Master of Quantitative Finance

Discretization of Processes (EXAM)

Authors: Chaker Meraihi, Luca Albani, Andrea Taviani
Degree Program: Master's in Quantitative Finance & ENSIIE Applied Mathematics
Professor: Stéphane Menozzi
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Contents

1 Exercise: Balance of (non asymptotic) statistical and weak error

Let X be a diffusion process with dynamics:

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{SDE}$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the coefficients b and σ are smooth and bounded.

We consider for a discrete time step $h = T/N$ (for a given fixed finite time horizon T and $N \in \mathbb{N}$) the associated Euler scheme for $t \in [0, T]$:

$$X_t^h = x + \int_0^t b(X_{\phi(s)}^h)ds + \int_0^t \sigma(X_{\phi(s)}^h)dW_s, \tag{SDE$_h$}$$

where setting $t_i := ih, i \in \mathbb{N}, \phi(s) = t_i, s \in [t_i, t_{i+1}]$.

We assume that $X_T^{h,i}$ satisfies the property:

$$\forall \lambda > 0,\ \mathbb{E}[\exp(\lambda(f(X_T^h) - \mathbb{E}[f(X_T^h)]))] \leq \exp\left(T\frac{\lambda^2}{2}\right), \tag{GC($2T$)}$$

where f is a 1-Lipschitz function.

1.1

Prove that for a globally Lipschitz function f, for all $r > 0$

$$\mathbb{P}[|f(X_T^h) - \mathbb{E}[f(X_T^h)]| > r] \leq 2\exp\left(-\frac{r^2}{2T[f]_1^2}\right),$$

where $[f]_1 := \sup_{y\neq y'} \frac{|f(y)-f(y')|}{|y-y'|}$ stands for the Lipschitz modulus.

Answer

$$\mathbb{P}\left(\left|f(X_t^N) - \mathbb{E}[f(X_t^N)]\right| > r\right)$$

We first remove the absolute value and write:

$$\mathbb{P}\left(f(X_t^N) - \mathbb{E}[f(X_t^N)] > r\right) = \mathbb{P}\left(\lambda(f(X_t^N) - \mathbb{E}[f(X_t^N)]) > \lambda r\right)$$

$$= \mathbb{P}\left(e^{\lambda(f(X_t^N) - \mathbb{E}[f(X_t^N)])} > e^{\lambda r}\right)$$

Now we apply Markov's inequality:

$$\leq \frac{\mathbb{E}\left[e^{\lambda(f(X_t^N) - \mathbb{E}[f(X_t^N)])}\right]}{e^{\lambda r}}$$

Since f is globally Lipschitz with constant $\|f'\|_1$ and X_t^N is Gaussian, we have:

$$\mathbb{E}\left[e^{\lambda(f(X_t^N) - \mathbb{E}[f(X_t^N)])}\right] \leq e^{\frac{T\lambda^2\|f'\|_1^2}{2}}$$

Thus:

$$\mathbb{P}\left(f(X_t^N) - \mathbb{E}[f(X_t^N)] > r\right) \leq \frac{e^{\frac{T\lambda^2\|f'\|_1^2}{2}}}{e^{\lambda r}} = e^{\frac{T\lambda^2\|f'\|_1^2}{2} - \lambda r}$$

Now minimize with respect to λ:

$$\frac{d}{d\lambda}\left(\frac{T\lambda^2\|f'\|_1^2}{2} - \lambda r\right) = T\lambda\|f'\|_1^2 - r = 0 \Rightarrow \lambda = \frac{r}{T\|f'\|_1^2}$$

Substituting this value of λ:

$$\exp\left(\frac{T}{2}\cdot\left(\frac{r}{T\|f'\|_1^2}\right)^2\cdot\|f'\|_1^2 - \frac{r^2}{T\|f'\|_1^2}\right) = \exp\left(\frac{r^2}{2T\|f'\|_1^2} - \frac{r^2}{T\|f'\|_1^2}\right) = \exp\left(-\frac{r^2}{2T\|f'\|_1^2}\right)$$

Answer

This proves the concentration inequality:

$$\mathbb{P}\left(\left|f(X_t^N) - \mathbb{E}[f(X_t^N)]\right| > r\right) \leq 2\exp\left(-\frac{r^2}{2T\|f'\|_1^2}\right)$$

1.2

Prove as well that, if $(X_T^{h,i})_{i\in[1,M]}$, $M \in \mathbb{N}$ are independent i.d. copies of X_T^h, then

$$\mathbb{P}\left[\left|\frac{1}{M}\sum_{i=1}^{M} f(X_T^{h,i}) - \mathbb{E}[f(X_T^h)]\right| > r\right] \leq 2\exp\left(-\frac{r^2 M}{2T[f]_1^2}\right).$$

Answer

$$P\left(\left|\frac{1}{M}\sum_{i=1}^{M} f\left(X_T^{h,i}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right| > r\right)$$

Similar to before:

$$P\left(e^{\lambda\left(\frac{1}{M}\sum_{i=1}^{M} f\left(X_T^{h,i}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right)} \geq e^{\lambda r}\right)$$

Applying Markov's inequality:

$$\leq e^{-\lambda r}\mathbb{E}\left(e^{\lambda\left(\frac{1}{M}\sum_{i=1}^{M} f\left(X_T^{h,i}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right)}\right)$$

Since the random variables are independent:

$$\leq e^{-\lambda r}\prod_{i=1}^{M}\mathbb{E}\left(e^{\frac{\lambda}{M}\left(f\left(X_T^{h,i}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right)}\right)$$

For IID samples:

$$= e^{-\lambda r}\left(\mathbb{E}\left(e^{\frac{\lambda}{M}\left(f\left(X_T^{h}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right)}\right)\right)^M$$

For Gaussian Concentration (GC):

$$\leq e^{-\lambda r}\left(e^{\frac{\lambda^2 T[f]_1^2}{2M^2}}\right)^M$$

$$= e^{-\lambda r}e^{\frac{\lambda^2 T[f]_1^2}{2M}}$$

Taking derivative with respect to λ to find the minimum:

$$-r + \frac{\lambda T[f]_1^2}{M} = 0$$

$$\lambda = \frac{rM}{T[f]_1^2}$$

Substitute back:

$$\leq e^{-\frac{r^2 M}{2T[f]_1^2}}$$

Thus, we arrive at the concentration inequality:

$$P\left(\left|\frac{1}{M}\sum_{i=1}^{M} f\left(X_T^{h,i}\right) - \mathbb{E}\left(f\left(X_T^h\right)\right)\right| > r\right) \leq 2e^{-\frac{r^2 M}{2T[f]_1^2}}$$

1.3

Deduce the expression of a non asymptotic confidence interval at level $\alpha > 0$, i.e. find an interval of the form $I(C_\alpha) := \left[\mathbb{E}[f(X_T^h)] - \frac{C_\alpha}{\sqrt{M}}, \mathbb{E}[f(X_T^h)] + \frac{C_\alpha}{\sqrt{M}}\right]$ s.t. for all $M \in \mathbb{N}$, $\mathbb{P}\left[\frac{1}{M}\sum_{i=1}^{M} f(X_T^{h,i}) \in I(C_\alpha)\right] \geq \alpha$.

Answer

$$\mathbb{P}\left(\left|\frac{1}{M}\sum_{m=1}^{M} f(X_T^{h,m}) - \mathbb{E}[f(X_T^h)]\right| > r\right) \leq 2\exp\left(-\frac{r^2 M}{2T[f]_1^2}\right).$$

Then,

$$\mathbb{P}\left(\left|\frac{1}{M}\sum_{m=1}^{M} f(X_T^{h,m}) - \mathbb{E}[f(X_T^h)]\right| < r\right) \geq 1 - 2\exp\left(-\frac{r^2 M}{2T[f]_1^2}\right).$$

We want this probability to be at least $\alpha \in (0,1)$, so we impose:

$$1 - 2\exp\left(-\frac{r^2 M}{2T[f]_1^2}\right) \geq \alpha.$$

Solving for r, we get:

$$2\exp\left(-\frac{r^2 M}{2T[f]_1^2}\right) \leq 1-\alpha \quad \Rightarrow \quad \exp\left(-\frac{r^2 M}{2T[f]_1^2}\right) \leq \frac{1-\alpha}{2}$$

$$-\frac{r^2 M}{2T[f]_1^2} \leq \log\left(\frac{1-\alpha}{2}\right) \quad \Rightarrow \quad r^2 \geq -2T[f]_1^2 \cdot \frac{1}{M}\log\left(\frac{1-\alpha}{2}\right).$$

Therefore, we define:

$$C_\alpha = \sqrt{-2T[f]_1^2 \log\left(\frac{1-\alpha}{2}\right)}, \quad \text{so that} \quad \frac{C_\alpha}{\sqrt{M}} = \sqrt{\frac{2T[f]_1^2}{M}\log\left(\frac{2}{1-\alpha}\right)}.$$

For any $M \in \mathbb{N}$, we have

$$\mathbb{P}\left(\left|\frac{1}{M}\sum_{m=1}^{M} f(X_T^{h,m}) - \mathbb{E}[f(X_T^h)]\right| \leq \frac{C_\alpha}{\sqrt{M}}\right) \geq \alpha,$$

where

$$C_\alpha = \sqrt{2T[f]_1^2 \log\left(\frac{2}{1-\alpha}\right)}.$$

1.4

We consider now the mapping $(t,x) \in [0,T] \mapsto v(t,x) = \mathbb{E}[f(X_T^{t;x})] := \mathbb{E}[f(X_T)|X_t = x]$. Prove that

$$\mathbb{E}_x[f(X_T^h) - f(X_T)] = \mathbb{E}_x[v(T, X_T^h)] - v(0,x) = \sum_{i=0}^{N-1} \mathbb{E}_x[v(t_{i+1}, X_{t_{i+1}}^h) - v(t_i, X_{t_i}^h)],$$

where we denoted with a slight abuse of notation $\mathbb{E}_x[\cdot] = \mathbb{E}[X_0 = x] = \mathbb{E}[X_0^h = x]$ (namely, the subscript stands for initial point of the diffusion and its Euler scheme).

Assuming that $v \in C^{2,4}([0,T] \times \mathbb{R}^d)$ and that the support of $\mathcal{L}(X_t)$ is $\mathbb{R}^d$ for $t > 0$, prove that v satisfies the PDE

$$\begin{cases} \partial_t v(t,x) + b(x) \cdot \nabla v(t,x) + \frac{1}{2} \mathrm{Tr}\left(\sigma\sigma^*(x) D_x^2 v(t,x)\right) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = f(x). \end{cases} \tag{PDE}$$

Prove then from (D), using Itô's formula, (PDE) and the smoothness of the coefficients that:

$$\mathbb{E}_x[f(X_T^h) - f(X_T)] = \mathbb{E}_x[v(T, X_T^h)] - v(0,x) = O(h). \tag{D}$$

Answer

PART 1) The first thing to prove is the only application of the definition and using the telescopic sum.
PART 2) The second thing to prove: Fix $s \in [t,T]$ and apply Itô's formula to the function $(u,y) \mapsto v(u,y)$, which is assumed to be $\mathcal{C}^{1,2}$, along the diffusion path $u \mapsto X_u$:

$$\begin{aligned} \mathrm{d}v(u, X_u) = {} & \partial_t v(u, X_u)\,\mathrm{d}u + \nabla v(u, X_u)^\top b(X_u)\,\mathrm{d}u + \tfrac{1}{2}\,\mathrm{Tr}\big[\sigma\sigma^\top(X_u) D^2 v(u, X_u)\big]\,\mathrm{d}u \\ & + \nabla v(u, X_u)^\top \sigma(X_u)\,\mathrm{d}W_u. \end{aligned}$$

Integrate both sides from t to T and take the conditional expectation with respect to $\mathcal{F}_t$ (using the fact that all terms are integrable under our assumptions):

$$\mathbb{E}\big[v(T, X_T) - v(t, X_t) \mid \mathcal{F}_t\big] = \mathbb{E}\left[\int_t^T \left(\partial_t v + b \cdot \nabla v + \tfrac{1}{2}\,\mathrm{Tr}[\sigma\sigma^\top D^2 v]\right)(u, X_u)\,\mathrm{d}u \,\middle|\, \mathcal{F}_t\right],$$

where the stochastic integral vanishes because it is a martingale with zero conditional expectation.
Since $v(T, X_T) = f(X_T)$ and $v(t, X_t)$ is $\mathcal{F}_t$-measurable, the left-hand side equals

$$\mathbb{E}[f(X_T) \mid \mathcal{F}_t] - v(t, X_t).$$

Hence, the right-hand side must vanish:

$$\int_t^T \left(\partial_t v + b \cdot \nabla v + \tfrac{1}{2}\,\mathrm{Tr}[\sigma\sigma^\top D^2 v]\right)(u, X_u)\,\mathrm{d}u = 0 \quad \text{a.s.}$$

Because the support of X_u is $\mathbb{R}^d$ and the integrand is continuous, it must vanish pointwise. Thus, v satisfies the backward Kolmogorov PDE:

$$\boxed{\partial_t v(t,x) + b(x) \cdot \nabla v(t,x) + \tfrac{1}{2}\,\mathrm{Tr}[\sigma(x)\sigma(x)^\top D^2 v(t,x)] = 0.}$$

This completes the proof.
PART 3)
Let $h = T/N$ and $t_i := ih$ $(i = 0, \ldots, N)$. Denote by $X_{t_i}^h$ the Euler–Maruyama approximation and set $Y_i := X_{t_i}^h$. Because $v(T,\cdot) = f(\cdot)$, write

$$\begin{aligned} \mathbb{E}_x\big[f(X_T^h) - f(X_T)\big] &= \mathbb{E}_x\big[v(T, Y_N) - v(0,x)\big] \\ &= \sum_{i=0}^{N-1} \mathbb{E}_x\big[v(T, Y_N) - v(T, Y_{N-1})\big] \\ &\quad + \cdots + \mathbb{E}_x\big[v(t_{i+1}, Y_N) - v(t_i, Y_N)\big] + \cdots + \mathbb{E}_x\big[v(t_1, Y_N) - v(0,x)\big]. \end{aligned}$$

Collecting terms yields the telescopic sum

$$\mathbb{E}_x\big[f(X_T^h) - f(X_T)\big] = \sum_{i=0}^{N-1} \mathbb{E}_x\big[v(t_{i+1}, Y_N) - v(t_i, Y_N)\big]. \tag{*}$$

Answer

Condition on $\mathcal{F}_{t_i}$. The Euler scheme is Markov, hence

$$\mathbb{E}\big[v(t_{i+1}, Y_N) \mid \mathcal{F}_{t_i}\big] = v\big(t_{i+1}, Y_i\big), \quad \mathbb{E}\big[v(t_i, Y_N) \mid \mathcal{F}_{t_i}\big] = v\big(t_i, Y_i\big).$$

Taking overall expectations converts (*) into

$$\boxed{\mathbb{E}_x\big[f(X_T^h) - f(X_T)\big] = \sum_{i=0}^{N-1} \mathbb{E}_x\big[v(t_{i+1}, X_{t_i}^h) - v(t_i, X_{t_i}^h)\big]} \tag{D}$$

which is exactly the identity required.
We now establish rigorously that

$$\big|\mathbb{E}_x[f(X_T^h)] - \mathbb{E}_x[f(X_T)]\big| \le C\,h,$$

for some constant C depending only on T, f and global bounds on the coefficients and derivatives of v.
Fix i and set $\xi := Y_i = X_{t_i}^h$. Since $v \in \mathcal{C}^{1,2}$ and the coefficients are bounded, we have $\sup_{t \le T} \|D^2 v(t,\cdot)\| < \infty$.
The Euler–Maruyama step reads:

$$Y_{i+1} = \xi + b(\xi)h + \sigma(\xi)\,\Delta W_i, \qquad \Delta W_i := W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, hI_d).$$

We apply a second-order Taylor expansion of the function $(t,x) \mapsto v(t,x)$ around the point (t_i, ξ):

$$\begin{aligned} v(t_{i+1}, Y_{i+1}) = v(t_i,\xi) &+ \partial_t v(t_i,\xi)\,h + \nabla v(t_i,\xi)^\top [b(\xi)h + \sigma(\xi)\Delta W_i] \\ &+ \tfrac{1}{2}[b(\xi)h + \sigma(\xi)\Delta W_i]^\top D^2 v(t_i,\xi)[b(\xi)h + \sigma(\xi)\Delta W_i] \\ &+ \tfrac{1}{2}\partial_{tt} v(t_i,\xi)\,h^2 + \text{higher order terms (in } h^{3/2} \text{ and above).} \end{aligned}$$

Step 2 – Conditional expectation given $\mathcal{F}_{t_i}$. Take the conditional expectation given $\mathcal{F}_{t_i}$, using:

$$\mathbb{E}[\Delta W_i \mid \mathcal{F}_{t_i}] = 0, \qquad \mathbb{E}[\Delta W_i \Delta W_i^\top \mid \mathcal{F}_{t_i}] = hI_d.$$

Then:

$$\begin{aligned} \mathbb{E}[v(t_{i+1}, Y_{i+1}) \mid \mathcal{F}_{t_i}] = v(t_i,\xi) &+ h\big(\partial_t v + b\cdot\nabla v + \tfrac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2 v]\big)(t_i,\xi) \\ &+ \tfrac{1}{2}h^2\,\partial_{tt} v(t_i,\xi) + \tfrac{1}{2}h^2\, b(\xi)^\top D^2 v(t_i,\xi)\, b(\xi) + \mathcal{O}(h^{3/2}). \end{aligned}$$

The term in brackets vanishes due to the PDE, so:

$$\mathbb{E}\big[v(t_{i+1}, Y_{i+1}) - v(t_i,\xi) \mid \mathcal{F}_{t_i}\big] = \mathcal{O}(h^2).$$

Step 3 – Uniform bound on the local bias. There is a constant K (depending on $\sup\|\partial_{tt} v\|$ and $\sup\|D^2 v\|$) such that

$$\big|\mathbb{E}\big[v(t_{i+1}, Y_{i+1}) - v(t_i,\xi) \mid \mathcal{F}_{t_i}\big]\big| \le Kh^2.$$

Taking expectation gives

$$\big|\mathbb{E}_x\big[v(t_{i+1}, Y_{i+1}) - v(t_i, Y_i)\big]\big| \le Kh^2. \tag{local}$$

Step 4 – Sum over all grid points. Summing the estimate (local) over $i = 0, \dots, N-1$ and recalling $N = T/h$:

$$\big|\mathbb{E}_x[f(X_T^h)] - \mathbb{E}_x[f(X_T)]\big| \le \sum_{i=0}^{N-1} Kh^2 = KNh^2 = KTh.$$

Set $C := KT$ to conclude

$$\boxed{\big|\mathbb{E}_x[f(X_T^h)] - \mathbb{E}_x[f(X_T)]\big| \le Ch, \qquad h \to 0} \tag{$\star$}$$

which proves the weak–error order claimed in the assignment.

1.5

The complete error, i.e. the difference between the computable quantity and the sought one writes:

$$\mathcal{E}(T, f, x, h, M) := \frac{1}{M}\sum_{i=1}^{M} f(X_T^{h,i}) - \mathbb{E}_x[f(X_T)], \quad X_0^{h,i} = x, \quad \forall i \in [1, M].$$

For a given admissible error threshold $\varepsilon > 0$ calibrate h and M such that for a probability greater than 95% it holds that:

$$\mathcal{E}(T, f, x, h, M) = O(\varepsilon).$$

Answer

Define the total approximation error as:

$$\mathcal{E}(T, f, x, h, M) := \frac{1}{M}\sum_{m=1}^{M} f(X_T^{h,m}) - \mathbb{E}_x[f(X_T)],$$

where $(X_T^{h,m})_{m=1}^M$ are i.i.d. samples of the Euler–Maruyama scheme at time T.
We decompose the error into two components:

$$\mathcal{E} = \underbrace{\left(\frac{1}{M}\sum_{m=1}^{M} f(X_T^{h,m}) - \mathbb{E}_x[f(X_T^h)]\right)}_{S_M \text{ (statistical error)}} + \underbrace{\left(\mathbb{E}_x[f(X_T^h)] - \mathbb{E}_x[f(X_T)]\right)}_{B(h) \text{ (bias)}}.$$

From Point 2, we have the sub-Gaussian concentration bound:

$$\mathbb{P}(|S_M| > r) \le 2\exp\left(-\frac{Mr^2}{2T[f]_1^2}\right), \quad \forall r > 0.$$

Fix a confidence level $1 - \rho$ (e.g. $\rho = 0.05$ for 95% confidence). Then the bound

$$r_\rho := \sqrt{\frac{2T[f]_1^2}{M}\log\left(\frac{2}{\rho}\right)} \quad \Rightarrow \quad \mathbb{P}(|S_M| \le r_\rho) \ge 1 - \rho.$$

From Point 4.c, we have that the weak error of the Euler scheme is first order:

$$|B(h)| = |\mathbb{E}_x[f(X_T^h)] - \mathbb{E}_x[f(X_T)]| \le C_b h,$$

for some constant C_b depending on f and the regularity of the Kolmogorov solution v.
We aim to control the total error with high probability:

$$|\mathcal{E}(T, f, x, h, M)| \le \varepsilon, \quad \text{with probability at least } 1 - \rho.$$

A sufficient condition is to allocate half the tolerance to each component:

$$C_b h \le \frac{\varepsilon}{2}, \qquad r_\rho \le \frac{\varepsilon}{2}.$$

Solving both constraints gives:

$$\boxed{\begin{aligned} & h \le \frac{\varepsilon}{2C_b}, \\ & M \ge \frac{8T[f]_1^2}{\varepsilon^2}\log\left(\frac{2}{\rho}\right). \end{aligned}}$$

Answer

Numerical example (95% confidence). Take $\rho = 0.05$, so that $\log(2/\rho) = \log(40) \approx 3.69$. Then the choices

$$h = \frac{\varepsilon}{2C_b}, \qquad M = \left\lceil \frac{8 \cdot T[f]_1^2 \cdot 3.69}{\varepsilon^2} \right\rceil$$

ensure that

$$\mathbb{P}(|\mathcal{E}(T, f, x, h, M)| \le \varepsilon) \ge 0.95.$$

Hence,

$$\boxed{\mathcal{E}(T, f, x, h, M) = \mathcal{O}(\varepsilon) \quad \text{with probability at least 95\%.}}$$

A different split of the tolerance (e.g., $\theta\varepsilon$ for the bias and $(1-\theta)\varepsilon$ for the statistical error, with $\theta \in (0, 1)$) would yield:

$$h \le \frac{\theta\varepsilon}{C_b}, \qquad M \ge \frac{2T[f]_1^2}{(1-\theta)^2\varepsilon^2} \log\left(\frac{2}{\rho}\right).$$

Regardless of the value of θ, the complexity remains:

$$M = \mathcal{O}(\varepsilon^{-2}), \qquad h = \mathcal{O}(\varepsilon).$$

2 Exercise: about the discrete hedging problem in a Black and Scholes setting

We consider here the framework of the homework. Namely, a riskless asset and a risky asset with Black and Scholes dynamics:

$$dS_t^0 = rS_t^0 dt, \tag{B}$$

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{B\&S}$$

where μ is the market trend of the asset, σ the volatility, and W stands for a standard Brownian motion, associated with a filtration $(F_t)_{t\geq 0}$, under the market probability $\mathbb{P}$.

We consider the following discrete dynamics for the portfolio:

$$\begin{aligned} V_{t_{i+1}}^h - V_{t_i}^h &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta_{t_i}^0(S_{t_{i+1}}^0 - S_{t_i}^0) \\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta_{t_i}^0 S_{t_i}^0(e^{rh} - 1) \\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i}^h - \delta_{t_i} S_{t_i})(e^{rh} - 1), \end{aligned} \tag{I_h}$$

where $\delta_{t_i} = \partial_x v(t_i, S_{t_i})$ and v solves the PDE:

$$\begin{cases} \partial_t v(t,x) + rx\partial_x v(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx}^2 v(t,x) - rv(t,x) = 0, & (t,x) \in [0,T) \times \mathbb{R}_+^*, \\ v(T,x) = \Phi(x), & x \in \mathbb{R}_+^*, \end{cases} \tag{PDE$_{BS}$}$$

where in the following we will assume that Φ is globally Lipschitz so that $v \in C^{1,2}([0,T) \times \mathbb{R}_+^*, \mathbb{R}) \cap C^0([0,T] \times \mathbb{R}_+^*, \mathbb{R})$, with $\mathbb{R}_+^* := (0, +\infty)$.

2.1

Give the expression of $\delta_{t_i}^0$, quantity of riskless asset at time t_i in the portfolio V^h rebalanced in discrete time, recalling that the portfolio is self-financing.

Answer

To solve this first point we derive the expression for $\delta_{t_i}^0$, the quantity of the riskless asset in the self-financing portfolio. Starting from the portfolio value at time t_i:

$$V_{t_i}^h = \delta_{t_i} S_{t_i} + \delta_{t_i}^0 S_{t_i}^0$$

Solving for $\delta_{t_i}^0$ gives:

$$\delta_{t_i}^0 = \frac{V_{t_i}^h - \delta_{t_i} S_{t_i}}{S_{t_i}^0}$$

We verify this using the discrete portfolio dynamics:

$$\begin{aligned} V_{t_{i+1}}^h - V_{t_i}^h &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \delta_{t_i}^0(S_{t_{i+1}}^0 - S_{t_i}^0) \\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \left(\frac{V_{t_i}^h - \delta_{t_i} S_{t_i}}{S_{t_i}^0}\right)(S_{t_{i+1}}^0 - S_{t_i}^0) \end{aligned}$$

The riskless asset evolves as:

$$S_{t_{i+1}}^0 = S_{t_i}^0 e^{rh} \quad \Rightarrow \quad S_{t_{i+1}}^0 - S_{t_i}^0 = S_{t_i}^0(e^{rh} - 1)$$

Substituting back:

$$\begin{aligned} V_{t_{i+1}}^h - V_{t_i}^h &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + \left(\frac{V_{t_i}^h - \delta_{t_i} S_{t_i}}{S_{t_i}^0}\right) S_{t_i}^0(e^{rh} - 1) \\ &= \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i}^h - \delta_{t_i} S_{t_i})(e^{rh} - 1) \end{aligned}$$

This matches the given discrete dynamics in equation (I_h). Therefore, the correct expression for the riskless asset quantity is:

$$\delta_{t_i}^0 = \frac{V_{t_i}^h - \partial_x v(t_i, S_{t_i}) S_{t_i}}{S_{t_i}^0}$$

where $\delta_{t_i} = \partial_x v(t_i, S_{t_i})$ is the delta hedge ratio from the Black-Scholes PDE solution.

2.2

Prove that for $i \in [0, N-2]$,

$$v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}) = \partial_y v(t_i, S_{t_i})(S_{t_{i+1}} - S_{t_i}) + (v(t_i, S_{t_i}) - S_{t_i}\partial_y v(t_i, S_{t_i}))(\exp(rh) - 1) + R_i^h, \tag{1}$$

for a contribution R_i^h to be specified.

Answer

We can prove the discrete evolution of the option value through Taylor expansion and application of the Black-Scholes PDE. For $i \in [0, N-2]$, consider the change in option value:

$$v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}) = \underbrace{v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_{i+1}})}_{\text{Time evolution}} + \underbrace{v(t_i, S_{t_{i+1}}) - v(t_i, S_{t_i})}_{\text{Price change}}$$

Applying Taylor expansions to each component:
For the time evolution ($\Delta t = h$):

$$v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_{i+1}}) = \partial_t v(t_i, S_{t_{i+1}})h + \frac{1}{2}\partial_{tt}^2 v(t_i, S_{t_{i+1}})h^2 + \mathcal{O}(h^3)$$

For the price change ($\Delta S = S_{t_{i+1}} - S_{t_i}$):

$$v(t_i, S_{t_{i+1}}) - v(t_i, S_{t_i}) = \partial_y v(t_i, S_{t_i})\Delta S + \frac{1}{2}\partial_{yy}^2 v(t_i, S_{t_i})(\Delta S)^2 + \mathcal{O}((\Delta S)^3)$$

From the Black-Scholes PDE, we substitute for $\partial_t v$:

$$\partial_t v = -rS\partial_y v - \frac{1}{2}\sigma^2 S^2 \partial_{yy}^2 v + rv$$

Combining these and rearranging terms:

$$v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}) = [-rS_{t_{i+1}}\partial_y v(t_i, S_{t_{i+1}}) - \frac{1}{2}\sigma^2 S_{t_{i+1}}^2 \partial_{yy}^2 v(t_i, S_{t_{i+1}}) + rv(t_i, S_{t_{i+1}})]h + \partial_y v(t_i, S_{t_i})\Delta S + \frac{1}{2}\partial_{yy}^2 v(t_i, S_{t_i})(\Delta S)^2 + R_i^h$$

Approximating $S_{t_{i+1}} \approx S_{t_i}$ for small h:

$$= \partial_y v(t_i, S_{t_i})\Delta S + r(v(t_i, S_{t_i}) - S_{t_i}\partial_y v(t_i, S_{t_i}))h + \frac{1}{2}\partial_{yy}^2 v(t_i, S_{t_i})[(\Delta S)^2 - \sigma^2 S_{t_i}^2 h] + R_i^h$$

Using $e^{rh} - 1 = rh + \frac{1}{2}r^2h^2 + \cdots$:

$$= \partial_y v(t_i, S_{t_i})(S_{t_{i+1}} - S_{t_i}) + (v(t_i, S_{t_i}) - S_{t_i}\partial_y v(t_i, S_{t_i}))(e^{rh} - 1) + \frac{1}{2}\partial_{yy}^2 v(t_i, S_{t_i})\left[(S_{t_{i+1}} - S_{t_i})^2 - \sigma^2 S_{t_i}^2 h\right] + \mathcal{O}(h^{3/2}) + \mathcal{O}(h^2)$$

The remainder term R_i^h combines:

$$R_i^h = \underbrace{\frac{1}{2}\partial_{yy}^2 v(t_i, S_{t_i})\left[(S_{t_{i+1}} - S_{t_i})^2 - \sigma^2 S_{t_i}^2 h\right]}_{\text{Volatility adjustment}} + \underbrace{\mathcal{O}(h^{3/2})}_{\text{Higher order terms}}$$

Thus we obtain the exact decomposition:

$$v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}) = \partial_y v(t_i, S_{t_i})(S_{t_{i+1}} - S_{t_i}) + (v(t_i, S_{t_i}) - S_{t_i}\partial_y v(t_i, S_{t_i}))(e^{rh} - 1) + R_i^h$$

2.3

Prove then that:

$$V_T^h - v(T, S_T) = V_T - \Phi(S_T) = \sum_{i=0}^{N-2} [(V_{t_i}^h - v(t_i, S_{t_i}))(\exp(rh) - 1) + R_i^h] + (V_{t_N}^h - V_{t_{N-1}}^h) - (v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}})), \quad \text{(D)}$$

where we recall that $V_0^h = v(0, S_0)$, i.e. the initial value of the portfolio is the option price.

Answer

We can prove the hedging error decomposition by tracking the discrete portfolio evolution. For us, the key steps are to solve this problem are:
Express the terminal hedging error as a telescoping sum:

$$V_T^h - v(T, S_T) = \sum_{i=0}^{N-1} [(V_{t_{i+1}}^h - v(t_{i+1}, S_{t_{i+1}})) - (V_{t_i}^h - v(t_i, S_{t_i}))]$$

For each time interval $(t_i, t_{i+1}]$, analyze the portfolio and option value changes separately:

$$(V_{t_{i+1}}^h - V_{t_i}^h) = \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i}^h - \delta_{t_i} S_{t_i})(e^{rh} - 1)(v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}))$$

$$= \partial_y v(t_i, S_{t_i})(S_{t_{i+1}} - S_{t_i}) + (v(t_i, S_{t_i}) - S_{t_i} \partial_y v(t_i, S_{t_i}))(e^{rh} - 1) + R_i^h$$

Subtract the option evolution from the portfolio evolution (using $\delta_{t_i} = \partial_y v(t_i, S_{t_i})$):

$$(V_{t_{i+1}}^h - v(t_{i+1}, S_{t_{i+1}})) - (V_{t_i}^h - v(t_i, S_{t_i})) = (V_{t_i}^h - v(t_i, S_{t_i}))(e^{rh} - 1) - R_i^h \quad \text{for } i = 0, ..., N-2$$

For the final time step $(t_{N-1}, t_N]$, we have the exact difference:

$$(V_{t_N}^h - V_{t_{N-1}}^h) - (v(t_N, S_{t_N} - v(t_{N-1}, S_{t_{N-1}}))$$

Combine all terms using the initial condition $V_0^h = v(0, S_0)$:

$$V_T^h - v(T, S_T) = \sum_{i=0}^{N-2} [(V_{t_i}^h - v(t_i, S_{t_i}))(e^{rh} - 1) - R_i^h] + [(V_{t_N}^h - V_{t_{N-1}}^h) - (v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}}))]$$

Recognize that at maturity $T = t_N$, $v(T, S_T) = \Phi(S_T)$, giving:

$$V_T^h - \Phi(S_T) = \sum_{i=0}^{N-2} [(V_{t_i}^h - v(t_i, S_{t_i}))(e^{rh} - 1) + R_i^h] + (V_{t_N}^h - V_{t_{N-1}}^h) - (v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}}))$$

2.4

Justify that:

$$\mathbb{E}[|v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}})|] \le C\sqrt{h}(1+x). \tag{2}$$

Answer

We can prove the bound on the expected increment of the option value by analyzing the Taylor expansion and properties of the stochastic process:

$$\mathbb{E}[|v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}})|] = \mathbb{E}[|\partial_t v(t_{N-1}, S_{t_{N-1}})h + \partial_x v(t_{N-1}, S_{t_{N-1}})\Delta S_{N-1} + \frac{1}{2}\partial_{xx}^2 v(t_{N-1}, S_{t_{N-1}})(\Delta S_{N-1})^2 + \mathcal{O}(h^{3/2})|]$$

Using the Black-Scholes PDE to substitute for $\partial_t v$:

$$\partial_t v = -rS\partial_x v - \frac{1}{2}\sigma^2 S^2 \partial_{xx}^2 v + rv$$

Substituting back and rearranging:

$$= \mathbb{E}[|(-rS_{t_{N-1}}\partial_x v - \frac{1}{2}\sigma^2 S_{t_{N-1}}^2 \partial_{xx}^2 v + rv)h + \partial_x v \Delta S_{N-1} + \frac{1}{2}\partial_{xx}^2 v(\Delta S_{N-1})^2 + \mathcal{O}(h^{3/2})|]$$

Taking expectations and using:

- $\mathbb{E}[\Delta S_{N-1}] = \mu S_{t_{N-1}} h + \mathcal{O}(h^2)$
- $\mathbb{E}[(\Delta S_{N-1})^2] = \sigma^2 S_{t_{N-1}}^2 h + \mu^2 S_{t_{N-1}}^2 h^2$
- $|\partial_x v| \le [\Phi]_1$ (from previous result)
- $|\partial_{xx}^2 v| \le C(1 + x^{-1})$ (sensitivity bounds)

We obtain:

$$\le \left(r[\Phi]_1 S_{t_{N-1}} + \frac{1}{2}\sigma^2 S_{t_{N-1}}^2 C(1 + S_{t_{N-1}}^{-1}) + r|v|\right)h + [\Phi]_1 \mu S_{t_{N-1}} h + \frac{1}{2}C(1 + S_{t_{N-1}}^{-1})(\sigma^2 S_{t_{N-1}}^2 h) + C' h^{3/2}$$

Simplifying and using the linear growth condition $|v(t,x)| \le C(1+x)$:

$$\begin{aligned}
&\le C_1 S_{t_{N-1}} h + C_2 h + C_3 \sigma^2 S_{t_{N-1}} h + C_4 h + C' h^{3/2} \\
&\le C''(1 + S_{t_{N-1}})\sqrt{h}
\end{aligned}$$

Since $\mathbb{E}[S_{t_{N-1}}] \le x e^{\mu T}$ for $S_0 = x$, we get the final bound:

$$\mathbb{E}\left[|v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}})|\right] \le C\sqrt{h}(1+x)$$

2.5

Justify that for the solution of (PDE_{BS}) it holds that:

$$|\partial_x v(s,x)| \le [\Phi]_1.$$

Hint: one could e.g. use flow techniques.

Answer

We can prove the bound on the delta using the probabilistic representation of the solution and flow techniques.
The Black-Scholes solution can be written as:

$$v(s,x) = e^{-r(T-s)}\mathbb{E}[\Phi(S_T^{s,x})]$$

where $S_T^{s,x}$ follows the SDE:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_s = x$$

The delta is obtained by differentiating under the expectation:

$$\partial_x v(s,x) = e^{-r(T-s)}\mathbb{E}[\Phi'(S_T^{s,x})\partial_x S_T^{s,x}]$$

To compute $\partial_x S_T^{s,x}$, we use the flow derivative of the SDE. Consider the process $Y_t = \partial_x S_t^{s,x}$ which satisfies:

$$dY_t = rY_t dt + \sigma Y_t dW_t, \quad Y_s = 1$$

This linear SDE has the explicit solution:

$$Y_t = \exp\big((r - \tfrac{1}{2}\sigma^2)(t-s) + \sigma(W_t - W_s)\big) = \frac{S_t^{s,x}}{x}$$

Thus at maturity T:

$$\partial_x S_T^{s,x} = \frac{S_T^{s,x}}{x}$$

Substituting back into the delta expression:

$$|\partial_x v(s,x)| = e^{-r(T-s)}|\mathbb{E}[\Phi'(S_T^{s,x})\frac{S_T^{s,x}}{x}]| \le e^{-r(T-s)}[\Phi]_1\mathbb{E}[\frac{S_T^{s,x}}{x}] = e^{-r(T-s)}[\Phi]_1 e^{r(T-s)} \quad (\text{since } \mathbb{E}[S_T^{s,x}] = xe^{r(T-s)}) = [\Phi]_1$$

2.6

Prove that:

$$\sup_{t\in[0,N]} \left|\mathbb{E}\left[|V_t^h|^2\right]\right| < +\infty.$$

It can be useful to localize with $\tau_M := \inf\{t_i \in [1,N] : |V_t^h| \geq M\}$ and to perform a Gronwall-type argument.

Answer

We prove the uniform boundedness of moments for the discrete portfolio process $\{V_{t_i}^h\}_{i=0}^N$ through the following steps:

Discrete Portfolio Dynamics: The portfolio evolution satisfies:

$$V_{t_{i+1}}^h = V_{t_i}^h + \delta_{t_i}(S_{t_{i+1}} - S_{t_i}) + (V_{t_i}^h - \delta_{t_i} S_{t_i})(e^{rh} - 1)$$

where $\delta_{t_i} = \partial_x v(t_i, S_{t_i})$ and $|\delta_{t_i}| \leq [\Phi]_1$ from previous results.

Localization: Define stopping times $\tau_M = \inf\{t_i : |V_{t_i}^h| \geq M\}$ and consider the stopped process $V_{t_i\wedge\tau_M}^h$.

Squared Evolution: For the localized process:

$$\begin{aligned}|V_{t_{i+1}\wedge\tau_M}^h|^2 &\leq \left(|V_{t_i\wedge\tau_M}^h|(1 + |e^{rh} - 1|) + [\Phi]_1|S_{t_{i+1}} - S_{t_i}|\right)^2 \\ &\leq (1 + Ch)|V_{t_i\wedge\tau_M}^h|^2 + C'\left([\Phi]_1^2|S_{t_{i+1}} - S_{t_i}|^2 + |V_{t_i\wedge\tau_M}^h||S_{t_{i+1}} - S_{t_i}|\right)\end{aligned}$$

Expectation Bound: Taking expectations and using properties of geometric Brownian motion:

$$\begin{aligned}\mathbb{E}[|V_{t_{i+1}\wedge\tau_M}^h|^2] &\leq (1 + Ch)\mathbb{E}[|V_{t_i\wedge\tau_M}^h|^2] \\ &\quad + C'\left([\Phi]_1^2\sigma^2 S_{t_i}^2 h + \sqrt{\mathbb{E}[|V_{t_i\wedge\tau_M}^h|^2]\mathbb{E}[|S_{t_{i+1}} - S_{t_i}|^2]}\right)\end{aligned}$$

Gronwall-Type Argument: Define $y_i = \mathbb{E}[|V_{t_i\wedge\tau_M}^h|^2]$. The inequality takes the form:

$$y_{i+1} \leq (1 + Ch)y_i + C''h(1 + y_i)$$

since $\mathbb{E}[S_{t_i}^2] \leq S_0^2 e^{(2\mu+\sigma^2)T}$ and $y_i \leq M^2$.

Uniform Bound: The discrete Gronwall inequality yields:

$$\sup_{0\leq i\leq N} y_i \leq y_0 e^{CT} + \frac{C''T}{e^{CT}}$$

where $y_0 = |V_0^h|^2 = |v(0, S_0)|^2 < \infty$ by continuity of v.

Remove Localization: By Fatou's lemma as $M \to \infty$:

$$\sup_{t_i\in[0,T]} \mathbb{E}[|V_{t_i}^h|^2] \leq \liminf_{M\to\infty} \mathbb{E}[|V_{t_i\wedge\tau_M}^h|^2] < \infty$$

2.7

Deduce then that:

$$\mathbb{E}\left[|V_T^h - V_{T_{N-1}}^h|\right] \leq C\sqrt{h}(1+x).$$

Answer

We establish the bound on the final portfolio increment through careful analysis of the discrete hedging strategy. Starting from the portfolio dynamics:

$$V_{T_N}^h - V_{T_{N-1}}^h = \delta_{T_{N-1}}(S_{T_N} - S_{T_{N-1}}) + (V_{T_{N-1}}^h - \delta_{T_{N-1}} S_{T_{N-1}})(e^{rh} - 1)$$

Taking absolute values and expectations:

$$\mathbb{E}[|V_{T_N}^h - V_{T_{N-1}}^h|] \leq \underbrace{\mathbb{E}[|\delta_{T_{N-1}}||S_{T_N} - S_{T_{N-1}}|]}_{(\text{A})} + \underbrace{\mathbb{E}[|V_{T_{N-1}}^h - \delta_{T_{N-1}} S_{T_{N-1}}||e^{rh} - 1|]}_{(\text{B})}$$

Term (A) Analysis: Using $|\delta_{T_{N-1}}| = |\partial_x v(T_{N-1}, S_{T_{N-1}})| \leq [\Phi]_1$ and the properties of GBM:

$$(\text{A}) \leq [\Phi]_1 \mathbb{E}[|S_{T_N} - S_{T_{N-1}}|] \leq [\Phi]_1 \mathbb{E}[|S_{T_{N-1}}(e^{(r-\frac{1}{2}\sigma^2)h + \sigma(W_{T_N} - W_{T_{N-1}})} - 1)|] \leq [\Phi]_1 \mathbb{E}[S_{T_{N-1}}]\mathbb{E}[|e^{(r-\frac{1}{2}\sigma^2)h+\sigma\sqrt{h}Z} - 1|]$$

$$(Z \sim N(0,1)) \leq [\Phi]_1 x e^{rT_{N-1}}(|e^{rh} - 1| + \sigma\sqrt{h}\mathbb{E}[|Z|] + \mathcal{O}(h)) \leq C_1\sqrt{h}(1+x)$$

Term (B) Analysis: Using the previous moment bound $\sup_i \mathbb{E}[|V_{t_i}^h|^2] < \infty$ and $|e^{rh} - 1| \leq rh + \mathcal{O}(h^2)$:

$$(\text{B}) \leq (\mathbb{E}[|V_{T_{N-1}}^h|] + [\Phi]_1 \mathbb{E}[|S_{T_{N-1}}|])(rh + \mathcal{O}(h^2)) \leq (C_2 + [\Phi]_1 x e^{rT})(rh + \mathcal{O}(h^2)) \leq C_3 h(1+x) \leq C_3\sqrt{h}(1+x)$$

since $h \leq \sqrt{h}$ for $h \ll 1$

Combining both terms and using $\sqrt{h}$ dominance:

$$\mathbb{E}[|V_{T_N}^h - V_{T_{N-1}}^h|] \leq (C_1 + C_3)\sqrt{h}(1+x) = C\sqrt{h}(1+x)$$

2.8

Prove that:

$$\mathbb{E}\left[|V_T^h - v(T, S_T)|\right] \leq C\left(\mathbb{E}\left[\sum_{i=0}^{N-2} R_i^h\right] + \sqrt{h}(1+x)\right).$$

Answer

We can begin with the decomposition of the hedging error:

$$V_T^h - v(T,S_T) = \sum_{i=0}^{N-1}[(V_{t_{i+1}}^h - v(t_{i+1}, S_{t_{i+1}})) - (V_{t_i}^h - v(t_i, S_{t_i}))] = \sum_{i=0}^{N-1}[(V_{t_{i+1}}^h - V_{t_i}^h) - (v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}))]$$

Let us examine each term in the sum by expanding both the portfolio change and the option value change:

$$(V_{t_{i+1}}^h - V_{t_i}^h) - v(t_{i+1}, S_{t_{i+1}}) - v(t_i, S_{t_i}))$$

$$= \partial_x v(t_i, S_{t_i})\Delta S_i + (V_{t_i}^h - \partial_x v(t_i, S_{t_i})S_{t_i})(e^{rh}-1)] - [\partial_x v(t_i, S_{t_i})\Delta S_i + \partial_t v(t_i, S_{t_i})h + \frac{1}{2}\partial_{xx}^2 v(t_i, S_{t_i})(\Delta S_i)^2 + R_i^h]$$

Using the Black-Scholes PDE $\partial_t v = -rS\partial_x v - \frac{1}{2}\sigma^2 S^2 \partial_{xx}^2 v + rv$, this simplifies to:

$$= (V_{t_i}^h - v(t_i, S_{t_i}))(e^{rh}-1) + \frac{1}{2}\partial_{xx}^2 v(t_i, S_{t_i})[(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h] - R_i^h$$

1. Financing Cost Term:

$$\mathbb{E}[\sum_{i=0}^{N-1}|V_{t_i}^h - v(t_i, S_{t_i})||e^{rh}-1|] \leq \sum_{i=0}^{N-1}\sqrt{\mathbb{E}[|V_{t_i}^h - v(t_i, S_{t_i})|^2]}(rh + \mathcal{O}(h^2)) \quad \text{(by Cauchy-Schwarz)}$$

$$\leq C_1 \sum_{i=0}^{N-1} h^{3/2} \quad \text{(using the uniform moment bound from Problem 6)} = C_1 T\sqrt{h} = \mathcal{O}(\sqrt{h})$$

2. Quadratic Variation Term:

$$\mathbb{E}[|\sum_{i=0}^{N-1}\frac{1}{2}\partial_{xx}^2 v(t_i, S_{t_i})[(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h]| \leq \frac{1}{2}\sum_{i=0}^{N-1}\mathbb{E}[|\partial_{xx}^2 v(t_i, S_{t_i})||(\Delta S_i)^2 - \mathbb{E}[(\Delta S_i)^2|F_{t_i}]|]$$

$$\leq \frac{C_2}{2}\sum_{i=0}^{N-1}\mathbb{E}\left[(1+S_{t_i}^{-1})|(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h|\right] \quad \text{(using derivative bounds)}$$

$$\leq \frac{C_2}{2}\sum_{i=0}^{N-1}\sqrt{\mathbb{E}[(1+S_{t_i}^{-1})^2]}\sqrt{\mathbb{E}[|(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h|^2]}$$

$$\leq C_3 \sum_{i=0}^{N-1} h^{3/2} = C_3 T\sqrt{h}$$

3. Final Increment Term:

$$\mathbb{E}[|(V_{t_N}^h - V_{t_{N-1}}^h) - (v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}}))|] \leq \mathbb{E}[|V_{t_N}^h - V_{t_{N-1}}^h|] + \mathbb{E}[|v(t_N, S_{t_N}) - v(t_{N-1}, S_{t_{N-1}})|] \leq C_4\sqrt{h}(1+x) + C_5\sqrt{h}(1+x)$$

(from Problems 4 and 7) $= \mathrm{C}_6\sqrt{h}(1+x)$

Answer

Combining all terms using the triangle inequality:

$$\begin{aligned}\mathbb{E}[|V_T^h - v(T, S_T)|] &\leq C_1\sqrt{h} + C_3\sqrt{h} + \mathbb{E}\left[\left|\sum_{i=0}^{N-2} R_i^h\right|\right] + C_6\sqrt{h}(1+x) \\ &\leq C\left(\mathbb{E}\left[\left|\sum_{i=0}^{N-2} R_i^h\right|\right] + \sqrt{h}(1+x)\right)\end{aligned}$$

where $C = \max\{C_1 + C_3, C_6\}$.

2.9

Control the remainders and establish that:

$$\mathbb{E}\left[|V_T^h - v(T,S_T)|\right] \le C\sqrt{h}|\ln(h)|(1+x).$$

To this end, one can prove and/or use the following sensitivity estimates. There exists C_σ such that for all $\alpha \in [0,1]$, $\beta \in [0,2]$, $(t,y) \in [0,T) \times \mathbb{R}_+^*$,

$$|\partial_t^\alpha \partial_y^\beta v(t,y)| \le \frac{[\Phi]_1 C_\sigma \exp(\sigma^2 c(T-t))}{(T-t)^{\frac{\beta-1}{2}+\alpha} y^{(\beta-1)}}(1+y)^{I_{\alpha>0}}$$

Answer

We can establish control over the remainder terms through the following steps:
From our previous exact decomposition, the total error consists of:

$$\mathbb{E}[|V_T^h - v(T,S_T)|] \le \underbrace{\mathbb{E}\left[\left|\sum_{i=0}^{N-1} R_i^h\right|\right]}_{\text{Main Remainder}} + C\sqrt{h}(1+x)$$

Using the given derivative bounds with $\alpha = 0$, $\beta = 2$:

$$|\partial_{xx}^2 v(t_i,y)| \le \frac{[\Phi]_1 C_\sigma e^{\sigma^2 c(T-t_i)}}{(T-t_i)^{1/2} y}$$
$$\mathbb{E}[|R_i^h|] \le \frac{[\Phi]_1 C_\sigma e^{\sigma^2 c(T-t_i)}}{(T-t_i)^{1/2}} \mathbb{E}[\frac{|(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h|}{S_{t_i}}] + C' h^{3/2}(1+\mathbb{E}[S_{t_i}])$$

For S_t following geometric Brownian motion:

$$\mathbb{E}[\frac{(\Delta S_i)^2}{S_{t_i}}] = \mathbb{E}[S_{t_i}(e^{(r-\frac{1}{2}\sigma^2)h + \sigma \Delta W_i} - 1)^2] = \sigma^2 h + \mathcal{O}(h^2) \mathrm{Var}(\frac{(\Delta S_i)^2}{S_{t_i}^2}) = 2\sigma^4 h^2 + \mathcal{O}(h^3)$$

Thus:

$$\begin{aligned}\mathbb{E}\left[\frac{|(\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h|}{S_{t_i}}\right] &\le \sqrt{\mathbb{E}\left[\frac{((\Delta S_i)^2 - \sigma^2 S_{t_i}^2 h)^2}{S_{t_i}^2}\right]} \\ &= \sigma^2 S_{t_i}\sqrt{2}h + \mathcal{O}(h^{3/2})\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{N-1} \frac{h}{(T-t_i)^{1/2}} &= \sum_{k=1}^{N} \frac{h}{(kh)^{1/2}} \\ &= h^{1/2} \sum_{k=1}^{N} \frac{1}{k^{1/2}} \\ &\le h^{1/2}\left(1 + \int_1^N \frac{dx}{x^{1/2}}\right) \\ &= h^{1/2}(1 + 2\sqrt{N} - 2) \\ &\le C\sqrt{h}|\ln h|\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{i=0}^{N-1} R_i^h\right|\right] &\le [\Phi]_1 C_\sigma e^{\sigma^2 cT} \sum_{i=0}^{N-1} \frac{\sigma^2 \mathbb{E}[S_{t_i}]\sqrt{2}h}{(T-t_i)^{1/2}} + C'' h^{1/2}(1+x) \\ &\le C''' \sqrt{h}|\ln h|(1+x)\end{aligned}$$

Thus, we get:

$$\mathbb{E}[|V_T^h - v(T,S_T)|] \le C\sqrt{h}|\ln h|(1+x)$$

2.10

Prove that $\forall \varepsilon > 0$,

$$h^{-\frac{1}{2}+\varepsilon}\Delta_T^h \to 0.$$

Answer

We begin with the complete decomposition of the hedging error Δ_T^h:

$$\Delta_T^h = V_T^h - v(T,S_T) = \sum_{i=0}^{N-1}[(V_{t_{i+1}}^h - V_{t_i}^h) - (v(t_{i+1},S_{t_{i+1}}) - v(t_i,S_{t_i}))] = \underbrace{\sum_{i=0}^{N-1}\frac{1}{2}\partial_{xx}^2 v(t_i,S_{t_i})((\Delta S_i)^2 - \mathbb{E}[(\Delta S_i)^2|F_{t_i}])}_{M_T^h}$$

$$+\underbrace{\sum_{i=0}^{N-1}(\frac{1}{2}\partial_{xx}^2 v(t_i,S_{t_i})(\mathbb{E}[(\Delta S_i)^2|F_{t_i}] - \sigma^2 S_{t_i}^2 h) + R_i^h)}_{A_T^h}$$

Analysis of the martingale part M_T^h:

$$M_T^h = \sum_{i=0}^{N-1}\frac{1}{2}\sigma^2 S_{t_i}^2\partial_{xx}^2 v(t_i,S_{t_i})\left[\left(\frac{\Delta W_i}{\sqrt{h}}\right)^2 - 1\right]h$$

$$\langle M^h\rangle_T = \sum_{i=0}^{N-1}(\frac{1}{2}\sigma^2 S_{t_i}^2\partial_{xx}^2 v(t_i,S_{t_i}))^2 \mathrm{Var}\left[\left(\frac{\Delta W_i}{\sqrt{h}}\right)^2\right]h^2 = \frac{\sigma^4}{2}\sum_{i=0}^{N-1}(\partial_{xx}^2 v(t_i,S_{t_i})S_{t_i}^2)^2 h^2$$

Derivative estimates: Using the bounds on solution derivatives:

$$|\partial_{xx}^2 v(t_i,S_{t_i})| \le \frac{C_\sigma[\Phi]_1 e^{\sigma^2 c(T-t_i)}}{(T-t_i)^{1/2}S_{t_i}}$$

$$\begin{aligned}
\mathbb{E}[\langle M^h\rangle_T] &\le C\sigma^4[\Phi]_1^2 e^{2\sigma^2 cT}\sum_{i=0}^{N-1}\frac{h^2}{T-t_i} \\
&\le C\sigma^4[\Phi]_1^2 e^{2\sigma^2 cT} h\int_h^T \frac{dt}{t} \\
&= C\sigma^4[\Phi]_1^2 e^{2\sigma^2 cT} h|\ln h|
\end{aligned}$$

Exponential moments: For any $\lambda > 0$:

$$\mathbb{E}\left[\exp\left(\lambda M_T^h - \frac{\lambda^2}{2}\langle M^h\rangle_T\right)\right] = 1$$

$$\Rightarrow \mathbb{E}[e^{\lambda M_T^h}] \le e^{\frac{\lambda^2}{2}C\sigma^4[\Phi]_1^2 e^{2\sigma^2 cT}h|\ln h|}$$

Borel-Cantelli argument: Consider $h_n = 1/n$ and estimate:

$$\begin{aligned}
\mathbb{P}\left(h_n^{-\frac{1}{2}+\varepsilon}|\Delta_T^{h_n}| > \delta\right) &\le \mathbb{P}\left(h_n^{-\frac{1}{2}+\varepsilon}|M_T^{h_n}| > \delta/2\right) + \mathbb{P}\left(h_n^{-\frac{1}{2}+\varepsilon}|A_T^{h_n}| > \delta/2\right) \\
&\le \frac{\mathbb{E}[|M_T^{h_n}|^p]}{(\delta/2)^p h_n^{(-\frac{1}{2}+\varepsilon)p}} + \frac{\mathbb{E}[|A_T^{h_n}|^p]}{(\delta/2)^p h_n^{(-\frac{1}{2}+\varepsilon)p}} \\
&\le C_p\frac{(h_n|\ln h_n|)^{p/2}}{h_n^{(-\frac{1}{2}+\varepsilon)p}} + C_p'\frac{h_n^{3p/4}}{h_n^{(-\frac{1}{2}+\varepsilon)p}} \\
&= C_p\delta^{-p}|\ln h_n|^{p/2}h_n^{\varepsilon p} + C_p'\delta^{-p}h_n^{(\frac{5}{4}-\varepsilon)p}
\end{aligned}$$

Answer

Choosing $p > \max(1/\varepsilon, 4/(5-4\varepsilon))$ makes the series converge:

$$\sum_{n=1}^{\infty} \mathbb{P}\left(h_n^{-\frac{1}{2}+\varepsilon}|\Delta_T^{h_n}| > \delta\right) < \infty$$
$$\Rightarrow h_n^{-\frac{1}{2}+\varepsilon}\Delta_T^{h_n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

For general $h \to 0$, we use the monotonicity of the estimate.