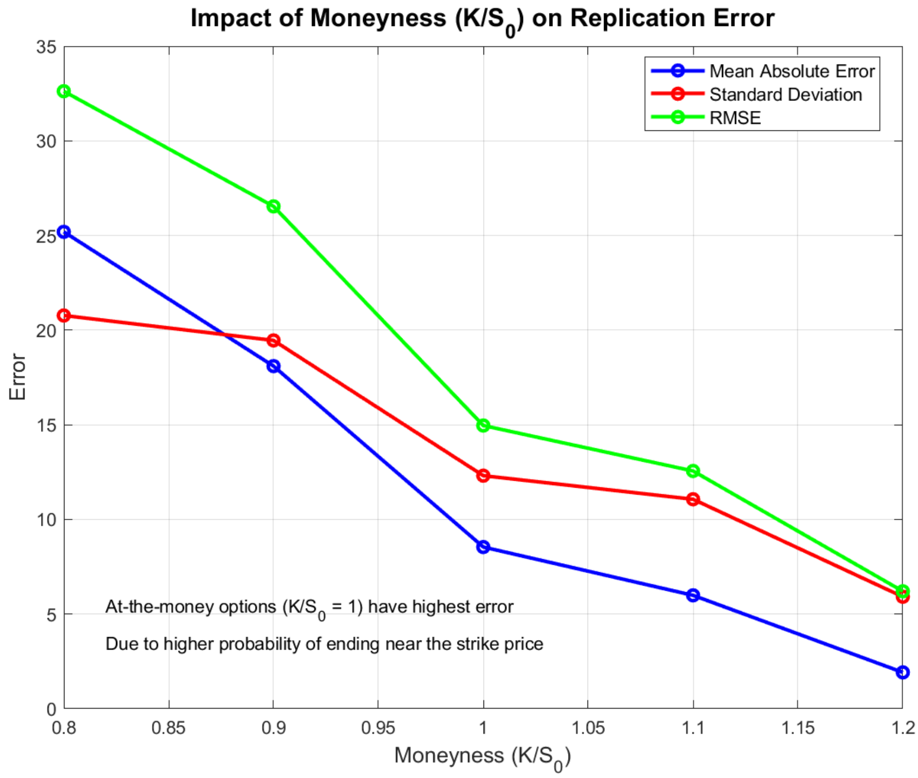

Figure 6: Impact of Moneyness on Replication Error. The graph shows that at-the-money options ($K/S_0 = 1$) have the highest replication error. This is because these options have a higher probability of ending near the strike price, where the payoff function is non-smooth, leading to larger hedging errors.

3.6 Probability of Ending Near the Strike

The probability of the stock price ending near the strike price is a key factor affecting the replication error. When the stock price is close to the strike at maturity, the non-smoothness of the option payoff leads to larger hedging errors.

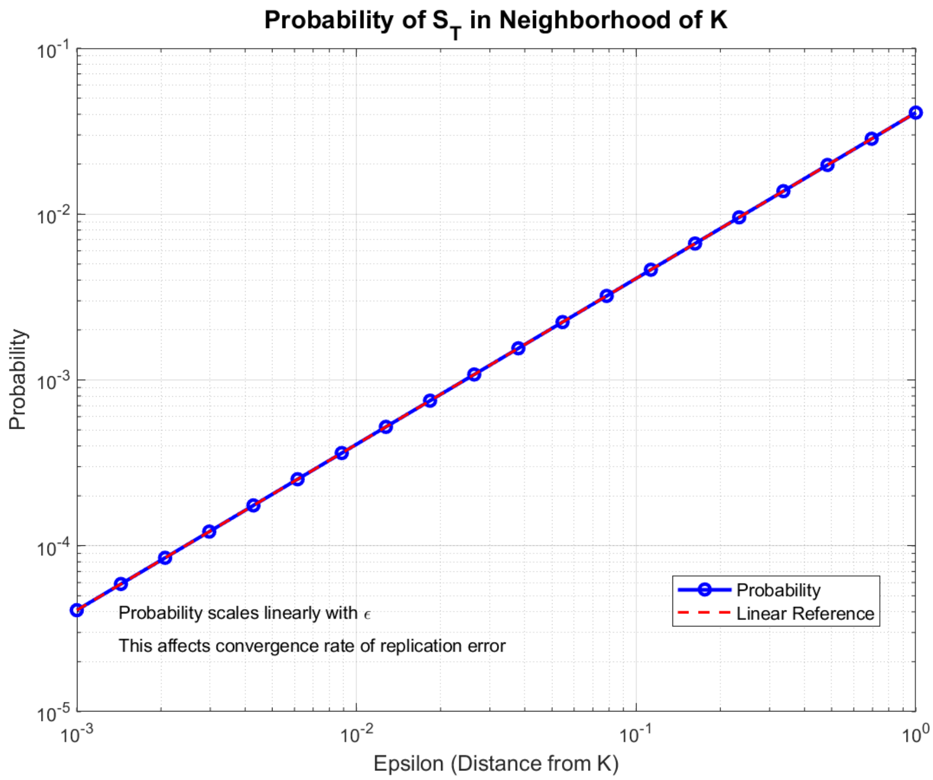

Figure 7: Probability of S_T in Neighborhood of K. The graph shows that the probability of the stock price ending within a distance ϵ of the strike price scales linearly with ϵ. This linear relationship affects the convergence rate of the replication error, limiting it to $O(N^{-0.5})$.

4 Conclusion

In this document, we have provided detailed solutions to problems related to the Black-Scholes model and option replication. We derived the Black-Scholes formula from first principles, explained the concept of risk-neutral pricing, and analyzed the performance of delta hedging as a replication strategy.

Key findings include:

- The replication error converges at a rate of $O(N^{-0.5})$ as the number of rebalancing points increases.
- The convergence rate is limited by the non-smoothness of the option payoff at the strike price.
- Higher volatility leads to larger replication errors.
- At-the-money options have higher replication errors due to the higher probability of ending near the strike price.

These results provide insights into the practical implementation of option replication strategies and the factors that affect their performance.

5 Appendix: Numerical Methods

5.1 Monte Carlo Simulation

Monte Carlo simulation is a powerful technique for pricing options and analyzing replication strategies. The basic idea is to generate multiple sample paths of the underlying asset price and compute the average payoff or replication error.

For option pricing, we generate M sample paths of the stock price under the risk-neutral measure:

$$S_T^{(i)} = S_0 \exp\left[\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z_i\right] \tag{131}$$

where $Z_i \sim \mathcal{N}(0,1)$ are independent standard normal random variables.

The Monte Carlo estimate of the option price is:

$$\hat{V}_0 = e^{-rT}\frac{1}{M}\sum_{i=1}^{M}\Phi(S_T^{(i)}) \tag{132}$$

For analyzing replication strategies, we simulate the stock price path and the replicating portfolio at discrete time points, and compute the replication error at maturity.

5.2 Finite Difference Methods

Finite difference methods are numerical techniques for solving the Black-Scholes PDE. The basic idea is to discretize the PDE in both time and space, and solve the resulting system of equations.

Let's define a grid in time and space:

$$t_j = j \cdot \Delta t \quad \text{for } j = 0, 1, \ldots, N_t \tag{133}$$

$$s_i = i \cdot \Delta s \quad \text{for } i = 0, 1, \ldots, N_s \tag{134}$$

where $\Delta t = \frac{T}{N_t}$ and Δs is the step size in the stock price dimension.

We denote the approximation of $v(t_j, s_i)$ as $v_{i,j}$. The Black-Scholes PDE can be discretized using finite differences:

$$\frac{v_{i,j+1} - v_{i,j}}{\Delta t} + rs_i\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta s} + \frac{1}{2}\sigma^2 s_i^2\frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{(\Delta s)^2} - rv_{i,j} = 0 \tag{135}$$

This can be rearranged to express $v_{i,j}$ in terms of $v_{i,j+1}$, $v_{i+1,j}$, and $v_{i-1,j}$. Starting from the terminal condition $v_{i,N_t} = \Phi(s_i)$, we can solve backwards in time to find the option price at earlier times.

5.3 Probability of Ending Near the Strike

The probability of the stock price ending within a small neighborhood of the strike price is an important factor affecting the replication error. For a log-normal process, this probability can be computed analytically.

Let's define the event $A_\epsilon = \{|S_T - K| < \epsilon\}$, which represents the stock price ending within a distance ϵ of the strike price. We want to compute $P(A_\epsilon)$.

Under the risk-neutral measure, $\ln(S_T)$ is normally distributed with mean $\ln(S_0) + (r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Therefore:

$$P(A_\epsilon) = P(K - \epsilon < S_T < K + \epsilon) \tag{136}$$
$$= P\left(\ln(K - \epsilon) < \ln(S_T) < \ln(K + \epsilon)\right) \tag{137}$$
$$= P\left(\frac{\ln(K - \epsilon) - \ln(S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} < Z < \frac{\ln(K + \epsilon) - \ln(S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{138}$$

where $Z \sim \mathcal{N}(0, 1)$ is a standard normal random variable.

For small ϵ, we can use the approximation $\ln(K \pm \epsilon) \approx \ln(K) \pm \frac{\epsilon}{K}$. This gives:

$$P(A_\epsilon) \approx P\left(\frac{\ln(K) - \frac{\epsilon}{K} - \ln(S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} < Z < \frac{\ln(K) + \frac{\epsilon}{K} - \ln(S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{139}$$
$$= P\left(d_2 - \frac{\epsilon}{K\sigma\sqrt{T}} < Z < d_2 + \frac{\epsilon}{K\sigma\sqrt{T}}\right) \tag{140}$$

Using the approximation of the standard normal density around d_2:

$$P(A_\epsilon) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \cdot \frac{2\epsilon}{K\sigma\sqrt{T}} \tag{141}$$
$$= \frac{2\epsilon}{K\sigma\sqrt{T}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \tag{142}$$

This shows that the probability scales linearly with ϵ for small ϵ, with a proportionality constant that depends on the option parameters.

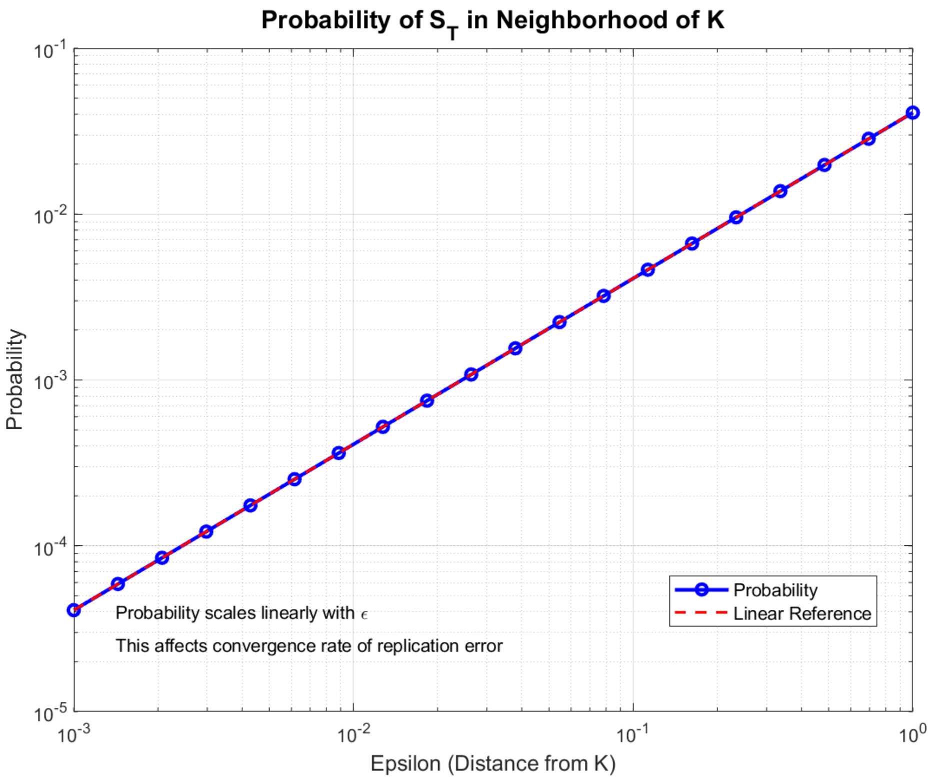

Figure 8: Probability of S_T in Neighborhood of K. The graph confirms the linear relationship between the probability and the distance ϵ from the strike price. This linear scaling is a key factor limiting the convergence rate of the replication error to $O(N^{-0.5})$.